QUANTIZATION DIMENSION FOR INFINITE CONFORMAL ITERATED FUNCTION SYSTEMS

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Abstract. The quantization dimension function for an F-conformal measure m_F generated by an infinite conformal iterated function system satisfying the strong open set condition and by a summable Hölder family of functions is expressed by a simple formula involving the temperature function of the system. The temperature function is commonly used to perform a multifractal analysis, in our context of the measure m_F. The result in this paper extends a similar result of Lindsay and Mauldin established for finite conformal iterated function systems.

1. Introduction

Various types of dimensions such as Hausdorff and packing dimension or the lower and upper box-counting dimension are important to characterize the complexity of highly irregular sets. In the past decades, a lot of research has been done aiming at the calculation of these dimensions for various special cases or establishing some significant properties. In recent years, paralleling methods have been adopted to study the corresponding dimensions for measures (cf. [F]). In this paper, we study the quantization dimension for probability measures. The quantization problem consists in studying the quantization error induced by the approximation of a given probability measure with discrete probability measures of finite supports. This problem originated in information theory and some engineering technology. A detailed account of this theory can be found in [GL1]. Given a Borel probability measure μ on $\mathbb{R}^d$, a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the nth *quantization error of order* r for μ is defined by

$$V_{n,r}(\mu) := \inf\{\int d(x, \alpha)^r d\mu(x) : \alpha \subset \mathbb{R}^d, \operatorname{card}(\alpha) \leq n\},$$

where $d(x, \alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^d$. We note that if $\int \|x\|^r d\mu(x) < \infty$ then there is some set α for which the infimum is achieved (cf. [GL1]). The set α for which the infimum is achieved is called an *optimal set of n-means* or *n-optimal set* of order r for $0 < r < +\infty$. The *upper and lower quantization dimension* of order r for μ is defined to be

$$\overline{D}_r(\mu) := \limsup_{n\to\infty} \frac{r \log n}{-\log V_{n,r}(\mu)}; \ \underline{D}_r(\mu) := \liminf_{n\to\infty} \frac{r \log n}{-\log V_{n,r}(\mu)}.$$

If $\overline{D}_r(\mu)$ and $\underline{D}_r(\mu)$ coincide, we call the common value the *quantization dimension* of order r for the probability measure μ, and is denoted by $D_r := D_r(\mu)$. One sees that the quantization dimension is actually a function $r \mapsto D_r$ which measures the asymptotic rate at which $V_{n,r}$ goes to zero. If D_r exists, then one can write

$$\log V_{n,r} \sim \log(\frac{1}{n})^{r/D_r}.$$

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Graf and Luschgy first determined the quantization dimension for a finite system of self-similar mappings associated with a probability vector (cf. [GL1, GL2]). Lindsay and Mauldin extended the above result to the F-conformal measure m associated with a conformal iterated function system determined by finitely many conformal mappings (cf. [LM]). Later, quantization dimension was determined for many other fractal probability measures, for example one could see [R1, R2, R3, R4, WD]. But, in each case the number of mappings considered was finite. Quantization dimension for a fractal probability measure generated by infinite mappings was a long time open problem. In [R6], Roychowdhury first determined the quantization dimension for a self-similar measure generated by infinite similitudes $(S_n)_{n\geq 1}$ with similarity ratios $(s_n)_{n\geq 1}$ respectively, associated with a probability vector $(p_n)_{n\geq 1}$, although an extra assumption is needed for the existence of the function $P(q,t) := \sum_{j=1}^{\infty} p_j^q s_j^t$ that occurred in that paper. In this paper, we give an extension of Lindsay and Mauldin's (cf. [LM]) results to the realm of itereted function systems with a countable infinite alphabet. The probability measure m_F considered here is the F-conformal measure associated with a summable Hölder family of functions $F := \{f^{(i)} : X \to \mathbb{R}, i \in I\}$ and a conformal iterated function system $\{\varphi_i : X \to X, i \in I\}$, where I is a countable set, called alphabet, with finitely many, or what we want to emphasize, infinitely many, elements. We show that for this measure m_F, the quantization dimension exists and is uniquely determined by the following formula:

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in I^n} \left(\| \exp(S_\omega(F))\| \|\varphi'_\omega\|^r\right)^{\frac{D_r}{r+D_r}} = 0. \tag{1}$$

The multifractal formalism for a probability measure corresponding to a two parameter family of Hölder continuous functions

$$G_{q,t} := \{g_{q,t}^{(i)} := qf^{(i)} + t \log |\varphi'_i| : i \in I\}$$

does indeed hold if I is a countable set (cf. [HMU, MU]). In particular, the singularity exponent $\beta(q)$ (also known as the temperature function) satisfies the usual equation

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in I^n} \| \exp(S_\omega(F))\|^q \|\varphi'_\omega\|^{\beta(q)} = 0, \tag{2}$$

and that the spectrum $f(\alpha)$ is the Legendre transform of $\beta(q)$. Comparing (1) and (2), we see that if $q_r = \frac{D_r}{r+D_r}$, then $\beta(q_r) = rq_r$, that is, the quantization dimension function for an infinite self-conformal measure has a relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis (for thermodynamic formalism, multifractal analysis and the Legendre transform one could see [F, HMU]). The result in this paper is an infinite extension of Lindsay and Mauldin (cf. [LM]).

2. Basic definitions and lemmas

In this paper, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space equipped with a metric d. Let us write,

$$V_{n,r} = V_{n,r}(\mu) := \inf\{\int d(x,\alpha)^r d\mu(x) : \alpha \subset \mathbb{R}^d, \ \text{card}(\alpha) \leq n\}.$$

A set $\alpha \subset \mathbb{R}^d$ with $\text{card}(\alpha) \leq n$ is called an *n-optimal set of centers for μ of order r* or *$V_{n,r}(\mu)$-optimal set* if

$$V_{n,r}(\mu) = \int d(x,\alpha)^r d\mu(x).$$

As stated before, n-optimal sets exist when $\int \|x\|^r d\mu(x) < \infty$.

Let X be a nonempty compact subset of $\mathbb{R}^d$ and I be a countable set with infinitely many elements. Without any loss of generality we can take $I = \{1, 2, \cdots\}$ the set of natural numbers.

Let $S = \{\varphi_i : i \in I\}$ be a collection of injective contractions from X into X for which there exists $0 < s < 1$ such that

$$d(\varphi_i(x), \varphi_i(y)) \leq sd(x, y) \tag{3}$$

for every $i \in I$ and every pair of points $x, y \in X$. Thus the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. Put $I^* = \bigcup_{n \geq 1} I^n$ and for $\omega = \omega_1\omega_2 \cdots \omega_n \in I^n$, $n \geq 1$, set

$$\varphi_\omega = \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n}.$$

If $\omega \in I^n$ we say n is the length of ω. We have made the convention that the empty word $\emptyset$ is the only word of length 0 and $\varphi_\emptyset = \mathrm{Id}_X$. If $\omega \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \cdots \omega_n$. Observe now that given $\omega \in I^\infty$, the compact sets $\varphi_{\omega|_n}(X)$, $n \geq 1$, are decreasing and their diameters converge to zero. In fact, by (3)

$$\mathrm{diam}(\varphi_{\omega|_n}(X)) \leq s^n \mathrm{diam}(X). \tag{4}$$

It implies that the set

$$\pi(\omega) = \bigcap_{n=0}^{\infty} \varphi_{\omega|_n}(X)$$

is a singleton and therefore this formula defines a map $\pi : I^\infty \to X$ which, in view of (4) is continuous. The main object of our interest will be the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega \in I^\infty} \bigcap_{n=0}^{\infty} \varphi_{\omega|_n}(X).$$

Let $\sigma : I^\infty \to I^\infty$ denote the left shift map (cutting out the first coordinate) on I^∞, i.e., $\sigma(\omega) = \omega_2\omega_3 \cdots$. Note that $\pi \circ \sigma(\omega) = \varphi_{\omega_1}^{-1} \circ \pi(\omega)$, and hence rewriting $\pi(\omega) = \varphi_{\omega_1}(\pi(\sigma(\omega)))$, we see that

$$J = \bigcup_{i \in I} \varphi_i(J).$$

The set J is called the *infinite self-conformal set* corresponding to the infinite conformal iterated function system S. Notice that if I is finite, then J is compact and this property fails for infinite systems. An iterated function system satisfies the Open Set Condition if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\varphi_i(U) \subset U$ for every $i \in I$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$. Furthermore, the system satisfies the strong open set condition if U can be chosen such that $U \cap J \neq \emptyset$.

An iterated function system satisfying the Open Set Condition is said to be conformal if the following conditions are satisfied:

(i) $U = \mathrm{Int}_{\mathbb{R}^d}(X)$.

(ii) There exists an open connected set V with $X \subset V \subset \mathbb{R}^d$ such that all maps φ_i, $i \in I$, extend to C^1-conformal diffeomorphisms of V into V.

(iii) There exist $\gamma, \ell > 0$ such that for every $x \in \partial X \subset \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \gamma, \ell) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude ℓ.

(iv) Bounded Distortion Property (BDP): There exists $C \geq 1$ such that

$$|\varphi'_\omega(y)| \leq C|\varphi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\varphi'_\omega(x)|$ means the norm of the derivative.

Inequality (3) implies that for every $i \in I$,

$$\|\varphi'_i\| = \sup_{x \in X} |\varphi'_i(x)| = \sup_{x \in X} \lim_{y \to x} \frac{d(\varphi_i(y), \varphi_i(x))}{d(y, x)} \leq \sup_{x \in X} \frac{sd(x, y)}{d(x, y)} = s,$$

and hence $\|\varphi'_\omega\| \leq s^n$ for every $\omega \in I^n$, $n \geq 1$. For $t \geq 0$, the topological pressure function for a conformal iterated function system $S = \{\varphi_i : i \in I\}$ is given by

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in I^n} \|\varphi'_\omega\|^t,$$

provided the limit exists. As it was shown in [MU1], there are two disjoint classes of conformal iterated function systems, regular and irregular. A system is regular if there exists $t \geq 0$ such that $P(t) = 0$. Otherwise the system is irregular. Moreover, if S is a conformal iterated function system, then

$$\dim_H(J) = \sup\{\dim_H(J_F) : F \subset I, F \text{ finite}\} = \inf\{t \geq 0 : P(t) \leq 0\},$$

where $\dim_H(J)$ represents the Hausdorff dimension of the limit set J and J_F is the limit set associated to the index set F. If a system is regular and $P(t) = 0$, then $t = \dim_H(J)$. Let us assume that the conformal iterated function system considered in this paper is regular.

Let $F = \{f^{(i)} : X \to \mathbb{R}\}_{i \in I}$ be a family of continuous functions such that if we define for each $n \geq 1$,

$$V_n(F) = \sup_{\omega \in I^n} \sup_{x,y \in X} \left\{|f^{(\omega_1)}(\varphi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\varphi_{\sigma(\omega)}(y))|\right\} e^{\beta(n-1)}$$

for some $\beta > 0$, then the following is satisfied:

$$V_\beta(F) = \sup_{n \geq 1} \{V_n(F)\} < \infty. \tag{5}$$

The collection F is called then a *Hölder family of functions* (of order β). Denote by $\|\cdot\|_0$ the supremum norm on the Banach space $\mathcal{C}(X)$ and by $1\!\!1$ the function with constant value 1 on X. If in addition to (5) we have

$$\sum_{i \in I} \|e^{f^{(i)}}\|_0 < \infty \text{ or equivalently } \mathcal{L}_F(1\!\!1) \in \mathcal{C}(X),$$

where

$$\mathcal{L}_F(g)(x) = \sum_{i \in I} e^{f^{(i)}(x)} g(\varphi_i(x)), \quad g \in \mathcal{C}(X),$$

is the associated Perron-Frobenius or transfer operator, then F is called a *summable Hölder family of functions* (of order β). It was originally in [HMU], and called a strongly Hölder family of functions. In our paper, we assume that F is a summable Hölder family of functions of order β.

For $n \geq 1$ and $\omega \in I^n$, set $S_\omega(F) := \sum_{j=1}^n f^{(\omega_j)} \circ \varphi_{\sigma^j(\omega)}$. Then following the classical thermodynamic formalism, the *topological pressure* of F is defined by

$$P(F) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in I^n} \|\exp(S_\omega(F))\|_0.$$

The limit above exists by the standard theory of sub-additive sequences. Subtracting from each of the functions $f^{(i)}$ the topological pressure of F we may assume that $P(F) = 0$. By [HMU], there exists a unique Borel probability measure m_F on X such that for any continuous function $g : X \to \mathbb{R}$ and $n \geq 1$,

$$\int g dm_F = \sum_{\omega \in I^n} \int \exp(S_\omega(F)) \cdot (g \circ \varphi_\omega) dm_F.$$

In particular, for any Borel set $A \subset J$ and $\tau \in I^n$, $n \geq 1$, we have

$$\begin{aligned} m_F(\varphi_\tau(A)) &= \sum_{\omega \in I^n} \int \exp(S_\omega(F)(x)) \cdot (I_{\varphi_\tau(A)} \circ \varphi_\omega(x)) dm_F(x) \\ &= \int \exp(S_\tau(F)(x)) \cdot (I_{\varphi_\tau(A)} \circ \varphi_\tau(x)) dm_F(x) \\ &= \int_A \exp(S_\tau(F)(x)) dm_F(x). \end{aligned}$$

Moreover, m_F satisfies $m_F(\varphi_\omega(X) \cap \varphi_\tau(X)) = 0$ for all incomparable words $\omega, \tau \in I^*$. The probability measure m_F is called the *F-conformal measure* of the (possibly) infinite conformal iterated function system S and the summable Hölder family of functions $F = \{f^{(i)} : X \to \mathbb{R}, i \in I\}$. Let us now consider a two-parameter family of Hölder continuous functions

$$G_{q,t} = \{g^{(i)}(q,t) := qf^{(i)} + t \log |\varphi_i'|\}_{i \in I}.$$

The *topological pressure* corresponding to $G_{q,t}$ is given by

$$P(q,t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \| \exp(S_\omega(F)) \|^q \|\varphi_\omega'\|^t. \tag{6}$$

The limit above exists by the standard theory of sub-additive sequences. We assume that for every $q \in [0,1]$ there exists $u \in \mathbb{R}$ such that

$$0 < P(q,u) < +\infty. \tag{7}$$

For $q = 0$ this simply means that the system S is strongly regular; see [MU1] for a detailed discussion of this concept. Let

$$\mathrm{Fin}(q) = \{t \in \mathbb{R} : \mathcal{L}_{G_{q,t}}(\mathbb{1}) < \infty\} = \{t \in \mathbb{R} : P(q,t) < \infty\} \text{ and } \theta(q) = \inf \mathrm{Fin}(q).$$

Notice that either $\mathrm{Fin}(q) = (\theta(q), +\infty)$ or $\mathrm{Fin}(q) = [\theta(q), +\infty)$ The following lemma is easy to prove (cf. [HMU], Lemma 7.1).

Lemma 2.1. *For every $q \in \mathbb{R}$ the function $(\theta(q), +\infty) \ni q \mapsto P(q,t)$ is strictly decreasing, convex and hence continuous.*

With the use of (7), the proof of [HMU, Lemma 7.2] gives the following.

Lemma 2.2. *If $q \in [0,1]$, then there exists a unique $t = \beta(q) \in \mathbb{R}$ such that $P(q, \beta(q)) = 0$.*

By [HMU, Theorem 7.4], the function β is strictly decreasing, convex and hence continuous on $[0,1]$. This function is commonly called the *temperature function* of the thermodynamic formalism under consideration.

Note 2.3. Since the system S is strongly regular $\beta(0) = \dim_{\mathrm{H}}(J)$ of the limit set J (cf. [HMU]). Moreover, $P(1,0) = 0$, which gives $\beta(1) = 0$ (see Figure 1).

3. Main result

For a given $r \in (0, +\infty)$ consider the function $g : (0,1] \to \mathbb{R}$ given by the formula

$$g(x) = \frac{\beta(x)}{rx}.$$

We know that $\beta(1) = 0$ and $\beta(0) = \dim_{\mathrm{H}}(J)$, and so $g(1) = 0$ and $\lim_{x \to 0+} g(x) = +\infty$. Moreover, the function g is continuous, even differentiable, and strictly decreasing (calculate its derivative which is negative since $\beta' < 0$) on $(0,1]$. Hence there exists a unique $q_r \in (0,1)$ such that $g(q_r) = 1$, i.e.,

$$\beta(q_r) = rq_r.$$

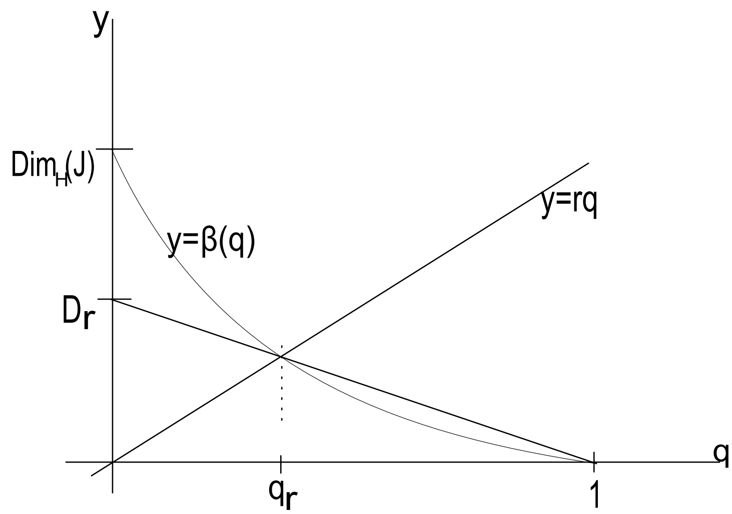

FIGURE 1. To determine D_r first find the point of intersection of $y = \beta(q)$ and the line $y = rq$. Then D_r is the y-intercept of the line through this point and the point (1, 0).

The relationship between the quantization dimension function and the temperature function $\beta(q)$ for the F-conformal measure m, where the temperature function is the Legendre transform of the $f(\alpha)$ curve (for the definitions of $f(\alpha)$ and the Legendre transform see [F, HMU]) is given by the following theorem which constitutes the main result of our paper. For its graphical description see Figure 1.

Theorem 3.1. *Let m be the infinite F-conformal measure associated with the infinite family of strongly Hölder functions $F := \{f^{(i)} : X \to \mathbb{R}\}_{i\in I}$ and the infinite conformal iterated function system S. For each $r \in (0, +\infty)$ the quantization dimension (of order r) of the probability measure m_F is given by*

$$D_r(m_F) = \frac{\beta(q_r)}{1 - q_r},$$

where, we recall β is the temperature function.

Lemma 3.2. *Let $0 < r < +\infty$ be given. Then there exists exactly one number $\kappa_r \in (0, +\infty)$ such that*

$$P\left(\frac{\kappa_r}{r + \kappa_r}, \frac{r\kappa_r}{r + \kappa_r}\right) = 0.$$

Proof. Just take $\kappa_r := \frac{rq_r}{1-q_r}$ and apply the definition of q_r. □

For every $M \geq 2$ write $I_M = \{1, 2, \cdots, M\}$, and consider the partial system

$$S_M = \{\varphi_i : X \to X : i \in I_M\}$$

of the infinite system S. Let J_M be its limit set. Consider also the partial Hölder family of functions

$$F_M = \{f^{(i)} : X \to \mathbb{R} : i \in I_M\}$$

of the infinite family F. Let m_M be the corresponding F_M-conformal measure on J_M. Furthermore, let $P_M(q, t)$ be the topological pressure and $\beta_M(q)$ be the temperature function associated with the system S_M. Note that for each $M \geq 2$, $P_M(q,t)$ is strictly decreasing, convex and hence continuous in each variable $q, t \in \mathbb{R}$ separately (cf. [F1, P]).

The following lemma is a special case of the Lemma 3.2.

Lemma 3.3. *Let $0 < r < +\infty$ and $M \geq 2$ be as before. Then there exists exactly one number $\kappa_{r,M} \in (0, +\infty)$ such that*

$$P_M\left(\frac{\kappa_{r,M}}{r+\kappa_{r,M}}, \frac{r\kappa_{r,M}}{r+\kappa_{r,M}}\right) = 0.$$

Lindsay and Mauldin showed that the above $\kappa_{r,M}$ is the quantization dimension for the probability measure m_M (cf. [LM]). We shall prove the following lemma.

Lemma 3.4. *If $(q_M)_{M\geq 2}$ is a sequence of elements in $(0,1)$ such that $q_M \to q$ for some $q \in (0,1]$, then $\beta_M(q_M) \to \beta(q)$ as $M \to \infty$.*

Proof. Let us first show that $\beta_M(q_M) \leq \beta(q_M)$ for all $M \geq 2$. If not then there exists a positive integer $M' \geq 2$ such that $\beta_{M'}(q_{M'}) > \beta(q_{M'})$. Then

$$0 = P_{M'}(q_{M'}, \beta_{M'}(q_{M'})) < P_{M'}(q_{M'}, \beta(q_{M'})) \leq P(q_{M'}, \beta(q_{M'})) = 0,$$

which is contradiction, and hence $\beta_M(q_M) \leq \beta(q_M)$ for all $M \geq 2$. By the hypothesis, $q_M \to q$ for some $q \in (0,1]$ as $M \to \infty$. Moreover, the temperature functions are continuous on $(0,1]$, and hence

$$\limsup_M \beta_M(q_M) \leq \limsup_M \beta(q_M) = \lim_M \beta(q_M) = \beta(q). \tag{8}$$

If $\liminf_M \beta_M(q_M) < \beta(q)$, then for some $\epsilon > 0$ there exists a subsequence $(\beta_{M_k}(q_{M_k}))_{k\geq 1}$ of the sequence $(\beta_M(q_M))_{M\geq 2}$ such that

$$\beta_{M_k}(q_{M_k}) \leq \beta(q) - \epsilon < \beta(q).$$

Then for all $k \geq 1$,

$$0 = P_{M_k}(q_{M_k}, \beta_{M_k}(q_{M_k})) \geq P_{M_k}(q_{M_k}, \beta(q) - \epsilon),$$

and hence for any positive integer ℓ if $k \geq \ell$,

$$0 \geq P_{M_k}(q_{M_k}, \beta(q) - \epsilon) \geq P_{M_\ell}(q_{M_k}, \beta(q) - \epsilon),$$

which implies

$$0 \geq \lim_{k\to\infty} P_{M_\ell}(q_{M_k}, \beta(q) - \epsilon) = P_{M_\ell}(q, \beta(q) - \epsilon),$$

and so we have

$$0 \geq \lim_{\ell\to\infty} P_{M_\ell}(q, \beta(q) - \epsilon) = P(q, \beta(q) - \epsilon) > P(q, \beta(q)) = 0,$$

which is a contradiction. Hence,

$$\liminf_M \beta_M(q_M) \geq \beta(q). \tag{9}$$

By (8) and (9), it follows that

$$\lim_{M\to\infty} \beta_M(q_M) = \beta(q),$$

and thus the lemma is yielded. □

Lemma 3.5. *Let $0 < r < +\infty$, and let κ_r and $\kappa_{r,M}$ be as in Lemma 3.2 and Lemma 3.3. Then $\kappa_{r,M} \to \kappa_r$ as $M \to \infty$.*

Proof. Let $q_{r,M} = \frac{\kappa_{r,M}}{r+\kappa_{r,M}}$ and $q_r = \frac{\kappa_r}{r+\kappa_r}$. It is enough to prove $q_{r,M} \to q_r$ as $M \to \infty$. Let $(q_{r,M_K})_{k\geq 1}$ be a subsequence of $(q_{r,M})_{M\geq 2}$ such that $q_{r,M_k} \to \hat{q}$ for some $\hat{q} \in (0,1]$. Then by Lemma 3.4,

$$r\hat{q} = \lim_{k\to\infty} rq_{r,M_k} = \lim_{k\to\infty} \beta_{M_k}(q_{r,M_k}) = \beta(\hat{q}).$$

So, by uniqueness of q_r, this implies that $\hat{q} = q_r$. Hence $q_{r,M} \to q_r$ as $M \to \infty$, i.e., $\kappa_{r,M} \to \kappa_r$ as $M \to \infty$. □

Lemma 3.6. *Let m_M and m_F be as defined before. Then m_M converges weakly to m_F as $M \to \infty$.*

Proof. Let $f : I^\infty \to I^\infty$ be such that $f(\omega) = f^{(\omega_1)}(\pi(\sigma(\omega)))$, i.e. f is an amalgamated function corresponding to the infinite family F of strongly Hölder functions. Then for each $M \geq 2$, $f|_{I_M^\infty}$ is an amalgamated function for the partial family F_M of strongly Hölder functions. Let $\tilde{m}_M$ be the conformal measure of the function f_M with respect to the dynamical system generated by the shift map $\sigma_M : I_M^\infty \to I_M^\infty$. Then (see Theorem 3.2.3 in [MU]) $m_M := \tilde{m}_M \circ \pi_M^{-1}$ is the unique F_M-conformal measure, where $\pi_M := \pi|_{I_M^\infty}$ is the restriction of the coding map π on the space I_M^∞. The proof of Theorem 2.7.3 along with Corollary 2.7.5 (especially its uniqueness part (a)) in [MU] gives that the sequence $(\tilde{m}_M)_{M=2}^\infty$ converges weakly to $\tilde{m}$, the unique conformal measure for $f : I^\infty \to \mathbb{R}$. Since the projection map $\pi : I^\infty \to X$ is continuous, we therefore have that the sequence $(m_M)_{M=2}^\infty$ converges weakly to the measure $m_F := \tilde{m} \circ \pi^{-1}$. Thus the proof of the lemma is complete. □

But because of Theorem 3.2.3 in [MU] again, so defined measure m_F is the unique conformal measure for the Hölder family F, i.e., $m_F = m$, where m is the F-conformal measure defined before.

Let $\mathcal{M}$ denote the set of all Borel probability measures on X. Then given $0 < r < +\infty$, we know L_r-minimal metric (also refereed as L_r-Wasserstein metric or L_r-Kantorovich metric) is given by

$$\rho_r(P_1, P_1) = \inf_\nu \left(\int \|x-y\|^r d\nu(x,y) \right)^{\frac{1}{r}},$$

where the infimum is taken over all Borel probabilities ν on $\mathbb{R}^d \times \mathbb{R}^d$ with fixed marginals P_1 and P_2. Note that weak topology and the topology induced by any L_r-minimal metric ρ_r coincide on $\mathcal{M}$.

The following lemma is well-known (cf. [GL1]).

Lemma 3.7. *Let $\mathcal{P}_n$ denote the set of all discrete probability measures Q on $\mathbb{R}^d$ with $|supp(Q)| \leq n$. Then*

$$V_{n,r}(P) = \inf_{Q\in\mathcal{P}_n} \rho_r^r(P, Q).$$

Let us now prove the following lemma.

Lemma 3.8. *Let $0 < r < +\infty$, and $m_M \to m$ with respect to the weak topology. Then for any $n \geq 1$,*

$$\lim_{M\to\infty} V_{n,r}(m_M) = V_{n,r}(m).$$

Proof. Since X is compact, for any Borel probability measure ν we have $\int \|x\|^r d\nu(x) < \infty$. Hence by Lemma 3.7, it follows that

$$|V_{n,r}^{1/r}(m_M) - V_{n,r}^{1/r}(m)| \leq \rho_r(m_M, m),$$

which yields the lemma. □

Now we prove the main result of our paper.

Proof of Theorem 3.1. To prove the theorem let us first prove

$$\kappa_r \leq \liminf_n \frac{r \log n}{-\log V_{n,r}(m)} \leq \limsup_n \frac{r \log n}{-\log V_{n,r}(m)} \leq \kappa_r. \tag{10}$$

If possible, let $\liminf_n \frac{r \log n}{-\log V_{n,r}(m)} < \kappa_r$. Then there exists a subsequence $\left(\frac{r \log n_k}{-\log V_{n_k,r}(m)}\right)_{k\geq 1}$ of the sequence $\left(\frac{r \log n}{-\log V_{n,r}(m)}\right)_{n\geq 1}$ such that $\lim_{k\to\infty} \frac{r \log n_k}{-\log V_{n_k,r}(m)} < \kappa_r$, which implies that there exists a positive integer K_0 such that $\frac{r \log n_k}{-\log V_{n_k,r}(m)} < \kappa_r$ for all $k \geq K_0$. Thus for $k \geq K_0$, using Lemma 3.5 and Lemma 3.8, we obtain

$$\lim_{M\to\infty} \frac{r \log n_k}{-\log V_{n_k,r}(m_M)} = \frac{r \log n_k}{-\log V_{n_k,r}(m)} < \kappa_r = \lim_{M\to\infty} \kappa_{r,M},$$

and so there exists a positive integer M' such that $\frac{r \log n_k}{-\log V_{n_k,r}(m_M)} < \kappa_{r,M}$ for all $M \geq M'$ and for all $k \geq K_0$. In particular, for all $k \geq K_0$ we have

$$\frac{r \log n_k}{-\log V_{n_k,r}(m_{M'})} < \kappa_{r,M'}. \tag{11}$$

Note that $V_{n_k,r}(m_{M'}) \to 0$ as $k \to \infty$, and $V_{n_k,r}(m_{M'}) \geq V_{n_{k+1},r}(m_{M'}) > 0$ for all $k \geq 1$, and thus there exists a positive integer K_0' such that for all $k \geq K_0'$,

$$1 > V_{n_k,r}(m_{M'}) \geq V_{n_{k+1},r}(m_{M'}) > 0,$$

Hence for all $k \geq K_0'$, we have

$$\frac{r \log n_k}{-\log V_{n_k,r}(m_{M'})} \geq \frac{r \log n_{k+1}}{-\log V_{n_{k+1},r}(m_{M'})}, \tag{12}$$

i.e., $\left(\frac{r \log n_k}{-\log V_{n_k,r}(m_{M'})}\right)_{k\geq K_0'}$ is a decreasing sequence of real numbers. Then by (11) and (12), we deduce

$$\lim_{k\to\infty} \frac{r \log n_k}{-\log V_{n_k,r}(m_{M'})} < \kappa_{r,M'}, \text{ i.e., } \liminf_n \frac{r \log n}{-\log V_{n,r}(m_{M'})} \leq \lim_{k\to\infty} \frac{r \log n_k}{-\log V_{n_k,r}(m_{M'})} < \kappa_{r,M'}.$$

$\kappa_{r,M'}$ is the quantization dimension for the probability measure $m_{M'}$, and so by the preceding inequality, we obtain

$$\kappa_{r,M'} = \lim_{n\to\infty} \frac{r \log n}{-\log V_{n,r}(m_{M'})} = \liminf_n \frac{r \log n}{-\log V_{n,r}(m_{M'})} < \kappa_{r,M'},$$

which is a contradiction. Hence

$$\kappa_r \leq \liminf_n \frac{r \log n}{-\log V_{n,r}(m)}, \text{ and similarly, } \limsup_n \frac{r \log n}{-\log V_{n,r}(m)} \leq \kappa_r.$$

Therefore, the inequalities in (10) are proved, and thus $\lim_{n\to\infty} \frac{r \log n}{-\log V_{n,r}(m)}$ exists and equals κ_r, i.e.,

$$D_r(m) = \lim_{n\to\infty} \frac{r \log n}{-\log V_{n,r}(m)} = \kappa_r.$$

Note that if $q_r = \frac{\kappa_r}{r+\kappa_r}$, by Lemma 3.2, we have $\beta(q_r) = rq_r$, and then $D_r = \frac{\beta(q_r)}{1-q_r}$. Thus the proof of the theorem is complete.

We would like to close this section with a class of examples of summable Hölder families of functions which fulfill our assumptions. Assume that an infinite conformal iterated function system $S = \{\varphi_i\}_{i\in I}$ is co-finitely (or hereditarily) regular. Let $\kappa : X \to \mathbb{R}$ be an arbitrary Hölder continuous function. Fix $s > \theta_S$. For every $i \in I$ define

$$f^{(i)}(x) = \kappa(x) + s \log |\varphi_i'(x)|.$$

Then $F := \{f^{(i)}\}_{i\in I}$ is a summable Hölder family of functions for which (7) holds, and in consequence, Theorem 3.1 is true.

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