DYNAMICAL RIGIDITY

OF

TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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ABSTRACT. We prove the form of dynamical rigidity of transcendental meromorphic functions which asserts that if two tame transcendental meromorphic functions restricted to their Julia sets are topologically conjugate via a locally bi–Lipschitz homeomorphism, then they, treated as functions defined on the entire complex plane $\mathbb{C}$, are topologically conjugate via an affine map, i.e. a map from $\mathbb{C}$ to $\mathbb{C}$ of the form $z \mapsto az + b$. As an intermediate step we show that no tame transcendental meromorphic function is essentially affine.

1. INTRODUCTION

Let X and Y be arbitrary metric spaces. We say that a homeomorphism $H : X \to Y$ is locally bi-Lipschitz if each point $x \in X$ has some open neighborhood U_x such that both the restriction $H|_{U_x} : U_x \to H(U_x)$ and its inverse $\left(H|_{U_x}\right)^{-1} : H(U_x) \to U_x$ are Lipschitz continuous. The main goal of this paper is to show that if two tame transcendental meromorphic functions restricted to their Julia sets are topologically conjugate via a locally bi–Lipschitz homeomorphism, then they, treated as functions defined on the entire complex plane $\mathbb{C}$, are topologically conjugate via an affine map, i.e. a map from $\mathbb{C}$ to $\mathbb{C}$ of the form $z \mapsto az + b$. As an intermediate step we show that no tame transcendental meromorphic function is essentially affine.

Our work stems from D. Sullivan article [10] treating among others the dynamical rigidity of conformal expanding repellers. Its systematical account can be found in [7]. We make also an essential use of the rigidity result for conformal iterated function systems from [5]. The case of tame rational functions is actually done in [6].

The structure of our argument is this. First we make use of the existence of nice sets for tame transcendental meromorphic functions as proved in [1]. This

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means we canonically associate, as in [9], to each nice set U a conformal iterated function system S_U in the sense of [3] and [4]. Then we show (Proposition 3.2) that no such systems S_U are essentially affine. Having this we strengthen the dynamical rigidity result from [5] to conclude that any locally bi–Lipschitz conjugacy between two tame transcendental meromorphic functions yields conformal conjugacy on some neighborhoods of the closures of the limit sets of the associated (via nice sets) iterated function systems. As the last step we prove that such conjugacy extends holomorphically to a holomorphic automorphism of the complex plane $\mathbb{C}$. It thus must be an affine map $z \mapsto az + b$.

2. Preliminaries

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. The Fatou set of f consists of all points $z \in \mathbb{C}$ that admit an open neighborhood U_z such that all the forward iterates f^n, $n \geq 0$, of f are well-defined on U_z and the family of maps $\{f^n|_{U_z} : U_z \to \mathbb{C}\}_{n=0}^{\infty}$ is normal. The Julia set of f, denoted by J_f, is then defined as the complement of the Fatou set of f in $\mathbb{C}$. By $\mathrm{Sing}(f^{-1})$ we denote the set of singularities of f^{-1}. We define the *postsingular* set of $f : \mathbb{C} \to \hat{\mathbb{C}}$ as

$$\mathrm{PS}(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{Sing}(f^{-1}))}.$$

Given a set $F \subset \hat{\mathbb{C}}$ and $n \geq 0$, by $\mathrm{Comp}(f^{-n}(F))$ we denote the collection of all connected components of the inverse image $f^{-n}(F)$. A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called *tame* if its postsingular set does not contain its Julia set. This is the primary object of our interest in this paper.

We make heavy use of the concept of *a nice set* which J. Rivera–Letelier introduced in [8] in the realm of the dynamics of rational maps of the Riemann sphere. In [1] N. Dobbs proved their existence for tame meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$. We quote now his theorem.

Theorem 2.1. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a tame meromorphic function. Fix $z \in \mathcal{J}(f) \setminus \mathcal{P}(f)$, $L > 1$ and $K > 1$. Then there exists $\kappa > 1$ such that for all $r > 0$ sufficiently small, there exists an open connected set $U = U(z, r) \subset \mathbb{C} \setminus \mathcal{P}(f)$ such that*

(a) *If $V \in Comp(f^{-n}(U))$ and $V \cap U \neq \emptyset$, then $V \subset U$.*
(b) *If $V \in Comp(f^{-n}(U))$ and $V \cap U \neq \emptyset$, then, for all $w, w' \in V$,*

$$|(f^n)'(w)| > L \text{ and } \frac{|(f^n)'(w)|}{|(f^n)'(w')|} < K.$$

(c) $\overline{B(z, r)} \subset U \subset B(z, \kappa r) \subset \mathbb{C} \setminus \mathcal{P}(f)$.

Let $\mathcal{U}$ be the collection of all nice sets of $f : \mathbb{C} \to \hat{\mathbb{C}}$, i.e. all the sets U satisfying the above proposition with some $z \in J_f \setminus \mathrm{PS}(f)$ and some $r > 0$. Note that if $U = U(z, r) \in \mathcal{U}$ and $V \in \mathrm{Comp}(f^{-n}(U)$ satisfies the requirements (a), (b) and (c) from Theorem 2.1 then there exists a unique holomorphic inverse branch $f_V^{-n} : B(z, \kappa r) \to \mathbb{C}$ such that $f_V^{-n}(U) = V$. As noted in [9] the collection S_U of all such inverse branches forms obviously an iterated function system in the sense of [3] and [4]. In particular, it clearly satisfies the Open Set Condition. We denote its limit set by J_U. We have just mentioned [3] and [4]. In what concerns iterated function systems we try our concepts and notation to be compatible with that of [4].

3. Essential Affinity

In this section we prove that the iterated function system corresponding to any nice set of a transcendental meromorphic function is not essentially affine. An important step in this proof is provided by the following abstract lemma.

Lemma 3.1. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function and let $A_j : \mathbb{C} \to \mathbb{C}$, $j = 1, 2$, be two affine maps, $A_j(z) = a_j z + b_j$ such that $0 < |a_j| < 1$ and $A_2^k \neq A_1^k$ for all integers $k \geq 1$. Let $U \subset \mathbb{C}$ be an open connected set such that $A_j(U) \subset U$ for $j = 1, 2$. If $\psi : U \to \hat{\mathbb{C}}$ is a non-constant meromorphic function, then there is no integer $q \geq 1$ such that*

$$\psi(z) = f^q \circ \psi \circ A_j(z)$$

for $j = 1, 2$ and all $z \in U$.

Before we prove the lemma let us show the following claim.

Claim. Suppose that

$$\psi(z) = f^q \circ \psi \circ A_j(z) \tag{3.1}$$

for some $q \geq 1$, all $j = 1, 2$ and all $z \in U$ with U, ψ, $f : \mathbb{C} \to \hat{\mathbb{C}}$ as in Lemma 3.1. Then the function $\psi : U \to \hat{\mathbb{C}}$ has a unique meromorphic extension to all of $\mathbb{C}$ such that (3.1) holds for $j = 1, 2$ and all $z \in U$.

Proof of Claim. First note that in order to prove the claim all what we need to show is that $\psi : U \to \hat{\mathbb{C}}$ has a meromorphic extension onto $\mathbb{C}$. We shall show by induction that for every $n \geq 0$ the map ψ has a unique meromorphic extension to $A_1^{-n}(U)$ such that (3.1) holds on $A_1^{-n}(U)$. Indeed, for $n = 0$ this is our seeking contradiction assumption. For the inductive step suppose it holds for some $n \geq 0$. Define then the function $\psi_* : A_1^{-(n+1)}(U) \to \hat{\mathbb{C}}$ as

$$\psi_*(z) = f^q \circ \psi \circ A_j(z). \tag{3.2}$$

Note that $A_1^{-n}(U) \subset A_1^{-(n+1)}(U)$ and, by our inductive assumption, $\psi_*(z) = \psi(z)$ for all $z \in A_1^{-n}(U)$. Renaming then ψ_* to ψ, (3.2) holds for all $z \in A_1^{-(n+1)}(U)$. The uniqueness part follows from the fact that two meromorphic functions defined on the open connected set $A_1^{-n}(U)$ and coinciding on its open subset U, are equal. The inductive proof is finished. Since $\bigcup_{n=0}^\infty A_1^{-n}(U) = \mathbb{C}$, the claim is proved.

Proof of Lemma 3.1. Seeking contradiction, we suppose that (3.1) holds. It follows from Claim that $\psi = f^q \circ \psi \circ A_1 = f^q \circ \psi \circ A_2$ on $\mathbb{C}$, and therefore $\psi \circ A_1^{-1} = f^q \circ \psi = \psi \circ A_2^{-1}$, which yields

$$\psi = \psi \circ (A_2^{-1} \circ A_1). \tag{3.3}$$

Seeking contradiction suppose that $a_1^n \neq a_2^n$ for all $n \geq 1$. Then the map $A_2^{-1} \circ A_1 : \mathbb{C} \to \mathbb{C}$ has a unique fixed point. Denote it by ξ. If ξ is neither a pole nor a critical point of ψ, then (3.3) entails $\psi'(\xi) = \psi'(\xi) a_2^{-1} a_1$, which yields $a_2 = a_1$. This contradiction entails ξ to be either a pole or a critical point of ψ. Replacing ψ by $1/\psi$ in the case when ξ is a pole, we may assume without loss of generality that that ξ is a critical point of ψ. Write $A_2^{-1} \circ A_1(z) = a_2^{-1} a_1 z + b$. Let $p \geq 2$ be the order of the critical point ξ of ψ. There then exists a holomorphic function $g : V \to \mathbb{C}$ defined on some sufficiently small neighborhood V of ξ such that

$$g(\xi) \neq 0$$

and

$$\psi(z) = \psi(\xi) + (z - \xi)^p g(z)$$

for all $z \in V$. Inserting this to (3.3) yields

$$(z - \xi)^p g(z) = (a_2^{-1} a_1 z + b - \xi)^p g(A_2^{-1} \circ A_1(z)). \tag{3.4}$$

**Changes in computation!

But $a_2^{-1} a_1 \xi + b = \xi$, so $b = \xi - a_2^{-1} a_1 \xi$, and (3.4) takes on the form

$$(z - \xi)^p g(z) = (a_2^{-1} a_1)(z - \xi)^p g(A_2^{-1} \circ A_1(z)).$$

So, for $z \neq \xi$ and close to ξ,

$$g(z) = (a_2^{-1} a_1)^p g(A_2^{-1} \circ A_1(z)).$$

Since $g(\xi) \neq 0$ and $A_2^{-1} \circ A_1(\xi) = \xi$, this gives

$$(a_2^{-1} a_1)^p = \lim_{z \to \xi} \frac{g(z)}{g(A_2^{-1} \circ A_1(z))} = 1.$$

This contradiction shows that

$$a_2^l = a_2^l$$

for some integer $l \geq 1$.

Therefore, we can write

$$(A_2^{-1} \circ A_1)^l = T_b$$

where $T_b(z) = z + b$ with some $b \in \mathbb{C} \setminus \{0\}$. Formula (3.3) then yields

$$\psi = \psi \circ (A_2^{-1} \circ A_1)^l = \psi \circ T_b.$$

Let

$$\Gamma = \{w \in \mathbb{C} : \psi = \psi \circ T_w\}.$$

Clearly Γ is a subgroup of $\mathbb{C}$ containing b. If the set Γ is dense in $\mathbb{C}$, then $\psi : \mathbb{C} \to \hat{\mathbb{C}}$ is a constant function, contrary to our hypothesis. So, Γ is generated either by one element or else by two elements linearly independent over $\mathbb{R}$. Suppose first that the latter case holds. Then Γ has a bounded fundamental domain, denote it by $\Re$. Fix a real number $\eta > 0$ so large that $B(0,\eta) \supset \Re$ and $A_1^{-1}(B(0,\eta)) \supset B(0,\eta)$. Then $\psi|_{A_1^{-1}(B(0,\eta))}$ is finite-to-one but $f^q \circ \psi \circ A_1|_{A_1^{-1}(B(0,\eta))}$ is infinite-to-one as f^q is and $\psi \circ A_1(A_1^{-1}(B(0,\eta))) = \psi(B(0,\eta) = \hat{\mathbb{C}}$. Thus, we are left to consider the case when Γ is generated by one element. Denote it by c. Fix a complex number $w \in \mathbb{C} \setminus (\mathbb{R}c \cup \mathbb{R}a_1c)$. Then

$$D = \mathbb{R}w + [0,1)c = \{sw + tc : s \in \mathbb{R},\ 0 \leq t < 1\}$$

is a fundamental domain for Γ. We have

$$A_1^{-1}(D) = \mathbb{R}(a_1^{-1}w) + [0,1)(a_1^{-1}c) + b_1',$$

where $b_1' = -a_1{-}1b_1$. Seeking contradiction suppose that, for some $\xi_0 \in A_1^{-1}(D)$, the set $A_1^{-1}(D) \cap \Gamma(\xi_0)$ is infinite, where $\Gamma(\xi_0) = \xi_0 + \mathbb{Z}c$. Write $\xi_0 = s_0a_1^{-1}w + t_0a_1^{-1}c + b_1$, where $s_0 \in \mathbb{R}$ and $t \in [0,1)$. If $\xi \in A_1^{-1}(D) \cap \Gamma(\xi_0)$, then **Changes

$$s_\xi a_1^{-1}w + t_\xi a_1^{-1}c + b_1' = \xi = \xi_0 + k_\xi c = s_0a_1^{-1}w + t_0a_1^{-1}c + b_1 + k_\xi c$$

with some $s_\xi \in \mathbb{R}$, $t_\xi \in [0,1)$ and $k_\xi \in \mathbb{Z}$. Hence

$$a_1^{-1}(s_\xi - s_0) = (t_0 - t_\xi)a_1^{-1}c + k_\xi c. \tag{3.5}$$

Since $A_1^{-1}(D) \cap \Gamma(\xi_0)$ is a countably infinite set, we can enumerate its elements as $(\xi_n)_{n=1}^\infty$ and require that the map $\mathbb{N} \ni n \mapsto \xi_n$ be 1-to-1. Since , we have

$$\lim_{n\to\infty} |\xi_n| = +\infty.$$

Put $s_n = s_{\xi_n}$, $t_n = t_{\xi_n}$, and $k_n = k_{\xi_n}$. It then follows from (3.5) that

$$\lim_{n\to\infty} |s_n| = +\infty = \lim_{n\to\infty} |k_n|$$

and

$$\left(\frac{s_n - s_0}{k_n}\right)\frac{w}{a_1} = \frac{(t_0 - t_n)a_1^{-1}c}{k_n} + c.$$

Letting $n \to \infty$ we from these formulas that $\lim_{n\to\infty} k_n^{-1}(s_n - s_0)$ exists. Denote it by α. Then $c = \alpha\frac{w}{a_1}$. Equivalently, $w = \alpha^{-1}a_1c \in \mathbb{R}a_1c$. This contradiction shows that for every $z \in \mathbb{C}$ the intersection $A_1^{-1}(D) \cap \psi^{-1}(z)$ is finite. In other words, $\psi|_{A_1^{-1}(D)}$ is finite-to-one. But on the other hand, $\psi|_{A_1^{-1}(D)} = f^q \circ \psi \circ A_1\|_{A_1^{-1}(D)}$ is infinite-to-one since f^q is and $\psi \circ A_1(A_1^{-1}(D)) = \psi(D)$ has at most two point large complement in $\hat{\mathbb{C}}$. This contradiction finishes the proof. $\square$

Let $S = \{\phi_e : W \to W\}_{e\in e}$, $W \subset \mathbb{C}$, be an arbitrary conformal iterated function system whose phase space W is contained in the complex plane $\mathbb{C}$. Recalling from [5] we say that the system S is *essentially affine* if the conformal structure on $\overline{J_S}$ admits a Euclidean isometries refinement so that all the maps ϕ_e, $e \in E$, become affine conformal. More precisely, there exists an atlas $\{\psi_t : U_t \to \mathbb{C}\}_{t\in T}$ with some parameter set T and some open connected simply connected sets U_t, $t \in T$, consisting of conformal univalent maps such that

(a) $\bigcup_{t\in T} U_t \supset \overline{J_S}$ and $\bigcup_{t\in T} U_t \subset W$.
(b) All the sets $U_t \cap U_s$ and $U_t \cap \phi_e(U_s)$, $s, t \in T$, $e \in E$, are connected.
(c) The compositions $\psi_t \circ \psi_s^{-1}$ and $\psi_t \circ \phi_e \psi_s^{-1}$, defined respectively on $\psi_s(U_t \cap U_s)$ and $\psi_s \circ \phi_e^{-1}(U_t \cap \phi_e(U_s))$, $s, t \in T$, $e \in E$, are all affine (of the form $z \mapsto az + b$) with $|(\psi_t \circ \psi_s^{-1})'| = 1$.

Our application of Lemma 3.1 to tame meromorphic functions is this.

Proposition 3.2. *If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a tame transcendental meromorphic function and $U \subset \mathbb{C}$ is a nice set for f, then the corresponding iterated function system $S_U = \{\phi_e\}_{e\in E}$ is not essentially affine.*

Proof. Suppose on the contrary that the iterated function system S_U is essentially affine and let $\{\psi_t : U_t \to \mathbb{C}\}_{t\in T}$ be the corresponding conformal atlas. Take an element $\psi_s : U_s \to \mathbb{C}$ from this atlas such that $U_S \cap J_S \neq \emptyset$. Take then two incomparable words $\omega, \tau \in E^*$ such that $x_\omega, x_\tau \in U_s$, where x_ω and x_τ are the unique fixed points respectively of ϕ_ω and ϕ_τ. Taking the words ω and τ sufficiently long, we may further assume that

$$\phi_\omega(U_s) \subset U_s \ \text{ and } \ \phi_\tau(U_s) \subset U_s.$$

Let $k = ||\omega||$ and $l = ||\tau||$, i.e. $f^k \circ \phi_\omega = \mathrm{Id}$ and $f^\circ \phi_\tau = \mathrm{Id}$. We then have

$$\phi_{\omega^l}(U_s) \subset U_s \ \text{ and } \ \phi_{\tau^k}(U_s) \subset U_s.$$

Set

$$U = \psi_s(U_s) \ \text{ and } \ \psi := \psi_s^{-1} : U \to \mathbb{C}.$$

It then follows from property (c) of essential affiness that

$$A_1 := \psi^{-1} \circ \phi_{\omega^l} \circ \psi : U \to U \ \text{ and } \ A_2 := \psi^{-1} \circ \phi_{\tau^k} \circ \psi : U \to U \tag{3.6}$$

are affine maps. Write $A_j(z) = a_j(z) + b_j$, $j = 1, 2$. Then

$$\begin{aligned}
|a_1| &= |A_1'(\psi^{-1}(x_\omega)| \\
&= |(\psi^{-1})'(\phi_\omega(\psi(\psi^{-1}(x_\omega))))| \cdot |\phi_{\omega^l}'(\psi(\psi^{-1}(x_\omega)))| \cdot |\psi'(\psi^{-1}(x_\omega))| \\
&= |(\psi^{-1})'(\phi_\omega(x_\omega))| \cdot |\phi_{\omega^l}'(x_\omega)||\psi'(\psi^{-1}(x_\omega))| \\
&= |(\psi^{-1})'(x_\omega)| \cdot |\phi_{\omega^l}'(x_\omega)| \cdot |\psi'(\psi^{-1}(x_\omega))| \\
&= |\phi_{\omega^l}'(x_\omega)| < 1
\end{aligned}$$

and likewise

$$|a_2| = |\phi_{\tau^k}'(x_\tau)| < 1.$$

Also, for each integer $n \geq 1$, the words $(\omega^l)^n$ and $(\tau^k)^n$ are extensions respectively of ω and τ, and are therefore different. Hence

$$A_1^n = \psi^{-1} \circ \phi_{\omega^l}^n \circ \psi = \psi^{-1} \circ \phi_{(\omega^l)^n} \circ \psi \neq \psi^{-1} \circ \phi_{\tau^k}^n \circ \psi = \psi^{-1} \circ \phi_{(\tau^k)^n} \circ \psi = A_2^n,$$

and the the assumptions of Lemma 3.1 are verified. But it follows from (3.6) that $\psi \circ A_1 = \phi_{\omega^l} \circ \psi$, and applying f^{kl} to both sides of this equality, we get that $f^{kl} \circ \psi \circ A_1 = f^{kl} \circ \phi_{\omega^l} \circ \psi = \psi$. Likewise $f^{kl} \circ \psi \circ A_2 = \psi$. This however contradicts Lemma 3.1 and ends the proof of our proposition. □

4. Conjugacies of Conformal Iterated Function Systems

In this section we deal with bi-Lipschitz conjugacies of conformal iterated function systems. We want to apply them to the systems generated by nice sets of tame meromorphic functions. This however causes two difficulties that has not been addressed in the literature yet. One is that, as noted in the proof of Claim 1 in Theorem 5.1 of [9], the systems thus emerging do not have to satisfy the Open Set Condition. The second difficulty is that they do not have to be regular. Both of these difficulties are taken care of below to the extent which is sufficient for our applications to meromorphic functions.

We call a conformal iterated function system $S = \{\phi_e\}_{e \in E}$ of W type if there exists a continuous map

$$F_S : \bigcup_{e \in E} \phi_e(\overline{J}_S) \to \overline{J}_S$$

such that

$$F_S \circ \phi_e = \mathrm{Id}_{\overline{J}_S}$$

for all $e \in E$. The map F_S then induces a conformal Walters expanding map as defined in [2]. For all conformal iterated function systems and all real numbers $t \geq 0$ the topological pressure $\mathrm{P}(t) \in \mathbb{R}$ is well-define (though can take up the value $+\infty$), however the existence of $e^{\mathrm{P}(t)}$-conformal measures requires either the Open Set Condition (see [3] and [4] or the W property. By an $e^{\mathrm{P}(t)}$-conformal measure we mean a Borel probability measure m_t on the limit set J_S such that

$$m_t(\phi_e(A)) = e^{-\mathrm{P}(t)} \int_A |\phi_e'|^t dm_t \tag{4.1}$$

for all $e \in E$ and all Borel sets $A \subset J_S$, and also that

$$m_t\big(\phi_a(J_S) \cap \phi_b(J_S)\big) = 0 \tag{4.2}$$

whenever $a, b \in E$ and $a \neq b$. Applying (4.1) and (4.2) inductively gives

$$m_t(\phi_\omega(A)) = e^{-\mathrm{P}(t)|\omega|} \int_A |\phi_\omega'|^t dm_t \tag{4.3}$$

for all $\omega \in E^*$ and all Borel sets $A \subset J_S$, and also that

$$m_t\big(\phi_\omega(J_S) \cap \phi_\tau(J_S)\big) = 0 \tag{4.4}$$

whenever $\omega, \tau \in E$ are incomparable. If an $e^{\mathrm{P}(t)}$-conformal measure exists, it is unique and by μ_t we denote the unique Borel probability measure on J_S that is absolutely continuous with respect to the measure m_t and such that

$$\sum_{e\in E} \mu_t(\phi_e(A)) = \mu_t(A)$$

for all Borel sets $A \subset J_S$. If the system S is of W type then this condition just means that the measure μ_t is F_S-invariant. We shall prove the following.

Proposition 4.1. *If two conformal iterated function systems $S = \{\phi_e\}_{e\in E}$ and $Q = \{\psi_e\}_{e\in E}$ are bi-Lipschitz conjugate via a bi-Lipschitz map $H : J_S \to J_Q$, then*

$$\mathrm{P}_Q(t) = \mathrm{P}_S(t)$$

for all $t \geq 0$. If in addition both systems either satisfy the Open Set Condition or the W property, then

$$\mu_{Q,t} = \mu_{S,t} \circ H^{-1}$$

for all $t \in \mathcal{F}in(S) = \mathcal{F}in(Q)$.

Proof. Since the systems S and Q are bi-Lipschitz conjugate and $||\phi_\omega'|| \asymp \mathrm{diam}(\phi_\omega(J_S))$, $||\psi_\omega'|| \asymp \mathrm{diam}(\psi_\omega(J_Q))$ for all $\omega \in E^*$, we conclude that

$$C^{-1} \leq \frac{||\psi_\omega'||}{||\phi_\omega'||} \leq C$$

for some constant $C \geq 1$ and all $\omega \in E^*$. Therefore, $\mathcal{F}in(S) = \mathcal{F}in(Q)$ and

$$\mathrm{P}_Q(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{|\omega|=n} ||\psi_\omega'||^t = \lim_{n\to\infty} \frac{1}{n} \sum_{|\omega|=n} ||\phi_\omega'||^t = \mathrm{P}_S(t). \tag{4.5}$$

Now let $t \in \mathcal{F}in(S) = \mathcal{F}in(Q)$. Assuming either the Open Set Condition or the W property (so in particular conformal measures $m_{Q,t}$ and $m_{S,t}$, as well as their invariant versions $\mu_{Q,t}$ are well-defined), it follows from (4.5) and (4.3) that

$$C^{-t}K^{-t} \le \frac{m_{Q,t}(\psi_\omega(J_Q))}{m_{S,t}(\phi_\omega(J_S))} \le C^t K^t$$

for all $\omega \in E^*$. Thus the measures $m_{Q,t}$ and $m_{S,t} \circ H^{-1}$ are equivalent (even with Radon Nikodym derivatives bounded by $(CK)^t$ and $(CK)^{-t}$ respectively from above and from below). Since also $\mu_{Q,t} \asymp m_{Q,t}$ and $\mu_{S,t} \asymp m_{S,t}$, we therefore conclude that the measures $\mu_{Q,t}$ and $\mu_{S,t} \circ H^{-1}$ are equivalent. Since they are ergodic (Theorem 2.2.9 and formula (3.10) in [4] if the Open Set Condition is satisfied and Theorem 2.5(b) in [2] if the W property holds), they must coincide. We are done. □

Now the proof of Theorem 3.1 in [5] goes through with Jacobians $\tilde{D}\phi_i$ replaced by $\tilde{D}^t\phi_i$ with respect to any measure $\mu_{S,t}$ with $t \in (0, +\infty) \cap \mathcal{F}in(S)$. This in turn permits to prove, with an analogous proof, the following refinement of Theorem 4.2 in [5].

Theorem 4.2. *Let $S = \{\phi_e\}_{e\in E}$ and $Q = \{\psi_e\}_{e\in E}$ be two complex plane conformal iterated function systems either of W type or satisfying the Open Set Condition. If at least one of these two systems is not essentially affine and they are topologically conjugate by a homeomorphism $H : J_S \to J_Q$, then the following conditions are equivalent.*

(a) *The conjugacy $H : J_S \to J_Q$ extends in a conformal manner to an open neighborhood of $\overline{J_S}$.*

(b) *The conjugacy $H : J_S \to J_Q$ extends in a real–analytic manner to an open neighborhood of $\overline{J_S}$.*

(c) *The conjugacy $H : J_S \to J_Q$ is bi-Lipschitz continuous.*

(d) $|\psi'_\omega(y_\omega)| = |\phi'_\omega(x_\omega)|$ *for all $\omega \in E^*$, where x_ω and y_ω are the only fixed points of ϕ_ω and ψ_ω respectively.*

(e) $\exists C \ge 1\ \forall \omega \in E^*$

$$C^{-1} \le \frac{\mathrm{diam}(\psi_\omega(J_Q))}{\mathrm{diam}(\phi_\omega(J_S))} \le C.$$

(f) $\exists D \ge 1\ \forall \omega \in E^*$

$$D^{-1} \le \frac{||\psi'_\omega||}{||\phi'_\omega||} \le D.$$

(g) *For every* $t \in (0, +\infty) \cap \mathcal{F}in(S)$ *the measures* $m_{Q,t}$ *and* $m_{S,t} \circ H^{-1}$ *are equivalent.*

(h) *For every* $t \in (0, +\infty) \cap \mathcal{F}in(S)$*,* $\mu_{Q,t} = \mu_{S,t} \circ H^{-1}$*.*

(i) *There exists* $t \in (0, +\infty) \cap \mathcal{F}in(S)$ *such that the measures* $m_{Q,t}$ *and* $m_{S,t} \circ H^{-1}$ *are equivalent.*

(j) *There exists* $t \in (0, +\infty) \cap \mathcal{F}in(S)$ *such that* $\mu_{Q,t} = \mu_{S,t} \circ H^{-1}$*.*

5. Conjugacies of tame Meromorphic Functions

Making use of the main results of the two previous sections we now shall prove the following main result of our paper.

Theorem 5.1. *If the restrictions to their Julia sets of two tame transcendental meromorphic functions* $f : \mathbb{C} \to \hat{\mathbb{C}}$ *and* $g : \mathbb{C} \to \hat{\mathbb{C}}$ *are topologically conjugate by a locally bi–Lipschitz homeomorphism* $H : J_f \to J_g$*, then this conjugacy extends to an affine linear* ($z \mapsto az+b$) *conjugacy from* $\mathbb{C}$ *to* $\mathbb{C}$ *between the meromorphic maps* $f, g : \mathbb{C} \to \hat{\mathbb{C}}$*.*

Proof. Let U be a nice set, for the map $f : \mathbb{C} \to \hat{\mathbb{C}}$, so small that $H : J_f \cap U \to H(J_f \cap U)$ is bi–Lipschitz. Let $S_U = \{\phi_e\}_{e \in E}$ be the iterated function system induced by the nice set U, and let

$$S_U^* = \{H \circ \phi_e \circ H^{-1} : H(J_U) \to H(J_U)\}_{e \in E}.$$

Then J_U satisfies the Open Set condition and S_U^* is a conformal iterated function system of W type. By Proposition 3.2 the system S_U is not essentially affine, and therefore, in virtue of Theorem 4.2, there exist an open set $V \supset \overline{J_U}$ and a conformal conjugacy map $\hat{H} : V \to \mathbb{C}$ such that $\hat{H}|_{\overline{J_U}} = H$ and $H(U \setminus \overline{J_U}) \subset \mathbb{C} \setminus J_g$. For every $e \in E$ let $||e|| \geq 1$ be uniquely determined by the requirement that $\phi_e : U \to U$ is a holomorphic inverse branch of $f^{||e||}$. Take now $a \in E$ with $k := ||a|| \geq 1$ so large that $\phi_a(U) \subset V$, and then define $\tilde{H} : U \to \mathbb{C}$ as

$$\tilde{H} = g^k \circ \hat{H} \circ \phi_a.$$

Then

(1) $\tilde{H} : U \to \mathbb{C}$ is a holomorphic map,
(2) $\tilde{H}|_{U \cap \overline{J_U}} = \hat{H}|_{U \cap \overline{J_U}}$,
(3) $\tilde{H}(U \setminus J_f) \subset \mathbb{C} \setminus J_g$.

The map $\tilde{H}$ is indeed well-defined and holomorphic as the Fatou set of g contains no inverse images of ∞ under any iterate of g, and on the Julia sets we have topological conjugacy between f and g which respects inverse images of ∞. Items (2) and (3) are now clear. Fix e, an arbitrary element in E, and put

$n = ||e||$. Then $g^n \circ \tilde{H} \circ \phi_e|_{J_U} = \tilde{H}|_{J_U}$, and as both maps $g^n \circ \tilde{H} \circ \phi_e : U \to \hat{\mathbb{C}}$ and $\tilde{H} : U \to \hat{\mathbb{C}}$ are holomorphic, we conclude that

$$g^n \circ \tilde{H} \circ \phi_e = \tilde{H}, \tag{5.1}$$

both maps defined on U. Let $\mathrm{Sing}(f^{-1})$ be the set of all singularities of f^{-1} and let

$$\mathrm{PS}(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{Sing}(f^{-1})}$$

In order to apply Kuratowski–Zorn Lemma, consider the family $\mathcal{F}$ of all open connected subsets W of $\mathbb{C} \setminus (\mathrm{PS}(f) \setminus U)$ containing U for which there exists a holomorphic function $\tilde{H}_W : W \to \hat{\mathbb{C}}$ with the following two properties.

**$\mathrm{PS}(f) \cap U = \emptyset$??, wiec moze trzeba usunac \U z calego dowodu

(a) $\tilde{H}_W|_U = \tilde{H}$.

(b) If $z \in U$ and $f^n(z) \in W$, then $g^n \circ \tilde{H}(z) = \tilde{H}_W \circ f^n(z)$.

The family $\mathcal{F}$ is partially ordered by inclusion and, by (5.1), it contains U, so $\mathcal{F}$ is not empty. If $\mathcal{C}$ is a linearly ordered subset of $\mathcal{F}$, then $\tilde{H}_{W_2}|_{W_1} = \tilde{H}_{W_1}$ whenever $W_1, W_2 \in \mathcal{C}$ and $W_1 \subset W_2$. This is so since $W_1 \supset U$ and (a) holds. Thus putting $W = \bigcup\{G : G \in \mathcal{C}\}$ and defining $\tilde{H}_W(z) = \tilde{H}_G(z)$ if $z \in G \in \mathcal{C}$, we see that $\tilde{H}_W : W \to \hat{\mathbb{C}}$ is a well-defined holomorphic function satisfying the requirements (a) and (b). So, W is an upper bound of $\mathcal{C}$. We therefore conclude from Kuratowski–Zorn Lemma that $\mathcal{F}$ contains a maximal element, and we denote it by G. We claim that

$$G = \mathbb{C} \setminus (\mathrm{PS}(f) \setminus U).$$

Indeed, if not, then there exists a point $w \in \partial G \setminus (\mathrm{PS}(f) \setminus U)$. Since $G \supset U$, we thus have that $w \notin \mathrm{PS}(f)$. Let

$$R_1 = \frac{1}{2}\mathrm{dist}(w, \mathrm{PS}(f)).$$

By Montel's Theorem $\bigcup_{j=0}^{\infty} f^j(U) \supset \hat{\mathbb{C}} \setminus \mathrm{PS}(f)$. In particular there exist $p \geq 0$ and $\xi \in U$ such that $f^p(\xi) = w$. Take then $0 < R_2 \leq R_1$ so small that

$$f_\xi^{-p}(B(w, R_2) \subset U,$$

where $f_\xi^{-p} : B(w, 2R_1) \to \mathbb{C}$ is the unique holomorphic branch of f^{-p} defined on $B(w, 2R_1)$ that sends w to ξ. Define the holomorphic map $\tilde{H}_{\xi,l} : B(w, R_2) \to \hat{\mathbb{C}}$ as

$$\tilde{H}_{\xi,l} := g^p \circ \tilde{H} \circ f_\xi^{-p}.$$

Since $\tilde{H}_{\xi,l}$ and $\tilde{H}_G$ coincide on $G \cap B(w, R_2)$, they glue together to a single holomorphic function $\tilde{H} : W \to \mathbb{C}$, where $W = G \cup B(w, R_2) \subset \mathbb{C} \setminus (\mathrm{PS}(f) \setminus U)$ is an open connected subset of $\mathbb{C} \setminus (\mathrm{PS}(f) \setminus U)$ containing properly G. Now (a) holds since $W \supset G \supset U$ and $\tilde{H}_G|_U = \tilde{H}$. In order to prove (b) consider and integer $n \geq 0$ and C, a connected component of $U \cap f^{-n}(W)$. If $f^n(C) \cap G \neq \emptyset$, then (b) holds for W because it holds for G. So, we may assume without loss of generality that

$$f^n(C) \cap G = \emptyset,$$

in particular

$$f^n(C) \subset B(w, R_2). \tag{5.2}$$

Let $f_C^{-n} : B(w, 2R_1) \to \mathbb{C}$ be the unique holomorphic branch of f^{-n} define on $B(w, 2R_1)$ and determined by the requirement that $f_C^{-n}(f^n(C)) = C$. Now fix a point $y \in J_f \cap U$. Let $r = (1 + 8K)^{-1}\text{dist}(y, U^c)$. Consider two cases . Assume first that

$$B(w, R_1) \cap J_f = \emptyset. \tag{5.3}$$

Since $y \in J_f$ and $w \notin \text{Sing}(f^{-1})$, it follows from Montel's Theorem that there exists $q_1 \geq 0$ such that $f^{q_1}(B(y, r)) \ni w$. Fix then $x_1 \in B(y, r)$ such that $f^{q_1}(x_1) = w$. Let $f_{x_1}^{-(n+q_1)} : B(w, 2R_1) \to \mathbb{C}$ be the unique holomorphic branch of $f^{-(n+q_1)}$ defined on $B(w, 2R_1)$ that sends w to x_1. Note that $f^{q_1} \circ f_{x_1}^{-(n+q_1)} = f_C^{-n}$. By virtue of Koebe's Distortion Theorem,

$$f_{x_1}^{-(n+q_1)}(B(w, R_1)) \supset B\big(x_1, 4^{-1}|(f^{n+q_1})'(x_1)|^{-1}R_1\big). \tag{5.4}$$

Now, it follows from (5.3) that $J_f \cap f_x^{-(n+q_1)}\big(B(w, 2R_1)\big) = \emptyset$, and, as $y \in J_f$, it follows from (5.4) that $y \notin B\big(x_1, 4^{-1}|(f^{n+q_1})'(x_1)|^{-1}R_1\big)$. Hence

$$r \geq \text{dist}(y, x_1) \geq 4^{-1}|(f^{n+q_1})'(x_1)|^{-1}R_1.$$

So, by Koebe's Distortion Theorem,

$$\begin{aligned}\text{diam}\big(f_{x_1}^{-(n+q_1)}(B(w, R_2))\big) &\leq \text{diam}\big(f_{x_1}^{-(n+q_1)}(B(w, R_1))\big) \\ &\leq 2K|(f^{n+q_1})'(x_1)|^{-1}R_1 \\ &\leq 8K\text{dist}(y, x_1).\end{aligned}$$

Thus,

$$\begin{aligned}f_x^{-(n+q_1)}(B(w, R_1)) &\subset B\big(y, \text{dist}(y, x_1) + 8K\text{dist}(y, x_1)\big) \\ &= B\big(y, (1 + 8K)\text{dist}(y, x_1)\big) \\ &\subset B(y, (1 + 8K)r) \\ &= B(y, \text{dist}(y, U^c)) \\ &\subset U\end{aligned} \tag{5.5}$$

Now assume that

$$B(w, R_1) \cap J_f \neq \emptyset.$$

Then the family of all holomorphic inverse branches $f_*^{-j}: B(w,2R_1)\to\mathbb{C}$, $j\geq o$, such that $f_*^{-j}(w)\in U$, is normal and all its limit points are constant functions. fix $a\in U$. There then exist $q_2\geq 0$ and $x_2\in B(a,2^{-1}\text{dist}(a,U^c))\cap f^{-(n+q_2)}(w)$ such that $f^{q_2}\circ f_{x_2}^{-(n+q_2)}=f_C^{-n}$ and $\text{diam}\big(f_{x_2}^{-(n+q_2)}(B(w,R_1))\big)<2^{-1}\text{dist}(a,U^c)$. Hence

$$
\begin{aligned}
f_{x_2}^{-(n+q_2)}(B(w,R_2)) &\subset f_{x_2}^{-(n+q_2)}(B(w,R_1)) \\
&\subset B\big(x_2,\text{diam}\big(f_{x_2}^{-(n+q_2)}(B(w,R_1))\big)\big) \\
&\subset B\big(x_2,2^{-1}\text{dist}(a,U^c)\big) \\
&\subset B(a,\text{dist}(a,U^c)) \\
&\subset U.
\end{aligned} \tag{5.6}
$$

Denote jointly q_1 and q_2 by q and x_1 and x_2 by x. The formulas (5.5) and (5.6) can be then jointly written as

$$
f_x^{-(n+q)}(B(w,R_2))\subset U. \tag{5.7}
$$

It follows from (b) applied to G that

$$
g^{n+q}\circ\tilde{H}\circ f_x^{-(n+q)}|_{B(w,R_2)\cap G}=\tilde{H}_G|_{B(w,R_2)\cap G}=\hat{H}|_{B(w,R_2)\cap G}.
$$

Hence

$$
g^{n+q}\circ\tilde{H}\circ f_x^{-(n+q)}=\hat{H}|_{B(w,R_2)}.
$$

Also since $C\subset U$ and, by (5.2) and (5.7), $f_x^{-(n+q)}\circ f^n(C)\subset f_x^{-(n+q)}(B(w,R_2))\subset U$, we have that

$$
g^q\circ\tilde{H}\circ\big(f_x^{-(n+q)}\circ f^n\big)|_C=\tilde{H}|_C.
$$

Therefore,

$$
\begin{aligned}
\hat{H}|_{f^n(C)} &= g^{n+q}\circ\tilde{H}\circ f_x^{-(n+q)}|_{f^n(C)} \\
&= g^n\circ\big(g^q\circ\tilde{H}\circ\big(f_x^{-(n+q)}\circ f^n\big)|_C\big)\circ f_C^{-n}|_{f^n(C)} \\
&= g^n\circ\tilde{H}|_C\circ f_C^{-n}|_{f^n(C)} \\
&= g^n\circ\tilde{H}\circ f_C^{-n}|_{f^n(C)}.
\end{aligned}
$$

This means that setting $\tilde{H}|_W:=\hat{H}$, condition (b) is satisfied and in consequence $W\in\mathcal{C}$, contrary to maximality of G. Thus, we have proved that there exists a holomorphic function $H_*:\mathbb{C}\setminus(\text{PS}(f)\setminus U)\to\mathbb{C}$ such that

$$
H_*|_U=\tilde{H}
$$

and

$$
g^n\circ\tilde{H}(z)=H_*\circ f^n(z) \tag{5.8}
$$

whenever $z \in U \cap f^{-n}\big(\mathbb{C} \setminus (\mathrm{PS}(f) \setminus U)\big)$. Our next step is to extend H_* beyond $\mathrm{PS}(f) \setminus U$. Set

$$V = \mathbb{C} \setminus (\mathrm{PS}(f) \setminus U).$$

Take an arbitrary point $w \in \mathrm{PS}(f) \setminus U$. Let

$$E_2(f) = \overline{\setminus} \bigcup_{k=0}^{\infty} f^k(U).$$

Montel's Theorem tells us that the set $E_2(f)$ consists of two points. Assume in addition that $w \notin E_2(f)$. Then there exist $k \geq 0$ and $\xi \in U$ such that $f^k(\xi) = w$. Take $R > 0$ so small that if $C_\xi(w, R)$ is the connected component of $f^{-k}(B(w, R))$ containing ξ, then

$$C_\xi(w, R) \cap f^{-k}(w) = \{\xi\}, \tag{5.9}$$

$$C_\xi(w, R) \subset U, \tag{5.10}$$

and the map $f^k|_{C_\xi(w,R)} : C_\xi(w, R) \to B(w, R)$ has no other critical points except possibly ξ. Let l be an arbitrary closed line segment joining w and $\partial B(w, R)$. There then exists $f_l^{-k} : B(w, R) \setminus l \to \mathbb{C}$, a holomorphic branch of f^{-k} such that

$$f_l^{-k}\big(B(w, R) \setminus l\big) \subset C_\xi(w, R).$$

Define the holomorphic map $\tilde{H} : B(w, R) \setminus l \to \mathbb{C}$ as

$$\tilde{H} = g^k \circ \tilde{H} \circ f_l^{-k}.$$

Because of (5.10) and (5.10),

$$\tilde{H}_l|_{V \cap (B(w,R) \setminus l)} = H_*|_{V \cap (B(w,R) \setminus l)},$$

and therefore, if q is another closed line segment joining w and $\partial B(w, R)$, then $\tilde{H}_l$ and $\tilde{H}_q$ coincide on the uncountable set $V \cap (B(w, R) \setminus (l \cup q))$. Hence, they glue together to a single holomorphic map $\tilde{H}_w : B(w, R) \setminus \{w\} \to \mathbb{C}$. In virtue of (5.9), $\lim_{z\to w} f_l^{-k}(z) = \xi$ and $\lim_{z\to w} f_q^{-k}(z) = \xi$. Therefore,

$$\lim_{z\to w} \tilde{H}_w(z) = g^k(\tilde{H}(\xi)).$$

Consequently $\tilde{H}_w$ extends holomorphically to a function from $B(w, R)$ to $\mathbb{C}$. Since $\tilde{H}_w$ and H_* coincide on $V \cap B(w, R)$, they glue together to a single holomorphic function $\tilde{H}_w^* : W_w \to \mathbb{C}$, where $W_w = V \cup B(w, R)$. We claim that all the maps $\tilde{H}_w^*$, $w \in (\mathrm{PS}(f) \setminus U) \setminus E_2(f)$, glue together to a single holomorphic function

$$H_f : \mathbb{C} \setminus E_2(f) = \bigcup_{(\mathrm{PS}(f) \setminus U) \setminus E_2(f)} W_w \to \mathbb{C}.$$

Indeed, let C be a connected component of $W_{w_1} \cap W_{w_2}$, $w_1, w_2 \in (\mathrm{PS}(f) \setminus U) \setminus E_2(f)$. Since the set V is dense in $\mathbb{C} \setminus E_2(F)$, the intersection $C \cap V$ is a non-empty open subset of V. But from our construction, $\tilde{H}^*_{w_2}|_V = \tilde{H}^*_{w_1}|_V$, whence $\tilde{H}_{w_2}|^*_{C\cap V} = \tilde{H}^*_{w_1}|_{C\cap V}$. Consequently $\tilde{H}^*_{w_2}|_C = \tilde{H}^*_{w_1}|_C$ and our claim is established. By the symmetry of the situation we also have now a holomorphic function $H_g^{-1} : \mathbb{C} \setminus E_2(f) \to \mathbb{C}$ which extends $H^{-1} : J_g \to J_f$ from some neighborhood of a point in J_g. But $H_g^{-1} \circ H_f$ is a holomorphic function well-defined on the set $\mathbb{C} \setminus \big(E_2(f) \cup H_f^{-1}(E_2(g))\big)$. We might have constructed H_g^{-1} starting with a nice set U_g such that $H^{-1}(J_g \cap U_g) \subset J_f \cap U$. Then

$$H_g^{-1} \circ H_f|_{H^{-1}\left(J_{U_g}\right)} = H_g^{-1}|_{J_{U_g}} \circ H_f|_{H^{-1}\left(J_{U_g}\right)} = \mathrm{Id}_{H^{-1}\left(J_{U_g}\right)}.$$

Thus,

$$H_g^{-1} \circ H_f = \mathrm{Id}_{\mathbb{C}\setminus\left(E_2(f)\cup H_f^{-1}(E_2(g))\right)}. \tag{5.11}$$

In particular, $H_f|_{\mathbb{C}\setminus\left(E_2(f)\cup H_f^{-1}(E_2(g))\right)}$ is one–to–one. Hence, $H_f : \mathbb{C} \setminus E_2(f) \to \mathbb{C}$ is one–to–one. Therefore $E_2(f)$ consists only of removable singularities of the function $H_f : \mathbb{C} \setminus E_2(f) \to \mathbb{C}$. Consequently H_f extends holomorphically to $\mathbb{C}$. The same holds for H_g^{-1}, and thus, because of (5.11), $H_g^{-1} \circ H_f = \mathrm{Id}_{\mathbb{C}}$. So, $H_f : \mathbb{C} \to \mathbb{C}$ is a holomorphic isomorphism and $H_f \circ f = g \circ H_f$ on $\mathbb{C}$. But then the map $H_f : \mathbb{C} \to \mathbb{C}$ must be affine linear, i.e. of the form $\mathbb{C} \ni z \mapsto az + b \in \mathbb{C}$. The proof is complete. □

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