Asymptotic degrees and measures on folded fractals

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Abstract

We study hyperbolic non-invertible maps f on fractal sets Λ of saddle type, preserving an equilibrium (Gibbs) measure μ_ϕ associated to an arbitrary Hölder potential ϕ. We prove a formula for a *measure-theoretic asymptotic logarithmic degree of* $f|_\Lambda$ with respect to μ_ϕ. This gives the average value of the logarithmic growth of the preimage counting functions of $f^n|_\Lambda, n \geq 1$, in the case when $f|_\Lambda$ is not constant-to-1. In particular we obtain an *asymptotic degree* of f_Λ, which takes into consideration the average behaviour on Λ of *all* the iterates. We study then weighted sums on preimage sets and show that a formula for the pressure $P(\phi)$, that holds in the expanding case, is no longer true on saddle sets. Non-invertibility creates new phenomena and demands techniques different than those commonly used in the diffeomorphism case. In the end we study concrete examples of hyperbolic basic sets for smooth endomorphisms.

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1 Introduction and outline of main results.

Stationary measures and Gibbs measures have a special role in thermodynamics (for instance [2], [3], [4], [17], [18], [20], etc.) We are concerned in this paper with equilibrium (Gibbs) measures of Hölder potentials on basic sets, and with several ergodic notions related to them, like the Jacobian of such a measure, and the "degree" of a map restricted to a folded fractal in the case when the number of f-preimages of a point is not constant.

The *Jacobian* of an invariant measure μ with respect to an endomorphism f of a Lebesgue space X (see Parry, [11]) describes locally the ratio between $\mu(f(A))$ and $\mu(A)$, given that an arbitrary point in X may have several f-preimages and that, by invariance $\mu(f(A)) = \mu(f^{-1}(f(A)))$.

Here we are concerned with the case when f is a $\mathcal{C}^2$ *endomorphism* (i.e a non-invertible map) on a manifold M, having a compact invariant set $\Lambda \subset M$. We assume that the endomorphism f is *hyperbolic* on Λ (see [13], [19]), i.e there exists a continuous splitting of the tangent bundle over the inverse limit $\hat\Lambda$, into stable and unstable directions. The map f is not assumed expanding on Λ, thus we do not have the machinery from the expanding case here. Hyperbolicity of f on Λ implies that we have local stable manifolds of type $W^s_r(x), x \in \Lambda$, and local unstable manifolds of type $W^u_r(\hat x)$ which depend on whole past trajectories $\hat x \in \hat\Lambda$. Through $x \in \Lambda$ there may pass even uncountably many local unstable manifolds corresponding to different prehistories of x in $\hat\Lambda$.

By **basic set** (or locally maximal set [5]) we shall mean a compact f-invariant set Λ s.t $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$ for a neighbourhood U of Λ, and such that f is topologically transitive on Λ. Such sets will also be referred to sometimes as *folded hyperbolic fractals*. We shall denote by μ_ϕ the (unique) *equilibrium measure* of a Hölder continuous potential ϕ on Λ. Examples of hyperbolic basic sets for endomorphisms appeared in many papers, for instance in [1], [4], [13], [12], [21], [22], [7], [9], etc.

The **main results** of the paper are the following:

In **Theorem 1** we will prove a formula and definition for the an *asymptotic logarithmic degree* with respect to μ_ϕ. This degree involves only those n-preimages of x (i.e preimages with respect to f^n) which behave well with respect to μ_ϕ; the number of these well-behaved n-preimages of x is denoted by $d_n(x, \mu_\phi, \tau)$ (see Definition 3). Notice that the dynamics of f on Λ is basically the same as that of f^n on Λ; the iterate f^n still invariates Λ and the measure μ_ϕ. Therefore in a sense one may take any iterate and study the preimages of points with respect to that iterate.

Theorem 1 (Asymptotic logarithmic degree in terms of the folding entropy for μ_ϕ)**.** *Let $f : M \to M$ be a $\mathcal{C}^2$ endomorphism and Λ a basic set for f so that f is hyperbolic on Λ and does not have critical points in Λ. Let also ϕ a Hölder continuous potential on Λ and μ_ϕ the equilibrium measure associated to ϕ. Then we have the following formula for the folding entropy of μ_ϕ:*

$$F_f(\mu_\phi) = \lim_{\tau\to 0} \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau) \ d\mu_\phi(x)$$

In **Corollary 1** we will use the formula proved in Theorem 1 in order to calculate the folding entropy of the measure of maximal entropy μ_0. This gives the average value with respect to μ_0 of the **logarithmic growth of the number of n-preimages** of x in Λ. As Λ is not necessarily totally invariant, the function $d_n(x) := Card(f^{-n}(x) \cap \Lambda), x \in \Lambda$ may be non-constant on Λ; see examples in [7]. So it is natural to study the average value of $\log d_n(\cdot)$.

Corollary 1 (Average value of $\log d_n(\cdot)$ for maps which are not constant-to-1 on Λ)**.** *In the setting of Theorem 1, denote by μ_0 the unique measure of maximal entropy for f on Λ. If $d_n(x)$ denotes the cardinality of $f^{-n}(f^n x) \cap \Lambda$ for $n \geq 1$, then the average logarithmic growth of d_n is given by*

$$\lim_{n\to\infty} \frac{1}{n} \int_\Lambda \log d_n(x) \ d\mu_0(x) = F_f(\mu_0)$$

In particular if $f|_\Lambda$ is d-to-1, then $F_f(\mu_0) = \log d$. Corollary 1 allows us to make the following:

Definition 1. In the setting of Theorem 1, define the **asymptotic logarithmic degree of** $f|_\Lambda$ (with respect to the measure of maximal entropy μ_0) by:

$$a_l(f, \Lambda) := \lim_n \frac{1}{n} \int_\Lambda \log d_n(x) d\mu_0(x)$$

The **asymptotic degree** of $f|_\Lambda$ is then defined as the number $d_\infty(f, \Lambda) := e^{a_l(f,\Lambda)}$.

In Section 3 we will give a class of skew-product endomorphisms hyperbolic on fractal sets, which are *not constant-to-one*. We show in **Corollary** 3 that, even though such maps are not constant 2-to-1, still their **asymptotic degree is equal to 2**. Thus we can compute the asymptotic degree in certain cases.

To prove Theorem 1 we will need **Proposition 1** which gives a formula for the **Jacobian** of an equilibrium measure μ_ϕ, with respect to an *arbitrary iterate* f^n; the estimates do not depend on n.

Proposition 1 (Jacobians of iterates of endomorphisms on folded sets)**.** *Let f be a $\mathcal{C}^2$ hyperbolic endomorphism on a folded basic set Λ, which has no critical points in Λ; let also ϕ a Hölder continuous potential on Λ and let μ_ϕ the unique equilibrium measure of ϕ on Λ. Then there exists a comparability constant $C > 0$ independent of m, x s.t for $\mu_\phi - a.e\ x \in \Lambda$ the Jacobian of μ_ϕ with respect to f^m satisfies:*

$$C^{-1} \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \leq \ J_{f^m}(\mu_\phi)(x) \leq \ C \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}}, \tag{1}$$

Recall now (see [14]) that in the expanding case we have the following formula for pressure:

Theorem (Relation between preimage sets and pressure in the expanding case, [14])**.** *Let $f : X \to X$ a topologically transitive open distance expanding map, then for every Hölder continuous potential $\phi : X \to \mathbb{R}$ and every $x \in X$ we have the equality*

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} e^{S_n\phi(y)} = P(\phi)$$

In our saddle set setting we obtain however the following different result:

Theorem 2 (Relation between preimage sets and pressure in the saddle case)**.** *In the setting of Proposition 1 and for an arbitrary Hölder continuous potential ϕ on Λ, we have for μ_ϕ-a.e $x \in \Lambda$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} e^{S_n\phi(y)} = P(\phi) + F_f(\mu_\phi) - h_{\mu_\phi}$$

The Remark after the proof of Theorem 2 shows that in general $F_f(\mu_\phi) \neq h_{\mu_\phi}$.

Another application is in the next Corollary, where we compute the μ_ϕ-measure of an **arbitrary ball** centered on Λ:

Corollary 2. *In the same setting as in Proposition 1, assuming f is conformal on both stable and unstable local manifolds, there is $C > 0$ such that the μ_ϕ-measure of an arbitrary ball is given by:*

$$\frac{1}{C} \int_{B_n(z,\varepsilon)} \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \, d\mu_\phi(x) \leq \mu_\phi(B(f^m z, \rho) \leq C \int_{B_n(z,\varepsilon)} \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \, d\mu_\phi(x),$$

where ε is fixed and m, n are the largest integers s.t $\varepsilon |Df_s^m(z)| \geq \rho$ and $\varepsilon |Df_u^{n-m}(f^m z)|^{-1} \geq \rho$, for any $z \in \Lambda, \rho > 0$.

In **Section 3** we will give several **examples** of hyperbolic basic fractal sets, and apply the results above to equilibrium measures on them. Such examples are obtained from perturbations of hyperbolic toral endomorphisms, or from skew products with Cantor sets of overlaps in fibers, saddle basic sets for holomorphic maps, or from solenoids with self-intersections.

2 Main results and proofs. Examples.

For the rest of the paper let us fix a smooth (say $\mathcal{C}^2$) non-invertible map $f : M \to M$ defined on a compact Riemannian manifold, and let Λ be a fixed basic set of f, such that f is hyperbolic on Λ. Sometimes Λ may be the whole manifold as in the case of Anosov endomorphisms. In general however Λ may not be totally invariant, i.e we do not always have $f^{-1}(\Lambda) = \Lambda$. As said before, hyperbolicity is understood here in the sense of *endomorphisms* (i.e non-invertible maps), i.e there exists a continuous splitting of the tangent bundle into stable and unstable directions, over the inverse limit $\hat\Lambda$ consisting of sequences of consecutive preimages, $\hat\Lambda = \{\hat x = (x, x_{-1}, x_{-2}, \dots,)$ with $x_{-i} \in \Lambda, f(x_{-i}) = x_{-i+1}, i \ge 1\}$. For any $\hat x \in \hat\Lambda$ we have a stable direction E^s_x and an unstable direction $E^u_{\hat x}$. There is a small $r > 0$ and local stable and local unstable manifolds, $W^s_r(x)$ and $W^u_r(\hat x)$ for any $\hat x \in \hat\Lambda$. Denote also

$$Df_s(x) := Df|_{E^s_x},\ x \in \Lambda \text{ and } Df_u(\hat x) := Df|_{E^u_{\hat x}},\ \hat x \in \hat\Lambda \tag{2}$$

The endomorphism f is assumed to have stable directions too, so it is non-expanding. More about hyperbolicity for endomorphisms can be found for example in [19], [10], etc. When the map is not invertible, there appear significantly different phenomena and different techniques than in the case of diffeomorphisms (as for example in [1], [17], [22], [7], [9], etc.)

We will use in the sequel the notions of *Jacobian of an invariant measure* introduced by Parry in [11]. Let $f : M \to M$ be a continuous endomorphism on the manifold M and μ an f-invariant probability on M; assume also that f is at most countable-to-one. Then as shown by Rohlin ([16], [11]), there exists a measurable partition $\xi = (A_0, A_1, \dots)$ so that f is injective on each A_i, and the push-forward measure $((f|_{A_i})^{-1})_*\mu$ is absolutely continuous on A_i with respect to μ. The respective Radon-Nykodim derivative, will be called the **Jacobian** of μ with respect to f:

$$J_f(\mu)(x) = \frac{d\mu \circ (f|_{A_i})}{d\mu}(x),\ \mu - \text{a.e on } A_i, i \ge 0$$

Notice that from the f-invariance of μ, we have $J_f(\mu)(x) \ge 1, \mu -$ a.e $x \in M$. Consider now in general $f : M \to M$ a $\mathcal{C}^1$ endomorphism and μ an f-invariant probability on the manifold M; then the *folding entropy* $F_f(\mu)$ of μ is the conditional entropy:

$$F_f(\mu) := H_\mu(\epsilon | f^{-1}\epsilon),$$

where ϵ is the partition into single points. From [16], we can use the measurable single point partition ϵ in order to desintegrate μ into a canonical family of conditional measures μ_x on the finite fiber $f^{-1}(x)$ for μ-a.e x. Hence the entropy of the conditional measure of μ restricted to $f^{-1}(x)$ is $H(\mu_x) = -\Sigma_{y\in f^{-1}(x)}\mu_x(y)\log\mu_x(y)$. From [11] we have also $J_f(\mu)(x) = \frac{1}{\mu_{f(x)}(x)}$, $\mu-$ a.e x, hence

$$F_f(\mu) = \int \log J_f(\mu)(x)d\mu(x) \tag{3}$$

Definition 2. Given two positive functions $Q_1(n,x), Q_2(n,x)$, we will say that they are **comparable** if there exists a positive constant C so that $\frac{1}{C} \le \frac{Q_1(n,x)}{Q_2(n,x)} \le C$ for all n, x.

Recall also (for example from [5]) that, given an expansive homeomorphism $f : X \to X$ on a compact metric space, having the specification property, the equilibrium measure μ_ϕ of the Hölder potential ϕ satisfies $A_\varepsilon e^{S_n\phi(x)-nP(\phi)} \le \mu_\phi(B_n(x,\varepsilon)) \le B_\varepsilon e^{S_n\phi(x)-nP(\phi)}$, where $B_n(x,\varepsilon) := \{y \in X, d(f^iy, f^ix) < \varepsilon, i = 0,\dots,n-1\}$, $P(\phi)$ denotes the topological pressure of ϕ with respect to f, and where the positive constants $A_\varepsilon, B_\varepsilon$ are independent of x, n.

The general homeomorphism framework above allows us to apply this result to equilibrium measures on the inverse limit $\hat\Lambda$. If $\pi : \hat\Lambda \to \Lambda, \pi(\hat x) := x, \hat x \in \hat\Lambda$ is the *canonical projection* and if ϕ is a Hölder potential on Λ, then μ_ϕ is the unique equilibrium measure for ϕ on Λ if and only if $\mu_\phi = \pi_*\mu_{\phi\circ\pi}$, where $\mu_{\phi\circ\pi}$ is the unique equilibrium measure of $\phi\circ\pi$ on the compact metric space $\hat\Lambda$; here the homeomorphism $\hat f : \hat\Lambda \to \hat\Lambda$ is the shift map defined by $\hat f(x, x_{-1}, x_{-2},\dots) = (f(x), x, x_{-1},\dots)$. So for the non-invertible map f and the measure μ_ϕ we obtain the same estimate as above:

$$A_\varepsilon e^{S_n\phi(x)-nP(\phi)} \le \mu_\phi(B_n(x,\varepsilon)) \le B_\varepsilon e^{S_n\phi(x)-nP(\phi)},$$

with positive constants $A_\varepsilon, B_\varepsilon$ independent of n, x, where the consecutive sum $S_n\phi$ is defined as

$$S_n\phi(x) := \phi(x) + \ldots + \phi(f^{n-1}(x)), \text{for } x \in \Lambda,\ n \in \mathbb{N}$$

Let us give now the proof of the main result of the paper, namely a formula for the asymptotic logarithmic degree with respect to μ_ϕ; this degree takes into consideration those n-preimages which behave well with respect to μ_ϕ. We assume for the moment that Proposition 1 is known; its proof is independent of Theorem 1 and will be given later in the paper.

First, for an f-invariant probability μ on Λ, $\tau > 0$ small, $n \in \mathbb{N}$ and $x \in \Lambda$ let us define the set

$$G_n(x,\mu,\tau) := \{y \in f^{-n}(f^nx) \cap \Lambda, \text{ s.t } |\frac{S_n\phi(y)}{n} - \int \phi d\mu| < \tau\}, \tag{4}$$

Definition 3. In the above setting, denote by $d_n(x,\mu,\tau) := \text{Card}G_n(x,\mu,\tau), x \in \Lambda, n > 0, \tau > 0$. The function $d_n(\cdot,\mu,\tau)$ is measurable, nonnegative and finite on Λ.

Proof of Theorem 1. First let us recall formula (3) for an arbitrary f-invariant measure μ,

$$F_f(\mu) = \int_\Lambda \log J_f(\mu)(x)d\mu_(x)$$

From the Chain Rule for Jacobians, $J_{f^n}(\mu)(x) = J_f(\mu)(x)\ldots J_f(\mu)(f^{n-1}(x))$ μ-a.e, for any $n \ge 1$. On the other hand, since μ is f-invariant, we have that

$$\int \log J_f(\mu)(x)d\mu(x) = \int \log J_f(\mu)(f(x))d\mu(x) = \int \log J_f(\mu)(f^kx)d\mu(x),$$

for all $k \geq 1$. These facts imply that for any $n \geq 1$,

$$F_f(\mu) = \frac{1}{n} \int \log J_{f^n}(\mu)(x) d\mu(x) \tag{5}$$

Therefore from Proposition 1, since the constant C is independent of n we obtain that:

$$F_f(\mu_\phi) = \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log \frac{\sum\limits_{y \in f^{-n}(f^n(x)) \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \tag{6}$$

Now since Λ is compact, each point $x \in \Lambda$ has only finitely many f-preimages in Λ, i.e there exists a positive integer d s.t $\mathrm{Card}(f^{-1}x) \leq d, x \in \Lambda$. Since μ_ϕ is an ergodic measure (as it is an equilibrium state) and from Birkhoff Ergodic Theorem we obtain that

$$\mu_\phi \left(x \in \Lambda, |\frac{S_n\phi(x)}{n} - \int \phi d\mu| > \tau/2 \right) \underset{n \to \infty}{\to} 0,$$

for any small $\tau > 0$. Thus for any $\eta > 0$ there exists a large integer $n(\eta)$ such that for $n \geq n(\eta)$,

$$\mu_\phi(x \in \Lambda, |\frac{S_n\phi(x)}{n} - \int \phi d\mu| > \tau/2) < \eta \tag{7}$$

Let us now take a point $x \in \Lambda$ with $|\frac{S_n\phi(x)}{n} - \int \phi d\mu| < \tau$. From Definition 3 we have

$$\frac{e^{n(\int \phi d\mu_\phi - \tau)} d_n(x, \mu_\phi, \tau) + r_n(x, \mu_\phi, \tau)}{e^{n(\int \phi d\mu + \tau)}} \leq \frac{\sum\limits_{y \in f^{-n}(f^n x) \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} \leq \frac{e^{n(\int \phi d\mu_\phi + \tau)} d_n(x, \mu_\phi, \tau) + r_n(x, \mu_\phi, \tau)}{e^{n(\int \phi d\mu_\phi - \tau)}}, \tag{8}$$

where $r_n(x, \mu_\phi, \tau)$ is the remainder $\sum\limits_{y \in f^{-n} f^n(x) \setminus G_n(x, \mu_\phi, \tau)} e^{S_n\phi(y)}$. In order to simplify notation, we will also denote $r_n(x, \mu_\phi, \tau)$ by r_n when no confusion can arise.

Given n large, let us consider now a partition $(A_i^n)_{1 \leq i \leq K}$ of Λ (modulo μ_ϕ) so that for each $0 \leq i \leq K$, there exists a point $z_i \in A_i^n$ so that for any n-preimage $\xi_{ij} \in f^{-n}(z_i) \cap \Lambda, 1 \leq j \leq d_{n,i}$, we have $A_i^n \subset f^n(B_n(\xi_{ij}, \varepsilon)), 1 \leq j \leq d_{n,i}, 1 \leq i \leq K$. For the above partition, let us denote by A_{ij}^n the part of the n-preimage of A_i^n which belongs to the Bowen ball $B_n(\xi_{ij}, \varepsilon)$, i.e

$$A_{ij}^n := f^{-n}(A_i^n) \cap B_n(\xi_{ij}, \varepsilon), 1 \leq j \leq d_{n,i}, 1 \leq i \leq K$$

Since the sets A_i^n were chosen disjoint, also the pieces of their preimages, namely A_{ij}^n, i, j, are mutually disjoint.

We will decompose the integral in (6) over the sets A_{ij}^n. Notice that if $y, z \in A_{ij}^n$, then since ϕ is Hölder continuous and $A_{ij}^n \subset B_n(\xi_{ij}, \varepsilon)$, it follows that we have

$$|S_n\phi(y) - S_n\phi(z)| \leq C(\varepsilon), \tag{9}$$

where $C(\varepsilon)$ is a positive function with $C(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$. So we will obtain:

$$\int_\Lambda \log \frac{\sum\limits_{y \in f^{-n} f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) = \sum_{0 \leq j \leq d_i, 0 \leq i \leq K} \int_{A_{ij}^n} \log \frac{\sum\limits_{y \in f^{-n} f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \tag{10}$$

Let us now denote by $R_n(i, \mu_\phi, \tau)$ the set of preimages ξ_{ij} with $\xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)$, and denote simply by $R_{n,i}$ the set of indices $j, 1 \le j \le d_{n,i}$ with $\xi_{ij} \in R_n(i, \mu_\phi, \tau)$ for every $1 \le i \le K$. Now in the decomposition from (10) we notice that the integral over those sets A^n_{ij} with $j \in R_{n,i}$ will not matter significantly. Indeed as $\text{Card}(f^{-1}x \cap \Lambda) \le d, x \in \Lambda$ and since $-M \le \phi(x) \le M, x \in \Lambda$ we have

$$1 \le \frac{\sum\limits_{y \in f^{-n}f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} \le d^n e^{2nM}$$

Now recall that each $A^n_{ij} \subset B_n(\xi_{ij}, \varepsilon)$ and the sets A^n_{ij}, i, j are mutually disjoint (with respect to μ_ϕ). Hence by using inequalities (7) and (9) and the fact that $\xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)$ whenever $j \in R_{n,i}$, we obtain:

$$\sum_{0 \le i \le K, j \in R_{n,i}} \frac{1}{n} \int_{A^n_{ij}} \log \frac{\sum\limits_{y \in f^{-n}f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \le \frac{1}{n} \log(d^n e^{2nM}) \cdot \eta = \eta(\log d + 2M) \tag{11}$$

But by using the comparison between different parts of the n-preimage of a small set from the proof of Proposition 1 (see (18)), we deduce that the last term of formula (10) is comparable to

$$\sum_{i,j} \mu_\phi(A^n_{ij}) \log \frac{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A^n_{ij}) + \tilde{r}_n(z_i, \mu_\phi, \tau)}{\mu_\phi(A^n_{ij})}, \tag{12}$$

where $\tilde{r}_n(z_i, \mu, \tau) := \sum\limits_{\xi_{ij} \in f^{-n}(z_i) \cap \Lambda,\ \xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)} \mu_\phi(A^n_{ij})$. Hence from (18), (11) and (12) we obtain:

$$\begin{aligned}
&\frac{1}{n} \sum_{i, j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \log d_n(z_i, \mu_\phi, \tau) + \frac{1}{n} \sum_{i, j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \log(1 + \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A^n_{ij})}) - \delta(\tau) - \eta C' \le \\
&\le \int_\Lambda \frac{1}{n} \log \frac{\sum\limits_{y \in f^{-n}f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \le \\
&\le \frac{1}{n} \sum_{i, j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \log d_n(z_i, \mu_\phi, \tau) + \frac{1}{n} \sum_{i, j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \log(1 + \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A^n_{ij})}) + \delta(\tau) + \eta C',
\end{aligned} \tag{13}$$

with $C' = \log d + 2M$ being the constant found in (11), and where the positive constant $\delta(\tau)$ comes from the uniformly bounded variation of $\frac{1}{n} S_n\phi(x)$ when x is in A^n_{ij} and when $1 \le i \le K, j \notin R_{n,i}$ vary; clearly we have $\delta(\tau) \underset{\tau \to 0}{\to} 0$.

Now we know that in general $\log(1 + x) \le x$, for $x > 0$. Thus $\log(1 + \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A^n_{ij})}) \le \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A^n_{ij})}, i, j$ and hence in (13) we have, for n large enough that:

$$\begin{aligned}
&\sum_{i, j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \log(1 + \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A^n_{ij})}) \le \sum_{i, j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A^n_{ij})} = \\
&= \sum_{1 \le i \le K} \tilde{r}_n(z_i, \mu_\phi, \tau) \le \eta,
\end{aligned} \tag{14}$$

where we used that by definition, there are $d_n(z_i, \mu_\phi, \tau)$ indices j in $\{1, \ldots, d_{n,i}\} \setminus R_{n,i}$ for any $1 \le i \le K$.

Therefore from the last displayed inequality and from (13) we obtain, for $n \ge n(\eta)$, that:

$$\left| \frac{1}{n} \int_\Lambda \log \frac{\sum\limits_{y \in f^{-n} f^n x \cap \Lambda} e^{S_n \phi(y)}}{e^{S_n \phi(x)}} d\mu_\phi(x) - \frac{1}{n} \int_\Lambda \log d_n(z, \mu_\phi, \tau) d\mu_\phi(z) \right| \le \delta(\tau) + \eta, \tag{15}$$

where $\delta(\tau) \underset{\tau \to 0}{\to} 0$. Then by taking $n \to \infty$ and $\tau \to 0$, we will obtain the conclusion of the Theorem from (6) and (15), namely that

$$F_f(\mu_\phi) = \lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau) d\mu_\phi(x)$$

□

We give now the proof of the auxilliary Proposition 1; this proof is independent of Theorem 1.

Proof of Proposition 1. We know from definition that the Jacobian $J_{f^m}(\mu_\phi)$ is the Radon-Nikodym derivative of $\mu_\phi \circ f^m$ with respect to μ_ϕ on sets of injectivity for f^m. In order to estimate the Jacobian of μ_ϕ with respect to f^m, we have to compare the measure μ_ϕ on different components of the preimage set $f^{-m}(B)$, for a small Borel set B, where $m \ge 1$ is fixed. Let us consider two subsets E_1, E_2 of Λ so that $f^m(E_1) = f^m(E_2) \subset B$ and E_1, E_2 belong to two disjoint balls $B_m(y_1, \varepsilon)$, respectively $B_m(y_2, \varepsilon)$. This happens if $\text{diam}(B)$ is small enough, since f has no critical points in Λ and thus there exists a positive distance ε_0 between any two different preimages from $f^{-1}(y)$ for $y \in \Lambda$. As in [5] since the borelian sets with boundaries of measure zero form a sufficient collection, we can assume that each of the sets E_1, E_2 have boundaries of μ_ϕ-measure zero. Recall that $f^m(E_1) = f^m(E_2)$. As in [5], μ_ϕ is the limit of the sequence of measures: $\tilde{\mu}_n := \frac{1}{P(f,\phi,n)} \cdot \sum\limits_{x \in \text{Fix}(f^n) \cap \Lambda} e^{S_n \phi(x)} \delta_x$, where $P(f, \phi, n) := \sum\limits_{x \in \text{Fix}(f^n) \cap \Lambda} e^{S_n \phi(x)}, n \ge 1$. So we obtain

$$\tilde{\mu}_n(E_1) = \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \text{Fix}(f^n) \cap E_1} e^{S_n \phi(x)}, n \ge 1 \tag{16}$$

Consider a periodic point $x \in \text{Fix}(f^n) \cap E_1$; it follows that $f^m(x) \in f^m(E_1)$, so there exists a point $y \in E_2$ such that $f^m(y) = f^m(x)$. However the point y is not necessarily periodic. We will use now the Specification Property ([5], [2]) on hyperbolic locally maximal sets. If $\varepsilon > 0$ is fixed, there exists a constant $M_\varepsilon > 0$ such that for all $n > M_\varepsilon$, there is a point $z \in \text{Fix}(f^n) \cap \Lambda$ which ε-shadows the $(n - M_\varepsilon)$-orbit of y. In particular $z \in B_m(y_2, 2\varepsilon)$, since $E_2 \subset B_m(y_2, \varepsilon)$. Let now $V \subset B_m(y_2, \varepsilon)$ be an arbitrary neighbourhood of the set E_2. Take two points $x, x' \in \text{Fix}(f^n) \cap E_1$ and assume the same periodic point $z \in V \cap \text{Fix}(f^n)$ corresponds to both of them through the previous shadowing procedure. Thus the $(n - M_\varepsilon - m)$-orbit of $f^m(z)$ ε-shadows the $(n - M_\varepsilon - m)$-orbit of $f^m(x)$ and also the $(n - M_\varepsilon - m)$-orbit of $f^m(x')$. So the $(n - M_\varepsilon - m)$-orbit of $f^m(x)$ 2ε-shadows the $(n - M_\varepsilon - m)$-orbit of $f^m(x')$. But we took $x, x' \in E_1 \subset B_m(y_1, \varepsilon)$, so $x' \in B_m(x, 2\varepsilon)$ and hence from above, $x' \in B_{n-M_\varepsilon}(x, 2\varepsilon)$. We partition now the set $B_{n-M_\varepsilon}(x, 2\varepsilon)$ into smaller Bowen balls of type $B_n(\zeta, 2\varepsilon)$, and let us denote their number by N_ε. In each of these $(n, 2\varepsilon)$-Bowen balls we may have at most one fixed point for f^n. Then if $d(f^i\xi, f^i\zeta) < 2\varepsilon, i = 0, \ldots, n-1$ and if ε is small enough, we can apply the Inverse Function Theorem at each step, and thus there exists only one fixed point for f^n in $B_n(\zeta, 2\varepsilon)$. So there exist at most N_ε periodic points in Λ from $\text{Fix}(f^n) \cap E_1$ having the same point $z \in V \cap \text{Fix}(f^n)$ associated to them by the above procedure. Notice also that if $x, x' \in \text{Fix}(f^n) \cap E_1$ have the same point $z \in V$ attached to them, then $x' \in B_{n-M_\varepsilon}(x, 2\varepsilon)$ and then, from the Hölder continuity of ϕ it follows $|S_n\phi(x) - S_n\phi(x')| \leq \hat{C}_\varepsilon$, for some positive constant $\hat{C}_\varepsilon$ depending on ϕ (but independent of n, m, x). This can be used then in the estimate for $\tilde{\mu}_n(E_1)$ from (16). Notice also that if $z \in B_{n-M_\varepsilon}(y, \varepsilon)$, then $f^m(z) \in B_{n-M_\varepsilon-m}(f^m(x), \varepsilon)$. Thus from the Hölder continuity of ϕ and since $x \in E_1 \subset B_m(y_1, \varepsilon)$, it follows that there exists a positive constant $\hat{C}'_\varepsilon$ satisfying: $|S_n\phi(z) - S_n\phi(x)| \leq |S_m\phi(y_1) - S_m\phi(y_2)| + \hat{C}'_\varepsilon$, for $n > n(\varepsilon, m)$. Then using also (16), and since there are at most N_ε points $x \in \text{Fix}(f^n) \cap E_1$ having the same $z \in V \cap \text{Fix}(f^n) \cap \Lambda$ corresponding to them, we obtain that there exists a constant $C_\varepsilon > 0$ s.t:

$$\tilde{\mu}_n(E_1) \leq C_\varepsilon \tilde{\mu}_n(V) \cdot \frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}}, \tag{17}$$

where we recall that $E_1 \subset B_m(y_1, \varepsilon), E_2 \subset B_m(y_2, \varepsilon)$ and $f^m(E_1) = f^m(E_2)$. But $\partial E_1, \partial E_2$ were assumed of μ_ϕ-measure zero, hence: $\mu_\phi(E_1) \leq C_\varepsilon \mu_\phi(V) \cdot \frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}}$. But V was chosen arbitrarily as a neighbourhood of E_2, and by applying the same procedure for E_1 we obtain:

$$\frac{1}{C}\mu_\phi(E_2)\frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}} \leq \mu_\phi(E_1) \leq C\mu_\phi(E_2)\frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}}, \tag{18}$$

where $C > 0$ does not depend on m, E_1, E_2.

Now the Jacobian $J_{f^m}(\mu_\phi)$ is the Radon-Nikodym derivative of $\mu_\phi \circ f^m$ with respect to μ_ϕ on sets of injectivity for f^m, hence

$$\mu_\phi(f^m(D)) = \int_D J_{f^m}(\mu_\phi)(x) d\mu_\phi(x),$$

for any borelian set D on which f^m is injective. Hence from the Lebesgue Density Theorem, we have that, by putting $D = B(x, r)$ for small $r > 0$, we obtain:

$$J_{f^m}(\mu_\phi)(x) = \lim_{r \to 0} \frac{\mu_\phi(f^m(B(x,r)))}{\mu_\phi(B(x,r))}, \tag{19}$$

for μ_ϕ-almost all $x \in \Lambda$. On the other hand from the invariance of μ_ϕ, we have for any Borel set D that:

$$\mu_\phi(f^m(D)) = \mu_\phi(f^{-m}(f^m D)) \tag{20}$$

Thus if D is a small ball around x, one has to consider the m-preimages y of x, belonging to Λ. Let us notice that if $\zeta \in B_m(y, \varepsilon)$ then, from the Hölder continuity of ϕ we have that $|S_m\phi(\zeta) - S_m\phi(y)| \leq \hat{C}_\varepsilon$, where the constant $\hat{C}_\varepsilon$ does not depend on $m > 0, y \in \Lambda$. So in the comparison inequlities of (18), we can take instead of y_1, y_2, the respective m-preimages of x belonging to Λ.

Therefore from relationship (19), the invariance in (20) and the comparison between different pieces of the m-preimage from (18), it follows that the Jacobian of μ_ϕ with respect to f^m satisfies:

$$J_{f^m}(\mu_\phi)(x) \approx \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x))\cap\Lambda} e^{S_m\phi(\zeta)}}{e^{S^m\phi(x)}},\ \mu_\phi - \text{a.e } x \in \Lambda,$$

where the comparability constant $C > 0$ is independent of $m > 1, x \in \Lambda$.

□

Let us recall now that in the expanding case we have a formula relating $P(\phi)$ to the preimage sets of $f^n, n \geq 1$ (given in Section 1, see [14]); however the proof for that result does not work in the saddle case. We give then in our saddle case, the proof of a formula for $P(\phi)$ in terms of the folding entropy and the preimage sets, announced in Theorem 2 in Section 1:

Proof of Theorem 2.

First recall that ϕ is a Hölder continuous function on the hyperbolic basic set Λ, then its unique equilibrium measure μ_ϕ is ergodic. Also from the properties of the Jacobian of μ_ϕ we know that it satisfies the Chain Rule, i.e $J_{f\circ g}(\mu_\phi)(x) = J_f(\mu_\phi)(g(x)) \cdot J_g(\mu_\phi)(x)$ for μ_ϕ-a.e $x \in \Lambda$. Hence μ_ϕ-a.e,

$$\log J_{f^m}(\mu_\phi)(x) = \log J_f(\mu_\phi)(x) + \ldots + \log J_f(\mu_\phi)(f^{m-1}(x))$$

This means that we can apply Birkhoff Ergodic Theorem and obtain that

$$\frac{\log J_{f^m}(\mu_\phi)}{m} \underset{m\to\infty}{\to} \int_\Lambda \log J_f(\mu_\phi) d\mu_\phi = F_f(\mu_\phi)$$

We apply now Proposition 1 to get μ_ϕ-a.e the convergence

$$\frac{\log \sum\limits_{y\in f^{-m}(f^m x)} e^{S_m\phi(y)} - \log e^{S_m\phi(x)}}{m} \underset{m\to\infty}{\to} F_f(\mu_\phi) \tag{21}$$

But again from Birkhoff Ergodic Theorem, $\frac{S_m\phi(x)}{m} \to \int \phi d\mu_\phi$ for μ_ϕ-a.e $x \in \Lambda$. Thus from (21) and the definition of equilibrium measure $P(\phi) = \int \phi d\mu_\phi + h_{\mu_\phi}$, we obtain:

$$\frac{\log \sum\limits_{y\in f^{-m}(f^m x)} e^{S_m\phi(y)}}{m} \underset{m\to\infty}{\to} F_f(\mu_\phi) + P(\phi) - h_{\mu_\phi}$$

□

Remark:

In general $F_f(\mu_\phi) \neq h_{\mu_\phi}$. Indeed consider the inverse SRB measure μ^- introduced on a d-to-1 hyperbolic repeller Λ in [8]. Then this measure μ^- satisfies

$$h_{\mu^-}(f) = \log d - \int_\Lambda \sum_{i, \lambda_i(\mu^-,x)<0} \lambda_i(\mu^-, x) m_i(\mu^-, x) d\mu^-(x),$$

where $\lambda_i(\mu^-, x)$ are the Lyapunov exponents of μ^- at x and $m_i(\mu^-, x)$ their respective multiplicities. Now if there are negative Lyapunov exponents on Λ, for instance for the hyperbolic repellers introduced in [8] and explained below in Example 3), then we obtain that $F_f(\mu^-) = \log d$, whereas $h_{\mu^-} > \log d$.

$\square$

We obtain then directly from Theorem 1 the proof for **Corollary 1**, thus giving the asymptotic degree of $f|_\Lambda$ in terms of the folding entropy of the measure of maximal entropy μ_0 on Λ. Corollary 1 expresses the average value of the **logarithmic growth of the preimage counting function of** $f^n|_\Lambda$; as Λ is not necessarily totally invariant, $d_n(\cdot)$ may be non-constant and it is difficult to obtain the number of preimages of x in Λ; see the examples from [7] which are not constant-to-one on their basic sets, and the effect of preimages on dynamics and the stable dimension.

A consequence of Proposition 1 is Corollary 2, which gives the measure μ_ϕ of **an arbitrary** ball in Λ. It can be proved by writing an arbitrary ball $B(y, \rho)$ as the iterate $f^m(B_n(z, \varepsilon))$ of a certain Bowen ball, in such a way that the iterate $f^m(B_n(z, \varepsilon)$ has roughly the same sides in the stable and unstable directions; here $y = f^m(z)$ and $\rho > 0$ is arbitrary.

Ruelle defined in [18] the *entropy production* of an f-invariant measure μ for a smooth endomorphism by $e_f(\mu) := F_f(\mu) - \int \log |\det Df(x)| d\mu(x)$. Then Theorem 1 gives immediately a formula for $e_f(\mu_\phi)$. Also due to Kingman Subadditive Theorem ([23]) we have $\int_\Lambda \log |\det Df(x)| d\mu_\phi(x) = \sum_i \lambda_i(\mu_\phi)$, where $\lambda_i(\mu_\phi)$ are the Lyapunov exponents of μ_ϕ. In particular for the measure of maximal entropy μ_0 we obtain that $e_f(\mu_0) \geq 0$ if and only if $d_\infty(f, \Lambda) \geq e^{\sum_i \lambda_i(\mu_0)}$.

3 Applications to examples of folded hyperbolic fractals.

1) Let us give now a class of examples of hyperbolic endomorphisms on saddle-type basic sets, **which are not repellers**. For a small positive α, take the subintervals $I_1^\alpha, I_2^\alpha \subset I := [0, 1]$, of small positive length, with $I_1^\alpha = [b_1(\alpha), b_2(\alpha)], I_2^\alpha = [b_3(\alpha), b_4(\alpha)]$; assume that $b_2(\alpha) < \frac{1}{2}$, $b_2(\alpha)$ is close to $\frac{1}{2}$, and that $b_4(\alpha)$ is close to $1-\alpha$ and $b_4(\alpha) < 1-\alpha$; we assume that $|b_1(\alpha) - \frac{1}{2}|$ and $|b_3(\alpha) - (1-\alpha)|$ are both much smaller than α, for instance $0 < \max\{|b_1(\alpha) - \frac{1}{2}|,\ |b_3(\alpha) - (1-\alpha)|\} =: \epsilon(\alpha) < \alpha^2$.

Let us take now $g : I_1^\alpha \cup I_2^\alpha \to I$, a strictly increasing smooth map which expands both I_1^α and I_2^α to I, i.e $g(I_1^\alpha) = g(I_2^\alpha) = I$. Assume that $g'(x) > \beta(\alpha) >> 1, x \in I_1^\alpha \cup I_2^\alpha$. Hence there exist subintervals $I_{11}^\alpha, I_{12}^\alpha \subset I_1^\alpha$ and $I_{21}^\alpha, I_{22}^\alpha \subset I_2^\alpha$ such that $g(I_{11}^\alpha) = g(I_{21}^\alpha) = I_1^\alpha$ and $g(I_{12}^\alpha) = g(I_{22}^\alpha) = I_2^\alpha$. Denote by $J^\alpha := I_{11}^\alpha \cup I_{12}^\alpha \cup I_{21}^\alpha \cup I_{22}^\alpha$ and $J_*^\alpha := \{x \in J^\alpha,\ g^i x \in J^\alpha, i \geq 0\}$. Define now for a small α the skew product with overlaps in fibers $f_\alpha : J_*^\alpha \times I \to J_*^\alpha \times I$, where

$$f_\alpha(x, y) = (g(x), h_\alpha(x, y)),$$

$$h_\alpha(x, y) = \begin{cases} x + \frac{y}{2},\ x \in I_{11}^\alpha \\ 1 - x + \frac{y}{2},\ x \in I_{21}^\alpha \\ 1 - \frac{y}{2},\ x \in I_{12}^\alpha \\ \frac{y}{2},\ x \in I_{22}^\alpha \end{cases} \qquad (22)$$

We denote also the function $h_\alpha(x, \cdot) : I \to I$ by $h_{x,\alpha}$ for $x \in J_*^\alpha$. From the definition of $h_\alpha(x, y)$ it can be seen that for $x \in J_*^\alpha \cap I_1$, there are two images of intervals intersecting inside $\{x\} \times I$, namely $h_{x_{-1},\alpha}(I)$ and $h_{\tilde{x}_{-1},\alpha}$, where x_{-1} denotes the g-preimage of x belonging to I_1^α, and $\tilde{x}_{-1}$ denotes the g-preimage of x belonging to I_2^α. Then, with $h^n_{y,\alpha} := h_{f^{n-1}y,\alpha} \circ \ldots \circ h_{y,\alpha}, n \geq 0$,

$$\Lambda_\alpha := \bigcup_{x\in J_*^\alpha} \bigcap_{n\geq 0} \bigcup_{y\in g^{-n}x\cap J_*^\alpha} h^n_{y,\alpha}(I), \tag{23}$$

It was proved in [7] that f_α is hyperbolic as an endomorphism on Λ_α and that there are Cantor sets of points in Λ_α where the unstable manifolds depend on whole prehistories, so f_α has a strong non-invertible character on Λ_α. Also f_α was proved **not** to be constant-to-one. In fact Λ_α can be partitioned as $V_1 \cup V_2$, where V_1 is the set of points with exactly one preimage in Λ_α and V_2 is the set of points with exactly two preimages in Λ_α; both V_1, V_2 are uncountable sets.

Corollary 3. *For the above endomorphism f_α and its hyperbolic basic set Λ_α we have that f_α is not constant-to-one on Λ_α, and that the asymptotic degree satisfies:*

$$d_\infty(f_\alpha, \Lambda_\alpha) = 2$$

Proof. By applying Corollary 1 to this example, it follows we can express the asymptotic logarithmic growth of the number $d_{n,\alpha}(x)$ of n-preimages of x with respect to $f_\alpha|_{\Lambda_\alpha}$ with the help of the folding entropy of the measure of maximal entropy $\mu_{0,\alpha}$ on Λ_α:

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Lambda_\alpha} \log d_{n,\alpha}(x) d\mu_{0,\alpha}(x) = F_{f_\alpha}(\mu_{0,\alpha}) \tag{24}$$

In our case, since f_α has an expanding map in the base and contracting vertical directions, it follows that the spanning sets are determined only by g, and thus $h_{top}(f_\alpha|_{\Lambda_\alpha}) = \log 4$, since the dynamics in the base is conjugated to the shift on the symbolic space Σ_4. On the other hand we can apply the formula from [6], namely $h_\mu \leq F_{f_\alpha}(\mu) - \sum\limits_{\lambda_i<0} m_i \lambda_i(\mu)$, where $\lambda_i(\mu)$ are the Lyapunov exponents of μ and m_i their respective multiplicities. In our case we have only one negative Lyapunov exponent, and since $|D(f_\alpha)_s| \equiv \frac{1}{2}$, we obtain that $\lambda_1 = -\log 2$; also $m_1 = 1$. Thus from the last displayed inequality applied to the measure of maximal entropy $\mu_{0,\alpha}$, it follows that

$$\log 4 \leq F_{f_\alpha}(\mu_{0,\alpha}) + \log 2$$

But recall Corollary 1 which says that $F_{f_\alpha}(\mu_{0,\alpha}) \leq \log 2$, since $d_n(x) \leq 2^n, x \in \Lambda_\alpha$. This implies that $F_{f_\alpha}(\mu_{0,\alpha}) = \log 2$, hence from Corollary 1, the asymptotic degree is $d_\infty(f_\alpha, \Lambda_\alpha) = 2$.

□

Also notice that Λ_α is not an attractor since from Theorem 4 of [7] the unstable dimension is strictly less than 1, so unstable manifolds cannot be contained in Λ_α. Thus from [15] it follows that f_α cannot have an SRB measure on Λ_α. We can also apply Proposition 1 to obtain the Jacobian $J_{f_\alpha}(\mu_\phi)$ and Theorem 1 to obtain the folding entropy $F_{f_\alpha}(\mu_\phi)$, for an equilibrium measure μ_ϕ of an arbitrary Hölder continuous potential ϕ on Λ_α. Notice also that for any $x \in J_*^\alpha$, $|\text{det}Df_\alpha|(x,y) = \frac{|g'(x)|}{2}$. However the subintervals I_1^α, I_2^α can be taken very small, thus $|g'(x)|$ will be large. Now from [7], each point in Λ_α has at most two f_α-preimages remaining in Λ_α, so for any invariant probability measure μ on Λ_α, we have $F_{f_\alpha}(\mu) = H_\mu(\epsilon | f_\alpha^{-1}\epsilon) \le \log 2$.

2) An important class of examples are given by hyperbolic toral (linear) endomorphisms $f_A : \mathbb{T}^m \to \mathbb{T}^m$ and their *perturbations*; they are Anosov endomorphisms. Notice that a small $\mathcal{C}^2$ perturbation g of f_A is not necessarily conjugated to f_A if f_A is not invertible ([13]). Also notice that f_A is $|\det(A)|$-to-1 on $\mathbb{T}^m$, and the same is true also for g; however given an equilibrium measure μ_ϕ for g, perhaps not all g-preimages are well behaved with respect to μ_ϕ.

Then by using Theorem 1 and Proposition 1 we can obtain the Jacobian of μ_ϕ and the asymptotic logarithmic degree with respect to μ_ϕ. Moreover by applying Corollary 2 to the perturbation g, we obtain the μ_ϕ-measure of any ball in $\mathbb{T}^2$.

3) Examples of *hyperbolic attractors* for endomorphisms can be obtained from *solenoids with self-intersections*, by the method of Bothe ([1]). We consider $f : \mathbb{D}^2 \times S^1 \to \mathbb{D}^2 \times S^1$ given by:

$$f(x, y, t) := (\lambda_1(t) \cdot x + z_1(t), \lambda_2(t) \cdot y + z_2(t), \phi(t)),$$

where $\phi, 0 < \lambda_i(t) < 1, z_i, i = 1, 2$ are $\mathcal{C}^1$ functions and where $\phi'(t) > 1$. For certain choice of ϕ, λ_i, z_i the map f is non-invertible. We obtain then the hyperbolic saddle-type fractal attractor

$$\Lambda = \bigcap_{j \ge 0} f^j(\mathbb{D}^2 \times S^1)$$

Then for an arbitrary Hölder potential ϕ on Λ one can obtain the measure μ_ϕ of an arbitrary ball in Λ by applying Corollary 2, in the case when $\lambda_1 = \lambda_2$. Also we obtain the asymptotic degree of $f|_\Lambda$ as $d_\infty(f, \Lambda)$ from Corollary 1 with the help of $F_f(\mu_0)$; and the Jacobian $J_{f^n}(\mu_\phi)$ with respect to an iterate $f^n, n \ge 1$.

4) One can consider also holomorphic endomorphisms f defined on $\mathbb{P}^k\mathbb{C}, k \ge 2$ and take basic sets on which f is hyperbolic. For instance, consider *holomorphic perturbations* g of a map $f(z, w) = (f_1(z), f_2(w))$ with $f_1(z), f_2(w)$ one-variable polynomials of the same degree, which are hyperbolic on their Julia sets; take then the associated basic set Λ_g of g, close to $\Lambda := \{p\} \times J(f_2)$, where p is a fixed attracting point of f_1 and $J(f_2)$ is the Julia set of f_2.

Then Λ_g is in general a fractal set in $\mathbb{C}^2$ and the number of g-preimages of a point in Λ_g is not necessarily constant, nor equal to the number of f-preimages in Λ (see for eg. [9] and references therein). By applying Theorem 1 we can express then the asymptotic degree of $g|_{\Lambda_g}$.

Moreover since g is conformal on both stable and unstable directions, we can apply Corollary 2 in order to obtain the measure μ_ϕ of an arbitrary ball, for any Hölder continuous potential ϕ on Λ_g. Namely for any $(z, w) \in \Lambda_g$ and $\rho > 0$, we take first m, n to be the largest integers s.t $\varepsilon |Dg_s^m(z, w)| \ge \rho$ and $\varepsilon |Dg_u^{n-m}(g^m(z, w))|^{-1} \ge \rho$. Then for $y = (z, w)$ we obtain:

$$\frac{1}{C} \int_{B_n(y,\varepsilon)} \frac{\sum\limits_{\zeta \in g^{-m}(g^m(x)) \cap \Lambda_g} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \, d\mu_\phi(x) \le \mu_\phi(B(g^m(y), \rho)) \le C \int_{B_n(y,\varepsilon)} \frac{\sum\limits_{\zeta \in g^{-m}(g^m(x)) \cap \Lambda_g} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \, d\mu_\phi(x),$$

where ε is fixed and $S_m\phi(x), B_n((z,w),\varepsilon)$ are taken with respect to g.

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