

for $n = 3, 4, 5, \dots$. This says, for example, that

$$D_3 = 4D_2 - D_1 = 4(15) - 4 = 56$$

$$D_4 = 4D_3 - D_2 = 4(56) - 15 = 209$$

and so on. Using a computer, check this result for $5 \leq n \leq 10$.

- (b) By writing

$$D_n = 4D_{n-1} - D_{n-2}$$

and the identity, $D_{n-1} = D_{n-1}$, in matrix form,

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

show that

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} D_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 15 \\ 4 \end{bmatrix}$$

- (c) Use the methods in Section 7.2 and a computer to show that

$$\begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix}^{n-2} = \frac{\begin{bmatrix} (2 + \sqrt{3})^{n-1} - (2 - \sqrt{3})^{n-1} & (2 - \sqrt{3})^{n-2} - (2 + \sqrt{3})^{n-2} \\ (2 + \sqrt{3})^{n-2} - (2 - \sqrt{3})^{n-2} & (2 - \sqrt{3})^{n-3} - (2 + \sqrt{3})^{n-3} \end{bmatrix}}{2\sqrt{3}}$$

and hence

$$D_n = \frac{(2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1}}{2\sqrt{3}}$$

for $n = 1, 2, 3, \dots$.

- (d) Using a computer, check this result for $1 \leq n \leq 10$.

T2. In this exercise, we determine a formula for calculating A_n^{-1} from D_k for $k = 0, 1, 2, 3, \dots, n$, assuming that D_0 is defined to be 1.

- (a) Use a computer to compute A_k^{-1} for $k = 1, 2, 3, 4$, and 5.
 (b) From your results in part (a), discover the conjecture that

$$A_n^{-1} = [\alpha_{ij}]$$

where $\alpha_{ij} = \alpha_{ji}$ and

$$\alpha_{ij} = (-1)^{i+j} \left(\frac{D_{n-j} D_{i-1}}{D_n} \right)$$

for $i \leq j$.

- (c) Use the result in part (b) to compute A_7^{-1} and compare it to the result obtained using the computer.

11.6 MARKOV CHAINS

In this section we describe a general model of a system that changes from state to state. We then apply the model to several concrete problems.

PREREQUISITES: Linear Systems
 Matrices
 Intuitive Understanding of Limits

A Markov Process

Suppose a physical or mathematical system undergoes a process of change such that at any moment it can occupy one of a finite number of states. For example, the weather in a certain city could be in one of three possible states: sunny, cloudy, or rainy. Or

an individual could be in one of four possible emotional states: happy, sad, angry, or apprehensive. Suppose that such a system changes with time from one state to another and at scheduled times the state of the system is observed. If the state of the system at any observation cannot be predicted with certainty, but the probability that a given state occurs can be predicted by just knowing the state of the system at the preceding observation, then the process of change is called a *Markov chain* or *Markov process*.

DEFINITION

If a Markov chain has k possible states, which we label as $1, 2, \dots, k$, then the probability that the system is in state i at any observation after it was in state j at the preceding observation is denoted by p_{ij} and is called the *transition probability* from state j to state i . The matrix $P = [p_{ij}]$ is called the *transition matrix of the Markov chain*.

For example, in a three-state Markov chain, the transition matrix has the form

Preceding State

$$\begin{array}{ccc} & \begin{matrix} 1 & 2 & 3 \end{matrix} & \\ \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{matrix} \text{New State} \end{matrix} \end{array}$$

In this matrix, p_{32} is the probability that the system will change from state 2 to state 3, p_{11} is the probability that the system will still be in state 1 if it was previously in state 1, and so forth.

EXAMPLE 1 Transition Matrix of the Markov Chain

A car rental agency has three rental locations, denoted by 1, 2, and 3. A customer may rent a car from any of the three locations and return the car to any of the three locations. The manager finds that customers return the cars to the various locations according to the following probabilities:

Rented from Location

$$\begin{array}{ccc} & \begin{matrix} 1 & 2 & 3 \end{matrix} & \\ \begin{bmatrix} .8 & .3 & .2 \\ .1 & .2 & .6 \\ .1 & .5 & .2 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{matrix} \text{Returned} \\ \text{to} \\ \text{Location} \end{matrix} \end{array}$$

This matrix is the transition matrix of the system considered as a Markov chain. From this matrix, the probability is .6 that a car rented from location 3 will be returned to location 2, the probability is .8 that a car rented from location 1 will be returned to location 1, and so forth. ♦

EXAMPLE 2 Transition Matrix of the Markov Chain

By reviewing its donation records, the alumni office of a college finds that 80% of its alumni who contribute to the annual fund one year will also contribute the next year,

and 30% of those who do not contribute one year will contribute the next. This can be viewed as a Markov chain with two states: state 1 corresponds to an alumnus giving a donation in any one year, and state 2 corresponds to the alumnus not giving a donation in that year. The transition matrix is

$$P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \blacklozenge$$

In the examples above, the transition matrices of the Markov chains have the property that the entries in any column sum to 1. This is not accidental. If $P = [p_{ij}]$ is the transition matrix of any Markov chain with k states, then for each j we must have

$$p_{1j} + p_{2j} + \cdots + p_{kj} = 1 \quad (1)$$

because if the system is in state j at one observation, it is certain to be in one of the k possible states at the next observation.

A matrix with property (1) is called a **stochastic matrix**, a **probability matrix**, or a **Markov matrix**. From the preceding discussion, it follows that the transition matrix for a Markov chain must be a stochastic matrix.

In a Markov chain, the state of the system at any observation time cannot generally be determined with certainty. The best one can usually do is specify probabilities for each of the possible states. For example, in a Markov chain with three states, we might describe the possible state of the system at some observation time by a column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

in which x_1 is the probability that the system is in state 1, x_2 the probability that it is in state 2, and x_3 the probability that it is in state 3. In general we make the following definition.

DEFINITION

The **state vector** for an observation of a Markov chain with k states is a column vector $\mathbf{x}$ whose i th component x_i is the probability that the system is in the i th state at that time.

Observe that the entries in any state vector for a Markov chain are nonnegative and have a sum of 1. (Why?) A column vector that has this property is called a **probability vector**.

Let us suppose now that we know the state vector $\mathbf{x}^{(0)}$ for a Markov chain at some initial observation. The following theorem will enable us to determine the state vectors

$$\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}, \dots$$

at the subsequent observation times.

THEOREM 11.6.1

If P is the transition matrix of a Markov chain and $\mathbf{x}^{(n)}$ is the state vector at the n th observation, then $\mathbf{x}^{(n+1)} = P\mathbf{x}^{(n)}$.

The proof of this theorem involves ideas from probability theory and will not be given here. From this theorem, it follows that

$$\begin{aligned} \mathbf{x}^{(1)} &= P\mathbf{x}^{(0)} \\ \mathbf{x}^{(2)} &= P\mathbf{x}^{(1)} = P^2\mathbf{x}^{(0)} \\ \mathbf{x}^{(3)} &= P\mathbf{x}^{(2)} = P^3\mathbf{x}^{(0)} \\ &\vdots \\ \mathbf{x}^{(n)} &= P\mathbf{x}^{(n-1)} = P^n\mathbf{x}^{(0)} \end{aligned}$$

In this way, the initial state vector $\mathbf{x}^{(0)}$ and the transition matrix P determine $\mathbf{x}^{(n)}$ for $n = 1, 2, \dots$

EXAMPLE 3 Example 2 Revisited

The transition matrix in Example 2 was

$$P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

We now construct the probable future donation record of a new graduate who did not give a donation in the initial year after graduation. For such a graduate the system is initially in state 2 with certainty, so the initial state vector is

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

From Theorem 11.6.1 we then have

$$\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .3 \\ .7 \end{bmatrix}$$

$$\mathbf{x}^{(2)} = P\mathbf{x}^{(1)} = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .3 \\ .7 \end{bmatrix} = \begin{bmatrix} .45 \\ .55 \end{bmatrix}$$

$$\mathbf{x}^{(3)} = P\mathbf{x}^{(2)} = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .45 \\ .55 \end{bmatrix} = \begin{bmatrix} .525 \\ .475 \end{bmatrix}$$

Thus, after three years the alumnus can be expected to make a donation with probability .525. Beyond three years, we find the following state vectors (to three decimal places):

$$\mathbf{x}^{(4)} = \begin{bmatrix} .563 \\ .438 \end{bmatrix}, \quad \mathbf{x}^{(5)} = \begin{bmatrix} .581 \\ .419 \end{bmatrix}, \quad \mathbf{x}^{(6)} = \begin{bmatrix} .591 \\ .409 \end{bmatrix}, \quad \mathbf{x}^{(7)} = \begin{bmatrix} .595 \\ .405 \end{bmatrix}$$

$$\mathbf{x}^{(8)} = \begin{bmatrix} .598 \\ .402 \end{bmatrix}, \quad \mathbf{x}^{(9)} = \begin{bmatrix} .599 \\ .401 \end{bmatrix}, \quad \mathbf{x}^{(10)} = \begin{bmatrix} .599 \\ .401 \end{bmatrix}, \quad \mathbf{x}^{(11)} = \begin{bmatrix} .600 \\ .400 \end{bmatrix}$$

For all n beyond 11, we have

$$\mathbf{x}^{(n)} = \begin{bmatrix} .600 \\ .400 \end{bmatrix}$$

to three decimal places. In other words, the state vectors converge to a fixed vector as the number of observations increases. (We shall discuss this further below.) ♦

EXAMPLE 4 Example 1 Revisited

The transition matrix in Example 1 was

$$\begin{bmatrix} .8 & .3 & .2 \\ .1 & .2 & .6 \\ .1 & .5 & .2 \end{bmatrix}$$

If a car is rented initially from location 2, then the initial state vector is

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Using this vector and Theorem 11.6.1, one obtains the later state vectors listed in Table 1.

Table 1

$\mathbf{x}^{(n)}$ \ n	0	1	2	3	4	5	6	7	8	9	10	11
$x_1^{(n)}$	0	.300	.400	.477	.511	.533	.544	.550	.553	.555	.556	.557
$x_2^{(n)}$	1	.200	.370	.252	.261	.240	.238	.233	.232	.231	.230	.230
$x_3^{(n)}$	0	.500	.230	.271	.228	.227	.219	.217	.215	.214	.214	.213

For all values of n greater than 11, all state vectors are equal to $\mathbf{x}^{(11)}$ to three decimal places.

Two things should be observed in this example. First, it was not necessary to know how long a customer kept the car. That is, in a Markov process the time period between observations need not be regular. Second, the state vectors approach a fixed vector as n increases, just as in the first example. ♦

EXAMPLE 5 Using Theorem 11.6.1

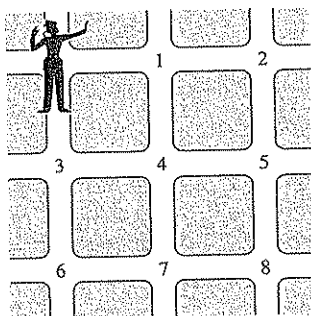


Figure 11.6.1

A traffic officer is assigned to control the traffic at the eight intersections indicated in Figure 11.6.1. She is instructed to remain at each intersection for an hour and then to either remain at the same intersection or move to a neighboring intersection. To avoid establishing a pattern, she is told to choose her new intersection on a random basis, with each possible choice equally likely. For example, if she is at intersection 5, her next intersection can be 2, 4, 5, or 8, each with probability $\frac{1}{4}$. Every day she starts at the location where she stopped the day before. The transition matrix for this Markov chain is

		Old Intersection									
		1	2	3	4	5	6	7	8	New Intersection	
$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{5} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{5} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & \frac{1}{4} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$	1										1
	2										2
	3										3
	4										4
	5										5
	6										6
	7										7
	8										8

If the traffic officer begins at intersection 5, her probable locations, hour by hour, are given by the state vectors given in Table 2. For all values of n greater than 22, all state vectors are equal to $\mathbf{x}^{(22)}$ to three decimal places. Thus, as with the first two examples, the state vectors approach a fixed vector as n increases. ♦

Table 2

$\mathbf{x}^{(n)}$ \ n	0	1	2	3	4	5	10	15	20	22
$x_1^{(n)}$	0	.000	.133	.116	.130	.123	.113	.109	.108	.107
$x_2^{(n)}$	0	.250	.146	.163	.140	.138	.115	.109	.108	.107
$x_3^{(n)}$	0	.000	.050	.039	.067	.073	.100	.106	.107	.107
$x_4^{(n)}$	0	.250	.113	.187	.162	.178	.178	.179	.179	.179
$x_5^{(n)}$	1	.250	.279	.190	.190	.168	.149	.144	.143	.143
$x_6^{(n)}$	0	.000	.000	.050	.056	.074	.099	.105	.107	.107
$x_7^{(n)}$	0	.000	.133	.104	.131	.125	.138	.142	.143	.143
$x_8^{(n)}$	0	.250	.146	.152	.124	.121	.108	.107	.107	.107

Limiting Behavior of the State Vectors

In our examples we saw that the state vectors approached some fixed vector as the number of observations increased. We now ask whether the state vectors always approach a fixed vector in a Markov chain. A simple example shows that this is not the case.

EXAMPLE 6 System Oscillates between Two State Vectors

Let

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then, because $P^2 = I$ and $P^3 = P$, we have that

$$\mathbf{x}^{(0)} = \mathbf{x}^{(2)} = \mathbf{x}^{(4)} = \cdots = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{x}^{(1)} = \mathbf{x}^{(3)} = \mathbf{x}^{(5)} = \cdots = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This system oscillates indefinitely between the two state vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so it does not approach any fixed vector. ♦

However, if we impose a mild condition on the transition matrix, we can show that a fixed limiting state vector is approached. This condition is described by the following definition.

DEFINITION

A transition matrix is **regular** if some integer power of it has all positive entries.

Thus, for a regular transition matrix P , there is some positive integer m such that all entries of P^m are positive. This is the case with the transition matrices of Examples 1 and 2 for $m = 1$. In Example 5 it turns out that P^4 has all positive entries. Consequently, in all three examples the transition matrices are regular.

A Markov chain that is governed by a regular transition matrix is called a **regular Markov chain**. We shall see that every regular Markov chain has a fixed state vector $\mathbf{q}$ such that $P^n \mathbf{x}^{(0)}$ approaches $\mathbf{q}$ as n increases for any choice of $\mathbf{x}^{(0)}$. This result is of major importance in the theory of Markov chains. It is based on the following theorem.

THEOREM 11.6.2**Behavior of P^n as $n \rightarrow \infty$**

If P is a regular transition matrix, then as $n \rightarrow \infty$,

$$P^n \rightarrow \begin{bmatrix} q_1 & q_1 & \cdots & q_1 \\ q_2 & q_2 & \cdots & q_2 \\ \vdots & \vdots & & \vdots \\ q_k & q_k & \cdots & q_k \end{bmatrix}$$

where the q_i are positive numbers such that $q_1 + q_2 + \cdots + q_k = 1$.

We will not prove this theorem here. The interested reader is referred to a more specialized text, such as J. Kemeny and J. Snell, *Finite Markov Chains* (New York: Springer-Verlag, 1976).

Let us set

$$Q = \begin{bmatrix} q_1 & q_1 & \cdots & q_1 \\ q_2 & q_2 & \cdots & q_2 \\ \vdots & \vdots & & \vdots \\ q_k & q_k & \cdots & q_k \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{bmatrix}$$

Thus, Q is a transition matrix, all of whose columns are equal to the probability vector $\mathbf{q}$. Q has the property that if $\mathbf{x}$ is any probability vector, then

$$\begin{aligned} Q\mathbf{x} &= \begin{bmatrix} q_1 & q_1 & \cdots & q_1 \\ q_2 & q_2 & \cdots & q_2 \\ \vdots & \vdots & & \vdots \\ q_k & q_k & \cdots & q_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} q_1 x_1 + q_1 x_2 + \cdots + q_1 x_k \\ q_2 x_1 + q_2 x_2 + \cdots + q_2 x_k \\ \vdots \\ q_k x_1 + q_k x_2 + \cdots + q_k x_k \end{bmatrix} \\ &= (x_1 + x_2 + \cdots + x_k) \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{bmatrix} = (1)\mathbf{q} = \mathbf{q} \end{aligned}$$

That is, Q transforms any probability vector $\mathbf{x}$ into the fixed probability vector $\mathbf{q}$. This result leads to the following theorem.

THEOREM 11.6.3**Behavior of $P^n \mathbf{x}$ as $n \rightarrow \infty$**

If P is a regular transition matrix and $\mathbf{x}$ is any probability vector, then as $n \rightarrow \infty$,

$$P^n \mathbf{x} \rightarrow \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{bmatrix} = \mathbf{q}$$

where $\mathbf{q}$ is a fixed probability vector, independent of n , all of whose entries are positive.

This result holds since Theorem 11.6.2 implies that $P^n \rightarrow Q$ as $n \rightarrow \infty$. This in turn implies that $P^n \mathbf{x} \rightarrow Q\mathbf{x} = \mathbf{q}$ as $n \rightarrow \infty$. Thus, for a regular Markov chain, the system eventually approaches a fixed state vector $\mathbf{q}$. The vector $\mathbf{q}$ is called the *steady-state vector* of the regular Markov chain.

For systems with many states, usually the most efficient technique of computing the steady-state vector $\mathbf{q}$ is simply to calculate $P^n \mathbf{x}$ for some large n . Our examples illustrate this procedure. Each is a regular Markov process, so that convergence to a steady-state vector is ensured. Another way of computing the steady-state vector is to make use of the following theorem.

THEOREM 11.6.4**Steady-State Vector**

The steady-state vector $\mathbf{q}$ of a regular transition matrix P is the unique probability vector that satisfies the equation $P\mathbf{q} = \mathbf{q}$.

To see this, consider the matrix identity $PP^n = P^{n+1}$. By Theorem 11.6.2, both P^n and P^{n+1} approach Q as $n \rightarrow \infty$. Thus, we have $PQ = Q$. Any one column of this matrix equation gives $P\mathbf{q} = \mathbf{q}$. To show that $\mathbf{q}$ is the only probability vector that satisfies this equation, suppose $\mathbf{r}$ is another probability vector such that $P\mathbf{r} = \mathbf{r}$. Then also $P^n \mathbf{r} = \mathbf{r}$ for $n = 1, 2, \dots$. When we let $n \rightarrow \infty$, Theorem 11.6.3 leads to $\mathbf{q} = \mathbf{r}$.

Theorem 11.6.4 can also be expressed by the statement that the homogeneous linear system

$$(I - P)\mathbf{q} = \mathbf{0}$$

has a unique solution vector $\mathbf{q}$ with nonnegative entries that satisfy the condition $q_1 + q_2 + \dots + q_k = 1$. We can apply this technique to the computation of the steady-state vectors for our examples.

EXAMPLE 7 Example 2 Revisited

In Example 2 the transition matrix was

$$P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

so the linear system $(I - P)\mathbf{q} = \mathbf{0}$ is

$$\begin{bmatrix} .2 & -.3 \\ -.2 & .3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2)$$

This leads to the single independent equation

$$.2q_1 - .3q_2 = 0$$

or

$$q_1 = 1.5q_2$$

Thus, when we set $q_2 = s$, any solution of (2) is of the form

$$\mathbf{q} = s \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

where s is an arbitrary constant. To make the vector $\mathbf{q}$ a probability vector, we set $s = 1/(1.5 + 1) = .4$. Consequently,

$$\mathbf{q} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$$

is the steady-state vector of this regular Markov chain. This means that over the long run, 60% of the alumni will give a donation in any one year, and 40% will not. Observe that this agrees with the result obtained numerically in Example 3. ♦

EXAMPLE 8 Example 1 Revisited

In Example 1 the transition matrix was

$$P = \begin{bmatrix} .8 & .3 & .2 \\ .1 & .2 & .6 \\ .1 & .5 & .2 \end{bmatrix}$$

so the linear system $(I - P)\mathbf{q} = \mathbf{0}$ is

$$\begin{bmatrix} .2 & -.3 & -.2 \\ -.1 & .8 & -.6 \\ -.1 & -.5 & .8 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The reduced row-echelon form of the coefficient matrix is (verify)

$$\begin{bmatrix} 1 & 0 & -\frac{34}{13} \\ 0 & 1 & -\frac{14}{13} \\ 0 & 0 & 0 \end{bmatrix}$$

so the original linear system is equivalent to the system

$$q_1 = \left(\frac{34}{13}\right)q_3$$

$$q_2 = \left(\frac{14}{13}\right)q_3$$

When we set $q_3 = s$, any solution of the linear system is of the form

$$\mathbf{q} = s \begin{bmatrix} \frac{34}{13} \\ \frac{14}{13} \\ 1 \end{bmatrix}$$

To make this a probability vector, we set

$$s = \frac{1}{\frac{34}{13} + \frac{14}{13} + 1} = \frac{13}{61}$$

Thus, the steady-state vector of the system is

$$\mathbf{q} = \begin{bmatrix} \frac{34}{61} \\ \frac{14}{61} \\ \frac{13}{61} \end{bmatrix} = \begin{bmatrix} .5573\dots \\ .2295\dots \\ .2131\dots \end{bmatrix}$$

This agrees with the result obtained numerically in Table 1. The entries of $\mathbf{q}$ give the long-run probabilities that any one car will be returned to location 1, 2, or 3, respectively. If the car rental agency has a fleet of 1000 cars, it should design its facilities so that there are at least 558 spaces at location 1, at least 230 spaces at location 2, and at least 214 spaces at location 3. ♦

EXAMPLE 9 Example 5 Revisited

We will not give the details of the calculations but simply state that the unique probability vector solution of the linear system $(I - P)\mathbf{q} = \mathbf{0}$ is

$$\mathbf{q} = \begin{bmatrix} \frac{3}{28} \\ \frac{3}{28} \\ \frac{3}{28} \\ \frac{5}{28} \\ \frac{4}{28} \\ \frac{3}{28} \\ \frac{4}{28} \\ \frac{3}{28} \end{bmatrix} = \begin{bmatrix} .1071... \\ .1071... \\ .1071... \\ .1785... \\ .1428... \\ .1071... \\ .1428... \\ .1071... \end{bmatrix}$$

The entries in this vector indicate the proportion of time the traffic officer spends at each intersection over the long term. Thus, if the objective is for her to spend the same proportion of time at each intersection, then the strategy of random movement with equal probabilities from one intersection to another is not a good one. (See Exercise 5.) ♦

EXERCISE SET 11.6

1. Consider the transition matrix

$$P = \begin{bmatrix} .4 & .5 \\ .6 & .5 \end{bmatrix}$$

- (a) Calculate $\mathbf{x}^{(n)}$ for $n = 1, 2, 3, 4, 5$, if $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

- (b) State why P is regular and find its steady-state vector.

2. Consider the transition matrix

$$P = \begin{bmatrix} .2 & .1 & .7 \\ .6 & .4 & .2 \\ .2 & .5 & .1 \end{bmatrix}$$

- (a) Calculate $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ to three decimal places if

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- (b) State why P is regular and find its steady-state vector.

3. Find the steady-state vectors of the following regular transition matrices:

$$(a) \begin{bmatrix} \frac{1}{3} & \frac{2}{4} \\ \frac{2}{3} & \frac{1}{4} \end{bmatrix} \quad (b) \begin{bmatrix} .81 & .26 \\ .19 & .74 \end{bmatrix} \quad (c) \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

4. Let P be the transition matrix

$$\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$$

- (a) Show that P is not regular.

$\mathbf{q}$ give the
spectively.
o that there
t least 214

- (b) Show that as n increases, $P^n \mathbf{x}^{(0)}$ approaches $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for any initial state vector $\mathbf{x}^{(0)}$.
- (c) What conclusion of Theorem 11.6.3 is not valid for the steady state of this transition matrix?
5. Verify that if P is a $k \times k$ regular transition matrix all of whose row sums are equal to 1, then the entries of its steady-state vector are all equal to $1/k$.
6. Show that the transition matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

is regular, and use Exercise 5 to find its steady-state vector.

7. John is either happy or sad. If he is happy one day, then he is happy the next day four times out of five. If he is sad one day, then he is sad the next day one time out of three. Over the long term, what are the chances that John is happy on any given day?
8. A country is divided into three demographic regions. It is found that each year 5% of the residents of region 1 move to region 2, and 5% move to region 3. Of the residents of region 2, 15% move to region 1 and 10% move to region 3. And of the residents of region 3, 10% move to region 1 and 5% move to region 2. What percentage of the population resides in each of the three regions after a long period of time?

SECTION 11.6



Technology Exercises

The following exercises are designed to be solved using a technology utility. Typically, this will be MATLAB, *Mathematica*, Maple, Derive, or Mathcad, but it may also be some other type of linear algebra software or a scientific calculator with some linear algebra capabilities. For each exercise you will need to read the relevant documentation for the particular utility you are using. The goal of these exercises is to provide you with a basic proficiency with your technology utility. Once you have mastered the techniques in these exercises, you will be able to use your technology utility to solve many of the problems in the regular exercise sets.

T1. Consider the sequence of transition matrices

$$\{P_2, P_3, P_4, \dots\}$$

with

$$P_2 = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{bmatrix}, \quad P_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{5} \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix},$$

and so on.

- (a) Use a computer to show that each of these four matrices is regular by computing their squares.
- (b) Verify Theorem 11.6.2 by computing the 100th power of P_k for $k = 2, 3, 4, 5$. Then make a conjecture as to the limiting value of P_k^n as $n \rightarrow \infty$ for all $k = 2, 3, 4, \dots$
- (c) Verify that the common column $\mathbf{q}_k$ of the limiting matrix you found in part (b) satisfies the equation $P_k \mathbf{q}_k = \mathbf{q}_k$, as required by Theorem 11.6.4.

- T2. A mouse is placed in a box with nine rooms as shown in the accompanying figure. Assume that it is equally likely that the mouse goes through any door in the room or stays in the room.
- Construct the 9×9 transition matrix for this problem and show that it is regular.
 - Determine the steady-state vector for the matrix.
 - Use a symmetry argument to show that this problem may be solved using only a 3×3 matrix.

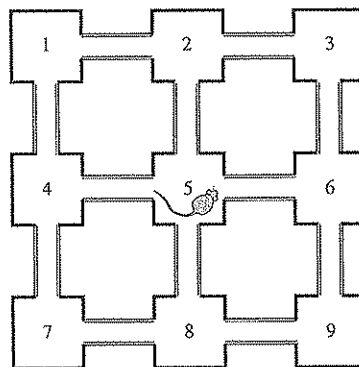


Figure Ex-T2

11.7 GRAPH THEORY

In this section we introduce matrix representations of relations among members of a set. We use matrix arithmetic to analyze these relationships.

PREREQUISITES: Matrix Addition and Multiplication

Relations among Members of a Set

There are countless examples of sets with finitely many members in which some relation exists among members of the set. For example, the set could consist of a collection of people, animals, countries, companies, sports teams, or cities; and the relation between two members, A and B , of such a set could be that person A dominates person B , animal A feeds on animal B , country A militarily supports country B , company A sells its product to company B , sports team A consistently beats sports team B , or city A has a direct airline flight to city B .

We shall now show how the theory of *directed graphs* can be used to mathematically model relations such as those in the preceding examples.

Directed Graphs

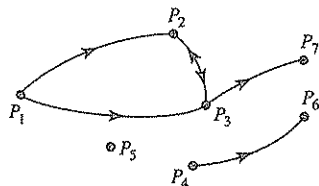


Figure 11.7.1

A *directed graph* is a finite set of elements, $\{P_1, P_2, \dots, P_n\}$, together with a finite collection of ordered pairs (P_i, P_j) of distinct elements of this set, with no ordered pair being repeated. The elements of the set are called *vertices*, and the ordered pairs are called *directed edges*, of the directed graph. We use the notation $P_i \rightarrow P_j$ (which is read " P_i is connected to P_j ") to indicate that the directed edge (P_i, P_j) belongs to the directed graph. Geometrically, we can visualize a directed graph (Figure 11.7.1) by representing the vertices as points in the plane and representing the directed edge $P_i \rightarrow P_j$ by drawing a line or arc from vertex P_i to vertex P_j , with an arrow pointing from P_i to P_j . If both $P_i \rightarrow P_j$ and $P_j \rightarrow P_i$ hold (denoted $P_i \leftrightarrow P_j$), we draw a single line between P_i and P_j with two oppositely pointing arrows (as with P_2 and P_3 in the figure).