using the nonlinear feasible region, $0 \le x$, $0 \le y$, and

$$x^2 + y^2 \le 1$$

(m) Let the results of parts (a) through (k) begin a sequence of values for $z_{\max}$. Do these values approach the value determined in part (l)? Explain.

T2. Repeat Exercise T1 using the objective function $z = x + y$.

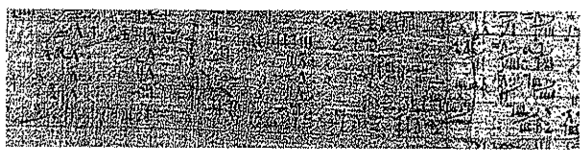
11.4 THE EARLIEST APPLICATIONS OF LINEAR ALGEBRA

Linear systems can be found in the earliest writings of many ancient civilizations. We give some examples of the types of problems that they used to solve.

PREREQUISITES: Linear Systems

The practical problems of early civilizations included the measurement of land, the distribution of goods, the tracking of resources such as wheat and cattle, and taxation and inheritance calculations. In many cases, these problems led to linear systems of equations since linearity is one of the simplest relationship that can exist among variables. In this section we present examples from five diverse ancient cultures illustrating how they used and solved systems of linear equations. We restrict ourselves to examples before A.D. 500. These examples consequently predate the development, by Islamic/Arab mathematicians, of the field of algebra, a field that ultimately led, in the nineteenth century, to the branch of mathematics now called linear algebra.

EXAMPLE 1 Egypt (about 1650 B.C.)



Problem 40 of the Ahmes Papyrus

The Ahmes (or Rhind) Papyrus is the source of most of our information about ancient Egyptian mathematics. This 5-meter-long papyrus contains 84 short mathematical problems, together with their solutions, and dates from about 1650 B.C. Problem 40 in this papyrus is the following:

Divide 100 hekats of barley among five men in arithmetic progression so that the sum of the two smallest is one-seventh the sum of the three largest.

Let a be the least amount that any man obtains, and let d be the common difference of the terms in the arithmetic progression. Then the other four men receive $a + d$, $a + 2d$, $a + 3d$, and $a + 4d$ hekats. The two conditions of the problem require that

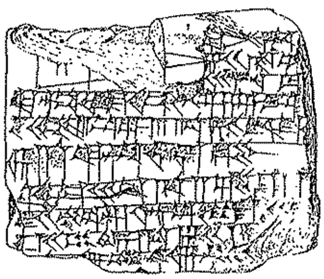
$$\begin{aligned} a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) &= 100 \\ \tfrac{1}{7}[(a + 2d) + (a + 3d) + (a + 4d)] &= a + (a + d) \end{aligned}$$

These equations reduce to the following system of two equations in two unknowns:

$$\begin{aligned} 5a + 10d &= 100 \\ 11a - 2d &= 0 \end{aligned} \tag{1}$$

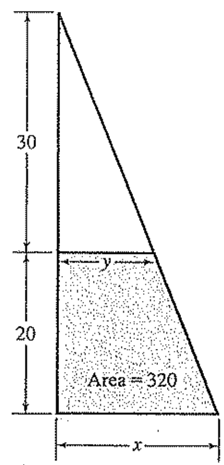
The solution technique described in the papyrus is known as the method of false position or false assumption. It begins by assuming some convenient value of a (in our case $a = 1$), substituting that value into the second equation, and obtaining $d = 11/2$. Substituting $a = 1$ and $d = 11/2$ into the left-hand side of the first equation gives 60, whereas the right-hand side is 100. Adjusting the initial guess for a by multiplying it by $100/60$ leads to the correct value $a = 5/3$. Substituting $a = 5/3$ into the second equation then gives $d = 55/6$, so the quantities of barley received by the five men are $10/6$, $65/6$, $120/6$, $175/6$, and $230/6$ hekats. This technique of guessing a value of an unknown and later adjusting it has been used by many cultures throughout the ages. ◈

EXAMPLE 2 Babylonia (1900–1600 B.C.)



Babylonian Clay Tablet Ca MLA 1950

The Old Babylonian Empire flourished in Mesopotamia between 1900 and 1600 B.C. Many clay tablets containing mathematical tables and problems survive from that period, one of which (designated Ca MLA 1950) contains the next problem. The statement of the problem is a bit muddled because of the condition of the tablet, but the diagram and the solution on the tablet indicate that the problem is as follows:



A trapezoid with an area of 320 square units is cut off from a right triangle by a line parallel to one of its sides. The other side has length 50 units, and the height of the trapezoid is 20 units. What are the upper and the lower widths of the trapezoid?

Let x be the lower width of the trapezoid and y its upper width. The area of the trapezoid is its height times its average width, so $20\left(\frac{x+y}{2}\right) = 320$. Using similar triangles, we also have $\frac{x}{50} = \frac{y}{30}$. The solution on the tablet uses these relations to generate the linear system

$$\begin{aligned} \tfrac{1}{2}(x + y) &= 16 \\ \tfrac{1}{2}(x - y) &= 4 \end{aligned} \tag{2}$$

Adding and subtracting these two equations then gives the solution $x = 20$ and $y = 12$. ◈

EXAMPLE 3 China (A.D. 263)

九章算術

Chiu Chang Suan Shu in Chinese characters

The most important treatise in the history of Chinese mathematics is the Chiu Chang Suan Shu, or "The Nine Chapters of the Mathematical Art." This treatise, which is a collection of 246 problems and their solutions, was assembled in its final form by Liu Hui in A.D. 263. Its contents, however, go back to at least the beginning of the Han dynasty in the second century B.C. The eighth of its nine chapters, entitled "The Way of Calculating by Arrays," contains 18 word problems that lead to linear systems in three to six unknowns. The general solution procedure described is almost identical to the Gaussian elimination technique developed in Europe in the nineteenth century by Carl Friedrich Gauss (see page 13 for his biography). The first problem in the eighth chapter is the following:

There are three classes of corn, of which three bundles of the first class, two of the second, and one of the third make 39 measures. Two of the first, three of the second, and one of the third make 34 measures. And one of the first, two of the second, and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class?

Let x, y, and z be the measures of the first, second, and third classes of corn. Then the conditions of the problem lead to the following linear system of three equations in three unknowns:

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned} \tag{3}$$

The solution described in the treatise represented the coefficients of each equation by an appropriate number of rods placed within squares on a counting table. Positive coefficients were represented by black rods, negative coefficients were represented by red rods, and the squares corresponding to zero coefficients were left empty. The counting table was laid out as follows so that the coefficients of each equation appear in columns with the first equation in the rightmost column:

1	2	3
2	3	2
3	1	1
26	34	39

Next the numbers of rods within the squares were adjusted to accomplish the following two steps: (1) two times the numbers of the third column were subtracted from three times the numbers in the second column and (2) the numbers in the third column were subtracted from three times the numbers in the first column. The result was the following array:

		3
4	5	2
8	1	1
39	24	39

In this array, four times the numbers in the second column were subtracted from five times the numbers in the first column, yielding

		3
	5	2
36	1	1
99	24	39

This last array is equivalent to the linear system

$$\begin{aligned} 3x + 2y + z &= 39 \\ 5y + z &= 24 \\ 36z &= 99 \end{aligned}$$

This triangular system was solved by a method equivalent to back-substitution to obtain $x = 37/4$, $y = 17/4$, and $z = 11/4$. ◈

EXAMPLE 4 Greece (third century B.C.)

Archimedes c. 287–212 B.C.

Perhaps the most famous system of linear equations from antiquity is the one associated with the first part of Archimedes' celebrated Cattle Problem. This problem supposedly was posed by Archimedes as a challenge to his colleague Eratosthenes. No solution has come down to us from ancient times, so that it is not known how, or even whether, either of these two geometers solved it.

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colors, one milk white, another glossy black, a third yellow, and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all of the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each color, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

The conventional designation of the eight variables in this problem is

$$\begin{aligned} W &= \text{number of white bulls} \\ B &= \text{number of black bulls} \\ Y &= \text{number of yellow bulls} \\ D &= \text{number of dappled bulls} \end{aligned}$$

w = number of white cows
b = number of black cows
y = number of yellow cows
d = number of dappled cows

The problem can now be stated as the following seven homogeneous equations in eight unknowns:

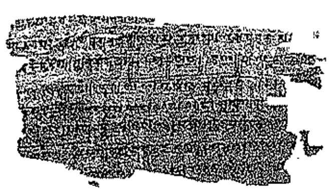
1. $W = \left(\frac{1}{2} + \frac{1}{3}\right) B + Y$ (The white bulls were equal to a half and a third of the black [bulls] together with the whole of the yellow [bulls].)
2. $B = \left(\frac{1}{4} + \frac{1}{5}\right) D + Y$ (The black [bulls] were equal to the fourth part of the dappled [bulls] and a fifth, together with, once more, the whole of the yellow [bulls].)
3. $D = \left(\frac{1}{6} + \frac{1}{7}\right) W + Y$ (The remaining bulls, the dappled, were equal to a sixth part of the white [bulls] and a seventh, together with all of the yellow [bulls].)
4. $w = \left(\frac{1}{3} + \frac{1}{4}\right)(B + b)$ (The white [cows] were precisely equal to the third part and a fourth of the whole herd of the black.)
5. $b = \left(\frac{1}{4} + \frac{1}{5}\right)(D + d)$ (The black [cows] were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together.)
6. $d = \left(\frac{1}{5} + \frac{1}{6}\right)(Y + y)$ (The dappled [cows] in four parts [that is, in totality] were equal in number to a fifth part and a sixth of the yellow herd.)
7. $y = \left(\frac{1}{6} + \frac{1}{7}\right)(W + w)$ (The yellow [cows] were in number equal to a sixth part and a seventh of the white herd.)

As we ask the reader to show in the exercises, this system has infinitely many solutions of the form

$$\begin{aligned} W &= 10{,}366{,}482k \\ B &= 7{,}460{,}514k \\ Y &= 4{,}149{,}387k \\ D &= 7{,}358{,}060k \\ w &= 7{,}206{,}360k \\ b &= 4{,}893{,}246k \\ y &= 5{,}439{,}213k \\ d &= 3{,}515{,}820k \end{aligned} \tag{4}$$

where k is any real number. The values $k = 1, 2, \ldots$ give infinitely many positive integer solutions to the problem, with $k = 1$ giving the smallest solution. ◈

EXAMPLE 5 India (fourth century A.D.)



Fragment III-5-3v of the Bakhshali Manuscript

The Bakhshali Manuscript is an ancient work of Indian/Hindu mathematics dating from around the fourth century A.D., although some of its materials undoubtedly come from many centuries before. It consists of about 70 leaves or sheets of birch bark containing mathematical problems and their solutions. Many of its problems are so-called equalization problems that lead to systems of linear equations. One such problem on the fragment shown is the following:

One merchant has seven asava horses, a second has nine haya horses, and a third has ten camels. They are equally well off in the value of their animals if each gives two animals, one to each of the others. Find the price of each animal and the total value of the animals possessed by each merchant.

Let x be the price of an asava horse, let y be the price of a haya horse, let z be the price of a camel, and the let K be the total value of the animals possessed by each merchant. Then the conditions of the problem lead to the following system of equations:

$$\begin{aligned} 5x + y + z &= K \\ x + 7y + z &= K \\ x + y + 8z &= K \end{aligned} \tag{5}$$

The method of solution described in the manuscript begins by subtracting the quantity $(x + y + z)$ from both sides of the three equations to obtain $4x = 6y = 7z = K - (x + y + z)$. This shows that if the prices x, y, and z are to be integers, then the quantity $K - (x + y + z)$ must be an integer that is divisible by 4, 6, and 7. The manuscript takes the product of these three numbers, or 168, for the value of $K - (x + y + z)$, which yields $x = 42$, $y = 28$, and $z = 24$ for the prices and $K = 262$ for the total value. (See Exercise 6 for more solutions to this problem.) ◈

EXERCISE SET 11.4

1. The following lines from Book 12 of Homer's *Odyssey* relate a precursor of Archimedes' Cattle Problem:

   Thou shalt ascend the isle triangular,
   Where many oxen of the Sun are fed,
   And fatted flocks. Of oxen fifty head
   In every herd feed, and their herds are seven;
   And of his fat flocks is their number even.

   The last line means that there are as many sheep in all the flocks as there are oxen in all the herds. What is the total number of oxen and sheep that belong to the god of the Sun? (This was a difficult problem in Homer's day.)

2. Solve the following problems from the Bakhshali Manuscript.

   (a) B possesses two times as much as A; C has three times as much as A and B together; D has four times as much as A, B, and C together. Their total possessions are 300. What is the possession of A?

   (b) B gives 2 times as much as A; C gives 3 times as much as B; D gives 4 times as much as C. Their total gift is 132. What is the gift of A?

3. A problem on a Babylonian tablet requires finding the length and width of a rectangle given that the length and the width add up to 10, while the length and one-fourth of the width add up to 7. The solution provided on the tablet consists of the following four statements:

   Multiply 7 by 4 to obtain 28.
   Take away 10 from 28 to obtain 18.
   Take one-third of 18 to obtain 6, the length.
   Take away 6 from 10 to obtain 4, the width.

   Explain how these steps lead to the answer.

4. The following two problems are from "The Nine Chapters of the Mathematical Art." Solve them using the array technique described in Example 3.

   (a) Five oxen and two sheep are worth 10 units and two oxen and five sheep are worth 8 units. What is the value of each ox and sheep?

(b) There are three kinds of corn. The grains contained in two, three, and four bundles, respectively, of these three classes of corn, are not sufficient to make a whole measure. However, if we added to them one bundle of the second, third, and first classes, respectively, then the grains would become on full measure in each case. How many measures of grain does each bundle of the different classes contain?

5. This problem in part (a) is known as the "Flower of Thymaridas," named after a Pythagorean of the fourth century B.C.

(a) Given the n numbers $a_1, a_2, \ldots, a_n$, solve for $x_1, x_2, \ldots, x_n$ in the following linear system:

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= a_1 \\ x_1 + x_2 &= a_2 \\ x_1 + x_3 &= a_3 \\ &\vdots \\ x_1 + x_n &= a_n \end{aligned}$$

(b) Identify a problem in this exercise set that fits the pattern in part (a), and solve it using your general solution.

6. For Example 5 from the Bakhshali Manuscript:

(a) Express Equations (5) as a homogeneous linear system of three equations in four unknowns (x, y, z, and K) and show that the solution set has one arbitrary parameter.

(b) Find the smallest solution for which all four variables are positive integers.

(c) Show that the solution given in Example 5 is included among your solutions.

7. Solve the problems posed in the following three epigrams, which appear in a collection entitled "The Greek Anthology," which was compiled in part by a scholar named Metrodorus around A.D. 500. Some of its 46 mathematical problems are believed to date as far back as 600 B.C. (Before solving parts (a) and (c), you will have to formulate the question.)

(a) I desire my two sons to receive the thousand staters of which I am possessed, but let the fifth part of the legitimate one's share exceed by ten the fourth part of what falls to the illegitimate one.

(b) Make me a crown weighing sixty minae, mixing gold and brass, and with them tin and much-wrought iron. Let the gold and brass together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold you must put in, how much brass, how much tin, and how much iron, so as to make the whole crown weigh sixty minae.

(c) First person: I have what the second has and the third of what the third has. Second person: I have what the third has and the third of what the first has. Third person: And I have ten minae and the third of what the second has.

SECTION 11.4 Technology Exercises

The following exercises are designed to be solved using a technology utility. Typically, this will be MATLAB, *Mathematica*, Maple, Derive, or Mathcad, but it may also be some other type of linear algebra software or a scientific calculator with some linear algebra capabilities. For each exercise you will need to read the relevant documentation for the particular utility you are using. The goal of these exercises is to provide you with a basic proficiency with your technology utility. Once you have mastered the techniques in these exercises, you will be able to use your technology utility to solve many of the problems in the regular exercise sets.

T1. (a) Solve Archimedes' Cattle Problem using a symbolic algebra program.

(b) The Cattle Problem has a second part in which two additional conditions are imposed. The first of these states that "When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth." This requires that $W + B$ be a square number, i.e., 1, 4, 9, 16, 25, etc. Show that this requires that the values of k in Eq. (4)

be restricted as follows:

$$k = 4{,}456{,}749r^2, \qquad r = 1, 2, 3, \ldots$$

and find the smallest total number of cattle that satisfies this second condition.

REMARK The second condition imposed in the second part of the Cattle Problem states that "When the yellow and the dappled bulls were gathered into one herd, they stood in such a manner that their number, beginning from one, grew slowly greater 'til it completed a triangular figure." This requires that the quantity $Y + D$ be a triangular number—that is, a number of the form $1, 1+2, 1+2+3, 1+2+3+4, \ldots$. This final part of the problem was not completely solved until 1965 when all 206,545 digits of the smallest number of cattle that satisfies this condition were found using a computer.

T2. The following problem is from "The Nine Chapters of the Mathematical Art" and determines a homogeneous linear system of five equations in six unknowns. Show that the system has infinitely many solutions, and find the one for which the depth of the well and the lengths of the five ropes are the smallest possible positive integers.

Suppose that five families share a well. Suppose further that

2 of A's ropes are short of the well's depth by one of B's ropes.

3 of B's ropes are short of the well's depth by one of C's ropes.

4 of C's ropes are short of the well's depth by one of D's ropes.

5 of D's ropes are short of the well's depth by one of E's ropes.

6 of E's ropes are short of the well's depth by one of A's ropes.

11.5 CUBIC SPLINE INTERPOLATION

In this section an artist's drafting aid is used as a physical model for the mathematical problem of finding a curve that passes through specified points in the plane. The parameters of the curve are determined by solving a linear system of equations.

PREREQUISITES: Linear Systems
Matrix Algebra
Differential Calculus

Curve Fitting

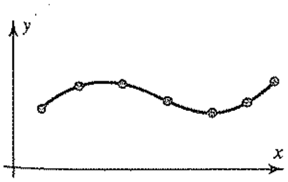


Figure 11.5.1

Fitting a curve through specified points in the plane is a common problem encountered in analyzing experimental data, in ascertaining the relations among variables, and in design work. In Figure 11.5.3 seven points in the xy-plane are displayed, and in Figure 11.5.2 a smooth curve has been drawn that passes through them. A curve that passes through a set of points in the plane is said to ***interpolate*** those points, and the curve is called an ***interpolating curve*** for those points. The interpolating curve in Figure 11.5.1 was drawn with the aid of a *drafting spline* (Figure 11.5.2). This drafting aid consists of a thin, flexible strip of wood or other material that is bent to pass through the points to be interpolated. Attached sliding weights hold the spline in position while the artist draws the interpolating curve. The drafting spline will serve as the physical model for a mathematical theory of interpolation that we will discuss in this section.

Statement of the Problem

Suppose that we are given n points in the xy-plane,

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

(b) Determine the equation of the hyperplane in R^9 that goes through the following ni points:

$$(1,2,3,4,5,6,7,8,9) \quad (2,3,4,5,6,7,8,9,1) \quad (3,4,5,6,7,8,9,1,2)$$
$$(4,5,6,7,8,9,1,2,3) \quad (5,6,7,8,9,1,2,3,4) \quad (6,7,8,9,1,2,3,4,5)$$
$$(7,8,9,1,2,3,4,5,6) \quad (8,9,1,2,3,4,5,6,7) \quad (9,1,2,3,4,5,6,7,8)$$

11.2 ELECTRICAL NETWORKS

In this section basic laws of electrical circuits are discussed, and it is show how these laws can be used to obtain systems of linear equations whos solutions yield the currents flowing in an electrical circuit.

PREREQUISITES: Linear Systems

The simplest electrical circuits consist of two basic components:

electrical sources	denoted by	
resistors	denoted by	

Electrical sources, such as batteries, create currents in an electrical circuit. Resistor such as lightbulbs, limit the magnitudes of the currents.

There are three basic quantities associated with electrical circuits: ***electrical potenti*** (E), ***resistance*** (R), and ***current*** (I). These are commonly measured in the followir units:

E	in volts	(V)
R	in ohms	(Ω)
I	in amperes	(A)

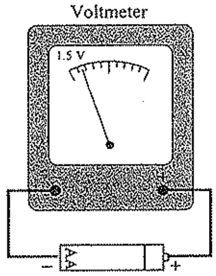


Figure 11.2.1

Electrical potential is associated with two points in an electrical circuit and is measur in practice by connecting those points to a device called a *voltmeter*. For example, common AA battery is rated at 1.5 volts, which means that this is the electrical potenti across its positive and negative terminals (Figure 11.2.1).

In an electrical circuit the electrical potential between two points is called the ***volta drop*** between these points. As we shall see, currents and voltage drops can be eith positive or negative.

The flow of current in an electrical circuit is governed by three basic principles:

1. ***Ohm's Law*** The voltage drop across a resistor is the product of the current passi through it and its resistance; that is, $E = IR$.
2. ***Kirchhoff's Current Law*** The sum of the currents flowing into any point equa the sum of the currents flowing out from the point.
3. ***Kirchhoff's Voltage Law*** Around any closed loop, the algebraic sum of the volta drops is zero.

EXAMPLE 1 Finding Currents in a Circuit

Find the unknown currents I_1, I_2, and I_3 in the circuit shown in Figure 11.2.2.

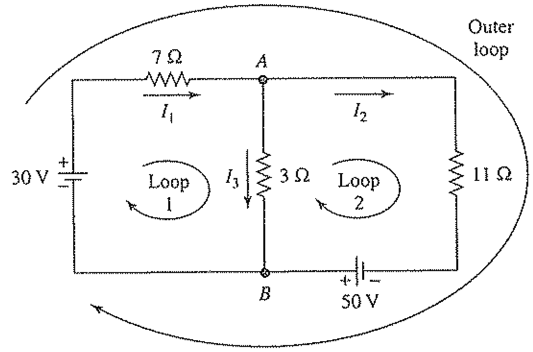


Figure 11.2.2

Solution

The flow directions for the currents I_1, I_2, and I_3 (marked by the arrowheads) were picked arbitrarily. Any of these currents that turn out to be negative actually flow opposite to the direction selected.

Applying Kirchhoff's current law to points A and B yields

$$\begin{aligned} I_1 &= I_2 + I_3 \quad &&\text{(Point } A\text{)} \\ I_3 + I_2 &= I_1 &&\text{(Point } B\text{)} \end{aligned}$$

Since these equations both simplify to the same linear equation

$$I_1 - I_2 - I_3 = 0 \tag{1}$$

we still need two more equations to determine I_1, I_2, and I_3 uniquely. We will obtain them using Kirchhoff's voltage law.

To apply Kirchhoff's voltage law to a loop, select a positive direction around the loop (say clockwise) and make the following sign conventions:

- A current passing through a resistor produces a positive voltage drop if it flows in the positive direction of the loop and a negative voltage drop if it flows in the negative direction of the loop.
- A current passing through an electrical source produces a positive voltage drop if the positive direction of the loop is from + to − and a negative voltage drop if the positive direction of the loop is from − to +.

Applying Kirchhoff's voltage law and Ohm's law to loop 1 in Figure 11.2.2 yields

$$7I_1 + 3I_3 - 30 = 0 \tag{2}$$

and applying them to loop 2 yields

$$11I_2 - 3I_3 - 50 = 0 \tag{3}$$

Combining (1), (2), and (3) yields the linear system

$$\begin{aligned} I_1 - \quad I_2 - \quad I_3 &= 0 \\ 7I_1 \qquad\qquad + 3I_3 &= 30 \\ 11I_2 - 3I_3 &= 50 \end{aligned}$$

Solving this linear system yields the following values for the currents:

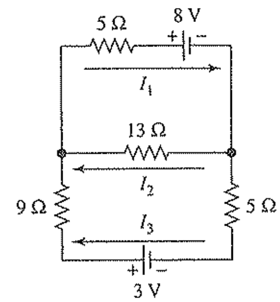
$$I_1 = \tfrac{570}{131}\ \text{(A)}, \qquad I_2 = \tfrac{590}{131}\ \text{(A)}, \qquad I_3 = -\tfrac{20}{131}\ \text{(A)}$$

Note that I_3 is negative, which means that this current flows opposite to the direction indicated in Figure 11.2.2. Also note that we could have applied Kirchhoff's voltage law to the outer loop of the circuit. However, this produces a redundant equation (try it). ◈

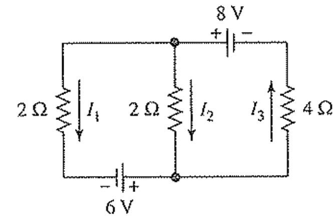
EXERCISE SET 11.2

In Exercises 1–4 find the currents in the circuits.

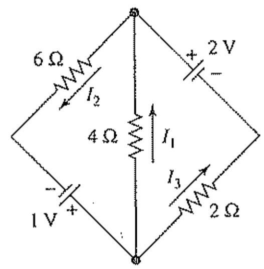
1.



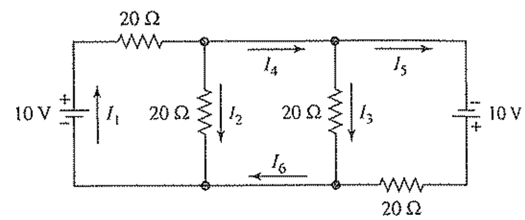
2.



3.



4.



5. Show that if the current I_5 in the circuit of the accompanying figure is zero, then $R_4 = R_3 R_2 / R_1$.

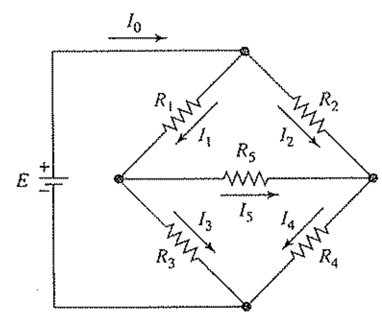


Figure Ex-5

REMARK This circuit, called a Wheatstone bridge circuit, is used for the precise measurement of resistance. Here, R_4 is an unknown resistance and R_1, R_2, and R_3 are adjustable calibrated resistors. R_5 represents a galvanometer—a device for measuring current. After the operator varies the resistances R_1, R_2, and R_3 until the galvanometer reading is zero, the formula $R_4 = R_3 R_2 / R_1$ determines the unknown resistance R_4.

6. Show that if the two currents labeled I in the circuits of the accompanying figure are equal, then $R = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$.

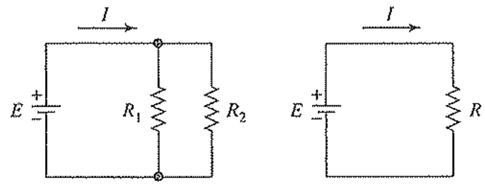


Figure Ex-6

SECTION 11.2 Technology Exercises

The following exercises are designed to be solved using a technology utility. Typically, this will be MATLAB, *Mathematica*, Maple, Derive, or Mathcad, but it may also be some other type of linear algebra software or a scientific calculator with some linear algebra capabilities. For each exercise you will need to read the relevant documentation for the particular utility you are using. The goal of these exercises is to provide you with a basic proficiency with your technology utility. Once you have mastered the techniques in these exercises, you will be able to use your technology utility to solve many of the problems in the regular exercise sets.

T1. The accompanying figure shows a sequence of different circuits.

(a) Solve for the current I_1, for the circuit in part (*a*) of the figure.

(b) Solve for the currents I_1 through I_3, for the circuit in part (*b*) of the figure.

(c) Solve for the currents I_1 through I_5, for the circuit in part (*c*) of the figure.

(d) Continue this process until you discover a pattern in the values of $I_1, I_2, I_3, \ldots$.

(e) Investigate the sequence of values for I_1 in each of the circuits in parts (a), (b), (c), and so on, and numerically show that the limit of this sequence approaches the value

$$\left(\frac{\sqrt{5}-1}{2}\right)\frac{E}{R} \approx (0.6180)\frac{E}{R}$$

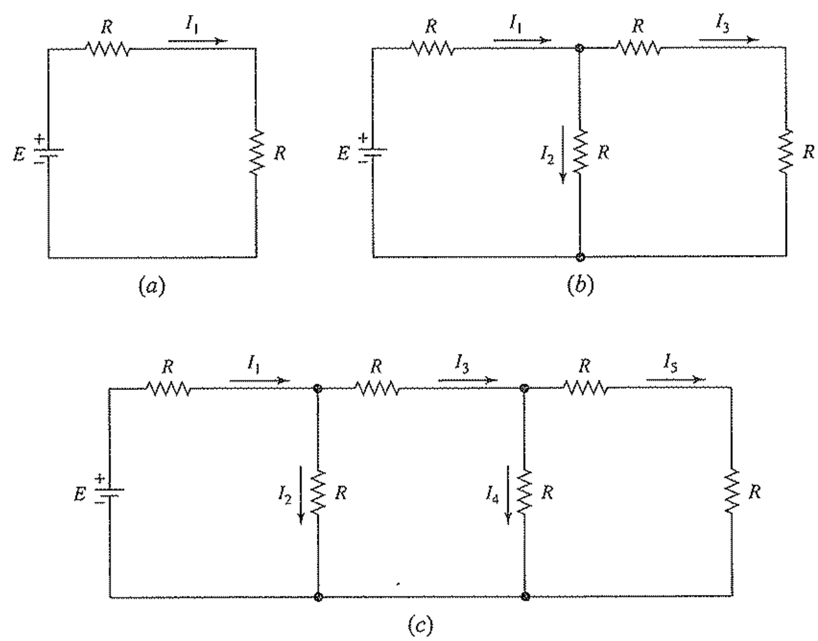


Figure Ex-T1

T2. The accompanying figure shows a sequence of different circuits.

(a) Solve for the current I_1, for the circuit in part (a) of the figure.

(b) Solve for the current I_1, for the circuit in part (b) of the figure.

(c) Solve for the current I_1, for the circuit in part (c) of the figure.

(d) Continue this process until you discover a pattern in the values of I_1.

(e) Investigate the sequence of values for I_1 in each of the circuits in parts (a), (b), (c), and so on, and numerically show that the limit of this sequence approaches the value

$$\left(\frac{\sqrt{5}-1}{2}\right)\frac{E}{R} \approx (0.6180)\frac{E}{R}$$

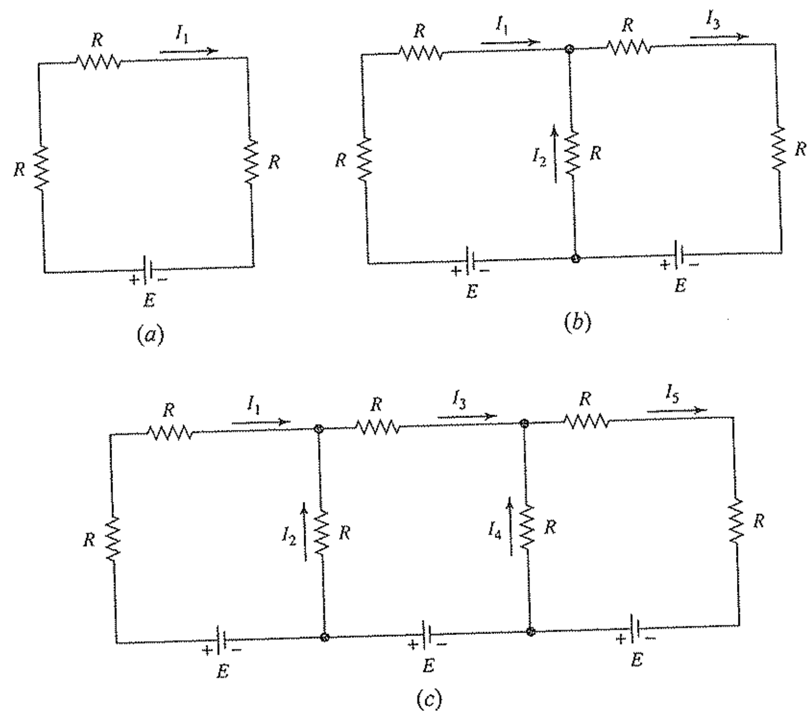


Figure Ex-T2

11.3 GEOMETRIC LINEAR PROGRAMMING

In this section we describe a geometric technique for maximizing or minimizing a linear expression in two variables subject to a set of linear constraints.

PREREQUISITES: Linear Systems
Linear Inequalities

Linear Programming

The study of linear programming theory has expanded greatly since the pioneering work of George Dantzig in the late 1940s. Today, linear programming is applied to a wide variety of problems in industry and science. In this section we present a geometric approach to the solution of simple linear programming problems. Let us begin with some examples.