T2. A mouse is placed in a box with nine rooms as shown in the accompanying figure. Assume that it is equally likely that the mouse goes through any door in the room or stays in the room.

(a) Construct the 9×9 transition matrix for this problem and show that it is regular.

(b) Determine the steady-state vector for the matrix.

(c) Use a symmetry argument to show that this problem may be solved using only a 3×3 matrix.

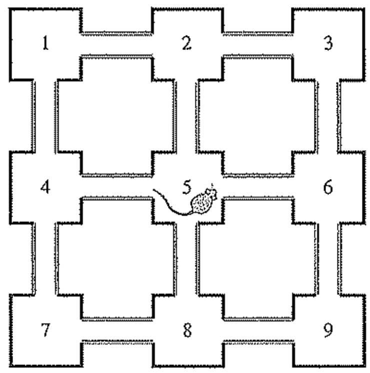


Figure Ex-T2

11.7 GRAPH THEORY

In this section we introduce matrix representations of relations among members of a set. We use matrix arithmetic to analyze these relationships.

PREREQUISITES: Matrix Addition and Multiplication

Relations among Members of a Set

There are countless examples of sets with finitely many members in which some relation exists among members of the set. For example, the set could consist of a collection of people, animals, countries, companies, sports teams, or cities; and the relation between two members, A and B, of such a set could be that person A dominates person B, animal A feeds on animal B, country A militarily supports country B, company A sells its product to company B, sports team A consistently beats sports team B, or city A has a direct airline flight to city B.

We shall now show how the theory of *directed graphs* can be used to mathematically model relations such as those in the preceding examples.

Directed Graphs

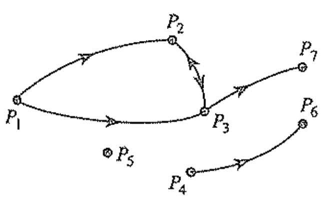


Figure 11.7.1

A ***directed graph*** is a finite set of elements, $\{P_1, P_2, \ldots, P_n\}$, together with a finite collection of ordered pairs (P_i, P_j) of distinct elements of this set, with no ordered pair being repeated. The elements of the set are called ***vertices***, and the ordered pairs are called ***directed edges***, of the directed graph. We use the notation $P_i \to P_j$ (which is read "P_i is connected to P_j") to indicate that the directed edge (P_i, P_j) belongs to the directed graph. Geometrically, we can visualize a directed graph (Figure 11.7.1) by representing the vertices as points in the plane and representing the directed edge $P_i \to P_j$ by drawing a line or arc from vertex P_i to vertex P_j, with an arrow pointing from P_i to P_j. If both $P_i \to P_j$ and $P_j \to P_i$ hold (denoted $P_i \leftrightarrow P_j$), we draw a single line between P_i and P_j with two oppositely pointing arrows (as with P_2 and P_3 in the figure).

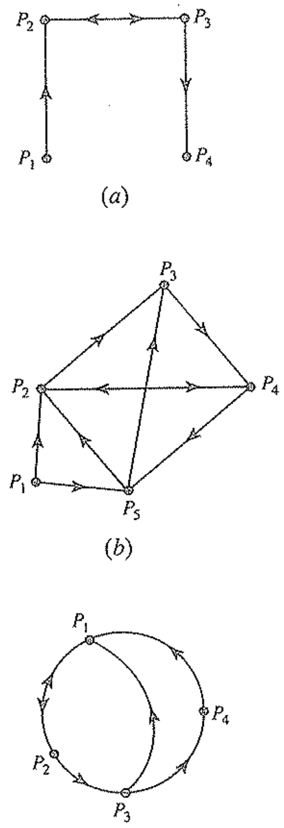


Figure 11.7.2

As in Figure 11.7.1, for example, a directed graph may have separate "components" of vertices that are connected only among themselves; and some vertices, such as P_5, may not be connected with any other vertex. Also, because $P_i \rightarrow P_i$ is not permitted in a directed graph, a vertex cannot be connected with itself by a single arc that does not pass through any other vertex.

Figure 11.7.2 shows diagrams representing three more examples of directed graphs. With a directed graph having n vertices, we may associate an $n \times n$ matrix $M = [m_{ij}]$, called the ***vertex matrix*** of the directed graph. Its elements are defined by

$$m_{ij} = \begin{cases} 1, & \text{if } P_i \rightarrow P_j \\ 0, & \text{otherwise} \end{cases}$$

for $i, j = 1, 2, \ldots, n$. For the three directed graphs in Figure 11.7.2, the corresponding vertex matrices are

$$\text{Figure 11.7.2}a: \qquad M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Figure 11.7.2}b: \qquad M = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{Figure 11.7.2}c: \qquad M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

By their definition, vertex matrices have the following two properties:

(i) All entries are either 0 or 1.

(ii) All diagonal entries are 0.

Conversely, any matrix with these two properties determines a unique directed graph having the given matrix as its vertex matrix. For example, the matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

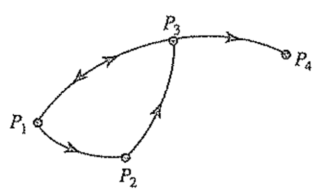


Figure 11.7.3

determines the directed graph in Figure 11.7.3.

EXAMPLE 1 Influences within a Family

A certain family consists of a mother, father, daughter, and two sons. The family members have influence, or power, over each other in the following ways: the mother can influence the daughter and the oldest son; the father can influence the two sons; the daughter can influence the father; the oldest son can influence the youngest son; and the youngest son can influence the mother. We may model this family influence pattern with a directed graph whose vertices are the five family members. If family member A influences family member B, we write $A \to B$. Figure 11.7.4 is the resulting directed graph, where we have used obvious letter designations for the five family members. The vertex matrix of this directed graph is

$$\begin{array}{c} \\ M \\ F \\ D \\ OS \\ YS \end{array} \begin{array}{c} \begin{array}{ccccc} M & F & D & OS & YS \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad ◈$$

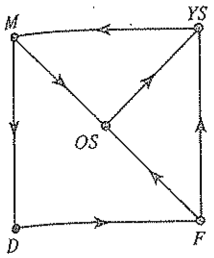


Figure 11.7.4

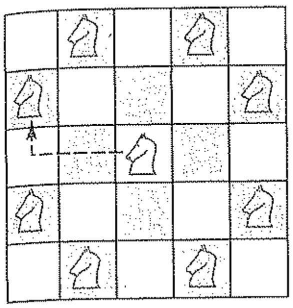


Figure 11.7.5

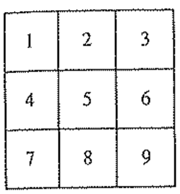


Figure 11.7.6

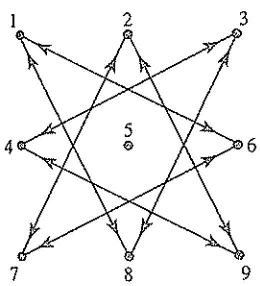


Figure 11.7.7

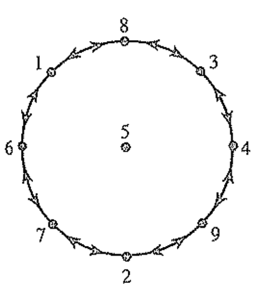


Figure 11.7.8

EXAMPLE 2 Vertex Matrix: Moves on a Chessboard

In chess the knight moves in an "L"-shaped pattern about the chessboard. For the board in Figure 11.7.5 it may move horizontally two squares and then vertically one square, or it may move vertically two squares and then horizontally one square. Thus, from the center square in the figure, the knight may move to any of the eight marked shaded squares. Suppose that the knight is restricted to the nine numbered squares in Figure 11.7.6. If by $i \to j$ we mean that the knight may move from square i to square j, the directed graph in Figure 11.7.7 illustrates all possible moves that the knight may make among these nine squares. In Figure 11.7.8 we have "unraveled" Figure 11.7.7 to make the pattern of possible moves clearer.

The vertex matrix of this directed graph is given by

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad ◈$$

In Example 1 the father cannot directly influence the mother; that is, $F \to M$ is not true. But he can influence the youngest son, who can then influence the mother. We write this as $F \to YS \to M$ and call it a ***2-step connection*** from F to M. Analogously, we call $M \to D$ a ***1-step connection***, $F \to OS \to YS \to M$ a ***3-step connection***, and so forth. Let us now consider a technique for finding the number of all possible r-step connections $(r = 1, 2, \ldots)$ from one vertex P_i to another vertex P_j of an arbitrary directed graph. (This will include the case when P_i and P_j are the same vertex.) The number of 1-step connections from P_i to P_j is simply m_{ij}. That is, there is either zero or one 1-step connection from P_i to P_j, depending on whether m_{ij} is zero or one. For the number of 2-step connections, we consider the square of the vertex matrix. If we let $m_{ij}^{(2)}$ be the (i, j)-th element of M^2, we have

$$m_{ij}^{(2)} = m_{i1}m_{1j} + m_{i2}m_{2j} + \cdots + m_{in}m_{nj} \tag{1}$$

Now, if $m_{i1} = m_{1j} = 1$, there is a 2-step connection $P_i \to P_1 \to P_j$ from P_i to P_j. But if either m_{i1} or m_{1j} is zero, such a 2-step connection is not possible. Thus $P_i \to P_1 \to P_j$

is a 2-step connection if and only if $m_{i1}m_{1j} = 1$. Similarly, for any $k = 1, 2, \ldots, n$, $P_i \to P_k \to P_j$ is a 2-step connection from P_i to P_j if and only if the term $m_{ik}m_{kj}$ on the right side of (1) is one; otherwise, the term is zero. Thus, the right side of (1) is the total number of two 2-step connections from P_i to P_j.

A similar argument will work for finding the number of 3-, 4-, ..., n-step connections from P_i to P_j. In general, we have the following result.

THEOREM 11.7.1

Let M be the vertex matrix of a directed graph and let $m_{ij}^{(r)}$ be the (i, j)-th element of M^r. Then $m_{ij}^{(r)}$ is equal to the number of r-step connections from P_i to P_j.

EXAMPLE 3 Using Theorem 11.7.1

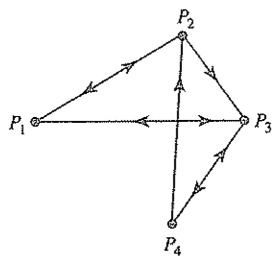


Figure 11.7.9

Figure 11.7.9 is the route map of a small airline that services the four cities P_1, P_2, P_3, P_4. As a directed graph, its vertex matrix is

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

We have that

$$M^2 = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M^3 = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 2 & 3 & 1 \\ 4 & 0 & 2 & 2 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

If we are interested in connections from city P_4 to city P_3, we may use Theorem 11.7.1 to find their number. Because $m_{43} = 1$, there is one 1-step connection; because $m_{43}^{(2)} = 1$, there is one 2-step connection; and because $m_{43}^{(3)} = 3$, there are three 3-step connections. To verify this, from Figure 11.7.9 we find

1-step connections from P_4 to P_3: $P_4 \to P_3$

2-step connections from P_4 to P_3: $P_4 \to P_2 \to P_3$

3-step connections from P_4 to P_3: $P_4 \to P_3 \to P_4 \to P_3$

$P_4 \to P_2 \to P_1 \to P_3$

$P_4 \to P_3 \to P_1 \to P_3$

Cliques

In everyday language a "clique" is a closely knit group of people (usually three or more) that tends to communicate within itself and has no place for outsiders. In graph theory this concept is given a more precise meaning.

DEFINITION

A subset of a directed graph is called a ***clique*** if it satisfies the following three conditions:

(i) The subset contains at least three vertices.

(ii) For each pair of vertices P_i and P_j in the subset, both $P_i \to P_j$ and $P_j \to P_i$ are true.

(iii) The subset is as large as possible; that is, it is not possible to add another vertex to the subset and still satisfy condition (ii).

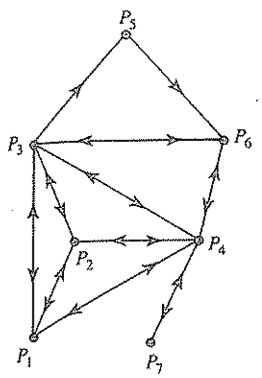


Figure 11.7.10

This definition suggests that cliques are maximal subsets that are in perfect "communication" with each other. For example, if the vertices represent cities, and $P_i \to P_j$ means that there is a direct airline flight from city P_i to city P_j, then there is a direct flight between any two cities within a clique in either direction.

EXAMPLE 4 A Directed Graph with Two Cliques

The directed graph illustrated in Figure 11.7.10 (which might represent the route map of an airline) has two cliques:

$$\{P_1, P_2, P_3, P_4\} \quad \text{and} \quad \{P_3, P_4, P_6\}$$

This example shows that a directed graph may contain several cliques and that a vertex may simultaneously belong to more than one clique. ◈

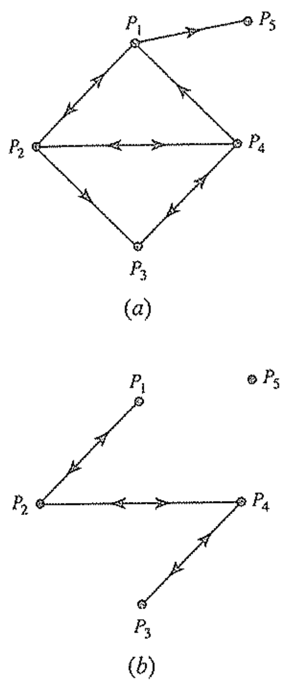


Figure 11.7.11

For simple directed graphs, cliques can be found by inspection. But for large directed graphs, it would be desirable to have a systematic procedure for detecting cliques. For this purpose, it will be helpful to define a matrix $S = [s_{ij}]$ related to a given directed graph as follows:

$$s_{ij} = \begin{cases} 1, & \text{if } P_i \leftrightarrow P_j \\ 0, & \text{otherwise} \end{cases}$$

The matrix S determines a directed graph that is the same as the given directed graph, with the exception that the directed edges with only one arrow are deleted. For example, if the original directed graph is given by Figure 11.7.11a, the directed graph that has S as its vertex matrix is given in Figure 11.7.11b. The matrix S may be obtained from the vertex matrix M of the original directed graph by setting $s_{ij} = 1$ if $m_{ij} = m_{ji} = 1$ and setting $s_{ij} = 0$ otherwise.

The following theorem, which uses the matrix S, is helpful for identifying cliques.

THEOREM 11.7.2

Identifying Cliques

Let $s_{ij}^{(3)}$ be the (i, j)-th element of S^3. Then a vertex P_i belongs to some clique if and only if $s_{ii}^{(3)} \neq 0$.

Proof If $s_{ii}^{(3)} \neq 0$, then there is at least one 3-step connection from P_i to itself in the modified directed graph determined by S. Suppose it is $P_i \to P_j \to P_k \to P_i$. In the modified directed graph, all directed relations are two-way, so we also have the connections $P_i \leftrightarrow P_j \leftrightarrow P_k \leftrightarrow P_i$. But this means that $\{P_i, P_j, P_k\}$ is either a clique or a subset of a clique. In either case, P_i must belong to some clique. The converse statement, "if P_i belongs to a clique, then $s_{ij}^{(3)} \neq 0$," follows in a similar manner. ∎

EXAMPLE 5 Using Theorem 11.7.2

Suppose that a directed graph has as its vertex matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S^3 = \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

Because all diagonal entries of S^3 are zero, it follows from Theorem 11.7.2 that the directed graph has no cliques. ◈

EXAMPLE 6 Using Theorem 11.7.2

Suppose that a directed graph has as its vertex matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then

$$S = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S^3 = \begin{bmatrix} 2 & 4 & 0 & 4 & 3 \\ 4 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 2 & 1 \\ 3 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The nonzero diagonal entries of S^3 are $s_{11}^{(3)}$, $s_{22}^{(3)}$, and $s_{44}^{(3)}$. Consequently, in the given directed graph, P_1, P_2, and P_4 belong to cliques. Because a clique must contain at least three vertices, the directed graph has only one clique, $\{P_1, P_2, P_4\}$. ◈

Dominance-Directed Graphs

In many groups of individuals or animals, there is a definite "pecking order" or dominance relation between any two members of the group. That is, given any two individuals A and B, either A dominates B or B dominates A, but not both. In terms of a directed graph in which $P_i \rightarrow P_j$ means P_i dominates P_j, this means that for all distinct pairs, either $P_i \rightarrow P_j$ or $P_j \rightarrow P_i$, but not both. In general, we have the following definition.

DEFINITION

A ***dominance-directed graph*** is a directed graph such that for any distinct pair of vertices P_i and P_j, either $P_i \rightarrow P_j$ or $P_j \rightarrow P_i$, but not both.

An example of a directed graph satisfying this definition is a league of n sports teams that play each other exactly one time, as in one round of a round-robin tournament in which no ties are allowed. If $P_i \to P_j$ means that team P_i beat team P_j in their single match, it is easy to see that the definition of a dominance-directed group is satisfied. For this reason, dominance-directed graphs are sometimes called ***tournaments***.

Figure 11.7.12 illustrates some dominance-directed graphs with three, four, and five vertices, respectively. In these three graphs, the circled vertices have the following interesting property: from each one there is either a 1-step or a 2-step connection to any other vertex in its graph. In a sports tournament, these vertices would correspond to the most "powerful" teams in the sense that these teams either beat any given team or beat some other team that beat the given team. We can now state and prove a theorem that guarantees that any dominance-directed graph has at least one vertex with this property.

THEOREM 11.7.3

Connections in Dominance-Directed Graphs

In any dominance-directed graph, there is at least one vertex from which there is a 1-step or 2-step connection to any other vertex.

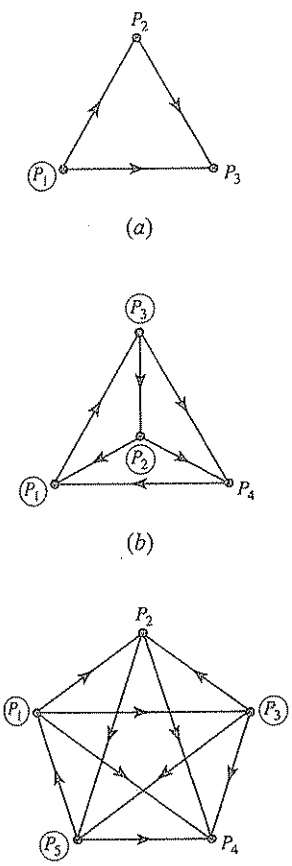


Figure 11.7.12

Proof Consider a vertex (there may be several) with the largest total number of 1-step and 2-step connections to other vertices in the graph. By renumbering the vertices, we may assume that P_1 is such a vertex. Suppose there is some vertex P_i such that there is no 1-step or 2-step connection from P_1 to P_i. Then, in particular, $P_1 \to P_i$ is not true, so that by definition of a dominance-directed graph, it must be that $P_i \to P_1$. Next, let P_k be any vertex such that $P_1 \to P_k$ is true. Then we cannot have $P_k \to P_i$, as then $P_1 \to P_k \to P_i$ would be a 2-step connection from P_1 to P_i. Thus, it must be that $P_i \to P_k$. That is, P_i has 1-step connections to all the vertices to which P_1 has 1-step connections. The vertex P_i must then also have 2-step connections to all the vertices to which P_1 has 2-step connections. But because, in addition, we have that $P_i \to P_1$, this means that P_i has more 1-step and 2-step connections to other vertices than does P_1. However, this contradicts the way in which P_1 was chosen. Hence, there can be no vertex P_i to which P_1 has no 1-step or 2-step connection. ∎

This proof shows that a vertex with the largest total number of 1-step and 2-step connections to other vertices has the property stated in the theorem. There is a simple way of finding such vertices using the vertex matrix M and its square M^2. The sum of the entries in the ith row of M is the total number of 1-step connections from P_i to other vertices, and the sum of the entries of the ith row of M^2 is the total number of 2-step connections from P_i to other vertices. Consequently, the sum of the entries of the ith row of the matrix $A = M + M^2$ is the total number of 1-step and 2-step connections from P_i to other vertices. In other words, a row of $A = M + M^2$ with the largest row sum identifies a vertex having the property stated in Theorem 11.7.3.

EXAMPLE 7 Using Theorem 11.7.3

Suppose that five baseball teams play each other exactly once, and the results are as indicated in the dominance-directed graph of Figure 11.7.13. The vertex matrix of the graph is

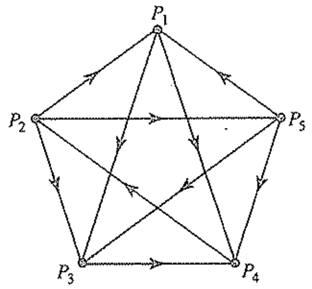


Figure 11.7.13

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

so

$$A = M + M^2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{bmatrix}$$

The row sums of A are

$$\begin{aligned} \text{1st row sum} &= 4 \\ \text{2nd row sum} &= 9 \\ \text{3rd row sum} &= 2 \\ \text{4th row sum} &= 4 \\ \text{5th row sum} &= 7 \end{aligned}$$

Because the second row has the largest row sum, the vertex P_2 must have a 1-step or 2-step connection to any other vertex. This is easily verified from Figure 11.7.13. ◈

We have informally suggested that a vertex with the largest number of 1-step and 2-step connections to other vertices is a "powerful" vertex. We can formalize this concept with the following definition.

DEFINITION

The ***power*** of a vertex of a dominance-directed graph is the total number of 1-step and 2-step connections from it to other vertices. Alternatively, the power of a vertex P_i is the sum of the entries of the ith row of the matrix $A = M + M^2$, where M is the vertex matrix of the directed graph.

EXAMPLE 8 Example 7 Revisited

Let us rank the five baseball teams in Example 7 according to their powers. From the calculations for the row sums in that example, we have

$$\begin{aligned} \text{Power of team } P_1 &= 4 \\ \text{Power of team } P_2 &= 9 \\ \text{Power of team } P_3 &= 2 \\ \text{Power of team } P_4 &= 4 \\ \text{Power of team } P_5 &= 7 \end{aligned}$$

Hence, the ranking of the teams according to their powers would be

P_2 (first), P_5 (second), P_1 and P_4 (tied for third), P_3 (last) ◈

EXERCISE SET 11.7

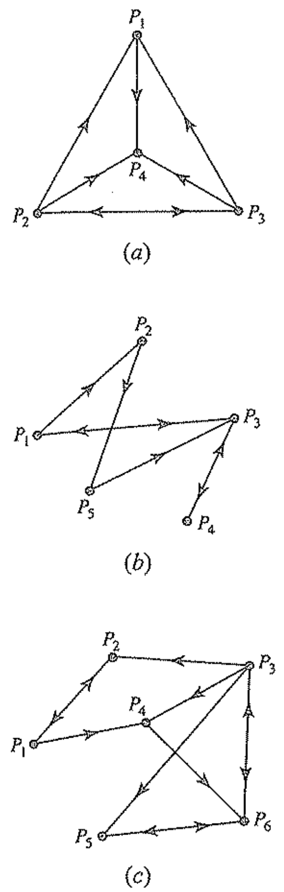


Figure Ex-1

1. Construct the vertex matrix for each of the directed graphs illustrated in the accompanying figure.

2. Draw a diagram of the directed graph corresponding to each of the following vertex matrices.

(a) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

3. Let M be the following vertex matrix of a directed graph:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(a) Draw a diagram of the directed graph.

(b) Use Theorem 11.7.1 to find the number of 1-, 2-, and 3-step connections from the vertex P_1 to the vertex P_2. Verify your answer by listing the various connections as in Example 3.

(c) Repeat part (b) for the 1-, 2-, and 3-step connections from P_1 to P_4.

4. (a) Compute the matrix product M^TM for the vertex matrix M in Example 1.

(b) Verify that the kth diagonal entry of M^TM is the number of family members who influence the kth family member. Why is this true?

(c) Find a similar interpretation for the values of the nondiagonal entries of M^TM.

5. By inspection, locate all cliques in each of the directed graphs illustrated in the accompanying figure.

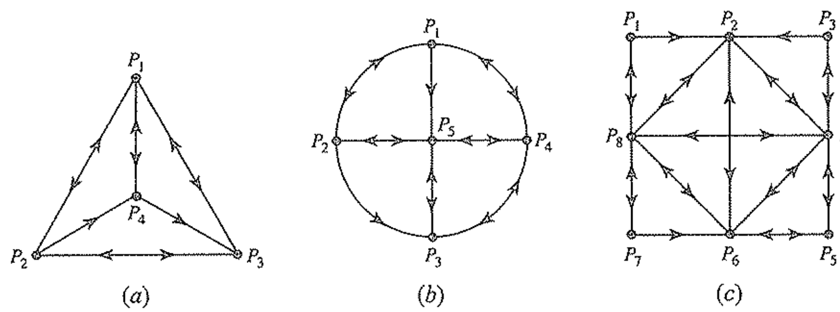


Figure Ex-5

6. For each of the following vertex matrices, use Theorem 11.7.2 to find all cliques in the corresponding directed graphs.

(a) $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$

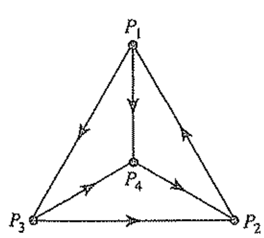


Figure Ex-7

7. For the dominance-directed graph illustrated in the accompanying figure, construct the vertex matrix and find the power of each vertex.

8. Five baseball teams play each other one time with the following results:

A beats B, C, D
B beats C, E
C beats D, E
D beats B
E beats A, D

Rank the five baseball teams in accordance with the powers of the vertices they correspond to in the dominance-directed graph representing the outcomes of the games.

SECTION 11.7 Technology Exercises

The following exercises are designed to be solved using a technology utility. Typically, this will be MATLAB, *Mathematica*, Maple, Derive, or Mathcad, but it may also be some other type of linear algebra software or a scientific calculator with some linear algebra capabilities. For each exercise you will need to read the relevant documentation for the particular utility you are using. The goal of these exercises is to provide you with a basic proficiency with your technology utility. Once you have mastered the techniques in these exercises, you will be able to use your technology utility to solve many of the problems in the regular exercise sets.

T1. A graph having n vertices such that every vertex is connected to every other vertex has vertex matrix given by

$$M_n = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

In this problem we develop a formula for M_n^k whose (i, j)-th entry equals the number of k-step connections from P_i to P_j.

(a) Use a computer to compute the eight matrices M_n^k for $n = 2, 3$ and for $k = 2, 3, 4, 5$.

(b) Use the results in part (a) and symmetry arguments to show that M_n^k can be written as

$$M_n^k = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \cdots & 0 \end{bmatrix}^k = \begin{bmatrix} \alpha_k & \beta_k & \beta_k & \beta_k & \beta_k & \cdots & \beta_k \\ \beta_k & \alpha_k & \beta_k & \beta_k & \beta_k & \cdots & \beta_k \\ \beta_k & \beta_k & \alpha_k & \beta_k & \beta_k & \cdots & \beta_k \\ \beta_k & \beta_k & \beta_k & \alpha_k & \beta_k & \cdots & \beta_k \\ \beta_k & \beta_k & \beta_k & \beta_k & \alpha_k & \cdots & \beta_k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_k & \beta_k & \beta_k & \beta_k & \beta_k & \cdots & \alpha_k \end{bmatrix}$$

(c) Using the fact that $M_n^k = M_n M_n^{k-1}$, show that

$$\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} 0 & n-1 \\ 1 & n-2 \end{bmatrix} \begin{bmatrix} \alpha_{k-1} \\ \beta_{k-1} \end{bmatrix}$$

with

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(d) Using part (c), show that

$$\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} 0 & n-1 \\ 1 & n-2 \end{bmatrix}^{k-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(e) Use the methods of Section 7.2 to compute

$$\begin{bmatrix} 0 & n-1 \\ 1 & n-2 \end{bmatrix}^{k-1}$$

and thereby obtain expressions for α_k and β_k, and eventually show that

$$M_n^k = \left(\frac{(n-1)^k - (-1)^k}{n}\right) U_n + (-1)^k I_n$$

where U_n is the $n \times n$ matrix all of whose entries are ones and I_n is the $n \times n$ identity matrix.

(f) Show that for $n > 2$, all vertices for these directed graphs belong to cliques.

T2. Consider a round-robin tournament among n players (labeled $a_1, a_2, a_3, \ldots, a_n$) where a_1 beats a_2, a_2 beats a_3, a_3 beats $a_4, \ldots, a_{n-1}$ beats a_n, and a_n beats a_1. Compute the "power" of each player, showing that they all have the same power; then determine that common power.

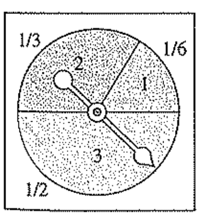
Hint Use a computer to study the cases $n = 3, 4, 5, 6$; then make a conjecture and prove your conjecture to be true.

11.8 GAMES OF STRATEGY

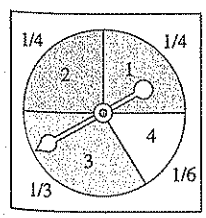
In this section we discuss a general game in which two competing players choose separate strategies to reach opposing objectives. The optimal strategy of each player is found in certain cases with the use of matrix techniques.

PREREQUISITES: Matrix Multiplication
Basic Probability Concepts

Game Theory



Row-wheel of player R



Column-wheel of player C

Figure 11.8.1

To introduce the basic concepts in the theory of games, we will consider the following carnival-type game that two people agree to play. We will call the participants in the game *player R* and *player C*. Each player has a stationary wheel with a movable pointer on it as in Figure 11.8.1. For reasons that will become clear, we will call player R's wheel the *row-wheel* and player C's wheel the *column-wheel*. The row-wheel is divided into three sectors numbered 1, 2, and 3, and the column-wheel is divided into four sectors numbered 1, 2, 3, and 4. The fractions of the area occupied by the various sectors are indicated in the figure. To play the game, each player spins the pointer of his or her wheel and lets it come to rest at random. The number of the sector in which each pointer comes to rest is called the *move* of that player. Thus, player R has three possible moves and player C has four possible moves. Depending on the move each player makes, player C then makes a payment of money to player R according to Table 1.

For example, if the row-wheel pointer comes to rest in sector 1 (player R makes move 1), and the column-wheel pointer comes to rest in sector 2 (player C makes move 2), then player C must pay player R the sum of \$5. Some of the entries in this table are negative, indicating that player C makes a negative payment to player R. By this we mean that player R makes a positive payment to player C. For example, if the row-wheel shows 2 and the column-wheel shows 4, then player R pays player C the sum of \$4, because the corresponding entry in the table is −\$4. In this way the positive entries of the table are the gains of player R and the losses of player C, and the negative entries are the gains of player C and the losses of player R.