(e) Use the methods of Section 7.2 to compute

$$\begin{bmatrix} 0 & n-1 \\ 1 & n-2 \end{bmatrix}^{k-1}$$

and thereby obtain expressions for α_k and β_k, and eventually show that

$$M_n^k = \left(\frac{(n-1)^k - (-1)^k}{n}\right) U_n + (-1)^k I_n$$

where U_n is the $n \times n$ matrix all of whose entries are ones and I_n is the $n \times n$ identity matrix.

(f) Show that for $n > 2$, all vertices for these directed graphs belong to cliques.

T2. Consider a round-robin tournament among n players (labeled $a_1, a_2, a_3, \ldots, a_n$) where a_1 beats a_2, a_2 beats a_3, a_3 beats $a_4, \ldots, a_{n-1}$ beats a_n, and a_n beats a_1. Compute the "power" of each player, showing that they all have the same power; then determine that common power.

Hint Use a computer to study the cases $n = 3, 4, 5, 6$; then make a conjecture and prove your conjecture to be true.

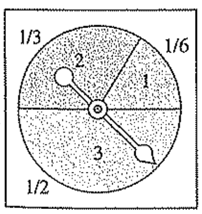
11.8 GAMES OF STRATEGY

In this section we discuss a general game in which two competing players choose separate strategies to reach opposing objectives. The optimal strategy of each player is found in certain cases with the use of matrix techniques.

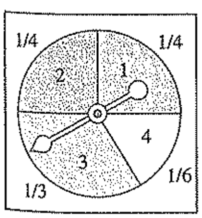
PREREQUISITES: Matrix Multiplication
Basic Probability Concepts

Game Theory

To introduce the basic concepts in the theory of games, we will consider the following carnival-type game that two people agree to play. We will call the participants in the game *player R* and *player C*. Each player has a stationary wheel with a movable pointer on it as in Figure 11.8.1. For reasons that will become clear, we will call player R's wheel the *row-wheel* and player C's wheel the *column-wheel*. The row-wheel is divided into three sectors numbered 1, 2, and 3, and the column-wheel is divided into four sectors numbered 1, 2, 3, and 4. The fractions of the area occupied by the various sectors are indicated in the figure. To play the game, each player spins the pointer of his or her wheel and lets it come to rest at random. The number of the sector in which each pointer comes to rest is called the *move* of that player. Thus, player R has three possible moves and player C has four possible moves. Depending on the move each player makes, player C then makes a payment of money to player R according to Table 1.



Row-wheel of player R



Column-wheel of player C

Figure 11.8.1

For example, if the row-wheel pointer comes to rest in sector 1 (player R makes move 1), and the column-wheel pointer comes to rest in sector 2 (player C makes move 2), then player C must pay player R the sum of \$5. Some of the entries in this table are negative, indicating that player C makes a negative payment to player R. By this we mean that player R makes a positive payment to player C. For example, if the row-wheel shows 2 and the column-wheel shows 4, then player R pays player C the sum of \$4, because the corresponding entry in the table is $-\$4$. In this way the positive entries of the table are the gains of player R and the losses of player C, and the negative entries are the gains of player C and the losses of player R.

Table 1 Payment to Player R

		Player C's Move			
		1	2	3	4
Player R's Move	1	\$3	\$5	−\$2	−\$1
	2	−\$2	\$4	−\$3	−\$4
	3	\$6	−\$5	\$0	\$3

In this game the players have no control over their moves; each move is determined by chance. However, if each player can decide whether he or she wants to play, then each would want to know how much he or she can expect to win or lose over the long term if he or she chooses to play. (Later in the section we will discuss this question and also consider a more complicated situation in which the players can exercise some control over their moves by varying the sectors of their wheels.)

Two-Person Zero-Sum Matrix Games

The game described above is an example of a ***two-person zero-sum matrix game.*** The term *zero-sum* means that in each play of the game, the positive gain of one player is equal to the negative gain (loss) of the other player. That is, the sum of the two gains is zero. The term *matrix game* is used to describe a two-person game in which each player has only a finite number of moves, so that all possible outcomes of each play, and the corresponding gains of the players, may be displayed in tabular or matrix form, as in Table 1.

In a general game of this type, let player R have m possible moves and let player C have n possible moves. In a play of the game, each player makes one of his or her possible moves, and then a *payoff* is made from player C to player R, depending on the moves. For $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$, let us set

$$a_{ij} = \text{payoff that player } C \text{ makes to player } R \text{ if player } R \text{ makes move } i \text{ and player } C \text{ makes move } j$$

This payoff need not be money; it may be any type of commodity to which we can attach a numerical value. As before, if an entry a_{ij} is negative, we mean that player C receives a payoff of $|a_{ij}|$ from player R. We arrange these mn possible payoffs in the form of an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

which we will call the ***payoff matrix*** of the game.

Each player is to make his or her moves on a probabilistic basis. For example, for the game discussed in the introduction, the ratio of the area of a sector to the area of the wheel would be the probability that the player makes the move corresponding to that sector. Thus, from Figure 11.8.1, we see that player R would make move 2 with probability $\frac{1}{3}$, and player C would make move 2 with probability $\frac{1}{4}$. In the general case we make the following definitions:

$$p_i = \text{probability that player } R \text{ makes move } i \quad (i = 1, 2, \ldots, m)$$
$$q_j = \text{probability that player } C \text{ makes move } j \quad (j = 1, 2, \ldots, n)$$

It follows from these definitions that

$$p_1 + p_2 + \cdots + p_m = 1$$

and

$$q_1 + q_2 + \cdots + q_n = 1$$

With the probabilities p_i and q_j we form two vectors:

$$\mathbf{p} = [p_1 \quad p_2 \quad \cdots \quad p_m] \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

We call the row vector $\mathbf{p}$ the ***strategy of player R*** and the column vector $\mathbf{q}$ the ***strategy of player C***. For example, from Figure 11.8.1 we have

$$\mathbf{p} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$$

for the carnival game described earlier.

From the theory of probability, if the probability that player R makes move i is p_i, and independently the probability that player C makes move j is q_j, then $p_i q_j$ is the probability that for any one play of the game, player R makes move i *and* player C makes move j. The payoff to player R for such a pair of moves is a_{ij}. If we multiply each possible payoff by its corresponding probability and sum over all possible payoffs, we obtain the expression

$$a_{11}p_1q_1 + a_{12}p_1q_2 + \cdots + a_{1n}p_1q_n + a_{21}p_2q_1 + \cdots + a_{mn}p_mq_n \tag{1}$$

Equation (1) is a weighted average of the payoffs to player R; each payoff is weighted according to the probability of its occurrence. In the theory of probability, this weighted average is called the ***expected payoff*** to player R. It can be shown that if the game is played many times, the long-term average payoff per play to player R is given by this expression. We denote this expected payoff by $E(\mathbf{p}, \mathbf{q})$ to emphasize the fact that it depends on the strategies of the two players. From the definition of the payoff matrix A and the strategies $\mathbf{p}$ and $\mathbf{q}$, it can be verified that we may express the expected payoff in matrix notation as

$$E(\mathbf{p}, \mathbf{q}) = [p_1 \quad p_2 \quad \cdots \quad p_m] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \mathbf{p}A\mathbf{q} \tag{2}$$

Because $E(\mathbf{p}, \mathbf{q})$ is the expected payoff to player R, it follows that $-E(\mathbf{p}, \mathbf{q})$ is the expected payoff to player C.

EXAMPLE 1 Expected Payoff to Player R

For the carnival game described earlier, we have

$$E(\mathbf{p}, \mathbf{q}) = \mathbf{p}A\mathbf{q} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 5 & -2 & -1 \\ -2 & 4 & -3 & -4 \\ 6 & -5 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix} = \frac{13}{72} = .1805\ldots$$

Thus, in the long run, player R can expect to receive an average of about 18 cents from player C in each play of the game. ◈

So far we have been discussing the situation in which each player has a predetermined strategy. We will now consider the more difficult situation in which both players may change their strategies independently. For example, in the game described in the introduction, we would allow both players to alter the areas of the sectors of their wheels and thereby control the probabilities of their respective moves. This qualitatively changes the nature of the problem and puts us firmly in the field of true game theory. It is understood that neither player knows what strategy the other will choose. It is also assumed that each player will make the best possible choice of strategy and that the other player knows this. Thus, player R attempts to choose a strategy $\mathbf{p}$ such that $E(\mathbf{p}, \mathbf{q})$ is as large as possible for the best strategy $\mathbf{q}$ that player C can choose; and similarly, player C attempts to choose a strategy $\mathbf{q}$ such that $E(\mathbf{p}, \mathbf{q})$ is as small as possible for the best strategy $\mathbf{p}$ that player R can choose. To see that such choices are actually possible, we shall need the following theorem, called the ***Fundamental Theorem of Two-Person Zero-Sum Games***. (The general proof, which involves ideas from the theory of linear programming, will be omitted. However, later we shall prove two special cases of the theorem.)

THEOREM 11.8.1

Fundamental Theorem of Zero-Sum Games

There exist strategies $\mathbf{p}^*$ *and* $\mathbf{q}^*$ *such that*

$$E(\mathbf{p}^*, \mathbf{q}) \geq E(\mathbf{p}^*, \mathbf{q}^*) \geq E(\mathbf{p}, \mathbf{q}^*) \tag{3}$$

for all strategies $\mathbf{p}$ *and* $\mathbf{q}$.

The strategies $\mathbf{p}^*$ and $\mathbf{q}^*$ in this theorem are the best possible strategies for players R and C, respectively. To see why this is so, let $v = E(\mathbf{p}^*, \mathbf{q}^*)$. The left-hand inequality of Equation (3) then reads

$$E(\mathbf{p}^*, \mathbf{q}) \geq v \quad \text{for all strategies } \mathbf{q}$$

This means that if player R chooses the strategy $\mathbf{p}^*$, then no matter what strategy $\mathbf{q}$ player C chooses, the expected payoff to player R will never be below v. Moreover, it is not possible for player R to achieve an expected payoff greater than v. To see why, suppose there is some strategy $\mathbf{p}^{**}$ that player R can choose such that

$$E(\mathbf{p}^{**}, \mathbf{q}) > v \quad \text{for all strategies } \mathbf{q}$$

Then, in particular,

$$E(\mathbf{p}^{**}, \mathbf{q}^*) > v$$

But this contradicts the right-hand inequality of Equation (3), which requires that $v \geq E(\mathbf{p}^{**}, \mathbf{q}^*)$. Consequently, the best player R can do is prevent his or her expected payoff from falling below the value v. Similarly, the best player C can do is ensure that player R's expected payoff does not exceed v, and this can be achieved by using strategy $\mathbf{q}^*$.

On the basis of this discussion, we arrive at the following definitions.

DEFINITION

If $\mathbf{p}^*$ and $\mathbf{q}^*$ are strategies such that

$$E(\mathbf{p}^*, \mathbf{q}) \geq E(\mathbf{p}^*, \mathbf{q}^*) \geq E(\mathbf{p}, \mathbf{q}^*) \tag{4}$$

for all strategies $\mathbf{p}$ and $\mathbf{q}$, then

(i) $\mathbf{p}^*$ is called an ***optimal strategy for player R***.

(ii) $\mathbf{q}^*$ is called an ***optimal strategy for player C***.

(iii) $v = E(\mathbf{p}^*, \mathbf{q}^*)$ is called the ***value*** of the game.

The wording in this definition suggests that optimal strategies are not necessarily unique. This is indeed the case, and in Exercise 2 we ask the reader to show this. However, it can be proved that any two sets of optimal strategies always result in the same value v of the game. That is, if $\mathbf{p}^*$, $\mathbf{q}^*$ and $\mathbf{p}^{**}$, $\mathbf{q}^{**}$ are optimal strategies, then

$$E(\mathbf{p}^*, \mathbf{q}^*) = E(\mathbf{p}^{**}, \mathbf{q}^{**}) \tag{5}$$

The value of a game is thus the expected payoff to player R when both players choose any possible optimal strategies.

To find optimal strategies, we must find vectors $\mathbf{p}^*$ and $\mathbf{q}^*$ that satisfy Equation (4). This is generally done by using linear programming techniques. Next, we discuss special cases for which optimal strategies may be found by more elementary techniques.

We now introduce the following definition.

DEFINITION

An entry a_{rs} in a payoff matrix A is called a ***saddle point*** if

(i) a_{rs} is the smallest entry in its row, and

(ii) a_{rs} is the largest entry in its column.

A game whose payoff matrix has a saddle point is called ***strictly determined.***

For example, the shaded element in each of the following payoff matrices is a saddle point:

$$\begin{bmatrix} 3 & 1 \\ -4 & 0 \end{bmatrix}, \quad \begin{bmatrix} 30 & -50 & -5 \\ 60 & 90 & 75 \\ -10 & 60 & -30 \end{bmatrix}, \quad \begin{bmatrix} 0 & -3 & 5 & -9 \\ 15 & -8 & -2 & 10 \\ 7 & 10 & 6 & 9 \\ 6 & 11 & -3 & 2 \end{bmatrix}$$

If a matrix has a saddle point a_{rs}, it turns out that the following strategies are optimal strategies for the two players:

$$\mathbf{p}^* = [0 \quad 0 \quad \cdots \quad \underset{r\text{th entry}}{1} \quad \cdots \quad 0], \qquad \mathbf{q}^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow s\text{th entry}$$

That is, an optimal strategy for player R is to always make the rth move, and an optimal strategy for player C is to always make the sth move. Such strategies for which only one move is possible are called ***pure strategies***. Strategies for which more than one move is possible are called ***mixed strategies***. To show that the above pure strategies are optimal, the reader may verify the following three equations (see Exercise 6):

$$E(\mathbf{p}^*, \mathbf{q}^*) = \mathbf{p}^* A\mathbf{q}^* = a_{rs} \tag{6}$$

$$E(\mathbf{p}^*, \mathbf{q}) = \mathbf{p}^* A\mathbf{q} \geq a_{rs} \quad \text{for any strategy } \mathbf{q} \tag{7}$$

$$E(\mathbf{p}, \mathbf{q}^*) = \mathbf{p} A\mathbf{q}^* \leq a_{rs} \quad \text{for any strategy } \mathbf{p} \tag{8}$$

Together, these three equations imply that

$$E(\mathbf{p}^*, \mathbf{q}) \geq E(\mathbf{p}^*, \mathbf{q}^*) \geq E(\mathbf{p}, \mathbf{q}^*)$$

for all strategies $\mathbf{p}$ and $\mathbf{q}$. Because this is exactly Equation (4), it follows that $\mathbf{p}^*$ and $\mathbf{q}^*$ are optimal strategies.

From Equation (6) the value of a strictly determined game is simply the numerical value of a saddle point a_{rs}. It is possible for a payoff matrix to have several saddle points, but then the uniqueness of the value of a game guarantees that the numerical values of all saddle points are the same.

EXAMPLE 2 Optimal Strategies to Maximize a Viewing Audience

Two competing television networks, R and C, are scheduling one-hour programs in the same time period. Network R can schedule one of three possible programs, and network C can schedule one of four possible programs. Neither network knows which program the other will schedule. Both networks ask the same outside polling agency to give them an estimate of how all possible pairings of the programs will divide the viewing audience. The agency gives them each Table 2, whose (i, j)-th entry is the percentage of the viewing audience that will watch network R if network R's program i is paired against network C's program j. What program should each network schedule in order to maximize its viewing audience?

Table 2 Audience Percentage for Network R

		Network C's Program			
		1	2	3	4
Network R's Program	1	60	20	30	55
	2	50	75	45	60
	3	70	45	35	30

Solution

Subtract 50 from each entry in Table 2 to construct the following matrix:

$$\begin{bmatrix} 10 & -30 & -20 & 5 \\ 0 & 25 & -5 & 10 \\ 20 & -5 & -15 & -20 \end{bmatrix}$$

This is the payoff matrix of the two-person zero-sum game in which each network is considered to start with 50% of the audience, and the (i, j)-th entry of the matrix is the percentage of the viewing audience that network C loses to network R if programs i and j are paired against each other. It is easy to see that the entry

$$a_{23} = -5$$

is a saddle point of the payoff matrix. Hence, the optimal strategy of network R is to schedule program 2, and the optimal strategy of network C is to schedule program 3. This will result in network R's receiving 45% of the audience and network C's receiving 55% of the audience. ◈

2 × 2 Matrix Games

Another case in which the optimal strategies can be found by elementary means occurs when each player has only two possible moves. In this case, the payoff matrix is a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

If the game is strictly determined, at least one of the four entries of A is a saddle point, and the techniques discussed above can then be applied to determine optimal strategies for the two players. If the game is not strictly determined, we first compute the expected payoff for arbitrary strategies $\mathbf{p}$ and $\mathbf{q}$:

$$\begin{aligned} E(\mathbf{p}, \mathbf{q}) = \mathbf{p}A\mathbf{q} &= [p_1 \quad p_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\ &= a_{11}p_1q_1 + a_{12}p_1q_2 + a_{21}p_2q_1 + a_{22}p_2q_2 \end{aligned} \tag{9}$$

Because

$$p_1 + p_2 = 1 \quad \text{and} \quad q_1 + q_2 = 1 \tag{10}$$

we may substitute $p_2 = 1 - p_1$ and $q_2 = 1 - q_1$ into (9) to obtain

$$E(\mathbf{p}, \mathbf{q}) = a_{11}p_1q_1 + a_{12}p_1(1 - q_1) + a_{21}(1 - p_1)q_1 + a_{22}(1 - p_1)(1 - q_1) \tag{11}$$

If we rearrange the terms in Equation (11), we may write

$$E(\mathbf{p}, \mathbf{q}) = [(a_{11} + a_{22} - a_{12} - a_{21})p_1 - (a_{22} - a_{21})]q_1 + (a_{12} - a_{22})p_1 + a_{22} \tag{12}$$

By examining the coefficient of the q_1 term in (12), we see that if we set

$$p_1 = p_1^* = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \tag{13}$$

then that coefficient is zero, and (12) reduces to

$$E(\mathbf{p}^*, \mathbf{q}) = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \tag{14}$$

Equation (14) is independent of $\mathbf{q}$; that is, if player R chooses the strategy determined by (13), player C cannot change the expected payoff by varying his or her strategy.

In a similar manner, it may be verified that if player C chooses the strategy determined by

$$q_1 = q_1^* = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \tag{15}$$

then substituting in (12) gives

$$E(\mathbf{p}, \mathbf{q}^*) = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \tag{16}$$

Equations (14) and (16) show that

$$E(\mathbf{p}^*, \mathbf{q}) = E(\mathbf{p}^*, \mathbf{q}^*) = E(\mathbf{p}, \mathbf{q}^*) \tag{17}$$

for all strategies $\mathbf{p}$ and $\mathbf{q}$. Thus, the strategies determined by (13), (15), and (10) are optimal strategies for players R and C, respectively, and so we have the following result.

THEOREM 11.8.2 **Optimal Strategies for a 2 × 2 Matrix Game**

For a 2×2 game that is not strictly determined, optimal strategies for players R and C are

$$\mathbf{p}^* = \begin{bmatrix} \dfrac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} & \dfrac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \end{bmatrix}$$

and

$$\mathbf{q}^* = \begin{bmatrix} \dfrac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \\ \dfrac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \end{bmatrix}$$

The value of the game is

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

In order to be complete, we must show that the entries in the vectors $\mathbf{p}^*$ and $\mathbf{q}^*$ are numbers strictly between 0 and 1. In Exercise 8 we ask the reader to show that this is the case as long as the game is not strictly determined.

Equation (17) is interesting in that it implies that either player may force the expected payoff to be the value of the game by choosing his or her optimal strategy, regardless of which strategy the other player chooses. This is not true, in general, for games in which either player has more than two moves.

EXAMPLE 3 Using Theorem 11.8.2

The federal government desires to inoculate its citizens against a certain flu virus. The virus has two strains, and the proportions in which the two strains occur in the virus population is not known. Two vaccines have been developed. Vaccine 1 is 85% effective against strain 1 and 70% effective against strain 2. Vaccine 2 is 60% effective against strain 1 and 90% effective against strain 2. What inoculation policy should the government adopt?

Solution

We may consider this a two-person game in which player R (the government) desires to make the payoff (the fraction of citizens resistant to the virus) as large as possible, and player C (the virus) desires to make the payoff as small as possible. The payoff matrix is

$$\begin{array}{cc} & \textbf{Strain} \\ & \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \textbf{Vaccine} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} & \begin{bmatrix} .85 & .70 \\ .60 & .90 \end{bmatrix} \end{array}$$

This matrix has no saddle points, so Theorem 11.8.2 is applicable. Consequently,

$$p_1^* = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{.90 - .60}{.85 + .90 - .70 - .60} = \frac{.30}{.45} = \frac{2}{3}$$

$$p_2^* = 1 - p_1^* = 1 - \frac{2}{3} = \frac{1}{3}$$

$$q_1^* = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{.90 - .70}{.85 + .90 - .70 - .60} = \frac{.20}{.45} = \frac{4}{9}$$

$$q_2^* = 1 - q_1^* = 1 - \frac{4}{9} = \frac{5}{9}$$

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{(.85)(.90) - (.70)(.60)}{.85 + .90 - .70 - .60} = \frac{.345}{.45} = .7666\ldots$$

Thus, the optimal strategy for the government is to inoculate $\frac{2}{3}$ of the citizens with vaccine 1 and $\frac{1}{3}$ of the citizens with vaccine 2. This will guarantee that about 76.7% of the citizens will be resistant to a virus attack regardless of the distribution of the two strains.

In contrast, a virus distribution of $\frac{4}{9}$ of strain 1 and $\frac{5}{9}$ of strain 2 will result in the same 76.7% of resistant citizens, regardless of the inoculation strategy adopted by the government (see Exercise 8). ◈

EXERCISE SET 11.8

1. Suppose that a game has a payoff matrix

$$A = \begin{bmatrix} -4 & 6 & -4 & 1 \\ 5 & -7 & 3 & 8 \\ -8 & 0 & 6 & -2 \end{bmatrix}$$

(a) If players R and C use strategies

$$\mathbf{p} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

respectively, what is the expected payoff of the game?

(b) If player C keeps his strategy fixed as in part (a), what strategy should player R choose to maximize his expected payoff?

(c) If player R keeps her strategy fixed as in part (a), what strategy should player C choose to minimize the expected payoff to player R?

2. Construct a simple example to show that optimal strategies are not necessarily unique. For example, find a payoff matrix with several equal saddle points.

3. For the strictly determined games with the following payoff matrices, find optimal strategies for the two players, and find the values of the games.

(a) $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & -2 \\ 2 & 4 \\ -4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -2 & 0 \\ -6 & 0 & -5 \\ 5 & 2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & 2 & -1 \\ -2 & -1 & 5 \\ -4 & 1 & 0 \\ -3 & 4 & 6 \end{bmatrix}$

4. For the 2×2 games with the following payoff matrices, find optimal strategies for the two players, and find the values of the games.

(a) $\begin{bmatrix} 6 & 3 \\ -1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 40 & 20 \\ -10 & 30 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 7 \\ -5 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 7 & -3 \\ -5 & -2 \end{bmatrix}$

5. Player R has two playing cards: a black ace and a red four. Player C also has two cards: a black two and a red three. Each player secretly selects one of his or her cards. If both selected cards are the same color, player C pays player R the sum of the face values in dollars. If the cards are different colors, player R pays player C the sum of the face values. What are optimal strategies for both players, and what is the value of the game?

6. Verify Equations (6), (7), and (8).

7. Verify the statement in the last paragraph of Example 3.

8. Show that the entries of the optimal strategies $\mathbf{p}^*$ and $\mathbf{q}^*$ given in Theorem 11.8.2 are numbers strictly between zero and one.

SECTION 11.8 Technology Exercises

The following exercises are designed to be solved using a technology utility. Typically, this will be MATLAB, *Mathematica*, Maple, Derive, or Mathcad, but it may also be some other type of linear algebra software or a scientific calculator with some linear algebra capabilities. For each exercise you will need to read the relevant documentation for the particular utility you are using. The goal of these exercises is to provide you with a basic proficiency with your technology utility. Once you have mastered the techniques in these exercises, you will be able to use your technology utility to solve many of the problems in the regular exercise sets.

T1. Consider a game between two players where each player can make up to n different moves ($n > 1$). If the ith move of player R and the jth move of player C are such that $i + j$ is even, then C pays R \$1. If $i + j$ is odd, then R pays C \$1. Assume that both players have the same strategy—that is, $\mathbf{p}_n = [\rho_i]_{1\times n}$ and $\mathbf{q}_n = [\rho_i]_{n\times 1}$, where $\rho_1 + \rho_2 + \rho_3 + \cdots + \rho_n = 1$. Use a computer to show that

$$\begin{aligned} E(\mathbf{p}_2, \mathbf{q}_2) &= (\rho_1 - \rho_2)^2 \\ E(\mathbf{p}_3, \mathbf{q}_3) &= (\rho_1 - \rho_2 + \rho_3)^2 \\ E(\mathbf{p}_4, \mathbf{q}_4) &= (\rho_1 - \rho_2 + \rho_3 - \rho_4)^2 \\ E(\mathbf{p}_5, \mathbf{q}_5) &= (\rho_1 - \rho_2 + \rho_3 - \rho_4 + \rho_5)^2 \end{aligned}$$

Using these results as a guide, prove in general that the expected payoff to player R is

$$E(\mathbf{p}_n, \mathbf{q}_n) = \left(\sum_{j=1}^{n} (-1)^{j+1}\rho_j\right)^2 \geq 0$$

which shows that in the long run, player R *will not lose* in this game.

T2. Consider a game between two players where each player can make up to n different moves ($n > 1$). If both players make the same move, then player C pays player R \$$(n-1)$. However, if both players make different moves, then player R pays player C \$1. Assume that both players have the same strategy—that is, $\mathbf{p}_n = [\rho_i]_{1\times n}$ and $\mathbf{q}_n = [\rho_i]_{n\times 1}$, where $\rho_1 + \rho_2 + \rho_3 + \cdots + \rho_n = 1$. Use a computer to show that

$$E(\mathbf{p}_2, \mathbf{q}_2) = \tfrac{1}{2}(\rho_1 - \rho_1)^2 + \tfrac{1}{2}(\rho_1 - \rho_2)^2 + \tfrac{1}{2}(\rho_2 - \rho_1)^2 + \tfrac{1}{2}(\rho_2 - \rho_2)^2$$

$$\begin{aligned} E(\mathbf{p}_3, \mathbf{q}_3) = {} & \tfrac{1}{2}(\rho_1 - \rho_1)^2 + \tfrac{1}{2}(\rho_1 - \rho_2)^2 + \tfrac{1}{2}(\rho_1 - \rho_3)^2 \\ & + \tfrac{1}{2}(\rho_2 - \rho_1)^2 + \tfrac{1}{2}(\rho_2 - \rho_2)^2 + \tfrac{1}{2}(\rho_2 - \rho_3)^2 \\ & + \tfrac{1}{2}(\rho_3 - \rho_1)^2 + \tfrac{1}{2}(\rho_3 - \rho_2)^2 + \tfrac{1}{2}(\rho_3 - \rho_3)^2 \end{aligned}$$

$$\begin{aligned} E(\mathbf{p}_4, \mathbf{q}_4) = {} & \tfrac{1}{2}(\rho_1 - \rho_1)^2 + \tfrac{1}{2}(\rho_1 - \rho_2)^2 + \tfrac{1}{2}(\rho_1 - \rho_3)^2 + \tfrac{1}{2}(\rho_1 - \rho_4)^2 \\ & + \tfrac{1}{2}(\rho_2 - \rho_1)^2 + \tfrac{1}{2}(\rho_2 - \rho_2)^2 + \tfrac{1}{2}(\rho_2 - \rho_3)^2 + \tfrac{1}{2}(\rho_2 - \rho_4)^2 \\ & + \tfrac{1}{2}(\rho_3 - \rho_1)^2 + \tfrac{1}{2}(\rho_3 - \rho_2)^2 + \tfrac{1}{2}(\rho_3 - \rho_3)^2 + \tfrac{1}{2}(\rho_3 - \rho_4)^2 \\ & + \tfrac{1}{2}(\rho_4 - \rho_1)^2 + \tfrac{1}{2}(\rho_4 - \rho_2)^2 + \tfrac{1}{2}(\rho_4 - \rho_3)^2 + \tfrac{1}{2}(\rho_4 - \rho_4)^2 \end{aligned}$$

Using these results as a guide, prove in general that the expected payoff to player R is

$$E(\mathbf{p}_n, \mathbf{q}_n) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\rho_i - \rho_j)^2 \geq 0$$

which shows that in the long run, player R *will not lose* in this game.

11.9 LEONTIEF ECONOMIC MODELS

In this section we discuss two linear models for economic systems. Some results about nonnegative matrices are applied to determine equilibrium price structures and outputs necessary to satisfy demand.

PREREQUISITES: Linear Systems
Matrices

Economic Systems

Matrix theory has been very successful in describing the interrelations among prices, outputs, and demands in economic systems. In this section we discuss some simple models based on the ideas of Nobel laureate Wassily Leontief. We examine two different but related models: the closed or input–output model, and the open or production model. In each, we are given certain economic parameters that describe the interrelations between the "industries" in the economy under consideration. Using matrix theory, we then evaluate certain other parameters, such as prices or output levels, in order to satisfy a desired economic objective. We begin with the closed model.

Leontief Closed (Input–Output) Model

First we present a simple example; then we proceed to the general theory of the model.

EXAMPLE 1 An Input–Output Model

Three homeowners—a carpenter, an electrician, and a plumber—agree to make repairs in their three homes. They agree to work a total of 10 days each according to the following schedule:

	Work Performed by		
	Carpenter	**Electrician**	**Plumber**
Days of Work in Home of Carpenter	2	1	6
Days of Work in Home of Electrician	4	5	1
Days of Work in Home of Plumber	4	4	3