Mathematical Research Problems

The problems in these notes were discussed in the Informal Mathematics Research Problem Session (IMRPS). The IMRPS is a weekly activity sponsored by the Research Training Group (RTG) in Logic and Dyanmics, Department of Mathematics, University of North Texas. The organizer for the IMRPS is Professor Dan Mauldin. For more information on the IMRPS, including its schedules and an archive of notes, visit the RTG website at `http://www.math.unt.edu/rtg/`.

Question: What is a continued fraction?

Before answering this question, let's look at the following expression,

$$\begin{aligned}\frac{49}{38} &= 1\frac{11}{38}\\ &= 1+\frac{1}{\frac{38}{11}}\\ &= 1+\frac{1}{3+\frac{5}{11}}\\ &= 1+\frac{1}{3+\frac{1}{\frac{11}{5}}}\\ &= 1+\frac{1}{3+\frac{1}{2+\frac{1}{5}}}.\end{aligned}$$

Definition: A finite continued fraction is an expression of the form

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}},$$

where a_0 is an integer, any other a_i members are positive integers, and n is a non-negative integer.

An infinite continued fraction can be written as

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\ddots}}}}.$$

Note that we have yet to make sense of this formal object! The previous expression illustrates a finite continued fraction. The integers a_0, a_1, etc., are called the partial quotients of the continued fraction. We prefer to call them partial denominators! One can abbreviate a continued fraction as

$$x = [a_0; a_1, a_2, a_3].$$

Therefore, $\frac{49}{38} = [1; 3, 2, 5]$ with our new notation.

The Euclidean algorithm has a close relationship with continued fractions. In fact, for rationals, the continued fraction algorithm is nothing but the Euclidan Algorithm! In the worked example above, the GCD(49,38) was calculated, and the quotients are $1, 3, 2$ and 5, respectively.

One may draw a picture of what is going on by the successive partitioning of a rectangle. What will happen if we vary some of those a_i's of a continued fraction? It may be inferred from the picture that if we increase/decrease the even/odd partial denominators, the number being represented will increase/decrease.

Let us now move towards making sense of our infinite continued fraction. Consider the following fact, which is natural when seen from the viewpoint of Möbius transformations or $SL(2, \mathbb{Z})$ and may be proved via an easy induction.

Definiton: If $(a_n)_{n\geq 0}$ is given, where a_0 is an integer and any other a_i's are positive integers, we define $\frac{p_n}{q_n} := [a_0; a_1, \cdots, a_n]$.

Then

$$(*) \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}.$$

Observation 1

$$\begin{bmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} a_{n+1} & 1 \\ 1 & 0 \end{bmatrix}.$$

This equation implies that,

$$p_{n+1} = a_{n+1}p_n + p_{n-1} \quad p_{-1} = 1 \quad p_{-2} = 0$$

$$q_{n+1} = a_{n+1}q_n + q_{n-1} \quad q_{-1} = 0 \quad q_{-2} = 1.$$

Observation 2 $1 = q_0 \leq q_1 < q_2 < q_3 < \cdots.$

Observation 3 $q_n \geq 2^{\frac{n-2}{2}}.$

Observation 4 *By taking determinants on both sides of* $(*)$, *we have*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}.$$

i.e.,

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1}q_n}.$$

One may now observe that we have

$$\frac{p_n}{q_n} = a_0 + \frac{1}{q_0 q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_2 q_3} + \cdots + \frac{(-1)^{n-1}}{q_{n-1} q_n}.$$

Thus we can define

$$\alpha := \lim_{n\to\infty} \frac{p_n}{q_n} = a_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{q_{n-1} q_n}.$$

Note that the alternating series in the definition converges due to Leibniz's Test, since the terms are decreasing in absolute value as can be see from **Observation 2**. Also note that

$$a_0 = \frac{p_0}{q_0} < \cdots < \frac{p_{2n}}{q_{2n}} < \cdots < \alpha < \cdots < \frac{p_{2n+1}}{q_{2n+1}} < \cdots < \frac{p_1}{q_1}.$$

One may again interpret these strings on inequalities via a picture. In general, I recommend you to draw as many pictures as possible to enhance our geometric intuition for these expansions!

Question: Is α rational? (will be answered later.)

Observation 5

$$\alpha - \frac{p_n}{q_n} = (-1)^n \left(\frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1} q_{n+2}} + \cdots \right).$$

Observation 6

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

Proof:

$$\left| \alpha - \frac{p_n}{q_n} \right| < \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}}.$$

□

Dirichlet's Theorem:

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2} \le \frac{1}{q_n^2}.$$

Example:

We study $\pi = [3; 7, 15, 1, 292, 1, \cdots]$. $\frac{22}{7} = [3; 7]$. $\frac{355}{113} = [3; 7, 15, 1]$. We have

$$\left| \pi - \frac{22}{7} \right| < \frac{1}{15 \cdot 7^2}$$

$$\left| \pi - \frac{355}{113} \right| < \frac{1}{292 \cdot 113^2}.$$

Every infinite continued fraction is irrational, and every irrational number can be represented in precisely one way as an infinite continued fraction. The first statement answers our question above and may be deduced via Dirichlet's Theorem.

Observation 7 $\alpha \notin \mathbb{Q}$.

Proof: We prove this by way of contradiction.

Suppose not. Let $\alpha = \frac{a}{b} \in \mathbb{Q}$. Since $\left|\frac{a}{b} - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$. $|q_n a - bp_n| \leq \frac{bq_n}{q_n^2} = \frac{b}{q_n}$. For n large enough, $\frac{a}{b} = \frac{p_n}{q_n}$. $(p_n, q_n) = 1, \forall n$. This contradicts that the sequence (q_n) continually increases.

□

One of the most important reasons for the utility of infinite continued fraction representations for an irrational number is due to its initial segments providing excellent rational approximations to the number. These rational numbers are called the convergents of the continued fraction. Remember that even-numbered convergents are smaller than the original number, while odd-numbered ones are bigger - see the string of inequalities above **Observation 5** and think about the picture that goes with it.

Convergents are locally best approximations. Let $\alpha = [a_0, a_1, a_2, \cdots] \notin \mathbb{Q}$. For any $n > 1$, p, q with $0 < q \leq q_n$, if $\frac{p}{q} \neq \frac{p_n}{q_n}$, then $|p_n - q_n\alpha| < |p - q\alpha|$, or $\left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{p}{q}\right|$.

We're almost out of time and so let me give you just a taste for yet another way to view continued fractions - this time from the perspective of dynamical systems. Let us make the following

Definition:(Gauss map) Define a map $G : [0, 1) \to [0, 1)$ by $x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$. The map G is called the Gauss map.

The partitioning into intervals of $[0, 1)$ via looking at higher iterates of this map is related to the Farey sequence, F_n, which may be defined as follows - for any positive integer n is the set of irreducible rational numbers $\frac{a}{b}$ with $0 \leq a \leq b \leq n$ and $(a, b) = 1$ arranged in increasing order. The first few are

$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$

$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$

$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$.

Unfortunately we don't have time to enter that story. More about the Gauss Map and the measure-theoretic/probabilistic study of continued fractions may be found in Khinchin's book. One will also find here the beautiful theorem of Lagrange on the periodicity of c.f. expansions for quadratic surds.

To end let me mention that there is no reason to stop at continued fraction expansions for real numbers! In 1761 Lambert proved that $\tan x \notin \mathbb{Q}$ for $x \in \mathbb{Q}\backslash\{0\}$ from which $\pi \notin \mathbb{Q}$ follows easily. He proved this using a remarkable continued fraction expansion for $\tan(z)$, viz.

$$\tan(z) = \cfrac{z}{1 - \cfrac{z^2}{3 - \cfrac{z^2}{5 - \cfrac{z^2}{\ddots}}}}.$$

More details in the paper by Laczkovich. In fact Euler deduced the irrationality of e from its continued fraction expansion - for more see the beautiful article by Shirali written for high school kids.

So we've gone over the time limit! One can't leave without recommending Wall's book *Analytic Theory of Continued Fractions.*

My apologies for not being able to discuss some of the current research interest in Diophantine Approximation and Continued Fractions - especially from a dynamical perspective, which is closest to my own. Please feel free to come by and ask me for details about such. Thank you for your indulgence!

(Discussed by Tushar Das on December 3rd, 2010. Notes taken by Xiaohui Shi.)

References

[1] Khinchin, A. Ya., Continued fractions. With a preface by B. V. Gnedenko. Translated from the third (1961) Russian edition. Reprint of the 1964 translation. Dover Publications, Inc., Mineola, NY, 1997.

[2] Laczkovich, M. On Lambert's proof of the irrationality of π. Amer. Math. Monthly 104 (1997), no. 5, 439443.

[3] Shirali, S. Continued fractions for e. Resonance 5(1),2000,14-28.

[4] Wall, H. S. Analytic Theory of Continued Fractions. D. Van Nostrand Company, Inc., New York, N. Y., 1948.