Rigidity in infinite-dimensional hyperbolic spaces.

Tushar Das

University of North Texas (UNT)

joint with
David Simmons (UNT)
Bernd Stratmann (Bremen)
Mariusz Urbański (UNT)

March 4, 2012

Geometry and Analysis on Fractal Spaces

2012 AMS Spring Western Section Meeting

University of Hawaii at Manoa

Outline

Gromov hyperbolic spaces

Classification of isometries

Kleinian groups

Theorems on the radial limit set

Sullivan Measurable Rigidity

Outline

Gromov hyperbolic spaces

Classification of isometries

Kleinian groups

Theorems on the radial limit set

Sullivan Measurable Rigidity

The birth of discrete actions of hyperbolic isometries

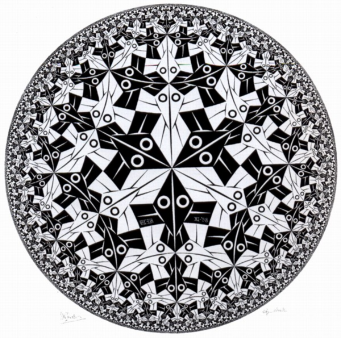
The study of discrete groups acting on two- and three-dimensional Euclidean open balls by hyperbolic isometries began at the end of 19th century with the works of Fuchs, Klein and Poincaré.

The birth of discrete actions of hyperbolic isometries

The study of discrete groups acting on two- and three-dimensional Euclidean open balls by hyperbolic isometries began at the end of 19th century with the works of Fuchs, Klein and Poincaré. The modern revival was led by Thurston, Sullivan and Gromov.

The birth of discrete actions of hyperbolic isometries

The study of discrete groups acting on two- and three-dimensional Euclidean open balls by hyperbolic isometries began at the end of 19th century with the works of Fuchs, Klein and Poincaré. The modern revival was led by Thurston, Sullivan and Gromov.



Circle Limit I, Escher (1958)

Outline

Gromov hyperbolic spaces

Classification of isometries

Kleinian groups

Theorems on the radial limit set

Sullivan Measurable Rigidity

...once more, with (an infinite-dimensional) feeling

"Séminaire Sullivan", IHÉS 1983.

Dennis Sullivan in the early '80s indicated a possibility of developing the theory of discrete groups acting by hyperbolic isometries on the open unit ball of a separable Hilbert space.

...once more, with (an infinite-dimensional) feeling

"Séminaire Sullivan", IHÉS 1983.

Dennis Sullivan in the early '80s indicated a possibility of developing the theory of discrete groups acting by hyperbolic isometries on the open unit ball of a separable Hilbert space.

Geometric Group Theory, Sussex 1991.

Asymptotic invariants of infinite groups

Misha Gromov refers to infinite dimensional hyperbolic spaces as …"cute and sexy" …"long neglected by geometers and algebraists alike."

Outline

Gromov hyperbolic spaces

Classification of isometries

Kleinian groups

Theorems on the radial limit set

Sullivan Measurable Rigidity

...once more, with (an infinite-dimensional) feeling

"Séminaire Sullivan", IHÉS 1983.

Dennis Sullivan in the early '80s indicated a possibility of developing the theory of discrete groups acting by hyperbolic isometries on the open unit ball of a separable Hilbert space.

Geometric Group Theory, Sussex 1991.

Asymptotic invariants of infinite groups

Misha Gromov refers to infinite dimensional hyperbolic spaces as ..."cute and sexy" ..."long neglected by geometers and algebraists alike."

Gromov+Sullivan IHÉS 2007, Copyright: MFO

Outline

Gromov hyperbolic spaces

Classification of ...etries

...ian groups

...rems on the ...l limit set

...van ...surable ...ity

Gromov hyperbolicity

Definition

Let (X, d) be a metric space. For three points $x, y, z \in X$, we define the Gromov product of x and y with respect to z by

$$\langle x|y\rangle_z := \frac{1}{2}[d(x,z) + d(y,z) - d(x,y)].$$

[1] Denotes an additive asymptotic.

Gromov hyperbolicity

Definition

Let (X, d) be a metric space. For three points $x, y, z \in X$, we define the Gromov product of x and y with respect to z by

$$\langle x|y\rangle_z := \frac{1}{2}[d(x,z) + d(y,z) - d(x,y)].$$

Definition

(X, d) is hyperbolic (or Gromov hyperbolic) if for every four points $x, y, z, w \in X$ we have

$$\langle x|z\rangle_w \gtrsim^1 \min(\langle x|y\rangle_w, \langle y|z\rangle_w).$$

We will refer to this inequality as Gromov's inequality.

[1] Denotes an additive asymptotic.

Examples of Gromov hyperbolic spaces

Every CAT(-1) space is Gromov hyperbolic. In particular, the following examples are Gromov hyperbolic:

Examples of Gromov hyperbolic spaces

Every CAT(-1) space is Gromov hyperbolic. In particular, the following examples are Gromov hyperbolic:

- Standard hyperbolic space $\mathbb{H}^n$

Examples of Gromov hyperbolic spaces

Every CAT(-1) space is Gromov hyperbolic. In particular, the following examples are Gromov hyperbolic:

- Standard hyperbolic space $\mathbb{H}^n$
- Complex and quaternionic hyperbolic space

Examples of Gromov hyperbolic spaces

Every CAT(-1) space is Gromov hyperbolic. In particular, the following examples are Gromov hyperbolic:

- Standard hyperbolic space $\mathbb{H}^n$
- Complex and quaternionic hyperbolic space
- Infinite-dimensional hyperbolic space $\mathbb{H}^\infty \subseteq \ell^2$

Examples of Gromov hyperbolic spaces

Every CAT(-1) space is Gromov hyperbolic. In particular, the following examples are Gromov hyperbolic:

- Standard hyperbolic space $\mathbb{H}^n$
- Complex and quaternionic hyperbolic space
- Infinite-dimensional hyperbolic space $\mathbb{H}^\infty \subseteq \ell^2$
- Any $\mathbb{R}$-tree

Examples of Gromov hyperbolic spaces

Every CAT(-1) space is Gromov hyperbolic. In particular, the following examples are Gromov hyperbolic:

- Standard hyperbolic space $\mathbb{H}^n$
- Complex and quaternionic hyperbolic space
- Infinite-dimensional hyperbolic space $\mathbb{H}^\infty \subseteq \ell^2$
- Any $\mathbb{R}$-tree
- A "generic" finitely presented group with its Cayley metric

Examples of Gromov hyperbolic spaces

Every CAT(-1) space is Gromov hyperbolic. In particular, the following examples are Gromov hyperbolic:

- Standard hyperbolic space $\mathbb{H}^n$
- Complex and quaternionic hyperbolic space
- Infinite-dimensional hyperbolic space $\mathbb{H}^\infty \subseteq \ell^2$
- Any $\mathbb{R}$-tree
- A "generic" finitely presented group with its Cayley metric

Definition

A Gromov hyperbolic space X is proper if the distance function $d(0,\cdot) : X \to \mathbb{R}$ is proper, or in other words if for all $r > 0$ the set

$$\overline{B}(0,r)$$

is compact. Here 0 is a distinguished point that we fix in X.

The boundary of a hyperbolic space

Any Gromov hyperbolic space X has a Gromov boundary ∂X, analogous to the sphere at infinity of standard hyperbolic space. It is defined in a similar way to the completion of a metric space, with the quantity

$$e^{-\langle x|y\rangle_0}$$

playing a role analogous to the distance function.

The boundary of a hyperbolic space

Any Gromov hyperbolic space X has a Gromov boundary ∂X, analogous to the sphere at infinity of standard hyperbolic space. It is defined in a similar way to the completion of a metric space, with the quantity

$$e^{-\langle x|y\rangle_0}$$

playing a role analogous to the distance function.

The most important fact about the Gromov boundary is the following heuristic:

The boundary of a hyperbolic space

Any Gromov hyperbolic space X has a Gromov boundary ∂X, analogous to the sphere at infinity of standard hyperbolic space. It is defined in a similar way to the completion of a metric space, with the quantity

$$e^{-\langle x|y\rangle_0}$$

playing a role analogous to the distance function.

The most important fact about the Gromov boundary is the following heuristic:
The Gromov product can be extended to the boundary in a way that preserves key formulas.

A metric on ∂X

Proposition

For each $a > 1$ sufficiently small, there exists a complete metric D_a on ∂X satisfying the following asymptotic:

$$D_a(\xi, \eta) \asymp_\times a^{-\langle \xi | \eta \rangle_0}. \tag{2.1}$$

If X is proper, then $(\partial X, D_a)$ is compact.

A metric on ∂X

Proposition

For each $a > 1$ sufficiently small, there exists a complete metric D_a on ∂X satisfying the following asymptotic:

$$D_a(\xi, \eta) \asymp_\times a^{-\langle \xi | \eta \rangle_0}. \tag{2.1}$$

If X is proper, then $(\partial X, D_a)$ is compact.

D_a is a generalization of the spherical metric on the Gromov boundary of standard hyperbolic space. It is often called a *visual metric*.

A metric on ∂X

Proposition

For each $a > 1$ sufficiently small, there exists a complete metric D_a on ∂X satisfying the following asymptotic:

$$D_a(\xi, \eta) \asymp_\times a^{-\langle \xi | \eta \rangle_0}. \tag{2.1}$$

If X is proper, then $(\partial X, D_a)$ is compact.

D_a is a generalization of the spherical metric on the Gromov boundary of standard hyperbolic space. It is often called a *visual metric*.

Remark

For CAT(-1) spaces, and in particular for the standard model of hyperbolic geometry, the above proposition holds for any $1 < a \leq e$. In particular, $a = e$ gives the spherical metric.

Convergence of sequences

Outline

Gromov hyperbolic spaces

Classification of isometries

Kleinian groups

Theorems on the radial limit set

Sullivan Measurable Rigidity

Definition

Fix a sequence $(x_n)_n$ in X and a point $\eta \in \partial X$. We say that $(x_n)_n$ converges to η if

$$\langle x_n | \eta \rangle_0 \underset{n}{\to} \infty.$$

[Idea: $a^{-\langle x_n | \eta \rangle_0} \underset{n}{\to} 0$]

In this case, we write $x_n \underset{n}{\to} \eta$.

A metric on $\partial X \setminus \{\xi\}$

Lemma

Let $E_\xi := \partial X \setminus \{\xi\}$ for some fixed $\xi \in \partial X$. If $x_n \to \xi$, then

$$e^{d(0,x_n)} D_{x_n}(\eta_1, \eta_2) \underset{n,\times}{\longrightarrow} e^{-[\langle \eta_1 | \eta_2 \rangle_0 - \sum_{i=1}^{2} \langle \eta_i | \xi \rangle_0]}.$$

A metric on $\partial X \setminus \{\xi\}$

Lemma

Let $E_\xi := \partial X \setminus \{\xi\}$ *for some fixed* $\xi \in \partial X$. *If* $x_n \to \xi$, *then*

$$e^{d(0,x_n)} D_{x_n}(\eta_1, \eta_2) \underset{n,\times}{\longrightarrow} e^{-[\langle \eta_1 | \eta_2 \rangle_0 - \sum_{i=1}^{2} \langle \eta_i | \xi \rangle_0]}.$$

Here $a_n \underset{n,\times}{\longrightarrow} b$ means $\frac{1}{K} \leq \frac{\liminf a_n}{b} \leq \frac{\limsup a_n}{b} \leq K$.

A metric on $\partial X \setminus \{\xi\}$

Lemma

Let $E_\xi := \partial X \setminus \{\xi\}$ for some fixed $\xi \in \partial X$. If $x_n \to \xi$, then

$$e^{d(0,x_n)} D_{x_n}(\eta_1, \eta_2) \xrightarrow[n,\times]{} e^{-[\langle \eta_1 | \eta_2 \rangle_0 - \sum_{i=1}^{2} \langle \eta_i | \xi \rangle_0]}.$$

Here $a_n \xrightarrow[n,\times]{} b$ means $\frac{1}{K} \leq \frac{\liminf a_n}{b} \leq \frac{\limsup a_n}{b} \leq K$.

Corollary

There exists a metric

$$D_{\xi,0}(\eta_1, \eta_2) \asymp_\times e^{-[\langle \eta_1 | \eta_2 \rangle_0 - \sum_{i=1}^{2} \langle \eta_i | \xi \rangle_0]}.$$

Note that in CAT(-1) you have a limit and equality respectively.

Classification of isometries

Theorem

Let g be an isometry preserving some $\xi \in \partial X$. Then $\exists t \in \mathbb{R}$ such that

A $B_\xi(x, g^n x) \asymp_+ nt$

B $D_{\xi,0}(g^n(\eta_1), g^n(\eta_2)) \asymp_\times e^{nt} D_{\xi,0}(\eta_1, \eta_2)$

We call e^{-t} the dynamical derivative of g at ξ.

Classification of isometries

Theorem

Let g be an isometry preserving some $\xi \in \partial X$. Then $\exists t \in \mathbb{R}$ such that

A $B_\xi(x, g^n x) \asymp_+ nt$

B $D_{\xi,0}(g^n(\eta_1), g^n(\eta_2)) \asymp_\times e^{nt} D_{\xi,0}(\eta_1, \eta_2)$

We call e^{-t} the dynamical derivative of g at ξ.

Here $B_\xi(x, y) := \liminf_{z \to \xi} [d(z, x) - d(z, y)]$ is the Busemann function. In Hilbert space, it describes the signed horospherical distance between horospheres centered at ξ through x and y respectively.

Classification of isometries

Theorem

Let g be an isometry preserving some $\xi \in \partial X$. Then $\exists t \in \mathbb{R}$ such that

A $B_\xi(x, g^n x) \asymp_+ nt$

B $D_{\xi,0}(g^n(\eta_1), g^n(\eta_2)) \asymp_\times e^{nt} D_{\xi,0}(\eta_1, \eta_2)$

We call e^{-t} the dynamical derivative of g at ξ.

Here $B_\xi(x,y) := \liminf_{z \to \xi}[d(z,x) - d(z,y)]$ is the Busemann function. In Hilbert space, it describes the signed horospherical distance between horospheres centered at ξ through x and y respectively.

Definition

Let g be an isometry preserving some $\xi \in \partial X$.

a ξ is called indifferent fixed point if $t = 0$

b ξ is called attracting fixed point if $t > 0$

c ξ is called repelling fixed point if $t < 0$

Definition

A group of isometries G is

a **elliptic** if the orbit of some base point is bounded.

b **parabolic** if there exists a ξ that is an indifferent fixed point for every element of the group and G is not elliptic.

c **hyperbolic** if there exists some attracting or repelling fixed point for the group.

Definition

A group of isometries G is

a **elliptic** if the orbit of some base point is bounded.

b **parabolic** if there exists a ξ that is an indifferent fixed point for every element of the group and G is not elliptic.

c **hyperbolic** if there exists some attracting or repelling fixed point for the group.

Theorem

Every group is exactly one of the 3 types above.

Definition

A group of isometries G is

a **elliptic** if the orbit of some base point is bounded.

b **parabolic** if there exists a ξ that is an indifferent fixed point for every element of the group and G is not elliptic.

c **hyperbolic** if there exists some attracting or repelling fixed point for the group.

Theorem

Every group is exactly one of the 3 types above.

Remark

- One can therefore classify isometries according to their cyclic group.

Definition

A group of isometries G is

a **elliptic** if the orbit of some base point is bounded.

b **parabolic** if there exists a ξ that is an indifferent fixed point for every element of the group and G is not elliptic.

c **hyperbolic** if there exists some attracting or repelling fixed point for the group.

Theorem

Every group is exactly one of the 3 types above.

Remark

- One can therefore classify isometries according to their cyclic group.
- Note that this proves the existence of fixed points for isometries with unbounded orbits. However in Gromov hyperbolic spaces, there may be an elliptic group without a fixed point.

Almost recurrent parabolics

Example

There are examples of parabolic isometries whose orbits accumulate at their fixed point on the boundary but recur infinitely often to some bounded region in the interior.

The earliest examples we could find were discovered in a different context by Edelstein in the '60s.

Kleinian groups

Definition

Let G be a group of isometries acting on a hyperbolic space X. We say that G is strongly discrete if for every $r > 0$

$$\#\{g \in G : g(0) \in B(0, r)\} < \infty.$$

We say that a group is Kleinian if it is strongly discrete.

Kleinian groups

Definition

Let G be a group of isometries acting on a hyperbolic space X. We say that G is strongly discrete if for every $r > 0$

$$\#\{g \in G : g(0) \in B(0, r)\} < \infty.$$

We say that a group is Kleinian if it is strongly discrete.

A Kleinian group G is non-elementary if there is no finite set $F \subseteq \partial X$ or bounded set $F \subseteq X$ such that $G(F) = F$.

Kleinian groups

Definition

Let G be a group of isometries acting on a hyperbolic space X. We say that G is strongly discrete if for every $r > 0$

$$\#\{g \in G : g(0) \in B(0, r)\} < \infty.$$

We say that a group is Kleinian if it is strongly discrete.

A Kleinian group G is non-elementary if there is no finite set $F \subseteq \partial X$ or bounded set $F \subseteq X$ such that $G(F) = F$.

Observation

If X is proper, then strong discreteness is equivalent to a variety of notions of *discreteness*; however this is not true in general.

Kleinian groups

Definition

Let G be a group of isometries acting on a hyperbolic space X. We say that G is strongly discrete if for every $r > 0$

$$\#\{g \in G : g(0) \in B(0,r)\} < \infty.$$

We say that a group is Kleinian if it is strongly discrete.

A Kleinian group G is non-elementary if there is no finite set $F \subseteq \partial X$ or bounded set $F \subseteq X$ such that $G(F) = F$.

Observation

If X is proper, then strong discreteness is equivalent to a variety of notions of *discreteness*; however this is not true in general. The simplest counterexample, say in Hilbert space, is an infinite-rank parabolic group.

Outline

Gromov hyperbolic spaces

Classification of isometries

Kleinian groups

Theorems on the radial limit set

Sullivan Measurable Rigidity

Radial convergence

Observation

Fix a sequence $(x_n)_n$ in X and a point $\eta \in \partial X$. Suppose that $d(0, x_n) \underset{n}{\to} \infty$, and that either of the following equivalent asymptotics holds:

$$\langle 0|\eta\rangle_{x_n} \asymp 0$$
$$\langle x_n|\eta\rangle_0 \asymp d(0, x_n).$$

Then $x_n \underset{n}{\to} \eta$.

Radial convergence

Observation

Fix a sequence $(x_n)_n$ in X and a point $\eta \in \partial X$. Suppose that $d(0, x_n) \underset{n}{\to} \infty$, and that either of the following equivalent asymptotics holds:

$$\langle 0|\eta\rangle_{x_n} \asymp 0$$
$$\langle x_n|\eta\rangle_0 \asymp d(0, x_n).$$

Then $x_n \underset{n}{\to} \eta$.

Definition

In the situation above, we say $(x_n)_n$ converges radially to η. We say that $(x_n)_n$ converges uniformly radially to η if it converges radially and if the distances $(d(x_n, x_{n+1}))_n$ remain bounded.

Equivalent definition of radial convergence

As in the case of standard hyperbolic space, radial convergence can also be defined in terms of shadows; however we must generalize what we mean by "shadow":

Equivalent definition of radial convergence

As in the case of standard hyperbolic space, radial convergence can also be defined in terms of shadows; however we must generalize what we mean by "shadow":

Definition

For each $\sigma > 0$ and $x \in X$, let

$$\mathrm{Shad}(x, \sigma) = \{\eta \in \partial X : \langle 0|\eta\rangle_x \leq \sigma\}.$$

Equivalent definition of radial convergence

As in the case of standard hyperbolic space, radial convergence can also be defined in terms of shadows; however we must generalize what we mean by "shadow":

Definition

For each $\sigma > 0$ and $x \in X$, let

$$\mathrm{Shad}(x, \sigma) = \{\eta \in \partial X : \langle 0|\eta\rangle_x \leq \sigma\}.$$

Observation

Suppose that $d(0, x_n) \underset{n}{\to} \infty$. Then $x_n \underset{n}{\to} \eta$ radially if and only if there exists $\sigma > 0$ such that for all $n \in \mathbb{N}$,

$$\eta \in \mathrm{Shad}(x_n, \sigma).$$

Limit sets of a Kleinian group

Definition

Let G be a Kleinian group. The sets

$$L(G) := \{\eta \in \partial X : \exists g_n(0) \underset{n}{\to} \eta\}$$

$$L_{\mathrm{r}}(G) := \{\eta \in \partial X : \exists g_n(0) \underset{n}{\to} \eta \text{ radially}\}$$

$$L_{\mathrm{ur}}(G) := \{\eta \in \partial X : \exists g_n(0) \underset{n}{\to} \eta \text{ uniformly radially}\}$$

denote the limit set, radial limit set, and uniformly radial limit set, respectively.

The theorem of Bishop and Jones

For each $s > 0$, we define the Poincaré series for G with exponent s to be the series

$$\Sigma_s(G) := \sum_{g \in G} a^{-sd(0,g(0))} .$$

The theorem of Bishop and Jones

For each $s > 0$, we define the Poincaré series for G with exponent s to be the series

$$\Sigma_s(G) := \sum_{g \in G} a^{-sd(0,g(0))} .$$

We define the critical exponent of G to be the number

$$\delta(G) := \inf\{s > 0 : \Sigma_s(G) < \infty\}.$$

The theorem of Bishop and Jones

For each $s > 0$, we define the Poincaré series for G with exponent s to be the series

$$\Sigma_s(G) := \sum_{g \in G} a^{-sd(0,g(0))} \ .$$

We define the critical exponent of G to be the number

$$\delta(G) := \inf\{s > 0 : \Sigma_s(G) < \infty\}.$$

Definition

A measure μ is Ahlfors s-regular if

$$\mu(B(x,r)) \asymp_\times r^s \ .$$

Theorem

Let G be a non-elementary Kleinian group. For every $s < \delta$, there exists μ supported on $L_{ur}(G)$ such that μ is Ahlfors s-regular.

Theorem

Let G be a non-elementary Kleinian group. For every $s < \delta$, there exists μ supported on $L_{ur}(G)$ such that μ is Ahlfors s-regular.

Corollary

For any nonelementary Kleinian group G,

$$\mathrm{HD}(L_{\mathrm{r}}(G)) = \mathrm{HD}(L_{\mathrm{ur}}(G)) = \delta(G).$$

Outline

Gromov hyperbolic spaces

Classification of isometries

Kleinian groups

Theorems on the radial limit set

Sullivan Measurable Rigidity

Theorem

Let G be a non-elementary Kleinian group. For every $s < \delta$, there exists μ supported on $L_{ur}(G)$ such that μ is Ahlfors s-regular.

Corollary

For any nonelementary Kleinian group G,

$$\mathrm{HD}(L_{\mathrm{r}}(G)) = \mathrm{HD}(L_{\mathrm{ur}}(G)) = \delta(G).$$

Bishop and Jones (Acta '97) proved this theorem in the case where X is a finite-dimensional hyperbolic space.

A group for which $\delta \neq \mathrm{HD}(L_{\mathrm{ur}})$

Outline

Gromov hyperbolic spaces

Classification of isometries

Kleinian groups

Theorems on the radial limit set

Sullivan Measurable Rigidity

A group for which $\delta \neq \mathrm{HD}(L_{\mathrm{ur}})$

Example

There exists a group G with:

a $\mathrm{HD}(L_{\mathrm{r}}(G)) < \infty$

b $\delta = \infty$

c G is "parametrically discrete"

d G acts irreducibly on $\mathbb{H}^\infty$

A group for which $\delta \neq \mathrm{HD}(L_{\mathrm{ur}})$

Example

There exists a group G with:

a $\mathrm{HD}(L_{\mathrm{r}}(G)) < \infty$

b $\delta = \infty$

c G is "parametrically discrete"

d G acts irreducibly on $\mathbb{H}^\infty$

Idea.

Start with a Schottky group H generated by two elements that are both "rotations" - i.e. cycle through all the coordinates, and let $G := \{g : g(H(0)) = H(0)\}$. Then $L(G) = L(H)$ but $\#\mathrm{Stab}_0(G) = \infty$ and so G is not strongly discrete. □

Quasiconformal measures

Definition

Fix $s > 0$. A measure μ on ∂X is said to be s-quasiconformal with respect to G if for every Borel set $A \subseteq \partial X$ and for every $g \in G$, we have

$$\mu(g(A)) \asymp_\times \int_A a^{sB_\eta(0,g^{-1}(0))} \mathrm{d}\mu(\eta).$$

Here $B_\eta(0, g^{-1}(0)) := \langle g^{-1}(0)|\eta\rangle_0 - \langle 0|\eta\rangle_{g^{-1}(0)}$.

Quasiconformal measures

Definition

Fix $s > 0$. A measure μ on ∂X is said to be s-quasiconformal with respect to G if for every Borel set $A \subseteq \partial X$ and for every $g \in G$, we have

$$\mu(g(A)) \asymp_\times \int_A a^{sB_\eta(0,g^{-1}(0))} \mathrm{d}\mu(\eta).$$

Here $B_\eta(0, g^{-1}(0)) := \langle g^{-1}(0)|\eta\rangle_0 - \langle 0|\eta\rangle_{g^{-1}(0)}$.

- Interpret the expression

  $$a^{B_\eta(0,g^{-1}(0))}$$

  as being "the derivative of g at η".

Quasiconformal measures

Definition

Fix $s > 0$. A measure μ on ∂X is said to be s-quasiconformal with respect to G if for every Borel set $A \subseteq \partial X$ and for every $g \in G$, we have

$$\mu(g(A)) \asymp_\times \int_A a^{sB_\eta(0,g^{-1}(0))} \mathrm{d}\mu(\eta).$$

Here $B_\eta(0, g^{-1}(0)) := \langle g^{-1}(0)|\eta\rangle_0 - \langle 0|\eta\rangle_{g^{-1}(0)}$.

- Interpret the expression
$$a^{B_\eta(0,g^{-1}(0))}$$
as being "the derivative of g at η".
- If X is a CAT(-1) space, then this interpretation can be made explicit, i.e.
$$a^{B_\eta(0,g^{-1}(0))} = \lim_{\xi\to\eta} \frac{d(g\xi, g\eta)}{d(\xi,\eta)} .$$

Existence and uniqueness of δ-quasiconformal measures

Definition

A Kleinian group G is of divergence type if its Poincaré series diverges at its critical exponent, i.e. if

$$\Sigma_\delta(G) = \infty.$$

Existence and uniqueness of δ-quasiconformal measures

Definition

A Kleinian group G is of divergence type if its Poincaré series diverges at its critical exponent, i.e. if

$$\Sigma_\delta(G) = \infty.$$

Theorem

If G is a nonelementary Kleinian group of divergence type, then there exists a δ-quasiconformal measure μ on ∂X.

Existence and uniqueness of δ-quasiconformal measures

Definition

A Kleinian group G is of divergence type if its Poincaré series diverges at its critical exponent, i.e. if

$$\Sigma_\delta(G) = \infty.$$

Theorem

If G is a nonelementary Kleinian group of divergence type, then there exists a δ-quasiconformal measure μ on ∂X. It is unique up to equivalence: if ν is another δ-quasiconformal measure then $\mu \asymp_\times \nu$.

Existence and uniqueness of δ-quasiconformal measures

Definition

A Kleinian group G is of divergence type if its Poincaré series diverges at its critical exponent, i.e. if

$$\Sigma_\delta(G) = \infty.$$

Theorem

If G is a nonelementary Kleinian group of divergence type, then there exists a δ-quasiconformal measure μ on ∂X. It is unique up to equivalence: if ν is another δ-quasiconformal measure then $\mu \asymp_\times \nu$. Furthermore, μ is supported on the radial limit set L_r.

There is a theorem “dual” to the Bishop-Jones result that is used when proving the last theorem, which may be interesting to highlight

There is a theorem “dual” to the Bishop-Jones result that is used when proving the last theorem, which may be interesting to highlight

Theorem

Given a non-elementary Kleinian group G and a conformal measure μ,

There is a theorem "dual" to the Bishop-Jones result that is used when proving the last theorem, which may be interesting to highlight

Theorem

Given a non-elementary Kleinian group G and a conformal measure μ,

$$G \text{ is of divergence type} \iff \mu(L_{\mathrm{r}}(G)) > 0.$$

Outline

Gromov hyperbolic spaces

Classification of isometries

Kleinian groups

Theorems on the radial limit set

Sullivan Measurable Rigidity

There is a theorem "dual" to the Bishop-Jones result that is used when proving the last theorem, which may be interesting to highlight

Theorem

Given a non-elementary Kleinian group G and a conformal measure μ,

$$G \text{ is of divergence type} \iff \mu(L_{\mathrm{r}}(G)) > 0.$$

Ahlfors provides Thurston's proof of this theorem in the case that X is standard hyperbolic space and G is of the first kind, i.e. $L(G) = \partial X = S^{\dim X - 1}$ and $\delta = \dim X - 1$.

There is a theorem "dual" to the Bishop-Jones result that is used when proving the last theorem, which may be interesting to highlight

Theorem

Given a non-elementary Kleinian group G and a conformal measure μ,

$$G \text{ is of divergence type} \iff \mu(L_{\mathrm{r}}(G)) > 0.$$

Ahlfors provides Thurston's proof of this theorem in the case that X is standard hyperbolic space and G is of the first kind, i.e. $L(G) = \partial X = S^{\dim X - 1}$ and $\delta = \dim X - 1$.

Idea.

There is a theorem "dual" to the Bishop-Jones result that is used when proving the last theorem, which may be interesting to highlight

Theorem

Given a non-elementary Kleinian group G and a conformal measure μ,

$$G \text{ is of divergence type} \iff \mu(L_{\mathrm{r}}(G)) > 0.$$

Ahlfors provides Thurston's proof of this theorem in the case that X is standard hyperbolic space and G is of the first kind, i.e. $L(G) = \partial X = S^{\dim X - 1}$ and $\delta = \dim X - 1$.

Idea.

- Move to $\check{\mathrm{C}}\mathrm{echStone}(\overline{X})$, where $\overline{X} = X \cup \partial X$

There is a theorem "dual" to the Bishop-Jones result that is used when proving the last theorem, which may be interesting to highlight

Theorem

Given a non-elementary Kleinian group G and a conformal measure μ,

$$G \text{ is of divergence type} \iff \mu(L_{\mathrm{r}}(G)) > 0.$$

Ahlfors provides Thurston's proof of this theorem in the case that X is standard hyperbolic space and G is of the first kind, i.e. $L(G) = \partial X = S^{\dim X - 1}$ and $\delta = \dim X - 1$.

Idea.

- Move to $\check{\mathrm{C}}\mathrm{echStone}(\overline{X})$, where $\overline{X} = X \cup \partial X$
- Standard Patterson-Sullivan theory constructs conformal μ on $\partial\check{\mathrm{C}}\mathrm{echStone}(\overline{X})$

There is a theorem "dual" to the Bishop-Jones result that is used when proving the last theorem, which may be interesting to highlight

Theorem

Given a non-elementary Kleinian group G and a conformal measure μ,

$$G \text{ is of divergence type} \iff \mu(L_{\mathrm{r}}(G)) > 0.$$

Ahlfors provides Thurston's proof of this theorem in the case that X is standard hyperbolic space and G is of the first kind, i.e. $L(G) = \partial X = S^{\dim X - 1}$ and $\delta = \dim X - 1$.

Idea.

- Move to $\check{\mathrm{C}}\mathrm{echStone}(\overline{X})$, where $\overline{X} = X \cup \partial X$
- Standard Patterson-Sullivan theory constructs conformal μ on $\partial\check{\mathrm{C}}\mathrm{echStone}(\overline{X})$
- By Ahlfors-Thurston we get that μ is supported on L_{r}

There is a theorem "dual" to the Bishop-Jones result that is used when proving the last theorem, which may be interesting to highlight

Theorem

Given a non-elementary Kleinian group G and a conformal measure μ,

$$G \text{ is of divergence type} \iff \mu(L_{\mathrm{r}}(G)) > 0.$$

Ahlfors provides Thurston's proof of this theorem in the case that X is standard hyperbolic space and G is of the first kind, i.e. $L(G) = \partial X = S^{\dim X - 1}$ and $\delta = \dim X - 1$.

Idea.

- Move to $\check{\mathrm{C}}\mathrm{echStone}(\overline{X})$, where $\overline{X} = X \cup \partial X$
- Standard Patterson-Sullivan theory constructs conformal μ on $\partial\check{\mathrm{C}}\mathrm{echStone}(\overline{X})$
- By Ahlfors-Thurston we get that μ is supported on L_{r}
- Show that $L_{\mathrm{r}} \subset \partial X$

A group without a conformal measure!

Example

There exists an infinitely generated Schottky group of convergence type with no conformal measure.

A group without a conformal measure!

Example

There exists an infinitely generated Schottky group of convergence type with no conformal measure.

Idea.

G will be a Schottky group. There exists a $B(0, R)$ such that any two geodesics between any two of the generating balls intersects $B(0, R)$. This gives us that $L_{\mathrm{r}}(G) = L(G)$. Heuristically, the diameters of the generating balls must converge to zero at a specific rate that forces the group to be of convergence type. Then Ahlfors-Thurston implies that there is no conformal measure. □

Sullivan Measurable Rigidity

Theorem

Start with a Borel conjugacy T between two actions of non-elementary strongly discrete groups Γ_1 and Γ_2.

Sullivan Measurable Rigidity

Theorem

Start with a Borel conjugacy T between two actions of non-elementary strongly discrete groups Γ_1 and Γ_2. Suppose T is nonsingular with respect to δ-quasiconformal measures μ_1 and μ_2, viz. $\mu_1(A) > 0 \Leftrightarrow \mu_2(TA) > 0$.

Sullivan Measurable Rigidity

Theorem

Start with a Borel conjugacy T between two actions of non-elementary strongly discrete groups Γ_1 and Γ_2.
Suppose T is nonsingular with respect to δ-quasiconformal measures μ_1 and μ_2, viz. $\mu_1(A) > 0 \Leftrightarrow \mu_2(TA) > 0$.
If the Poincaré series for Γ_1 diverges at δ, then T agrees (μ_1-a.e.) with a conformal conjugacy between Γ_1 and Γ_2.

Sullivan Measurable Rigidity

Theorem

Start with a Borel conjugacy T between two actions of non-elementary strongly discrete groups Γ_1 and Γ_2. Suppose T is nonsingular with respect to δ-quasiconformal measures μ_1 and μ_2, viz. $\mu_1(A) > 0 \Leftrightarrow \mu_2(TA) > 0$. If the Poincaré series for Γ_1 diverges at δ, then T agrees (μ_1-a.e.) with a conformal conjugacy between Γ_1 and Γ_2.

Note that if we assume T to be Lipschitz, then it suffices to assume that Γ_1 is of divergence type and that μ_1 is δ-quasiconformal. In such a case μ_2 will turn out to be δ-quasiconformal as well.

C.Bishop & P. Jones, Hausdorff dimension and Kleinian groups, Acta Math. 179 (1997), 1-39.

T. Das, Kleinain groups in Hilbert spaces, Ph. D. Thesis, expected May 2012.

T. Das, D. Simmons, B. O. Stratmann & M. Urbański, Kleinian groups in hyperbolic spaces I, in preparation.

M. Gromov, *Asymptotic Invariants of Infinite Groups*, Geometric group theory, Vol. 2 (Sussex, 1991), Cambridge Univ. Press, Cambridge, 1993.

R. D. Mauldin & M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge Univ. Press (2003).

R. D. Mauldin, T. Szarek & M. Urbański, Graph Directed Markov Systems on Hilbert Spaces, Math. Proc. Cambridge Phil. Soc. 147 (2009), 455-488.

B. O. Stratmann & M. Urbański, Pseudo-Markov Systems and Infinitely Generated Schottky Groups, Amer. J. Math. 129 (2007), 1019-1062.

D. Sullivan, The density at infinity of a discrete group, IHÉS Publ. Math. No. 50 (1979), 171-202.

Many thanks to our organizers

Michel, Hung, John and Machiel

for inviting us to this fabulous island :-)

. . . and for a wonderfully thought-out Special Session.

Encore!

Groups of compact type

Definition

A properly discontinuous group G is said to be of compact type when $L(G)$ is compact.

Groups of compact type

Definition

A properly discontinuous group G is said to be of compact type when $L(G)$ is compact.

Theorem

For a properly discontinuous group G, the following are equivalent:

1. *G is of compact type.*
2. *Every infinite subset of $G(0)$ contains an accumulation point.*
3. *Each sequence $(g_n(0))_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|g_n(0)\| = 1$ has a converging subsequence, which necessarily accumulates at an element in $L(G)$.*
4. *Every infinite subset of $G(0)$ contains a sequence $(z_n)_n$ such that $\langle z_n, z_m\rangle_0 \to \infty$ as $n, m \to \infty$.*

Groups of compact type

Definition

A properly discontinuous group G is said to be of compact type when $L(G)$ is compact.

Theorem

For a properly discontinuous group G, the following are equivalent:

1. *G is of compact type.*
2. *Every infinite subset of $G(0)$ contains an accumulation point.*
3. *Each sequence $(g_n(0))_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|g_n(0)\| = 1$ has a converging subsequence, which necessarily accumulates at an element in $L(G)$.*
4. *Every infinite subset of $G(0)$ contains a sequence $(z_n)_n$ such that $\langle z_n, z_m\rangle_0 \to \infty$ as $n, m \to \infty$.*

In CAT(-1) spaces, any group of compact type acting properly discontinuously is strongly discrete.

Convex-cobounded groups

Say we're in a geodesic Gromov-hyperbolic space. For $w, z \in \overline{X}$, let $\gamma_{w,z}$ be the unique geodesic joining w and z.

$$C_\Delta(G) := \bigcup_{\xi_1, \xi_2 \in L(G)} \gamma^\circ_{\xi_1,\xi_2} .$$

Convex-cobounded groups

Say we're in a geodesic Gromov-hyperbolic space. For $w, z \in \overline{X}$, let $\gamma_{w,z}$ be the unique geodesic joining w and z.

$$C_\Delta(G) := \bigcup_{\xi_1,\xi_2 \in L(G)} \gamma^\circ_{\xi_1,\xi_2} .$$

Then notice that $C_\Delta(G)$ is G-invariant, i.e. for any $g \in G$,

$$g(C_\Delta(G)) = C_\Delta(G).$$

Convex-cobounded groups

Say we're in a geodesic Gromov-hyperbolic space. For $w, z \in \overline{X}$, let $\gamma_{w,z}$ be the unique geodesic joining w and z.

$$C_\Delta(G) := \bigcup_{\xi_1, \xi_2 \in L(G)} \gamma^\circ_{\xi_1, \xi_2} .$$

Then notice that $C_\Delta(G)$ is G-invariant, i.e. for any $g \in G$,

$$g(C_\Delta(G)) = C_\Delta(G).$$

Definition

A properly discontinuous group G is convex cobounded if there exists a ball about the origin $B(0, r)$ such that $q[B(0, r) \cap C_\Delta(G)] = q[C_\Delta(G)]$.

Convex-cobounded groups

Say we're in a geodesic Gromov-hyperbolic space. For $w, z \in \overline{X}$, let $\gamma_{w,z}$ be the unique geodesic joining w and z.

$$C_\Delta(G) := \bigcup_{\xi_1,\xi_2 \in L(G)} \gamma^\circ_{\xi_1,\xi_2} .$$

Then notice that $C_\Delta(G)$ is G-invariant, i.e. for any $g \in G$,

$$g(C_\Delta(G)) = C_\Delta(G).$$

Definition

A properly discontinuous group G is convex cobounded if there exists a ball about the origin $B(0,r)$ such that $q[B(0,r) \cap C_\Delta(G)] = q[C_\Delta(G)]$.

Theorem

Let G be properly discontinuous and of compact type. TFAE:

1. $L_r(G) = L(G)$.
2. $L_{ur}(G) = L(G)$.
3. *The group is convex-cobounded.*

A "remarkable" description of $L(G)$.

Definition

A group G is called elementary whenever $\#L(G) \in \{0, 1, 2\}$.

A "remarkable" description of $L(G)$.

Definition

A group G is called elementary whenever $\#L(G) \in \{0, 1, 2\}$.

Theorem (It's elementary dear ...)

For every G the following are equivalent:

1. $\#[L(G)] < \infty$.
2. *Either*
    - $G = < e >$.
    - $\exists! \xi \in \partial\mathbb{B}_\infty$ *parabolic with* $G(\xi) = \xi$ *and* G *consists entirely of parabolics.*
    - $G = < g >$, *with* g *hyperbolic.*
3. $\nexists g, h \in G$ *hyperbolic with* $Fix(g) \cap Fix(h) = \varnothing$.

A “remarkable” description of $L(G)$.

Definition

A group G is called elementary whenever $\#L(G) \in \{0, 1, 2\}$.

Theorem (It's elementary dear ...)

For every G the following are equivalent:

1. $\#[L(G)] < \infty$.
2. *Either*
   - $G = < e >$.
   - $\exists ! \xi \in \partial \mathbb{B}_\infty$ *parabolic with* $G(\xi) = \xi$ *and* G *consists entirely of parabolics.*
   - $G = < g >$, *with* g *hyperbolic.*
3. $\nexists g, h \in G$ *hyperbolic with* $Fix(g) \cap Fix(h) = \varnothing$.

Theorem (Minimality)

For every non-elementary group G, $L(G)$ is the smallest closed G-invariant subset of $\partial \mathbb{B}_\infty$ that contains at least 2 points.

Hilbert version

To fix ideas, let's start with a real separable Hilbert space, $\mathcal{H} \equiv \ell_2$ with the standard orthonormal basis denoted by $(e_n)_n$.

Hilbert version

To fix ideas, let's start with a real separable Hilbert space, $\mathcal{H} \equiv \ell_2$ with the standard orthonormal basis denoted by $(e_n)_n$. Of the many models of hyperbolic space, let's focus on the Poincaré ball $\mathbb{B}_\infty$ and the Upper-half space $\mathbb{H}_\infty$.

Hilbert version

To fix ideas, let's start with a real separable Hilbert space, $\mathcal{H} \equiv \ell_2$ with the standard orthonormal basis denoted by $(e_n)_n$. Of the many models of hyperbolic space, let's focus on the Poincaré ball $\mathbb{B}_\infty$ and the Upper-half space $\mathbb{H}_\infty$.

$$\mathbb{B}_\infty := \{x \in \mathcal{H} : \|x\| < 1\}$$
$$\partial\mathbb{B}_\infty = \{x \in \mathcal{H} : \|x\| = 1\}$$
$$\mathbb{H}_\infty = \{x \in \mathcal{H} : x_0 > 0\}$$
$$\partial\mathbb{H}_\infty = \{x \in \mathcal{H} : x_0 = 0\} \cup \{\infty\}$$

Hilbert version

To fix ideas, let's start with a real separable Hilbert space, $\mathcal{H} \equiv \ell_2$ with the standard orthonormal basis denoted by $(e_n)_n$. Of the many models of hyperbolic space, let's focus on the Poincaré ball $\mathbb{B}_\infty$ and the Upper-half space $\mathbb{H}_\infty$.

$$\mathbb{B}_\infty := \{x \in \mathcal{H} : \|x\| < 1\}$$
$$\partial\mathbb{B}_\infty = \{x \in \mathcal{H} : \|x\| = 1\}$$
$$\mathbb{H}_\infty = \{x \in \mathcal{H} : x_0 > 0\}$$
$$\partial\mathbb{H}_\infty = \{x \in \mathcal{H} : x_0 = 0\} \cup \{\infty\}$$

- Consider $\mathbb{B}_\infty, \mathbb{H}_\infty \subseteq \widehat{\mathcal{H}} := \mathcal{H} \cup \{\infty\}$. The topology on $\widehat{\mathcal{H}}$ is defined as follows: $U \subseteq \widehat{\mathcal{H}}$ open if and only if $U \cap \mathcal{H}$ is open and $\mathcal{H} \setminus U$ is bounded if $\infty \in U$.

Hilbert version

To fix ideas, let's start with a real separable Hilbert space, $\mathcal{H} \equiv \ell_2$ with the standard orthonormal basis denoted by $(e_n)_n$. Of the many models of hyperbolic space, let's focus on the Poincaré ball $\mathbb{B}_\infty$ and the Upper-half space $\mathbb{H}_\infty$.

$$\mathbb{B}_\infty := \{x \in \mathcal{H} : \|x\| < 1\}$$
$$\partial\mathbb{B}_\infty = \{x \in \mathcal{H} : \|x\| = 1\}$$
$$\mathbb{H}_\infty = \{x \in \mathcal{H} : x_0 > 0\}$$
$$\partial\mathbb{H}_\infty = \{x \in \mathcal{H} : x_0 = 0\} \cup \{\infty\}$$

- Consider $\mathbb{B}_\infty, \mathbb{H}_\infty \subseteq \widehat{\mathcal{H}} := \mathcal{H} \cup \{\infty\}$. The topology on $\widehat{\mathcal{H}}$ is defined as follows: $U \subseteq \widehat{\mathcal{H}}$ open if and only if $U \cap \mathcal{H}$ is open and $\mathcal{H} \setminus U$ is bounded if $\infty \in U$.
- $\partial\mathbb{B}_\infty$, $\overline{\mathbb{B}}_\infty$, $\partial\mathbb{H}_\infty$ and $\overline{\mathbb{H}}_\infty$ refer to the boundary and closure with respect to the topology on $\widehat{\mathcal{H}}$ and so $\overline{\mathbb{B}}_\infty = \{x \in \mathcal{H} : \|x\| \leq 1\}$ and $\overline{\mathbb{H}}_\infty = \{x \in \mathcal{H} : x_0 \geq 0\} \cup \{\infty\}$.

Just as in finite dimensions, we have the following formulae for the associated length elements:

$$ds_{\mathbb{B}}^2 = \frac{4\|dx\|^2}{(1-\|x\|^2)^2}$$

and

$$ds_{\mathbb{H}}^2 = \frac{\|dx\|^2}{x_0^2}.$$

Just as in finite dimensions, we have the following formulae for the associated length elements:

$$ds^2_{\mathbb{B}} = \frac{4\|dx\|^2}{(1 - \|x\|^2)^2}$$

and

$$ds^2_{\mathbb{H}} = \frac{\|dx\|^2}{x_0^2} .$$

Here $\|\cdot\|$ refers to the usual norm in Hilbert space, i.e.

$$\|dx\|^2 = \sum_i dx_i^2 .$$

Classification of isometries, à lá Klein

Theorem (Classification of isometries of hyperbolic space)

Any isometry of hyperbolic space is conjugate to exactly one of the following:

Classification of isometries, à lá Klein

Theorem (Classification of isometries of hyperbolic space)

Any isometry of hyperbolic space is conjugate to exactly one of the following:

(1) *Elliptic case: A bijective linear isometry on* $\mathbb{B}_\infty$*, i.e.* $T \restriction \mathbb{B}_\infty$ *for some* $T \in \mathscr{O}$.

Classification of isometries, à lá Klein

Theorem (Classification of isometries of hyperbolic space)

Any isometry of hyperbolic space is conjugate to exactly one of the following:

(1) *Elliptic case: A bijective linear isometry on $\mathbb{B}_\infty$, i.e. $T \upharpoonright \mathbb{B}_\infty$ for some $T \in \mathscr{O}$.*

(2) *Parabolic case: A bijective affine euclidean isometry on $\mathbb{H}_\infty$ with no fixed points in the interior.*

Classification of isometries, à lá Klein

Theorem (Classification of isometries of hyperbolic space)

Any isometry of hyperbolic space is conjugate to exactly one of the following:

(1) *Elliptic case: A bijective linear isometry on $\mathbb{B}_\infty$, i.e. $T \upharpoonright \mathbb{B}_\infty$ for some $T \in \mathscr{O}$.*

(2) *Parabolic case: A bijective affine euclidean isometry on $\mathbb{H}_\infty$ with no fixed points in the interior.*

(3) *Hyperbolic case: A map of the form $g = \lambda M : \mathbb{H}_\infty \to \mathbb{H}_\infty$, where $0 < \lambda < 1$ and M is a bijective linear isometry of $\mathbb{H}_\infty$.*

Can one make sense of *orientation preserving* transformations in infinite dimensions?

Can one make sense of *orientation preserving* transformations in infinite dimensions?

This does *NOT* make sense in infinite dimensions.

Can one make sense of *orientation preserving* transformations in infinite dimensions?

This does *NOT* make sense in infinite dimensions.
If one wanted to define orientation-preserving via the kernel of a continuous homomorphism $\mathcal{O} : \mathscr{O}(\mathcal{H}) \to \mathbb{Z}_2$ one would easily fall into a trap ... For example, any reflection in a hyperplane on $\ell_2(\mathbb{Z})$ would be orientation-preserving.

Can one make sense of *orientation preserving* transformations in infinite dimensions?

This does *NOT* make sense in infinite dimensions.
If one wanted to define orientation-preserving via the kernel of a continuous homomorphism $\mathcal{O} : \mathscr{O}(\mathcal{H}) \to \mathbb{Z}_2$ one would easily fall into a trap . . . For example, any reflection in a hyperplane on $\ell_2(\mathbb{Z})$ would be orientation-preserving.

Example

For example, for $v, w \in \mathcal{H}$, let $H_{v,w} := \{x + w | x \in v^\perp\}$ be the hyperplane determined by v and w and let $r_{v,w}$ be reflection in this hyperplane given by $z \mapsto \big(\mathrm{id} - 2P_v\big)(z - w) + w$, where P_v is the projection onto the hyperplane $v^\perp$. Then $\mathcal{O}(r_{v,w})$ can be shown to equal 1, i.e. be orientation-preserving.