Dynamics and geometry in infinite-dimensional hyperbolic spaces

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Outline

Discrete actions of hyperbolic isometries

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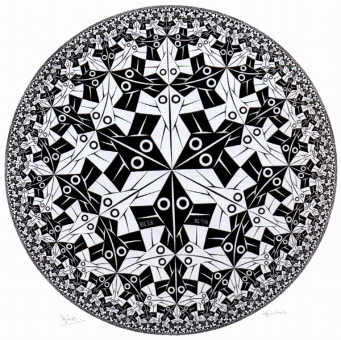
Greatly popularized in scientific and artistic circles via the seminal work of Mandelbrot and Escher.



Circle Limit I, Escher (1958)

...once more, with (an infinite-dimensional) feeling

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Dennis Sullivan in the early '80s indicated a possibility of developing the theory of discrete groups acting by hyperbolic isometries on the open unit ball of a separable Hilbert space.

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Misha Gromov lamented the paucity of results regarding such actions and encouraged their investigation:

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> *The spaces like this . . . look as cute and sexy to me as their finite-dimensional siblings but they have been for years shamefully neglected by geometers and algebraists alike.*

Gromov hyperbolicity

Definition

Let (X, d) be a metric space. For three points $x, y, z \in X$, we define the Gromov product of x and y with respect to z by

$$\langle x|y\rangle_z := \frac{1}{2}[d(x,z) + d(y,z) - d(x,y)].$$

[1] Denotes an additive asymptotic.

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Definition

(X, d) is hyperbolic (or Gromov hyperbolic) if for every four points $x, y, z, w \in X$ we have

$$\langle x|z\rangle_w \gtrsim^1 \min(\langle x|y\rangle_w, \langle y|z\rangle_w).$$

We will refer to this inequality as Gromov's inequality.

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Definition

A Gromov hyperbolic space X is **proper** if $d(0, \cdot) : X \to \mathbb{R}$ is proper. In other words, if for all $r > 0$ the set $\overline{B}(0, r)$ is compact.
Here 0 is a distinguished point that we fix in X.

The boundary of a hyperbolic space

Any Gromov hyperbolic space X has a Gromov boundary ∂X, analogous to the sphere at infinity of standard hyperbolic space. It is defined in a similar way to the completion of a metric space, with the quantity

$$e^{-\langle x|y\rangle_0}$$

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The most important fact about the Gromov boundary is the following heuristic: The Gromov product may be extended to the boundary while preserving key formulas!

A metric on ∂X

Proposition

For each $a > 1$ sufficiently small, there exists a complete metric D_a on ∂X satisfying the following asymptotic:

$$D_a(\xi, \eta) \asymp_\times a^{-\langle \xi | \eta \rangle_0}. \tag{2.1}$$

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Remark

For CAT(-1) spaces, and in particular for the standard model of hyperbolic geometry, the above proposition holds for any $1 < a \leq e$. In particular, $a = e$ gives the spherical metric.

Convergence of sequences

Definition

Fix a sequence $(x_n)_n$ in X and a point $\eta \in \partial X$. We say that $(x_n)_n$ converges to η if

$$\langle x_n | \eta \rangle_0 \underset{n}{\to} \infty.$$

[Idea: $a^{-\langle x_n|\eta\rangle_0} \underset{n}{\to} 0$]

In this case, we write $x_n \underset{n}{\to} \eta$.

A metric on $\partial X \setminus \{\xi\}$

Lemma

Let $E_\xi := \partial X \setminus \{\xi\}$ for some fixed $\xi \in \partial X$. If $x_n \to \xi$, then

$$e^{d(0,x_n)} D_{x_n}(\eta_1, \eta_2) \underset{n,\times}{\longrightarrow} e^{-[\langle \eta_1 | \eta_2 \rangle_0 - \sum_{i=1}^{2} \langle \eta_i | \xi \rangle_0]}.$$

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Here $a_n \underset{n,\times}{\longrightarrow} b$ means $\frac{1}{K} \leq \frac{\liminf a_n}{b} \leq \frac{\limsup a_n}{b} \leq K$.

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Corollary

There exists a metric

$$D_{\xi,0}(\eta_1, \eta_2) \asymp_\times e^{-[\langle \eta_1 | \eta_2 \rangle_0 - \sum_{i=1}^{2} \langle \eta_i | \xi \rangle_0]}.$$

Note that in CAT(-1) you have a limit and equality respectively.

Classification of isometries

Theorem

Let g be an isometry preserving some $\xi \in \partial X$. Then $\exists t \in \mathbb{R}$ such that

A $B_\xi(x, g^n x) \asymp_+ nt$

B $D_{\xi,0}(g^n(\eta_1), g^n(\eta_2)) \asymp_\times e^{nt} D_{\xi,0}(\eta_1, \eta_2)$

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Here $B_\xi(x, y) := \liminf_{z \to \xi}[d(z, x) - d(z, y)]$ is the Busemann function. In Hilbert space, it describes the signed horospherical distance between horospheres centered at ξ through x and y respectively.

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Let g be an isometry preserving some $\xi \in \partial X$.

a ξ is called indifferent fixed point if $t = 0$

b ξ is called attracting fixed point if $t > 0$

c ξ is called repelling fixed point if $t < 0$

Definition

A group of isometries G is

a **elliptic** if the orbit of some base point is bounded.

b **parabolic** if there exists a ξ that is an indifferent fixed point for every element of the group and G is not elliptic.

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Remark

- One can therefore classify isometries according to their cyclic group.
- Note that this proves the existence of fixed points for isometries with unbounded orbits.

Almost recurrent parabolics

Example

There are examples of parabolic isometries whose orbits accumulate at their fixed point on the boundary but recur infinitely often to some bounded region in the interior.

The earliest examples we could find were discovered in a different context by Edelstein in the '60s.

Kleinian groups

Definition

Let G be a group of isometries acting on a hyperbolic space X. We say that G is strongly discrete if for every $r > 0$

$$\#\{g \in G : g(0) \in B(0, r)\} < \infty.$$

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Observation

If X is proper, then strong discreteness is equivalent to a variety of notions of *discreteness*; however this is not true in general. The simplest counterexample, say in Hilbert space, is an infinite-rank parabolic group.

Radial convergence

Observation

Fix a sequence $(x_n)_n$ in X and a point $\eta \in \partial X$. Suppose that $d(0, x_n) \underset{n}{\to} \infty$, and that either of the following equivalent asymptotics holds:

$$\langle 0|\eta\rangle_{x_n} \asymp 0$$
$$\langle x_n|\eta\rangle_0 \asymp d(0, x_n).$$

Then $x_n \underset{n}{\to} \eta$.

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Definition

In the situation above, we say $(x_n)_n$ converges radially to η. We say that $(x_n)_n$ converges uniformly radially to η if it converges radially and if the distances $(d(x_n, x_{n+1}))_n$ remain bounded.

Equivalent definition of radial convergence

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Observation

Suppose that $d(0, x_n) \underset{n}{\to} \infty$. Then $x_n \underset{n}{\to} \eta$ radially if and only if there exists $\sigma > 0$ such that for all $n \in \mathbb{N}$,

$$\eta \in \mathrm{Shad}(x_n, \sigma).$$

Limit sets of a Kleinian group

Definition

Let G be a Kleinian group. The sets

$$L(G) := \{\eta \in \partial X : \exists g_n(0) \underset{n}{\to} \eta\}$$
$$L_{\mathrm{r}}(G) := \{\eta \in \partial X : \exists g_n(0) \underset{n}{\to} \eta \text{ radially}\}$$
$$L_{\mathrm{ur}}(G) := \{\eta \in \partial X : \exists g_n(0) \underset{n}{\to} \eta \text{ uniformly radially}\}$$

denote the limit set, radial limit set, and uniformly radial limit set, respectively.

The theorem of Bishop and Jones

For each $s > 0$, we define the Poincaré series for G with exponent s to be the series

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Definition

A measure μ is Ahlfors s-regular if

$$\mu(B(x,r)) \asymp_\times r^s .$$

Theorem

Let G be a non-elementary Kleinian group. For every $s < \delta$, there exists μ supported on $L_{ur}(G)$ such that μ is Ahlfors s-regular.

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Bishop and Jones (Acta '97) proved this theorem in the case where X is a finite-dimensional hyperbolic space.

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Example

There exists a group G with:

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Idea.
Start with a Schottky group H generated by two elements that are both "rotations" - i.e. cycle through all the coordinates, and let $G := \{g : g(H(0)) = H(0)\}$. Then $L(G) = L(H)$ but $\#\mathrm{Stab}_0(G) = \infty$ and so G is not strongly discrete. □

Quasiconformal measures

Definition

Fix $s > 0$. A measure μ on ∂X is said to be *s-quasiconformal* with respect to G if for every Borel set $A \subseteq \partial X$ and for every $g \in G$, we have

$$\mu(g(A)) \asymp_\times \int_A a^{sB_\eta(0,g^{-1}(0))} \mathrm{d}\mu(\eta).$$

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- Interpret the expression
$$a^{B_\eta(0,g^{-1}(0))}$$
as being "the derivative of g at η".
- If X is a CAT(-1) space, then this interpretation can be made explicit, i.e.
$$a^{B_\eta(0,g^{-1}(0))} = \lim_{\xi\to\eta} \frac{d(g\xi, g\eta)}{d(\xi,\eta)} .$$

Existence and uniqueness of δ-quasiconformal measures

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A Kleinian group G is of divergence type if its Poincaré series diverges at its critical exponent, i.e. if

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If X is a CAT(-1) space, then μ can be made conformal, and is unique up to a constant.

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Ahlfors provides Thurston's proof of this theorem in the case that X is standard hyperbolic space and G is of the first kind, i.e. $L(G) = \partial X = S^{\dim X - 1}$ and $\delta = \dim X - 1$.

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Idea for existence of Patterson-Sullivan measure.

- Move to $\check{\mathrm{C}}\mathrm{echStone}(\overline{X})$, where $\overline{X} = X \cup \partial X$

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Ahlfors provides Thurston's proof of this theorem in the case that X is standard hyperbolic space and G is of the first kind, i.e. $L(G) = \partial X = S^{\dim X - 1}$ and $\delta = \dim X - 1$.

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- Show that $L_{\mathrm{r}} \subset \partial X$

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Idea.

G is constructed so that there exists a $B(0, R)$ such that any two geodesics between any two of the generating balls intersects $B(0, R)$. This gives us that $L_{\mathrm{r}}(G) = L(G)$. Heuristically, the diameters of the generating balls must converge to zero at a specific rate that forces the group to be of convergence type. Then Ahlfors–Thurston implies that there is no conformal measure. □

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Note that if we assume T to be Lipschitz, then it suffices to assume that Γ_1 is of divergence type and that μ_1 is δ-quasiconformal. In such a case μ_2 will turn out to be δ-quasiconformal as well.

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Note that if we assume T to be Lipschitz, then it suffices to assume that Γ_1 is of divergence type and that μ_1 is δ-quasiconformal. In such a case μ_2 will turn out to be δ-quasiconformal as well. We are currently investigating various ramifications for groups of convergence type and Hopf's dichotomy in our setting.

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This theorem may be proved without resorting to the thermodynamic formalism à la Bowen-Ruelle-Sinai. However, access to similar geometric results as well as strong stochastic properties for systems associated to large classes of infinitely-generated Schottky groups are via extensions of the thermodynamic formalism to such settings and *fine inducing* on (Lai-San) Young towers.

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for this wonderfully well thought out workshop.

Encore!

Groups of compact type

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A properly discontinuous group G is said to be of compact type when $L(G)$ is compact.

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Theorem

For a properly discontinuous group G, the following are equivalent:

1. *G is of compact type.*
2. *Every infinite subset of $G(0)$ contains an accumulation point.*
3. *Each sequence $(g_n(0))_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|g_n(0)\| = 1$ has a converging subsequence, which necessarily accumulates at an element in $L(G)$.*
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In CAT(-1) spaces, any group of compact type acting properly discontinuously is strongly discrete.

Convex-cobounded groups

Say we're in a geodesic Gromov-hyperbolic space. For $w, z \in \overline{X}$, let $\gamma_{w,z}$ be the unique geodesic joining w and z.

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Theorem

Let G be properly discontinuous and of compact type. TFAE:

1. $L_r(G) = L(G)$.
2. $L_{ur}(G) = L(G)$.
3. *The group is convex-cobounded.*

A "remarkable" description of $L(G)$.

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Theorem (It's elementary dear . . .)

For every G the following are equivalent:

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Theorem (Minimality)

For every non-elementary group G, $L(G)$ is the smallest closed G-invariant subset of $\partial\mathbb{B}_\infty$ that contains at least 2 points.

Hilbert version

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- Just as in finite dimensions, we have the following formulae for the associated length elements:

$$ds_{\mathbb{B}}^2 := \frac{4\|dx\|^2}{(1 - \|x\|^2)^2} \quad \text{and} \quad ds_{\mathbb{H}}^2 := \frac{\|dx\|^2}{x_0^2}.$$

Conformal maps and Liouville's Theorem

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Definition

Let $f : M \to N$ be a diffeomorphism. Then f is called conformal when there exists a differentiable positive function $\alpha : M \to \mathbb{R}$ such that for all $x \in M$ and for all $v, w \in T_x M$

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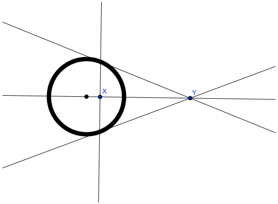
Let $x \in \mathcal{H}$ and $\alpha > 0$. Then the inversion with respect to the sphere $S(x, \alpha)$ is the map

$$i_{x,\alpha} : z \mapsto \alpha^2 \frac{z - x}{\|z - x\|^2} + x$$



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Joseph Liouville
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Theorem (Liouville, 1850)

Let $U \in \mathcal{H}$ be a non-empty domain, $\phi : U \to \mathcal{H}$ be a conformal map. Then either there exists a unique quadruple (λ, x, y, M) with $\lambda > 0$; $x, y \in \mathcal{H}$ and $M \in \mathscr{O}(\mathcal{H})$ such that

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Theorem: The conformal map $i_{-e_0,\sqrt{2}} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ is a homeomorphism between $\overline{\mathbb{H}}_\infty$ and $\overline{\mathbb{B}}_\infty$ and an isometry between $\mathbb{H}_\infty$ and $\mathbb{B}_\infty$.

Theorem (Hyperbolic isometry inside, conformal upto the boundary)

Let $\mathbb{K}_\infty$ be either $\mathbb{B}_\infty$ or $\mathbb{H}_\infty$. Then TFAE:

(1) *g is the restriction of a conformal map $\hat{g} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ which preserves $\mathbb{K}_\infty$.*

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Isometries of hyperbolic space

Let us define the Möbius group as follows:

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Classification of isometries, à la Klein

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(3) *Hyperbolic case: A map of the form* $g = \lambda M : \mathbb{H}_\infty \to \mathbb{H}_\infty$*, where* $0 < \lambda < 1$ *and* M *is a bijective linear isometry on* $\mathbb{H}_\infty$.

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Example

For example, for $v, w \in \mathcal{H}$, let $H_{v,w} := \{x + w | x \in v^\perp\}$ be the hyperplane determined by v and w and let $r_{v,w}$ be reflection in this hyperplane given by $z \mapsto (\mathrm{id} - 2P_v)(z - w) + w$, where P_v is the projection onto the hyperplane $v^\perp$. Then $\mathcal{O}(r_{v,w})$ can be shown to equal 1, i.e. be orientation-preserving.