Intersections and joins of subgroups in free groups

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Let F be a free group, H, K subgroups in F.

Define the **join** $H \vee K = \langle H, K \rangle$, and the **reduced rank:**

$$\operatorname{rr} H = \max(\operatorname{rank} H - 1, 0).$$

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Expectation: the bigger $\operatorname{rr}(H \cap K)$ is, the smaller $\operatorname{rr}(H \vee K)$.

Results and Conjectures:

Imrich–Müller '94: If H or K has finite index in $H \vee K$, then

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(False if H or K has infinite index.)

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- $m \geq 6$: **false (S. '20)**
- $m = 4$: **true (S. '20)**

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Geometric Conjecture

Let M be a closed, orientable, hyperbolic 3-manifold. If $\pi_1(M)$ is k-free for $k \geq 3$ then there exists a point P in M such that the set of all elements of $\pi_1(M, P)$ represented by loops of length less than $\log(2k - 1)$ is contained in a (free) subgroup of $\pi_1(M)$ of rank $\leq k - 3$.

(A group is called *k-free* if all of its k-generator subgroups are free.)

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Corollary (S. '20)

The Geometric Conjecture is true for $k = 6$.

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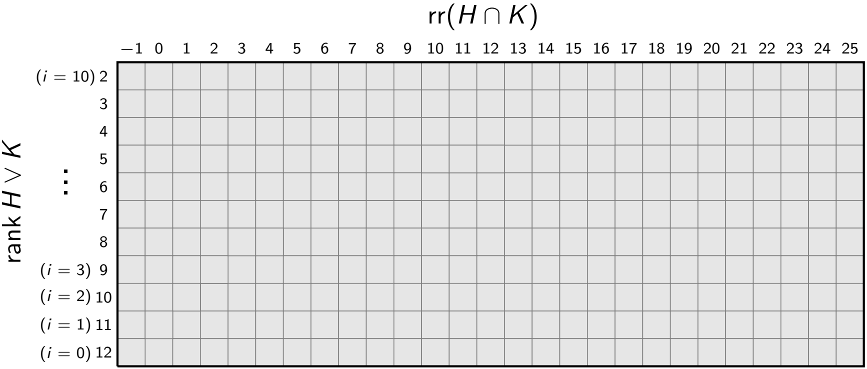
Note: Guzman and Shalen recently proved GC in full generality without dependence on GTC.


rr(H ∩ K)
−1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
rank H ∨ K
(i = 10) 2
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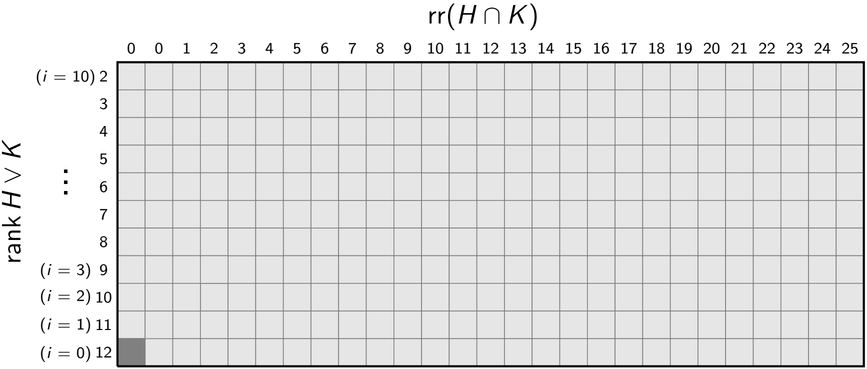
(i = 0) 12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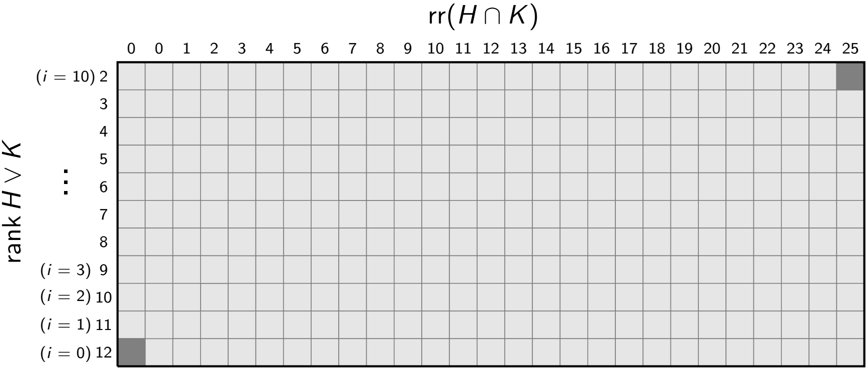
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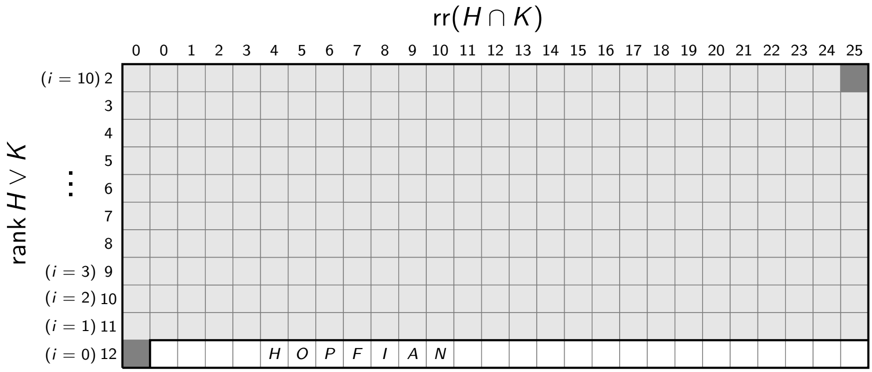

rr($H \cap K$)
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rank $H \vee K$
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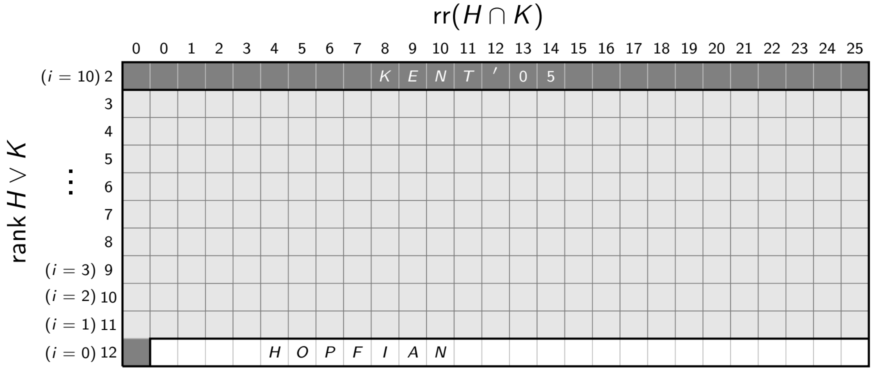
H O P F I A N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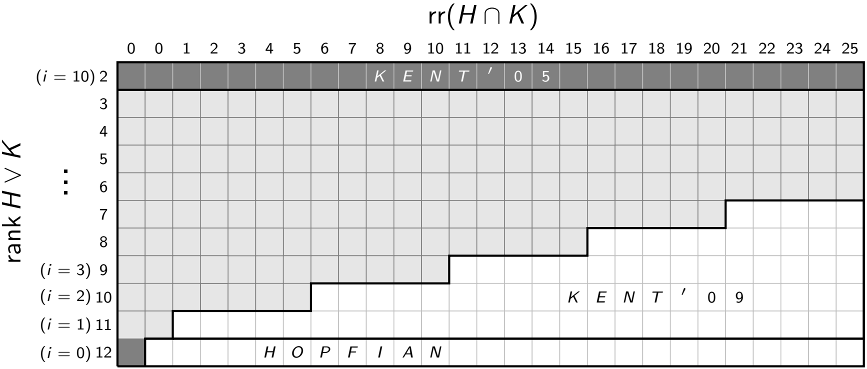
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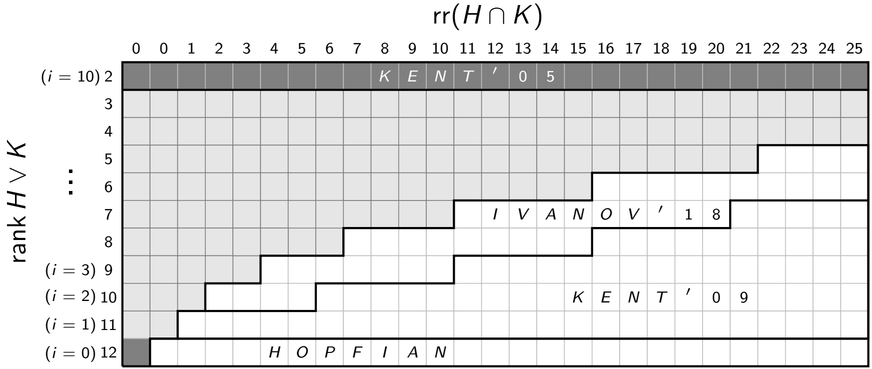
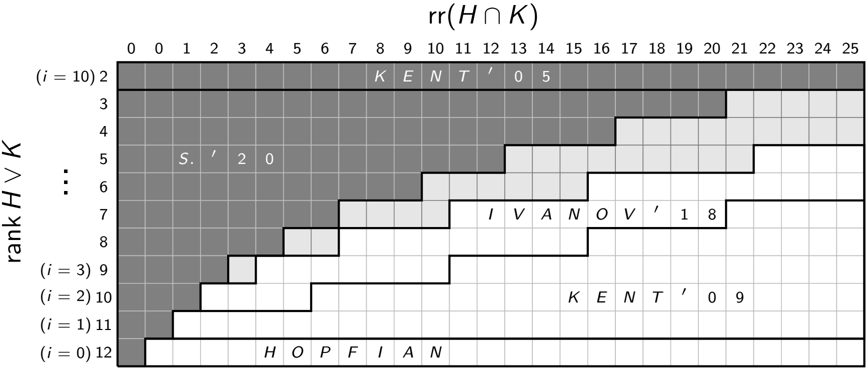
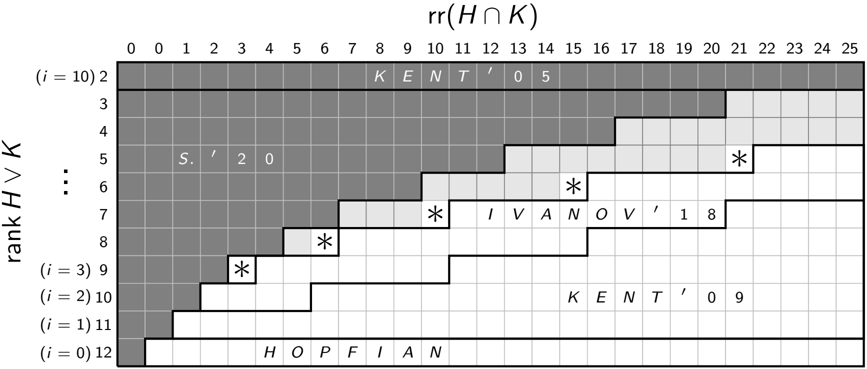


Ivanov: $\mathrm{rr}(H \cap K) \leq \frac{i(i-1)}{2}$.



Legend: dark grey: realizable; white: non-realizable; light-grey: conjecturally non-realizable values, for $\operatorname{rank} H = \operatorname{rank} K = 6$.

Ivanov: $\operatorname{rr}(H \cap K) \leq \frac{i(i-1)}{2}$. S.: $\operatorname{rr}(H \cap K) \leq \left\lfloor \frac{i^2}{4} \right\rfloor$.



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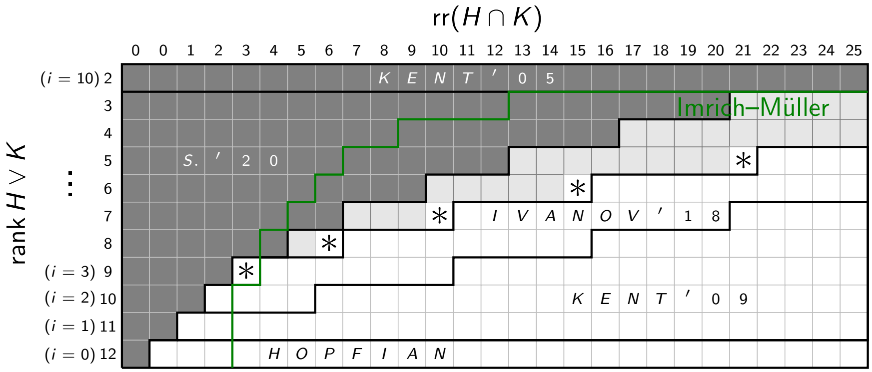
Conjecture (S. '20): Light-gray area is non-realizable.



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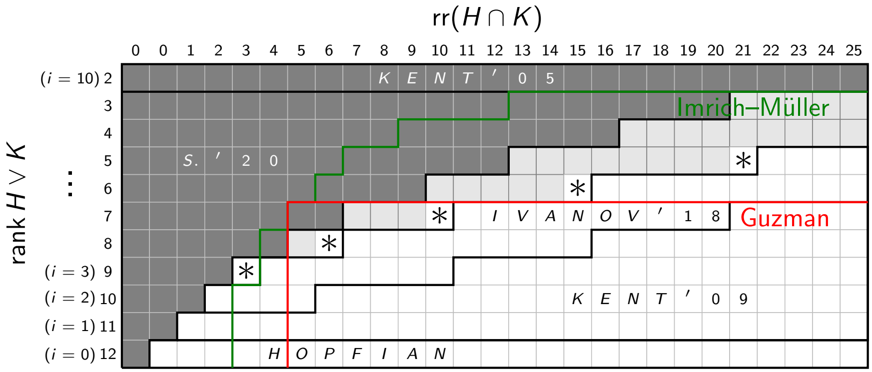

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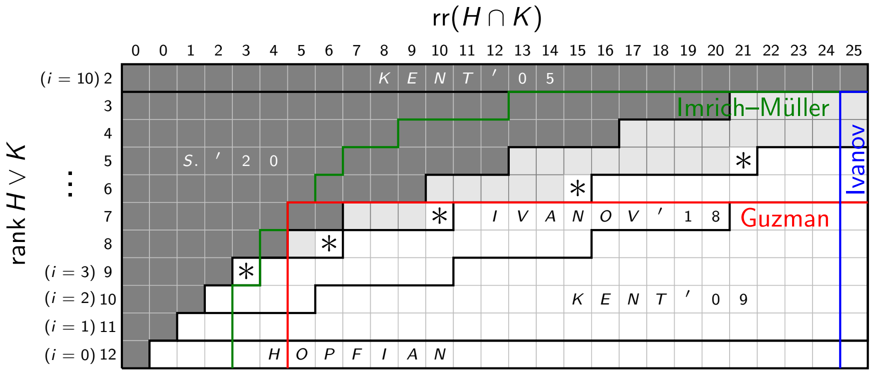

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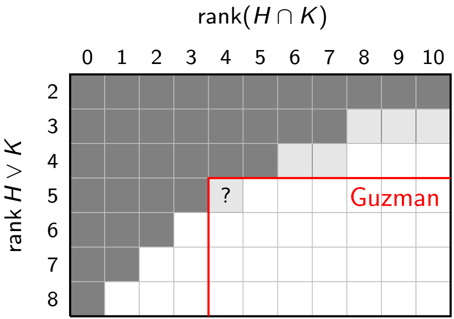
Guzman's GTC for $m = 4$



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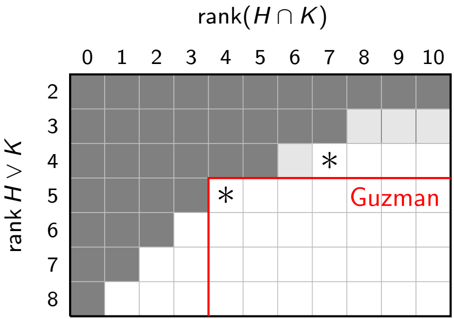

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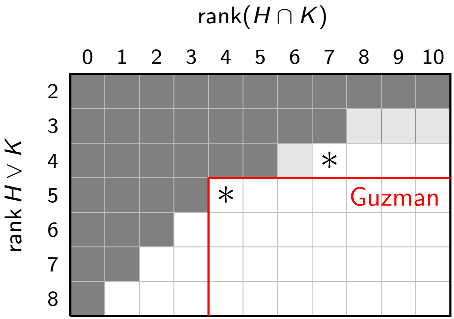


Theorem (S. '20)

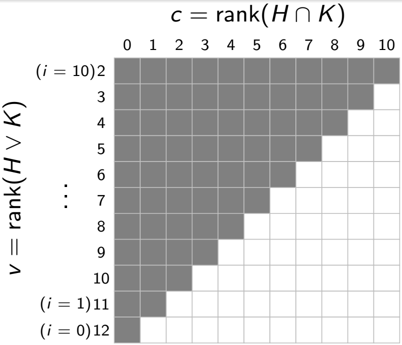
Suppose $H, K \leq F$ *with* $h = \operatorname{rank} H$, $k = \operatorname{rank} K$. *Then if* $\operatorname{rank}(H \vee K) = h + k - i$ *for some* $i \geq 3$, *then* $\operatorname{rank}(H \cap K) \leq \binom{i}{2}$.

Hence Guzman's GTC for rank 4 is true.

Case $\text{rank}(H) = 2$:

Theorem (S. '20)

Let integers k, v, c satisfy: $k \geq 2$, $2 \leq v \leq k+2$ and $0 \leq c \leq k$. Then there exist subgroups $H, K \leq F$ such that $\text{rank}(H) = 2$, $\text{rank}(K) = k$, $\text{rank}(H \vee K) = v$, and $\text{rank}(H \cap K) = c$ if and only if $c + v \leq k + 2$.



All realizable values for $\text{rank}(H) = 2$, $\text{rank}(K) = 10$.

Methods:

Stallings' core graphs: Γ_H, Γ_K, $\Gamma_{H\cap K}$, $\Gamma_{H\vee K}$.
$\Gamma_{H\cap K}$ is the pullback of Γ_H, Γ_K
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To relate $\Gamma_{H\cap K}$ and $\Gamma_{H\vee K}$ consider **topological pushout** $\mathcal{T}$:
$\mathcal{T}$ is obtained from $\Gamma_H \sqcup \Gamma_K$ by identifying $x \sim y$ if and only if (x, y) belongs to $\Gamma_{H\cap K}$.
Alternatively, $\mathcal{T}$ is the join of Γ_H, Γ_K, **folded only along edges of** $\Gamma_{H\cap K}$.

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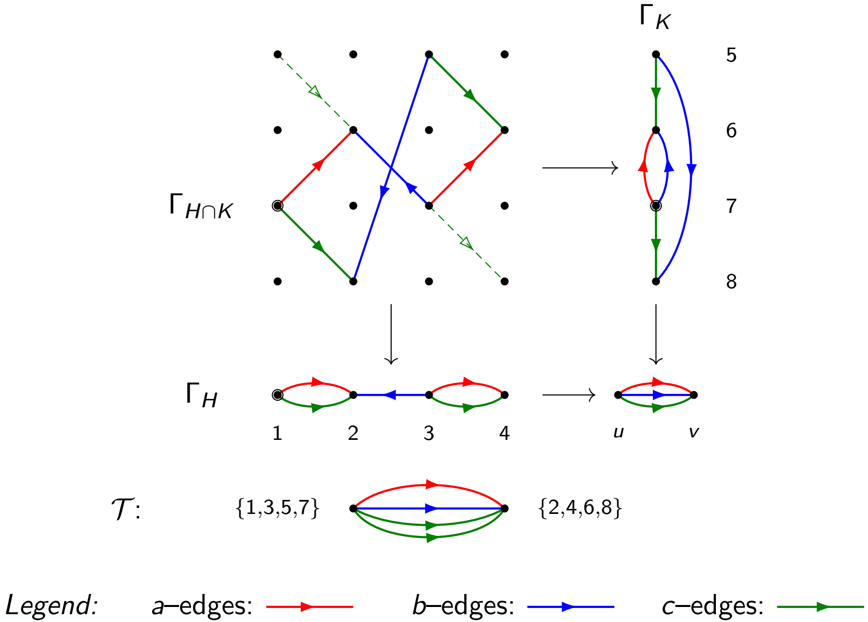
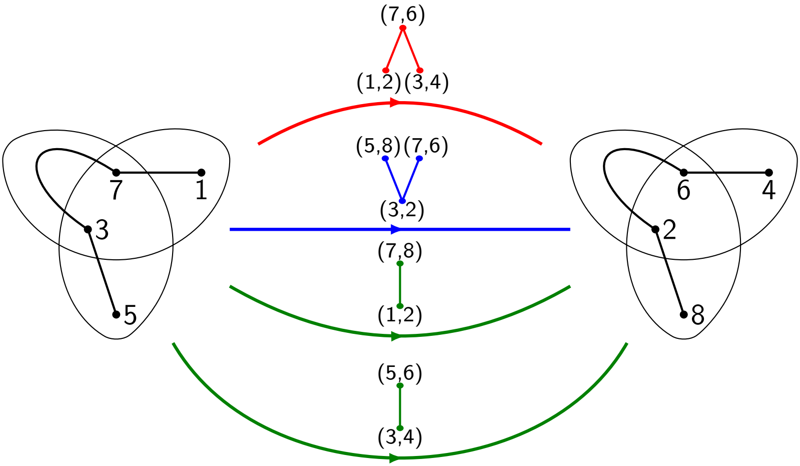
Describe $\mathcal{T}$ in terms of Dicks' graphs from his "Amalgamated graphs conjecture" paper in *Inventiones*. This translates the problem into (classical) graph theory.

Example: $H = \langle ca^{-1}, cb^{-1}ca^{-1}bc^{-1} \rangle$, $K = \langle ba^{-1}, cb^{-1}ca^{-1} \rangle$:





The topological pushout modeled on the Dicks graphs.

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