

LINEARITY OF SOME LOW-COMPLEXITY MAPPING CLASS GROUPS

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ABSTRACT. By analyzing known presentations of the pure mapping groups of orientable surfaces of genus g with b boundary components and n punctures in the cases when $g = 0$ with b and n arbitrary, and when $g = 1$ and $b + n$ is at most 3, we show that these groups are isomorphic to some groups related to the braid groups and the Artin group of type D_4 . As a corollary, we conclude that the pure mapping class groups are linear in these cases.

1. INTRODUCTION

Let $S_{g,b,n}$ denote the orientable surface of genus g with b boundary components and n punctures. Let $\text{PMod}_{g,b,n}$ denote the pure mapping class group of orientation-preserving diffeomorphisms of $S_{g,b,n}$ identical on the boundary and not permuting punctures, up to the isotopies identical on the boundary and not permuting punctures.

Presentations of $\text{PMod}_{g,b,n}$ are known. Gervais [Ger] discovered a symmetrical presentation for $\text{PMod}_{g,b,0}$ for $g > 0$ in terms of commutator, braid, and star relations only. Labruère and Paris have obtained in [LP] the presentations for $\text{PMod}_{g,b,n}$ with arbitrary $g > 0$, b and n in terms of quotients of Artin groups. However, neither of the authors have explicitly laid out to which known groups their presentation simplifies to when the genus is equal to 1 and the number of punctures and boundary components is small. As a result, a series of remarkable isomorphisms between these mapping class groups and some groups closely related to the braid group B_4 and to the Artin group of type D_4 have remained unknown to the general mathematical community. The purpose of this note is to make these beautiful connections to be broadly known.

We also provide a simple proof for the presentation of $\text{PMod}_{0,m,n}$, ($m > 1$), the pure mapping class group of the sphere with an arbitrary number of boundary components and punctures, which also seems to be missing in the literature.

Let B_n denote the braid group on n strands, PB_n the pure braid group on n strands, $A(D_4)$ the Artin group of type D_4 , and $Z(G)$ the center of a group G . We establish the following theorem.

Theorem 1. *The following table lists the isomorphism types of groups $\text{PMod}_{g,b,n}$ for the given triples (g, b, n) :*

(g, b, n)	$\text{PMod}_{g,b,n}$
$(0, m, n), m > 1$	$PB_{n+m-1} \times \mathbb{Z}^{m-1}$
$(1, 2, 0)$	$B_4 \times \mathbb{Z}$
$(1, 1, 1)$	B_4
$(1, 0, 2)$	$B_4/Z(B_4)$
$(1, 3, 0)$	$A(D_4) \times \mathbb{Z}^2$
$(1, 2, 1)$	$A(D_4) \times \mathbb{Z}$
$(1, 1, 2)$	$A(D_4)$
$(1, 0, 3)$	$A(D_4)/Z(A(D_4))$

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The genus 0 case of Theorem 1 is proved in Section 2 (Proposition 3) and all other cases in Section 3 (Propositions 8, 10, Corollaries 9, 11).¹

As a useful application we mention a well-known open problem to find out for which values of g, b, n the group $\text{PMod}_{g,b,n}$ admits a faithful linear representation. In her collection of open problems [Bir], Joan Birman mentions that for the following triples (g, b, n) :

$$(1, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, n), (0, 0, n), (2, 0, 0)$$

the mapping class group $\text{PMod}_{g,b,n}$ is known to be linear, and she asks if this list can be extended to contain any other triples. We extend this list by the triples from Theorem 1:

Corollary 2. *The pure mapping class group $\text{PMod}_{g,b,n}$ is linear for any of the triples (g, b, n) from Theorem 1:*

$$(0, m, n), m > 1, \quad (1, 2, 0), (1, 1, 1), (1, 0, 2), (1, 3, 0), (1, 2, 1), (1, 1, 2), (1, 0, 3).$$

Proof. Braid groups (and hence pure braid groups) are linear by the results of Krammer [Kra] and Bigelow [Big]. That Artin groups of spherical type are linear (in particular, $A(D_4)$), was proved by Cohen and Wales [CW] and independently by Digne [Dig]. The fact that if a group G is linear then the quotient by its center $G/Z(G)$ is linear, follows from Theorem 6.4 in [Wehr]. And, of course, if two groups are linear, their direct product is linear as well. \square

As another application, in Section 4 we answer in the negative a question of Hamidi-Tehrani [HT] whether a group generated by certain positive multi-twists in $S_{1,2,0}$ is free of rank 2.

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2. GENUS 0 CASE: $(g, b, n) = (0, m, n)$, $m > 1$

In this section we obtain the following description of the mapping class group $\text{PMod}_{0,m,n}$:

Proposition 3. *For $m > 1$, $n \geq 0$,*

$$\text{PMod}_{0,m,n} \cong \mathbb{Z}^{m-1} \times PB_{m+n-1}.$$

If, in addition, $m + n \geq 3$, then

$$\text{PMod}_{0,m,n} \cong \mathbb{Z}^m \times PB_{m+n-1}/Z(PB_{m+n-1}).$$

This result is mentioned in the literature (see e.g. [Har, Lemma 3.4]), but we were unable to find a detailed proof of it anywhere, so we provide it here.²

For completeness, we recall a few known results on $\text{PMod}_{0,m,n}$, see [FM, 9.3, 4.2.4, Prop. 2.4]:

$$\text{PMod}_{0,0,n} \cong PB_{n-1}/Z(PB_{n-1}), \quad \text{PMod}_{0,1,n} \cong PB_n \cong PB_n/Z(PB_n) \times \mathbb{Z} \quad (\text{if } n \geq 2), \quad (1)$$

$$\text{PMod}_{0,0,0} = \text{PMod}_{0,0,1} = \text{PMod}_{0,0,2} = \text{PMod}_{0,0,3} = \text{PMod}_{0,1,0} = \text{PMod}_{0,1,1} = 1, \quad (2)$$

$$\text{PMod}_{0,0,4} = F_2, \quad \text{PMod}_{0,1,2} = \mathbb{Z}, \quad \text{PMod}_{0,1,3} = F_2 \times \mathbb{Z}, \quad \text{PMod}_{0,2,0} = \mathbb{Z}. \quad (3)$$

We will make use of the following lemma:

Lemma 4 (Capping the boundary, [FM, Prop. 3.19]). *Let S' be the surface obtained from a surface S by capping one boundary component β with a once-punctured disk. Let $\text{Cap}: \text{PMod}(S) \rightarrow \text{PMod}(S')$ be the induced homomorphism obtained by extending homeomorphisms of S to the once-punctured disk by the identity. Then the following sequence is exact:*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \text{PMod}(S) \xrightarrow{\text{Cap}} \text{PMod}(S') \longrightarrow 1$$

where \mathbb{Z} is generated by the twist T_β around β . \square

¹(Added June 6, 2024) After this paper was published, it came to the author's attention that the fact that $\text{PMod}_{1,0,2} \cong B_4/Z(B_4)$ was proved in [FM, Section 9.4.1].

²(Added June 6, 2024) After this paper was published the author discovered that a stronger result was stated and its proof was outlined in the Appendix to the paper Clay, Matt; Leininger, Christopher J.; Margalit, Dan; "Abstract commensurators of right-angled Artin groups and mapping class groups." Math. Res. Lett. 21 (2014), no. 3, 461–467.

Proof of Proposition 3. The second statement of the Proposition follows from the first one and from the fact that $PB_n \cong PB_n/Z(PB_n) \times \mathbb{Z}$ if $n \geq 2$, see (1) above. To prove the first statement, we argue by induction. Again, by (1), $\text{PMod}_{0,1,n} \cong PB_n \cong \mathbb{Z}^{1-1} \times PB_{1+n-1}$, which gives us the case of $m = 1$ (and arbitrary $n \geq 1$). Also, we know that $\text{PMod}_{0,2,0} = \mathbb{Z} = \mathbb{Z}^{2-1} \times PB_1$. Suppose that the statement is true for $\text{PMod}_{0,m,n+1}$ for some $m \geq 1, n \geq 0$; we are going to prove it for $\text{PMod}_{0,m+1,n}$. Let $\delta_1, \dots, \delta_{m+1}$ denote the boundary circles of $S_{0,m+1,n}$. By Lemma 4, we have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \text{PMod}_{0,m+1,n} \xrightarrow{\text{Cap}} \text{PMod}_{0,m,n+1} \longrightarrow 1,$$

where the kernel \mathbb{Z} is generated by the twist $T_{\delta_{m+1}}$ around δ_{m+1} . It is sufficient to prove that this sequence splits as a direct product. For that we exhibit an epimorphism $f: \text{PMod}_{0,m+1,n} \longrightarrow \mathbb{Z}$ such that the composition $\mathbb{Z} \longrightarrow \text{PMod}_{0,m+1,n} \longrightarrow \mathbb{Z}$ is identical on \mathbb{Z} . Recall that the forgetful homomorphism $\text{PMod}(S \setminus \{x\}) \longrightarrow \text{PMod}(S)$, which treats a mapping class of a surface S without a point x as a mapping class of S , is surjective [FM, 4.2.1]. We construct $f: \text{PMod}_{0,m+1,n} \longrightarrow \text{PMod}_{0,2,0} \cong \mathbb{Z}$ as a composition of $m-1$ capping epimorphisms, followed by $m-1+n$ forgetful epimorphisms, as shown in Figure 1. Obviously,

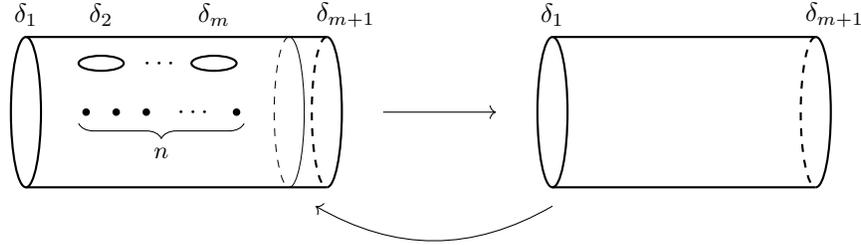


FIGURE 1. The inclusions between surfaces $S_{0,m+1,n}$ and $S_{0,2,0}$ which induce the epimorphism $f: \text{PMod}_{0,m+1,n} \longrightarrow \text{PMod}_{0,2,0}$ and the identification $\text{PMod}_{0,2,0} \cong \langle T_{\delta_{m+1}} \rangle$.

the boundary twist $T_{\delta_{m+1}}$ maps under f to the generator of $\text{PMod}_{0,2,0}$, which can be identified with the boundary twist $T_{\delta_{m+1}}$ of $S_{0,m+1,n}$ itself. This shows that $\text{PMod}_{0,m+1,n} \cong \mathbb{Z} \times \text{PMod}_{0,m,n+1}$, and hence that $\text{PMod}_{0,m+1,n} \cong \mathbb{Z}^m \times PB_{m+n}$ by induction. \square

3. GENUS 1 CASE

In this section we analyze the Gervais presentation for $\text{PMod}_{1,3,0}$ and show that it defines the group $A(D_4) \times \mathbb{Z}^2$. Then we apply Lemma 4 to obtain the descriptions of $\text{PMod}_{1,2,1}$, $\text{PMod}_{1,1,2}$, and $\text{PMod}_{1,0,3}$ from Theorem 1. In the end, we do the same for $\text{PMod}_{1,2,0}$, $\text{PMod}_{1,1,1}$, and $\text{PMod}_{1,0,2}$.

Recall that the Artin group $A(D_4)$ is defined by the Coxeter graph D_4 depicted in Figure 2 on the right, in the following way. The generators of $A(D_4)$ are vertices a_1, a_2, a_3, b of graph D_4 , with the following relations: any two of these generators x, y commute if the corresponding vertices are disjoint, and they form a braid relation $xyx = yxy$ if the corresponding vertices are connected with an edge.

In a similar way, the Artin group $A(A_3)$ is defined by the graph A_3 on the right of Figure 3, and this presentation identifies it with the braid group on four strands B_4 .

The Gervais presentation for $\text{PMod}_{1,3,0}$. In [Ger], Gervais obtained a remarkable finite presentation for $\text{PMod}_{g,m,0}$ which is very symmetrical, though it is admittedly not the most economical in terms of the total number of generators and relations involved. For $\text{PMod}_{1,3,0}$ it specifies to the following presentation, generated by the twists around the curves depicted in Figure 2. For simplicity, we denote a curve and its twist by the same letter.

Generators: $b, a_1, a_2, a_3, c_{12}, c_{21}, c_{13}, c_{31}, c_{23}, c_{32}$;

Relations:

- *Handles:* \emptyset (present only when $g \geq 2$);
- *Commutators:* Any two of the above generators commute if and only if their curves are disjoint;
- *Braids:* Any two of the above generators x, y form a braid relation $xyx = yxy$ if and only if their curves intersect only once;

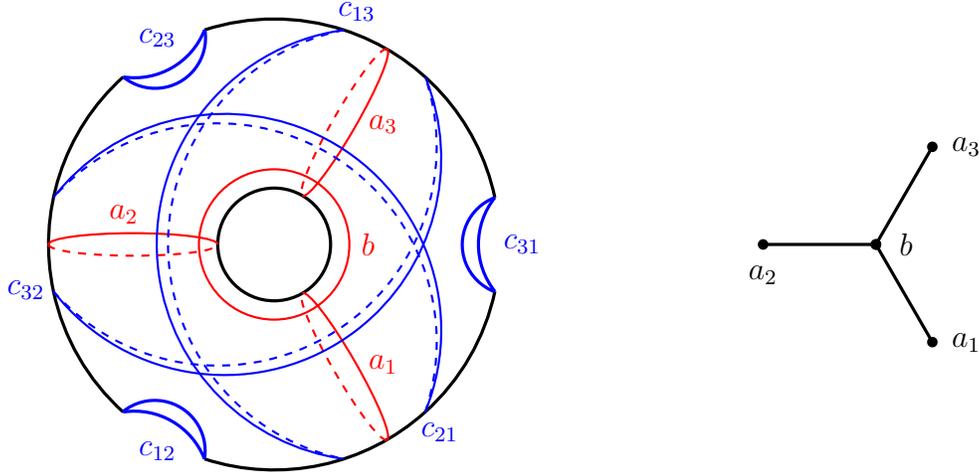


FIGURE 2. Gervais' generators of $\text{PMod}_{1,3,0}$ and the Coxeter graph D_4 .

- *Stars*: For any good triple $(i, j, k) \in \{1, 2, 3\}^3$, there is a star relation:

$$(a_i a_j a_k b)^3 = c_{ij} c_{jk} c_{ki}, \quad (E_{i,j,k})$$

(with the assumption that $c_{\ell\ell} = 1$);

A triple (i, j, k) is called *good*, if $(i, j, k) \neq (x, x, x)$ for all $x \in \{1, 2, 3\}$ and either $i \leq j \leq k$ or $j \leq k \leq i$ or $k \leq i \leq j$.

Reducing the number of star relations. Notice that, when all i, j, k are distinct, the element $\Delta_{ijk} := (a_i a_j a_k b)^3$ in the left-hand side of the star relation $(E_{i,j,k})$ is the fundamental element of the Artin group of type D_4 on the generators a_i, a_j, a_k, b . In particular, Δ_{ijk} does not depend on the order of the generators inside the third power, see [BS, Satz 7.2 and Zusatz]. Hence, $\Delta_{ijk} = \Delta_{jki} = \Delta_{kij}$ and the corresponding star relations $(E_{i,j,k})$, $(E_{j,k,i})$ and $(E_{k,i,j})$ coincide since c_{ij} , c_{jk} and c_{ki} pairwise commute. (Notice though, that this does not give us the equivalence of relations $(E_{i,j,k})$ corresponding to good triples with the ones corresponding to non-good triples, since the right-hand side of a non-good triple relation, say, $(E_{j,i,k})$, will be a product of completely different curves c_{ji} , c_{ik} , c_{kj} , which do not commute pairwise.)

If in a good triple (i, j, k) two elements are equal, say, $i = k$, then, by using only braid and commutation relations (underlined below), we obtain (cf. Lemma 2 (iii) of [Ger]):

$$\begin{aligned} \Delta_{iij} &= (a_i a_i a_j b)^3 = a_i \underline{a_i a_j} b \cdot a_i \underline{a_i a_j} b \cdot a_i a_i a_j b = a_i a_j \underline{a_i b a_i} a_j \underline{a_i b a_i} a_i a_j b = \\ & a_i a_j b a_i \underline{b a_j b} a_i b a_i a_j b = a_i a_j b \cdot a_i a_j b \cdot \underline{a_j a_i} b \cdot a_i a_j b = (a_i a_j b)^4. \end{aligned} \quad (4)$$

In particular, since a_i and a_j commute, we have $(a_i a_j b)^4 = (a_j a_i b)^4$, and, reading the above equation backwards, we conclude that $\Delta_{iij} = \Delta_{ijj}$. We summarize our observations in the following remark.

Remark 5. In the Gervais presentation, one only needs the good triples (i, j, k) with $i \leq j < k$.

(This was probably supposed to be the content of Remark 3 in [Ger], but due to an unfortunate typo, this information was not properly conveyed to the reader.)

Remark 6. When all i, j, k are distinct, Δ_{ijk} is the generator of the center of $A(D_4)$ ([BS]). Similarly, Δ_{iij} is the generator of the center of $A(A_3)$. Indeed, as above, using the braid and commutation relations only, we get (as in the proof of Lemma 2 of [Ger]):

$$\begin{aligned} (a_i a_j b)^4 &= a_i a_j b \cdot \underline{a_i a_j} b \cdot \underline{a_i a_j} b \cdot a_i a_j b = a_i \underline{a_j b a_j} a_i b a_j \underline{a_i b a_i} a_j b = \\ & a_i b a_j \underline{b a_i b} a_j b a_i \underline{b a_j b} = a_i b a_j \cdot a_i b \underline{a_i a_j} b \underline{a_i a_j} b a_j = (a_i b a_j)^4, \end{aligned} \quad (5)$$

which is an expression for the generator of the center of the Artin group $A(A_3)$ on the generators a_i, a_j, b , see [BS]. We will denote this element as Δ_{ij}^2 , since it is equal to the square of the fundamental element of the respective group.

Remark 7. The Dehn twists involved in the Gervais presentation can be either all right or all left uniformly in all the relations. Indeed, right and left twists are inverses of each other, and braid and commutation relations are invariant under taking inverses. The same is true for the star relation, since its left-hand side doesn't depend on the order of the generators inside the third power ([BS]), and its right-hand side consists of pairwise commuting twists. (One may argue that this phenomenon should hold for arbitrary presentations of the mapping class group of an orientable surface in terms of Dehn twists as generators, namely, the presentation should remain invariant under changing all twists from the left ones to the right ones uniformly in all relations. Indeed, the definition of what is a left twist and what is a right twist around a curve depends on the way we look at the surface: 'from the outside' or 'from the inside'. Since homeomorphisms of the surface are defined intrinsically, the presentations of the mapping class groups should also be independent of the way we look at the surface, and hence changing all twists from the left ones to the right ones or vice versa should produce an equivalent presentation.)

Proposition 8. *In the group $\text{PMod}_{1,3,0}$, the elements a_1, a_2, a_3 , and b generate a subgroup isomorphic to $A(D_4)$, and the elements c_{12}, c_{23}, c_{31} generate a free abelian subgroup \mathbb{Z}^3 . The group $\text{PMod}_{1,3,0}$ is isomorphic to their direct product modulo one star relation:*

$$\text{PMod}_{1,3,0} \cong A(D_4) \times \mathbb{Z}^3 / \langle c_{12}c_{23}c_{31} = (a_1a_2a_3b)^3 \rangle.$$

In particular, $\text{PMod}_{1,3,0} \cong A(D_4) \times \mathbb{Z}^2$.

Proof. We write down the Gervais presentation for $\text{PMod}_{1,3,0}$, taking into account Remark 5.

Generators: $b, a_1, a_2, a_3, c_{12}, c_{21}, c_{13}, c_{31}, c_{23}, c_{32}$.

Relations:

- Commutators (we write $x \rightleftharpoons y$ for $xy = yx$):
 - $a_i \rightleftharpoons a_j$ for all i, j ;
 - $a_1 \rightleftharpoons c_{ij}$ for all c_{ij} , except c_{32} ;
 - $a_2 \rightleftharpoons c_{ij}$ for all c_{ij} , except c_{13} ;
 - $a_3 \rightleftharpoons c_{ij}$ for all c_{ij} , except c_{21} ;
 - $b \rightleftharpoons c_{ij}$ for all i, j ;
 - $c_{12}, c_{23}, c_{31} \rightleftharpoons c_{ij}$ for all i, j .
- Braid relations: any two generators $x, y \in \{a_1, a_2, a_3, b\}$ form a braid relation $xyx = yxy$ if and only if the corresponding vertices in the Coxeter graph D_4 are connected with an edge, see Figure 2.
- Stars (only for triples (i, j, k) with $i \leq j < k$, using (4), (5)):
 - (1,1,2): $c_{12}c_{21} = \Delta_{12}^2$,
 - (1,1,3): $c_{13}c_{31} = \Delta_{13}^2$,
 - (1,2,3): $c_{12}c_{23}c_{31} = \Delta_{123}$,
 - (2,2,3): $c_{23}c_{32} = \Delta_{23}^2$.

To simplify this presentation, we eliminate three of c_{ij} 's using the degenerate star relations above:

$$c_{21} = c_{12}^{-1} \Delta_{12}^2, \quad c_{13} = \Delta_{13}^2 c_{31}^{-1}, \quad c_{32} = c_{23}^{-1} \Delta_{23}^2.$$

Let's show that, with these eliminations, all the commutator and braid relations involving the eliminated letters are consequences of the relations in the group $A(D_4) \times \mathbb{Z}^3(c_{12}, c_{23}, c_{31})$. For example, $a_1 \rightleftharpoons c_{21}$ is true since $a_1 \rightleftharpoons c_{12}$ and $a_1 \rightleftharpoons \Delta_{12}^2$ (the last element being central in $\langle a_1, b, a_2 \rangle$). A similar reasoning shows that $a_1 \rightleftharpoons c_{13}$, $a_2 \rightleftharpoons c_{21}$, $a_2 \rightleftharpoons c_{32}$, $a_3 \rightleftharpoons c_{13}$, $a_3 \rightleftharpoons c_{32}$, and also that $b \rightleftharpoons c_{ij}$ and $c_{12}, c_{23}, c_{31} \rightleftharpoons c_{13}, c_{32}, c_{21}$.

This shows that $\text{PMod}_{1,3,0}$ is isomorphic to $A(D_4) \times \mathbb{Z}^3$ modulo a single star relation for the triple (1, 2, 3).

To show that $\text{PMod}_{1,3,0}$ is isomorphic to $A(D_4) \times \mathbb{Z}^2$, we eliminate one of the generators of \mathbb{Z}^3 , say, c_{31} from the star relation:

$$c_{31} = c_{23}^{-1} c_{12}^{-1} \Delta_{123},$$

and observe (using the fact that Δ_{123} is central in $A(D_4)$) that all the commutation relations involving c_{31} in $A(D_4) \times \mathbb{Z}^3$ still hold in $A(D_4) \times \mathbb{Z}^2(c_{12}, c_{23})$. \square

Corollary 9. *We have the following isomorphisms:*

- $\text{PMod}_{1,2,1} \cong A(D_4) \times \mathbb{Z}$;
- $\text{PMod}_{1,1,2} \cong A(D_4)$;
- $\text{PMod}_{1,0,3} \cong A(D_4)/Z(A(D_4))$.

Proof. By applying Lemma 4 three times to the presentation

$$\text{PMod}_{1,3,0} \cong A(D_4) \times \mathbb{Z}^3 \langle c_{12}, c_{23}, c_{31} \rangle / \langle c_{12}c_{23}c_{31} = (a_1a_2a_3b)^3 \rangle,$$

i.e. capping the boundary twists c_{31} , c_{23} and c_{12} with a punctured disk (in that order), and arguing as in the end of the proof of Proposition 8, we get:

$$\text{PMod}_{1,2,1} \cong A(D_4) \times \mathbb{Z}^2 \langle c_{12}, c_{23} \rangle / \langle c_{12}c_{23} = (a_1a_2a_3b)^3 \rangle \cong A(D_4) \times \mathbb{Z};$$

$$\text{PMod}_{1,1,2} \cong A(D_4) \times \mathbb{Z} \langle c_{12} \rangle / \langle c_{12} = (a_1a_2a_3b)^3 \rangle \cong A(D_4);$$

$$\text{PMod}_{1,0,3} \cong A(D_4) / \langle (a_1a_2a_3b)^3 \rangle \cong A(D_4)/Z(A(D_4)),$$

the last isomorphism taking place since $(a_1a_2a_3b)^3$ generates the center of $A(D_4)$, see [BS, Satz 7.2 and Zusatz]. \square

Now we will deal with the group $\text{PMod}_{1,2,0}$ and its relatives in a similar, but simpler, way. Let generators a_1 , a_2 , b , c_{12} and c_{21} be as in the Figure 3. Notice that, by definition, the Artin group $A(A_3)$ is isomorphic to the braid group on four strands B_4 .

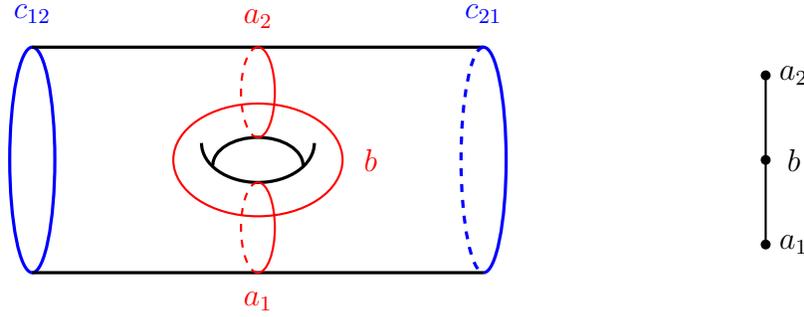


FIGURE 3. Gervais' generators of $\text{PMod}_{1,2,0}$ and the Coxeter graph A_3 .

Proposition 10. *In the group $\text{PMod}_{1,2,0}$, the elements a_1 , a_2 , and b generate a subgroup isomorphic to B_4 , and the elements c_{12} , c_{21} generate a free abelian subgroup \mathbb{Z}^2 . The whole group $\text{PMod}_{1,2,0}$ is isomorphic to their direct product modulo one (degenerate) star relation:*

$$\text{PMod}_{1,2,0} \cong B_4 \times \mathbb{Z}^2 / \langle c_{12}c_{21} = (a_1ba_2)^4 \rangle.$$

In particular, $\text{PMod}_{1,2,0} \cong B_4 \times \mathbb{Z}$.

Proof. The proof is analogous to the proof of Proposition 8. The Gervais presentation for $\text{PMod}_{1,2,0}$ is the following:

Generators: b , a_1 , a_2 , c_{12} , c_{21} .

Relations:

- Commutators:
 - $a_1 \rightleftharpoons a_2$; $a_1, a_2, b \rightleftharpoons c_{12}, c_{21}$; $c_{12} \rightleftharpoons c_{21}$.
- Braid relations: any two generators $x, y \in \{a_1, a_2, b\}$ form a braid relation $xyx = yxy$ if and only if the corresponding vertices in the Coxeter graph A_3 are connected with an edge, see Figure 3.
- Stars (only for triples (i, j, k) with $i \leq j < k$, using (4), (5)):
 - $(1,1,2)$: $c_{12}c_{21} = \Delta_{12}^2$,

We see that the Gervais presentation gives us the presentation

$$\text{PMod}_{1,2,0} \cong B_4 \times \mathbb{Z}^2 / \langle c_{12}c_{21} = (a_1ba_2)^4 \rangle$$

on the nose. To show that $\text{PMod}_{1,2,0} \cong B_4 \times \mathbb{Z}$, we eliminate c_{21} from the star relation:

$$c_{21} = c_{12}^{-1} \Delta_{12}^2,$$

and check that all the commutation relations from $B_4 \times \mathbb{Z}^2(c_{12}, c_{21})$ involving c_{21} hold true in $B_4 \times \mathbb{Z}$. Again, as above, we are using the fact that Δ_{12}^2 is central in B_4 . \square

Corollary 11. *We have the following isomorphisms:*

- $\text{PMod}_{1,1,1} \cong B_4$;
- $\text{PMod}_{1,0,2} \cong B_4/Z(B_4)$.

Proof. Again, as above, we apply Lemma 4 two times to the presentation

$$\text{PMod}_{1,2,0} \cong B_4 \times \mathbb{Z}^2(c_{12}, c_{21}) / \langle c_{12}c_{21} = (a_1ba_2)^4 \rangle.$$

which amounts to capping the boundary twists c_{21} , c_{12} with punctured disks. We get:

$$\text{PMod}_{1,1,1} \cong B_4 \times \mathbb{Z}(c_{12}) / \langle c_{12} = (a_1ba_2)^4 \rangle \cong B_4;$$

$$\text{PMod}_{1,0,2} \cong B_4 / \langle (a_1ba_2)^4 \rangle \cong B_4/Z(B_4),$$

the last isomorphism taking place since $(a_1ba_2)^4$ generates the center of B_4 . \square

Remark 12. One can obtain the isomorphisms from Theorem 1 “from first principles”, starting with the presentation for $\text{PMod}_{1,1,0} \cong B_3$ and reverting the process of capping a boundary component with a disk. The basic tools here are Lemma 4 together with the Birman exact sequence [FM, 4.2] (which describes the kernel of the forgetful map), and the method of constructing a finite presentation for an extension of one finitely presented group by another [John, Prop. 10.2.1]. The author undertook such an approach in an early version of this paper, which resulted in a much longer text full of computations. An interested reader is welcome to request it from the author.

Remark 13. The smallest triples (g, b, n) for which the linearity of $\text{PMod}_{g,b,n}$ is yet unknown, are $(1, b, n)$ with $b + n = 4$, $(2, 1, 0)$, $(2, 0, 1)$ and $(3, 0, 0)$. For $g = 1$, the presentation looks the simplest when there is only one boundary component, i.e. for $(g, b, n) = (1, 1, 3)$. Both the Gervais presentation and the Labruère–Paris one realize the group $\text{PMod}_{1,1,3}$ as a quotient of the (euclidean) Artin group $A(\tilde{D}_4)$ whose defining graph is the complete bipartite graph $K_{1,4}$. The simplification of the Gervais presentation in the same way as above yields three additional relations (besides the ones defining $A(\tilde{D}_4)$) involving different elements Δ_{ij}^2 , Δ_{ijk} , whereas the Labruère–Paris presentation has just one additional relation, of the form of a double commutator $[[\Delta_{13}, a_2], a_4] = 1$ (see relation (PR3) in [LP, Prop.3.3]). It is an interesting challenge to establish the equivalence of the two presentations directly.

4. ON A QUESTION OF HAMIDI-TEHRANI

As an application, we answer a question of Hamidi-Tehrani. In [HT], he studied subgroups of mapping class groups generated by positive multi-twists and asked the following question:

Question 1. If a_1 , a_2 , b are the Dehn twists as in the Figure 3, is it true that the group $\langle a_1^2 a_2, b \rangle$ is free of rank 2?

The answer to this question is negative. Indeed, we saw in (4) that the element $\Delta_{112} = (a_1^2 a_2 \cdot b)^3 = \Delta_{12}^2$ is central in B_4 , and hence is central and nontrivial in the group in question. Hence, this group cannot be free non-abelian.

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