Property R_∞ for new classes of Artin groups

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Abstract We establish property R_∞ for Artin groups of spherical type D_n, $n \geqslant 6$, their central quotients, and also for large hyperbolic-type free-of-infinity Artin groups and some other classes of large-type Artin groups. The key ingredients are recent descriptions of the automorphism groups for these Artin groups and their action on suitable Gromov-hyperbolic spaces.

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1 Introduction

Let G be a group and φ be an automorphism of G. We say that elements $g, h \in G$ are *φ-twisted conjugate* if there exists $x \in G$ such that $h = x\,g\,\varphi(x)^{-1}$. This defines an equivalence relation on G, and the number of its equivalence classes is called the *Reidemeister number* of φ, denoted by $R(\varphi)$. We say that G *has property R_∞* if $R(\varphi) = \infty$ for all $\varphi \in \mathrm{Aut}(G)$.

The notion of twisted conjugacy arises in Nielsen–Reidemeister fixed point theory, where under certain natural conditions the Reidemeister number serves as an upper bound for a homotopy invariant called the Nielsen number. Also, twisted conjugacy classes appear naturally in Arthur–Selberg theory, Galois cohomology, the twisted Burnside–Frobenius theory, nonabelian cohomology, and in some topics of algebraic geometry. See [TW11, FT15] and references therein.

The problem of determining which groups have property R_∞ was started in [FH94], and has since been an area of active research. The list of groups known to have property R_∞ is quite large and contains non-elementary Gromov hyperbolic and relatively hyperbolic groups, non-abelian generalized Baumslag–Solitar groups, many weakly branch groups, many arithmetic linear groups, mapping class groups of surfaces with large enough complexity, and some other non-amenable groups, see [FT15, FN16] for the references. On the other hand, the free nilpotent group $N_{r,c}$ of rank r and the nilpotency class c has property R_∞ if and only if $c \geqslant 2r$, see [DG14]. For some other recent developments, see [DL24, SSV23, Tro23, Tro24].

Let S be a finite set. A *Coxeter matrix* over S is a symmetric matrix $(m_{st})_{s,t \in S}$ with entries in $\{1, 2, \dots\} \cup \{\infty\}$, such that $m_{ss} = 1$ for all $s \in S$ and $m_{st} = m_{ts} \geqslant 2$ if $s \neq t$. A Coxeter matrix can be encoded by the corresponding *Coxeter graph* Γ having S as the set of vertices. Two distinct vertices $s, t \in S$ are connected with an edge in Γ if $m_{st} \geqslant 3$, and this edge is labeled with m_{st} if $m_{st} \geqslant 4$. The *Artin group of type* Γ is the group $A[\Gamma]$ given by the presentation:

$$A[\Gamma] = \langle S \mid \Pi(s, t, m_{st}) = \Pi(t, s, m_{ts}),\ \text{for all } s \neq t,\ m_{st} \neq \infty \rangle,$$

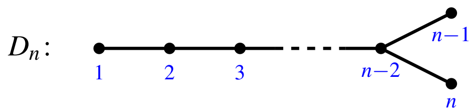

Figure 1: The Coxeter graph of type D_n, $n \geqslant 4$.

where $\Pi(s,t,m_{st})$ is the word $stst\ldots$ of length $m_{st} \geqslant 2$. The *Coxeter group* $W[\Gamma]$ *of type* Γ is the quotient of $A[\Gamma]$ by all relations of the form $s^2 = 1$, $s \in S$. We denote $P[\Gamma]$ the kernel of the natural epimorphism $A[\Gamma] \to W[\Gamma]$. It is called the *pure Artin group of type* Γ. An Artin group $A[\Gamma]$ is called *spherical*, if $W[\Gamma]$ is finite.

Concerning Artin groups, property R_∞ was established for braid groups (i.e. Artin groups of type A_n) in [FGD10], for pure braid groups $P[A_n]$ in [DGO21], for some classes of large-type Artin groups in [Juh12] (see the discussion of these results in Remark 21 of Section 4), for (non-abelian) right-angled Artin groups in [Wit23] (see also [DS21]), and for the spherical Artin groups of types B_n, D_4, $I_2(m)$ for $m \geqslant 5$, their pure subgroups, and for the affine Artin groups of types $\tilde{A}_n$, $\tilde{C}_n$, in [CS22].

In this article we extend the list of groups having property R_∞ to include Artin groups of a few new classes. The first one is the class of Artin groups of spherical type D_n, for $n \geqslant 6$, for which the Coxeter graph of type D_n is depicted in Figure 1. Namely, we prove the following theorem.

Theorem 1 *Let $n \geqslant 6$. Then the Artin group $A[D_n]$ and its central quotient $A[D_n]/Z(A[D_n])$ have property R_∞.*

The proof is based on the approach used in [FGD10] and [CS22] to establish property R_∞ for some Artin groups whose automorphism groups can be embedded into mapping class groups of certain surfaces with punctures. The new ingredient that made this approach possible for Artin groups of type D_n is the recent description of automorphisms of Artin groups of type D_n for $n \geqslant 6$, obtained in [CP23]. Using this result, we establish an embedding of $\mathrm{Aut}(A[D_n])$ into the extended mapping class group of a suitable punctured surface, in Proposition 10, which may be of independent interest. We use this embedding to show that $\mathrm{Aut}(A[D_n])$ acts in a non-elementary way on the Gromov-hyperbolic complex of curves for the surface, after that applying Delzant's Lemma (see Section 2) allows us to detect infinitely many twisted conjugacy classes.

We notice that property R_∞ for the Artin group of type D_4 was established earlier in [CS22], so the only unresolved case in the series $A[D_n]$ is $A[D_5]$. In this regard we formulate the following conjecture.

Conjecture 2 *The Artin group $A[D_5]$ and its central quotient $A[D_5]/Z(A[D_5])$ also have property R_∞.*

The second class of Artin groups for which we establish property R_∞ is the class of *large-type hyperbolic-type free-of-infinity* Artin groups. An Artin group $A[\Gamma]$ is called *large* (or *large-type*) if all $m_{st} \geqslant 3$ for all $s,t \in S$; $A[\Gamma]$ is called *hyperbolic-type* if its Coxeter group $W[\Gamma]$ is word-hyperbolic, and $A[\Gamma]$ is called *free-of-infinity*, if $m_{st} \neq \infty$ for all $s,t \in S$. We note that for the class of large-type Artin groups, being hyperbolic-type is equivalent to the absence of triangles with edge labels $(3,3,3)$ in its Coxeter graph Γ (see [Dav08, Corollary 12.6.3]). We also note that Artin groups of hyperbolic-type are not word-hyperbolic themselves, unless they are free. We prove the following theorem (see Corollary 16).

Theorem 3 *Let $A[\Gamma]$ be a large hyperbolic-type free-of-infinity Artin group. Then it has property R_∞.*

We also establish property R_∞ for some other subclasses of large hyperbolic-type Artin groups, whose definitions are deferred to Section 4, in the following theorem (whose proof is given in Corollaries 18 and 20).

Theorem 4 *Let $A[\Gamma]$ be a large hyperbolic-type Artin group. If either*

- *$A[\Gamma]$ has XXXL type and Γ is twistless, or*
- *Γ admits a twistless hierarchy terminating in twistless stars,*

then $A[\Gamma]$ has property R_∞.

The proofs of Theorems 3 and 4 are based on the recent description of the automorphism groups for these classes of Artin groups [Vas25, BMV24, HOV24], and on their action on the Deligne complex, which is known to be Gromov-hyperbolic.

This paper is organized as follows. In Section 2 we give preliminary results necessary for proving Theorem 1 in Section 3 and Theorems 3 and 4 in Section 4. In Section 5 we recall some famous conjectures concerning property R_∞, which we believe should be more broadly known.

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2 Preliminaries on twisted conjugacy

Recall that for a group G with center $Z(G)$, $\mathrm{Inn}(G)$ denotes the subgroup of $\mathrm{Aut}(G)$ consisting of all inner automorphisms. For $g \in G$, we denote by conj_g the inner automorphism of conjugation by g: $x \mapsto gxg^{-1}$. The map $\mathrm{conj}\colon g \mapsto \mathrm{conj}_g$ identifies $\mathrm{Inn}(G)$ with $G/Z(G)$ and $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$. The *outer automorphism group* of G is the quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

To determine property R_∞ for a group G one can proceed as follows.

Step 1: Reduce the problem to $G/Z(G)$:

Lemma 5 (See e.g. [FGD10, Section 2]) *Let G be a group, φ be an automorphism of G, and H be a normal φ-invariant subgroup of G. Denote by $\bar{\varphi}$ the automorphism induced by φ on G/H. Then the condition $R(\bar{\varphi}) = \infty$ implies $R(\varphi) = \infty$. In particular, if $G/Z(G)$ has property R_∞, then so does G.* □

Step 2: Reduce the problem to detecting the usual (non-twisted) conjugacy classes:

Let G be a group with $Z(G) = 1$ and φ be an automorphism of G. Let $m \in \{1, 2, 3, \dots\} \cup \{\infty\}$ be the order of φ in $\mathrm{Out}(G)$; if $m < \infty$, let $p \in G$ be the unique element such that $\varphi^m = \mathrm{conj}_p$. Consider the group

$$G_\varphi = \begin{cases} G * \langle t \rangle / \langle tgt^{-1} = \varphi(g) \text{ for all } g \in G \rangle, & \text{if } m = \infty, \\ G * \langle t \rangle / \langle tgt^{-1} = \varphi(g) \text{ for all } g \in G,\ t^m = p \rangle, & \text{if } m < \infty. \end{cases}$$

We have the following useful properties of G_φ:

Lemma 6 (Calvez–Soroko [CS22, Lemma 3]) *Let G be a group with $Z(G) = 1$, $\varphi \in \mathrm{Aut}(G)$, and G_φ defined as above. Then G is a normal subgroup of G_φ, the quotient G/G_φ is cyclic, and the assignment $g \mapsto \mathrm{conj}_g$ and $t \mapsto \varphi$ defines an injective homomorphism from G_φ to $\mathrm{Aut}(G)$.* □

We observe that if $m = \infty$, G_φ is the semidirect product $G \rtimes_\varphi \mathbb{Z}$, and if $m < \infty$, we have a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & G_\varphi & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow \simeq & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 1 \end{array}$$

in which the top short exact sequence does not split in general (for example, if G is the free group of rank 2, there are elements of order 6 in $\mathrm{Out}(G)$, but not in $\mathrm{Aut}(G)$, see [LS01, Proposition 4.6]). We also note that if $m = 1$ then G_φ is isomorphic to G.

It turns out that φ-twisted conjugacy classes in G bijectively correspond to the usual conjugacy classes in the coset Gt of G_φ:

Lemma 7 (Calvez–Soroko [CS22, Proposition 4]) *Let G be a group with $Z(G) = 1$, $\varphi \in \mathrm{Aut}(G)$, and G_φ defined as above. Two elements $g, h \in G$ are φ-twisted conjugate in G if and only if the elements gt and ht of G_φ are conjugate in G_φ. In particular, $R(\varphi) = \infty$ if and only if the coset Gt in G_φ contains infinitely many conjugacy classes.* □

Step 3: Detect infinitely many conjugacy classes using Delzant's lemma:

In case when a group Γ acts by isometries in a non-elementary way on a Gromov hyperbolic space, the very useful lemma of Delzant allows us to detect infinitely many conjugacy classes under certain conditions, which will be satisfied in our case. We recall the relevant definitions.

Let Γ be a group acting by isometries on a Gromov hyperbolic space X. For any $x \in X$ consider the set $\Lambda(\Gamma)$ of accumulation points of the orbit Γx on the boundary ∂X. The action of Γ on X is called *non-elementary*, if $\Lambda(\Gamma)$ contains at least three points; see [Osi16, Section 3].

Delzant's Lemma A ([LL00, Lemma 3.4]) *Let Γ be a non-elementary hyperbolic group. Let K be a normal subgroup of Γ such that Γ/K is abelian. Then every coset of Γ/K contains infinitely many conjugacy classes.* □

Delzant's Lemma B ([FGD10, Lemma 6.3]) *Let Γ be a group acting non-elementarily by isometries on a Gromov hyperbolic space, and let K be a normal subgroup of Γ such that the quotient Γ/K is abelian. Then every coset of K contains infinitely many conjugacy classes.* □

Corollary 8 *Let G be a group with $Z(G) = 1$. If $\mathrm{Aut}(G)$ acts by isometries on a Gromov-hyperbolic space X in such a way that the action of the subgroup $G \simeq \mathrm{Inn}(G) \leqslant \mathrm{Aut}(G)$ is non-elementary, then G has property R_∞.*

Proof By Lemma 6, for any $\varphi \in \mathrm{Aut}(G)$ we have embeddings $G \leqslant G_\varphi \leqslant \mathrm{Aut}(G)$, which imply that G_φ also acts on X non-elementarily. We apply Delzant's Lemma to $\Gamma = G_\varphi$, $K = G$, and conclude that the coset Gt of G_φ contains infinitely many conjugacy classes. Lemma 7 now implies that G has property R_∞. □

3 A geometric representation of $\mathrm{Aut}(A[D_n])$

In this section, we describe an embedding of $\mathrm{Aut}(A[D_n])$ into the (extended) mapping class group of a suitable surface with marked points, and use it to prove Theorem 1.

First, we introduce some notation. We denote by $t_1, \ldots, t_n$ the standard generators of $A = A[D_n]$ corresponding to the numbering of vertices of the Coxeter graph in Figure 1. The element

$$\Delta = (t_1 \ldots t_{n-2}t_{n-1}t_nt_{n-2} \ldots t_1)(t_2 \ldots t_{n-2}t_{n-1}t_nt_{n-2} \ldots t_2) \ldots (t_{n-2}t_{n-1}t_nt_{n-2})(t_{n-1}t_n)$$

is the so-called Garside element of A (see [Par97b, Lemma 5.1]). If n is even, then $\Delta t_i \Delta^{-1} = t_i$ for all $1 \leqslant i \leqslant n$, and the center $Z(A)$ is generated by Δ. If n is odd, then $\Delta t_i \Delta^{-1} = t_i$ for all $1 \leqslant i \leqslant n-2$, $\Delta t_{n-1} \Delta^{-1} = t_n$, $\Delta t_n \Delta^{-1} = t_{n-1}$, and $Z(A)$ is generated by Δ^2, see [BS72, Satz 7.2].

We define automorphisms $\zeta, \chi \in \mathrm{Aut}(A[D_n])$ as:

$$\zeta(t_i) = t_i \quad \text{for } 1 \leqslant i \leqslant n-2, \quad \zeta(t_{n-1}) = t_n, \quad \zeta(t_n) = t_{n-1},$$
$$\chi(t_i) = t_i^{-1} \quad \text{for } 1 \leqslant i \leqslant n.$$

Clearly, ζ and χ commute and both have order 2, and hence they generate a subgroup of $\mathrm{Aut}(A[D_n])$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. If n is odd, then ζ is the conjugation automorphism by Δ. If n is even, then ζ is not inner, as can be seen from the proof of Satz 7.1 of [BS72]. On the other hand, the automorphism χ is never inner. (Indeed, consider the homomorphism $\xi\colon A[D_n] \to \mathbb{Z}$, $t_i \mapsto 1$ for all $1 \leqslant i \leqslant n$. For every inner automorphism ϕ, we have $\xi(\phi(t_i)) = \xi(t_i)$ for all i. For the automorphism χ we have: $\xi(\chi(t_i)) = -1 = -\xi(t_i)$ for all i.)

Denote for brevity $\overline{A[D_n]} = A[D_n]/Z(A[D_n])$ and let $\pi\colon A[D_n] \to \overline{A[D_n]}$ be the canonical projection. It is well known (and easy to prove) that $Z(\overline{A[D_n]}) = 1$. For each $1 \leqslant i \leqslant n$, we set $\bar{t}_i = \pi(t_i)$ and denote $\bar{\zeta}$, $\bar{\chi}$ the automorphisms of $\overline{A[D_n]}$ induced by ζ, χ, respectively. In [CP23], Castel and Paris obtained the following description of $\mathrm{Aut}(A[D_n])$ and $\mathrm{Aut}(\overline{A[D_n]})$.

Theorem 9 (Castel–Paris [CP23, Corollary 2.10]) *Let $n \geqslant 6$.*

(1) *If n is even, then*

$$\mathrm{Aut}(\overline{A[D_n]}) = \mathrm{Inn}(\overline{A[D_n]}) \rtimes \langle \bar{\zeta}, \bar{\chi} \rangle \simeq \overline{A[D_n]} \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}),$$

and $\mathrm{Out}(\overline{A[D_n]}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(2) *If n is odd, then*

$$\mathrm{Aut}(\overline{A[D_n]}) = \mathrm{Inn}(\overline{A[D_n]}) \rtimes \langle \bar{\chi} \rangle \simeq \overline{A[D_n]} \rtimes \mathbb{Z}/2\mathbb{Z},$$

and $\mathrm{Out}(\overline{A[D_n]}) \simeq \mathbb{Z}/2\mathbb{Z}$.

Moreover, for any $n \geqslant 6$ we have $\mathrm{Aut}(A[D_n]) \simeq \mathrm{Aut}(\overline{A[D_n]})$ *and* $\mathrm{Out}(A[D_n]) \simeq \mathrm{Out}(\overline{A[D_n]})$. □

We now recall the relevant notions related to mapping class groups of surfaces.

Let Σ be an orientable surface with or without boundary, and let $\mathcal{P}$ be a finite collection of different points in the interior of Σ. The *mapping class group* $\mathcal{M}(\Sigma, \mathcal{P})$ of the pair $(\Sigma, \mathcal{P})$ is the group of orientation-preserving homeomorphisms of Σ, identical on the boundary and permuting the set $\mathcal{P}$, considered up to isotopies identical on the boundary and fixing $\mathcal{P}$ pointwise. If we allow orientation-reversing homeomorphisms in the above definition, we get the notion of the *extended mapping class group* of the pair $(\Sigma, \mathcal{P})$, which is denoted $\mathcal{M}^*(\Sigma, \mathcal{P})$. If a surface Σ has nonempty boundary, then $\mathcal{M}^*(\Sigma, \mathcal{P}) = \mathcal{M}(\Sigma, \mathcal{P})$, otherwise $\mathcal{M}(\Sigma, \mathcal{P})$ is a subgroup of index 2 in $\mathcal{M}^*(\Sigma, \mathcal{P})$. The *pure mapping class group* of the pair $(\Sigma, \mathcal{P})$ is the finite index subgroup $\mathcal{PM}(\Sigma, \mathcal{P})$ of $\mathcal{M}(\Sigma, \mathcal{P})$ which fixes the set $\mathcal{P}$ pointwise. These groups fit into the short exact sequence:

$$1 \longrightarrow \mathcal{PM}(\Sigma, \mathcal{P}) \longrightarrow \mathcal{M}(\Sigma, \mathcal{P}) \longrightarrow \mathfrak{S}_N \longrightarrow 1,$$

which does not split in general. Here $N = |\mathcal{P}|$ is the cardinality of $\mathcal{P}$ and $\mathfrak{S}_N$ is the symmetric group on N letters. We refer the reader to [FM12] for more information on the mapping class groups.

Let $n \geqslant 3$ and Σ_n be the orientable surface without boundary of genus $\lfloor (n-1)/2 \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer which is less than or equal to x, and let $\mathcal{P}_n$ be a finite set of punctures in Σ_n, with $|\mathcal{P}_n| = 2$, if n is odd, and $|\mathcal{P}_n| = 3$, if n is even, see Figure 2.

Proposition 10 *Let $n = 4$ or $n \geqslant 6$. There exists an embedding $\overline{A[D_n]} \hookrightarrow \mathcal{M}(\Sigma_n, \mathcal{P}_n)$ such that every automorphism of $\overline{A[D_n]}$ is induced by a conjugation with an element of $\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)$. In particular, we have inclusions:*

$$\overline{A[D_n]} \leqslant \mathrm{Aut}(\overline{A[D_n]}) \leqslant \mathrm{Inn}\big(\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)\big) \simeq \mathcal{M}^*(\Sigma_n, \mathcal{P}_n).$$

Proof The case $n = 4$ is special in the sense that $\overline{A[D_4]}$ has an outer automorphism of order 3, in addition to $\bar{\zeta}$ and $\bar{\chi}$. It was proved in [Sor21, Corollary 6], that

$$\mathrm{Aut}(A[D_4]) \simeq \mathrm{Aut}(\overline{A[D_4]}) \simeq \overline{A[D_4]} \rtimes \big(\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}\big) \simeq \mathcal{M}^*(\Sigma_4, \mathcal{P}_4),$$

where $(\Sigma_4, \mathcal{P}_4)$ is the three-times punctured torus, and $\mathfrak{S}_3$ is the symmetric group on 3 letters, which is the group of graph automorphisms of the Coxeter graph of type D_4. So for the rest of the proof we assume that $n \geqslant 6$.

There is a well-known embedding $\rho\colon A[D_n] \hookrightarrow \mathcal{M}(\Sigma_n^\partial)$ of Perron and Vannier [PV96], where Σ_n^∂ is the surface with boundary obtained from $(\Sigma_n, \mathcal{P}_n)$ by making every puncture in $\mathcal{P}_n$ into a boundary

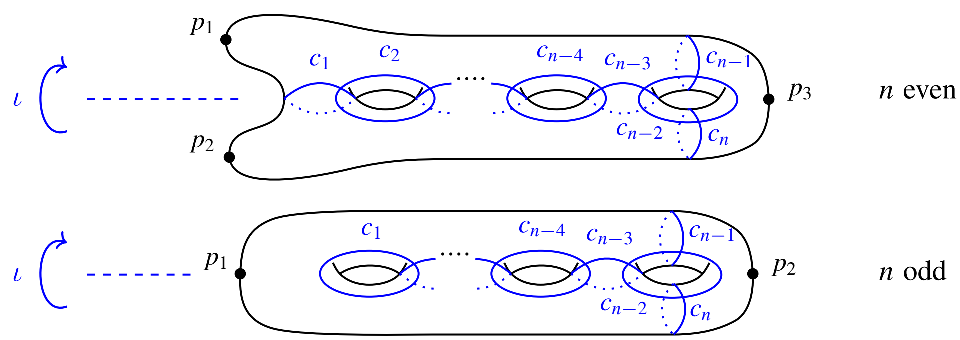

Figure 2: Surface $(\Sigma_n, \mathcal{P}_n)$ with $\mathcal{P}_n = \{p_1, p_2\}$ for n odd and $\mathcal{P}_n = \{p_1, p_2, p_3\}$ for n even. The image of the standard generator $\bar{t}_i$ of $\overline{A[D_n]}$ under the embedding of Proposition 10 is the Dehn twist about the circle c_i. The hyperelliptic involution ι rotates the surface by angle π in the ambient space.

component, see Figure 3. (A more streamlined proof for a similar embedding, for the surface obtained from $(\Sigma_n, \mathcal{P}_n)$ by making all but one punctures of $\mathcal{P}_n$ into boundary components, was recently given in [CP23, Theorem 3.6].)

On the other hand, the inclusion of surfaces $\Sigma_n^\partial \hookrightarrow \Sigma_n$ induces the homomorphism $\eta\colon \mathcal{M}(\Sigma_n^\partial) \to \mathcal{M}(\Sigma_n, \mathcal{P}_n)$ with kernel $\ker\eta \simeq \mathbb{Z}^N$ generated by boundary twists T_{∂_i}, where $N = |\mathcal{P}_n| \in \{2, 3\}$, see [FM12, Theorem 3.18]. Moreover, $\ker\eta$ lies in the center of $\mathcal{M}(\Sigma_n^\partial)$, since boundary twists commute with all mapping classes in $\mathcal{M}(\Sigma_n^\partial)$.

We claim that $\rho(A[D_n]) \cap \ker\eta = \rho(Z(A[D_n]))$. Indeed, since $\ker\eta$ is central in $\mathcal{M}(\Sigma_n^\partial)$, we have inclusion $\rho(A[D_n]) \cap \ker\eta \subseteq \rho(Z(A[D_n]))$. On the other hand, we know the image of the generator of the center $Z(A[D_n])$ under ρ by [LP01, Proposition 2.12] (see also [Mat00, Table 1]):

$$\rho(\Delta^2) = T_{\partial_1} T_{\partial_2}^{n-2}, \qquad \text{if } n \text{ is odd;}$$

$$\rho(\Delta) = T_{\partial_1} T_{\partial_2} T_{\partial_3}^{n/2-1}, \qquad \text{if } n \text{ is even.}$$

This shows that $\rho(Z(A[D_n]))$ lies in $\ker\eta = \langle T_{\partial_i} \rangle$, and we conclude that $\rho(A[D_n]) \cap \ker\eta = \rho(Z(A[D_n]))$.

Thus we can define an embedding $\bar{\rho}\colon \overline{A[D_n]} \hookrightarrow \mathcal{M}(\Sigma_n, \mathcal{P}_n)$ induced by ρ:

$$\bar{\rho}\colon \overline{A[D_n]} = A[D_n]/Z(A[D_n]) \longrightarrow \mathcal{M}(\Sigma_n^\partial)/\ker\eta = \mathcal{PM}(\Sigma_n, \mathcal{P}_n) \leqslant \mathcal{M}(\Sigma_n, \mathcal{P}_n).$$

We see that

$$\bar{\rho}(\bar{t}_i) = T_{c_i}, \quad \text{for all } 1 \leqslant i \leqslant n,$$

where T_{c_i} are the Dehn twists about the curves c_i shown in Figure 2. For the rest of the proof we identify $\overline{A[D_n]}$ with its image under $\bar{\rho}$ in $\mathcal{M}(\Sigma_n, \mathcal{P}_n)$.

Recall that $\mathrm{Aut}(\overline{A[D_n]})$ is generated by $\mathrm{Inn}(\overline{A[D_n]})$, $\bar{\zeta}$, and $\bar{\chi}$, by Theorem 9. Naturally, any inner automorphism of $\overline{A[D_n]} \leqslant \mathcal{M}(\Sigma_n, \mathcal{P}_n)$ is induced by the corresponding inner automorphism of the ambient group $\mathcal{M}(\Sigma_n, \mathcal{P}_n)$, hence we just need to realize $\bar{\zeta}$ and $\bar{\chi}$ as conjugations inside $\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)$.

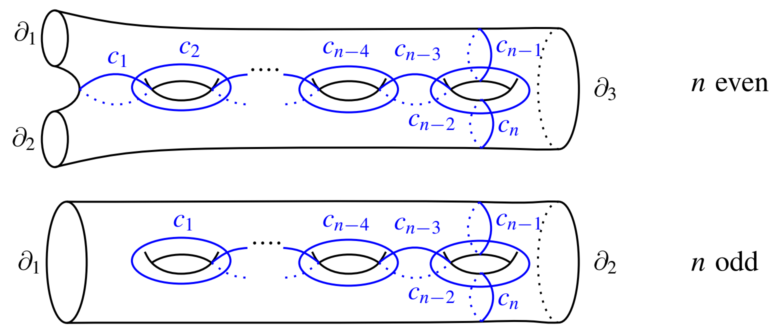

Figure 3: Surface Σ_n^∂ with the images of standard generators t_i of $A[D_n]$ under the Perron–Vannier embedding. (The image of t_i is the Dehn twist about the circle marked c_i.)

For automorphism $\bar{\zeta}$, consider the mapping class ι, which (viewed on the closed surface) is called *a hyperelliptic involution*, and which can be described as the rotation of the surface in the ambient space by π radian about the horizontal axis, see Figure 2. Clearly,

$$\begin{aligned} \iota \circ T_{c_i} \circ \iota = T_{\iota(c_i)} &= T_{c_i}, \qquad \text{for} \quad 1 \leqslant i \leqslant n-2, \quad \text{and} \\ \iota \circ T_{c_{n-1}} \circ \iota = T_{\iota(c_{n-1})} &= T_{c_n}, \\ \iota \circ T_{c_n} \circ \iota = T_{\iota(c_n)} &= T_{c_{n-1}}. \end{aligned}$$

(Here we treat each c_i as the isotopy class of unoriented curves containing the circle c_i drawn in Figure 2.) We see that, indeed, the conjugation by ι induces the same automorphism on $\overline{A[D_n]}$ as $\bar{\zeta}$.

If n is odd, then ι is equal to $\bar{\rho}(\pi(\Delta))$ as mapping classes in $\mathcal{M}(\Sigma_n, \mathcal{P}_n)$. Indeed, from the properties of Δ mentioned in the beginning of the section, it follows that conjugating $\overline{A[D_n]}$ by $\pi(\Delta)$ induces automorphism $\bar{\zeta}$. Thus, $\phi = \iota \circ \bar{\rho}(\pi(\Delta))$ leaves all curves c_i invariant (viewed as isotopy classes of unoriented curves). Now we notice that cutting the punctured surface $(\Sigma_n, \mathcal{P}_n)$ along the curves $\{c_i\}_{i=1}^n$ results in two once-punctured disks such that the homeomorphism of their boundary induced by ϕ is identity. Applying the Alexander method ([FM12, Proposition 2.8]), we conclude that $\phi = \mathrm{id}$, and hence $\iota = \bar{\rho}(\pi(\Delta))$. This proves that, for n odd, the conjugation by ι is an inner automorphism of $\bar{\rho}(\overline{A_n})$.

If n is even, then ι does not belong to $\bar{\rho}(\overline{A[D_n]})$, since ι interchanges punctures p_1 and p_2, whereas the image of $\bar{\rho}$ lies entirely in $\mathcal{PM}(\Sigma_n, \mathcal{P}_n)$, so conjugating by ι induces the outer automorphism $\bar{\zeta}$ of $\bar{\rho}(\overline{A[D_n]})$.

For automorphism $\bar{\chi}$, consider the orientation-reversing mapping class $X \in \mathcal{M}^*(\Sigma_n, \mathcal{P}_n)$ which can be described as the reflection of the surface Σ_n in the "plane of the picture" in Figure 2. Since X reverses orientation, we notice that

$$X \circ T_{c_i} \circ X = T^{-1}_{X(c_i)} = T^{-1}_{c_i}, \qquad \text{for} \ \ 1 \leqslant i \leqslant n.$$

This means that the conjugation by X induces automorphism $\bar{\chi}$ on $\bar{\rho}(\overline{A[D_n]})$. Since X reverses orientation, and all mapping classes from $\bar{\rho}(\overline{A[D_n]})$ preserve it, we see that the conjugation by X is an outer automorphism of $\bar{\rho}(\overline{A[D_n]})$, as expected.

We check by inspection that $X \circ \iota = \iota \circ X$, and that $\iota \circ X$ does not belong to $\bar{\rho}(\overline{A[D_n]})$ since it reverses orientation. This means that we have the following group isomorphisms:

$$\begin{aligned} &\operatorname{Aut}(\overline{A[D_n]}) \simeq \bar{\rho}(\overline{A[D_n]}) \rtimes \big(\langle \operatorname{conj}_\iota \rangle \times \langle \operatorname{conj}_X \rangle\big) \leqslant \operatorname{Inn}\big(\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)\big), \qquad \text{if } n \text{ is even},\\ &\operatorname{Aut}(\overline{A[D_n]}) \simeq \bar{\rho}(\overline{A[D_n]}) \rtimes \langle \operatorname{conj}_X \rangle \leqslant \operatorname{Inn}\big(\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)\big), \qquad \text{if } n \text{ is odd}. \end{aligned}$$

It remains to show the last isomorphism: $\operatorname{Inn}\big(\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)\big) \simeq \mathcal{M}^*(\Sigma_n, \mathcal{P}_n)$. Let z be a nontrivial central element in $\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)$. By the classification of centers of $\mathcal{M}(\Sigma, \mathcal{P})$ for various surfaces with punctures $(\Sigma, \mathcal{P})$ given in [FM12, p. 77], we conclude that z cannot belong to $\mathcal{M}(\Sigma_n, \mathcal{P}_n)$, if $n \geqslant 4$. Hence z reverses orientation. Since z commutes with every Dehn twist T_c, we have:

$$T_c = z\, T_c\, z^{-1} = T_{z(c)}^{-1},$$

which contradicts the property of Dehn twists:

$$T_a^k = T_b^\ell \quad \Longleftrightarrow \quad a = b \quad \text{and} \quad k = \ell,$$

see [FM12, p. 75]. Hence $Z\big(\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)\big) = 1$, and the result follows. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1 Our goal is to apply Corollary 8.

As was mentioned before, it is known that the center of $\overline{A[D_n]}$ is trivial. By Proposition 10, we have an embedding

$$\overline{A[D_n]} \leqslant \operatorname{Aut}(\overline{A[D_n]}) \leqslant \mathcal{M}^*(\Sigma_n, \mathcal{P}_n).$$

We use the well-known fact that $\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)$ acts by isometries on the curve complex $\mathcal{C} = \mathcal{C}(\Sigma_n, \mathcal{P}_n)$ of the surface. This complex is Gromov-hyperbolic by the result of Masur and Minsky [MM99, Theorem 1.1], if the surface has big enough complexity, i.e. if $3g + p - 4 > 0$, where g denotes the genus of the surface, and p the number of punctures. Recalling that Σ_n has genus $g = \lfloor (n-1)/2 \rfloor$, with $p = 2$ for n odd and $p = 3$ for n even, we see that the required inequality is satisfied already for $n \geqslant 3$ and a fortiori for $n \geqslant 6$.

We need to prove that the action of $\overline{A[D_n]}$ on $\mathcal{C}$ is non-elementary. By [MM99, Proposition 4.6], an element g of $\mathcal{M}^*(\Sigma_n, \mathcal{P}_n)$ acts loxodromically on $\mathcal{C}$ if and only if g is pseudo-Anosov. It will be sufficient to produce two pseudo-Anosov elements $g, h \in \overline{A[D_n]}$ which generate a rank 2 free group. To produce pseudo-Anosov elements we use Penner's construction [Pen88], [FM12, Theorem 14.4]:

Penner's Construction *Let $A = \{\alpha_1, \dots, \alpha_n\}$ and $B = \{\beta_1, \dots, \beta_m\}$ be multicurves in a surface Σ that together fill Σ. Any product of positive powers of the T_{α_i} and negative powers of the T_{β_i}, where each α_i and each β_i appear at least once, is pseudo-Anosov.* □

Recall that a collection of curves *fills* a surface Σ, if the surface obtained from Σ by cutting along all the curves from the collection, is a disjoint union of disks and once-punctured disks.

To choose multicurves A and B for application of Penner's Construction in the surface $(\Sigma_n, \mathcal{P}_n)$ we notice that the family of circles $\{c_i\}_{i=1}^n$ in Figure 2 cuts the surface Σ_n into two or three once-punctured disks, i.e. it fills the surface Σ_n. This allows us to set:

$$A = \{c_n, c_{n-1}, c_{n-3}, \dots, c_3, c_1\}, \quad B = \{c_{n-2}, c_{n-4}, \dots, c_4, c_2\}, \qquad \text{if } n \text{ is even, and}$$
$$A = \{c_n, c_{n-1}, c_{n-3}, \dots, c_4, c_2\}, \quad B = \{c_{n-2}, c_{n-4}, \dots, c_3, c_1\}, \qquad \text{if } n \text{ is odd.}$$

Next, we introduce elements

$$f_A = \prod_{c_i \in A} T_{c_i}^2, \qquad f_B = \prod_{c_j \in B} T_{c_j}^2,$$

and set

$$g = f_A^2 f_B^{-1}, \qquad h = f_A f_B^{-2}.$$

By Penner's Construction, g and h are pseudo-Anosov.

Now consider elements $Q_i = T_{c_i}^2$ as mapping classes on the surface Σ_n^∂, see Figure 3. They belong to the subgroup $A[D_n] \leqslant \mathcal{M}(\Sigma_n^\partial)$, by the Perron–Vannier embedding [PV96]. By the solution of Tits' conjecture by Crisp and Paris [CP01], we know that elements Q_i generate a right-angled Artin subgroup $K = \langle Q_i \mid 1 \leqslant i \leqslant n \rangle$ of $A[D_n]$, which has trivial center $Z(K) = 1$, see e.g. [Cha07, Section 2.3]. Thus, K maps isomorphically into $\mathcal{M}(\Sigma_n, \mathcal{P}_n)$ via $\eta\colon \mathcal{M}(\Sigma_n^\partial) \to \mathcal{M}(\Sigma_n, \mathcal{P}_n)$ (since $\ker \eta$ is central in $\mathcal{M}(\Sigma_n^\partial)$, see the proof of Proposition 10), and g, h belong to $\eta(K)$ by construction.

By the Centralizer Theorem of Servatius [Ser89, Cha07], we see that g and h do not commute. Hence, by the result of Baudisch [Bau81, Theorem 1.2], g and h generate a rank 2 free group. This shows that the action of $\overline{A[D_n]}$ on $\mathcal{C}$ is non-elementary.

Applying Corollary 8, we conclude that $\overline{A[D_n]}$ has property R_∞, and hence, by Lemma 5, $A[D_n]$ has property R_∞ as well. □

4 Results for large-type Artin groups

In this section Γ denotes the presentation graph of $A[\Gamma]$, not its Coxeter graph, see the definition below.

Let $A[\Gamma]$ be a large-type Artin group. Throughout this section we suppose that Γ has at least 3 vertices, ensuring that $A[\Gamma]$ is non-spherical. As is customary when working with general (i.e. not spherical and not affine) Artin groups, we consider Γ to be the presentation graph of the Artin group (as opposed to its Coxeter graph), which is defined as follows. Let S be a standard generating set for the Artin group. Then the *presentation graph* associated with S is the graph Γ whose vertex set $V(\Gamma)$ is S, and for which there is an edge with label m_{st} between s and t if and only if $m_{st} \neq \infty$. (I.e. in the presentation graph, two vertices s, t with $m_{st} = 2$ are connected with an edge labeled 2, and two vertices s, t with $m_{st} = \infty$ are disconnected, whereas in the Coxeter graph the situation is reversed: vertices s, t with $m_{st} = 2$ are disconnected, but the ones with $m_{st} = \infty$ are connected with an edge labeled ∞.)

We consider the subgroup $\mathrm{Aut}_\Gamma(A[\Gamma])$ of $A[\Gamma]$ generated by the following automorphisms:

- *inner automorphisms*, i.e. conjugations of the form $\mathrm{conj}_h \colon g \mapsto hgh^{-1}$;
- *graph-induced automorphisms*, i.e. automorphisms σ that consist of a permutation of the standard generators induced by a label-preserving graph automorphism of Γ;
- *the global inversion*, i.e. the automorphism χ that sends every standard generator to its inverse.

Let $A[\Gamma]$ be an Artin group. It is known by [Lek83] that for every full subgraph $\Gamma' \subseteq \Gamma$, the subgroup generated by the vertices of Γ' is isomorphic to the Artin group $A[\Gamma']$. Such subgroups are called *standard parabolic subgroups*.

Definition 11 (Charney–Davis [CD95]) Let $A[\Gamma]$ be any Artin group. The *Deligne complex* D_Γ is the simplicial complex defined as follows:

- The vertex set of D_Γ is the set left cosets of the form $gA[\Gamma']$ where $g \in A[\Gamma]$ and $A[\Gamma']$ is a spherical standard parabolic subgroup of $A[\Gamma']$;
- For every string of strict inclusion of the form $g_0A[\Gamma_0] \subsetneq \cdots \subsetneq g_nA[\Gamma_n]$, we put an n-simplex between the associated n vertices.

The group $A[\Gamma]$ acts on D_Γ by left-multiplication, and this action is by simplicial isomorphisms.

It is a classical result for Coxeter group that if $W[\Gamma]$ is large-type and Γ has at least three vertices then $W[\Gamma]$ is infinite. In particular, if $A[\Gamma]$ is large-type then for any induced subgraph $\Gamma' \subseteq \Gamma$ with at least three vertices the subgroup $A[\Gamma']$ is non-spherical. Consequently, a spherical standard parabolic subgroup $A[\Gamma']$ of a large-type Artin group $A[\Gamma]$ necessarily satisfies $|V(\Gamma')| \leqslant 2$.

In particular, Charney–Davis prove that the associated Deligne complex D_Γ is a 2-dimensional simplicial complex which, when endowed with the *Moussong metric*, becomes a CAT(0) metric space, see [CD95, p. 623].

One can naturally construct an action of $\mathrm{Inn}(A[\Gamma])$ on D_Γ by declaring that the inner automorphism conj_g acts as the element g. One can actually extend this to a compatible action of $\mathrm{Aut}_\Gamma(A[\Gamma])$ on D_Γ as follows:

Definition 12 (Jones–Vaskou [JV24, Definition 2.13]) The group $\mathrm{Aut}_\Gamma(A_\Gamma)$ acts by simplicial isomorphisms on D_Γ as follows. Let $gA[\Gamma']$ be a vertex of D_Γ, where $g \in A_\Gamma$ and $\Gamma' \subseteq \Gamma$ is an induced subgraph. Then:

- (inner automorphisms) $\mathrm{conj}_h \cdot gA[\Gamma'] := hgA[\Gamma']$;
- (graph automorphisms) $\sigma \cdot gA[\Gamma'] := \sigma(g)A[\sigma(\Gamma')]$;
- (global inversion) $\chi \cdot gA[\Gamma'] := \chi(g)A[\Gamma']$.

The following properties of the Deligne complex are known to experts, however they do not seem to be explicitly stated in the literature, so we provide clear statements and proofs here.

Lemma 13 *Let $A[\Gamma]$ be a large-type Artin group, and let $f \colon D_\Gamma \to D_\Gamma$ be a simplicial automorphism of D_Γ. Then:*

(1) *For every vertex* $gA[\Gamma']$*, the image* $f(gA[\Gamma']) = g'A[\Gamma'']$ *satisfies* $|V(\Gamma')| = |V(\Gamma'')|$*;*

(2) f *is a simplicial isometry, i.e. it sends simplices to simplices isometrically;*

(3) f *is an isometry.*

Proof (1) Recall that for a simplicial complex X, the *link* of a vertex v is the subcomplex $\mathrm{lk}(v) \subseteq X$ obtained as the union of all the simplices that are disjoint from v but which belong to simplices that contain v. Also recall that because $A[\Gamma]$ is large-type, any spherical standard parabolic subgroup $A[\Gamma']$ satisfies $|V(\Gamma')| \leqslant 2$. The first statement directly follows from the fact that the link $\mathrm{lk}(gA[\Gamma'])$ is:

- finite, if and only if $|V(\Gamma')| = 0$;
- infinite but bounded, if and only if $|V(\Gamma')| = 1$;
- unbounded, if and only if $|V(\Gamma')| = 2$,

where the above follows from [Vas22, Remark 3.4, Proposition E].

(2) It is enough to prove the second statement for 2-simplices, as the other types of simplices isometrically embed in them. If Δ is such a simplex, we notice that Δ must have the form $\Delta = \big(g_0A[\Gamma_0], g_1A[\Gamma_1], g_2A[\Gamma_2]\big)$, where for each $i \in \{0, 1, 2\}$ we have $|V(\Gamma_i)| = i$. Recall that the Moussong metric d is defined on Δ by letting (Δ, d) be isometric to the only Euclidean triangle satisfying:

$$d\big(g_0A[\Gamma_0], g_1A[\Gamma_1]\big) = 1, \quad \angle_{g_1A[\Gamma_1]}\big(g_0A[\Gamma_0], g_2A[\Gamma_2]\big) = \frac{\pi}{2}, \quad \angle_{g_2A[\Gamma_2]}\big(g_0A[\Gamma_0], g_1A[\Gamma_1]\big) = \frac{\pi}{2m},$$

where m is the coefficient of the dihedral Artin group $A[\Gamma_2]$, see [CD95].

The map f is a simplicial automorphism, so the image $f(\Delta)$ is a 2-simplex. As for Δ, it takes the form $f(\Delta) = \big(g'_0A[\Gamma'_0], g'_1A[\Gamma'_1], g'_2A[\Gamma'_2]\big)$, where $|V(\Gamma'_i)| = i$ for $i \in \{0, 1, 2\}$. By (1), we know that $f(g_iA[\Gamma_i]) = g'_iA[\Gamma'_i]$ for $i \in \{0, 1, 2\}$. In particular, by construction of d we have

$$d\big(g'_0A[\Gamma'_0], g'_1A[\Gamma'_1]\big) = 1, \quad \angle_{g'_1A[\Gamma'_1]}\big(g'_0A[\Gamma'_0], g'_2A[\Gamma'_2]\big) = \frac{\pi}{2}, \quad \angle_{g'_2A[\Gamma'_2]}\big(g'_0A[\Gamma'_0], g'_1A[\Gamma'_1]\big) = \frac{\pi}{2m'},$$

where m' is the coefficient of the dihedral Artin group $A[\Gamma'_2]$. So all that is left to show is that $m = m'$. For a graph G, let us denote by $sys(G)$ the *systole* of G, that is the smallest simplicial length of a non-contractible loop in G (also known as the girth of G). Since f acts by simplicial automorphisms, it is clear that $sys(\mathrm{lk}(g_2A[\Gamma_2])) = sys(\mathrm{lk}(g'_2A[\Gamma'_2]))$. We will show that $sys(\mathrm{lk}(g_2A[\Gamma_2])) = 4m$, and that, similarly, $sys(\mathrm{lk}(g'_2A[\Gamma'_2])) = 4m'$. Together, this will imply that $m = m'$.

Let $V(\Gamma_2) = \{a, b\}$. It was proved in [Vas22, Lemma 4.2]) that every non-backtracking loop γ in $\mathrm{lk}(g_2A[\Gamma_2])$ corresponds to a reduced word w in $\{a, b\}$ such that:

- The word w projects to the trivial element in the dihedral Artin group $\langle a, b\rangle$;
- Without loss of generality we can write $w = a^{n_1}b^{n_2}\dots x^{n_k}$ for some $x \in \{a, b\}$ and $k \geqslant 1$;
- The simplicial length $\ell(\gamma)$ of γ is equal to the integer $2k$.

It was also proved in [AS83, Lemma 6] that we have $k \geqslant 2m$. Moreover, this lower bound is reached (for instance, take w to be the relator of the dihedral Artin group $\langle a, b\rangle$). It follows that $sys(\mathrm{lk}(g_2A[\Gamma_2])) = 4m$, as desired.

(3) We now prove the third statement. Let x and y be any two points (not necessarily vertices) in D_Γ. Recall that, by definition, the Moussong metric d can be extended from simplices to the whole of D_Γ by letting $d(x,y)$ be the infimum of the lengths of the paths connecting x and y, where the length of a path γ is computed as the sum of the lengths of the sub-paths $\{\gamma_i\}_{i\in I}$ obtained by restricting γ to the various simplices $\{\Delta_i\}_{i\in I}$ it travels through. The fact that (D_Γ, d) is a CAT(0) metric space ensures that there is a unique geodesic $\gamma^{x,y}$ connecting x and y (in particular, the above infimum is always attained). Let us decompose $\gamma^{x,y} = \bigcup_{i\in I} \gamma_i^{x,y}$ as before. The map $f\colon D_\Gamma \to D_\Gamma$ is a simplicial isometry, so the image $f(\gamma^{x,y}) = \bigcup_{i\in I} f(\gamma_i^{x,y})$ is a path between $f(x)$ and $f(y)$ for which the length of $f(\gamma_i^{x,y})$ is the same as the length of $\gamma_i^{x,y}$ for all $i\in I$. It follows that $f(\gamma^{x,y})$ is a path of length $d(x,y)$ between $f(x)$ and $f(y)$, which shows that $d(f(x),f(y)) \leqslant d(x,y)$. Since f is bijective, the same argument works for f^{-1}, so that $d(x,y) \leqslant d(f(x),f(y))$. This yields $d(f(x),f(y)) = d(x,y)$ for all $x,y\in D_\Gamma$, i.e. f is an isometry. □

Recall that an Artin group $A[\Gamma]$ is said to be hyperbolic-type, if its Coxeter group $W[\Gamma]$ is Gromov-hyperbolic, and for the large-type Artin groups the requirement to be hyperbolic-type is equivalent to the absence of triangles in Γ with all edge labels 3.

Theorem 14 *Let $A[\Gamma]$ be an Artin group of large and hyperbolic type, and suppose that* $\mathrm{Aut}(A[\Gamma]) = \mathrm{Aut}_\Gamma(A[\Gamma])$. *Then $A[\Gamma]$ has property R_∞.*

Proof If Γ has 2 vertices, then $A[\Gamma]$ is a dihedral Artin group, which has property R_∞ by [CS22, Theorem 1]. So we assume that Γ has at least 3 vertices.

Since $A[\Gamma]$ is of large and hyperbolic type, the Deligne complex D_Γ (endowed with the Moussong metric) is Gromov-hyperbolic by [Cri05, Lemma 5]. By Lemma 13 we also know that $\mathrm{Aut}(A[\Gamma])$ acts on D_Γ by isometries. Also note that the group $A[\Gamma]$ has trivial center. This fact can be distilled from the work of Godelle [God07]; an explicit proof can be found in [Vas22, Corollary C].

Next we are going to show that the action of $A[\Gamma]$ on D_Γ is non-elementary. In order to do this, we choose two elements $x,y\in A[\Gamma]$ acting elliptically on D_Γ with disjoint fixed-point sets. In what follows, we denote by $\mathrm{Fix}(g)$ the fixed-point set of an element g acting on D_Γ. We split the argument in two cases.

<u>Case 1: Γ is connected</u>. There are three standard generators $a,b,c\in V(\Gamma)$ satisfying $m_{ab}, m_{ac} < \infty$, and we pick $x = z_{ab}$ and $y = z_{ac}$, where z_{st} denotes an element generating the center of $\langle s,t\rangle$. It follows from [Cri05, Lemma 8] that the $\mathrm{Fix}(x)$ is a single vertex: it is the coset that corresponds to the standard parabolic subgroup generated by a and b. The same applies to y with the generators a and c. In particular, $\mathrm{Fix}(x)$ and $\mathrm{Fix}(y)$ are disjoint.

<u>Case 2: Γ is not connected</u>. Then there are two standard generators $a,b\in V(\Gamma)$ that lie in distinct connected components of Γ. It follows from [Cri05, Lemma 8] that the fixed-point sets $\mathrm{Fix}(a)$ and $\mathrm{Fix}(b)$ are convex trees in D_Γ (often called *standard trees*). It is a standard result that $\mathrm{Fix}(a)$ and $\mathrm{Fix}(b)$ do not intersect in that case. Indeed, suppose that they do. By [HMS24, Corollary 2.18] $\mathrm{Fix}(a)\cap\mathrm{Fix}(b)$ must be a single vertex v. We consider the triangle T with vertices $\langle a\rangle$, $\langle b\rangle$ and v. Let us denote by K_Γ

the fundamental domain of the action of $A[\Gamma]$ on D_Γ. Note that K_Γ is convex, so the geodesic between $\langle a\rangle$ and $\langle b\rangle$ lies inside K_Γ. Because a and b lie in different connected components of Γ, the vertex corresponding to the coset $\{1\}$ disconnects K_Γ, and $\langle a\rangle$ and $\langle b\rangle$ lie in different connected components of $K_\Gamma \setminus \{1\}$. Consequently, $\{1\}$ lies on the geodesic from $\langle a\rangle$ to $\langle b\rangle$. It follows that

$$\angle_{\langle a\rangle}\big(\langle b\rangle, v\big) = \angle_{\langle a\rangle}\big(\{1\}, v\big).$$

The geodesic from $\langle a\rangle$ to v is contained in the standard tree Fix(a), the above angle is exactly $\pi/2$, by construction. The same applies to the angle $\angle_{\langle b\rangle}\big(\langle a\rangle, v\big)$. Because Fix($a$) and Fix($b$) intersect at a single point, it also follows that $\angle_v\big(\langle a\rangle, \langle b\rangle\big) > 0$. Altogether, this shows that the sum of the angles in T is strictly more than π, which contradicts [BH99, Chapter II, Exercise 2.12 (1)]. This contradiction shows that Fix(a) and Fix(b) are disjoint, and we can set $x = a$, $y = b$ in Case 2.

So now we have constructed two elliptic elements $x, y \in A[\Gamma]$ with disjoint fixed-point sets. By [Mar24, Proposition C], there is an integer $n > 0$ such that $H = \langle x^n, y^n\rangle$ is a non-abelian free group, and, moreover, an element $g \in H$ acts loxodromically on D_Γ if and only if g is not conjugate in $A[\Gamma]$ to a power of x^n or y^n. We can exhibit a pair of such elements in H, for example, $g = x^n y^n$ and $h = y^n x^n$, by [Mar24, Remark 3.3]. Since g and h do not commute, they generate a rank 2 free subgroup of H, and hence they act as independent loxodromic elements on D_Γ. This proves that the action of $A[\Gamma]$ on D_Γ is non-elementary.

Now we apply Corollary 8 and conclude that $A[\Gamma]$ has property R_∞. □

There has been recent progress on determining the automorphism group of Artin groups, and in particular on finding classes of Artin groups for which the automorphism group Aut($A[\Gamma]$) is as small as $\mathrm{Aut}_\Gamma(A[\Gamma])$. A first result is the following:

Theorem 15 (Vaskou [Vas25, Theorem A]) *Let $A[\Gamma]$ be a large-type free-of-infinity Artin group of rank at least 3. Then $\mathrm{Aut}(A[\Gamma]) = \mathrm{Aut}_\Gamma(A[\Gamma])$.* □

In particular, the known property R_∞ for dihedral Artin groups [CS22, Theorem 1] and Theorem 14 yield the following:

Corollary 16 *(=Theorem 3) Let $A[\Gamma]$ be a free-of-infinity Artin group of large and hyperbolic type. Then $A[\Gamma]$ has property R_∞.* □

Theorem 15 was recently extended to allow for some graphs that are not necessarily free-of-infinity. We introduce a few relevant definitions.

An Artin group $A[\Gamma]$ is said to be of *XXXL type* if every coefficient m_{st} is $\geqslant 6$. A graph Γ is called *twistless* if it is connected and has no cut-vertex and no separating edge. (We call an edge e between two vertices s, t of a connected graph Γ *separating* if the graph $\Gamma \setminus \{s, e, t\}$ obtained from Γ by removing the edge e and both vertices s, t incident to it, is disconnected.)

Theorem 17 (Blufstein–Martin–Vaskou [BMV24, Corollary 1.7]) *Let $A[\Gamma]$ be an Artin group of XXXL-type with Γ a twistless graph of rank at least 3. Then $\mathrm{Aut}(A[\Gamma]) = \mathrm{Aut}_\Gamma(A[\Gamma])$.* □

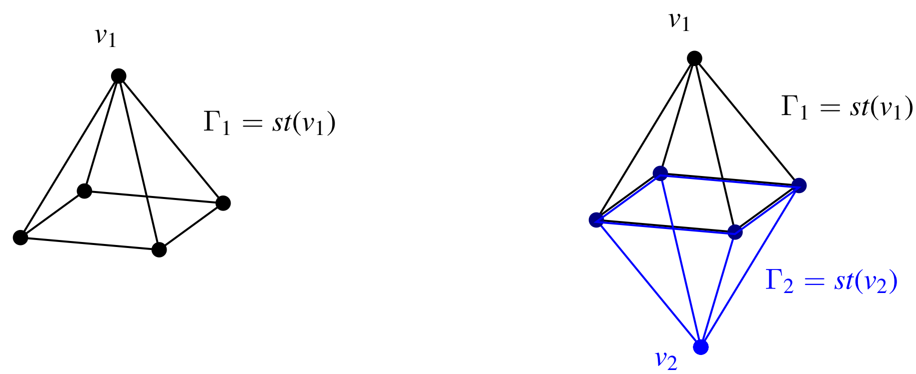

Figure 4: The graph Γ_1 is a twistless star, and $\Gamma = \Gamma_1 \cup \Gamma_2$ is a twistless hierarchy terminating in twistless stars (here $st(v_i)$ denotes the star of vertex v_i).

We immediately obtain the following corollary.

Corollary 18 *(=Theorem 4, part 1) Let $A[\Gamma]$ be an Artin group of XXXL-type with Γ a twistless graph. Then $A[\Gamma]$ has property R_∞.* □

We now come to the second extension of Theorem 15.

An *admissible decomposition* of a graph Γ is a pair of induced subgraphs Γ_1 and Γ_2 or Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ (in the sense that $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$ for the respective sets of vertices and edges). We say that an admissible decomposition is *twistless* if $\Gamma_1 \cap \Gamma_2$ is not empty, and is not a single vertex nor a single edge. Notice that a graph Γ is twistless if and only if every admissible decomposition of Γ is twistless.

We say that a graph Γ is a *twistless star* if Γ is the star of a vertex $v \in \Gamma$ and Γ is twistless. Note that complete graphs are examples of twistless stars.

We say that Γ is has a *twistless hierarchy terminating in twistless stars* if it is possible to start from Γ and perform finitely many successive admissible decompositions until we end up with a collection of graphs that are twistless stars. As examples of such graphs one can take 1-skeleta of triangulations of surfaces, see [HOV24, Remark 5.12].

In Figure 4 the graph Γ_1 is a twistless star and the graph $\Gamma = \Gamma_1 \cup \Gamma_2$, which is the 1-skeleton of an octahedron, is an example of a twistless hierarchy terminating in twistless stars, since it is obtained by identifying two copies of Γ_1 over a 4-cycle.

Theorem 19 (Huang–Osajda–Vaskou [HOV24, Theorem 1.7]) *Let $A[\Gamma]$ be a large-type Artin group and suppose that Γ admits a twistless hierarchy terminating in twistless stars. Then $\mathrm{Aut}(A[\Gamma]) = \mathrm{Aut}_\Gamma(A[\Gamma])$.* □

Corollary 20 *(=Theorem 4, part 2) Let $A[\Gamma]$ be an Artin group of large and hyperbolic type, and suppose that Γ admits a twistless hierarchy terminating in twistless stars. Then $A[\Gamma]$ has property R_∞.* □

Remark 21 In [Juh12], Juhász studied the following subclasses of large-type Artin groups:

- *extra-large-type* Artin groups, i.e. such Artin groups for which $m_{st} \geqslant 4$ for all $s, t \in S$, and
- the so-called *CLTTF* Artin groups, which were first introduced in [Cri05], and are defined as large-type Artin groups for which the presentation graph Γ is connected and has no triangles.

For these Artin groups Juhász proved that if ϕ is a length-preserving automorphism, then $R(\phi) = \infty$. (An automorphism is called *length-preserving*, if the word length of $\phi(g)$ with respect to a given generating set of a group is equal to the word length of g, for each element g of the group.) In particular, if $\mathrm{Aut}(A[\Gamma]) = \mathrm{Aut}_\Gamma(A[\Gamma])$, then each $\phi \in \mathrm{Out}(A[\Gamma])$ has a length-preserving representative in $\mathrm{Aut}(A[\Gamma])$. On the other hand, Lemma 2.1 of [FGD10] implies that if $R(\phi_\alpha) = \infty$ for all representatives $\{\phi_\alpha\}_{\alpha \in \mathrm{Out}(G)} \subseteq \mathrm{Aut}(G)$ of $\mathrm{Out}(G)$, then G has property R_∞. Combining this with our knowledge of automorphism groups given in Theorems 15 and 17 and the above-cited results of Juhász, we get another proof that the following classes of Artin groups have property R_∞:

(1) extra-large-type free-of-infinity Artin groups;

(2) XXXL-type Artin groups with Γ twistless.

Notice that these classes are subsumed in Corollaries 16 and 18. Remarkably, Juhász was using techniques of small cancellation, which capture the non-positive geometry of the presentation complex of a group. Our Theorem 14 uses action of groups on Gromov-hyperbolic spaces instead.

5 Some conjectures

A careful analysis of known groups that have property R_∞ and those that do not have it, reveals an interesting fact: all known finitely generated residually finite groups without property R_∞ are solvable-by-finite. Fel'shtyn and Troitsky expressed this observation as the following optimistic conjecture [FT15].

Conjecture (R) *A finitely generated residually finite group either has property R_∞ or is solvable-by-finite.*

Note that the conjunction 'or' in this statement is not exclusive, since there are many groups known to have property R_∞ while being solvable-by-finite (e.g. the free nilpotent groups $N_{r,c}$ of rank r and nilpotency class $c \geqslant 2r$ [DG14]). In a recent paper [Tro24] this conjecture was proved for all groups with finite upper (Prüfer) rank.

Since solvable-by-finite groups are amenable, a weaker variant of Conjecture (R) would be

Conjecture (NA) *A finitely generated residually finite non-amenable group has property R_∞.*

It was first stated as a theorem in the preprint [FT12], but according to Evgenij Troitsky [Tro24a], a mistake was found in the proof, and this statement is still a conjecture.

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