Endomorphisms of Artin groups of type $\tilde{A}_n$

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Abstract We determine a classification of the endomorphisms of the Artin group of affine type $\tilde{A}_n$ for $n \geq 4$.

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1 Introduction

Let S be a finite set. A *Coxeter matrix* over S is a square matrix $M = (m_{s,t})_{s,t \in S}$ indexed by the elements of S, with coefficients in $\mathbb{N} \cup \{\infty\}$, such that $m_{s,s} = 1$ for all $s \in S$, and $m_{s,t} = m_{t,s} \geq 2$ for all $s, t \in S$, $s \neq t$. Such a matrix is usually represented by a labeled graph, Γ, called a *Coxeter graph*, defined by the following data. The set of vertices of Γ is S. Two vertices $s, t \in S$ are connected by an edge if $m_{s,t} \geq 3$, and this edge is labeled with $m_{s,t}$ if $m_{s,t} \geq 4$.

If a, b are two letters and m is an integer ≥ 2, then we denote by $\Pi(a, b, m)$ the alternating word $aba\dots$ of length m. In other words, $\Pi(a, b, m) = (ab)^{\frac{m}{2}}$ if m is even, and $\Pi(a, b, m) = (ab)^{\frac{m-1}{2}}a$ if m is odd. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. With Γ we associate a group, $A[\Gamma]$, called the *Artin group* of Γ, defined by the presentation

$$A[\Gamma] = \langle S \mid \Pi(s, t, m_{s,t}) = \Pi(t, s, m_{s,t}) \text{ for } s, t \in S\,,\ s \neq t\,,\ m_{s,t} \neq \infty \rangle\,.$$

The *Coxeter group* of Γ, denoted by $W[\Gamma]$, is the quotient of $A[\Gamma]$ by the relations $s^2 = 1$, $s \in S$.

Despite the popularity of Artin groups, little is known about their automorphisms and even less about their endomorphisms. The most studied cases are the braid groups, corresponding to the Coxeter graphs A_n ($n \geq 1$), and the right-angled Artin groups. The automorphism group of $A[A_n]$ was determined in [DG81] and the set of its endomorphisms in [Cas09] for $n \geq 5$, in [CKM19] for $n \geq 4$ and in [Ore24] for $n \geq 2$ (see also [BM06, KM22]). On the other hand, there are many papers dealing with automorphism groups of right-angled Artin groups (see [BCV23, CV09, CV11, Day09, Day11, Lau95] for example), but little is known about their endomorphisms.

Apart from these two classes, the Artin groups for which the automorphism group is determined are the 2-generator Artin groups [GHMR00], some 2-dimensional Artin groups [Cri05, AC23], the large-type free-of-infinity Artin groups [Vas23], the Artin groups of type B_n, $\tilde{A}_n$ and $\tilde{C}_n$ [CC05], the Artin group of type D_4 [Sor21], and the Artin groups of type D_n for $n \geq 6$ [CP23]. On the other hand, apart from Artin groups of type A_n, the set of endomorphisms is determined only for Artin groups of type D_n for $n \geq 6$ [CP23].

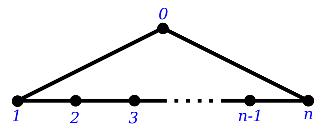

Figure 1.1: Coxeter graph $\tilde{A}_n$, $n \geq 2$

The aim of the present paper is to determine a classification of the endomorphisms of the Artin group of type $\tilde{A}_n$ for $n \geq 4$, where $\tilde{A}_n$ is the affine Coxeter graph depicted in Figure 1.1 (see Theorem 2.1).

A group G is called *Hopfian* if every surjective homomorphism $\psi\colon G \to G$ is injective, and it is called *co-Hopfian* if every injective homomorphism $\psi\colon G \to G$ is surjective. An easy consequence of our main theorem is that $A[\tilde{A}_n]$ is both Hopfian and co-Hopfian (see Corollary 2.5). The fact that all finite index subgroups (including $A[\tilde{A}_n]$) of the mapping class group of the punctured sphere are co-Hopfian was proved in [BM07, Corollary 3]. It is also known that the mapping class group of a compact connected orientable surface of genus $g \geq 2$ is co-Hopfian [IM99], and that a virtual braid group is co-Hopfian [BP20]. On the other hand, spherical type Artin groups are never co-Hopfian, since they admit transvection endomorphisms which are injective but not surjective, see [BM06], but some of them (of types A_n, B_n, D_n) are known to be almost co-Hopfian, i.e. their quotients by the center are co-Hopfian [BM06, BM07, CP23].

The paper is organized as follows. In Section 2 we give definitions and precise statements of our results. Section 3 contains preliminaries and Section 4 contains the proofs.

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2 Definitions and statements

Two other Coxeter graphs play an important role in our study: the Coxeter graphs A_n and B_n depicted in Figure 2.1.

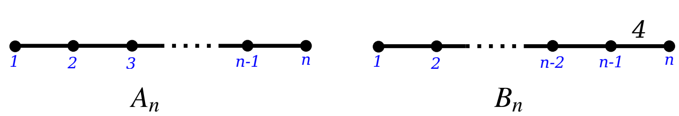

Figure 2.1: Coxeter graphs A_n and B_n

The standard generators of $A[A_n]$ will be denoted by $s_1, \dots, s_n$, the standard generators of $A[B_n]$ will be denoted by $r_1, \dots, r_n$, and the standard generators of $A[\tilde{A}_n]$ will be denoted by $t_0, t_1, \dots, t_n$.

Let Γ be a Coxeter graph. For $X \subset S$ we denote by Γ_X the full subgraph of Γ spanned by X, by $A_X[\Gamma]$ the subgroup of $A[\Gamma]$ generated by X, and by $W_X[\Gamma]$ the subgroup of $W[\Gamma]$ generated by X. We know by [Lek83] that $A_X[\Gamma]$ is naturally isomorphic to $A[\Gamma_X]$, and we know by [Bou68, Chapter 4, Section 1.8, Theorem 2(i)] that $W_X[\Gamma]$ is naturally isomorphic to $W[\Gamma_X]$. A subgroup of the form $A_X[\Gamma]$ is called a *standard parabolic subgroup* of $A[\Gamma]$, and a subgroup of the form $W_X[\Gamma]$ is called a *standard parabolic subgroup* of $W[\Gamma]$.

The word length of an element w in $W[\Gamma]$ with respect to S is denoted by $\lg(w)$. A *reduced expression* of w is a word $s_1 s_2 \dots s_\ell$ over S representing w such that $\ell = \lg(w)$. We denote by $\omega\colon A[\Gamma] \to W[\Gamma]$ the natural epimorphism which sends s to s for all $s \in S$. This epimorphism admits a natural set-section $\tau\colon W[\Gamma] \to A[\Gamma]$ defined as follows. Let $w \in W[\Gamma]$, and let $s_1 s_2 \dots s_\ell$ be a reduced expression of w. Then $\tau(w)$ is the element of $A[\Gamma]$ represented by $s_1 s_2 \dots s_\ell$. By [Tit69] the definition of $\tau(w)$ does not depend on the choice of the reduced expression.

We say that Γ is of *spherical type* if $W[\Gamma]$ is finite. In this case $W[\Gamma]$ has a unique element of maximal length, denoted by w_0, and this element satisfies $w_0^2 = 1$ and $w_0 S w_0 = S$ (see [Dav08, Lemma 4.6.1]). If Γ is of spherical type, then the *Garside element* of $A[\Gamma]$, denoted by $\Delta = \Delta[\Gamma]$, is defined to be $\tau(w_0)$. More generally, if Γ is any Coxeter graph and $X \subset S$ is such that Γ_X is of spherical type, then we denote by $\Delta_X = \Delta_X[\Gamma]$ the Garside element of $A_X[\Gamma] = A[\Gamma_X] \subset A[\Gamma]$. If Γ is connected and of spherical type, then the center of $A[\Gamma]$, denoted by $Z(A[\Gamma])$, is an infinite cyclic group generated by either Δ or Δ^2 (see [BS72]). If Γ is connected and not of spherical type, then it is conjectured that $A[\Gamma]$ has trivial center. This conjecture is proved in many cases but remains open in the whole generality.

The Coxeter graphs B_n and A_n are of spherical type, while the Coxeter graph $\tilde{A}_n$ is not of spherical type (it is of *affine* type). We also know that the center of $A[\tilde{A}_n]$ is trivial (see [CP03, Proposition 1.3]). The Garside element of $A[A_n]$ is

$$\Delta = \Delta[A_n] = (s_1 s_2 \dots s_n)(s_1 s_2 \dots s_{n-1}) \dots (s_1 s_2) s_1 \,,$$

see [KT08, Theorem 1.24]; we have $\Delta s_i \Delta^{-1} = s_{n+1-i}$ for all $1 \le i \le n$, and $Z(A[A_n])$ is generated by Δ^2. The Garside element of $A[B_n]$ is

$$\Delta[B_n] = (r_1 \dots r_n)^n \,,$$

and it generates the center of $A[B_n]$, see [BS72, Satz 7.2] and [Bou68, Chapter VI, Section 4, n°5, (III)].

If G is a group and $g \in G$, then we denote by $\mathrm{conj}_g\colon G \to G$, $h \mapsto g h g^{-1}$, the conjugation by g. We have a homomorphism $\mathrm{conj}\colon G \to \mathrm{Aut}(G)$, $g \mapsto \mathrm{conj}_g$, whose image is the group $\mathrm{Inn}(G)$ of *inner automorphisms*, and whose kernel is the center of G. The group $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, and the quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is the *outer automorphism group* of G. Two homomorphisms $\varphi_1, \varphi_2\colon G \to H$ are said to be *conjugate* if there exists $h \in H$ such that $\varphi_2 = \mathrm{conj}_h \circ \varphi_1$. Our classification of endomorphisms of $A[\tilde{A}_n]$ will be made up to conjugation.

Three automorphisms $\zeta, \eta, \mu \in \mathrm{Aut}(A[\tilde{A}_n])$ play a central role in our study. These are defined by

$$\zeta(t_i) = t_{i+1} \text{ for } 0 \le i \le n \,,\ \eta(t_i) = t_{n-i} \text{ for } 0 \le i \le n \,,\ \mu(t_i) = t_i^{-1} \text{ for } 0 \le i \le n \,,$$

where the indices are taken modulo $n+1$. Note that ζ, η generate the automorphism group of the Coxeter graph $\tilde{A}_n$, denoted by $\mathrm{Aut}(\tilde{A}_n)$, which is isomorphic to the dihedral group $D_{2(n+1)}$ of order

$2(n+1)$. The subgroup of $\mathrm{Aut}(A[\tilde{A}_n])$ generated by ζ, η, μ is isomorphic to $\mathrm{Aut}(\tilde{A}_n) \times \mathbb{Z}/2\mathbb{Z}$. It will be denoted by $\mathrm{Aut}^*(\tilde{A}_n)$.

Remark 1 We know that $A[\tilde{A}_n]$ embeds into $A[B_{n+1}]$ (see [KP02] and/or Proposition 3.1 below), and that $A[B_{n+1}]$ is torsion-free (see [BS72, Del72]), hence $A[\tilde{A}_n]$ is torsion-free. We also know that the center of $A[\tilde{A}_n]$ is trivial, hence $\mathrm{Inn}(A[\tilde{A}_n]) \simeq A[\tilde{A}_n]$ is torsion-free. Since $\mathrm{Aut}^*(\tilde{A}_n)$ is finite, it follows that $\mathrm{Aut}^*(\tilde{A}_n) \cap \mathrm{Inn}(A[\tilde{A}_n]) = \{1\}$, hence $\mathrm{Aut}^*(\tilde{A}_n)$ embeds into $\mathrm{Out}(A[\tilde{A}_n])$ via the projection $\mathrm{Aut}(A[\tilde{A}_n]) \to \mathrm{Out}(A[\tilde{A}_n])$.

A homomorphism $\varphi\colon G \to H$ is called *abelian* (resp. *cyclic*) if its image is an abelian (resp. cyclic) subgroup of H. A homomorphism $\varphi\colon A[A_n] \to H$ is abelian, if and only if it is cyclic, if and only if there exists $h \in H$ such that $\varphi(s_i) = h$ for all $1 \le i \le n$. Similarly, a homomorphism $\varphi\colon A[\tilde{A}_n] \to H$ is abelian, if and only if it is cyclic, if and only if there exists $h \in H$ such that $\varphi(t_i) = h$ for all $0 \le i \le n$.

We set $Y = \{t_1, t_2, \dots, t_n\}$. Notice that the full subgraph of $\tilde{A}_n$ spanned by Y is isomorphic to A_n. We denote by $\Delta_Y = \Delta_Y[\tilde{A}_n]$ the Garside element of $A_Y[\tilde{A}_n]$. Furthermore, we set $\rho = t_1 t_2 \dots t_n \in A_Y[\tilde{A}_n]$ and $\rho' = t_1^{-1} t_2^{-1} \dots t_n^{-1} \in A_Y[\tilde{A}_n]$. A direct calculation shows that $\rho t_i \rho^{-1} = \rho' t_i \rho'^{-1} = t_{i+1}$ for all $1 \le i \le n-1$ and $\rho^2 t_n \rho^{-2} = \rho'^2 t_n \rho'^{-2} = t_1$. Let $v_0 = \rho t_n \rho^{-1}$ and $v_1 = \rho' t_n \rho'^{-1}$. Then, for each $p \in \mathbb{Z}$, we have endomorphisms $\alpha_p, \beta_p\colon A[\tilde{A}_n] \to A[\tilde{A}_n]$ defined by

$$\alpha_p(t_i) = \beta_p(t_i) = t_i \Delta_Y^{2p} \text{ for } 1 \le i \le n\,,\ \alpha_p(t_0) = v_0 \Delta_Y^{2p}\,,\ \beta_p(t_0) = v_1 \Delta_Y^{2p}\,.$$

Note that $\mathrm{Im}(\alpha_p), \mathrm{Im}(\beta_p) \subset A_Y[\tilde{A}_n] \simeq A[A_n]$. Figure 2.2 depicts the standard generators $t_1, \dots, t_n$ and elements v_0 and v_1 interpreted as braids on $n+1$ strands. We note that, by Corollary 2.3 below, endomorphisms α_p and β_p are not injective.

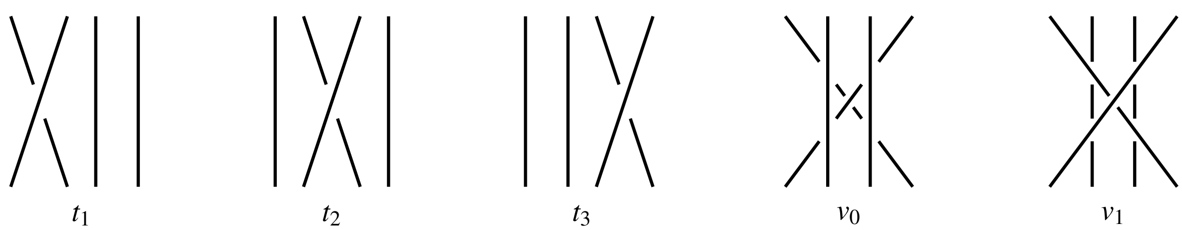

Figure 2.2: Braid pictures of the standard generators $t_1, \dots, t_n$ and elements v_0 and v_1 used in the definition of endomorphisms α_p and β_p, depicted here for $n = 3$.

The main result of the present paper is the following.

Theorem 2.1 *Let $n \ge 4$. Let $\varphi\colon A[\tilde{A}_n] \to A[\tilde{A}_n]$ be an endomorphism. Then we have one of the following four possibilities up to conjugation.*

(1) φ *is cyclic.*

(2) $\varphi \in \mathrm{Aut}^*(\tilde{A}_n)$.

(3) There exist $p \in \mathbb{Z}$ and $\psi \in \mathrm{Aut}^*(\tilde{A}_n)$ such that $\varphi = \psi \circ \alpha_p$.

(4) There exist $p \in \mathbb{Z}$ and $\psi \in \mathrm{Aut}^*(\tilde{A}_n)$ such that $\varphi = \psi \circ \beta_p$.

The reader may wonder whether there is redundancy in Theorem 2.1. The answer is essentially no. Indeed, it will be shown in Corollary 2.3 that neither α_p nor β_p is surjective, hence we cannot have both Case (2) and Case (3) together, or both Case (2) and Case (4) together. Neither α_p nor β_p is cyclic, hence we cannot have both Case (1) and Case (3) together, or both Case (1) and Case (4) together. Also, an automorphism of $A[\tilde{A}_n]$ is never cyclic, so we cannot have both Case (1) and Case (2) together. It remains to understand Case (3) and Case (4).

Let $\mathrm{Fix}(Y)$ be the subgroup of $\mathrm{Aut}(A[\tilde{A}_n])$ of automorphisms of $A[\tilde{A}_n]$ which restrict to the identity on $A_Y[\tilde{A}_n]$. If $\psi \in \mathrm{Fix}(Y)$, then $\psi \circ \alpha_p = \alpha_p$ and $\psi \circ \beta_p = \beta_p$. We note that $\mathrm{Fix}(Y)$ is not trivial since, for example, it contains $\mathrm{conj}_{\Delta_Y} \circ \zeta \circ \eta$. In particular, we have $\mathrm{conj}_{\Delta_Y} \circ \zeta \circ \eta \circ \alpha_p = \alpha_p$, which means that α_p is conjugate to $(\zeta \circ \eta) \circ \alpha_p$ with $\zeta \circ \eta \in \mathrm{Aut}^*(\tilde{A}_n)$ (and the same reasoning applies for β_p). This shows that the automorphism ψ in Case (3) or in Case (4) is not unique. However, the number p in Case (3) or in Case (4) is unique, and we cannot have both Case (3) and Case (4) together, as shown by the following proposition.

Proposition 2.2 *Let $n \geq 4$.*

(1) Let $\psi, \psi' \in \mathrm{Aut}(A[\tilde{A}_n])$ and $p, q \in \mathbb{Z}$. If $\psi \circ \alpha_p = \psi' \circ \alpha_q$, then $p = q$.

(2) Let $\psi, \psi' \in \mathrm{Aut}(A[\tilde{A}_n])$ and $p, q \in \mathbb{Z}$. If $\psi \circ \beta_p = \psi' \circ \beta_q$, then $p = q$.

(3) Let $\psi, \psi' \in \mathrm{Aut}(A[\tilde{A}_n])$ and $p, q \in \mathbb{Z}$. Then $\psi \circ \alpha_p \neq \psi' \circ \beta_q$.

The proof of Proposition 2.2 will follow that of Theorem 2.1 in Section 4.

We turn now to show some notable consequences of Theorem 2.1 before moving on to the preliminaries in Section 3 and to the proofs in Section 4.

Corollary 2.3 *Let $n \geq 4$. Let $\varphi\colon A[\tilde{A}_n] \to A[\tilde{A}_n]$ be an endomorphism.*

(1) φ is injective if and only if φ is conjugate to an element of $\mathrm{Aut}^*(\tilde{A}_n)$.

(2) φ is surjective if and only if φ is conjugate to an element of $\mathrm{Aut}^*(\tilde{A}_n)$.

Proof Let $\varphi\colon A[\tilde{A}_n] \to A[\tilde{A}_n]$ be an endomorphism. By Theorem 2.1 we have one of the following four possibilities up to conjugation.

(1) φ is cyclic.

(2) $\varphi \in \mathrm{Aut}^*(\tilde{A}_n)$.

(3) There exist $p \in \mathbb{Z}$ and $\psi \in \mathrm{Aut}^*(\tilde{A}_n)$ such that $\varphi = \psi \circ \alpha_p$.

(4) There exist $p \in \mathbb{Z}$ and $\psi \in \mathrm{Aut}^*(\tilde{A}_n)$ such that $\varphi = \psi \circ \beta_p$.

Clearly, if $\varphi \in \mathrm{Aut}^*(\tilde{A}_n)$, then φ is both injective and surjective. It remains to show that φ is neither injective nor surjective in the other three cases.

Suppose φ is cyclic. Then φ is not injective because $\varphi(t_1) = \varphi(t_2)$. It is not surjective either because $A[\tilde{A}_n]$ is not cyclic.

In the following we assume the reader to know how to decide whether an element of a Coxeter group belongs to a given standard parabolic subgroup (see [Bou68, Chapter 4, Section 1.8] for instance). Let

$p \in \mathbb{Z}$ and $\psi \in \mathrm{Aut}^*(\tilde{A}_n)$. Recall that $v_0 = t_1 \dots t_{n-1} t_n t_{n-1}^{-1} \dots t_1^{-1}$ and $\omega\colon A[\tilde{A}_n] \to W[\tilde{A}_n]$ denotes the epimorphism which sends t_i to t_i for all $0 \le i \le n$. We have $t_0 \neq v_0$, because $\omega(t_0) \notin W_Y[\tilde{A}_n]$ and $\omega(v_0) \in W_Y[\tilde{A}_n]$, and we have $\alpha_p(t_0) = \alpha_p(v_0) = v_0 \Delta_Y^{2p}$, hence α_p is not injective, and therefore $\psi \circ \alpha_p$ is not injective. On the other hand, $\mathrm{Im}(\alpha_p) \subset A_Y[\tilde{A}_n]$ and $t_0 \notin A_Y[\tilde{A}_n]$ (because $\omega(t_0) \notin W_Y[\tilde{A}_n]$), hence α_p is not surjective, and therefore $\psi \circ \alpha_p$ is not surjective. We prove in the same way that $\psi \circ \beta_p$ is neither injective nor surjective. □

A first consequence of Corollary 2.3 is the determination of the automorphism group and of the outer automorphism group of $A[\tilde{A}_n]$ for $n \ge 4$. Note that this result is already known and proved in [CC05] for $n \ge 2$.

Corollary 2.4 (Charney–Crisp [CC05]) *Let $n \ge 4$. Then*

$$\mathrm{Aut}(A[\tilde{A}_n]) = \mathrm{Inn}(A[\tilde{A}_n]) \rtimes \mathrm{Aut}^*(\tilde{A}_n) \simeq A[\tilde{A}_n] \rtimes (D_{2(n+1)} \times \mathbb{Z}/2\mathbb{Z}),$$

and

$$\mathrm{Out}(A[\tilde{A}_n]) = \mathrm{Aut}^*(\tilde{A}_n) \simeq D_{2(n+1)} \times \mathbb{Z}/2\mathbb{Z},$$

where $D_{2(n+1)}$ denotes the dihedral group of order $2(n+1)$. □

Recall that a group G is called *Hopfian* if every surjective homomorphism $\psi\colon G \to G$ is injective, and it is called *co-Hopfian* if every injective homomorphism $\psi\colon G \to G$ is surjective. Another straightforward consequence of Corollary 2.3 is the following.

Corollary 2.5 *Let $n \ge 4$. Then $A[\tilde{A}_n]$ is Hopfian and co-Hopfian.* □

This result is also known. That $A[\tilde{A}_n]$ is co-Hopfian was established in [BM07, Corollary 3] for $n \ge 2$, and that $A[\tilde{A}_n]$ is Hopfian follows from the fact that it is a finitely generated residually finite group (as a subgroup of a linear group $A[B_{n+1}]$, see Section 3).

Remark 2 The proofs of Corollaries 2.4, 2.5 given in [CC05] and [BM07], respectively, rely on works of Ivanov [Iva97] and Korkmaz [Kor99] on the automorphism group of the mapping class group of a surface, and these techniques cannot be used to determine the whole set of endomorphisms of $A[\tilde{A}_n]$. Our proof of Theorem 2.1 uses other tools, notably the study of the homomorphisms from $A[A_n]$ to $A[A_{n+1}]$ made in [Cas09] and [CKM19].

3 Preliminaries

The first ingredient in the proof of Theorem 2.1 is the following well-known chain of embeddings

$$A[A_n] \hookrightarrow A[\tilde{A}_n] \hookrightarrow A[B_{n+1}] \hookrightarrow A[A_{n+1}],$$

which is described as follows.

The first embedding $\iota_Y\colon A[A_n] \hookrightarrow A[\tilde{A}_n]$ is defined by $\iota_Y(s_i) = t_i$ for all $1 \le i \le n$.

Now, we describe the second embedding, $\iota_{\tilde{A}}\colon A[\tilde{A}_n] \hookrightarrow A[B_{n+1}]$. Let $\rho_B = r_1 \dots r_n r_{n+1} \in A[B_{n+1}]$. A direct calculation shows that $\rho_B r_i \rho_B^{-1} = r_{i+1}$ for all $1 \le i \le n-1$ and $\rho_B^2 r_n \rho_B^{-2} = r_1$. So if we set $r_0 = \rho_B r_n \rho_B^{-1}$, then $\rho_B r_i \rho_B^{-1} = r_{i+1}$ for all $0 \le i \le n-1$ and $\rho_B r_n \rho_B^{-1} = r_0$. (We note, however, that r_{n+1} and r_0 are two different elements of $A[B_{n+1}]$, since their images in the abelianization of $A[B_{n+1}]$ are distinct.) On the other hand, recall the automorphism $\zeta \in \mathrm{Aut}(\tilde{A}_n)$ which sends t_i to t_{i+1} for all $0 \le i \le n$. We consider the action of an infinite cyclic group $\langle u \rangle \simeq \mathbb{Z}$ on $A[\tilde{A}_n]$ defined by $u \cdot h = \zeta(h)$ for all $h \in A[\tilde{A}_n]$, and we consider the semi-direct product $A[\tilde{A}_n] \rtimes \langle u \rangle$ defined by this action. The following is proved in [KP02] (see also [CC05]).

Proposition 3.1 (Kent–Peifer [KP02]) *There exists an isomorphism* $\gamma\colon A[\tilde{A}_n] \rtimes \langle u \rangle \to A[B_{n+1}]$ *which sends* t_i *to* r_i *for all* $0 \le i \le n$ *and sends* u *to* ρ_B. □

The embedding $\iota_{\tilde{A}}\colon A[\tilde{A}_n] \hookrightarrow A[B_{n+1}]$ is defined to be the restriction of γ to $A[\tilde{A}_n]$. From now on we identify $A[\tilde{A}_n]$ with the image of $\iota_{\tilde{A}}$. So, $A[\tilde{A}_n]$ is viewed as the subgroup of $A[B_{n+1}]$ generated by $r_0, r_1, \dots, r_n$, and we have $t_i = r_i$ for all $0 \le i \le n$ via this identification.

Remark 3 We have a homomorphism $z\colon A[B_{n+1}] \to \mathbb{Z}$ defined by $z(r_i) = 0$ for all $1 \le i \le n$ and $z(r_{n+1}) = 1$. As $z(\rho_B) = 1$, we have $\mathrm{Ker}(z) = \mathrm{Im}(\iota_{\tilde{A}}) = A[\tilde{A}_n]$.

In what follows it will be useful to differentiate the generators of $A[A_n]$ from those of $A[A_{n+1}]$, hence we will denote by $s'_1, \dots, s'_n, s'_{n+1}$ the standard generators of $A[A_{n+1}]$. The following proposition has been observed independently by several authors (see [Lam94, Cri99], for example) and it is implicitly proved in [Bri73]. A simple combinatorial proof of it is given in [Man97, Proposition 1].

Proposition 3.2 *There exists an injective homomorphism* $\iota_B\colon A[B_{n+1}] \hookrightarrow A[A_{n+1}]$ *which sends* r_i *to* s'_i *for all* $1 \le i \le n$ *and sends* r_{n+1} *to* $(s'_{n+1})^2$. □

From now on we identify $A[B_{n+1}]$ with the image of ι_B. So, $A[B_{n+1}]$ is viewed as the subgroup of $A[A_{n+1}]$ generated by $s'_1, \dots, s'_n, (s'_{n+1})^2$, and we have $r_i = s'_i$ for all $1 \le i \le n$ and $r_{n+1} = (s'_{n+1})^2$ via this identification.

Remark 4 Recall that $\omega\colon A[A_{n+1}] \to W[A_{n+1}]$ denotes the epimorphism which sends s'_i to s'_i for all $1 \le i \le n+1$. Then $A[B_{n+1}] = \omega^{-1}(W_Y[A_{n+1}])$, where $Y = \{s'_1, \dots, s'_n\}$. This is proved in [Man97, Proposition 1], but also can be seen directly as follows. Observe that $A[B_{n+1}]$ maps onto $W_Y[A_{n+1}]$ under ω (since $\omega(s'^2_{n+1}) = 1$), and that $A[B_{n+1}]$ contains $\mathrm{Ker}\,\omega$. Indeed, $A[B_{n+1}]$ contains all normal generators $s'^2_1, \dots, s'^2_{n+1}$ of $\mathrm{Ker}\,\omega$ and all their conjugates by the generators $(s'_i)^{\pm 1}$ of $A[A_{n+1}]$. This is clear for $(s'_i)^{\pm 1}(s'_j)^2(s'_i)^{\mp 1}$ if $(i,j) \ne (n+1, n)$. For the remaining case we see that: $s'_{n+1} s'^2_n (s'_{n+1})^{-1} = (s'_{n+1} s'_n (s'_{n+1})^{-1})^2 = ((s'_n)^{-1} s'_{n+1} s'_n)^2 = (s'_n)^{-1} s'^2_{n+1} s'_n \in A[B_{n+1}]$, and similarly for the conjugation by s'^{-1}_{n+1}. Thus, $A[B_{n+1}]$ contains $\mathrm{Ker}\,\omega$, which finishes the proof.

The second ingredient in the proof of Theorem 2.1 is the following.

Theorem 3.3 (Castel [Cas09], Chen–Kordek–Margalit [CKM19], Orevkov [Ore24]) *Let* $n \ge 4$. *Let* $\varphi\colon A[A_n] \to A[A_{n+1}]$ *be a homomorphism. Then we have one of the following two possibilities up to conjugation.*

(1) φ is cyclic.

(2) There exist $\varepsilon \in \{\pm 1\}$ and $p, q \in \mathbb{Z}$ such that $\varphi(s_i) = s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q}$, where $Y = \{s_1', \dots, s_n'\}$, $\Delta_Y = \Delta_Y[A_{n+1}]$ and $\Delta = \Delta[A_{n+1}]$. □

The strategy for proving Theorem 2.1 is now clear. Firstly, we use Theorem 3.3 to determine the set of homomorphisms from $A[\tilde{A}_n]$ to $A[A_{n+1}]$ (see Proposition 4.1). Secondly, we determine which of these homomorphisms have an image contained in $A[\tilde{A}_n] \subset A[A_{n+1}]$.

To pass from homomorphisms from $A[A_n]$ to $A[A_{n+1}]$ to homomorphisms from $A[\tilde{A}_n]$ to $A[A_{n+1}]$, we use a third ingredient: the mapping class groups. So, we give below the information on these groups that we will need, and we refer to [FM12] for a complete exposition on the subject.

Let Σ be a compact oriented surface with or without boundary, and let $\mathcal{P}$ be a finite family of punctures in the interior of Σ. We denote by $\mathrm{Homeo}^+(\Sigma, \mathcal{P})$ the group of homeomorphisms of Σ which preserve the orientation, which are the identity on a neighborhood of the boundary, and which leave set-wise invariant the set $\mathcal{P}$. The *mapping class group* of the pair $(\Sigma, \mathcal{P})$, denoted by $\mathcal{M}(\Sigma, \mathcal{P})$, is the group of isotopy classes of elements of $\mathrm{Homeo}^+(\Sigma, \mathcal{P})$, where isotopies are required to leave the set $\partial\Sigma \cup \mathcal{P}$ point-wise invariant. If $\mathcal{P} = \varnothing$, then we write $\mathrm{Homeo}^+(\Sigma) = \mathrm{Homeo}^+(\Sigma, \varnothing)$ and $\mathcal{M}(\Sigma) = \mathcal{M}(\Sigma, \varnothing)$.

A *circle* of $(\Sigma, \mathcal{P})$ is the image of an embedding $a\colon \mathbb{S}^1 \hookrightarrow \Sigma \setminus (\partial\Sigma \cup \mathcal{P})$. It is called *generic* if it does not bound any disk containing 0 or 1 puncture, and if it is not parallel to any boundary component. The isotopy class of a circle a is denoted by $[a]$. We emphasize that isotopies are considered in $\Sigma \setminus (\partial\Sigma \cup \mathcal{P})$, i.e. circles are not allowed to pass through points of $\partial\Sigma \cup \mathcal{P}$ under isotopies. We denote by $\mathcal{C}(\Sigma, \mathcal{P})$ the set of isotopy classes of generic circles. The *intersection index* of two classes $[a], [b] \in \mathcal{C}(\Sigma, \mathcal{P})$ is $i([a], [b]) = \min\{|a' \cap b'| \mid a' \in [a] \text{ and } b' \in [b]\}$. The set $\mathcal{C}(\Sigma, \mathcal{P})$ is endowed with a structure of simplicial complex, where a non-empty finite subset $\mathcal{F} \subset \mathcal{C}(\Sigma, \mathcal{P})$ is a simplex if $i([a], [b]) = 0$ for all $[a], [b] \in \mathcal{F}$. This complex is called the *curve complex* of $(\Sigma, \mathcal{P})$.

In the present paper the (right) Dehn twist along a circle a is denoted by T_a. The following is important to make the link between Artin groups and mapping class groups. Its proof can be found in [FM12, Section 3.5].

Proposition 3.4 *Let Σ be an oriented compact surface and let $\mathcal{P}$ be a finite collection of punctures in the interior of Σ. Let a, b be two generic circles of $(\Sigma, \mathcal{P})$ which we assume to be in minimal position, that is, such that $i([a], [b]) = |a \cap b|$.*

(1) We have $T_a T_b = T_b T_a$ if $a \cap b = \varnothing$.

(2) We have $T_a T_b T_a = T_b T_a T_b$ if $|a \cap b| = 1$.

(3) The set $\{T_a, T_b\}$ generates a rank 2 free group if $|a \cap b| \geq 2$. □

An *arc* of $(\Sigma, \mathcal{P})$ is the image of an embedding $a\colon [0,1] \hookrightarrow \Sigma \setminus \partial\Sigma$ such that $a([0,1]) \cap \mathcal{P} = \{a(0), a(1)\}$, with $a(0) \neq a(1)$. We consider arcs up to isotopies under which interior points of arcs are mapped into $\Sigma \setminus (\partial\Sigma \cup \mathcal{P})$. We denote the isotopy class of an arc a by $[a]$, and the set of isotopy classes of arcs by $\mathcal{A}(\Sigma, \mathcal{P})$. The *intersection index* of two classes $[a], [b] \in \mathcal{A}(\Sigma, \mathcal{P})$ is $i([a], [b]) = \min\{|a' \cap b'| \mid a' \in [a] \text{ and } b' \in [b]\}$.

With an arc a of $(\Sigma, \mathcal{P})$ we associate an element $H_a \in \mathcal{M}(\Sigma, \mathcal{P})$, called the *(right) half-twist* along a. This element is the identity outside a regular neighborhood of a, it exchanges the two ends of a, and $H_a^2 = T_{\hat{a}}$, where $\hat{a}$ is the boundary of a regular neighborhood of a, see Figure 3.1. We refer the reader to [KT08, Section 1.6.2] for more information about properties of half-twists.

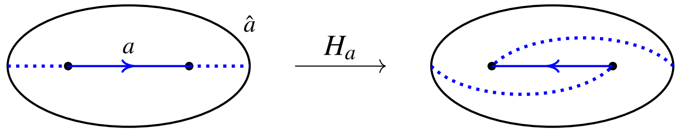

Figure 3.1: A half-twist

The following is known to experts but, as far as we know, the proof is not readily available in the literature, so we provide it here.

Proposition 3.5 *Let Σ be an oriented compact surface and let $\mathcal{P}$ be a finite collection of punctures in the interior of Σ. If $|\mathcal{P}| = 3$, we require additionally that $\partial\Sigma \neq \varnothing$. Let a, b be two non-isotopic arcs of $(\Sigma, \mathcal{P})$ which we assume to be in minimal position, that is, such that $i([a],[b]) = |a \cap b|$.*

(1) We have $H_a H_b = H_b H_a$ if $a \cap b = \varnothing$.

(2) We have $H_a H_b H_a = H_b H_a H_b$ if $|a \cap b| = 1$ and $a \cap b \subset \mathcal{P}$.

(3) The set $\{H_a, H_b\}$ generates a rank 2 free group if $|a \cap b| \geq 2$, or if $|a \cap b| = 1$ and $a \cap b \not\subset \mathcal{P}$.

Proof Proposition 3.5 follows from the Birman–Hilden theory combined with Proposition 3.4. So, we begin by recalling the results of Birman–Hilden [BH73] that we need and we refer to [MW21] or [FM12, Chapter 9] for more details.

Since arcs a, b are assumed to exist in $(\Sigma, \mathcal{P})$, the cardinality of $\mathcal{P}$ is at least 2. We form a set $\mathcal{P}' \subseteq \mathcal{P}$ as follows. If the cardinality of $\mathcal{P}$ is even, we set $\mathcal{P}' = \mathcal{P}$. If $|\mathcal{P}|$ is odd but there exists a point $x \in \mathcal{P} \setminus (a \cup b)$, we set $\mathcal{P}' = \mathcal{P} \setminus \{x\}$. This accounts for all situations except when $|\mathcal{P}| = 3$ and $|a \cap b \cap \mathcal{P}| = 1$, which we call the *exceptional case*, and we will deal with it later. For now assume that we are not in the exceptional case, so that $\mathcal{P}'$ is defined and the cardinality of $\mathcal{P}'$ is even.

Let $p\colon \tilde{\Sigma} \to \Sigma$ be the two-sheeted branched covering whose set of branch points is precisely $\mathcal{P}'$. To describe p explicitly, we choose an embedded disk $D \subset \Sigma$ such that $\mathcal{P}'$ lies in the interior of D and denote $\Sigma_0 = (\Sigma \setminus D) \cup \partial D$. Then $\tilde{\Sigma}$ is built as the union of two copies of Σ_0 attached to a surface Σ_D which has genus $\frac{1}{2}\,\mathrm{Card}\,\mathcal{P}' - 1$ and two boundary components. The branched covering p restricted to Σ_D is the quotient map $\Sigma_D \to D$ by the action of the hyperelliptic involution on Σ_D, and p sends each copy of $\Sigma_0 \subset \tilde{\Sigma}$ homeomorphically onto $\Sigma_0 \subset \Sigma$. Notice that $\tilde{\Sigma}$ is a surface with two punctures (and possibly with boundary) if $\mathcal{P}' \neq \mathcal{P}$.

If $\mathcal{P}' \neq \mathcal{P}$, then we set $\tilde{\mathcal{P}} = p^{-1}(x)$, where x is the element of $\mathcal{P}$ which does not belong to $\mathcal{P}'$, and, if $\mathcal{P}' = \mathcal{P}$, then we set $\tilde{\mathcal{P}} = \varnothing$. Let $\sigma\colon \tilde{\Sigma} \to \tilde{\Sigma}$ be the non-trivial deck transformation of p. Note that σ is not necessarily an element of $\mathrm{Homeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}})$ in the sense that it does not necessarily restrict to the identity on a neighborhood of the boundary of $\tilde{\Sigma}$. In fact, we have $\sigma \in \mathrm{Homeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}})$ if and only if $\partial\Sigma = \varnothing$. Let $\mathrm{SHomeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}})$ denote the subgroup of $\mathrm{Homeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}})$ consisting of the elements

$\tilde{f} \in \mathrm{Homeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}})$ which commute with σ, and let $\mathcal{SM}(\tilde{\Sigma}, \tilde{\mathcal{P}})$ denote the subgroup of $\mathcal{M}(\tilde{\Sigma}, \tilde{\mathcal{P}})$ consisting of the isotopy classes of elements of $\mathrm{SHomeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}})$.

Let $f \in \mathrm{Homeo}^+(\Sigma, \mathcal{P})$. We say that f is *liftable* if there exists $\tilde{f} \in \mathrm{Homeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}})$ such that $p \circ \tilde{f} = f \circ p$. In this case $\tilde{f}$ is an element of $\mathrm{SHomeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}})$. Note that, if $\partial\tilde{\Sigma} \neq \varnothing$, then $\tilde{f}$ is unique, and if $\partial\tilde{\Sigma} = \varnothing$, then f has exactly two lifts, $\tilde{f}$ and $\tilde{f} \circ \sigma = \sigma \circ \tilde{f}$. We denote by $\mathrm{LHomeo}^+(\Sigma, \mathcal{P})$ the subgroup of liftable elements of $\mathrm{Homeo}^+(\Sigma, \mathcal{P})$, and we denote by $\mathcal{LM}(\Sigma, \mathcal{P})$ the subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ consisting of the isotopy classes of elements of $\mathrm{LHomeo}^+(\Sigma, \mathcal{P})$.

We have a surjective homomorphism $\Phi\colon \mathrm{SHomeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}}) \to \mathrm{LHomeo}^+(\Sigma, \mathcal{P})$ defined as follows. Let $\tilde{f} \in \mathrm{SHomeo}^+(\tilde{\Sigma}, \tilde{\mathcal{P}})$. Then $\Phi(\tilde{f})$ is the unique homeomorphism $f \in \mathrm{LHomeo}^+(\Sigma, \mathcal{P})$ satisfying $p \circ \tilde{f} = f \circ p$. We have $\mathrm{Ker}(\Phi) = G$, where $G = \langle \sigma \rangle = \{\mathrm{id}, \sigma\}$ if $\partial\Sigma = \varnothing$, and $G = \{\mathrm{id}\}$ if $\partial\Sigma \neq \varnothing$. By [BH73, Theorem 2], Φ induces a surjective homomorphism $\Phi^*\colon \mathcal{SM}(\tilde{\Sigma}, \tilde{\mathcal{P}}) \to \mathcal{LM}(\Sigma, \mathcal{P})$ whose kernel is G', where $G' = \langle [\sigma] \rangle = \{\mathrm{id}, [\sigma]\} \simeq \mathbb{Z}/2\mathbb{Z}$ if $\partial\Sigma = \varnothing$, and $G' = \{\mathrm{id}\}$ if $\partial\Sigma \neq \varnothing$.

We observe that the full preimage $p^{-1}(a)$ is a circle $\tilde{a}$ in $(\tilde{\Sigma}, \tilde{\mathcal{P}})$. Since by assumption at least two non-isotopic arcs exist in $(\Sigma, \mathcal{P})$, we conclude that Σ is not a sphere or a disk with $\mathrm{Card}(\mathcal{P}') = 2$. Hence $\tilde{a}$ does not bound a disk with 0 or 1 puncture in $\tilde{\Sigma}$, and neither is $\tilde{a}$ parallel to a boundary component of $\tilde{\Sigma}$. We conclude that $\tilde{a}$ is a generic circle in $(\tilde{\Sigma}, \tilde{\mathcal{P}})$, with $H_a \in \mathcal{LM}(\Sigma, \mathcal{P})$, $T_{\tilde{a}} \in \mathcal{SM}(\tilde{\Sigma}, \tilde{\mathcal{P}})$, and $\Phi_*(T_{\tilde{a}}) = H_a$ (see [FM12, Section 9.4.1]). Similarly, $\tilde{b} = p^{-1}(b)$ is a generic circle with $H_b \in \mathcal{LM}(\Sigma, \mathcal{P})$, $T_{\tilde{b}} \in \mathcal{SM}(\tilde{\Sigma}, \tilde{\mathcal{P}})$, and $\Phi_*(T_{\tilde{b}}) = H_b$.

Arguing as in the proof of [FM12, Lemma 9.3], we see that the circles $\tilde{a} = p^{-1}(a)$ and $\tilde{b} = p^{-1}(b)$ are in minimal position. If $a \cap b = \varnothing$, then $\tilde{a} \cap \tilde{b} = \varnothing$, hence, by Proposition 3.4, $T_{\tilde{a}} T_{\tilde{b}} = T_{\tilde{b}} T_{\tilde{a}}$, and therefore, by applying Φ_* we get $H_a H_b = H_b H_a$. If $|a \cap b| = 1$ and $a \cap b \subset \mathcal{P}$, then $|\tilde{a} \cap \tilde{b}| = 1$, hence, by Proposition 3.4, $T_{\tilde{a}} T_{\tilde{b}} T_{\tilde{a}} = T_{\tilde{b}} T_{\tilde{a}} T_{\tilde{b}}$, and therefore, by applying Φ_* we get $H_a H_b H_a = H_b H_a H_b$. Suppose that $|a \cap b| \geq 2$, or $|a \cap b| = 1$ and $a \cap b \not\subset \mathcal{P}$. Then clearly $|\tilde{a} \cap \tilde{b}| \geq 2$, and since $\tilde{a}$ and $\tilde{b}$ are in minimal position, $i([\tilde{a}], [\tilde{b}]) \geq 2$. Hence, by Proposition 3.4, $\{T_{\tilde{a}}, T_{\tilde{b}}\}$ generates a rank 2 free group. Note that $\langle T_{\tilde{a}}, T_{\tilde{b}} \rangle \cap G' = \{\mathrm{id}\}$, because $\langle T_{\tilde{a}}, T_{\tilde{b}} \rangle$ is torsion-free and G' is finite. Since $G' = \mathrm{Ker}(\Phi_*)$, it follows that the restriction of Φ_* to $\langle T_{\tilde{a}}, T_{\tilde{b}} \rangle$ is injective, hence $\{\Phi_*(T_{\tilde{a}}), \Phi_*(T_{\tilde{b}})\} = \{H_a, H_b\}$ generates a rank 2 free group.

It remains to deal with the exceptional case, i.e. with the situation when $|\mathcal{P}| = 3$, $|a \cap b \cap \mathcal{P}| = 1$, and $\partial\Sigma \neq \varnothing$. Let c be a boundary component of Σ. We glue an annulus A to c and we take a puncture x inside this annulus. We denote by Σ'' the surface obtained in this way and we set $\mathcal{P}'' = \mathcal{P} \cup \{x\}$. Then a and b are still in minimal position in $(\Sigma'', \mathcal{P}'')$ and the embedding of Σ into Σ'' induces an injective homomorphism $\mathcal{M}(\Sigma, \mathcal{P}) \hookrightarrow \mathcal{M}(\Sigma'', \mathcal{P}'')$. So, we can restrict ourselves to the case where $|\mathcal{P}| = 4$. □

Remark 5 The only situation in the above proof where we used the assumption $\partial\Sigma \neq \varnothing$ was the case where $|\mathcal{P}| = 3$, $|a \cap b \cap \mathcal{P}| = 1$. We believe that the conclusions of Proposition 3.5 are true in this case even without the assumption $\partial\Sigma \neq \varnothing$.

We record an important corollary, which is an analog for half-twists of the corresponding fact about Dehn twists, stating that the two Dehn twists uniquely determine their respective circles up to isotopy (see [FM12, Fact 3.6]).

Corollary 3.6 *Let $(\Sigma, \mathcal{P})$ be as in Proposition 3.5. If a and b are two non-isotopic arcs in $(\Sigma, \mathcal{P})$, then their respective half-twists H_a and H_b are different elements of $\mathcal{M}(\Sigma, \mathcal{P})$.*

Proof Assume that $H_a = H_b$ in $\mathcal{M}(\Sigma, \mathcal{P})$. Observe that H_a interchanges points $a(0)$ and $a(1)$ and fixes $\mathcal{P} \setminus \{a(0), a(1)\}$ pointwise, and, similarly, H_b interchanges points $b(0)$, $b(1)$ and fixes $\mathcal{P} \setminus \{b(0), b(1)\}$ pointwise. Since isotopies in the definition of $\mathcal{M}(\Sigma, \mathcal{P})$ are required to fix the set $\mathcal{P}$ pointwise, we conclude that the two sets are equal: $\{a(0), a(1)\} = \{b(0), b(1)\}$. But then, by part (3) of Proposition 3.5, H_a and H_b generate a rank 2 free group in $\mathcal{M}(\Sigma, \mathcal{P})$, which is a contradiction with the assumption $H_a = H_b$. □

We denote by $\mathbb{D}$ the standard disk, and we choose a collection $\mathcal{P}_{n+2} = \{p_0, p_1, \dots, p_n, p_{n+1}\}$ of $n+2$ punctures in the interior of $\mathbb{D}$ (see Figure 3.2). Let $a_1, \dots, a_n, a_{n+1}$ be the arcs drawn in Figure 3.2. Then, by Proposition 3.5, we have a homomorphism $\Psi \colon A[A_{n+1}] \to \mathcal{M}(\mathbb{D}, \mathcal{P}_{n+2})$ which sends s_i' to H_{a_i} for all $1 \le i \le n+1$. By [KT08, Theorem 1.33], this homomorphism is an isomorphism. From now on we will identify $A[A_{n+1}]$ with $\mathcal{M}(\mathbb{D}, \mathcal{P}_{n+2})$ via Ψ. In particular, we assume that $s_i' = H_{a_i}$ for all $1 \le i \le n+1$. We claim that, via this identification, we have $t_0 = H_{b_0}$, where b_0 is the arc drawn in Figure 3.2. Indeed, recall that $t_0 \in A[\tilde{A}_n]$ is identified under the embedding $\iota_{\tilde{A}}$ with the element $r_0 = \rho_B r_n \rho_B^{-1}$ of $A[B_{n+1}]$, where $\rho_B = r_1 \dots r_n r_{n+1}$. In its turn, under the embedding ι_B, elements r_n and ρ_B get identified with s_n' and $s_1' s_2' \dots s_n' (s_{n+1}')^2$, respectively. Hence, the image of t_0 in $\mathcal{M}(\mathbb{D}, \mathcal{P}_{n+2})$ under Ψ is $\Psi(\rho_B) H_{a_n} \Psi(\rho_B)^{-1} = H_{\Psi(\rho_B)(a_n)}$. One easily checks by drawing pictures that $\Psi(\rho_B)(a_n) = H_{a_1} H_{a_2} \dots H_{a_n} H_{a_{n+1}}^2 (a_n) = b_0$, up to isotopy.

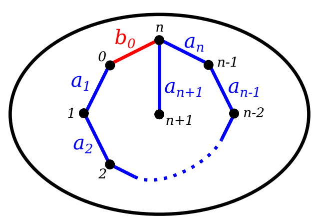

Figure 3.2: Punctured disk

Let Σ be an oriented compact surface, and let $\mathcal{P}$ be a finite collection of punctures in the interior of Σ. Assume the Euler characteristic of $\Sigma \setminus \mathcal{P}$ is negative. Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$. We say that a simplex $\mathcal{F}$ of $\mathcal{C}(\Sigma, \mathcal{P})$ is a *reduction system* for f if $f(\mathcal{F}) = \mathcal{F}$. In this case any element of $\mathcal{F}$ is called a *reduction class* for f. A reduction class $[a]$ is an *essential reduction class* if, for each $[b] \in \mathcal{C}(\Sigma, \mathcal{P})$ such that $i([a], [b]) \neq 0$ and for each $m \in \mathbb{Z} \setminus \{0\}$, we have $f^m([b]) \neq [b]$. In particular, if $[a]$ is an essential reduction class and $[b]$ is any reduction class, then $i([a], [b]) = 0$. We denote by $\mathcal{S}(f)$ the set of essential reduction classes for f. The following gathers together some results on $\mathcal{S}(f)$ that we will use in the proofs.

Theorem 3.7 (Birman–Lubotzky–McCarthy [BLM83]) *Let Σ be an oriented compact surface, and let $\mathcal{P}$ be a finite collection of punctures in the interior of Σ. Assume the Euler characteristic of $\Sigma \setminus \mathcal{P}$ is negative. Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$.*

(1) If $\mathcal{S}(f) \neq \varnothing$, then $\mathcal{S}(f)$ is a reduction system for f.

(2) *We have* $\mathcal{S}(f^n) = \mathcal{S}(f)$ *for all* $n \in \mathbb{Z} \setminus \{0\}$.

(3) *We have* $\mathcal{S}(gfg^{-1}) = g(\mathcal{S}(f))$ *for all* $g \in \mathcal{M}(\Sigma, \mathcal{P})$. □

In addition to Theorem 3.7 we have the following which is well-known and which is a direct consequence of [BLM83].

Proposition 3.8 *Let* Σ *be an oriented compact surface, and let* $\mathcal{P}$ *be a finite collection of punctures in the interior of* Σ. *Assume the Euler characteristic of* $\Sigma \setminus \mathcal{P}$ *is negative. Let* $f_0 \in Z(\mathcal{M}(\Sigma, \mathcal{P}))$ *be a central element of* $\mathcal{M}(\Sigma, \mathcal{P})$, *let* $\mathcal{F} = \{[a_1], [a_2], \dots, [a_p]\}$ *be a simplex of* $\mathcal{C}(\Sigma, \mathcal{P})$, *and let* $k_1, k_2, \dots, k_p$ *be non-zero integers. Let* $g = T_{a_1}^{k_1} T_{a_2}^{k_2} \dots T_{a_p}^{k_p} f_0$. *Then* $\mathcal{S}(g) = \mathcal{F}$. □

4 Proofs

As pointed out in Section 3, we start by determining the homomorphisms from $A[\tilde{A}_n]$ to $A[A_{n+1}]$. As before, we set $Y = \{s'_1, \dots, s'_n\} = \{t_1, \dots, t_n\}$, $\Delta_Y = \Delta_Y[A_{n+1}] = \Delta_Y[\tilde{A}_n]$ and $\Delta = \Delta[A_{n+1}]$. Consider the following elements of $A[\tilde{A}_n] \subset A[A_{n+1}]$.

$$u_0 = t_0\,,\ u_1 = \Delta_Y^{-1} t_0 \Delta_Y\,,\ v_0 = \rho t_n \rho^{-1} = t_1 \dots t_{n-1} t_n t_{n-1}^{-1} \dots t_1^{-1}\,,$$
$$v_1 = \rho' t_n \rho'^{-1} = t_1^{-1} \dots t_{n-1}^{-1} t_n t_{n-1} \dots t_1\,,$$

where $\rho = t_1 t_2 \dots t_n$ and $\rho' = t_1^{-1} t_2^{-1} \dots t_n^{-1}$. One can easily see that the above elements are identified under the isomorphism $\Psi \colon A[A_{n+1}] \to \mathcal{M}(\mathbb{D}, \mathcal{P}_{n+2})$ with the following half-twists: $u_0 = H_{b_0}$, $u_1 = H_{b_1}$, $v_0 = H_{c_0}$ and $v_1 = H_{c_1}$, where b_0, b_1, c_0, c_1 are the arcs drawn in Figure 4.1. (This can be checked using the property $fH_af^{-1} = H_{f(a)}$ for an orientation-preserving mapping class f.)

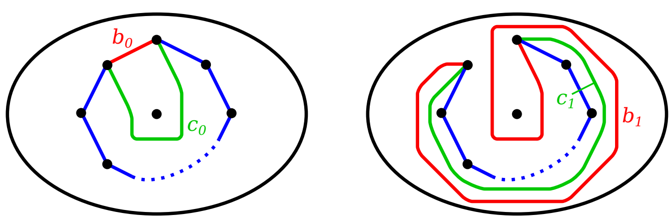

Figure 4.1: Arcs in the punctured disk

Proposition 4.1 *Let* $n \geq 4$. *Let* $\varphi \colon A[\tilde{A}_n] \to A[A_{n+1}]$ *be a homomorphism. Then we have one of the following three possibilities.*

(1) φ *is cyclic.*

(2) *There exist* $g \in A[A_{n+1}]$, $k \in \{0, 1\}$, $\varepsilon \in \{\pm 1\}$ *and* $q \in \mathbb{Z}$ *such that* $\varphi(t_i) = g s_i'^{\varepsilon} \Delta^{2q} g^{-1}$ *for all* $1 \leq i \leq n$, *and* $\varphi(t_0) = g u_k^{\varepsilon} \Delta^{2q} g^{-1}$.

(3) *There exist* $g \in A[A_{n+1}]$, $k \in \{0, 1\}$, $\varepsilon \in \{\pm 1\}$ *and* $p, q \in \mathbb{Z}$ *such that* $\varphi(t_i) = g s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$ *for all* $1 \leq i \leq n$, *and* $\varphi(t_0) = g v_k^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$.

Remark 6 It can be shown that a homomorphism $\varphi\colon A[\tilde{A}_n] \to A[A_{n+1}]$, $n \ge 4$, is never surjective, and it is injective if and only if it belongs to Case (2) of Proposition 4.1.

Proof of Proposition 4.1 Let $\varphi\colon A[\tilde{A}_n] \to A[A_{n+1}]$ be a homomorphism. Recall the embedding $\iota_Y\colon A[A_n] \hookrightarrow A[\tilde{A}_n]$ which sends s_i to t_i for all $1 \le i \le n$. By Theorem 3.3 we have one of the following two possibilities.

(1) $\varphi \circ \iota_Y$ is cyclic.

(2) There exist $g \in A[A_{n+1}]$, $\varepsilon \in \{\pm 1\}$, and $p, q \in \mathbb{Z}$ such that $(\varphi \circ \iota_Y)(s_i) = g s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$ for all $1 \le i \le n$.

Suppose $\varphi \circ \iota_Y$ is cyclic. There exists $h \in A[A_{n+1}]$ such that $(\varphi \circ \iota_Y)(s_i) = \varphi(t_i) = h$ for all $1 \le i \le n$. We also have

$$\varphi(t_0) = \varphi(t_1 t_0 t_1 t_0^{-1} t_1^{-1}) = \varphi(t_1 t_0)\, \varphi(t_1)\, \varphi(t_0^{-1} t_1^{-1}) = \varphi(t_1 t_0)\, \varphi(t_3)\, \varphi(t_0^{-1} t_1^{-1}) = \varphi(t_1 t_0 t_3 t_0^{-1} t_1^{-1}) = \varphi(t_3) = h\,,$$

hence φ is cyclic.

Now, assume that there exist $g \in A[A_{n+1}]$, $\varepsilon \in \{\pm 1\}$, and $p, q \in \mathbb{Z}$ such that $(\varphi \circ \iota_Y)(s_i) = g s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$ for all $1 \le i \le n$. From here the proof is divided into two cases depending on whether $p \neq 0$ or $p = 0$.

Case 1: $p \neq 0$. Let $\varphi' = \mathrm{conj}_{g^{-1}} \circ \varphi$. We have $\varphi'(t_i) = s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q}$ for all $1 \le i \le n$. It is easily seen that $\Delta_Y^2 = T_d$, where d is the circle drawn in Figure 4.2, $\Delta^2 = T_{\partial\mathbb{D}}$, where $T_{\partial\mathbb{D}}$ denotes the Dehn twist along the boundary of $\mathbb{D}$, and $s_i' = H_{a_i}$, where a_i is the arc drawn in Figure 3.2, for all $1 \le i \le n$. So, $\varphi'(t_i) = H_{a_i}^{\varepsilon} T_d^p T_{\partial\mathbb{D}}^q$ for all $1 \le i \le n$. For each $1 \le i \le n$ we denote by $\hat{a}_i$ the boundary of a regular neighborhood of a_i. Then $\varphi'(t_i)^2 = T_{\hat{a}_i}^{\varepsilon} T_d^{2p} T_{\partial\mathbb{D}}^{2q}$, hence, by Proposition 3.8, $\mathcal{S}(\varphi'(t_i)) = \mathcal{S}(\varphi'(t_i)^2) = \{[\hat{a}_i], [d]\}$ for all $1 \le i \le n$.

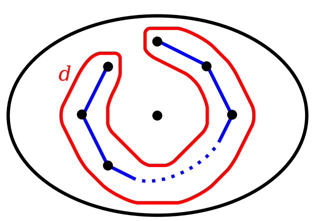

Figure 4.2: A circle in the punctured disk

Since t_0 is conjugate to t_1 in $A[\tilde{A}_n]$, $\varphi'(t_0)$ is conjugate to $\varphi'(t_1)$ in $A[A_{n+1}]$, hence $\varphi'(t_0)$ is of the form $\varphi'(t_0) = H_{a'}^{\varepsilon} T_{d'}^p T_{\partial\mathbb{D}}^q$, where d' is a circle which bounds a disk containing $n+1$ marked punctures, and a' is an arc inside this disk. Moreover, if $\hat{a}'$ is the boundary of a regular neighborhood of a', then $\mathcal{S}(\varphi'(t_0)) = \{[\hat{a}'], [d']\}$. Since $t_0 t_2 = t_2 t_0$, we have $\varphi'(t_0)\, \varphi'(t_2)\, \varphi'(t_0)^{-1} = \varphi'(t_2)$, hence, by Theorem 3.7 (3), $\varphi'(t_0)(\mathcal{S}(\varphi'(t_2))) = \mathcal{S}(\varphi'(t_2))$, and therefore $\varphi'(t_0)([d]) = [d]$. So, $[d]$ is a reduction class for $\varphi'(t_0)$. Since $[d']$ is an essential reduction class for $\varphi'(t_0)$, it follows that $i([d'], [d]) = 0$. So, we can assume without loss of generality that $d \cap d' = \varnothing$. Each of the two circles d and d' bounds a disk containing $n+1$ marked punctures, and $d \cap d' = \varnothing$, hence d' should be isotopic to d. So, we can assume without loss of generality that $d = d'$. Then $\varphi'(t_0) = H_{a'}^{\varepsilon} T_d^p T_{\partial\mathbb{D}}^q$.

Let $i \in \{2, \dots, n-1\}$. We have $t_i t_0 = t_0 t_i$, hence $\varphi'(t_i)\,\varphi'(t_0) = \varphi'(t_0)\,\varphi'(t_i)$. Since, moreover, both T_d and $T_{\partial\mathbb{D}}$ commute with H_{a_i} and with $H_{a'}$, we have $H_{a_i}H_{a'} = H_{a'}H_{a_i}$. By Proposition 3.5 we deduce that $a_i \cap a' = \varnothing$. Let $i \in \{1, n\}$. We have $t_i t_0 t_i = t_0 t_i t_0$, hence $\varphi'(t_i)\,\varphi'(t_0)\,\varphi'(t_i) = \varphi'(t_0)\,\varphi'(t_i)\,\varphi'(t_0)$. Since, moreover, both T_d and $T_{\partial\mathbb{D}}$ commute with H_{a_i} and with $H_{a'}$, we have $H_{a_i}H_{a'}H_{a_i} = H_{a'}H_{a_i}H_{a'}$. By Proposition 3.5 we deduce that $|a_i \cap a'| = 1$ and $a_i \cap a' \subset \mathcal{P}_{n+2}$. It is easily seen that up to isotopy there exist exactly two arcs a' satisfying $a' \cap d = \varnothing$, $a_i \cap a' = \varnothing$ for all $2 \le i \le n-1$, $|a_1 \cap a'| = |a_n \cap a'| = 1$, $a_1 \cap a' \subset \mathcal{P}_{n+2}$ and $a_n \cap a' \subset \mathcal{P}_{n+2}$. These two arcs are c_0 and c_1. So, either $\varphi'(t_0) = H_{c_0}^{\varepsilon} T_d^p T_{\partial\mathbb{D}}^q$ or $\varphi'(t_0) = H_{c_1}^{\varepsilon} T_d^p T_{\partial\mathbb{D}}^q$, that is, either $\varphi(t_0) = g v_0^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$ or $\varphi(t_0) = g v_1^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$.

Case 2: $p = 0$. Let $\varphi' = \mathrm{conj}_{g^{-1}} \circ \varphi$. We have $\varphi'(t_i) = s_i'^{\varepsilon} \Delta^{2q} = H_{a_i}^{\varepsilon} T_{\partial\mathbb{D}}^q$ for all $1 \le i \le n$. Since t_0 is conjugate to t_1 in $A[\tilde{A}_n]$, $\varphi'(t_0)$ is conjugate to $\varphi'(t_1)$ in $A[A_{n+1}]$, hence $\varphi'(t_0)$ is of the form $\varphi'(t_0) = H_{a'}^{\varepsilon} T_{\partial\mathbb{D}}^q$, where a' is an arc of $(\mathbb{D}, \mathcal{P}_{n+2})$.

Let $i \in \{2, \dots, n-1\}$. We have $t_i t_0 = t_0 t_i$, hence $\varphi'(t_i)\,\varphi'(t_0) = \varphi'(t_0)\,\varphi'(t_i)$. Since, moreover, $T_{\partial\mathbb{D}}$ commutes with H_{a_i} and with $H_{a'}$, we have $H_{a_i}H_{a'} = H_{a'}H_{a_i}$. By Proposition 3.5 we deduce that $a_i \cap a' = \varnothing$. Let $i \in \{1, n\}$. We have $t_i t_0 t_i = t_0 t_i t_0$, hence $\varphi'(t_i)\,\varphi'(t_0)\,\varphi'(t_i) = \varphi'(t_0)\,\varphi'(t_i)\,\varphi'(t_0)$. Since, moreover, $T_{\partial\mathbb{D}}$ commutes with H_{a_i} and with $H_{a'}$, we have $H_{a_i}H_{a'}H_{a_i} = H_{a'}H_{a_i}H_{a'}$. By Proposition 3.5 we deduce that $|a_i \cap a'| = 1$ and $a_i \cap a' \subset \mathcal{P}_{n+2}$. Since a' does not intersect a_i for any $2 \le i \le n-1$, we conclude that $a' \cap a_1$ is the singleton $\{p_0\}$, and $a' \cap a_n$ is the singleton $\{p_n\}$, so that a' is an arc connecting p_0 to p_n.

The union $\left(\bigcup_{i=1}^n a_i\right) \cup a'$ bounds a disk D embedded in $\mathbb{D}$. This disk either contains p_{n+1}, or does not contain p_{n+1}. If D does not contain p_{n+1}, then it is easily seen that a' is isotopic to c_0 or to c_1, hence either $\varphi'(t_0) = H_{c_0}^{\varepsilon} T_{\partial\mathbb{D}}^q$ or $\varphi'(t_0) = H_{c_1}^{\varepsilon} T_{\partial\mathbb{D}}^q$, that is, either $\varphi(t_0) = g v_0^{\varepsilon} \Delta^{2q} g^{-1}$ or $\varphi(t_0) = g v_1^{\varepsilon} \Delta^{2q} g^{-1}$.

Suppose now that D contains p_{n+1}. For the rest of the proof we "unmark" point p_{n+1} and consider isotopies of $\Omega = \left(\mathbb{D}, \{p_0, \dots, p_n\}\right)$, which are allowed to move point p_{n+1}. Under such isotopies, a' is isotopic to either c_0 or c_1, as above. Let $\{F_t \colon \Omega \to \Omega\}_{t \in [0,1]}$ be such an isotopy, for which $F_0 = \mathrm{id}$, $F_1(a') = c_k$, ($k \in \{0, 1\}$), and F_t is the identity on $\left(\bigcup_{i=1}^n a_i\right) \cup \partial\mathbb{D}$ for all $t \in [0, 1]$. The reader should keep in mind that the disk D can be embedded in $\mathbb{D}$ in a highly complicated manner, and the isotopy F_t "unwinds" it to one of the two standard positions: $F_1(D)$ is bounded by $\bigcup_{i=1}^n a_i$ and c_k (for $k = 0$ or 1), with $F_1(p_{n+1})$ belonging to $F_1(D)$. Now we choose another isotopy $\{F_t' \colon \Omega \to \Omega\}_{t \in [0,1]}$ such that $F_0' = \mathrm{id}$, $F_1'(c_k) = b_k$, F_t' is the identity on $\left(\bigcup_{i=1}^n a_i\right) \cup \partial\mathbb{D}$ for all $t \in [0, 1]$ and $F_1'(F_1(p_{n+1})) = p_{n+1}$. Obviously such isotopies exist. Let $\tilde{F} = F_1' \circ F_1$. Then $\tilde{F} \in \mathrm{Homeo}^+(\mathbb{D}, \mathcal{P}_{n+2})$ and we let $h \in \mathcal{M}(\mathbb{D}, \mathcal{P}_{n+2}) = A[A_{n+1}]$ be the mapping class $[\tilde{F}]$ represented by $\tilde{F}$. We see that $h([a_i]) = [a_i]$ for all $1 \le i \le n$ and $h([a']) = [b_k]$, for $k \in \{0, 1\}$. Note that, by [Par97b, Theorem 1.1], we must have $h \in \langle \Delta_Y^2, \Delta^2 \rangle = \langle T_d, T_{\partial\mathbb{D}} \rangle$, but this fact is not needed for the proof. We have $h H_{a_i} h^{-1} = H_{a_i}$ for all $1 \le i \le n$ and $h H_{a'} h^{-1} = H_{b_k}$, hence $(\mathrm{conj}_h \circ \varphi')(t_i) = s_i'^{\varepsilon} \Delta^{2q}$ for all $1 \le i \le n$ and $(\mathrm{conj}_h \circ \varphi')(t_0) = u_k^{\varepsilon} \Delta^{2q}$. So, $\varphi(t_i) = g h^{-1} s_i'^{\varepsilon} \Delta^{2q} h g^{-1}$ for all $1 \le i \le n$ and $\varphi(t_0) = g h^{-1} u_k^{\varepsilon} \Delta^{2q} h g^{-1}$. □

Proof of Theorem 2.1 Let $\varphi \colon A[\tilde{A}_n] \to A[\tilde{A}_n]$ be a homomorphism. Recall that $A[\tilde{A}_n]$ is viewed as a subgroup of $A[A_{n+1}]$, where $t_i = s_i'$ for all $1 \le i \le n$ and $t_0 = s_1' \dots s_n' s_{n+1}'^2 s_n' s_{n+1}'^{-2} s_n'^{-1} \dots s_1'^{-1}$.

Recall also that $Y = \{s'_1, \dots, s'_n\} = \{t_1, \dots, t_n\}$, $\Delta_Y = \Delta_Y[A_{n+1}] = \Delta_Y[\tilde{A}_n]$ and $\Delta = \Delta[A_{n+1}]$. By Proposition 4.1 we have one of the following three possibilities.

(1) φ is cyclic.

(2) There exist $g \in A[A_{n+1}]$, $k \in \{0,1\}$, $\varepsilon \in \{\pm 1\}$ and $q \in \mathbb{Z}$ such that $\varphi(t_i) = g s_i'^{\varepsilon} \Delta^{2q} g^{-1}$ for all $1 \le i \le n$ and $\varphi(t_0) = g u_k^{\varepsilon} \Delta^{2q} g^{-1}$.

(3) There exist $g \in A[A_{n+1}]$, $k \in \{0,1\}$, $\varepsilon \in \{\pm 1\}$ and $p, q \in \mathbb{Z}$ such that $\varphi(t_i) = g s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$ for all $1 \le i \le n$ and $\varphi(t_0) = g v_k^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$.

If φ is cyclic, then there is nothing to prove. So, we can assume that we are in Case (2) or in Case (3) of Proposition 4.1. More precisely, we suppose that there exist $g \in A[A_{n+1}]$, $w \in \{u_0, u_1, v_0, v_1\}$, $\varepsilon \in \{\pm 1\}$ and $p, q \in \mathbb{Z}$ such that $\varphi(t_i) = g s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$ for all $1 \le i \le n$, $\varphi(t_0) = g w^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$, and $p = 0$ if $w \in \{u_0, u_1\}$.

Recall the inclusions $A[\tilde{A}_n] \subset A[B_{n+1}] \subset A[A_{n+1}]$, where $r_i = s'_i$ for all $1 \le i \le n$ and $r_{n+1} = s_{n+1}'^2$. Recall also that $A[B_{n+1}] = \omega^{-1}(W_Y[A_{n+1}])$, where $\omega\colon A[A_{n+1}] \to W[A_{n+1}]$ is the standard epimorphism which sends s'_i to s'_i for all $1 \le i \le n+1$ (see Remark 4 after Proposition 3.2). If we identify $A[A_{n+1}]$ with $\mathcal{M}(\mathbb{D}, \mathcal{P}_{n+2})$, $W[A_{n+1}]$ can be identified with the symmetric group $\mathrm{Sym}(\mathcal{P}_{n+2})$ permuting the punctures, and hence $A[B_{n+1}]$ is isomorphic to the stabilizer of one puncture in $\mathcal{M}(\mathbb{D}, \mathcal{P}_{n+2})$, i.e. $A[B_{n+1}] = \{f \in \mathcal{M}(\mathbb{D}, \mathcal{P}_{n+2}) \mid f(p_{n+1}) = p_{n+1}\}$. (In the braid group interpretation of $A[A_{n+1}]$, the subgroup $A[B_{n+1}]$ is exactly the subgroup that permutes all strands but one.)

We first prove that $g \in A[B_{n+1}]$. By the above, it suffices to show that $g(p_{n+1}) = p_{n+1}$, or, equivalently, that $g^{-1}(p_{n+1}) = p_{n+1}$. Suppose not, that is, suppose $g^{-1}(p_{n+1}) \neq p_{n+1}$. Then there exists $i \in \{0, 1, \dots, n\}$ such that $g^{-1}(p_{n+1}) = p_i$. Suppose $1 \le i \le n$. On the one hand, $\varphi(t_i)(p_{n+1}) = p_{n+1}$, because $\varphi(t_i) \in A[\tilde{A}_n] \subset A[B_{n+1}]$. On the other hand, $\varphi(t_i)$ is of the form $\varphi(t_i) = g s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$. Note that, in the notation given in Figures 3.2 and 4.2, $s_i'^{\varepsilon}(p_i) = H_{a_i}^{\varepsilon}(p_i) = p_{i-1}$, $\Delta_Y^2(p_i) = T_d(p_i) = p_i$ and $\Delta^2(p_i) = T_{\partial\mathbb{D}}(p_i) = p_i$, hence

$$\varphi(t_i)(p_{n+1}) = (g s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1})(p_{n+1}) = (g s_i'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q})(p_i) = g(p_{i-1}) \neq p_{n+1}\,,$$

because $g^{-1}(p_{n+1}) = p_i \neq p_{i-1}$. This is a contradiction. Now suppose $i = 0$. On the one hand, $\varphi(t_1)(p_{n+1}) = p_{n+1}$, because $\varphi(t_1) \in A[\tilde{A}_n] \subset A[B_{n+1}]$. On the other hand, $\varphi(t_1)$ is of the form $\varphi(t_1) = g s_1'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1}$. Note that $s_1'^{\varepsilon}(p_0) = H_{a_1}^{\varepsilon}(p_0) = p_1$, $\Delta_Y^2(p_0) = T_d(p_0) = p_0$ and $\Delta^2(p_0) = T_{\partial\mathbb{D}}(p_0) = p_0$, hence

$$\varphi(t_1)(p_{n+1}) = (g s_1'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q} g^{-1})(p_{n+1}) = (g s_1'^{\varepsilon} \Delta_Y^{2p} \Delta^{2q})(p_0) = g(p_1) \neq p_{n+1}\,,$$

because $g^{-1}(p_{n+1}) = p_0 \neq p_1$. This is a contradiction. So, $g^{-1}(p_{n+1}) = p_{n+1}$, hence $g \in A[B_{n+1}]$.

Now, we prove that $q = 0$. We need the following lemma.

Lemma 4.2 $\Delta^2 = \Delta[A_{n+1}]^2 = \Delta[B_{n+1}]$.

Proof of Lemma 4.2 It is known that

$$\Delta[A_{n+1}]^2 = (s'_1 \dots s'_n s_{n+1}'^2 s'_n \dots s'_1)(s'_2 \dots s'_n s_{n+1}'^2 s'_n \dots s'_2) \dots (s'_n s_{n+1}'^2 s'_n) s_{n+1}'^2\,,$$
$$\Delta[B_{n+1}] = (r_1 \dots r_n r_{n+1} r_n \dots r_1)(r_2 \dots r_n r_{n+1} r_n \dots r_2) \dots (r_n r_{n+1} r_n) r_{n+1}\,,$$

(see [CP23, Lemma 5.1] for the first equality and [Par97b, Lemma 4.1] for the second one), hence $\Delta^2 = \Delta[A_{n+1}]^2 = \Delta[B_{n+1}]$. □

Let $z\colon A[B_{n+1}] \to \mathbb{Z}$ be the homomorphism which sends r_i to 0 for all $1 \le i \le n$ and sends r_{n+1} to 1. From the formula given in Section 2 we know that $\Delta[B_{n+1}] = (r_1 \dots r_n r_{n+1})^{n+1}$, and hence, by Lemma 4.2, $z(\Delta^2) = z(\Delta[B_{n+1}]) = (n+1)z(r_{n+1}) = n+1$. Recall that $\mathrm{Ker}(z) = A[\tilde{A}_n]$. We have $z(\Delta_Y^2) = 0$, because $\Delta_Y^2 \in \langle r_1, \dots, r_n \rangle \subset A[\tilde{A}_n]$, and we have $z(\varphi(t_1)) = 0$, since $\varphi(t_1) \in A[\tilde{A}_n]$ as well. The element $\varphi(t_1)$ is of the form $\varphi(t_1) = g r_1 \Delta_Y^{2p} \Delta^{2q} g^{-1}$, hence we also have

$$0 = z(\varphi(t_1)) = z(g) + q(n+1) - z(g) = q(n+1)\,,$$

which is possible only if $q = 0$.

Let $\rho_B = r_1 \dots r_n r_{n+1}$ be the element of $A[B_{n+1}]$ defined at the beginning of Section 3. By Proposition 3.1, there exist $g_1 \in A[\tilde{A}_n]$ and $m \in \mathbb{Z}$ such that $g = g_1 \rho_B^m$. Moreover, $\rho_B^m f \rho_B^{-m} = \zeta^m(f)$ for all $f \in A[\tilde{A}_n]$. We set $\varphi' = \zeta^{-m} \circ \mathrm{conj}_{g_1^{-1}} \circ \varphi$. Then $\varphi'(t_i) = s_i'^{\varepsilon} \Delta_Y^{2p} = t_i^{\varepsilon} \Delta_Y^{2p}$ for all $1 \le i \le n$, and $\varphi'(t_0) = w^{\varepsilon} \Delta_Y^{2p}$.

From Proposition 4.1 we see that if $w = u_0$ or u_1, then there is no Δ_Y^{2p} factor in the formulas for φ, i.e. $p = 0$. Now, if $w = u_0$ and $\varepsilon = 1$, then $\varphi' = \mathrm{id}$, hence $\varphi = \mathrm{conj}_{g_1} \circ \zeta^m \circ \varphi' = \mathrm{conj}_{g_1} \circ \zeta^m$. If $w = u_0$ and $\varepsilon = -1$, then $\varphi' = \mu$, hence $\varphi = \mathrm{conj}_{g_1} \circ \zeta^m \circ \varphi' = \mathrm{conj}_{g_1} \circ (\zeta^m \circ \mu)$. If $w = u_1$ and $\varepsilon = 1$, then $\varphi' = \mathrm{conj}_{\Delta_Y^{-1}} \circ \zeta \circ \eta$, hence $\varphi = \mathrm{conj}_{g_1} \circ \zeta^m \circ \varphi' = \mathrm{conj}_h \circ (\zeta^{m+1} \circ \eta)$, where $h = g_1 \cdot \zeta^m(\Delta_Y^{-1})$. If $w = u_1$ and $\varepsilon = -1$, then $\varphi' = \mathrm{conj}_{\Delta_Y^{-1}} \circ \zeta \circ \eta \circ \mu$, hence $\varphi = \mathrm{conj}_{g_1} \circ \zeta^m \circ \varphi' = \mathrm{conj}_h \circ (\zeta^{m+1} \circ \eta \circ \mu)$, where $h = g_1 \cdot \zeta^m(\Delta_Y^{-1})$. We conclude that if $w \in \{u_0, u_1\}$, then φ has the form stated in Case (2) of the theorem.

To understand the case $w \in \{v_0, v_1\}$ we must first observe that $\mu(v_0) = v_1^{-1}$ and $\mu(v_1) = v_0^{-1}$. If $w = v_0$ and $\varepsilon = 1$, then $\varphi' = \alpha_p$, hence $\varphi = \mathrm{conj}_{g_1} \circ \zeta^m \circ \varphi' = \mathrm{conj}_{g_1} \circ \zeta^m \circ \alpha_p$. If $w = v_0$ and $\varepsilon = -1$, then $\varphi' = \mu \circ \beta_{-p}$, hence $\varphi = \mathrm{conj}_{g_1} \circ \zeta^m \circ \varphi' = \mathrm{conj}_{g_1} \circ (\zeta^m \circ \mu) \circ \beta_{-p}$. If $w = v_1$ and $\varepsilon = 1$, then $\varphi' = \beta_p$, hence $\varphi = \mathrm{conj}_{g_1} \circ \zeta^m \circ \varphi' = \mathrm{conj}_{g_1} \circ \zeta^m \circ \beta_p$. If $w = v_1$ and $\varepsilon = -1$, then $\varphi' = \mu \circ \alpha_{-p}$, hence $\varphi = \mathrm{conj}_{g_1} \circ \zeta^m \circ \varphi' = \mathrm{conj}_{g_1} \circ (\zeta^m \circ \mu) \circ \alpha_{-p}$. We conclude that if $w \in \{v_0, v_1\}$, then φ has the form stated in Cases (3) and (4) of the theorem. □

Proof of Proposition 2.2 Let $\psi, \psi' \in \mathrm{Aut}(A[\tilde{A}_n])$ and $p, q \in \mathbb{Z}$ such that $\psi \circ \alpha_p = \psi' \circ \alpha_q$. Up to replacing ψ with $\psi'^{-1} \circ \psi$ if necessary, we can assume that $\psi' = \mathrm{id}$. Let $x\colon A[\tilde{A}_n] \to \mathbb{Z}$ be the homomorphism which sends t_i to 1 for all $0 \le i \le n$. By Theorem 2.1 there exist $\psi_1 \in \mathrm{Aut}^*(\tilde{A}_n)$ and $g \in A[\tilde{A}_n]$ such that $\psi = \mathrm{conj}_g \circ \psi_1$. From this decomposition it follows that either $x(\psi(t_i)) = 1$ for all $0 \le i \le n$, or $x(\psi(t_i)) = -1$ for all $0 \le i \le n$. This implies that either $(x \circ \psi \circ \alpha_p)(t_1) = 1 + pn(n+1)$, or $(x \circ \psi \circ \alpha_p)(t_1) = -1 - pn(n+1)$. Since $n(n+1) \ge 20$ and $(x \circ \alpha_q)(t_1) = 1 + qn(n+1)$, it follows that $p = q$ and $(x \circ \psi \circ \alpha_p)(t_1) = 1 + pn(n+1)$.

Let $\psi, \psi' \in \mathrm{Aut}(A[\tilde{A}_n])$ and $p, q \in \mathbb{Z}$. We prove in the same way that, if $\psi \circ \beta_p = \psi' \circ \beta_q$, then $p = q$. Similarly, if $\psi \circ \alpha_p = \psi' \circ \beta_q$, then $p = q$. It remains to show that $\psi \circ \alpha_p \neq \psi' \circ \beta_p$.

Suppose $\psi \circ \alpha_p = \psi' \circ \beta_p$. Up to replacing ψ with $\psi'^{-1} \circ \psi$ if necessary, we can assume that $\psi' = \mathrm{id}$. For each $1 \le i \le n$ we have

$$\psi(t_i \Delta_Y^{2p}) = (\psi \circ \alpha_p)(t_i) = \beta_p(t_i) = t_i \Delta_Y^{2p}\,,$$

hence, since Δ_Y^2 commutes with t_i for all $1 \le i \le n$,

$$(\psi \circ \alpha_p)(t_0) = \psi(v_0 \Delta_Y^{2p}) = \psi\big((t_1 \dots t_{n-1} t_n t_{n-1}^{-1} \dots t_1^{-1})\Delta_Y^{2p}\big) =$$
$$\psi\big((t_1 \Delta_Y^{2p}) \dots (t_{n-1}\Delta_Y^{2p})(t_n \Delta_Y^{2p})(t_{n-1}\Delta_Y^{2p})^{-1} \dots (t_1 \Delta_Y^{2p})^{-1}\big) =$$
$$(t_1 \Delta_Y^{2p}) \dots (t_{n-1}\Delta_Y^{2p})(t_n \Delta_Y^{2p})(t_{n-1}\Delta_Y^{2p})^{-1} \dots (t_1 \Delta_Y^{2p})^{-1} =$$
$$(t_1 \dots t_{n-1} t_n t_{n-1}^{-1} \dots t_1^{-1})\Delta_Y^{2p} = v_0 \Delta_Y^{2p}\,.$$

This is a contradiction because $\beta_p(t_0) = v_1 \Delta_Y^{2p}$, and $v_0 \neq v_1$ by Corollary 3.6, since the respective half-twists $H_{c_0} = v_0$ and $H_{c_1} = v_1$ correspond to non-isotopic arcs c_0 and c_1 in $(\mathbb{D}, \mathcal{P}_{n+2})$, see Figure 4.1. □

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