LINEARITY CRITERIA FOR POLY-FREE GROUPS

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It is an interesting open problem to give criteria of when an extension of two linear groups is linear. In this note we address this question for the following case:

Let F be a free group of finite rank, and A a finitely generated subgroup of $\mathrm{Aut}(F)$. *Give necessary and sufficient conditions for the existence of a faithful finite dimensional representation over a field of characteristic* 0 *for the semidirect product* $F \rtimes A$.

Definition ([DDMS, B.1])**.** Let G be a group. A *p-congruence system* in G is a descending chain $(N_i)_{i\in\mathbb{N}}$ of normal subgroups of G such that:

(i) G/N_1 is finite;
(ii) N_1/N_i is a finite p-group for all $i \geq 1$; and
(iii) $\bigcap_{i=1}^{\infty} N_i = 1$.

If in addition, a pro-p group $E = \varprojlim_{i\to\infty} N_1/N_i$ is an analytic pro-p group, we call $(N_i)_{i\in\mathbb{N}}$ an *analytic p-congruence system* in G.

Remark. Analytic pro-p groups can be defined in many equivalent ways, see [DDMS, Interlude A]. One of possible defining conditions for a pro-p group G to be analytic, is to have finite rank, i.e. the number

$$\mathrm{rk}(G) = \sup\{\, \mathrm{d}(H) \mid H \leq_c G \,\}$$

should be finite. Here $H \leq_c G$ means that H is a closed subgroup of G, and $\mathrm{d}(H)$ denotes the minimal cardinality of a topological generating set for H. Other equivalent conditions for a pro-p group G to be analytic include:

- G is isomorphic to a closed subgroup of $GL_d(\mathbb{Z}_p)$;
- G is a p-adic analytic Lie group.

We prove the following theorem, which strengthens the known criteria for linearity of semidirect products [CCLP, Th. 2.7] for the case when the normal factor of a semidirect product is free.

Theorem. *Let F be a free group of finite rank and A a finitely generated subgroup of* $\mathrm{Aut}(F)$. *Then the group $F \rtimes A$ is linear over a field of characteristic* 0 *if and only if A preserves some analytic p-congruence system $(N_i)_{i\in\mathbb{N}}$ in F.*

Here 'A preserves a p-congruence system $(N_i)_{i\in\mathbb{N}}$' means that for each automorphism $a \in A$, and for each $i \in \mathbb{N}$, we have: $a(N_i) \subseteq N_i$. Since all N_i's have finite index in F, this condition is equivalent to $a(N_i) = N_i$ for all $i \in \mathbb{N}$.

Proof of Theorem.
The proof in one direction follows the lines of [DDMS, B.6, B.5]. Let $G = F \rtimes A$ be linear over a field k of characteristic 0. Since G is finitely generated, we may assume that $G = \langle x_1, \ldots, x_m \rangle \subseteq GL_n(k)$ for some m, n. Let R be a subring of k generated by the entries

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of the matrices x_i, x_i^{-1} $(1 \leq i \leq m)$. Then $G \subseteq GL_n(R)$. By Lemma B.4 of [DDMS], there exist numbers s and ℓ such that R can be embedded into $\mathrm{Mat}_s(\mathbb{Z}_p)$ for every prime $p \nmid \ell$. Then $GL_n(R)$, and hence G, embeds into $GL_d(\mathbb{Z}_p)$, where $d = ns$.

For each i, let Γ_i denote the principal congruence subgroup modulo p^i in $GL_d(\mathbb{Z}_p)$:

$$\Gamma_i = \{\, g \in GL_d(\mathbb{Z}_p) \mid g \equiv 1 \pmod{p^i} \,\}.$$

Put $\varepsilon = 0$ if p is odd, $\varepsilon = 1$, if $p = 2$, and for each i let $N_i = F \cap \Gamma_{i+\varepsilon}$. Then arguing the same way as in the proof of Proposition B.5 of [DDMS], we conclude that $(N_i)_{i\in\mathbb{N}}$ is a p-congruence system in F such that $\mathrm{d}(N_i/N_j) \leq d^2$ for all $j > i \geq 1$. (We treat finite groups as discrete topological groups, so $\mathrm{d}(N_i/N_j)$ is just the minimal cardinality of a generating set for N_i/N_j.) Consider $E = \varprojlim_{j\to\infty} N_1/N_j$. If N is an open normal subgroup of E, then E/N is a homomorphic image of N_1/N_j for some $j \geq 1$, so $\mathrm{rk}(E/N) \leq d^2$. By Proposition 3.11 of [DDMS], $\mathrm{rk}(E) \leq d^2$, and hence E is an analytic pro-p group, thus making $(N_i)_{i\in\mathbb{N}}$ an analytic p-congruence system in F.

Finally, the action of A on F is induced by conjugation in $F \rtimes A \subseteq GL_d(\mathbb{Z}_p)$, and conjugation by matrices from $GL_d(\mathbb{Z}_p)$ preserves principal congruence subgroups $\Gamma_i \subseteq GL_d(\mathbb{Z}_p)$, and hence the action of A on F preserves the analytic p-congruence system $(N_i)_{i\in\mathbb{N}}$.

In the other direction, let $(N_i)_{i\in\mathbb{N}}$ be an analytic p-congruence system in F preserved by A. Consider $E = \varprojlim_{i\to\infty} N_1/N_i$. Since A preserves each subgroup N_i, every element $a \in A$ extends to an automorphism $\kappa(a)$ of E. Indeed, E can be identified with a subspace E_Π of Cartesian product $\prod_{i=1}^{\infty} N_1/N_i$:

$$E_\Pi = \{\, (g_iN_i) \in \textstyle\prod_{i=1}^{\infty} N_1/N_i \mid \forall i > j, \quad g_iN_j = g_jN_j \,\},$$

and we define $\kappa\colon A \to \mathrm{Aut}_{abs}(E_\Pi)$ as $\kappa(a)((g_iN_i)_i) = (a(g_i)N_i)_i$. Here $\mathrm{Aut}_{abs}(E_\Pi)$ denotes the group of all abstract, i.e. not necessarily continuous, automorphisms of group E_Π. Clearly, for any $a \in A$, $\kappa(a)$ preserves subspace E_Π inside $\prod_{i=1}^{\infty} N_1/N_i$, since the defining condition of E_Π is equivalent to $g_i^{-1}g_j \in N_j$ for all $i > j$.

Since E is by assumption an analytic pro-p group, it has finite rank, and in particular, it is finitely generated as a pro-p group. By Corollary 1.22 of [DDMS], every abstract automorphism of a finitely generated pro-p group is its topological automorphism (as "the topology of a finitely generated pro-p group is determined by its group structure", see Corollary 1.21 of [DDMS]). This shows that $\kappa(A)$ lies in $\mathrm{Aut}(E)$, the group of all continuous automorphisms of pro-p group E. Let's endow $\mathrm{Aut}(E)$ with the compact-open topology. Then $\mathrm{Aut}(E)$ becomes a topological group with a continuous action on E [RZ, Th. 4.4.2]), and moreover, with this topology, $\mathrm{Aut}(E)$ is virtually a pro-p group of finite rank [DDMS, Th.5.7]. This means that there exists an open subgroup (and hence, of finite index) B in $\mathrm{Aut}(E)$ which is a pro-p group of finite rank. Consider $S = E \rtimes B$. This is a profinite group by Lemma 1.3.6(b) of [Wil], with E a closed normal subgroup of S. By Ex. 3.1 of [DDMS], $\mathrm{rk}(S) \leq \mathrm{rk}(E) + \mathrm{rk}(B)$, a finite number. Therefore, $S = E \rtimes B$ is a pro-p group of finite rank, and hence, by Th. 7.19 of [DDMS], it admits a faithful linear representation over $\mathbb{Z}_p$.

Now we determine the kernel of κ. By definition of κ, $\ker\kappa = \{\, a \in A \mid \forall \bar{g} \in E_\Pi,\ k(a)(\bar{g}) = \bar{g} \,\}$. Here $\bar{g} \in E_\Pi$ if and only if $\bar{g} = (g_iN_i)_{i=1}^{\infty}$, $g_i \in N_1$, such that $g_i^{-1}g_j \in N_j$ whenever $i > j$. The condition $\kappa(a)(\bar{g}) = \bar{g}$ is equivalent to $(a(g_i)N_i)_{i=1}^{\infty} = (g_iN_i))_{i=1}^{\infty}$, or

$$\text{for all } i, \text{ for all } g_i \in N_1, \text{ we have } g_i^{-1}a(g_i) \in N_i.$$

Fix arbitrary $g \in N_1$. By setting $g_i = g$ for all i, we get: $a \in \ker\kappa$ implies $g^{-1}a(g) \in N_i$ for all i. So $g^{-1}a(g) \in \bigcap_{i=1}^{\infty} N_i = 1$, and $a|_{N_1} = \mathrm{id}_{N_1}$. Clearly, the converse is also true, and we conclude that

$$\ker\kappa = \{\, a \in A \mid a|_{N_1} = \mathrm{id}_{N_1} \,\}.$$

The following lemma is the only place in the whole proof which uses the fact that F is a free group.

Lemma. *Let F be a free group of finite rank and α an automorphism of F such that $\alpha|_N = \mathrm{id}_N$ for some normal subgroup of finite index N in F. Then $\alpha = \mathrm{id}_F$.*

Proof of Lemma. UPDATE 4/4/2018: Let m be the index of N in F. Then for every $x \in F$, we have $x^m \in N$, so $\alpha(x)^m = \alpha(x^m) = x^m$. But in a free group, roots are unique, so $\alpha(x) = x$. □

Old proof. Let m be the index of N in F. Then for every $x \in F$, we have $x^m \in N$, and for all $g \in F$ we have $gx^mg^{-1} \in N$, and $gx^mg^{-1} = \alpha(gx^mg^{-1}) = \alpha(g)x^m\alpha(g)^{-1}$, so that $g^{-1}\alpha(g)$ centralizes x^m.

If rank of F is less than 2, then $\alpha = \mathrm{id}_F$ trivially, otherwise we can take x, y to be two distinct free generators of F. Then the centralizer of x^m is the cyclic subgroup generated by x: $C(x^m) = \langle x\rangle$, and similarly, $C(y^m) = \langle y\rangle$. Now, for all $g \in F$, $g^{-1}\alpha(g)$ should centralize both x^m and y^m, so

$$g^{-1}\alpha(g) \in C(x^m) \cap C(y^m) = \langle x\rangle \cap \langle y\rangle = 1,$$

and so $\alpha = \mathrm{id}_F$. □

The lemma shows that $\ker\kappa$ is trivial, so that $\kappa\colon A \to \mathrm{Aut}(E)$ is injective.

Let $A_1 = \kappa^{-1}(B)$. Then $\kappa(A_1) = \kappa(A) \cap B$, and

$$|A : A_1| = |\kappa(A) : \kappa(A_1)| = |\kappa(A) : \kappa(A) \cap B| = |\kappa(A)B : B| \le |\,\mathrm{Aut}(E) : B|,$$

which is a finite number since B is a finite index subgroup in $\mathrm{Aut}(E)$. So A_1 has finite index in A, and $E \rtimes A_1$ is a linear group over $\mathbb{Z}_p$ as it is isomorphic to $E \rtimes \kappa(A_1) \subseteq E \rtimes B$. In particular, $N_1 \rtimes A_1$ is a linear group over $\mathbb{Z}_p$, as N_1 embeds in E due to condition (iii) of the definition of a p-congruence system.

Finally, the index $|(F \rtimes A) : (N_1 \rtimes A_1)|$ is finite since

$$|(F \rtimes A) : (N_1 \rtimes A_1)| = |(F \rtimes A) : (N_1 \rtimes A)| \cdot |(N_1 \rtimes A) : (N_1 \rtimes A_1)| = |F : N_1| \cdot |A : A_1|.$$

Therefore, $F \rtimes A$ is also a linear group over $\mathbb{Z}_p$ via the induced representation construction. And $\mathbb{Z}_p$ is a subring of $\mathbb{Q}_p$, a field of characteristic 0.

□

References

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