PART I. PRESENTATIONS AND LINEARITY OF SOME LOW GENUS MAPPING CLASS GROUPS

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CONTENTS

1. INTRODUCTION

Let $S_{g,b,n}$ denote the compact orientable surface of genus g with b boundary components and n punctures. Let $\mathrm{PMod}_{g,b,n}$ denote the (pure) mapping class group of orientation-preserving diffeomorphisms of $S_{g,b,n}$ identical on the boundary and not permuting punctures, via the isotopies identical on the boundary and not permuting punctures.

It is a well-known open problem to find out if the mapping class group $\mathrm{PMod}_{g,b,n}$ admits a faithful linear representation for arbitrary values of g, b, n. In her article [BIR, Problem 18], Joan Birman mentions that for the following triples (g,b,n):

$$(1,0,0), (1,1,0), (1,0,1), (0,1,n), (0,0,n), (2,0,0)$$

the mapping class group $\mathrm{PMod}_{g,b,n}$ is known to be linear, and she asks if this list can be extended to contain any other triples.

In Part I, by following the approach of [LP], we provide detailed proofs of the known presentations for mapping class groups with some small values of (g,b,n). Then, by analyzing these presentations, we show that the list of mapping class groups which are linear can be extended to include the following triples (g,b,n):

$$(0,m,n),\ m>1, \quad (1,2,0), (1,1,1), (1,0,2), (1,3,0), (1,2,1), (1,1,2), (1,0,3).$$

The main result of this Part can be stated as follows:

Theorem. *The following table lists the isomorphism types of mapping class groups* $\mathrm{PMod}_{g,b,n}$ *for some small values of* g, b, n*:*

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(g, b, n)	$\mathrm{PMod}_{g,b,n}$
$(0, m, n), \quad m > 1$	$\mathbb{Z}^{m-1} \times PB_{n+m-1}$
$(1, 2, 0)$	$\mathbb{Z} \times B_4$
$(1, 1, 1)$	B_4
$(1, 0, 2)$	$B_4/Z(B_4)$
$(1, 3, 0)$	$\mathbb{Z}^2 \times A(D_4)$
$(1, 2, 1)$	$\mathbb{Z} \times A(D_4)$
$(1, 1, 2)$	$A(D_4)$
$(1, 0, 3)$	$A(D_4)/Z(A(D_4))$

Here B_n denotes the braid group on n strands, PB_n is the pure braid group on n strands, $A(D_4)$ is the Artin group of type D_4, and $Z(G)$ denotes the center of a group G.

The genus 0 case is proved in section 3 below (proposition 1), the case of $g = 1$ and $b + n = 2$ is proved in section 4 (propositions 2, 3, 4) and the case of $g = 1$ and $b + n = 3$ in section 5 (propositions 5, 6, 7).

Corollary. *All mapping class groups from the Theorem are linear.*

Proof. Braid groups are linear by the results of Krammer [KRA] and Bigelow [BIG]. That Artin groups of spherical type are linear (in particular, $A(D_4)$), was proved by Cohen–Wales [CW] and Digne [DIG]. The fact that if a group G is linear then the quotient group $G/Z(G)$ is also linear, follows from Theorem 6.4 in [WEHR] (see lemma 6 below). □

2. Preliminary lemmas

We will use a few well-known results.

Lemma 1 (Birman exact sequence, [FM, Th. 4.6]). *Let S be a surface with $\chi(S) < 0$, and let S^* be $S \setminus \{x\}$ for a point x in the interior of S. Then the following sequence is exact:*

$$1 \longrightarrow \pi_1(S, x) \xrightarrow{p} \mathrm{PMod}(S^*) \xrightarrow{\mathrm{F}} \mathrm{PMod}(S) \longrightarrow 1$$

where p is the "point-pushing map" and F is the forgetful map which treats all maps S^* to S^* as maps S to S which send x to x. □

Lemma 2 (Properties of the point-pushing map, [IVA, Lemma 4.1.I, 4.1.C], [FM, Fact 4.7, 4.8]).

(a) Let α be a simple loop in a surface S representing an element of $\pi_1(S, x)$. Then

$$p([\alpha]) = t_a^{-1} t_b$$

where $[\alpha]$ is the class of the loop α in $\pi_1(S, x)$, a and b are the isotopy classes of the simple closed curves in $S^* = S \setminus \{x\}$ obtained by pushing α off itself to the left and right, respectively, and t_a, t_b denote the right Dehn twists about a, b.

(b) For any $h \in \mathrm{PMod}(S^*)$ and any $\alpha \in \pi_1(S, x)$ we have

$$p(h_*(\alpha)) = h p(\alpha) h^{-1}.$$

□

Lemma 3 (Capping the boundary, [FM, Prop. 3.19]). *Let S' be the surface obtained from a surface S by capping the boundary component β with a once-punctured disk. Let*

Cap: $\mathrm{PMod}(S) \longrightarrow \mathrm{PMod}(S')$ *be the induced homomorphism obtained by extending homeomorphisms of* S *to the once-punctured disk by the identity. Then the following sequence is exact:*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{PMod}(S) \xrightarrow{\mathrm{Cap}} \mathrm{PMod}(S') \longrightarrow 1$$

where $\mathbb{Z}$ *is generated by the twist around* β. □

The following group-theoretic lemma describes the presentation of an extension of two finitely presented groups. Suppose we have a short exact sequence of groups:

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\sigma} Q \longrightarrow 1$$

and suppose that groups K and Q have presentations $K = \langle S_K \mid R_K \rangle$, $Q = \langle S_Q \mid R_Q \rangle$. For each $x \in S_Q$ consider an element $\tilde{x} \in G$ such that $\sigma(\tilde{x}) = x$ and let

$$\tilde{S}_Q = \{\tilde{x} \mid x \in S_Q\}.$$

Also set $\tilde{S}_K = \iota(S_K)$.

For every relator $r \in R_Q$, consider its decomposition $r = x_1^{n_1} \dots x_k^{n_k}$ where all $x_i \in S_Q$, and the corresponding element $\tilde{r} = \tilde{x}_1^{n_1} \dots \tilde{x}_k^{n_k} \in G$. Since r represents 1 in Q, we have $\sigma(\tilde{r}) = r = 1$, and therefore $\tilde{r} \in \iota(K)$, so we can express $\tilde{r}$ as a word w_r in elements of $\tilde{S}_K$. Set

$$R_{\text{quo}} = \{\tilde{r} w_r^{-1} \mid r \in R_Q\}.$$

For every $\tilde{x} \in \tilde{S}_Q$ and for every $y \in \tilde{S}_K$ we can consider element $\tilde{x} y \tilde{x}^{-1}$. Since $\iota(K)$ is normal in G, this element belongs to $\iota(K)$, so we can represent $\tilde{x} y \tilde{x}^{-1}$ as a word $v_{x,y}$ in the elements of $\tilde{S}_K$. Set

$$R_{\text{conj}} = \{\tilde{x} y \tilde{x}^{-1} v_{x,y}^{-1} \mid \tilde{x} \in \tilde{S}_Q, y \in \tilde{S}_K\}.$$

Finally, set R_{ker} be the set of words in $\iota(K)$ obtained from R_K by replacing each $y \in S_K$ by $\iota(y) \in \tilde{S}_K$ wherever it appears.

Lemma 4 ([JOHN, Proposition 10.2.1]). *With the above notation, the group* G *has a presentation:*

$$G = \langle \tilde{S}_K \cup \tilde{S}_Q \mid R_{\text{ker}} \cup R_{\text{quo}} \cup R_{\text{conj}} \rangle.$$

□

We will also make use of the following obvious restatements of the braid relation $aba = bab$.

Lemma 5. *Let* a, b *be two elements of a group, and let* $\bar{a}, \bar{b}$ *denote* a^{-1}, b^{-1}, *respectively. Then the following relations are equivalent:*

(b1) $aba = bab$;
(b2) $\bar{a}\bar{b}\bar{a} = \bar{b}\bar{a}\bar{b}$;
(b3) $\bar{a}ba = ba\bar{b}$;
(b4) $ab\bar{a} = \bar{b}ab$;
(b5) $a\bar{b}\bar{a} = \bar{b}\bar{a}b$;
(b6) $\bar{a}\bar{b}a = b\bar{a}\bar{b}$.

□

The following fact is a very useful tool to establish linearity:

Lemma 6. *If* G *is a linear group then the quotient group by its center* $G/Z(G)$ *is also linear.*

Proof. The Theorem 6.4 of [WEHR] states that if G is a linear group and H is a Zariski closed normal subgroup then G/H isomorphic to a linear group. Since the center of a linear group is closed in Zariski topology, the result follows. □

3. GENUS 0: $(g, b, n) = (0, m, n)$, $m > 1$

In this section, by using the known presentation of the pure braid group on n strands PB_n we obtain the following description of the mapping class group $\mathrm{PMod}_{0,m,n}$:

Proposition 1. *For* $m > 1$, $n \geq 0$,

$$\mathrm{PMod}_{0,m,n} \cong \mathbb{Z}^{m-1} \times PB_{m+n-1} \cong \mathbb{Z}^m \times PB_{m+n-1}/Z(PB_{m+n-1}).$$

This result is not new and there are articles where similar results are mentioned (see e.g. [HAR, Lemma 3.4]). But we were unable to find a detailed proof of it anywhere, so we supply it in this section.

We will make use of the following results.

Lemma 7 ([FM, 9.3]).

$$\mathrm{PMod}_{0,1,n} \cong PB_n.$$

□

Lemma 8 ([FM, 9.2, 9.3]).

$$\mathrm{PMod}_{0,0,n} \cong PB_{n-1}/Z(PB_{n-1}).$$

□

Remark 1. For small n, the mapping class groups from lemmas 7 and 8 are the following (see [FM, 4.2.4, 9.3]):

$$\mathrm{PMod}_{0,0,0} = \mathrm{PMod}_{0,0,1} = \mathrm{PMod}_{0,0,2} = \mathrm{PMod}_{0,0,3} = \mathrm{PMod}_{0,1,0} = \mathrm{PMod}_{0,1,1} = 1,$$
$$\mathrm{PMod}_{0,0,4} = F_2, \qquad \mathrm{PMod}_{0,1,2} = \mathbb{Z}, \qquad \mathrm{PMod}_{0,1,3} = F_2 \times \mathbb{Z}$$

where F_2 denotes the free group of rank 2.

We reproduce here a remarkable result about pure braid groups.

Lemma 9. *The extension*

$$1 \longrightarrow Z(PB_n) \longrightarrow PB_n \longrightarrow PB_n/Z(PB_n) \longrightarrow 1$$

is split, i.e.

$$PB_n \cong PB_n/Z(PB_n) \times \mathbb{Z}.$$

Proof. ([FM, 9.3], for algebraic proof see [DM, Lemma 1.5].) It is sufficient to exhibit an epimorphism $\phi\colon PB_n \longrightarrow \mathbb{Z} = Z(PB_n)$ such that the composition $\mathbb{Z} \longrightarrow PB_n \longrightarrow \mathbb{Z}$ is identical on $\mathbb{Z}$. Consider $f\colon PB_n \longrightarrow PB_2 \cong \mathbb{Z}$, the homomorphism of forgetting the last $n-2$ strands, and $g\colon Z = \langle \sigma_1^2 \rangle \longrightarrow \mathbb{Z} = \langle \Delta_n^2 \rangle$, an isomorphism sending the generator σ_1^2 of PB_2 onto the generator Δ_n^2 of the center of PB_n. Then the composition $\phi = g \circ f\colon PB_n \longrightarrow Z(PB_n)$ will have the desired properties. Indeed, we can see from the geometric description of the central element that $f(\Delta_n^2) = \sigma_1^2$ and $g(\sigma_1^2) = \Delta_n^2$ by definition. □

Corollary.

$$PB_3 \cong F_2 \times \mathbb{Z}$$

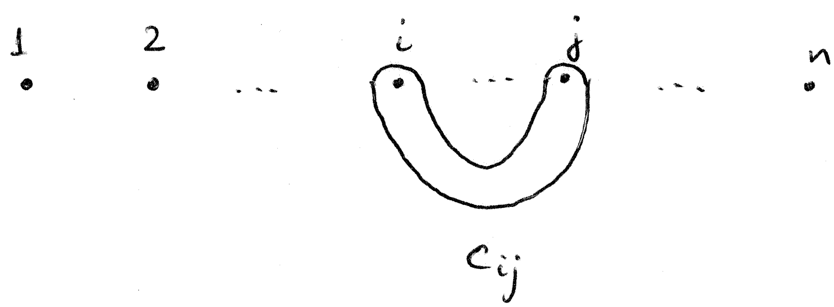
where F_2 *denotes the free group of rank* 2.

Proof. The proof follows from lemma 8 and the fact that $\mathrm{PMod}_{0,0,4} \cong F_2$ (see Remark 1 above). □

We will also need the following presentation of the pure braid group PB_n (which is a slight modification of the Artin's presentation for PB_n):

Lemma 10 ([MMcC, Th. 2.3]). *The pure braid group PB_n has the following presentation:*
Generators: a_{ij}, $\quad 1 \le i < j \le n$.
Relations:
(a) $[a_{pq}, a_{rs}] = 1 \quad$ *for all* $\quad p < q < r < s$;
(b) $[a_{ps}, a_{qr}] = 1 \quad$ *for all* $\quad p < q < r < s$;
(c) $a_{pr}a_{qr}a_{pq} = a_{qr}a_{pq}a_{pr} = a_{pq}a_{pr}a_{qr} \quad$ *for all* $\quad p < q < r$;
(d) $[a_{rs}a_{pr}a_{rs}^{-1}, a_{qs}] = 1 \quad$ *for all* $\quad p < q < r < s$. □

If we identify PB_n with $\mathrm{PMod}_{0,1,n}$ as in lemma 7, then the generator a_{ij} will correspond to the right Dehn twist around the curve c_{ij} which encircles i-th and j-th punctures only:



We treat all words in generators a_{ij} as written in the functional notation, i.e. the rightmost element is applied first. Now we are ready to prove the

Proposition 1. *For $m > 1$, $n \ge 0$,*

$$\mathrm{PMod}_{0,m,n} \cong \mathbb{Z}^{m-1} \times PB_{m+n-1} \cong \mathbb{Z}^m \times PB_{m+n-1}/Z(PB_{m+n-1}).$$

Proof. The second isomorphism follows from lemma 9. To prove the first isomorphism, we argue by induction. By lemma 7, $\mathrm{PMod}_{0,1,n} \cong PB_n \cong \mathbb{Z}^{1-1} \times PB_{1+n-1}$, which gives us the case of $m = 1$ (and arbitrary $n > 0$). Suppose that the statement is true for $\mathrm{PMod}_{0,m,n}$ (with m fixed and n arbitrary) and we prove it for $\mathrm{PMod}_{0,m+1,n}$. By lemma 3, we have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{PMod}_{0,m+1,n} \xrightarrow{\mathrm{Cap}} \mathrm{PMod}_{0,m,n+1} \longrightarrow 1$$

where the map Cap: $\mathrm{PMod}_{0,m+1,n} \longrightarrow \mathrm{PMod}_{0,m,n+1}$ is obtained by attaching a once-punctured disk to one of the boundary components and extending all maps to it by the identity. By the inductive assumption, $\mathrm{PMod}_{0,m,n+1} \cong \mathbb{Z}^{m-1} \times PB_{m+n}$ where the group PB_{m+n} is generated by elements a_{ij} from lemma 10, and $\mathbb{Z}^{m-1}$ is generated by the boundary twists around any $m-1$ out of m boundary components. By using lemma 4, we obtain the presentation for $\mathrm{PMod}_{0,m+1,n}$. In the notation of lemma 4 we have:

$K = \mathbb{Z} = \langle d_{m+1} \rangle$, where d_{m+1} is the right Dehn twist about the $(m+1)$-st boundary component;

$Q = \mathrm{PMod}_{0,m,n+1}$ with the presentation:

Generators:

a_{ij}, $1 \le i < j \le n$;
d_k, $2 \le k \le m$, (d_k is the right Dehn twist around the k-th boundary component).
Relations: (a),(b),(c),(d) from lemma 10, plus the commutation relations:
(e) $[d_k, a_{ij}] = 1$ $[d_k, d_l] = 1$ for all k, l, all i, j.

According to lemma 4, the group $G = \mathrm{PMod}_{0,m+1,n}$ will have generators:

$$\tilde{S}_Q \cup S_K$$

where

$$\tilde{S}_Q = \{\tilde{a}_{ij}\,, \tilde{d}_k \mid 1 \le i, j \le n,\, 2 \le k \le m\}, \quad S_K = \{d_{m+1}\}.$$

and relations

$$R_K \cup R_{\text{quo}} \cup R_{\text{conj}}.$$

It is easy to notice, that we can choose as $\tilde{a}_{ij}$ the Dehn twists about the same curves c_{ij} but now considered as transformations of the surface $S_{0,m+1,n}$. So it will be convenient for us to denote them with the same letters a_{ij}. Similarly, $\tilde{d}_k$, $(k = 2, \ldots, m)$ can be chosen as boundary twists of the surface $S_{0,m+1,n}$ so it is convenient to identify $\tilde{d}_k$ with d_k.

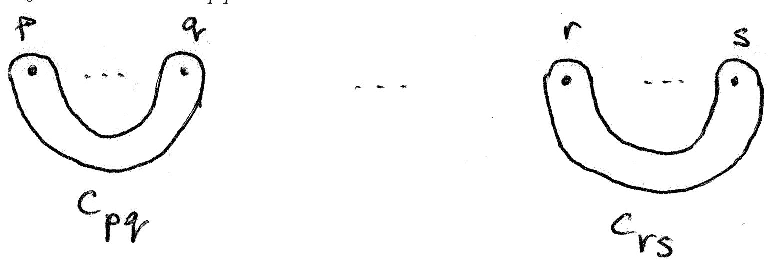
Therefore we may assume that $\mathrm{PMod}_{0,m+1,n}$ is generated by all a_{ij} and d_k, $k = 2, \ldots, m+1$.

To find relations $R_K \cup R_{\text{quo}} \cup R_{\text{conj}}$, we notice that $R_K = \varnothing$ since $K = \mathbb{Z}$, a free group. Since K is generated by the boundary twist d_{m+1}, which commutes with all other mapping classes, relations R_{conj} are:

$$R_{\text{conj}} = \{[d_{m+1}, a_{ij}] = 1, \quad [d_{m+1}, d_k] = 1, \quad 1 \le i < j \le n, \quad 2 \le k \le m\}.$$

To obtain relations R_{quo}, we need to 'lift' all relations (a)–(e) above from $Q = \mathrm{PMod}_{0,m,n+1}$ to $G = \mathrm{PMod}_{0,m+1,n}$:
(a): $[a_{pq}, a_{rs}] = 1$ for all $p < q < r < s$.

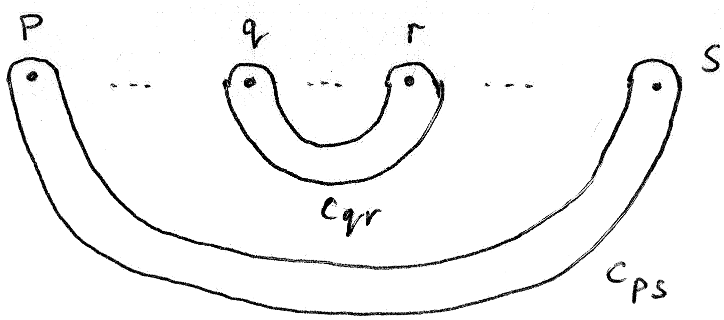
The relations of type (a) are disjointness relations, since a_{pq} and a_{rs} are right Dehn twists around disjoint curves c_{pq} and c_{rs}:



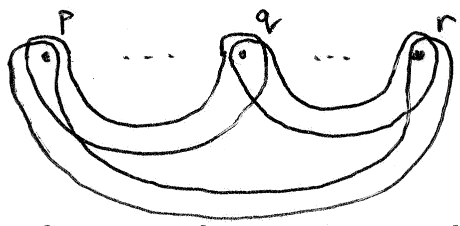
The corresponding Dehn twists in G will also commute so the same relation holds in G.

(b) $[a_{ps}, a_{qr}] = 1$ for all $p < q < r < s$.

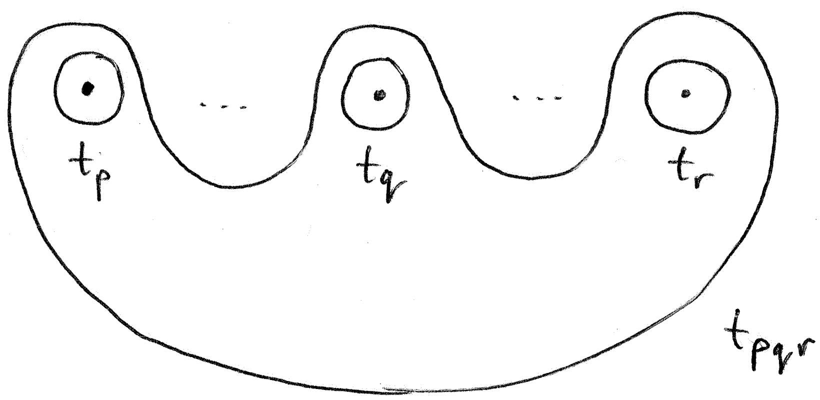
These relations are also disjointness relations, as the following picture shows.



(c) $a_{pr}a_{qr}a_{pq} = a_{qr}a_{pq}a_{pr} = a_{pq}a_{pr}a_{qr}$ for all $p < q < r$.



The configuration of curves c_{pq}, c_{pr} and c_{qr} is exactly the configuration of curves from the lantern relation (see e.g. [IVA, Lemma 4.1.H] or [FM, Prop. 5.1]) which says that these triple products are all equal to the product of the four right Dehn twists t_p, t_q, t_r and the twist t_{pqr} around the curve enclosing all three punctures p, q, r:

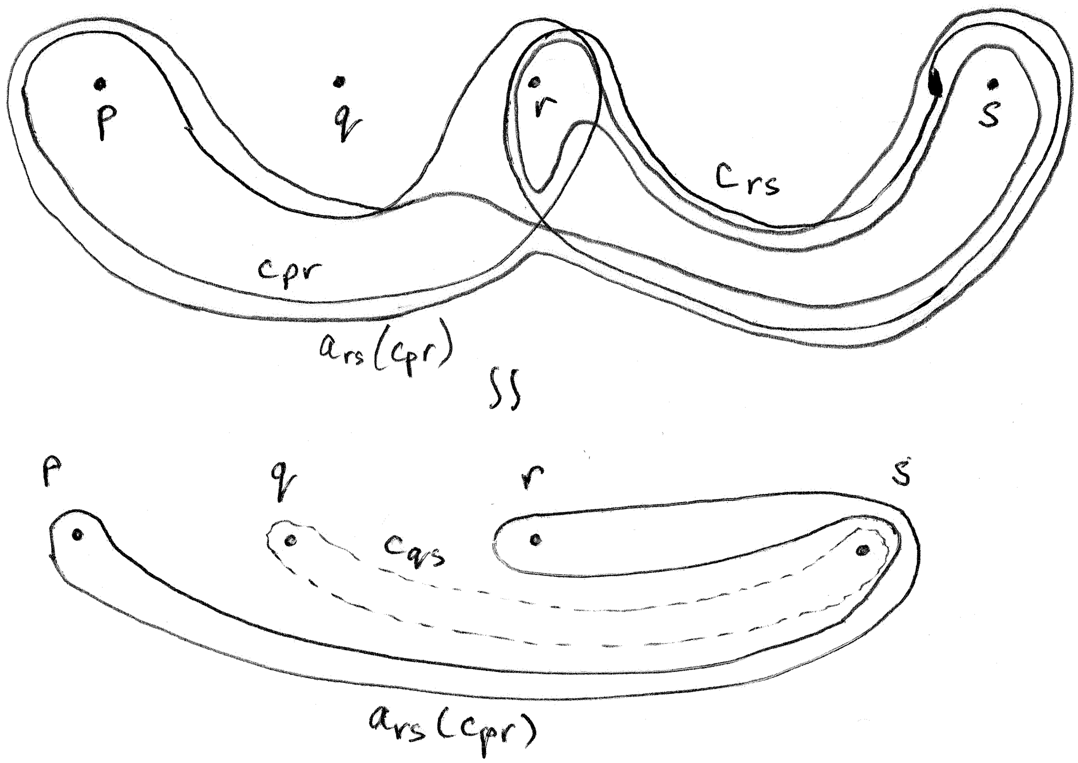


$$a_{pr}a_{qr}a_{pq} = a_{qr}a_{pq}a_{pr} = a_{pq}a_{pr}a_{qr} = t_pt_qt_rt_{pqr} \qquad \text{(lantern relation)}$$

In particular, relations of type (c) hold in the group G as well.

(d) $[a_{rs}a_{pr}a_{rs}^{-1}, a_{qs}] = 1$ for all $p < q < r < s$.

These relations are also disjointness relations: the map $a_{rs}a_{pr}a_{rs}^{-1}$ is the right Dehn twist around the curve $a_{rs}(c_{pr})$ (according to the well-known formula $ft_\alpha f^{-1} = t_{f(\alpha)}$) which is disjoint from the curve c_{qs}:



Therefore the relations of type (d) also hold in the group G.

(e) $[d_k, a_{ij}] = 1, \quad [d_k, d_l] = 1 \quad$ for all k, l, all i, j.
These relations hold in G since all twists d_k are boundary twists.
To summarize, we just established that all relations (a)–(e) hold as they are in the group $G = \mathrm{PMod}_{0,m+1,n}$ as well. Thus, the set of relations R_{quo} is just the relations (a)–(e) above. Therefore, the group G has all the generators and relations of the group $H = \mathrm{PMod}_{0,m,n+1}$ plus one additional generator d_{m+1} which commutes with all other generators.
Thus, $\mathrm{PMod}_{0,m+1,n} \cong \mathbb{Z} \times \mathrm{PMod}_{0,m,n+1} \cong \mathbb{Z}^m \times PB_{m+n}$, which finishes the proof. □

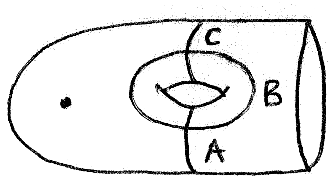
4. Genus 1, $b + n = 2$: $(g, b, n) = (1, 2, 0), (1, 1, 1), (1, 0, 2)$

In this section we prove that $\mathrm{PMod}_{1,1,1}$ is isomorphic to the braid group on four strands, and obtain presentations for $\mathrm{PMod}_{1,2,0}$ and $\mathrm{PMod}_{1,0,2}$ as corollaries.

Proposition 2.

$$\mathrm{PMod}_{1,1,1} \cong B_4 = \langle A, B, C \mid ABA = BAB, AC = CA, BCB = CBC \rangle,$$

where A,B,C are right Dehn twists about the curves shown in the picture:



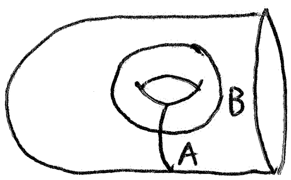
Proof. According to lemma 1, there is a short exact sequence

$$1 \longrightarrow \pi_1(S_{1,1,0}) \xrightarrow{p} \mathrm{PMod}_{1,1,1} \longrightarrow \mathrm{PMod}_{1,1,0} \longrightarrow 1$$

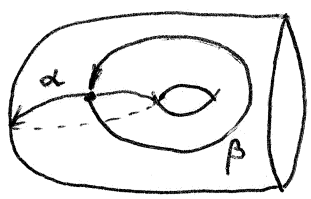
where p denotes the point-pushing map. It is a well-known fact (see for example [FM, 3.6.4]) that

$$\mathrm{PMod}_{1,1,0} \cong B_3 = \langle A, B \mid ABA = BAB \rangle$$

where A and B denote the right Dehn twists as in the picture below:



The fundamental group $\pi_1(S_{1,1,0})$ is a free group on two generators α, β, shown in the picture below:



and the Birman exact sequence takes the form:

$$1 \longrightarrow \langle \alpha, \beta \mid \rangle \xrightarrow{p} \mathrm{Mod}_{1,1,1} \longrightarrow \langle A, B \mid ABA = BAB \rangle \longrightarrow 1$$

Generators. We will adopt now notation from lemma 4.

Clearly, as a set $\tilde{S}_Q$ of representatives of twists A, $B \in \mathrm{PMod}_{1,1,0}$ (see picture 2) we can take the corresponding twists A, $B \in \mathrm{PMod}_{1,1,1}$ (see picture 1). As a set $\tilde{S}_K$ of generators of $\pi_1(S_{1,1,0})$, viewed as elements of $\mathrm{PMod}_{1,1,1}$, we take $\tilde{S}_K = \{a, b \mid a = p(\alpha),\ b = p(\beta)\}$.

Relations. According to lemma 4, the set of relations of $\mathrm{PMod}_{1,1,1}$ is the union of three sets: $R_{\text{ker}} \cup R_{\text{quo}} \cup R_{\text{conj}}$.

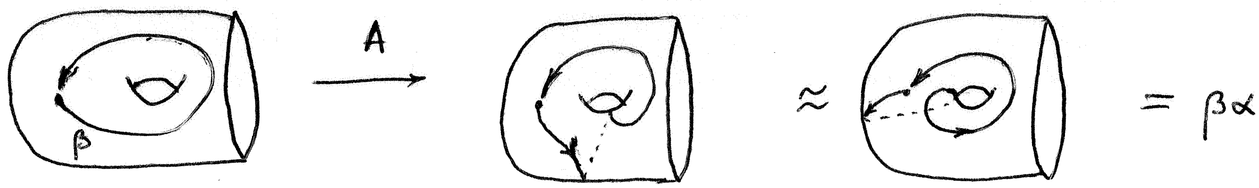
Clearly, $R_{\text{ker}} = \varnothing$ since $\pi_1(S_{1,1,0})$ is a free group.

Since the only relation in $\mathrm{PMod}_{1,1,0}$ is the braid relation $ABA = BAB$, the corresponding twists in $\mathrm{PMod}_{1,1,1}$ also satisfy it, so we see that it lifts to $\mathrm{PMod}_{1,1,1}$ without any change, and we have $R_{\text{quo}} = \{ABA = BAB\}$.

To obtain R_{conj}, we need to conjugate generators a, b of $p(\pi_1(S_{1,1,0}))$ with right Dehn twists A, B. By lemma 2(b), we have:

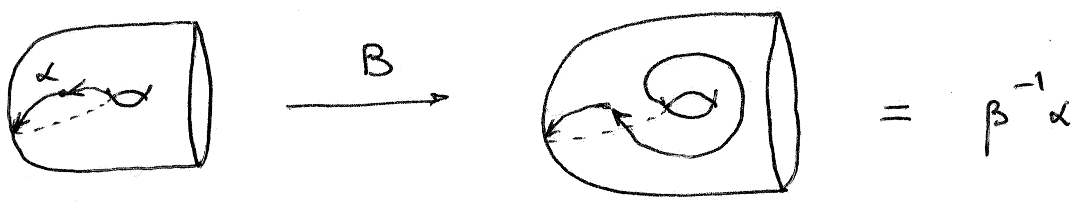
$$AaA^{-1} = p(A(\alpha)) = p(\alpha) = a$$

$$AbA^{-1} = p(A(\beta)) = p(\beta\alpha) = ba:$$



We adopt here the functional notation for multiplication of loops: in the notation $\beta\alpha$, loop α is traversed first.

$$BaB^{-1} = p(B(\alpha)) = p(\beta^{-1}\alpha) = b^{-1}a:$$



$$BbB^{-1} = p(B(\beta)) = p(\beta) = b.$$

Therefore, by lemma 4, the group $\mathrm{PMod}_{1,1,1}$ has a presentation:

Generators: a, b, A, B.

Relations: (i) $ABA = BAB$;

(1) $AaA^{-1} = a$;
(2) $AbA^{-1} = ba$;
(3) $BaB^{-1} = b^{-1}a$;
(4) $BbB^{-1} = b$.

Simplification of the presentation. In what follows, the bar ($\bar{\ }$) will denote the inverse element in a group.

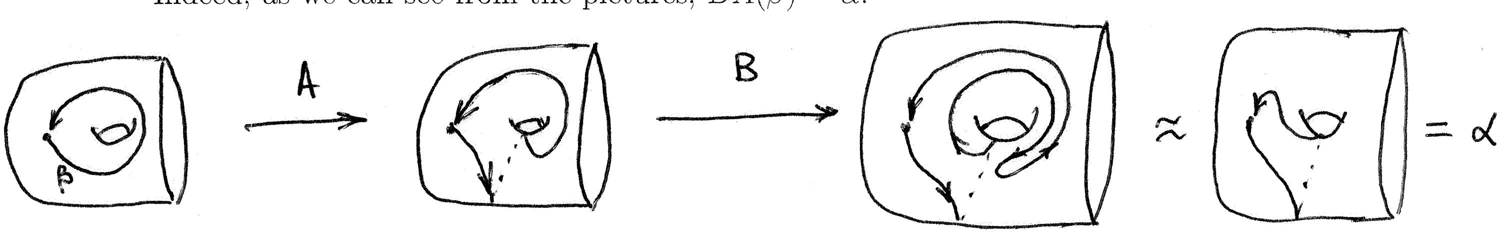
Let's introduce a new generator C which is the right Dehn twist denoted by the letter C in picture 1. Then, by lemma 2(a),

$$a = p(\alpha) = A^{-1}C = \bar{A}C$$

and this allows us to eliminate generator a from the presentation.

We can also eliminate generator b by expressing it in terms of A, B, and a.

Indeed, as we can see from the pictures, $BA(\beta) = \alpha$:



Thus, $\beta = (BA)^{-1}(\alpha)$, and, by lemma 2(a),

$$b = p(\beta) = (BA)^{-1}p(\alpha)(BA) = (BA)^{-1}a(BA) = \bar{A}\bar{B}\cdot\bar{A}C\cdot BA.$$

Thus we can assume that our group is generated by A, B, C only.

Now plugging the expression $a = \bar{A}C$ into relation (1) and performing obvious simplifications, we see that (1) is equivalent to the following sequence of relations:

$$\begin{aligned} AaA^{-1} &= a \\ \underline{A \cdot \bar{A}}C \cdot \bar{A} &= \bar{A}C \qquad \text{(cancel } A \cdot \bar{A}) \\ C\bar{A} &= \bar{A}C \\ AC &= CA \end{aligned}$$

So we obtained a new relation:

$$\text{(ii)} \qquad AC = CA.$$

Similarly, plugging the expressions for a and b into relation (3), we see that (3) is equivalent to the following relations:

$$\begin{aligned} BaB^{-1} &= b^{-1}a \\ B \cdot \bar{A}C \cdot \bar{B} &= \bar{A}\bar{B} \cdot \underline{\bar{C}A} \cdot B\underline{A \cdot \bar{A}}C \qquad \text{(exchange } \bar{C} \text{ and } A \text{ by (ii), cancel } A \cdot \bar{A}) \\ B\bar{A} \cdot C\bar{B} &= \underline{\bar{A}\bar{B}A} \cdot \bar{C}BC \qquad \text{(apply (i) and lemma 5, (b6))} \\ \underline{B\bar{A}} \cdot C\bar{B} &= \underline{B\bar{A}}\bar{B} \cdot \bar{C}BC \qquad \text{(cancel)} \\ C\bar{B} &= \bar{B}\bar{C}BC \\ BC\bar{B} &= \bar{C}BC \\ BCB &= CBC \end{aligned}$$

Thus we got yet another relation which we denote

$$\text{(iii)} \qquad BCB = CBC.$$

Let us show now that the remaining relations (2) and (4) are consequences of relations (i), (ii), (iii).

Relation (2) holds true since it is equivalent to the following sequence of relations:

$$\begin{aligned} AbA^{-1} &= ba \\ \underline{A(\bar{A}}\bar{B} \cdot \bar{A}C \cdot B\underline{A)\bar{A}} &= (\bar{A}\bar{B} \cdot \bar{A}C \cdot B\underline{A)\bar{A}}C \qquad \text{(cancel)} \\ \bar{B} \cdot \bar{A}C \cdot B &= \underline{\bar{A}\bar{B}\bar{A}} \cdot C \cdot BC \qquad \text{(apply (i) and lemma 5, (b2))} \\ \bar{B}\bar{A}CB &= \bar{B}\bar{A}\underline{\bar{B} \cdot B}CB \qquad \text{(cancel)} \\ \bar{B}\bar{A}CB &= \bar{B}\bar{A}CB \qquad \text{(holds true)} \end{aligned}$$

In a similar vein, relation (4) is true since it is equivalent to the following relations:

$$\begin{aligned} Bb\bar{B} &= b \\ B \cdot \bar{A}\bar{B} \cdot \bar{A}C \cdot \underline{BA \cdot \bar{B}} &= \bar{A}\bar{B} \cdot \underline{\bar{A}C} \cdot BA \qquad \text{(apply (i)+(b3) and (ii))} \\ B\bar{A}\bar{B} \cdot \bar{A}\underline{C \cdot \bar{A}BA} &= \bar{A}\bar{B} \cdot \underline{C\bar{A} \cdot BA} \qquad \text{(cancel on both sides)} \\ \bar{A}\bar{B}\underline{A \cdot \bar{A}} &= \bar{A}\bar{B} \qquad \text{(cancel)} \\ \bar{A}\bar{B} &= \bar{A}\bar{B} \qquad \text{(holds true)} \end{aligned}$$

This finishes the proof that

$$\mathrm{PMod}_{1,1,1} = \langle A, B, C \mid ABA = BAB, AC = CA, BCB = CBC \rangle \cong B_4.$$

□

Corollary. *There exist a split short exact sequence:*

$$1 \longrightarrow F_2 \xrightarrow{i} B_4 \xrightarrow{\pi} B_3 \longrightarrow 1$$

where $F_2 = F(\alpha, \beta)$ is a free group on two generators, and

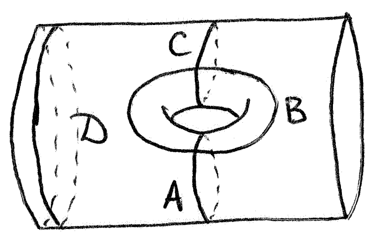
$$B_3 = \langle x, y \mid xyx = yxy \rangle,$$

$$B_4 = \langle A, B, C \mid ABA = BAB, AC = CA, BCB = CBC \rangle$$

are braid groups on 3 and 4 strands respectively. The inclusion i is given by $\alpha \mapsto A^{-1}C$, $\beta \mapsto (BA)^{-1} \cdot (A^{-1}C) \cdot (BA)$ (so that $i(F_2)$ is the normal closure in B_4 of element $A^{-1}C$), and the projection π is given by $A \mapsto x$, $B \mapsto y$, $C \mapsto x$. □

Remark 2. We just obtained a topological proof of the remarkable fact that $\ker(B_4 \xrightarrow{\pi} B_3) \cong F_2$. For algebraic proofs of this result, see [GAS, Th. 7], [GL, Th. 2.6] or [KR, Prop. 2.13].

Proposition 3. $\mathrm{PMod}_{1,2,0}$ *is isomorphic to $\mathbb{Z} \times B_4 = \langle A, B, C, D \mid ABA = BAB, AC = CA, BCB = CBC, AD = DA, BD = DB, CD = DC \rangle$ where A,B,C,D are right Dehn twists about the curves shown in the picture:*



Proof. By lemma 3, we have a short exact sequence:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{PMod}_{1,2,0} \longrightarrow \mathrm{PMod}_{1,1,1} \longrightarrow 1$$

where $\mathbb{Z}$ is generated by the boundary twist D.

Using the presentation from proposition 2 for $\mathrm{PMod}_{1,1,1}$ and applying lemma 4, we get:

$$\mathrm{PMod}_{1,2,0} = \langle A, B, C, D \mid R_{\mathrm{ker}} \cup R_{\mathrm{quo}} \cup R_{\mathrm{conj}} \rangle.$$

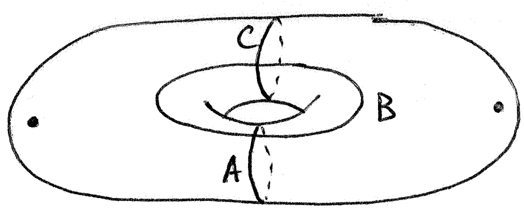
Again, $R_{\mathrm{ker}} = \varnothing$ since $\mathbb{Z}$ is a free group.

Relations from $\mathrm{PMod}_{1,1,1}$ are all braid relations, so they lift to $\mathrm{PMod}_{1,2,0}$ without changes, and we have $R_{\mathrm{quo}} = \{ABA = BAB, AC = CA, BCB = CBC\}$.

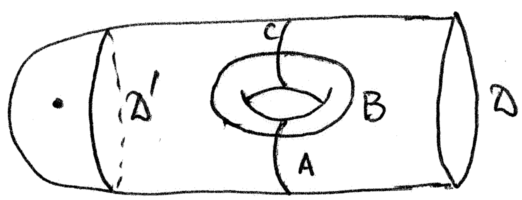
The conjugation relations R_{conj} are all trivial since the boundary twist D commutes with all other generators A,B,C: $R_{\mathrm{conj}} = \{ADA^{-1} = D, BDB^{-1} = D, CDC^{-1} = D\}$.

Therefore, the presentation of $\mathrm{PMod}_{1,2,0}$ is as stated in the Proposition. □

Proposition 4. $\mathrm{PMod}_{1,0,2}$ *is isomorphic to* $B_4/Z(B_4) \cong \langle A, B, C \mid ABA = BAB, AC = CA, BCB = CBC, (ABC)^4 = 1\rangle$, *where* A, B, C *are right Dehn twists about the curves shown in the picture:*



Proof. By lemma 3, $\mathrm{PMod}_{1,0,2}$ is isomorphic to $\mathrm{PMod}_{1,1,1}/\langle D\rangle$ where D is the boundary twist in the picture below:



According to the well-known 3-chain relation (see e.g. [FM, Prop. 4.12]), the element $(ABC)^4$ is equal to the product DD' where D' denotes a right Dehn twist shown on the picture above. As $D' = 1$ in $\mathrm{PMod}_{1,1,1}$, we have $D = (ABC)^4$.

Notice that the element $(ABC)^4$ generates the center $Z(B_4)$ of the braid group B_4 (see [KT, Ex. 1.3.2]).

□

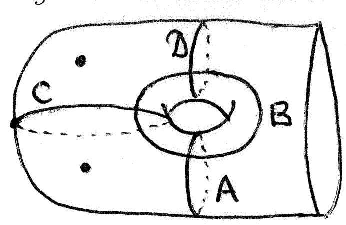
5. Genus 1, $b + n = 3$: $(g, b, n) = (1, 3, 0), (1, 2, 1), (1, 1, 2), (1, 0, 3)$

In this section, in a way similar to the one in section 4, we show that the group $\mathrm{PMod}_{1,1,2}$ is isomorphic to the Artin group $A(D_4)$ and obtain presentations for the groups $\mathrm{PMod}_{1,0,3}$, $\mathrm{PMod}_{1,2,1}$ and $\mathrm{PMod}_{1,3,0}$ as corollaries.

Proposition 5.

$$\mathrm{PMod}_{1,1,2} \cong A(D_4) = \langle A, B, C, D \mid ABA = BAB, AC = CA, AD = DA, \\ BCB = CBC, BDB = DBD, CD = DC\rangle,$$

where A, B, C, D *are right Dehn twists about the curves shown in the picture:*



Picture 3

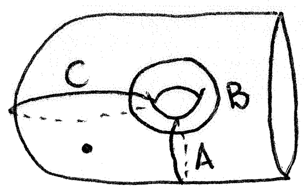
Proof. According to lemma 1, there is a short exact sequence

$$1 \longrightarrow \pi_1(S_{1,1,1}) \xrightarrow{p} \mathrm{PMod}_{1,1,2} \longrightarrow \mathrm{PMod}_{1,1,1} \longrightarrow 1$$

where p denotes the point-pushing map. As we proved in section 4, proposition 2,

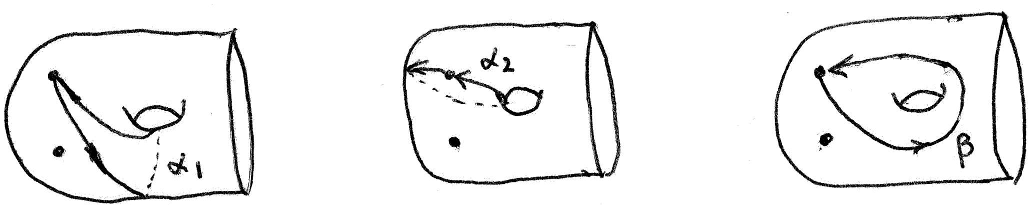
$$\mathrm{PMod}_{1,1,1} \cong B_4 = \langle A, B, C \mid ABA = BAB, AC = CA, BCB = CBC\rangle$$

where A, B, C denote the right Dehn twists as in the picture below:



Picture 4

The fundamental group $\pi_1(S_{1,1,1})$ is a free group on three generators $\alpha_1, \alpha_2, \beta$ shown in the pictures below:



and the Birman exact sequence (see lemma 1) takes the form:

$$1 \longrightarrow \langle \alpha_1, \alpha_2, \beta \mid \rangle \xrightarrow{p} \mathrm{PMod}_{1,1,2} \longrightarrow B_4 \longrightarrow 1$$

As in the proof of proposition 2, we adopt the notation from lemma 4.

Generators. Again, as before, as a set $\tilde{S}_Q$ of representatives of twists $A, B, C \in \mathrm{PMod}_{1,1,1}$ (see picture 4) in the group $\mathrm{PMod}_{1,1,2}$ we can choose the corresponding twists A, B, C (see picture 3). As a set $\tilde{S}_K$ of generators of $\pi_1(S_{1,1,1})$, viewed as belonging to $\mathrm{PMod}_{1,1,2}$, we take $\tilde{S}_K = \{a_1, a_2, b \mid a_1 = p(\alpha_1), a_2 = p(\alpha_2), b = p(\beta)\}$.

Relations. Again, according to lemma 4, the set of relations of $\mathrm{PMod}_{1,1,2}$ is the union of three sets: $R_{\mathrm{ker}} \cup R_{\mathrm{quo}} \cup R_{\mathrm{conj}}$.

As before, it is clear that $R_{\mathrm{ker}} = \varnothing$ since $\pi_1(S_{1,1,1})$ is a free group.

Since the only relations in $\mathrm{PMod}_{1,1,1}$ are the braid relations, the corresponding twists in $\mathrm{PMod}_{1,1,2}$ also satisfy them, so we see that they lift to $\mathrm{PMod}_{1,1,2}$ without any change, and we have $R_{\mathrm{quo}} = \{ABA = BAB, AC = CA, BCB = CBC\}$.

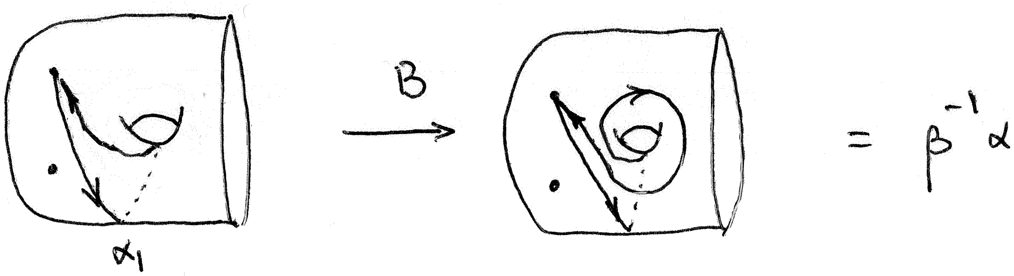
To obtain R_{conj}, we need to conjugate generators a_1, a_2, b of $p(\pi_1(S_{1,1,1}))$ with right Dehn twists A, B, C of $\mathrm{PMod}_{1,1,1}$.

By lemma 2(b) we have the following relations because of the disjointness of the underlying curves:

$$\begin{aligned} Aa_1A^{-1} &= p(A(\alpha_1)) = p(\alpha_1) = a_1 \\ Aa_2A^{-1} &= p(A(\alpha_2)) = p(\alpha_2) = a_2 \\ Ca_2C^{-1} &= p(C(\alpha_2)) = p(\alpha_2) = a_2 \\ BbB^{-1} &= p(B(\beta)) = p(\beta) = b \end{aligned}$$

We also see from the pictures that

$$Ba_1B^{-1} = p(B(\alpha_1)) = p(\beta^{-1}\alpha_1) = b^{-1}a_1;$$



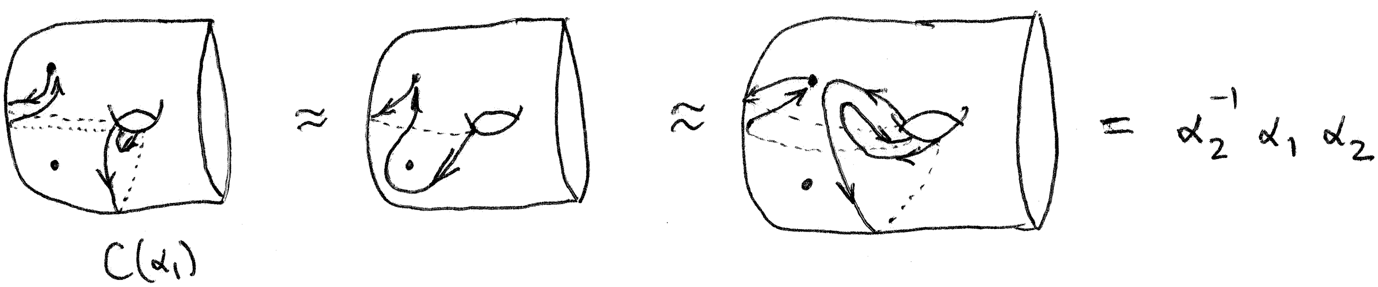
and similarly,

$$Ba_2B^{-1} = p(B(\alpha_2)) = p(\beta^{-1}\alpha_2) = b^{-1}a_2.$$

(Recall that we use functional notation for the multiplication of loops.)
Also,

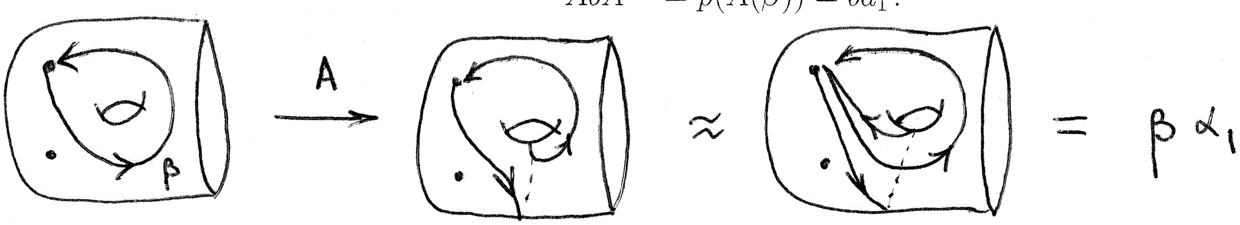
$$Ca_1C^{-1} = p(C(\alpha_1)) = p(\alpha_2^{-1}\alpha_1\alpha_2) = a_2^{-1}a_1a_2$$

since $C(\alpha_1) = \alpha_2^{-1}\alpha_1\alpha_2$. Indeed:



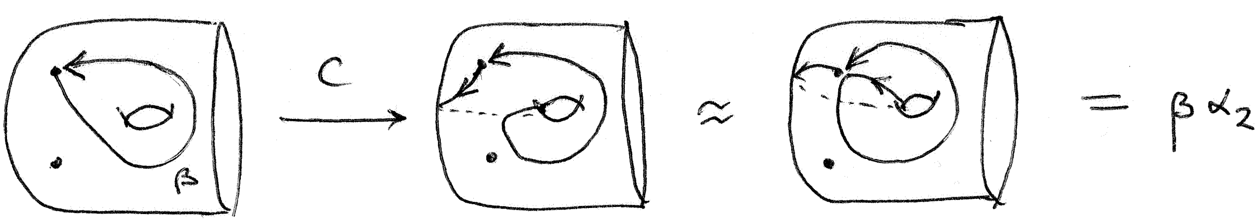
And also,

$$AbA^{-1} = p(A(\beta)) = ba_1:$$



and similarly,

$$CbC^{-1} = p(C(\beta)) = p(\beta\alpha_2) = ba_2:$$



As a result, we get a presentation:

Generators: a_1, a_2, b, A, B, C.

Relations: (i) $ABA = BAB$;
(ii) $BCB = CBC$;
(iii) $AC = CA$;

(1) $Aa_1A^{-1} = a_1$;
(2) $Aa_2A^{-1} = a_2$;
(3) $Ca_2C^{-1} = a_2$;
(4) $BbB^{-1} = b$;
(5) $Ca_1C^{-1} = a_2^{-1}a_1a_2$;
(6) $Ba_1B^{-1} = b^{-1}a_1$;
(7) $Ba_2B^{-1} = b^{-1}a_2$;
(8) $AbA^{-1} = ba_1$;
(9) $CbC^{-1} = ba_2$.

Simplification of the presentation. As before, we will use the bar ($\bar{\ }$) to denote the inverse element in a group.

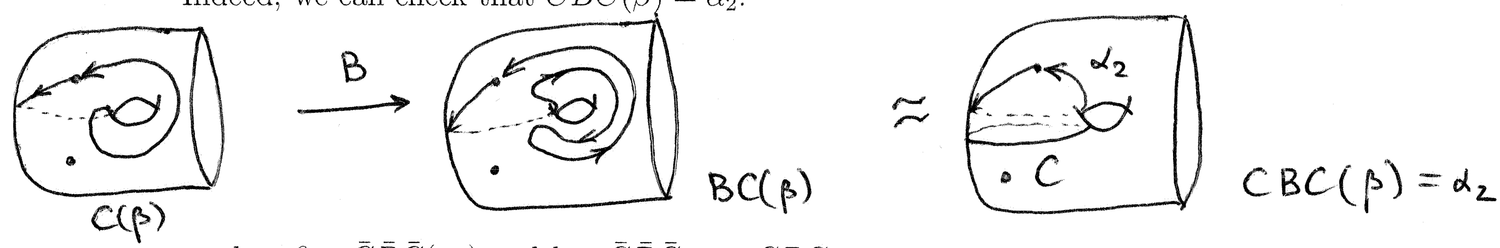
Let's introduce a new generator D which corresponds to the right Dehn twist denoted by the same letter D in picture 3. Then, by lemma 2(a),

$$a_2 = p(\alpha_2) = C^{-1}D = \bar{C}D$$

and we can eliminate generator a_2 from the presentation.

We can also eliminate generators a_1 and b by expressing them in terms of A, B, C, D and a_2. (The formulas below are taken from [LP, p.88].)

Indeed, we can check that $CBC(\beta) = \alpha_2$:



so that $\beta = \bar{C}\bar{B}\bar{C}(\alpha_2)$ and $b = \bar{C}\bar{B}\bar{C} \cdot a_2 \cdot CBC$.

Similarly, we can check that $BDABC(\alpha_1) = \alpha_2^{-1}$:

(see next page)

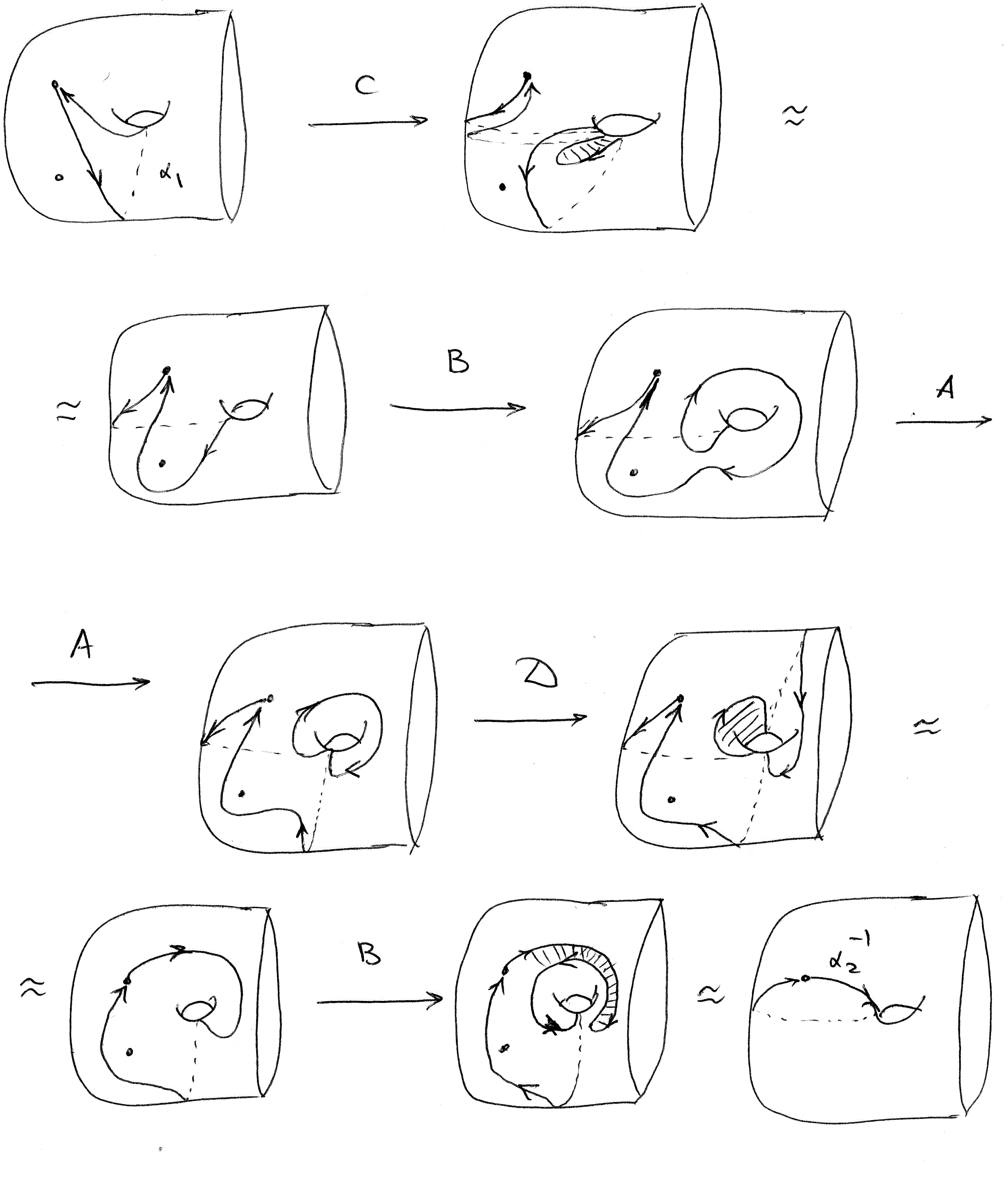
Therefore, $\alpha_1 = (BDABC)^{-1}(\alpha_2^{-1})$ and $a_1 = \bar{C}\bar{B}\bar{A}\bar{D}\bar{B} \cdot a_2^{-1} \cdot BDABC$.

We got the following expressions for a_1, a_2, b in terms of A,B,C,D:

$$\begin{aligned} a_2 &= \bar{C}D \\ b &= \bar{C}\bar{B}\bar{C} \cdot (\bar{C}D) \cdot CBC \\ a_1 &= \bar{C}\bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDABC \end{aligned} \qquad (*)$$

Let's now modify the relations (1)–(9) above to distill from them the presentation of the group $A(D_4)$.

$BDABC(\alpha_1) = \alpha_2^{-1}$:



Relation (2) is equivalent to:

$$\begin{aligned} Aa_2A^{-1} &= a_2 \\ \underline{A \cdot \bar{C}}D \cdot A^{-1} &= \bar{C}D \qquad \text{(use (iii))} \\ \underline{\bar{C}} \cdot AD &= \underline{\bar{C}} \cdot DA \\ AD &= DA \end{aligned}$$

We got a new relation which we denote

$$\text{(iv)} \qquad AD = DA.$$

Similarly, relation (3) is equivalent to:

$$\begin{aligned} Ca_2C^{-1} &= a_2 \\ \underline{C \cdot \bar{C}}D \cdot \bar{C} &= \bar{C}D \\ D\bar{C} &= \bar{C}D \\ CD &= DC \end{aligned}$$

So we got a new relation

$$\text{(v)} \qquad CD = DC.$$

Relation (7) is equivalent to:

$$\begin{aligned} Ba_1B^{-1} &= ba_1 \\ B \cdot \bar{C}D \cdot \bar{B} &= \bar{C}\bar{B}\underline{\bar{C} \cdot \bar{D}C} \cdot CB\underline{C \cdot \bar{C}}D \qquad \text{(cancel } \bar{C} \text{ and } C \text{ through } D \text{ using (v))} \\ B \cdot \bar{C}D \cdot \bar{B} &= \bar{C}\bar{B} \cdot \underline{\bar{D}C}B \cdot D \qquad \text{(use (v) again)} \\ B \cdot \bar{C}D \cdot \bar{B} &= \underline{\bar{C}\bar{B}C} \cdot \bar{D}BD \qquad \text{(use (ii) and lemma 5, (b6))} \\ \underline{B \cdot \bar{C}}D \cdot \bar{B} &= \underline{B\bar{C}}\bar{B} \cdot \bar{D}BD \qquad \text{(cancel on both sides)} \\ D\bar{B} &= \bar{B}\bar{D}BD \\ DBD &= BDB \end{aligned}$$

and we got yet another relation which we denote

$$\text{(vi)} \qquad BDB = DBD.$$

Let's now show that the remaining relations (1), (4)–(6), (8)–(9) are consequences of relations (i)–(vi).

Relation (1) is equivalent to the following relations:

$$Aa_1A^{-1} = a_1 \qquad \text{(substitute expression (*) for } a_1\text{)}$$

$$A \cdot (BDABC)^{-1} \cdot \bar{D}C \cdot BDABC \cdot \bar{A} = (BDABC)^{-1} \cdot \bar{D}C \cdot BDABC$$

$$(**) \qquad (BDABC) \cdot A \cdot (BDABC)^{-1} \cdot \bar{D}C = \bar{D}C \cdot BDABC \cdot A \cdot (BDABC)^{-1}$$

Let's simplify expression $(BDABC) \cdot A \cdot (BDABC)^{-1}$:

$$\begin{aligned} (BDABC) \cdot A \cdot (BDABC)^{-1} &= BDAB\underline{C \cdot A \cdot \bar{C}}\bar{B}\bar{A}\bar{D}\bar{B} \overset{\text{(iii)}}{=} BDA\underline{B \cdot A \cdot \bar{B}}\bar{A}\bar{D}\bar{B} \overset{\text{(i)}}{=} \\ &= BD\underline{A \cdot \bar{A}}B\underline{A \cdot \bar{A}}\bar{D}\bar{B} = \underline{BD \cdot B} \cdot \bar{D}\bar{B} \overset{\text{(vi)}}{=} D\underline{BD \cdot \bar{D}\bar{B}} = D. \end{aligned}$$

Therefore, relation (**) is equivalent to:

$$D \cdot \bar{D}C = \bar{D}C \cdot D, \quad \text{which holds true due to (v).}$$

Relation (4) is equivalent to:

$$\begin{aligned}
BbB^{-1} &= b \\
B \cdot \bar{C}\bar{B}\bar{C} \cdot \underline{\bar{C}D \cdot C}BC \cdot \bar{B} &= \bar{C}\bar{B}\bar{C} \cdot \underline{\bar{C}D \cdot C}BC && \text{(use (v) and cancel } \bar{C} \text{ and } C) \\
B \cdot \underline{\bar{C}\bar{B}\bar{C}} \cdot D \cdot \underline{BC \cdot \bar{B}} &= \bar{C}\bar{B}\bar{C} \cdot D \cdot BC && \text{(use (ii) and lemma 5, (b2),(b4))} \\
\underline{B \cdot \bar{B}}\bar{C}\bar{B} \cdot D \cdot \bar{C}\underline{BC} &= \bar{C}\bar{B}\bar{C} \cdot D \cdot \underline{BC} && \text{(cancel)} \\
\bar{C}\bar{B} \cdot D \cdot \bar{C} &= \bar{C}\bar{B}\bar{C} \cdot D && \text{(holds true due to (v)).}
\end{aligned}$$

Relation (5) is equivalent to:

$$\begin{aligned}
Ca_1C^{-1} &= a_2^{-1}a_1a_2 \\
\underline{C \cdot \bar{C}}\bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDAB\underline{C \cdot \bar{C}} &= \bar{D}\underline{C \cdot \bar{C}}\bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDAB\underline{C \cdot \bar{C}}D \\
\bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDAB &= \underline{\bar{D}} \cdot \bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDABD && \text{(multiply by } D \text{ on the left)} \\
D \cdot \bar{B}\underline{\bar{A}\bar{D}}\bar{B} \cdot \bar{D}C \cdot B\underline{DA}B &= \bar{B}\underline{\bar{A}\bar{D}}\bar{B} \cdot \bar{D}C \cdot B\underline{DA}BD && \text{(use (iv))} \\
\underline{D \cdot \bar{B}\bar{D}}\bar{A}\bar{B} \cdot \bar{D}C \cdot BADB &= \bar{B}\bar{D}\bar{A}\bar{B} \cdot \bar{D}C \cdot BA\underline{DBD} && \text{(use (vi))} \\
\bar{B}\bar{D}\underline{B \cdot \bar{A}\bar{B}} \cdot \bar{D}C \cdot BADB &= \bar{B}\bar{D}\bar{A}\bar{B} \cdot \bar{D}C \cdot \underline{BA \cdot B}DB && \text{(use (i))} \\
\underline{\bar{B}\bar{D} \cdot \bar{A}\bar{B}}A \cdot \bar{D}C \cdot \underline{BADB} &= \underline{\bar{B}\bar{D}\bar{A}\bar{B}} \cdot \bar{D}C \cdot A\underline{BA \cdot DB} && \text{(cancel on both sides)} \\
A\bar{D}C &= \bar{D}CA && \text{(holds true due to (iii) and (iv)).}
\end{aligned}$$

Relation (6) is equivalent to to:

$$\begin{aligned}
Ba_1B^{-1} &= b^{-1}a_1 \\
B \cdot \bar{C}\bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDABC \cdot \bar{B} &= \bar{C}\bar{B}\underline{\bar{C} \cdot \bar{D}C} \cdot C\underline{BC \cdot \bar{C}\bar{B}}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDABC \\
B \cdot \bar{C}\bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDABC \cdot \bar{B} &= \bar{C}\bar{B} \cdot \underline{\bar{D} \cdot C}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDABC && \text{(use (v))} \\
B \cdot \bar{C}\bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDABC \cdot \bar{B} &= \bar{C}\bar{B}C \cdot \bar{D}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDABC && \text{(use (v))} \\
B \cdot \bar{C}\bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}C \cdot BDABC \cdot \bar{B} &= \underline{\bar{C}\bar{B}C} \cdot \underline{\bar{D}\bar{A}} \cdot \underline{\bar{D}\bar{B}\bar{D}} \cdot C \cdot BDABC && \text{(use (ii),(iv),(vi))} \\
\underline{B \cdot \bar{C}\bar{B}\bar{A}\bar{D}\bar{B} \cdot \bar{D}}C \cdot BDABC \cdot \bar{B} &= \underline{B\bar{C}\bar{B} \cdot \bar{A}\bar{D} \cdot \bar{B}\bar{D}}\bar{B} \cdot C \cdot BDABC && \text{(cancel)} \\
CB \cdot DA \cdot \underline{BC\bar{B}} &= \underline{\bar{B}CB} \cdot DA \cdot BC && \text{(use (ii))} \\
\underline{CB} \cdot DA\bar{C} \cdot \underline{BC} &= \underline{CB} \cdot \bar{C}DA \cdot \underline{BC} && \text{(cancel)} \\
DA\bar{C} &= \bar{C}DA && \text{(holds true due to (iii) and (v)).}
\end{aligned}$$

Relation (8) is equivalent to:

$$
\begin{aligned}
AbA^{-1} &= ba_1 \\
A\cdot\bar C\bar B\bar C\cdot\underline{\bar C D\cdot C}BC\cdot\bar A &= \bar C\bar B\bar C\cdot\bar C D\cdot C\underline{BC\cdot\bar C\bar B}\bar A\bar D\bar B\cdot\bar D C\cdot BDABC && \text{(use (v))}\\
A\cdot\bar C\bar B\bar C\cdot D\cdot BC\cdot\bar A &= \bar C\bar B\bar C\cdot\underline{\bar C D\cdot C\cdot\bar A\bar D}\cdot\bar B\bar D C\cdot BDABC && \text{(use (iv),(v))}\\
\underline{A\cdot\bar C}\bar B\bar C\cdot D\cdot B\underline{C\cdot\bar A} &= \bar C\bar B\bar C\cdot\bar A\cdot\bar B\bar D C\cdot BDABC && \text{(use (iii))}\\
\underline{\bar C}A\cdot\bar B\bar C\cdot D\cdot B\cdot\bar A\underline{C} &= \underline{\bar C}\cdot\bar B\bar C\cdot\bar A\cdot\bar B\bar D C\cdot BDAB\cdot\underline{C} && \text{(cancel)}\\
A\bar B\bar C\cdot DB\cdot\bar A &= \underline{\bar B}\cdot\bar C\bar A\bar B\cdot\bar D C\cdot BD\cdot\underline{AB} && \text{(multiply by } B \text{ and by } \bar B\bar A)\\
\underline{BA\bar B}\cdot\bar C DB\cdot\underline{\bar A\bar B\bar A} &= \underline{\bar C\bar A}\bar B\cdot\bar D C\cdot BD && \text{(use (i) and (iii))}\\
\underline{\bar A}BA\cdot\bar C D\underline{B\cdot\bar B}\bar A\bar B &= \underline{\bar A}\bar C\cdot\bar B\bar D C\cdot BD && \text{(cancel)}\\
B\underline{A\cdot\bar C D\cdot\bar A}\bar B &= \bar C\cdot\bar B\underline{\bar D C}\cdot BD && \text{(cancel } A \text{ and } \bar A\text{, exchange } \bar D \text{ and } C)\\
B\cdot\bar C D\cdot\bar B &= \underline{\bar C\bar B C}\cdot\underline{\bar D BD} && \text{(use (ii) and (vi))}\\
B\cdot\bar C D\cdot\bar B &= B\bar C\underline{\bar B\cdot B}D\bar B && \text{(cancel)}\\
B\cdot\bar C D\cdot\bar B &= B\bar C D\bar B && \text{(holds true).}
\end{aligned}
$$

Relation (9) is equivalent to:

$$
\begin{aligned}
CbC^{-1} &= ba_2\\
\underline{C\cdot\bar C}\bar B\bar C\cdot\bar C D\cdot CB\underline{C\cdot\bar C} &= \underline{\bar C\bar B\bar C}\cdot\bar C D\cdot CB\underline{C\cdot\bar C}D && \text{(cancel and apply (ii))}\\
\underline{\bar B\bar C}\cdot\underline{\bar C D\cdot C}B &= \underline{\bar B\bar C}\bar B\cdot\underline{\bar C D\cdot C}B\cdot D && \text{(cancel and use (v))}\\
DB &= \bar B\cdot\underline{D\cdot BD} && \text{(use (vi))}\\
DB &= \underline{\bar B\cdot B}DB && \text{(holds true).}
\end{aligned}
$$

This finishes the proof that relations (1)–(9) are equivalent to relations (i)–(vi), and hence the proof of the proposition. □

Proposition 6.

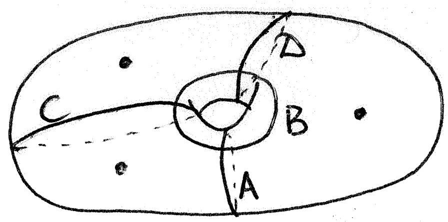
(a) $\mathrm{PMod}_{1,2,1} \cong \mathbb{Z} \times A(D_4)$;
(b) $\mathrm{PMod}_{1,3,0} \cong \mathbb{Z}^2 \times A(D_4)$;

Proof. The proof is almost a verbatim reiteration of the proof of proposition 3. □

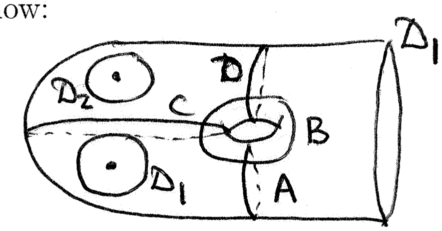
Proposition 7.

$$\mathrm{PMod}_{1,0,3} \cong A(D_4)/Z(A(D_4)) = \langle A, B, C, D \mid ABA = BAB, BCB = CBC, BDB = DBD, AC = CA, AD = DA, CD = DC, (ABCD)^3 = 1\rangle,$$

where A, B, C, D are right Dehn twists about the curves shown in the picture:



Proof. By lemma 3, $\mathrm{PMod}_{1,0,3}$ is isomorphic to $\mathrm{PMod}_{1,1,2}/\langle D_1\rangle$ where D_1 is the boundary twist in the picture below:



According to the well-known star relation (see e.g. [FM, 5.2.3], [GER]), the element $(ACDB)^3$ is equal to the product $D_1D_2D_3$ where D_2,D_3 denote right Dehn twists shown on the picture above. As $D_2 = D_3 = 1$ in $\mathrm{PMod}_{1,1,2}$, we have $D_1 = (ACDB)^3$.

Notice that the element $(ACDB)^3$ is the fundamental element ("Garside element") of the Artin group $A(D_4)$. It is also equal to $(ABCD)^3$ and it generates the center of $A(D_4)$ (see [BS, Lemma 5.8 and Zusatz,(ii)]).

□

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