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# A Liouville type theorem for higher order Hardy–Hénon equation in $R^n$

Tingzhi Cheng [a,*], Shuang Liu [b]

[a] Department of Mathematics, Shanghai Jiaotong University, 200240, PR China
[b] Department of Mathematics, University of South Carolina, United States

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ABSTRACT

In this paper, we consider the following higher order Hardy–Hénon type equations in $R^n$:

$$(-\Delta)^m u(x) = |x|^a u^p(x), \ x \in R^n \tag{1}$$

in subcritical cases with $a > 0$, and in particular, we focus on the non-existence of positive solutions.

First, under some very mild growth conditions, we show that problem (1) is equivalent to the integral equation

$$u(x) = \int_{R^n} G(x,y)|y|^a u^p(y)dy \tag{2}$$

where $G(x,y)$ is the Green's function associated with $(-\Delta)^m$ in $R^n$.

Then by using the method of moving planes in integral forms, we prove that there is no positive solution for integral equation (2) in subcritical cases $\frac{n}{n-2m} < p < \frac{n+2m+a}{n-2m}$. For the non-existence of positive radially symmetric solutions, we can extend the range to subcritical cases $1 < p < \frac{n+2m+2a}{n-2m}$. This partially solves an open conjecture posed by Quoc Hung Phan and Philippe Souplet [21].



## 1. Introduction

This article is devoted to the study of positive solutions of the following elliptic equation

$$(-\Delta)^m u(x) = |x|^a u^p(x), \ x \in R^n, \tag{3}$$

where $a > 0$, $2m < n$, and $1 < p < \tau := \frac{n+2m+2a}{n-2m}$.

* Corresponding author.
E-mail addresses: nowitzki1989@126.com (T. Cheng), shuang@math.sc.edu (S. Liu).

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This equation is related to the weighted Hardy–Littlewood inequality which has been widely studied in [1], [7], [10], [17], [21]. Recently, the Hardy–Hénon equation has been researched extensively, for example [4], [5], [6], [8], [9], [24]. Our primary interest is in the Liouville property, i.e. the non-existence of positive solutions in $R^n$. It is well-known that this kind of Liouville theorem plays an important role in a priori estimates of solutions for the corresponding family of equations either on domains or on Riemannian manifolds with boundaries. Because of sigficant role the Liouville type theorem plays, there have already existed some Liouville type results for Hardy–Hénon equation in [18], [19], [25], [26]. However, the similar results for the high order equation are relatively rare, so in this paper we will try to achieve a Liouville type theorem for a type of high order equation.

To prove such a non-existence result, an effective approach is to consider the corresponding integral equation

$$u(x) = \int_{R^n} G(x,y)|y|^a u^p(y)dy, \tag{4}$$

where

$$G(x,y) = \frac{C_n}{|x-y|^{n-2m}}$$

is the Green's function associated with $(-\Delta)^m$ in $R^n$.

Whether or not, the results on integral equations can be applied to PDEs, depends on whether one can prove the equivalence between the two. We say that (3) and (4) are equivalent, if $u$ is a solution of (3), then it is also a solution of (4), and vice versa.

We first establish the equivalence between PDE (3) and integral equation (4).

**Theorem 1.** *If $u$ is a classical positive solution of (3), then $u$ satisfies integral equation (4). If $u \in C^{2m}(R^n)$ is a solution of (4), then $u$ satisfies (3).*

The proof of Theorem 1 is based on the following super poly-harmonic property of solutions.

**Theorem 2.** *If $u$ is a positive solution of*

$$(-\Delta)^m u(x) = |x|^a u^p(x),\ x \in R^n,$$

*then*

$$(-\Delta)^i u(x) > 0,\ x \in R^n \setminus \{0\},\ (-\Delta)^i u(0) \geq 0,\ i = 1, \cdots, m-1. \tag{5}$$

By the results in [11], [16], [27], it is well-known that the super poly-harmonic property has many important applications in PDEs and ODEs.

Due to the equivalence between (3) and (4), in order to derive the properties of solutions of (3), we only need to deal with integral equation (4).

By using the method of moving planes in integral forms, we prove

**Theorem 3.** *For $a > 0$, $\frac{n}{n-2m} < p < \frac{n+2m+a}{n-2m}$, let $|x|^a u^{p-1} \in L_{loc}^{\frac{n}{2m}}(R^n)$. Assume that $u \in L_{loc}^q(R^n)$ for some $q > \frac{n}{n-2m}$. Then each nonnegative solution $u(x)$ of (4) is radically symmetric and monotone decreasing in $x$ about the origin.*

Then by using the *Pohozaev identity*, we prove

**Theorem 4.** *For* $1 < p < \frac{n+2m+2a}{n-2m}$*, (3) has no radially symmetric nonnegative solutions.*

The method of moving planes in integral forms which has been used in [12], [15] has become a powerful tool in studying qualitative properties for solutions of integral equations and systems. It is quite different from the traditional methods for PDEs. Instead of relying on maximum principles, one estimates integral norms. A remarkable advantage is that it treats all powers of Laplacians indiscriminately.

**Corollary 1.** *For* $\frac{n}{n-2m} < p < \frac{n+2m+a}{n-2m}$*, let* $|x|^a u^{p-1} \in L_{loc}^{\frac{n}{2m}}(R^n)$*. Assume that* $u \in L_{loc}^q(R^n)$ *for some* $q > \frac{n}{n-2m}$*. If* $u$ *is a nonnegative solution of (3), then* $u(x) \equiv 0$*.*

**Remark.** In particular, the case $m = 1$ has been widely studied by many authors.

In this case, (3) turns out to be

$$(-\Delta)u(x) = |x|^a u^p(x), \ x \in R^n. \tag{6}$$

Denote the Sobolev exponent by

$$p_s := \frac{N+2}{N-2},$$

and the Hardy–Sobolev exponent by

$$p_s(a) := \frac{N+2+2a}{N-2} (= \infty \text{ if } N = 2).$$

Let us introduce the following results in the case of radial solutions (stated in [23]; see [2] for a detailed proof).

**Proposition A.** *Let* $N \geq 2, a > -2$ *and* $p > 1$*.*

(i) *If* $p < p_s(a)$*, then (6) has no positive radial solution in* $R^n$*.*
(ii) *If* $p \geq p_s(a)$*, then (6) possesses bounded, positive radial solution in* $R^n$*.*

So far, for the popular radial solutions, the results have been clean and neat. One can see that the Hardy–Sobolev exponent $p_s(a)$ plays a critical role in the radial case and this leads to the following natural conjecture [21]:

**Conjecture.** *If* $N \geq 2, a > -2,$ *and* $1 < p < p_s(a)$*, then (6) has no positive solutions in* $R^n$*.*

The condition $p < p_s(a)$ is the best possible due to Proposition A(ii). However, apart from the radial case, the best available non-existence result up to now are the following:

**Proposition B.** *Let* $N \geq 2, a > -2$ *and* $p > 1$*.*

(i) *If*

$$p < \min(p_s, p_s(a)),$$

*then (6) has no positive solution in* $R^n$*.*

(ii) *The conclusion in Part (i) remains true if*

$$p \le \frac{N+a}{N-2}.$$

Proposition B(i) was proved by Marie Françoise Bidaut-Veron and Hector Giacomini in [3]. As for Proposition B(ii), it can be found in e.g. [22, Example 3.2]. For a particular dimension $N = 3$, Quoc Hung Phan and Philippe Souplet [21] proved

**Proposition C.** *Let $N = 3, a > 0, p > 1$. If $p < p_s(a)$, then (6) has no positive bounded solution in $R^n$.*

Here, combining Theorem 3 and Proposition B for $m = 1$ and $a > 0$, we conclude the following:

**Corollary 2.** *If $N \ge 2, a > 0$, and $1 < p < \frac{n+2+a}{n-2}$, then (6) has no positive solutions in $R^n$.*

The Corollary 2 partially solved the Conjecture. For more related results, please see [2,20,21] and the references therein.

This paper is arranged as follows. In Section 2, we will obtain the super poly-harmonic properties of the positive solutions of PDEs, and thus prove Theorem 2. In Section 3, based on Theorem 2, we will establish the equivalence between the integral equations and PDEs and thus prove Theorem 1. In Section 4, we will use the method of moving planes in integral forms and Kelvin transforms to prove Theorem 3 – the non-existence of positive solutions for integral equation (4). In Section 5, we will show that in subcritical cases, PDEs have no positive radially symmetric solutions and thus prove Theorem 4.

## 2. Super poly-harmonic properties

In this section, we prove Theorem 2, that is

**Theorem 2.** *If $u$ is a positive solution of*

$$(-\Delta)^m u(x) = |x|^a u^p(x), \ x \in R^n,$$

then

$$(-\Delta)^i u(x) > 0, \ i = 1, \cdots, m-1, \ x \in R^n \setminus \{0\},$$
$$(-\Delta)^i u(0) \ge 0, \ i = 1, \cdots, m-1.$$

In the following, $C$, $c_0$, $c_1$ and $c_2$ denote positive constants whose values may be different from line to line.

**Proof.** Let

$$u_i(x) = (-\Delta)^i u(x), \ i = 1, \cdots, m-1, \ x \in R^n.$$

***Part 1.*** We first show that

$$u_{m-1}(x) > 0, \ x \in R^n \setminus \{0\}, \ u_{m-1}(0) \ge 0. \tag{7}$$

Suppose on the contrary, then there are two possible cases.

Case i) There exists $x^1 \in R^n$, such that

$$u_{m-1}(x^1) < 0. \tag{8}$$

Case ii) $u_{m-1}(x) \geq 0, x \in R^n$, and there exists a point $\tilde{x} \in R^n \setminus \{0\}$, such that

$$u_{m-1}(\tilde{x}) = 0.$$

In this case, $\tilde{x}$ is a local minimum of $u_{m-1}$, and we must have $-\Delta u_{m-1}(\tilde{x}) \leq 0$. This contradicts with

$$-\Delta u_{m-1}(\tilde{x}) = |\tilde{x}|^a u^p(\tilde{x}) > 0,\ \tilde{x} \in R^n \setminus \{0\}.$$

Therefore we only need to consider Case i).

*Step 1.* In this step, we will show that $m$ must be even. If not, we assume that $m$ is odd. Let

$$\bar{u}(r) = \frac{1}{|\partial B_r(x^1)|} \int\limits_{\partial B_r(x^1)} u(x) d\sigma \tag{9}$$

be the spherical average of $u$. Then by the well-known property that

$$\overline{\Delta u} = \Delta \bar{u},$$

we have

$$\begin{cases} -\Delta \bar{u}_{m-1} &= \overline{|x|^a u^p(x)}, \\ -\Delta \bar{u}_{m-2} &= \bar{u}_{m-1}, \\ & \cdots \\ -\Delta \bar{u} &= \bar{u}_1. \end{cases} \tag{10}$$

From the first equation in (10), by Jensen's inequality, we have

$$\begin{aligned} -\Delta \bar{u}_{m-1} &= \frac{1}{|\partial B_r(x^1)|} \int\limits_{\partial B_r(x^1)} |x|^a u^p(x) d\sigma \\ &\geq (||r - |x^1||)^a \frac{1}{|\partial B_r(x^1)|} \int\limits_{\partial B_r(x^1)} u^p(x) d\sigma \\ &\geq (||r - |x^1||)^a \Big(\frac{1}{|\partial B_r(x^1)|} \int\limits_{\partial B_r(x^1)} u(x) d\sigma\Big)^p \\ &= (||r - |x^1||)^a \bar{u}^p(x) \\ &\geq 0, \quad \forall r > 0. \end{aligned} \tag{11}$$

Then integrating both sides from 0 to $r$ yields

$$\bar{u}'_{m-1}(r) \leq 0,\ \text{and } \bar{u}_{m-1}(r) \leq \bar{u}_{m-1}(0) = u_{m-1}(x^1) := -c < 0,\ r > 0. \tag{12}$$

Then from the second equation in (10), we have

$$-\frac{1}{r^{n-1}}(r^{n-1}\bar{u}'_{m-2})' = \bar{u}_{m-1}(r) \leq -c,\ \forall r > 0.$$

That is

$$(r^{n-1}\bar{u}'_{m-2})' \geq r^{n-1}c, \ \forall r > 0.$$

Integrating yields

$$\bar{u}'_{m-2}(r) \geq \frac{c}{n}r, \text{ and } \bar{u}_{m-2}(r) \geq \bar{u}_{m-2}(0) + \frac{c}{2n}r^2, \ \forall r > 0. \tag{13}$$

Hence, $\exists r_1 > 0$, such that

$$\bar{u}_{m-2}(r_1) > 0.$$

Making average at a new center $x^2$ with $|x^1 - x^2| = r_1, i.e.$

$$\bar{\bar{u}}(r) = \frac{1}{|\partial B_r(x^2)|} \int\limits_{\partial B_r(x^2)} \bar{u}(x) d\sigma,$$

we have

$$\bar{\bar{u}}_{m-2}(0) = \bar{u}_{m-2}(x^2) = \bar{u}_{m-2}(r_1) > 0. \tag{14}$$

Then by (11), we chose $r = |x^1 - x^2|$, then $(\bar{\bar{u}}, \bar{\bar{u}}_1, \cdots, \bar{\bar{u}}_{m-1})$ satisfies

$$\begin{cases} -\Delta \bar{\bar{u}}_{m-1} \geq & \overline{(||x^1 - x^2| - |x^1||)^a \bar{u}^p(x)}, \\ -\Delta \bar{\bar{u}}_{m-2} = & \bar{\bar{u}}_{m-1}, \\ & \cdots \\ -\Delta \bar{\bar{u}} = & \bar{\bar{u}}_1. \end{cases} \tag{15}$$

By (15) and Jensen's inequality, we obtain

$$\begin{aligned} -\Delta \bar{\bar{u}}_{m-1}(r) &\geq \overline{(||x - x^1| - |x^1||)^a \bar{u}^p(x)} \\ &= \frac{1}{|\partial B_r(x^2)|} \int\limits_{\partial B_r(x^2)} (||x - x^1| - |x^1||)^a \bar{u}^p(x) d\sigma \\ &\geq C^{*a} \bar{\bar{u}}^p(x) \\ &\geq 0, \quad \forall r > 0. \end{aligned}$$

Here $C^*$ is a constant and $C^* \geq 0$. By the same arguments as in deriving (13), we conclude

$$\bar{\bar{u}}_{m-2}(r) \geq \bar{\bar{u}}_{m-2}(0) + \frac{c_0}{2n}r^2, \ \forall r \geq 0. \tag{16}$$

By (12), (14), and (16), we have

$$\bar{\bar{u}}_{m-1}(r) < 0,$$
$$bar\bar{u}_{m-2}(r) > 0, \ \forall r \geq 0.$$

Continuing this way, after $m - 1$ steps of re-centers (denotes the results by $\tilde{u}$), we conclude, for any $r \geq 0$,

$$-\Delta \tilde{u}_{m-1}(r) \geq C^{*a} \tilde{u}^p(r) \geq 0 \tag{17}$$

and

$$(-1)^i \tilde{u}_{m-i}(r) > 0,\ i = 1, \cdots, m-1,\ \forall r \geq 0. \tag{18}$$

Since $m$ is odd, (18) implies

$$(-1)^{m-2}\tilde{u}_2(r) > 0,\ \text{i.e. } \tilde{u}_2(r) < 0,\ \forall r \geq 0.$$

And then we derive

$$\tilde{u}_1'(r) > 0,\ \text{and } \tilde{u}_1(r) \geq \tilde{u}_1(0) := c_1 > 0,\ \forall r \geq 0.$$

From the last equation in (15), we deduce

$$\tilde{u}(r) \leq \tilde{u}(0) - \frac{c_1}{2n} r^2 \to -\infty \text{ as } r \to \infty.$$

Which contradicts to the positivity of $u$. Hence $m$ must be even.

*Step 2.* Let

$$u_\lambda(x) = \lambda^{\frac{2m+a}{p-1}} u(\lambda x)$$

be the rescaling of $u$. It can be easily checked that

$$(-\Delta)^m u_\lambda(x) = |x|^a u_\lambda^p(x), \forall \lambda > 0.$$

By (18), we derive

$$\tilde{u}(r) \geq \tilde{u}(0) > 0,\ \forall r \geq 0.$$

Then we choose a sufficiently large $\lambda$ such that for any $a_0 > 0$

$$\tilde{u}(r) \geq a_0 \geq a_0 r^{\sigma_0},\ \forall r \in [0,1], \tag{19}$$

where $\sigma_0 > 1$ and $\sigma_0 p \geq 2m + n$. By (17) and (19), we have

$$\begin{aligned} -\Delta \tilde{u}_{m-1}(r) &\geq C^{*a} \tilde{u}^p(r) \\ &\geq C^{*a} a_0^p r^{p\sigma_0} \\ &:= C a_0^p r^{p\sigma_0}. \end{aligned}$$

It follows that

$$\tilde{u}_{m-1}(r) \leq \tilde{u}_{m-1}(0) - \frac{C a_0^p r^{\sigma_0 p + 2}}{(\sigma_0 p + n)(\sigma_0 p + 2)}.$$

Since $m$ is even, by (18), we obtain

$$\tilde{u}_{m-1}(r) \leq -\frac{C a_0^p r^{\sigma_0 p + 2}}{(\sigma_0 p + n)(\sigma_0 p + 2)} \leq -\frac{C a_0^p r^{\sigma_0 p + 2}}{(2\sigma_0 p)^2},\ \forall r \in [0,1]. \tag{20}$$

Similar to (20), by the second equation in (15), (18) and (20), we deduce

$$\tilde{u}_{m-2}(r) \geq \frac{C a_0^p r^{\sigma_0 p + 4}}{(2\sigma_0 p)^4},\ \forall r \in [0,1].$$

Continuing this way, we derive

$$\tilde{u}(r) \geq \frac{Ca_0^p r^{\sigma_0 p+2m}}{(2\sigma_0 p)^{2m}} \geq \frac{Ca_0^p r^{2\sigma_0 p}}{(2\sigma_0 p)^{2m}},\ \forall r \in [0,1].$$

Set

$$\sigma_1 = 2\sigma_0 p,\ \sigma_k = 2\sigma_{k-1}p,\ k = 2, \cdots,$$
$$a_1 = \frac{Ca_0^p}{(2\sigma_0 p)^{2m}},\ a_k = \frac{Ca_{k-1}^p}{(2\sigma_{k-1}p)^{2m}},\ k = 2, \cdots.$$

Repeating the above arguments, by the induction, one can prove

$$\tilde{u}(r) \geq a_k r^{\sigma_k},\ \forall r \in [0,1]. \tag{21}$$

Through elementary calculations, we have

$$\sigma_k = 2\sigma_{k-1}p = (2p)^2\sigma_{k-2} = \cdots = (2p)^k\sigma_0,$$

$$\begin{aligned} a_k &= \frac{C \cdot C^p \cdot C^{p^2} \cdot \cdots \cdot C^{p^{k-1}} a_0^{p^k}}{(2p)^{2m(k+(k-1)p+(k-2)p^2+\cdots+p^{k-1})}\sigma_0^{\frac{2m(p^k-1)}{p-1}}} \\ &\geq \frac{C \cdot C^p \cdot C^{p^2} \cdot \cdots \cdot C^{p^{k-1}} a_0^{p^k}}{(2p)^{2m\frac{p^{k+1}-p}{(p-1)^2}}\sigma_0^{\frac{2m(p^k-1)}{p-1}}} \\ &\geq C^{\frac{-1}{p-1}}\Big(\frac{C^{\frac{1}{p-1}}a_0}{(2p)^{\frac{2mp}{(p-1)^2}}\sigma_0^{\frac{2m}{p-1}}}\Big)^{p^k},\ k = 1, \cdots. \end{aligned}$$

We take

$$a_0 = 2C^{-\frac{1}{p-1}}(2p)^{\frac{2mp}{(p-1)^2}}\sigma_0^{\frac{2m}{p-1}}.$$

Then by (21), we deduce

$$\tilde{u}(1) \geq C^{\frac{-1}{p-1}}2^{p^k} \to \infty,\ \text{as } k \to \infty.$$

This is impossible. Hence (7) must hold.

***Part 2.*** Now we show that all other $u_k(x)$ must be nonnegative, where $k = 1, 2, \cdots, m-2$, $x \in R^n$. On the contrary, suppose for some $i, 2 \leq i \leq m-1$, $\exists x^0 \in R^n$, such that

$$u_{m-1}(x) \geq 0,\ u_{m-2}(x) \geq 0,\ \cdots,\ u_{m-i+1}(x) \geq 0,\ x \in R^n, \tag{22}$$
$$u_{m-i}(x^0) < 0. \tag{23}$$

Repeating the similar arguments as in *Step 1* of Part 1, we conclude that $m-i$ must be odd and thus derive

$$(-1)^{m-j}\tilde{u}_{m-j}(r) > 0,\ j = i, \cdots, m-1,\ \forall r \geq 0.$$

Then

$$\tilde{u}_1(r) < 0, \ \forall r \geq 0.$$

Therefore

$$\tilde{u}(r) \geq \tilde{u}(0) := c > 0 \, \forall r \geq 0.$$

By (17), we obtain

$$\begin{aligned} -\Delta \tilde{u}_{m-1}(r) &\geq C^{*a} \tilde{u}^p(r) \\ &\geq c_0 c^p := C > 0. \end{aligned}$$

Integrating both sides from 0 to $r$ yields

$$\tilde{u}_{m-1}(r) \leq \tilde{u}_{m-1}(0) - \frac{Cr^2}{2n} \to -\infty, \ \text{as } r \to \infty.$$

This contradicts with (7), and therefore (5) must be true. This completes the proof of Theorem 2.

## 3. The proof of the equivalence between (3) and (4)

We now prove the equivalence between (3) and (4). Here, the proof consists of two steps. In step (i), we show that if $u$ is a positive solution of (3), then $u$ satisfies (4). In step (ii), we prove that if $u \in C^{2m}(R^n)$ is a solution of (4), then $u$ satisfies (3).

(i) We first assume that $u$ is a positive solution of

$$(-\triangle)^m u(x) = |x|^a u^p(x), \ x \in R^n, \tag{24}$$

where $m$ is a positive integer and $2m < n$.

Let $\delta(x-y)$ be the Dirac Delta function for fixed $x$. $G_r(x,y)$ is the Green's function

$$\begin{cases} (-\triangle)^m G_r(x,y) = \delta(x-y), & \text{in } B_r(x) \\ G_r(x,y) = \triangle G_r(x,y) = \cdots = \triangle^{m-1} G_r(x,y) = 0, & \text{on } \partial B_r(x). \end{cases} \tag{25}$$

By the Hopf Lemma, one can easily verify that the outward normal derivative

$$\frac{\partial}{\partial \nu_y}[(-\triangle)^i G_r(x,y)] \leq 0, i = 0, \ldots, m-1, \ \text{on } \partial B_r(x). \tag{26}$$

Multiply both sides of (24) by $G_r(x,y)$ and integrate on $B_r(x)$. After integrating by parts, and due to Theorem 3, (25) and (26), we arrive at

$$\begin{aligned} \int_{B_r(x)} G_r(x,y)|y|^a u^p(y)dy &= \int_{B_r(x)} G_r(x,y)(-\triangle)^m u(y)dy \\ &= u(x) + \sum_{i=0}^{m-1} \int_{\partial B_r(x)} [(-\triangle)^i u] \frac{\partial}{\partial \nu_y}[(-\triangle)^{m-1-i} G_r(x,y)] d\sigma \\ &\leq u(x). \end{aligned} \tag{27}$$

Solving equations (25) directly and letting $r \to \infty$, we have

$$G_r(x,y) \to \frac{C}{|x-y|^{n-2m}}, \tag{28}$$

$$(-\triangle)^i G_r(x,y) \to \frac{C}{|x-y|^{n-2m+2i}},\ i=1,\cdots,m-1, \tag{29}$$

and

$$|\frac{\partial}{\partial \nu_y}[(-\triangle)^{m-1-i}G_r(x,y)]| \le \frac{C}{|x-y|^{n-2i-1}}, \quad i=0,\cdots,m-1. \tag{30}$$

It follows from (27) that

$$\int\limits_{R^n} \frac{1}{|x-y|^{n-2m}}|y|^a u^p(y)dy < \infty. \tag{31}$$

By (31), there exists $r_k \to \infty$, such that

$$0 < \frac{|r_k - |x||^a}{r_k^{n-2m-1}} \int\limits_{\partial B_{r_k}(x)} u^p(y)d\sigma \le \frac{1}{r_k^{n-2m-1}} \int\limits_{\partial B_{r_k}(x)} |y|^a u^p(y)d\sigma \to 0.$$

We further deduce that

$$\frac{1}{r_k^{n-2m-1-a}} \int\limits_{\partial B_{r_k}(x)} u^p(y)d\sigma \to 0,\ \text{as } r_k \to \infty.$$

Then by Jensen's inequality, we have

$$\frac{1}{r_k^{n-1-\frac{2m+a}{p}}} \int\limits_{\partial B_{r_k}(x)} u(y)d\sigma \to 0,\ \text{as } r_k \to \infty. \tag{32}$$

For $r^{n-1-\frac{2m+a}{p}} < r^{n-1}$, it is easy to see

$$\frac{1}{r_k^{n-1}} \int\limits_{\partial B_{r_k}(x)} u(y)d\sigma \to 0,\ \text{as } r_k \to \infty. \tag{33}$$

Set

$$(-\triangle)^i u = u_i,\ i=1,\cdots,m-1, \tag{34}$$

$$(-\triangle)^i G_r(x,y) = G_i(x,y),\ i=1,\cdots,m-1. \tag{35}$$

Multiply both sides of (34) by $G_{m-i}(x,y)$ and integrate on $B_r(x)$. After integrating by parts, by Theorem 3 and (26) again, we deduce

$$\begin{aligned}\int\limits_{B_r(x)} u_i(y)G_{m-i}(x,y)dy &= u(x) + \sum_{j=0}^{i-1} \int\limits_{\partial B_r(x)} [(-\triangle)^j u]\frac{\partial}{\partial \nu_y}[(-\triangle)^{m-1-j}G_r(x,y)]d\sigma \\ &\le u(x), \quad i=1,\cdots,m-1.\end{aligned} \tag{36}$$

Equations (29) and (36) imply

$$\int_{R^n} \frac{u_i(y)}{|x-y|^{n-2m+2(m-i)}} dy = \int_{R^n} \frac{u_i(y)}{|x-y|^{n-2i}} dy < \infty, \ i = 1, \cdots, m-1.$$

Then there exists $r_k \to \infty$, such that

$$\frac{1}{r_k^{n-2i-1}} \int_{\partial B_{r_k}(x)} u_i(y) d\sigma \to 0, \ i = 1, \cdots, m-1. \tag{37}$$

From equation (27), by (30), (31), (33) and (37), there exists $r_k \to \infty$, such that

$$u(x) = C \int_{R^n} \frac{1}{|x-y|^{n-2m}} |y|^a u^p(y) dy. \tag{38}$$

Here we finish the proof that if $u$ is a positive solution of (3), then $u$ satisfies (4). (ii) If $u \in C^{2m}(R^n)$ is a solution of (4), then

$$\begin{aligned}(-\triangle)^m u(x) &= \int_{R^n} (-\triangle)^m G(x,y) |y|^a u^p(y) dy \\ &= \int_{R^n} \delta(x-y) |y|^a u^p(y) dy \\ &= |x|^a u^p(x).\end{aligned}$$

It's easy to conclude the proof that if $u \in C^{2m}(R^n)$ is a solution of (4), then $u$ satisfies (3). This completes the proof of Theorem 1.

## 4. The proof of Theorem 3

For any real number $\lambda$, let the moving plane be

$$T_\lambda = \{x = (x_1, \cdots, x_n) | x_1 = \lambda\}.$$

We denote

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_n) | x_1 \leq \lambda\}.$$

Set $x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$, $u_\lambda(x) = u(x^\lambda)$, and $w_\lambda(x) = u_\lambda(x) - u(x)$.

The following lemma is key inequality in integral estimates.

**Lemma 1** *(An equivalent form of the Hardy–Littlewood–Sobolev inequality). Let $g \in L^p$ for $\frac{n}{n-\alpha} < p < \infty$. Define*

$$Tg(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} g(y) dy,$$

*then*

$$\|Tg\|_{L^p} \leq C(n, p, \alpha) \|g\|_{L^{\frac{np}{n+\alpha p}}}.$$

For $\frac{n}{n-2m} < p < \frac{n+2m+a}{n-2m}$, let

$$v(x) = \frac{1}{|x|^{n-2m}} u(\frac{x}{|x|^2}) \tag{39}$$

be the Kelvin transform of $u$. We calculate $y = \frac{\widetilde{y}}{|\widetilde{y}|^2}$, $dy = \frac{d\widetilde{y}}{|\widetilde{y}|^{2n}}$, and

$$\begin{aligned}
v(x) &= \frac{1}{|x|^{n-2m}} u(\frac{x}{|x|^2}) \\
&= \frac{1}{|x|^{n-2m}} \int\limits_{R^n} \frac{1}{\left|\frac{x}{|x|^2} - y\right|^{n-2m}} |y|^a u(y)^p dy \\
&= \frac{1}{|x|^{n-2m}} \int\limits_{R^n} \frac{1}{\left|\frac{x}{|x|^2} - \frac{\widetilde{y}}{|\widetilde{y}|^2}\right|^{n-2m}} \left|\frac{\widetilde{y}}{|\widetilde{y}|^2}\right|^a u^p(\frac{\widetilde{y}}{|\widetilde{y}|^2}) \frac{1}{|\widetilde{y}|^{2n}} d\widetilde{y} \\
&= \int\limits_{R^n} \frac{1}{\left|\frac{x}{|x|^2} - \frac{\widetilde{y}}{|\widetilde{y}|^2}\right|^{n-2m}} \frac{|\widetilde{y}|^a}{|x|^{n-2m}|\widetilde{y}|^{n-2m}} \left[\frac{u(\frac{\widetilde{y}}{|\widetilde{y}|^2})}{|\widetilde{y}|^{n-2m}}\right]^p \frac{1}{|\widetilde{y}|^b} d\widetilde{y} \\
&= \int\limits_{R^n} \frac{1}{|\widetilde{y} - x|^{n-2m}} v^p(\widetilde{y}) \frac{1}{|\widetilde{y}|^{b-a}} d\widetilde{y},
\end{aligned}$$

where $b = (n-2m)(\tau - p) > a$, $\tau = \frac{n+2m+2a}{n-2m}$.

Then we deal with $v(x)$. By our assumption on $u$ that $|x|^a u^{p-1} \in L_{loc}^{\frac{n}{2m}}(R^n)$, we can derive that $|x|^{a-b} v^{p-1} \in L_{loc}^{\frac{n}{2m}}(R^n \backslash 0)$. Equivalently for any domain $\Omega$ of a positive distance away from the origin, we have

$$\int\limits_{\Omega} \left(\frac{v^{p-1}(y)|y|^a}{|y|^b}\right)^{\frac{n}{2m}} dy < \infty. \tag{40}$$

Let $p = \frac{n+2m+a-\varepsilon}{n-2m}$, where $0 < \varepsilon < 2m + a$, we have $v(x) = \int_{R^n} \frac{1}{|x-y|^{n-2m}} \frac{v^p(y)}{|y|^\varepsilon} dy$.
Since

$$\begin{aligned}
v(x) &= \int\limits_{\Sigma_\lambda} G(x,y) \frac{1}{|y|^\varepsilon} v^p(y) dy + \int\limits_{\Sigma_\lambda} G(x^\lambda, y) \frac{1}{|y^\lambda|^\varepsilon} v_\lambda^p(y) dy \\
v(x^\lambda) &= \int\limits_{\Sigma_\lambda} G(x^\lambda, y) \frac{1}{|y|^\varepsilon} v^p(y) dy + \int\limits_{\Sigma_\lambda} G(x^\lambda, y^\lambda) \frac{1}{|y^\lambda|^\varepsilon} v_\lambda^p(y) dy
\end{aligned}$$

then we have

$$\begin{aligned}
v(x) - v_\lambda(x) &= \int\limits_{\Sigma_\lambda} [G(x,y) - G(x^\lambda, y)] \frac{1}{|y|^\varepsilon} v^p(y) dy + \int\limits_{\Sigma_\lambda} [G(x^\lambda, y) - G(x^\lambda, y^\lambda)] \frac{1}{|y^\lambda|^\varepsilon} v_\lambda^p(y) dy \\
&= \int\limits_{\Sigma_\lambda} [G(x,y) - G(x^\lambda, y)] \left[\frac{1}{|y|^\varepsilon} v^p(y) - \frac{1}{|y^\lambda|^\varepsilon} v_\lambda^p(y)\right] dy
\end{aligned} \tag{41}$$

We now prove that $v(x)$ must be radially symmetric and decreasing about $O$. Here, the proof consists of two steps. In step 1, we show that for any sufficiently negative $\lambda$,

$$v(x) \le v(x^\lambda) \quad \text{in} \quad \Sigma_\lambda \backslash B_\delta(0^\lambda). \tag{42}$$

In step 2, we move the plane $T_\lambda$ along the $x_1$ direction continuously from near negative infinity to the right as long as (42) holds. We show that the plane can be moved all the way to $x_1 = 0$. Hence

$$v(x) \le v(x^0), \ \forall x \in \Sigma_0.$$

If we move the plane from positive infinity to the left and carry on the same procedure as in step 1 and step 2, we can also prove that

$$v(x) \ge v(x^0) \quad \text{in} \quad \Sigma_0.$$

Therefore $v(x)$ is symmetric about the plane $T_0$. Since the direction of $x_1$ can be chosen arbitrarily, we deduce that $v(x)$ is symmetric and decreasing about $O$. It follows that $u(x)$ is also symmetric about $O$. For more related results and details, please see [13,14,27] and the references therein.

Step 1. Define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \backslash B_\delta(x^\lambda) | w_\lambda(x) < 0\}.$$

We show that for $\lambda$ sufficiently negative, $\Sigma_\lambda^-$ must be measure zero. By the Mean Value Theorem, we have, for sufficiently negative values of $\lambda$, and for any $x \in \Sigma_\lambda^-$,

$$\begin{aligned}
0 < v(x) - v_\lambda(x) &= \int_{\Sigma_\lambda} [G(x,y) - G(x^\lambda, y)] \left[\frac{1}{|y|^\varepsilon} v^p(y) - \frac{1}{|y^\lambda|^\varepsilon} v_\lambda^p(y)\right] dy \\
&= \int_{\Sigma_\lambda^-} [G(x,y) - G(x^\lambda, y)] \left[\frac{1}{|y|^\varepsilon} v^p(y) - \frac{1}{|y^\lambda|^\varepsilon} v_\lambda^p(y)\right] dy \\
&\quad + \int_{\Sigma_\lambda \backslash \Sigma_\lambda^-} [G(x,y) - G(x^\lambda, y)] \left[\frac{1}{|y|^\varepsilon} v^p(y) - \frac{1}{|y^\lambda|^\varepsilon} v_\lambda^p(y)\right] dy \\
&\le \int_{\Sigma_\lambda^-} [G(x,y) - G(x^\lambda, y)] \left[\frac{1}{|y|^\varepsilon} v^p(y) - \frac{1}{|y^\lambda|^\varepsilon} v_\lambda^p(y)\right] dy \\
&\le \int_{\Sigma_\lambda^-} [G(x,y) - G(x^\lambda, y)] \left[\frac{v^p(y) - v_\lambda^p(y)}{|y|^\varepsilon}\right] dy \\
&= p \int_{\Sigma_\lambda^-} [G(x,y) - G(x^\lambda, y)] \frac{\psi^{p-1}(y)}{|y|^\varepsilon} [v(y) - v_\lambda(y)] dy \\
&\le p \int_{\Sigma_\lambda^-} G(x,y) \frac{v^{p-1}(y)}{|y|^\varepsilon} [v(y) - v_\lambda(y)] dy.
\end{aligned}$$

Here the value of $\psi^{p-1}(y)$ is between the value of $v_\lambda^{p-1}(y)$ and that of $v^{p-1}(y)$. We apply Hardy–Littlewood–Sobolev inequality to obtain for any $q > \frac{n}{n-2m}$,

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \le \|C \int_{\Sigma_\lambda^-} G(x,y) \frac{v^{p-1}(y)}{|y|^\varepsilon} [v(y) - v_\lambda(y)] dy\|_{L^q(\Sigma_\lambda^-)}$$

$$\leq C \left\| \frac{v^{p-1} w_\lambda}{|y|^\varepsilon} \right\|_{L^{\frac{nq}{n+2mq}}(\Sigma_\lambda^-)}. \tag{43}$$

Then we apply Hödler inequality to (43), we first choose the appropriate number

$$s = \frac{n + 2mq}{n}$$

such that

$$\frac{nq}{n + 2mq} \cdot s = q,$$

then we choose

$$r = \frac{n + 2mq}{2mq}$$

to ensure that

$$\frac{1}{r} + \frac{1}{s} = 1.$$

We can get the following

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C \left\| \frac{v^{p-1}}{|y|^\varepsilon} \right\|_{L^{\frac{n}{2m}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}. \tag{44}$$

We can choose $N$ sufficiently large, such that for $\lambda \leq -N$,

$$C \left\| \frac{v^{p-1}}{|y|^\varepsilon} \right\|_{L^{\frac{n}{2m}}(\Sigma_\lambda^-)} \leq \frac{1}{2}.$$

Now inequality (44) implies $\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0$, and therefore $\Sigma_{\lambda^-}$ must be measure zero.

This implies

$$\omega_\lambda(x) \geq 0, \text{ a.e.} x \in \Sigma_\lambda. \tag{45}$$

Step 2. Move the plane to the origin to derive symmetry.

Inequality (45) provides a starting point to move the plane

$$T_\lambda = \{x \in R^n | x_1 = \lambda\}.$$

Now we start from the neighborhood of $x_1 = -\infty$ and move the plane to the right as long as (44) holds. We can see that by moving the plane this way, the plane will not stop before hitting the origin of $R^n$. Define

$$\lambda_0 = sup\{\lambda \leq 0 | w_\rho(x) \geq 0, \ \rho \leq \lambda, \ \forall x \in \Sigma_\rho \setminus \{0^\rho\}\},$$

where $0^\rho$ is the reflection of 0 about the plane $T_\rho$. We will prove that

$$\lambda_0 = 0. \tag{46}$$

Suppose $\lambda_0 < 0$, we first show that $v(x)$ is symmetric about the plane $T_{\lambda_0}$, i.e.

$$w_{\lambda_0} \equiv 0, \text{ a.e. } \forall x \in \Sigma_{\lambda_0}. \tag{47}$$

Suppose the contrary, then we have $w_{\lambda_0} \geq 0$, but $w_{\lambda_0} \neq 0$ a.e. on $\Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}$. We show that the plane can be moved further. More precisely, there exists an $\varepsilon > 0$ such that for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$,

$$v(x) \leq v_\lambda(x) \text{ a.e. on } \Sigma_\lambda \setminus \{0^\lambda\}.$$

By inequality (44), we have

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C \left\| \frac{v^{p-1}}{|y|^\varepsilon} \right\|_{L^{\frac{n}{2m}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}. \tag{48}$$

Again by (40), we choose $\varepsilon$ sufficiently small so that for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$,

$$C \left\| \frac{v^{p-1}}{|y|^\varepsilon} \right\|_{L^{\frac{n}{2m}}(\Sigma_\lambda^-)} \leq \frac{1}{2}. \tag{49}$$

We postpone the proof of (49) for a moment. Now by (48) and (49), we have $\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0$, therefore $\Sigma_\lambda^-$ must be measure zero. Hence for these values of $\lambda > \lambda_0$, we have

$$w_\lambda(x) \geq 0 \text{ a.e. } \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}.$$

This contradicts with the definition of $\lambda_0$, therefore (47) must hold.

Next, we show that (46) is true. Otherwise, if the plane stops at $x_1 = \lambda_0 < 0$, then

$$v_{\lambda_0}(x) = v(x) \quad \forall x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}.$$

Then it is easy to see that

$$0 = v(x) - v_{\lambda_0}(x) = \int_{\Sigma_{\lambda_0}} [G(x, y) - G(x^{\lambda_0}, y)] \left[ \frac{v^p(y)}{|y|^\varepsilon} - \frac{v_{\lambda_0}^p(y)}{|y^{\lambda_0}|^\varepsilon} \right] dy > 0.$$

This is obviously a contradiction. Hence

$$w_0(x) \geq 0 \text{ a.e. } \forall x \in \Sigma_0.$$

If we move the plane from the positive infinity to the left and carry on the same procedure as done above in steps 1 and 2, we can also prove that $w_0(x) \leq 0$ *a.e.* $\forall x \in \Sigma_0$. Therefore $v(x)$ is symmetric about the plane $T_0$, then we deduce that (47) must hold.

Since the direction of $x_1$ in $R^n$ can be chosen arbitrarily, we deduce that $v(x)$ must be radically symmetric in $x \in R^n$ about $x = 0$ and decreasing about the origin in $R^n$. By expression (39), we conclude that $u(x)$ must be radially symmetric in $x \in R^n$ about $x = 0$.

We further deduce that $u(x)$ decreases about the origin in $R^n$. Suppose on the contrary, then there exists $r_2 > r_1$, s.t. $u(r_2) > u(r_1)$, where $r_1 \neq 0$, we deduce that the minimum value must be in $B_{r_2}$. While, according to Theorem 3 $-\Delta u(x) > 0$, by the maximum principle, we conclude that it is impossible for $u(x)$ to have a minimum in $B_{r_2}$. This leads to contradiction, and therefore the conclusion that $u(x)$ decreases about the origin in $R^n$ is true.

Now we prove inequality (49). For any small $\eta > 0$, we can chose $r$ sufficiently large so that

$$\left\| \frac{v^{p-1}}{|y|^{\varepsilon}} \right\|_{L^{\frac{n}{2m}}(R^n \setminus \{0\} \setminus B_r(0))} < \eta. \tag{50}$$

Fix this $r$ and then we show that the measure of $\Sigma_\lambda^- \bigcap B_r(0)$ is sufficiently small for $\lambda$ close to $\lambda_0$. Then by Lemma 1, it is easy to see

$$w_{\lambda_0} > 0 \text{ in the interior of } \Sigma_{\lambda_0} \setminus \{0\}. \tag{51}$$

For any $\gamma > 0$, let

$$E_\gamma = \{x \in (\Sigma_{\lambda_0} \setminus \{0\}) \cap B_r(0) | w_{\lambda_0}(x) > \gamma\}, \; F_\gamma = ((\Sigma_{\lambda_0} \setminus \{0\}) \cap B_r(0)) \setminus E_\gamma.$$

It is obvious that

$$\lim_{\gamma \to 0} \mu(F_\gamma) = 0. \tag{52}$$

For $\lambda > \lambda_0$, let

$$D_\lambda = ((\Sigma_\lambda \setminus \{0\}) \setminus \Sigma_{\lambda_0}) \cap B_r(0).$$

Then it is obvious that

$$(\Sigma_\lambda^- \cap B_r(0)) \subset (\Sigma_\lambda^- \cap E_\gamma) \cup F_\gamma \cup D_\lambda. \tag{53}$$

It is easy to see that, the measure of $D_\lambda$ is small for $\lambda$ close to $\lambda_0$. Then we only need to show that the measure of $\Sigma_\lambda^- \cap E_\gamma$ can be sufficiently small as $\lambda$ close to $\lambda_0$. In fact, for any $x \in \Sigma_\lambda^- \cap E_\gamma$, we have

$$w_\lambda(x) = v_\lambda(x) - v(x) = v_\lambda(x) - v_{\lambda_0}(x) + v_{\lambda_0}(x) - v(x) < 0.$$

Hence

$$v_{\lambda_0}(x) - v_\lambda(x) > v_{\lambda_0}(x) - v(x) = w_{\lambda_0}(x) > \gamma.$$

It follows that

$$(\Sigma_\lambda^- \cap E_\gamma) \subset G_\gamma \equiv \{x \in B_R(0) | v_{\lambda_0}(x) - v_\lambda(x) > \gamma\}. \tag{54}$$

By the well-known Chebyshev inequality, we have

$$\begin{aligned} \mu(G_\gamma) &\leq \frac{1}{\gamma^{p+1}} \int\limits_{G_\gamma} |v_{\lambda_0}(x) - v_\lambda(x)|^{p+1} dx \\ &\leq \frac{1}{\gamma^{p+1}} \int\limits_{B_R(0)} |v_{\lambda_0}(x) - v_\lambda(x)|^{p+1} dx. \end{aligned}$$

For each fixed $\gamma$, as $\lambda$ close to $\lambda_0$, the right hand side of the above inequality can be made as small as we wish. Therefore by (53) and (54), the measure of $\Sigma_\lambda^- \cap B_r(0)$ can also be made sufficiently small. Combining this with (50), we obtain (49).

This completes the proof of Theorem 3. For more related results and details, please see [13,14,27] and the references therein.

## 5. The proof of Theorem 4

Now, we consider

$$u(x) = \int_{R^n} \frac{u^p(y)|y|^a}{|x-y|^{n-2m}} dy, \quad x \in R^n. \tag{55}$$

From (55), we have

$$u(\mu x) = \int_{R^n} \frac{u^p(y)|y|^a}{|\mu x-y|^{n-2m}} dy. \tag{56}$$

Since

$$\begin{aligned}
\frac{d}{d\mu}(|\mu x-y|^{2m-n}) &= \frac{d}{d\mu}(|\mu x-y|^2)^{\frac{2m-n}{2}} \\
&= \frac{2m-n}{2}(|\mu x-y|^2)^{\frac{2m-n}{2}-1}\frac{d}{d\mu}|\mu x-y|^2 \\
&= \frac{2m-n}{2}(|\mu x-y|^2)^{\frac{2m-n}{2}-1}(2(\mu x_1-y_1)x_1+\cdots+2(\mu x_n-y_n)x_n) \\
&= (2m-n)|\mu x-y|^{2m-n-2}\sum_{i=1}^{n}(\mu x_i-y_i)x_i \\
&= (2m-n)|\mu x-y|^{2m-n-2}x\cdot(\mu x-y).
\end{aligned}$$

We derivative (56), with respect to $\mu$:

$$x\cdot\nabla u(\mu x) = (2m-n)\int_{R^n} \frac{x\cdot(\mu x-y)u^p(y)|y|^a}{|\mu x-y|^{n-2m+2}} dy, \ x \neq 0.$$

Letting $\mu = 1$ yields

$$x\cdot\nabla u(x) = (2m-n)\int_{R^n} \frac{x\cdot(x-y)u^p(y)|y|^a}{|x-y|^{n-2m+2}} dy, \ x \neq 0. \tag{57}$$

Multiplying both sides of (57) by $|x|^a u^p(x)$ and integrating on $B_r \setminus B_\varepsilon := B_r(0) \setminus B_\varepsilon(0)$, we integrate by parts to obtain

$$\begin{aligned}
\text{The left} &= \int_{B_r\setminus B_\varepsilon} u^p(x)|x|^a(x\cdot\nabla u(x))dx \\
&= \int_\varepsilon^r \int_{\partial B_R} u^p(R)R^a(x\cdot\frac{x}{R}\frac{\partial u(R)}{\partial R})d\sigma dR \\
&= \frac{1}{p+1}\int_\varepsilon^r nw_n R^{n+a} du^{p+1}(x) \\
&= \frac{1}{p+1}\int_{\partial B_r} u^{p+1}(x)r^{a+1}d\sigma + \frac{1}{p+1}\int_{\partial B_\varepsilon} u^{p+1}(x)\varepsilon^{a+1}d\sigma
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{p+1} \int_{\varepsilon}^{r} n w_n u^{p+1}(x) dR^{n+a} \\
&= \frac{1}{p+1} \int_{\partial B_r} u^{p+1}(x) r^{a+1} d\sigma + \frac{1}{p+1} \int_{\partial B_\varepsilon} u^{p+1}(x) \varepsilon^{a+1} d\sigma \\
&- \frac{n+a}{p+1} \int_{B_r \setminus B_\varepsilon} u^{p+1}(x)|x|^a dx
\end{aligned} \tag{58}
$$

and

$$
\text{The right} = (2m-n) \int_{B_r \setminus B_\varepsilon} \int_{R^n} \frac{x \cdot (x-y) u^p(y)|y|^a |x|^a u^p(x)}{|x-y|^{n-2m+2}} dy dx. \tag{59}
$$

By (55), we have

$$
\begin{aligned}
u(r) \;=\; u(re) &= \int_{R^n} \frac{u^p(y)|y|^a}{|x-y|^{n-2m}} dy \\
&\geq \int_0^r \int_{\partial B_s} \frac{u^p(y)|y|^a}{|re-y|^{n-2m}} d\sigma ds \\
&\geq C \int_0^r \int_{\partial B_1} \frac{u^p(r)}{|re - s\omega|^{n-2m}} s^{n-1+a} d\omega ds \\
&= C \frac{u^p(r)}{r^{n-2m}} \int_0^r \int_{\partial B_1} \frac{s^{n-1+a}}{|e - \frac{s}{r}\omega|^{n-2m}} d\omega ds \\
&=: C \frac{u^p(r)}{r^{n-2m}} \int_0^r \int_{\partial B_1} s^{n-1+a} f(s) ds.
\end{aligned} \tag{60}
$$

Obviously, for each fixed $0 \leq s \leq r$, $f(s) > 0$. Set $t = \frac{s}{r}$, then $0 \leq t \leq 1$, $g(t) := f(s)$. Since $[0,1]$ is a compact set, $g(t)$ is continuous in $t$, we must have $g(t) \geq C_0 > 0$. Then by (60), we deduce

$$
u(r) \geq c u^p(r) r^{2m+a}.
$$

This implies

$$
u(r) \leq \frac{c}{r^{\frac{2m+a}{p-1}}}, \text{ as } r \to \infty, \tag{61}
$$

and by (61), we deduce

$$
\int_{R^n} u^{p+1}(y)|y|^a dy < \infty. \tag{62}
$$

Then there exists a sequence $r_j \to \infty$ as $j \to \infty$, such that

$$r_j^{1+a} \int\limits_{\partial B_{r_j}} u^{p+1}(x) d\sigma \to 0. \tag{63}$$

Since

$$x(x-y) + y(y-x) = |x-y|^2.$$

By symmetry, we have

$$\begin{aligned}
\frac{2m-n}{2} \int\limits_{R^n} u^{p+1}(x)|x|^a dx &= \frac{2m-n}{2} \int\limits_{R^n}\int\limits_{R^n} \frac{u^p(y)u^p(x)|x|^a|y|^a}{|x-y|^{n-2m}} dxdy \\
&= \frac{2m-n}{2} \int\limits_{R^n}\int\limits_{R^n} \frac{x\cdot(x-y)u^p(y)u^p(x)|x|^a|y|^a}{|x-y|^{n-2m+2}} dxdy \\
&\quad + \frac{2m-n}{2} \int\limits_{R^n}\int\limits_{R^n} \frac{y\cdot(y-x)u^p(y)u^p(x)|x|^a|y|^a}{|x-y|^{n-2m+2}} dxdy \\
&= (2m-n) \int\limits_{R^n}\int\limits_{R^n} \frac{x\cdot(x-y)u^p(y)u^p(x)|x|^a|y|^a}{|x-y|^{n-2m+2}} dxdy.
\end{aligned}$$

Letting $\varepsilon \to 0$ in (58) and (59), we derive

$$\begin{aligned}
&\frac{1}{p+1} \int\limits_{\partial B_r} r^{1+a} u^{p+1}(x) d\sigma - \frac{n+a}{p+1} \int\limits_{B_r} u^{p+1}(x) r^a dx \\
&= (2m-n) \int\limits_{B_r}\int\limits_{R^n} \frac{x\cdot(x-y)u^p(y)u^p(x)|x|^a|y|^a}{|x-y|^{n-2m+2}} dydx.
\end{aligned}$$

We arrive at

$$-\frac{n+a}{p+1} \int\limits_{B_r} u^{p+1}(x)|x|^a dx = \frac{2m-n}{2} \int\limits_{R^n} u^{p+1}(x)|x|^a dx. \tag{64}$$

Letting $r \to \infty$,

$$p+1 < \frac{2(n+a)}{n-2m},$$
$$\frac{n+a}{p+1} > \frac{n-2m}{2}$$

(64) implies $u \equiv 0$ in $R^n$. This completes the proof of Theorem 4.

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