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Paramodular forms of level 16 and supercuspidal representations

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This work bridges the abstract representation theory of GSp(4) with recent computational techniques. We construct four examples of paramodular newforms whose associated automorphic representations have local representations at $p = 2$ that are supercuspidal. We classify all relevant irreducible, admissible, supercuspidal representations of GSp(4, $\mathbb{Q}_2$), and show that our examples occur at the lowest possible paramodular level, 16. The required theoretical and computational techniques include paramodular newform theory, Jacobi restriction, bootstrapping and Borcherds products.

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Introduction

This paper consists of a local part and a global part. In the local part we classify irreducible, admissible, supercuspidal representations of $\mathrm{GSp}(4, \mathbb{Q}_2)$ with trivial central character and small conductor. In particular, we prove that there exists a unique such supercuspidal $\mathrm{sc}(16)$ with (the exponent of the) conductor $a(\mathrm{sc}(16)) = 4$. In the global part we construct Siegel paramodular cusp forms of weights 9, 11, 13, and 14 and paramodular level 16 generating an automorphic representation with $\mathrm{sc}(16)$ as its 2-component. To the best of our knowledge, these are the first examples of Siegel paramodular forms generating automorphic representations with a supercuspidal component. Other types of local representations can be seen in [PSY 2018].

We give two approaches to the construction of $\mathrm{sc}(16)$. The first approach relies on the local Langlands correspondence for the groups GL(2), GL(4) and GSp(4). We first construct, via automorphic induction, a set of six supercuspidals of $\mathrm{GL}(2, E)$, where $E = \mathbb{Q}_2(\sqrt{5})$ is the unramified quadratic extension of $\mathbb{Q}_2$. Up to unramified twists, these are precisely the depth zero supercuspidals of $\mathrm{GL}(2, E)$. We automorphically induce again to obtain three supercuspidals of $\mathrm{GL}(4, \mathbb{Q}_2)$. These are precisely the three depth zero supercuspidals of $\mathrm{GL}(4, \mathbb{Q}_2)$ with trivial central character. Of these three, exactly one is a transfer from a representation of $\mathrm{GSp}(4, \mathbb{Q}_2)$. This representation of $\mathrm{GSp}(4, \mathbb{Q}_2)$ is the unique generic supercuspidal $\mathrm{sc}(16)$ with trivial central character and conductor 4. As a corollary to our construction, we obtain a complete list of all supercuspidals of $\mathrm{GSp}(4, \mathbb{Q}_2)$ with trivial central character and conductor ≤ 4; see Table 2. We also determine, via direct calculation, that the value of the ε-factor at 1/2 of $\mathrm{sc}(16)$ is -1. This sign is important to know for global applications, as it will help us to identify $\mathrm{sc}(16)$ within the automorphic representations generated by paramodular forms.

Our second approach to $\mathrm{sc}(16)$ is via compact induction. The Langlands parameter $\mathrm{sc}(16)$, known from the first construction, is of the kind considered in [DeBacker and Reeder 2009]. The results of this paper then exhibit $\mathrm{sc}(16)$ as being compactly induced from a cuspidal representation κ_0 of $\mathrm{GSp}(4, \mathbb{Z}_2/2\mathbb{Z}_2)$ (inflated to $\mathrm{GSp}(4, \mathbb{Z}_2)$ and extended trivially to include the center). Since $\mathrm{GSp}(4, \mathbb{Z}/2\mathbb{Z}) \cong S_6$, the irreducible characters of this group are in bijection with the partitions of 6. The representation κ_0 corresponds to (2, 2, 1, 1) and has dimension 9. It is the unique cuspidal, generic character of $\mathrm{GSp}(4, \mathbb{Z}/2\mathbb{Z})$.

We describe the passage from global paramodular forms to local supercuspidal representations. The automorphic representations studied here are generated by the adelic function canonically associated to a paramodular eigenform $f \in S_k(K(N))^{\mathrm{new}}$. The interesting local representations are classified by computing the Hecke eigenvalues of f at primes dividing the level N. In order to rigorously compute these eigenvalues, we span the Fricke eigenspace containing f, $S_k(K(N))^{\epsilon}$. Accurate upper bounds for the dimension of $S_k(K(N))^{\epsilon}$ are provided by *Jacobi restriction*, which classifies all possible Fourier–Jacobi coefficients from $S_k(K(N))^{\epsilon}$ to some sufficient order. Lower bounds are created by the technique of *bootstrapping*. Bootstrapping seeds the target space with a Borcherds product, and then generates a subspace that contains the seed and is stable under a good Hecke operator. Bootstrapping is run modulo an auxiliary prime, and the subtle point is that it does not directly compute the action of a good Hecke operator $T(q)$ on $S_k(K(N))^{\epsilon}$, but rather of a formal Hecke operator $\mathcal{T}(q)$ on the Jacobi restriction space of initial Fourier–Jacobi expansions.

Even with the relevant spaces spanned, the eigenvalues at the bad primes resist direct computation because they involve Fourier coefficients from more than one 1-dimensional cusp. As in [PSY 2018], this is overcome using the technique of *restriction to a modular curve*. We found symmetric f with a supercuspidal local component early on, but only found the antisymmetric example in $S_{14}(K(16))^-$ as the computations were becoming prohibitive.

1. Notation

For any commutative ring R, let

$$\mathrm{GSp}(4, R) = \{g \in \mathrm{GL}(4, R) : {}^t g J g = \mu(g) J, \text{ for some } \mu(g) \in R^\times\}, \qquad J = \begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix}.$$

The kernel of the multiplier homomorphism $\mu : \mathrm{GSp}(4, R) \to R^\times$ is the group $\mathrm{Sp}(4, R)$. The $\mathbb{C}$-vector space of Siegel modular forms of weight $k \in \mathbb{Z}$ for a subgroup $\Gamma \subseteq \mathrm{GSp}(4, \mathbb{R})$ commensurable with $\mathrm{Sp}(4, \mathbb{Z})$ is denoted by $M_k(\Gamma)$, the subspace of cusp forms by $S_k(\Gamma)$.

2. Supercuspidal representations of GSp(4, $\mathbb{Q}_2$) of small conductor

Let F be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$ be its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and q the cardinality of the residue class field $\mathfrak{o}/\mathfrak{p}$. When there is more than one field involved, we sometimes write $\mathfrak{o}_F$, $\mathfrak{p}_F$, and q_F for clarity.

Let W_F be the Weil group of F, and W_F' the Weil–Deligne group. We refer to [Rohrlich 1994] or [Gross and Reeder 2010] for basic facts about the Weil and Weil–Deligne groups and their representations. If $\phi : W_F' \to \mathrm{GL}(n, \mathbb{C})$ is a representation of W_F', then we define the (exponent of the) *conductor* $a(\phi)$ of ϕ as in §10 of [Rohrlich 1994]. If π is an irreducible, admissible representation of $\mathrm{GL}(n, F)$, then the conductor of π is defined as $a(\pi) = a(\phi)$, where $\phi : W_F' \to \mathrm{GL}(n, \mathbb{C})$ is the Weil–Deligne representation corresponding to π via the local Langlands correspondence.

2.1. ***Discrete series parameters for* GSp(4).** The local Langlands correspondence (LLC) for $\mathrm{GL}(n)$ states that there is a bijection between isomorphism classes of irreducible, admissible representations π of $\mathrm{GL}(n, F)$ and Langlands parameters, i.e., conjugacy classes of admissible homomorphisms $\phi : W_F' \to \mathrm{GL}(n, \mathbb{C})$. This bijection satisfies a number of desirable properties. For example, if π corresponds to ϕ, then the central character of π corresponds to $\det(\phi)$ under the LLC for $\mathrm{GL}(1)$ (which is essentially the reciprocity law of local class field theory). Another property is that π is an essentially discrete series representation if and only if the image of ϕ is not contained in a proper Levi subgroup; such ϕ are therefore called discrete series parameters. Moreover, supercuspidal π correspond to irreducible ϕ.

The local Langlands correspondence is also a theorem for GSp(4); see [Gan and Takeda 2011]. The Langlands parameters are now admissible homomorphisms $\phi : W_F' \to \mathrm{GSp}(4, \mathbb{C})$, taken up to conjugacy by elements of $\mathrm{GSp}(4, \mathbb{C})$. A new phenomenon is that to one ϕ there now corresponds either a single representation π, as in the $\mathrm{GL}(n)$ case, or a set of two representations $\{\pi_1, \pi_2\}$. In either case we speak of the L-packet corresponding to ϕ. The size of the L-packet corresponding to ϕ equals the cardinality of $S_\phi / S_\phi^0 Z$, where S_ϕ is the centralizer of the image of ϕ, S_ϕ^0 is its identity component, and Z is the center of $\mathrm{GSp}(4, \mathbb{C})$.

The LLC for GSp(4) is such that the central character of the representations in the L-packet of ϕ corresponds to the multiplier $\mu \circ \phi$. As in the $\mathrm{GL}(n)$ case, the L-packet corresponding to ϕ consists of essentially discrete series representations if and only if the image of ϕ is not contained in a proper Levi subgroup of $\mathrm{GSp}(4, \mathbb{C})$. It is also true that irreducible $\phi : W_F' \to \mathrm{GSp}(4, \mathbb{C})$ correspond to singleton supercuspidal L-packets. However, there are plenty of supercuspidals whose Langlands parameter is not irreducible.

To better understand L-parameters for supercuspidals, we recall some of the discussion of Section 7 of [Gan and Takeda 2011]. Let $\phi : W_F' \to \mathrm{GSp}(4, \mathbb{C})$ be a discrete series parameter for GSp(4), meaning the image of ϕ is not contained in a proper Levi subgroup of $\mathrm{GSp}(4, \mathbb{C})$. Such parameters are of one of two types (A) or (B).

Type (A): Viewed as a four-dimensional representation of W_F', the map ϕ decomposes as $\phi_1 \oplus \phi_2$, where ϕ_1 and ϕ_2 are inequivalent indecomposable two-dimensional representations of W_F' with $\det(\phi_1) = \det(\phi_2)$. Explicitly, if $\phi_i(w) = \left[\begin{smallmatrix} a_i(w) & b_i(w) \\ c_i(w) & d_i(w) \end{smallmatrix}\right]$, then

$$\phi(w) = \begin{bmatrix} a_1(w) & & b_1(w) & \\ & a_2(w) & & b_2(w) \\ c_1(w) & & d_1(w) & \\ & c_2(w) & & d_2(w) \end{bmatrix}.$$

In this case the packet associated to ϕ consists of two elements, a generic representation π^{gen} and a non-generic π^{ng}. The common central character of these two representations corresponds to $\det(\phi_1) = \det(\phi_2)$. There are three subcases:

- **(A$_1$)**: Both ϕ_1 and ϕ_2 are irreducible. In this case π^{gen} and π^{ng} are both supercuspidal.
- **(A$_2$)**: One of ϕ_1, ϕ_2 is irreducible, and the other is reducible (but indecomposable). In this case π^{gen} is a representation of type XIa in the classification of [Roberts and Schmidt 2007]; it sits inside a representation induced from a supercuspidal representation of the Levi component of the Siegel parabolic subgroup. The non-generic π^{ng} is supercuspidal; it is a representation of type XIa* in the notation of [Roberts and Schmidt 2016].
- **(A$_3$)**: Both ϕ_1 and ϕ_2 are reducible (but indecomposable). In this case π^{gen} is a representation of type Va in the classification of [Roberts and Schmidt 2007]; it sits inside a representation induced from the Borel subgroup. The non-generic π^{ng} is supercuspidal; it is a representation of type Va* in the notation of [Roberts and Schmidt 2016].

Hence π^{ng} is always supercuspidal, but π^{gen} is only supercuspidal for class (A$_1$). Note that, by Theorem 3.4.3 of [Roberts and Schmidt 2007], non-generic supercuspidals do not contain paramodular vectors of any level. Hence, supercuspidals of the form π^{ng} cannot occur as local components in automorphic representations attached to paramodular cusp forms.

Type (B): Viewed as a four-dimensional representation of W_F', the map ϕ is indecomposable. In this case there is a single representation π attached to ϕ, and this π is generic. Via the inclusion $\mathrm{GSp}(4, \mathbb{C}) \hookrightarrow \mathrm{GL}(4, \mathbb{C})$ we may view ϕ as the Langlands parameter of a discrete series representation Π of $\mathrm{GL}(4, F)$. By the definitions involved, Π is the image of π under the functorial lifting from GSp(4) to GL(4) coming from the embedding $\mathrm{GSp}(4, \mathbb{C}) \hookrightarrow \mathrm{GL}(4, \mathbb{C})$ of dual groups. Again there are three subcases:

- **(B$_1$)**: ϕ is irreducible as a four-dimensional representation. In this case π is supercuspidal.
- **(B$_2$)**: $\phi = \varphi \otimes \mathrm{sp}(2)$ with an irreducible two-dimensional representation φ of W_F, and sp(2) being the special indecomposable two-dimensional representation of W_F'. In this case π is a representation of

type IXa; see Section 2.4 of [Roberts and Schmidt 2007]. This π sits inside a representation induced from a supercuspidal representation of the Levi component of the Klingen parabolic subgroup.

- **(B_3)**: $\phi = \xi \otimes \mathrm{sp}(4)$ with a one-dimensional representation ξ of W_F. Then π is a twist of the Steinberg representation $\mathrm{St}_{\mathrm{GSp}(4)}$ (type IVa in the classification of [loc. cit.]).

Hence π is supercuspidal only for class (B_1), i.e., if ϕ is irreducible. In this case π transfers to a supercuspidal representation Π of $\mathrm{GL}(4, F)$.

2.2. ***Counting supercuspidals for* GL(2) *and* GL(4).** We see from the parameters exhibited in the previous section that, in order to understand supercuspidal representations of $\mathrm{GSp}(4, F)$, we need to understand supercuspidal representations of $\mathrm{GL}(2, F)$ and $\mathrm{GL}(4, F)$, or equivalently, two-dimensional and four-dimensional irreducible representations of W_F. In this section we count the number of supercuspidals of $\mathrm{GL}(2, F)$ and $\mathrm{GL}(4, F)$ with small conductor.

The *conductor* $a(\pi)$ of an irreducible, admissible representation of $\mathrm{GL}(n, F)$ is by definition the Artin conductor $a(\phi)$ of its Langlands parameter ϕ; see §10 of [Rohrlich 1994]. Here, we always mean the *exponent* of the conductor, so that $a(\pi) = a(\phi)$ is a non-negative integer. Another measure of complexity is the *depth* $d(\pi)$, as defined in [Moy and Prasad 1994; 1996]. For supercuspidals, there is an easy relationship between depth and conductor, given by

$$d(\pi) = \frac{a(\pi) - n}{n}; \tag{1}$$

see Proposition 2.2 of [Lansky and Raghuram 2003]. The set of supercuspidals of a fixed conductor is invariant under unramified twisting.

The smallest conductor that can occur for a supercuspidal representation of $\mathrm{GL}(n, F)$ is $a(\pi) = n$. By (1), these are the depth zero supercuspidals. If π is one such supercuspidal, and χ is an unramified character, then the twist $\chi\pi$ is also a depth zero supercuspidal. For a positive integer n, let Z_n be the (finite) set of isomorphism classes of depth zero supercuspidals of $\mathrm{GL}(n, F)$ up to unramified twists. It is known that Z_n is in bijection with the set of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ orbits of length n in the group of characters of $\mathbb{F}_{q^n}^\times$; see Section 8 of [Deligne and Lusztig 1976] and Section 6 of [Moy and Prasad 1996]. It is an exercise to show that

$$\#Z_2 = \tfrac{1}{2}q(q-1), \qquad \#Z_4 = \tfrac{1}{4}q^2(q^2-1). \tag{2}$$

Note that if $q = 2$, then every tamely ramified character of $F^\times$ is unramified. Hence, in this case, every element of Z_n is represented by a unique depth zero supercuspidal with trivial central character. The reason is that depth zero supercuspidals are compactly induced from representations of ZK, where the representation on $K = \mathrm{GL}(2, \mathfrak{o})$ is inflated from a cuspidal representation of $\mathrm{GL}(2, \mathfrak{o}/\mathfrak{p})$. If $\mathfrak{o}/\mathfrak{p}$ has only two elements, then every representation of K thus obtained has trivial central character. In particular, we see from (2) that $\mathrm{GL}(2, \mathbb{Q}_2)$ has exactly one depth zero supercuspidal with trivial central character, and $\mathrm{GL}(4, \mathbb{Q}_2)$ has exactly three depth zero supercuspidals with trivial central character.

For a unitary character ω of $F^\times$, let S_ω be the set of isomorphism classes of depth zero supercuspidals of $\mathrm{GL}(2, F)$ with central character ω. By Proposition 3.4 of [Tunnell 1978], $\#S_\omega = 0$ if $a(\omega) \geq 2$. If

$a(\omega) \le 1$, then, by (4-1) of [Knightly and Ragsdale 2014],

$$\#S_\omega = \begin{cases} \frac{1}{2}(q-1) & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is trivial,} \\ \frac{1}{2}(q+1) & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is nontrivial,} \\ \frac{1}{2}q & \text{if } q \text{ is even.} \end{cases} \tag{3}$$

2.3. ***Depth zero supercuspidals of*** **$\mathrm{GL}(2, \mathbb{Q}_2(\sqrt{5}))$.** In this section let $E = \mathbb{Q}_2(\sqrt{5})$ be the unramified quadratic extension of $\mathbb{Q}_2$, and let L be the unramified quadratic extension of E. Note that 2 is a uniformizer both in E and in L. Let $\mathbb{F}_{p^n}$ be the field with p^n elements. The residue class field of E is $\mathbb{F}_4$, and the residue class field of L is $\mathbb{F}_{16}$. The polynomial $X^4 + X + 1 \in \mathbb{F}_2[X]$ is irreducible, so that

$$\mathbb{F}_{16} \cong \mathbb{F}_2[X]/(X^4 + X + 1).$$

Let $\bar{y}$ be the image of X via this isomorphism. Then $\mathbb{F}_{16} = \mathbb{F}_2(\bar{y})$, and $\bar{y}$ satisfies $\bar{y}^4 = \bar{y} + 1$. Clearly, the order of $\bar{y}$ in $\mathbb{F}_{16}^\times$ is not 3 or 5, so that $\bar{y}$ is a generator of the cyclic group $\mathbb{F}_{16}^\times$. The element $\bar{y}^5$ is then a generator of the cyclic group $\mathbb{F}_4^\times$. Let y be an element of $\mathfrak{o}_L^\times$ mapping to $\bar{y}$ under the projection $\mathfrak{o}_L^\times \to \mathbb{F}_{16}^\times$.

Let $\bar{\eta}$ be the character of $\mathbb{F}_{16}^\times$ determined by $\bar{\eta}(\bar{y}) = e^{2\pi i/15}$. For $r \in \mathbb{Z}/15\mathbb{Z}$ we define a character η_r of $L^\times$ by lifting $\bar{\eta}^r$ to $\mathfrak{o}_L^\times$ and setting $\eta_r(2) = -1$.

Let θ be the generator of $\mathrm{Gal}(L/\mathbb{Q}_2)$ that induces the map $x \mapsto x^2$ on $\mathbb{F}_{16}$. Then θ^2 generates $\mathrm{Gal}(L/E)$. We have $\eta_r^\theta = \eta_{2r}$. (If $\sigma \in \mathrm{Gal}(L/F)$ and π is a representation of $\mathrm{GL}(n, F)$, then π^σ is the representation of $\mathrm{GL}(n, F)$ defined by $\pi^\sigma(g) = \pi(\sigma(g))$.)

Consider automorphic induction $AI = AI_{L/E}$; see [Henniart and Herb 1995]. Recall that AI takes characters ξ of $L^\times$ to irreducible, admissible representations ρ of $\mathrm{GL}(2, E)$. By Proposition 4.5 of the same reference, the central character of ρ is given by $\chi_{L/E}(\xi|_{E^\times})$, where $\chi_{L/E}$ is the quadratic character of $E^\times$ corresponding to the extension L/E. On the Galois side, AI corresponds to induction of parameters, i.e., the parameter of ρ is

$$\phi_\rho = \mathrm{ind}_{W_L}^{W_E}(\xi).$$

This parameter is irreducible, i.e., ρ is supercuspidal, if and only if ξ is not $\mathrm{Gal}(L/E)$-invariant. We have $a(\rho) = 2a(\xi)$ by the conductor formula (a2) in §10 of [Rohrlich 1994].

We now consider $AI_{L/E}(\eta_r)$ for $r \in \{1, \dots, 15\}$. This representation is supercuspidal if and only if $\eta_{4r} \ne \eta_r$, which translates into $5 \nmid r$. Since $a(\eta) = 1$, we have $a(AI_{L/E}(\eta_r)) = 2$ for $r \ne 0$. The central character ω of $AI_{L/E}(\eta_r)$ is determined by $\omega(2) = 1$ and $\omega(y^5) = \eta_{5r}(y) = e^{2\pi i r/3}$. Hence, if we let ω_j be the character of $E^\times$ which is trivial on $1 + \mathfrak{p}_E$ and satisfies $\omega_j(2) = 1$ and $\omega_j(y^5) = e^{2\pi i(j-1)/3}$, then $\omega_1, \omega_2, \omega_3$ are the possible central characters of the $AI_{L/E}(\eta_r)$. We have $\omega_1^\theta = \omega_1$ and $\omega_2^\theta = \omega_3$. Considering Langlands parameters, it is easy to see that the $\mathrm{Gal}(E/\mathbb{Q}_2)$-conjugate of $AI(\xi)$ is given by $AI(\xi)^\theta = AI(\xi^\theta)$, and the contragredient is $AI(\xi)^\vee = AI(\xi^{-1})$.

Table 1 lists the supercuspidal representations of the form $AI(\eta_r)$. For each possible central character ω_j, there are two supercuspidals, which we denote by ρ_{ja} and ρ_{jb}. Note from (2) that there are exactly six depth zero supercuspidals of $\mathrm{GL}(2, E)$ up to unramified twists. The following lemma implies that the six representations $\{\rho_{1a}, \rho_{1b}, \rho_{2a}, \rho_{2b}, \rho_{3a}, \rho_{3b}\}$ represent these six classes of depth zero supercuspidals up to unramified twists. Note that having exactly two depth zero supercuspidals for a given central character ω_j is consistent with (3).

ξ	$AI(\xi)$	ω	$AI(\xi)^\theta$	$AI(\xi)^\vee$
η_3 or η_{12}	ρ_{1a}	ω_1	ρ_{1b}	ρ_{1a}
η_6 or η_9	ρ_{1b}	ω_1	ρ_{1a}	ρ_{1b}
η or η_4	ρ_{2a}	ω_2	ρ_{3a}	ρ_{3b}
η_7 or η_{13}	ρ_{2b}	ω_2	ρ_{3b}	ρ_{3a}
η_2 or η_8	ρ_{3a}	ω_3	ρ_{2a}	ρ_{2b}
η_{11} or η_{14}	ρ_{3b}	ω_3	ρ_{2b}	ρ_{2a}

Table 1. Representatives for the depth zero supercuspidals of $\mathrm{GL}(2, E)$ up to unramified twists. The first column shows $\mathrm{Gal}(L/E)$-orbits of length 2 of the characters $\xi = \eta_r$. The ω column shows the central character of the representation $AI_{L/E}(\xi)$. The columns $AI(\xi)^\theta$ and $AI(\xi)^\vee$ show the $\mathrm{Gal}(E/\mathbb{Q}_2)$-conjugate and contragredient of $AI_{L/E}(\xi)$, respectively.

Lemma 2.3.1. *Let $j \in \{1, 2, 3\}$.*

i) *The representation ρ_{ja} is not a twist of ρ_{jb}.*

ii) *Let $\rho = \rho_{ja}$ or $\rho = \rho_{jb}$. Then ρ^θ is not isomorphic to a twist of $\rho^\vee$.*

iii) *Let $\rho, \rho' \in \{\rho_{1a}, \rho_{1b}, \rho_{2a}, \rho_{2b}, \rho_{3a}, \rho_{3b}\}$. Then ρ is not an unramified twist of ρ', unless $\rho = \rho'$.*

Proof. i) Assume that $\rho_{ja} = \chi \otimes \rho_{jb}$ for some character χ of $E^\times$; we will obtain a contradiction. Taking central characters on both sides, we see that $\chi^2 = 1$. We have $a(\chi) \le 1$ by Proposition 3.4 of [Tunnell 1978].

Assume that $a(\chi) = 0$. Then χ is either the trivial character, or $\chi = \chi_{L/E}$, the unique nontrivial, unramified, quadratic character of $E^\times$. In either case $\chi \otimes \rho_{jb} = \rho_{jb}$, a contradiction.

Assume that $a(\chi) = 1$. Then χ induces a nontrivial character of $\mathfrak{o}_E^\times/(1+\mathfrak{p}_E)$. In particular, the image of $\chi|_{\mathfrak{o}_E^\times}$ consists of the third roots of unity, contradicting $\chi^2 = 1$.

ii) follows from i) and Table 1.

iii) Assume that ρ is an unramified twist of ρ'. Then the restrictions of the central characters of ρ and ρ' to $\mathfrak{o}_E^\times$ coincide. Hence $\rho = \rho_{j*}$ and $\rho' = \rho_{j*}$ with the same j. By i), we conclude $\rho = \rho'$. □

Lemma 2.3.2. *Let L be the unramified extension of degree 4 over $\mathbb{Q}_2$. Let the characters η_r of $L^\times$ be defined as above. Then, for $\xi \in \{\eta_3, \eta_6, \eta_9, \eta_{12}\}$,*

$$\varepsilon(1/2, \xi, \psi_L) = -1. \tag{4}$$

Here, $\psi_L = \psi \circ \mathrm{tr}_{L/\mathbb{Q}_2}$, where ψ is a character of $\mathbb{Q}_2$ that is trivial on $\mathbb{Z}_2$ but not on $2^{-1}\mathbb{Z}_2$.

Proof. Let

$$\mathbb{F}_{16} \cong \mathbb{F}_2[X]/(X^4 + X + 1),$$

and let $\bar{y}$ be the element corresponding to X, as at the beginning of this section. The Frobenius of the extension $\mathbb{F}_{16}/\mathbb{F}_2$ is given by squaring, so that

$$\mathrm{tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(x) = x + x^2 + x^4 + x^8$$

for any $x \in \mathbb{F}_{16}$. Using this formula and $\bar{y}^4 = \bar{y} + 1$, it is easy to calculate the trace of any element of $\mathbb{F}_{16}$. The results are as follows:

$$\begin{array}{c|ccccccccccccccc} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \hline \mathrm{tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(\bar{y}^i) & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{array} \tag{5}$$

Let $\xi \in \{\eta_3, \eta_6, \eta_9, \eta_{12}\}$. By the formula ($\epsilon$3) in §11 of [Rohrlich 1994],

$$\varepsilon(1/2, \xi, \psi_L) = q_L^{-a(\xi)/2} \int_{\varpi_L^{-a(\xi)} \mathfrak{o}_L^\times} \xi^{-1}(x) \psi_L(x)\, dx. \tag{6}$$

For this formula to hold, it is important that ψ_L has conductor $\mathfrak{o}_L$, which is the case for our additive character. The element ϖ_L is a uniformizer; in our case we may take $\varpi_L = 2$. We further have $a(\xi) = 1$ and $q_L = 16$, so that

$$\begin{aligned} \varepsilon(1/2, \xi, \psi_L) &= \frac{1}{4} \int_{2^{-1}\mathfrak{o}_L^\times} \xi^{-1}(x)\psi_L(x)\, dx = \frac{1}{4} |2^{-1}|_L \int_{\mathfrak{o}_L^\times} \xi^{-1}(2^{-1}x)\psi_L(2^{-1}x)\, dx \\ &= -4 \int_{\mathfrak{o}_L^\times} \xi^{-1}(x)\psi_L(2^{-1}x)\, dx = -4 \operatorname{vol}(1+\mathfrak{p}_L) \sum_{x \in \mathfrak{o}_L^\times/(1+\mathfrak{p}_L)} \xi^{-1}(x)\psi_L(2^{-1}x) \\ &= -\frac{1}{4} \sum_{x \in \mathfrak{o}_L^\times/(1+\mathfrak{p}_L)} \xi^{-1}(x)\psi(2^{-1}\mathrm{tr}_{L/\mathbb{Q}_2}(x)). \end{aligned}$$

We have

$$\psi(2^{-1}\mathrm{tr}_{L/\mathbb{Q}_2}(x)) = \begin{cases} 1 & \text{if } \mathrm{tr}_{L/\mathbb{Q}_2}(x) \in 2\mathbb{Z}_2 \quad \text{(equivalently, if } \mathrm{tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(\bar{x}) = 0\text{)}, \\ -1 & \text{if } \mathrm{tr}_{L/\mathbb{Q}_2}(x) \in \mathbb{Z}_2^\times \quad \text{(equivalently, if } \mathrm{tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(\bar{x}) = 1\text{)}. \end{cases}$$

Hence, using (5),

$$\begin{aligned} \varepsilon(1/2, \xi, \psi_L) &= -\frac{1}{4}\Bigg(\sum_{i \in \{1,2,4,5,8,10,15\}} \xi^{-1}(y^i) - \sum_{i \in \{3,6,7,9,11,12,13,14\}} \xi^{-1}(y^i) \Bigg) \\ &= -\tfrac{1}{4}\big(\zeta + \zeta^2 + \zeta^4 + \zeta^5 + \zeta^8 + \zeta^{10} + \zeta^{15} - \zeta^3 - \zeta^6 - \zeta^7 - \zeta^9 - \zeta^{11} - \zeta^{12} - \zeta^{13} - \zeta^{14}\big), \end{aligned}$$

where $\zeta = \xi^{-1}(y)$, a primitive fifth root of unity. Using $\zeta^5 = 1$ and $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$, this simplifies to

$$\begin{aligned} \varepsilon(1/2, \xi, \psi_L) &= -\tfrac{1}{4}\big(\zeta + \zeta^2 + \zeta^4 + 1 + \zeta^3 + 1 + 1 - \zeta^3 - \zeta - \zeta^2 - \zeta^4 - \zeta - \zeta^2 - \zeta^3 - \zeta^4\big) \\ &= -\tfrac{1}{4}\big(3 - \zeta - \zeta^2 - \zeta^3 - \zeta^4\big) = -1. \end{aligned}$$

This concludes the proof. □

2.4. ***Supercuspidals of* GSp(4, $\mathbb{Q}_2$) *with small conductor.*** As in the previous section, let E be the unramified quadratic extension of $\mathbb{Q}_2$. Let θ be the nontrivial element of $\mathrm{Gal}(E/\mathbb{Q}_2)$. We now consider automorphic induction $AI = AI_{E/\mathbb{Q}_2}$. Recall that AI takes irreducible, admissible representations ρ of $\mathrm{GL}(2, E)$ to irreducible, admissible representations π of $\mathrm{GL}(4, \mathbb{Q}_2)$. By Proposition 4.5 of [Henniart and Herb 1995], the central characters ω_ρ and ω_π are related by $\omega_\pi = \omega_\rho|_{\mathbb{Q}_2^\times}$. If ϕ_ρ is the parameter of ρ, then the parameter of π is

$$\phi_\pi = \mathrm{ind}_{W_E}^{W_{\mathbb{Q}_2}}(\phi_\rho).$$

Assume that ρ is supercuspidal, or equivalently, that ϕ_ρ is irreducible. Then π is supercuspidal if and only if $\rho \neq \rho^\theta$, where the Galois conjugate ρ^θ is defined by $\rho^\theta(g) = \rho(g^\theta)$ for $g \in \mathrm{GL}(2, E)$. In other words, ϕ_π is irreducible if and only if $\phi_\rho \neq \phi_\rho^\theta$, where $\phi_\rho^\theta(w) = \phi_\rho(\theta w \theta^{-1})$ for $w \in W_E$ (here we think of θ as an element of $W_{\mathbb{Q}_2}$ that is not in W_E). Also, we have $AI(\rho) = AI(\rho^\theta)$.

We apply $AI = AI_{E/\mathbb{Q}_2}$ to the supercuspidal representations of $\mathrm{GL}(2, E)$ listed in Table 1. It follows from this table that

$$AI(\rho_{1a}) = AI(\rho_{1b}), \qquad AI(\rho_{2a}) = AI(\rho_{3a}), \qquad AI(\rho_{2b}) = AI(\rho_{3b}), \tag{7}$$

and these are supercuspidal representations of $\mathrm{GL}(4, \mathbb{Q}_2)$. They all have trivial central character. By the conductor formula for induced representations of the Weil group, see (a2) in §10 of [Rohrlich 1994], they have conductor 4. It follows that the representations in (7) are precisely the three depth zero supercuspidals of $\mathrm{GL}(4, \mathbb{Q}_2)$ with trivial central character; see Section 2.2.

We will next determine which of the three supercuspidals in (7) are transfers from GSp(4). For any p-adic field F, an irreducible, admissible representation π of $\mathrm{GL}(4, F)$ is a transfer from $\mathrm{GSp}(4, F)$ if and only if its parameter $\phi_\pi : W_F' \to \mathrm{GL}(4, \mathbb{C})$, after suitable conjugation, has image in $\mathrm{GSp}(4, \mathbb{C})$. Assume this is the case, and consider the exterior square map $\bigwedge^2 : \mathrm{GL}(4, \mathbb{C}) \to \mathrm{GL}(6, \mathbb{C})$. Since the composition of $\bigwedge^2$ with the inclusion $\mathrm{GSp}(4, \mathbb{C}) \hookrightarrow \mathrm{GL}(4, \mathbb{C})$ decomposes as the direct sum of a five-dimensional and a one-dimensional representation of $\mathrm{GSp}(4, \mathbb{C})$, it follows that $\bigwedge^2 \circ \phi_\pi$ contains a one-dimensional representation of W_F'.

The following lemma was spelled out in a preprint version of [Gan and Takeda 2011] but not in the published version. We include a proof here.

Lemma 2.4.1. *Let E/F be a quadratic extension of p-adic fields. Let θ be an element of W_F that is not in W_E. Let (ϕ, V) be an irreducible two-dimensional representation of W_E, and let $\phi^\theta(w) = \phi(\theta w \theta^{-1})$ for $w \in W_E$. Then*

$$\textstyle\bigwedge^2 \left(\mathrm{ind}_{W_E}^{W_F}(\phi)\right) = U \oplus \mathrm{ind}_{W_E}^{W_F}(\det(\phi)),$$

where U is a 4-dimensional representation of W_F whose restriction to W_E is isomorphic to $\phi \otimes \phi^\theta$.

Proof. As a model for $\phi := \mathrm{ind}_{W_E}^{W_F}(\phi)$, we may take $V \oplus V$, with action

$$\phi(w)(v_1 \oplus v_2) = \phi(w)v_1 \oplus \phi^\theta(w)v_2 \quad (w \in W_E), \qquad \phi(\theta)(v_1 \oplus v_2) = v_2 \oplus \phi(\theta^2)v_1. \tag{8}$$

If spaces V_1 and V_2 carry an action of a group G, then

$$\textstyle\bigwedge^2 (V_1 \oplus V_2) \cong \bigwedge^2 V_1 \oplus (V_1 \otimes V_2) \oplus \bigwedge^2 V_2$$

as G-spaces. It follows that, as a W_E-representation,

$$\textstyle\bigwedge^2 \left(\mathrm{ind}_{W_E}^{W_F}(\phi)\right) = \det(\phi) \oplus (\phi \otimes \phi^\theta) \oplus \det(\phi)^\theta,$$

It is easy to see that $\det(\phi)\oplus\det(\phi)^\theta$ is invariant under the action of θ, and that in fact this two-dimensional space is isomorphic to $\mathrm{ind}_{W_E}^{W_F}(\det(\phi))$ as a W_F-representation. The space U realizing $\phi\otimes\phi^\theta$ is also invariant under θ. □

Lemma 2.4.2. *The representations* $AI_{E/\mathbb{Q}_2}(\rho_{2a})$ *and* $AI_{E/\mathbb{Q}_2}(\rho_{2b})$ *appearing in* (7) *are not transfers from* $\mathrm{GSp}(4, \mathbb{Q}_2)$.

Proof. Let $\rho = \rho_{2a}$ or ρ_{2b}. Let $\phi : W_E \to \mathrm{GL}(2, \mathbb{C})$ be the parameter of ρ. Then the parameter of $AI_{E/\mathbb{Q}_2}(\rho)$ is $\mathrm{ind}_{W_E}^{W_{\mathbb{Q}_2}}(\phi)$. By Lemma 2.4.1,

$$\bigwedge^2\big(\mathrm{ind}_{W_E}^{W_{\mathbb{Q}_2}}(\phi)\big) = U\oplus \mathrm{ind}_{W_E}^{W_{\mathbb{Q}_2}}(\det(\phi)),$$

where U is isomorphic to $\phi\otimes\phi^\theta$ as a W_E-representation. By Lemma 2.3.1 ii), the space U is irreducible, even as a W_E-representation. Since $\det(\phi) = \omega_2$ is not $\mathrm{Gal}(E/\mathbb{Q}_2)$-invariant, the two-dimensional $\mathrm{ind}_{W_E}^{W_{\mathbb{Q}_2}}(\det(\phi))$ is irreducible as a $W_{\mathbb{Q}_2}$-representation. Hence $\bigwedge^2\big(\mathrm{ind}_{W_E}^{W_{\mathbb{Q}_2}}(\phi)\big)$ does not contain any one-dimensional component. By our remarks above, $AI_{E/\mathbb{Q}_2}(\rho)$ cannot be a transfer from $\mathrm{GSp}(4, \mathbb{Q}_2)$. □

Theorem 2.4.3. *The group* $\mathrm{GSp}(4, \mathbb{Q}_2)$ *admits a unique generic supercuspidal representation* $\mathsf{sc}(16)$ *with conductor* $a(\mathsf{sc}(16)) = 4$ *and trivial central character. As a four-dimensional representation of* $W_{\mathbb{Q}_2}$, *the Langlands parameter of* $\mathsf{sc}(16)$ *is*

$$\phi_{\mathsf{sc}(16)} = \mathrm{ind}_{W_L}^{W_{\mathbb{Q}_2}}(\xi), \tag{9}$$

where L is the unramified extension of $\mathbb{Q}_2$ *of degree* 4, *and* ξ *is any character of* $L^\times$ *with the following properties*: ξ *is trivial on* $1+\mathfrak{p}_L$; *the values of the restriction of* ξ *to* $\mathfrak{o}_L^\times$ *are the fifth roots of unity*; $\xi(2) = -1$. *We have* $\varepsilon(1/2, \mathsf{sc}(16), \psi) = -1$, *where* ψ *is a character of* $\mathbb{Q}_2$ *which is trivial on* $\mathbb{Z}_2$ *but not on* $2^{-1}\mathbb{Z}_2$.

Proof. Let π be a generic supercuspidal representation of $\mathrm{GSp}(4, \mathbb{Q}_2)$ with $a(\pi) = 4$ and trivial central character. The requirement that π be generic excludes supercuspidals of type Va* and XIa*; these are the ones with parameters of type (A_2) and (A_3), as defined in Section 2.1. Assume that π has a parameter of type (A_1); we will obtain a contradiction. Parameters of type (A_1) are of the form $\phi_1\oplus\phi_2$, where ϕ_1, ϕ_2 are inequivalent irreducible, two-dimensional representations of $W_{\mathbb{Q}_2}$ with $\det(\phi_1) = \det(\phi_2) = 1$. Since $a(\pi) = 4$, we must have $a(\phi_1) = a(\phi_2) = 2$. Hence ϕ_1 and ϕ_2 correspond to supercuspidals of $\mathrm{GL}(2, \mathbb{Q}_2)$ with conductor 2 and trivial central character. By (3), there exists only one such supercuspidal. Hence $\phi_1 \cong \phi_2$, a contradiction.

By our considerations in Section 2.1, the parameter of π is of type (B_1), i.e., irreducible as a four-dimensional representation. Hence π transfers to a supercuspidal representation π' on $\mathrm{GL}(4, \mathbb{Q}_2)$ with trivial central character and $a(\pi') = 4$. It follows that π' is one of the representations in (7). By Lemma 2.4.2 we must have $\pi' = AI_{E/\mathbb{Q}_2}(\rho_{1a}) = AI_{E/\mathbb{Q}_2}(\rho_{1b})$, where E is the unramified quadratic extension of $\mathbb{Q}_2$. This shows that, as a four-dimensional representation, the parameter of π is

$$\mathrm{ind}_{W_E}^{W_{\mathbb{Q}_2}}(\phi_a) = \mathrm{ind}_{W_E}^{W_{\mathbb{Q}_2}}(\phi_b), \tag{10}$$

where ϕ_* is the parameter of ρ_{1*}. By the considerations on p. 284/285 of [Roberts 2001], there exists a

$a(\pi)$	π	type	generic	$\varepsilon(1/2,\pi)$	$L(s,\pi)^{-1}$
2	$\delta^*([\xi,\nu\xi],\nu^{-1/2})$	Va*	no	-1	$1-q^{-2s-1}$
3	$\delta^*(\nu^{1/2}\tau_2,\nu^{-1/2})$	XIa*	no	1	$1-q^{-s-1/2}$
	$\delta^*(\nu^{1/2}\tau_2,\nu^{-1/2}\xi)$	XIa*	no	-1	$1+q^{-s-1/2}$
4	$\delta^*(\nu^{1/2}\tau_3,\nu^{-1/2})$	XIa*	no	-1	$1-q^{-s-1/2}$
	$\delta^*(\nu^{1/2}\tau_3,\nu^{-1/2}\xi)$	XIa*	no	-1	$1+q^{-s-1/2}$
	$\delta^*(\nu^{1/2}\xi\tau_3,\nu^{-1/2})$	XIa*	no	1	$1-q^{-s-1/2}$
	$\delta^*(\nu^{1/2}\xi\tau_3,\nu^{-1/2}\xi)$	XIa*	no	1	$1+q^{-s-1/2}$
	sc(16)		yes	-1	1

Table 2. The supercuspidals π of $\mathrm{GSp}(4,\mathbb{Q}_2)$ with conductor $a(\pi)\le 4$ and trivial central character. The character ξ is the unique nontrivial, unramified, quadratic character of $\mathbb{Q}_2^\times$. The representation τ_2 is the unique supercuspidal of $\mathrm{GL}(2,\mathbb{Q}_2)$ with trivial central character and conductor 2. The representation τ_3 (resp. $\xi\tau_3$) is the unique supercuspidal of $\mathrm{GL}(2,\mathbb{Q}_2)$ with trivial central character, conductor 3 and root number 1 (resp. -1). The representation sc(16) is the one from Theorem 2.4.3. The non-generic supercuspidals $\delta^*(\dots)$ share an L-packet with the generic square-integrable representations $\delta(\dots)$ of type Va resp. XIa; see Section 4.6 of [Roberts and Schmidt 2016].

unique symplectic structure on the space of $\mathrm{ind}_{W_E}^{W_{\mathbb{Q}_2}}(\phi_*)$ for which $W_{\mathbb{Q}_2}$ acts with trivial similitude. We proved that the parameter of π is uniquely determined. The uniqueness and existence of π now follows from the local Langlands correspondence for $\mathrm{GSp}(4,\mathbb{Q}_2)$.

Let η_j be as in Table 1. Then $\eta_3,\eta_6,\eta_9,\eta_{12}$ are precisely the characters ξ of $L^\times$ with $\xi(2)=-1$, trivial on $1+\mathfrak{p}_L$, and such that the values of the restriction of ξ to $\mathfrak{o}_L^\times$ are the fifth roots of unity. Inducing η_3 or η_{12} to W_E gives the parameter ϕ_a of ρ_{1a}, and inducing η_6 or η_9 to W_E gives the parameter ϕ_b of ρ_{1b}. Hence (9) follows by transitivity of induction.

We have $\varepsilon(1/2,\pi,\psi)=\varepsilon(1/2,\xi,\psi_L)$ by Corollary 4 to Theorem 5.6 of [Henniart and Herb 1995], or by (ϵ2) in §11 of [Rohrlich 1994]. Hence the assertion about $\varepsilon(1/2,\pi,\psi)$ follows from Lemma 2.3.2. ☐

Corollary 2.4.4. *Table 2 contains a complete list of all the irreducible, admissible, supercuspidal representations π of* $\mathrm{GSp}(4,\mathbb{Q}_2)$ *with trivial central character and conductor $a(\pi)\le 4$.*

Proof. Let π be an irreducible, admissible, supercuspidal representations of $\mathrm{GSp}(4,\mathbb{Q}_2)$ with trivial central character and conductor $a(\pi)\le 4$. Assume first that π is generic. Then π cannot be of type Va* or XIa*. Equivalently, the Langlands parameter ϕ of π cannot be of type (A_2) or (A_3). Assume that ϕ is of type (A_1), so that $\phi=\phi_1\oplus\phi_2$ with irreducible, two-dimensional, inequivalent representations ϕ_1,ϕ_2 of $W_{\mathbb{Q}_2}$ for which $\det(\phi_1)=\det(\phi_2)=1$. Since $a(\phi_1),a(\phi_2)\ge 2$ and $a(\phi)\le 4$, we have $a(\phi_1)=a(\phi_2)=2$ and $a(\phi)=4$. It follows that π must be the representation sc(16) of Theorem 2.4.3. But then π transfers to a supercuspidal on $\mathrm{GL}(4,\mathbb{Q}_2)$, contradicting the reducibility of ϕ. This contradiction shows that ϕ cannot be of type (A_1). Alternatively, one can argue that, by (3), there is only one supercuspidal τ_2 of $\mathrm{GL}(2,\mathbb{Q}_2)$ with conductor 2 and trivial central character, contradicting the inequivalence of ϕ_1 and ϕ_2.

We proved that a generic supercuspidal π of $\mathrm{GSp}(4, \mathbb{Q}_2)$ with trivial central character and conductor $a(\pi) \le 4$ must have a parameter ϕ of type (B_1). Hence π transfers to a supercuspidal on $\mathrm{GL}(4, \mathbb{Q}_2)$ and must have $a(\pi) = 4$. Thus π is the representation sc(16) of Theorem 2.4.3.

Next assume that π is a non-generic supercuspidal of $\mathrm{GSp}(4, \mathbb{Q}_2)$ with trivial central character and conductor $a(\pi) \le 4$. Then π must have a parameter ϕ of type (A). Since $a(\phi) \le 4$, the argument above shows that ϕ cannot be of type (A_1), so that ϕ is of type (A_2) or (A_3).

Assume that ϕ is of type (A_3). By definition, $\phi = \phi_1 \oplus \phi_2$, where ϕ_1, ϕ_2 are reducible but indecomposable, inequivalent, and satisfy $\det(\phi_1) = \det(\phi_2) = 1$. Hence ϕ_i is the parameter of $\sigma_i \mathrm{St}_{\mathrm{GL}(2)}$ for distinct quadratic characters σ_1, σ_2 of $\mathbb{Q}_2^\times$. The restrictions on the conductors imply that σ_1 and σ_2 must both be unramified; see the proposition in §10 of [Rohrlich 1994]. Hence one of σ_1, σ_2 is trivial, and the other is the unique nontrivial, unramified, quadratic character ξ of $\mathbb{Q}_2^\times$. (This ξ is given by the local Hilbert symbol $(\cdot, 5)$.) The corresponding π is the representation $\delta^*([\xi, \nu\xi], \nu^{-1/2})$ of type Va*.

Assume that ϕ is of type (A_2). By definition, $\phi = \phi_1 \oplus \phi_2$, where ϕ_1 is irreducible and ϕ_2 is the parameter of $\sigma \mathrm{St}_{\mathrm{GL}(2)}$ for some character σ of $\mathbb{Q}_2^\times$. Moreover $\det(\phi_1) = \det(\phi_2) = 1$. Since $a(\phi) \le 4$, the character σ must be unramified, so that either $\sigma = 1$ or $\sigma = \xi$. In both cases $a(\phi_2) = 1$, which implies $a(\phi_1) \in \{2, 3\}$. There is only one possible ϕ_1 with $a(\phi_1) = 2$, namely the parameter of τ_2, the unique supercuspidal of $\mathrm{GL}(2, \mathbb{Q}_2)$ with trivial central character and conductor 2; see (3). From this ϕ_1 we therefore obtain two supercuspidals π with $a(\pi) = 3$. Using the notation of [Roberts and Schmidt 2016], these are the representations $\delta^*(\nu^{1/2}\tau_2, \nu^{-1/2})$ and $\delta^*(\nu^{1/2}\tau_2, \xi\nu^{-1/2})$ of type XIa*.

Finally, consider the case $a(\phi_1) = 3$. By Theorem 3.9 of [Tunnell 1978], there are exactly two possibilities for ϕ_1. One corresponds to a supercuspidal representation τ_3 of $\mathrm{GL}(2, \mathbb{Q}_2)$ with trivial central character, $a(\tau_3) = 3$ and $\varepsilon(1/2, \tau_3) = 1$. The other corresponds to the twist $\xi\tau_3$, which is distinguished from τ_3 by the value of the ε-factor $\varepsilon(1/2, \xi\tau_3) = -1$. The two possibilities of ϕ_1, together with the two possibilities for σ, lead to four supercuspidals π of type XIa*.

For the non-generic representations, the values of the L- and ϵ-factors in Table 2 can be read off Tables A.8 and A.9 of [Roberts and Schmidt 2007]. Note that Va* has the same factors as Va, since they constitute a two-element L-packet; similarly for XIa and XIa*. The ε-factor for sc(16) is given in Theorem 2.4.3. The L-factor for sc(16) is 1, since the parameter of sc(16) is irreducible. □

We refer to Section 4 of [Roberts and Schmidt 2016] for a construction of the representations of type Va* and XIa* in terms of the theta correspondence. Note that the representation of type Va* occurring in Table 2 is invariant under twisting by the unramified character ξ.

2.5. ***The representation* sc(16) *via compact induction.*** We give an alternative construction of the supercuspidal representation sc(16) by employing compact induction. Consider the Langlands parameter $\phi_{\mathsf{sc}(16)}$ of sc(16) given in (9). After choosing a suitable basis of $\mathrm{ind}_{W_L}^{W_{\mathbb{Q}_2}}(\xi)$ we may think of $\phi_{\mathsf{sc}(16)}$ as a map $W_{\mathbb{Q}_2} \to \mathrm{GSp}(4, \mathbb{C})$. The image lies in fact in $\mathrm{Sp}(4, \mathbb{C})$, the dual group of $G = \mathrm{SO}(5) \cong \mathrm{PGSp}(4)$, so that, if we wish, we may work in a semisimple context.

In this section we consider the Vogan L-packet of $\phi_{\mathsf{sc}(16)}$. Recall that a Vogan L-packet may contain representations across all *pure inner forms* of a group; see [Vogan 1993] or the overview in Section 3 of [Gross and Prasad 1992]. As explained in Section 8 of [Gross and Reeder 2010], the split group $\mathrm{SO}(2n+1)$ has a unique non-split pure inner form $\mathrm{SO}^*(2n+1)$. We will see that the L-packet of $\phi_{\mathsf{sc}(16)}$

has two elements, one being a representation of $SO(5, \mathbb{Q}_2) \cong PGSp(4, \mathbb{Q}_2)$ (this is our sc(16)), the other one a representation of $SO^*(5, \mathbb{Q}_2)$.

The parameter $\phi_{\mathsf{sc}(16)} : W_{\mathbb{Q}_2} \to Sp(4, \mathbb{C})$ is *discrete* in the sense that its image has finite centralizer. It is *tame* in the sense that the image of wild inertia is trivial; this is because the character $\xi : L^\times \to \mathbb{C}^\times$ is trivial on $1 + \mathfrak{p}_L$. Moreover, $\phi_{\mathsf{sc}(16)}$ is in *general position*, meaning the image of tame inertia is generated by a regular, semisimple element. Hence $\phi_{\mathsf{sc}(16)}$ is among the Langlands parameters considered in [DeBacker and Reeder 2009]. The construction in [DeBacker and Reeder 2009] attaches a Vogan L-packet to each tame, discrete Langlands parameter in general position. In the context of GSp(4), the paper [Lust 2013] assures that the packets thus obtained coincide with the L-packets defined in [Gan and Takeda 2011] and [Gan and Tantono 2014].

The centralizer C_ϕ of the image of $\phi_{\mathsf{sc}(16)} : W_{\mathbb{Q}_2} \to Sp(4, \mathbb{C})$ is precisely the center $\pm I_4$ of $Sp(4, \mathbb{C})$. The work [DeBacker and Reeder 2009] attaches to each irreducible character ρ of C_ϕ a depth-zero supercuspidal representation on a pure inner form of the group under consideration. In our case, going through the definitions shows that the trivial character of C_ϕ gives rise to a representation of $SO(5, \mathbb{Q}_2)$, and the nontrivial character to a representation of $SO^*(5, \mathbb{Q}_2)$. We will concentrate on the former, since (by [Lust 2013]) this is our supercuspidal sc(16).

As explained in Section 4.4 of [DeBacker and Reeder 2009], each irreducible character ρ of C_ϕ gives rise to an orbit of vertices in the Bruhat–Tits building of $G = PGSp(4)$ over $\mathbb{Q}_2$. By Lemma 6.2.1 of [DeBacker and Reeder 2009], these vertices are hyperspecial if and only if ρ is trivial. It is exactly the hyperspecial vertices that lead to *generic* depth-zero supercuspidals, consistent with the fact that sc(16) is generic.

We may work with the hyperspecial vertex x_0 whose associated parahoric subgroup is $p(K)$, where $K = GSp(4, \mathbb{Z}_2)$ and $p : GSp(4, \mathbb{Q}_2) \to G(\mathbb{Q}_2)$ is the projection. Let G_0 be the reductive group over the residue class field $\mathfrak{f} = \mathbb{F}_2$ attached to x_0, so that $\mathsf{G}_0(\mathfrak{f}) \cong p(K)/p(K)^+$, where $p(K)^+$ is the pro-unipotent radical of $p(K)$. In our case $p(K)^+$ is a principal congruence subgroup, and $\mathsf{G}_0 = Sp(4)$. The construction of sc(16) is then as follows. The parameter $\phi_{\mathsf{sc}(16)}$ determines an $\mathfrak{f}$-minisotropic maximal torus T_0 in G_0. The restriction of $\phi_{\mathsf{sc}(16)}$ to tame inertia defines a character θ of $\mathsf{T}_0(\mathfrak{f})$ via the tame local Langlands correspondence for tori. Since $\phi_{\mathsf{sc}(16)}$ is in general position, the character θ will be in general position in the sense of Definition 5.15 of [Deligne and Lusztig 1976]. Deligne–Lusztig induction therefore yields an irreducible, cuspidal character

$$\kappa_0 = \pm R_{T,\theta} \tag{11}$$

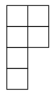
of $\mathsf{G}_0(\mathfrak{f}) \cong Sp(4, \mathfrak{f})$. Let κ be the inflation of κ_0 to $p(K)$ via $\mathsf{G}_0(\mathfrak{f}) \cong p(K)/p(K)^+$. Then

$$\mathsf{sc}(16) = \text{c-Ind}_{p(K)}^{G(\mathbb{Q}_2)}(\kappa), \tag{12}$$

where we identify representations of $G(\mathbb{Q}_2)$ with representations of $GSp(4, \mathbb{Q}_2)$ with trivial central character. Alternatively, we can first pull back κ to a character of K, extend it trivially to ZK, where Z is the center of $GSp(4, \mathbb{Q}_2)$, and compactly induce to $GSp(4, \mathbb{Q}_2)$. By Proposition 6.6 of [Moy and Prasad 1996], the induced representation in (12) is irreducible and supercuspidal.

Making things explicit, one finds that T_0 is the maximal torus corresponding to the conjugacy class consisting of length 2 elements in the 8-element Weyl group of G_0; see Section 3.3 of [Carter 1985] for the correspondence between conjugacy classes in the Weyl group and maximal tori. The group $\mathsf{T}_0(\mathfrak{f})$ is cyclic of order 5. The characters θ of $\mathsf{T}_0(\mathfrak{f})$ in general position are precisely the isomorphisms of this group with the fifth roots of unity. By Corollary 7.2 of [Deligne and Lusztig 1976], the character κ_0 in (11) has degree 9.

It is an exercise in elementary character theory to show that $\mathrm{Sp}(4, \mathfrak{f})$ has exactly one irreducible, cuspidal representation κ_0 of dimension 9, and that this representation is generic; see [Enomoto 1972] for information on the characters of $\mathrm{Sp}(4, \mathbb{F}_{2^n})$. This κ_0 corresponds to the irreducible character with Young diagram



(13)

under the isomorphism of $\mathrm{Sp}(4, \mathfrak{f})$ with the symmetric group S_6 described in Section 3.5.2 of [Wilson 2009]. There is in fact only one other irreducible, cuspidal character of $\mathrm{Sp}(4, \mathfrak{f})$, namely the one-dimensional sign character under the isomorphism $\mathrm{Sp}(4, \mathfrak{f}) \cong S_6$.

To summarize, $\mathrm{sc}(16)$ is a depth-zero supercuspidal representation of $\mathrm{GSp}(4, \mathbb{Q}_2)$ which may be constructed as follows. Take the unique irreducible, cuspidal character κ_0 of $\mathrm{Sp}(4, \mathfrak{f})$ that is not one-dimensional; it has dimension 9 and is generic. Inflate κ_0 to a representation κ of $K = \mathrm{GSp}(4, \mathbb{Z}_2)$ and extend it to ZK by making it trivial on the center Z of $\mathrm{GSp}(4, \mathbb{Q}_2)$. Then we have $\mathrm{sc}(16) = \text{c-Ind}_{ZK}^{\mathrm{GSp}(4,\mathbb{Q}_2)}(\kappa)$. The Vogan L-packet of $\mathrm{sc}(16)$ contains an additional representation which lives on the non-split inner form of $\mathrm{GSp}(4)$.

3. Paramodular cusp forms of weight $k \le 14$ and level $N = 16$

A good reference for the notation in this section and hereafter is [PSY 2018]. For each $N \in \mathbb{N}$, the paramodular group, $K(N)$, and its normalizing Fricke involution, μ_N, are defined by

$$K(N) = \begin{bmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{bmatrix} \cap \mathrm{Sp}(4, \mathbb{Q}),\ * \in \mathbb{Z}\,; \qquad \mu_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 0 & -N & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & N & 0 \end{bmatrix}.$$

Let $S_k(K(N))^\epsilon$ for $\epsilon = \pm$ denote the Fricke eigenspace of $S_k(K(N))$ with eigenvalue ± 1, so that we have the decomposition $S_k(K(N)) = S_k(K(N))^+ \oplus S_k(K(N))^-$. In the case where N is a power of a prime, the Fricke sign is also the Atkin–Lehner sign at that prime. The Gritsenko lift is an injective linear map from $J_{k,N}^{\mathrm{cusp}}$ to $S_k(K(N))^\epsilon$ for $\epsilon = (-1)^k$. Paramodular forms that are not Gritsenko lifts will be called *nonlifts*.

We are searching for a supercuspidal paramodular form, i.e., a newform $f \in S_k(K(N))$ whose associated adelic representation has a supercuspidal local component. Since non-generic supercuspidals do not admit non-zero paramodular vectors by Theorem 3.4.3 of [Roberts and Schmidt 2007], a supercuspidal coming from a paramodular newform f is necessarily generic. In particular, f must be a nonlift. By Table 2, among 2-powers, the smallest N for which f can be supercuspidal is $N = 16$. By Corollary 7.5.5 of [Roberts and Schmidt 2007], the value of the ε-factor at $1/2$ of an irreducible, admissible, generic representation coincides with the eigenvalue of the Atkin–Lehner involution on the newform. It therefore follows from Table 2 that if $S_k(K(16))$ contains a supercuspidal form, it must occur in $S_k(K(16))^-$. Hence, we pay special attention to these spaces.

Our first goal is to find all the nonlift newforms in $S_k(K(16))^{\pm}$ for $k \le 14$. In order to separate the nonlift newforms from the nonlift oldforms, we also find all the nonlift eigenforms in $S_k(K(N))$ for $k \le 14$ and $N \in \{1, 2, 4, 8\}$; we separate these eigenforms into their Fricke eigenspaces as well. The dimensions of $S_k(K(N))$ are known for $N \in \{1, 2, 4\}$, see [Igusa 1962; Ibukiyama and Onodera 1997; Poor and Yuen 2013]. Comparing with the known [Skoruppa and Zagier 1989] dimensions of Jacobi cusp forms $J_{k,N}^{\text{cusp}}$, we see that $S_k(K(N))$ for $N \in \{1, 2\}$ and $k \le 14$ does not have any nonlifts. Thus we need only consider $N \in \{4, 8, 16\}$ in this section. Our first task is to compute the dimension of each of these spaces, and this will entail finding upper and lower bounds that are equal.

3.1. ***Paramodular forms and Fourier expansions.*** A paramodular form $f \in S_k(K(N))$ has a Fourier expansion

$$f(\Omega) = \sum_t a(t; f) e(\langle \Omega, t \rangle)$$

where the sum is over $t \in \mathcal{X}_2(N) = \left\{ \left[\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right] > 0 : n, r, m \in \mathbb{Z} \right\}$ and where $\langle \Omega, t \rangle = \operatorname{tr}(\Omega t)$. The similarity group $\left\{ u \in \mathrm{GL}(2, \mathbb{R}) : \left[\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right] \in K(N) \right\}$ equals $\hat{\Gamma}^0(N) = \left\langle \Gamma^0(N), \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \right\rangle$, where, as usual, $\Gamma^0(N) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z}) : b \equiv 0 \bmod N \right\}$, and hence the Fourier coefficients satisfy the following relations amongst themselves: for $t[u] = {}^t u t u$,

$$a(t[u]; f) = \det(u)^k a(t; f), \quad \text{for all} \quad u \in \hat{\Gamma}^0(N). \tag{14}$$

Another set of important relations among the Fourier coefficients comes from the Fricke involution μ_N; we have $a(t; f|\mu_N) = a(\operatorname{Twin}(t); f)$ for

$$t = \left[\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right], \qquad \operatorname{Twin}(t) = \left[\begin{smallmatrix} m & -r/2 \\ -r/2 & Nn \end{smallmatrix}\right], \tag{15}$$

so that $t \mapsto \operatorname{Twin}(t)$ gives the action of μ_N on the Fourier coefficients. Therefore Fricke eigenforms obey the additional conditions

$$a(\operatorname{Twin}(t); f) = \epsilon\, a(t; f), \qquad \text{for } f \in S_k(K(N))^{\epsilon}. \tag{16}$$

Note that twinning stabilizes $\mathcal{X}_2(N)$ and respects $\hat{\Gamma}^0(N)$-classes. These observations follow from the equation $\operatorname{Twin}(t) = F_N t\, {}^t F_N$, for $F_N = \frac{1}{\sqrt{N}} \left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right]$, the elliptic Fricke involution on $\Gamma_0(N)$. We may view the Fourier expansion as a map $\mathrm{FE} : S_k(K(N)) \to \prod_{t \in \mathcal{X}_2(N)} \mathbb{C}$ that sends f to $(a(t; f))_{t \in \mathcal{X}_2(N)}$. Relations (14) and (16) above show that the image of $S_k(K(N))^{\epsilon}$ under FE lies in a very special subspace.

For a ring $R \subseteq \mathbb{C}$, we define $S_k(K(N))(R) \subseteq S_k(K(N))$ to be the R-module of paramodular cusp forms $f \in S_k(K(N))$ with $a(t; f) \in R$ for all $t \in \mathcal{X}_2(N)$. Fundamental results of Shimura [1975] show that general spaces of modular forms have integral bases, i.e., a basis with integral Fourier coefficients.

The natural reduction map $\mathrm{R_p} : \mathbb{Z} \to \mathbb{F}_p$ allows us to define modular forms over $\mathbb{F}_p$, a concept useful for both theory and computations: $S_k(K(N))(\mathbb{F}_p) = \mathrm{R_p} \circ \mathrm{FE}\,(S_k(K(N))(\mathbb{Z}))$. Thus paramodular forms over $\mathbb{F}_p$ are formal series with coefficients in $\mathbb{F}_p$ and the Fourier expansion map $\mathrm{FE} : S_k(K(N))(\mathbb{F}_p) \to \prod_{t \in \mathcal{X}_2(N)} \mathbb{F}_p$ is really the identity map. From the existence of an integral basis, it follows from the structure theorem for finitely generated $\mathbb{Z}$-modules that

$$\dim_{\mathbb{C}} S_k(K(N))^{\epsilon} = \operatorname{rank}_{\mathbb{Z}} S_k(K(N))^{\epsilon}(\mathbb{Z}) = \dim_{\mathbb{F}_p} S_k(K(N))^{\epsilon}(\mathbb{F}_p).$$

For odd primes p, we have the direct sum $S_k(K(N))(\mathbb{F}_p) = S_k(K(N))^{+}(\mathbb{F}_p) \oplus S_k(K(N))^{-}(\mathbb{F}_p)$.

3.2. ***Good Hecke operators and their action on Fourier coefficients.*** A Hecke operator is called *good* when its similitude is prime to the level. For each prime q not dividing N, we use the good Hecke operator $T(q): S_k(K(N)) \to S_k(K(N))$ defined as follows. Decompose $K(N)\,\mathrm{diag}(1,1,q,q)K(N) = \cup_j K(N)\gamma_j$ into a union of distinct cosets. For $f \in S_k(K(N))$, set $f|T(q) = \sum_j f|\gamma_j$, which is again in $S_k(K(N))$. Since $T(q)$ commutes with the Fricke involution μ_N, $T(q)$ also stabilizes $S_k(K(N))^\epsilon$. The action of $T(q)$ on the Fourier expansion of f is given by

$$a(t; f|T(q)) = a(qt; f) + q^{k-2} a\big(q^{-1} t\big[\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\big]; f\big) + q^{k-2} \sum_{j \bmod q} a\big(q^{-1} t\big[\begin{smallmatrix} 1 & 0 \\ j & q \end{smallmatrix}\big]; f\big) + q^{2k-3} a(q^{-1} t; f). \tag{17}$$

For $k \geq 2$, this equation shows that $T(q)$ stabilizes $S_k(K(N))^\epsilon(R)$ and is R-linear for subrings R of $\mathbb{C}$. On $S_k(K(N))^\epsilon(\mathbb{F}_p)$, the reduction of $T(q)$, $T(q)_p$, is defined by $\big(\mathrm{R_p} \circ \mathrm{FE}(f)\big)\,|T(q)_p = \mathrm{R_p} \circ \mathrm{FE}\,(f|T(q))$ and also obeys equation (17).

A possible source of confusion is that equation (17) is valid for the *classical* normalization of the slash, setting $\sigma = \big[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\big] \in \mathrm{GSp}(4,\mathbb{R})^+$ with similitude $\mu = \mu(\sigma) = \det(\sigma)^{1/2}$,

$$(f|_k\sigma)(\Omega) = \mu^{2k-3} \det(C\Omega + D)^{-k} f\left((A\Omega + B)(C\Omega + D)^{-1}\right).$$

In contrast, representation theory employs the *scalar invariant* slash where the power of the similitude is μ^k instead of μ^{2k-3}. The tension between these normalizations is real because local Euler factors depend only upon the local representation for the scalar invariant action of the Hecke algebra, whereas $T(q)$ is uniformly defined over $\mathbb{Z}$ for weights $k \geq 2$ only for the classical action. Our concession to this tension is to write the scalar invariant action of the left and the classical action on the right, so that $f|T(q) = q^{k-3} T(q) f$.

3.3. ***Fourier–Jacobi expansions, Jacobi forms, and Jacobi Hecke operators.*** The Fourier expansion of a paramodular cusp form $f \in S_k(K(N))$ may be rearranged to give the Fourier–Jacobi expansion, setting $\Omega = \big[\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\big] \in \mathcal{H}_2$, and $q = e(\tau)$, $\zeta = e(z)$,

$$f(\Omega) = \sum_{j=1}^{\infty} \phi_j(\tau, z) e(Nj\omega), \tag{18}$$

$$\phi_j(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nNj > r^2}} a\big(\big[\begin{smallmatrix} n & r/2 \\ r/2 & Nj \end{smallmatrix}\big]; f\big) q^n \zeta^r. \tag{19}$$

When we want to indicate the dependence of the ϕ_j on f we will write $\phi_j(\tau, z; f)$ instead of $\phi_j(\tau, z)$, or $\phi_j(f)$ instead of ϕ_j. We recall the definition of a Jacobi form and the following subgroups, for rings $R \subseteq \mathbb{C}$,

$$P_{2,1}(R) = \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \cap \mathrm{Sp}(4, R); \quad GP_{2,1}(R) = \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \cap \mathrm{GSp}(4, R).$$

A *Jacobi form* $\phi \in J_{k,m}$ of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}_{\geq 0}$ is a holomorphic function $\phi: \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ such that the associated function $E_m\phi : \mathcal{H}_2 \to \mathbb{C}$ given by $(E_m\phi)(\Omega) = \phi(\tau, z)e(m\omega)$ is invariant under $P_{2,1}(\mathbb{Z})$, and is bounded on domains of the type $\{\Omega \in \mathcal{H}_2 : \mathrm{Im}\,\Omega > Y_o\}$. The boundedness condition is essential and, given the other assumptions, is equivalent to a Fourier expansion for ϕ of the form $\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}:\, n \geq 0,\, 4nm \geq r^2} c(n, r; \phi) q^n \zeta^r$. For *Jacobi cusp forms* $\phi \in J_{k,m}^{\mathrm{cusp}}$, we require $4mn > r^2$. For a *weakly holomorphic* $\psi \in J^{\text{wh}}_{k,m}$ we drop the boundedness condition and require $n \gg -\infty$. Indices with $4mn \le r^2$ are called *singular*. Spaces of Jacobi forms have integral bases by [Eichler and Zagier 1985] and so we may define $J^{\text{cusp}}_{k,m}(R)$ for R a subring of $\mathbb{C}$ or for $\mathbb{F}_p$ as in the case of paramodular forms.

The subgroup $K_\infty(N) = P_{2,1}(\mathbb{Q}) \cap K(N)$ stabilizes the Fourier–Jacobi expansion (18) term by term, so that each $\phi_j \in J^{\text{cusp}}_{k,Nj}$ is a Jacobi form and the Fourier coefficients of the ϕ_j are

$$c(n, r; \phi_j) = a\left(\left[\begin{smallmatrix} n & r/2 \\ r/2 & Nj \end{smallmatrix}\right]; f\right). \tag{20}$$

The Fourier–Jacobi expansion defines a map

$$\text{FJ} : S_k(K(N)) \to \prod_{j=1}^{\infty} J^{\text{cusp}}_{k,Nj}, \qquad f \mapsto \sum_{j=1}^{\infty} \phi_j\, \xi^{Nj}, \tag{21}$$

where we have let $\xi = e(\omega)$ and identified the sum on the right with the vector (ϕ_j).

The infinite direct product $\prod_{j=1}^{\infty} J^{\text{cusp}}_{k,Nj}$ is an inverse limit with respect to the projection maps

$$\text{proj}^u_d : \bigoplus_{j=1}^{u} J^{\text{cusp}}_{k,Nj} \to \bigoplus_{j=1}^{d} J^{\text{cusp}}_{k,Nj}, \qquad \text{for } d \le u.$$

We also define $\text{proj}^\infty_d : \prod_{j=1}^{\infty} J^{\text{cusp}}_{k,Nj} \to \bigoplus_{j=1}^{d} J^{\text{cusp}}_{k,Nj}$. The projection onto the first u Fourier–Jacobi coefficients

$$\text{proj}^\infty_u \circ \text{FJ} : S_k(K(N))^\epsilon \to \bigoplus_{j=1}^{u} J^{\text{cusp}}_{k,Nj} \tag{22}$$

injects for sufficiently large u and algorithms to find u_0 such that the map (22) injects for $u \ge u_0$ may be found in [Breeding et al. 2016]. When N is a prime power for example, u_0 is roughly $Nk/5$ and Table 3 displays u_0 for $1 \le k \le 14$ and $N \in \{4, 8, 16\}$. We write $S_k(K(N))^\epsilon[u]$ for the projection of $S_k(K(N))^\epsilon$ onto its first u Fourier–Jacobi coefficients, i.e.,

$$S_k(K(N))^\epsilon[u] = \text{proj}^\infty_u \circ \text{FJ}\left(S_k(K(N))^\epsilon\right).$$

One cannot take an arbitrary sequence of Jacobi forms ϕ_j and obtain the Fourier–Jacobi expansion $\sum_{j=1}^{\infty} \phi_j \xi^{Nj}$ of some paramodular form. Indeed, the Fourier–Jacobi coefficients of a paramodular Fricke eigenform satisfy the following symmetries. Let $f \in S_k(K(N))^\epsilon$ have the Fourier–Jacobi expansion $\sum_{j=1}^{\infty} \phi_j \xi^{Nj}$. Then

for all $t_1 = \left[\begin{smallmatrix} n_1 & r_1/2 \\ r_1/2 & Nm_1 \end{smallmatrix}\right]$, $t_2 = \left[\begin{smallmatrix} n_2 & r_2/2 \\ r_2/2 & Nm_2 \end{smallmatrix}\right] \in \mathcal{X}_2(N)$, and $u \in \hat{\Gamma}^0(N)$,

$$t_1[u] = t_2 \quad \Longrightarrow \quad c(n_1, r_1; \phi_{m_1}) = \det(u)^k c(n_2, r_2; \phi_{m_2}), \tag{23}$$

and

$$\text{for all } t = \left[\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right] \in \mathcal{X}_2(N), \quad c(n, r; \phi_m) = (-1)^k \epsilon\, c(m, r; \phi_n). \tag{24}$$

Equations (23) and (24) are consequences of (14) and (16). We refer to equation (24) as the *involution conditions*. Formal series of Jacobi forms that satisfy (23) and (24) and converge in an appropriate sense are in fact Fourier–Jacobi expansions of paramodular forms; see [Ibukiyama et al. 2013].

u_0				$u_1^+,\ u_1^-$			
k	$K(4)$	$K(8)$	$K(16)$	k	$K(4)$	$K(8)$	$K(16)$
1	0	0	0	1	0	0	0
2	0	0	2	2	0	0	0
3	0	1	4	3	0	0	0
4	0	2	7	4	0	0	1, 0
5	1	3	9	5	0	0, 1	0, 1
6	1	4	11	6	0	1, 0	2, 0
7	2	5	14	7	0, 1	0, 1	0, 2
8	3	6	16	8	1, 0	1, 0	2, 0
9	4	8	18	9	0, 1	0, 2	1, 3
10	4	9	21	10	1, 0	2, 0	3, 1
11	5	10	23	11	0, 2	1, 2	2, 4
12	5	11	25	12	2, 0	3, 1	4, 2
13	6	12	28	13	0, 2	2, 3	3, 4
14	6	13	30	14	2, 0	3, 2	5, 3

Table 3. A sufficient number u_0 to make projection from $S_k(K(N))^\epsilon$ onto the first u_0 Jacobi coefficients injective. An improved number u_1^ϵ is given in the second set.

Following [Gritsenko 1995], we present the action of $T(q)$ on the Fourier–Jacobi coefficients of a paramodular cusp form in terms of the Jacobi raising and lowering operators, V_q and W_q. The raising operator $V_q : J_{k,m} \to J_{k,mq}$ is defined, for primes q, by

$$\left(\phi|V_q\right)(\tau,z) = q^{k-1}\phi(q\tau,qz) + \frac{1}{q}\sum_{\lambda \bmod q}\phi\left(\frac{\tau+\lambda}{q}, z\right),$$

or equivalently by

$$c(n,r;\phi|V_q) = q^{k-1}c(\frac{n}{q},\frac{r}{q};\phi) + c(qn,r;\phi), \tag{25}$$

as in [Eichler and Zagier 1985]. The lowering operators $W_q : J_{k,m} \to J_{k,\frac{m}{q}}$ were introduced in a special case in [Kohnen and Skoruppa 1989]. Their image is zero when the prime q does not divide m. When q divides m, we have

$$(\phi|W_q)(\tau,z) = q^{k-2}\sum_{\lambda \bmod q}\phi(q\tau, z+\lambda\tau)e\left(\frac{m}{q}(2\lambda z+\lambda^2\tau)\right) + q^{-2}\sum_{\lambda,\mu \bmod q}\phi\left(\frac{\tau+\lambda}{q},\frac{z+\mu}{q}\right),$$

or equivalently

$$c(n,r;\phi|W_q) = c(qn,qr;\phi) + q^{k-2}\sum_{\lambda \bmod q} c\left(\frac{n+\lambda r+\frac{m}{q}\lambda^2}{q}, \frac{r+2\frac{m}{q}\lambda}{q};\phi\right). \tag{26}$$

The invariance properties of the raising and lowering operators, i.e., that they send Jacobi forms to Jacobi forms, can be obtained by considering them as the Hecke operators $V_q = K_\infty(N)\,\mathrm{diag}(q, q, 1, 1)K_\infty(N)$ and $W_q = K_\infty(N)\,\mathrm{diag}(1, 1, q, q)K_\infty(N)$ for the noncommutative Jacobi Hecke algebra for $K_\infty(N)$ inside $GP_{2,1}(\mathbb{Q})$, see [Gritsenko 1995]. The action of $T(q)$ on the Fourier–Jacobi expansion of an $f \in S_k(K(N))$ is given by

$$\mathrm{FJ}(f) = \sum_{j=1}^{\infty} \phi_j \xi^{Nj}; \quad \mathrm{FJ}(f|T(q)) = \sum_{j=1}^{\infty} \left(\phi_{qj}|W_q + q^{k-2}\phi_{j/q}|V_q\right) \xi^{Nj}, \tag{27}$$

as can be directly verified by comparing equations (25) and (26) with (17) using (20).

3.4. *Jacobi restriction and upper bounds.* In this section we define the Jacobi restriction spaces $\mathcal{J}_u^\epsilon(R)$ for R being $\mathbb{F}_p$ or a subring of $\mathbb{C}$. Jacobi restriction is described in [Ibukiyama et al. 2013; Breeding et al. 2016] but we cover it here in further detail because the extension of $T(q)$ to $\mathcal{J}_u^\epsilon(\mathbb{F}_p)$ in Section 3.7 is subtle.

By collectively ordering the index sets of the Fourier expansions of $J_{k,Nj}^{\mathrm{cusp}}$ for all $j \in \mathbb{N}$ in some way, we view $\prod_{j=1}^{\infty} J_{k,Nj}^{\mathrm{cusp}}(R) \subseteq R^\infty$.

Definition 3.4.1. Let $N, u, D_0 \in \mathbb{N}$, $k \in \mathbb{Z}$, and $\epsilon \in \{-1, 1\}$. Let R be $\mathbb{F}_p$ or a subring of $\mathbb{C}$. The R-module

$$\mathcal{J}_u^\epsilon(R) \subseteq \bigoplus_{j=1}^{u} J_{k,Nj}^{\mathrm{cusp}}(R) \subseteq R^\infty$$

consists of the $\mathfrak{f} = \sum_{j=1}^{u} \mathfrak{f}_j\, \xi^{Nj} \in \bigoplus_{j=1}^{u} J_{k,Nj}^{\mathrm{cusp}}(R)$ that satisfy the following conditions:

$$\text{for all } t_1 = \begin{bmatrix} n_1 & r_1/2 \\ r_1/2 & Nm_1 \end{bmatrix},\ t_2 = \begin{bmatrix} n_2 & r_2/2 \\ r_2/2 & Nm_2 \end{bmatrix} \in \mathcal{X}_2(N) \text{ and } U \in \hat{\Gamma}^0(N),$$

$$t_1[U] = t_2 \text{ and } \det(2t_1), \det(2t_2) \le D_0 \text{ and } m_1, m_2 \le u \implies c(n_1, r_1; \mathfrak{f}_{m_1}) = \det(U)^k c(n_2, r_2; \mathfrak{f}_{m_2}), \tag{28}$$

and

$$\text{for all } t = \begin{bmatrix} n & r/2 \\ r/2 & Nm \end{bmatrix} \in \mathcal{X}_2(N), \quad \det(2t) \le D_0 \text{ and } n, m \le u \implies c(n, r; \mathfrak{f}_m) = (-1)^k \epsilon\, c(m, r; \mathfrak{f}_n). \tag{29}$$

This important construction calls for a number of comments. The defining equations in Definition 3.4.1 are truly elementary, one coordinate in R^∞ equals ± 1 times another, so that $\mathcal{J}_u^\epsilon(R)$ is defined over the various commutative rings R. The R-module $\mathcal{J}_u^\epsilon(R)$ also depends on N, k, and D_0 so that $\mathcal{J}_u^\epsilon(R, N, k, D_0)$ would be more proper, but we supress N, k, and D_0 to lighten the notation somewhat. When no ring is indicated the field of complex numbers is meant, so $\mathcal{J}_u^\epsilon = \mathcal{J}_u^\epsilon(\mathbb{C})$. We have written a program, which we call Jacobi restriction, for the cases $R = \mathbb{Z}$ and $R = \mathbb{F}_p$. This program accepts input $(N, k, \epsilon, D_0, u, R)$ and returns initial expansions, out to (n, r) satisfying $4nNj - r^2 \le D_0$, of an R-basis of $\mathcal{J}_u^\epsilon(R)$. We always choose D_0 large enough so that elements of $J_{k,Nj}^{\mathrm{cusp}}(R)$ for $j \le u$ are determined by their initial expansions out to $4nNj - r^2 \le D_0$; thus, the output characterizes a basis of $\mathcal{J}_u^\epsilon(R)$, and $\mathcal{J}_u^\epsilon(R)$ is an R-module of finite rank very amenable to computation. In particular, $\mathrm{rank}_R\, \mathcal{J}_u^\epsilon(R)$ is always known. Finally, because the spaces $J_{k,m}^{\mathrm{cusp}}$ have integral bases, the output for $R = \mathbb{Z}$ also works for any subring $R \subseteq \mathbb{C}$.

The next lemma shows that $\mathcal{J}_u^\epsilon$ is an upper approximation of the space $S_k(K(N))^\epsilon[u]$.

Lemma 3.4.2. *Let $N, u \in \mathbb{N}$, $k \in \mathbb{Z}$, and $\epsilon \in \{-1, 1\}$. We have*

$$\mathrm{proj}_u^\infty \circ \mathrm{FJ} : S_k(K(N))^\epsilon \to S_k(K(N))^\epsilon[u] \subseteq \mathcal{J}_u^\epsilon.$$

Proof. By equations (23) and (24), the Fourier–Jacobi expansion of an $f \in S_k(K(N))^\epsilon$ satisfies the conditions in Definition 3.4.1 for all choices of indices. The conditions defining $\mathcal{J}_u^\epsilon$ are thus a subset of the conditions satisfied by $(\mathrm{proj}_u^\infty \circ \mathrm{FJ})(f)$. □

Corollary 3.4.3. *Let $u \in \mathbb{N}$ be such that $\mathrm{proj}_u^\infty \circ \mathrm{FJ} : S_k(K(N))^\epsilon \to S_k(K(N))^\epsilon[u]$ injects. Then $\dim S_k(K(N))^\epsilon \le \dim \mathcal{J}_u^\epsilon$.*

3.5. ***Jacobi restriction modulo p.*** Jacobi restriction can also be run modulo a prime p. As in the appendix of [Berger and Klosin 2017], for a subset $H \subseteq \mathbb{C}^\infty$, let $H_p = \mathrm{R_p}(H \cap \mathbb{Z}^\infty) \subseteq \mathbb{F}_p^\infty$ denote the reduction of $H \cap \mathbb{Z}^\infty$ mod p. If $H_1, H_2 \subseteq \mathbb{C}^\infty$ are subspaces with integral bases and $L : H_1 \to H_2$ is a linear map whose matrix in these bases is integral, then L also has a reduction, $L_p : H_{1p} \to H_{2p}$, with the defining property that $(L(h))_p = L_p(h_p)$ for $h \in H_1$. To give some examples, for paramodular forms we have $(\mathrm{FE}(S_k(K(N))))_p = S_k(K(N))(\mathbb{F}_p)$ and for Jacobi forms $(\mathrm{FE}(J_{k,m}^{\mathrm{cusp}}))_p = J_{k,m}^{\mathrm{cusp}}(\mathbb{F}_p)$. The good Hecke operator $T(q) : S_k(K(N))^\epsilon(\mathbb{Z}) \to S_k(K(N))^\epsilon(\mathbb{Z})$ has, for $k \ge 2$, an integral matrix by (17), and so induces a map $T(q)_p : S_k(K(N))^\epsilon(\mathbb{F}_p) \to S_k(K(N))^\epsilon(\mathbb{F}_p)$ given by: $\mathfrak{f}|T(q)_p = \mathfrak{g}$ means there exists an $f \in S_k(K(N))^\epsilon(\mathbb{Z})$ such that $\mathrm{R_p}(\mathrm{FE}(f)) = \mathfrak{f}$ and $\mathrm{R_p}(\mathrm{FE}(f|T(q))) = \mathfrak{g}$.

Because spaces of modular forms have integral bases, important information survives the reduction mod p. For example, $\dim_{\mathbb{C}} S_k(K(N))^\epsilon[u] = \dim_{\mathbb{F}_p} S_k(K(N))^\epsilon[u]_p \le \dim \mathcal{J}_{u,p}^\epsilon$. Hence if $u \ge u_0$, for some basic u_0 making $\mathrm{proj}_{u_0}^\infty \circ \mathrm{FJ}$ injective, we have $\dim_{\mathbb{C}} S_k(K(N))^\epsilon \le \dim \mathcal{J}_{u,p}^\epsilon$ as well. We easily have $\mathcal{J}_{u,p}^\epsilon \subseteq \mathcal{J}_u^\epsilon(\mathbb{F}_p)$ and examples show that the containment can be proper. Noting Lemma 3.4.2, the *hope* when we run Jacobi restriction is that all the following spaces have the same dimension:

$$S_k(K(N))^\epsilon \xrightarrow{\mathrm{proj}_u^\infty \circ \mathrm{FJ}} S_k(K(N))^\epsilon[u] \xrightarrow{\mathrm{mod}\, p} S_k(K(N))^\epsilon[u]_p \subseteq \mathcal{J}_{u,p}^\epsilon \subseteq \mathcal{J}_u^\epsilon(\mathbb{F}_p). \tag{30}$$

When these spaces do have the same dimension we can, in retrospect, regard the computations as having been perfomed in any one of them; however it is the space $\mathcal{J}_u^\epsilon(\mathbb{F}_p)$ that is most amenable to computation, being a finite-dimensional $\mathbb{F}_p$-vector space with a known basis. Especially, we can row reduce and compute the smallest u_1^ϵ for which the projection

$$\mathrm{proj}_{u_1^\epsilon}^u : \mathcal{J}_u^\epsilon(\mathbb{F}_p) \to \bigoplus_{j=1}^{u_1^\epsilon} J_{k,Nj}^{\mathrm{cusp}}(\mathbb{F}_p)$$

is injective. For $u = u_0$, Table 3 also gives particular values of u_1^ϵ with this property for $1 \le k \le 14$, $N \in \{4, 8, 16\}$, $p = 12347$, and various D_0. The choice of D_0 was 400 for $K(4)$, 800 for $K(8)$ when $k \le 10$ and 1000 for larger k, and 1600 for $K(16)$ when $k \le 10$ and 2000 for larger k. The caption of Table 3, however, instead reports that the projection from $S_k(K(N))^\epsilon$ to $S_k(K(N))^\epsilon[u_1^\epsilon]$ is injective. The injectivity in these cases follows from the proof in Section 3.10 that $\dim S_k(K(N))^\epsilon = \dim \mathcal{J}_{u_1^\epsilon}^\epsilon(\mathbb{F}_p)$, and so p and D_0 are not reported in Table 3.

3.6. *Extending $T(q)$ to $\mathcal{J}_u^\epsilon(\mathbb{C})$.* Our goal in this section is to lift the map $T(q) : S_k(K(N))^\epsilon \to S_k(K(N))^\epsilon$ to another map $\hat{T}(q) : \mathcal{J}_u^\epsilon \to \mathcal{J}_u^\epsilon$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{J}_u^\epsilon & \xrightarrow{\hat{T}(q)} & \mathcal{J}_u^\epsilon \\ {\scriptstyle \mathrm{proj}_u^\infty \circ \mathrm{FJ}} \uparrow & & \uparrow {\scriptstyle \mathrm{proj}_u^\infty \circ \mathrm{FJ}} \\ S_k(K(N))^\epsilon & \xrightarrow{T(q)} & S_k(K(N))^\epsilon \end{array} \tag{31}$$

Admittedly, this diagram will only be useful for u large enough to make the vertical map injective. We proceed in two steps and need to make certain assumptions about the space $\mathcal{J}_u^\epsilon$. Because we can compute with $\mathcal{J}_u^\epsilon$ it is reasonable to impose needed conditions on $\mathcal{J}_u^\epsilon$ as long as they can be checked in practice. First, define a map

$$\tilde{T}(q) : \bigoplus_{j=1}^{u} J_{k,Nj}^{\mathrm{cusp}} \to \bigoplus_{j=1}^{\lfloor u/q \rfloor} J_{k,Nj}^{\mathrm{cusp}}, \qquad \sum_{j=1}^{u} \phi_j \, \xi^{Nj} \mapsto \sum_{j=1}^{\lfloor u/q \rfloor} \left(q^{k-2} \phi_{j/q} | V_q + \phi_{qj} | W_q \right) \xi^{Nj}. \tag{32}$$

This definition reflects the computational fact that the operator $T(q)$ returns shorter Fourier–Jacobi expansions than it receives. Since the above action agrees with equation (27) we have

$$\mathrm{proj}_{\lfloor u/q \rfloor}^{u} \, \mathrm{proj}_u^\infty \, \mathrm{FJ}(f | T(q)) = \left(\mathrm{proj}_u^\infty \, \mathrm{FJ}(f) \right) | \tilde{T}(q).$$

We introduce the notion of one map being relatively stable with respect to another. Let $\pi : A \to \pi A$ and $T : A \to \pi A$ be maps and $B \subseteq A$. We say T is *relatively stable on B with respect to π* when $T(B) \subseteq \pi(B)$. This is equivalent to saying that $T : A \to \pi A$ extends to a relative map $T : (A, B) \to \pi(A, B)$. We will require that $\tilde{T}(q)$ be relatively stable on $\mathcal{J}_u^\epsilon$ with respect to $\mathrm{proj}_{\lfloor u/q \rfloor}^{u}$. When $\mathcal{J}_u^\epsilon$ has successfully been computed, we will need to check whether or not $\tilde{T}(q) : \mathcal{J}_u^\epsilon \to \mathrm{proj}_{\lfloor u/q \rfloor}^{u} \mathcal{J}_u^\epsilon \subseteq \bigoplus_{j=1}^{\lfloor u/q \rfloor} J_{k,Nj}^{\mathrm{cusp}}$. We will also require that $\lfloor u/q \rfloor \ge u_1^\epsilon$, so that $\mathrm{proj}_{\lfloor u/q \rfloor}^{u}$ injects on $\mathcal{J}_u^\epsilon$.

Proposition 3.6.1. *Let $N, u \in \mathbb{N}$, $k \in \mathbb{Z}$, and $\epsilon \in \{-1, 1\}$. Let q be a prime with $q \nmid N$. Assume that*:

i) *$\tilde{T}(q)$ is relatively stable on $\mathcal{J}_u^\epsilon$ with respect to* $\mathrm{proj}_{\lfloor u/q \rfloor}^{u}$.
ii) *The restriction of* $\mathrm{proj}_{\lfloor u/q \rfloor}^{u}$ *to $\mathcal{J}_u^\epsilon$ is injective.*

Then $\hat{T}(q) : \mathcal{J}_u^\epsilon \to \mathcal{J}_u^\epsilon$ is well-defined by: $\mathfrak{f} | \hat{T}(q) = \mathfrak{g}$ *means* $\mathfrak{f} | \tilde{T}(q) = \mathrm{proj}_{\lfloor u/q \rfloor}^{u} \mathfrak{g}$. *Under these hypotheses, diagram* (31) *commutes.*

Proof. Assume that $\mathfrak{f} \in \mathcal{J}_u^\epsilon$. Because $\tilde{T}(q)$ is relatively stable there exists a $\mathfrak{g} \in \mathcal{J}_u^\epsilon$ such that $\mathfrak{f} | \tilde{T}(q) = \mathrm{proj}_{\lfloor u/q \rfloor}^{u} \mathfrak{g}$. Because $\mathrm{proj}_{\lfloor u/q \rfloor}^{u}$ is injective, this $\mathfrak{g}$ is unique, and thus $\hat{T}(q)$ is well-defined. The linearity of $\hat{T}(q)$ follows from the equation $\mathfrak{f} | \tilde{T}(q) = \mathrm{proj}_{\lfloor u/q \rfloor}^{u} \mathfrak{g}$ and the uniqueness of $\mathfrak{g}$.

In order to show the commutativity of the diagram we must check

$$\left(\mathrm{proj}_u^\infty (\mathrm{FJ}(f)) \right) | \hat{T}(q) = (\mathrm{proj}_u^\infty \circ \mathrm{FJ})(f | T(q)),$$

or, by definition of $\hat{T}(q)$, we must show that

$$\left(\mathrm{proj}_u^\infty (\mathrm{FJ}(f)) \right) | \tilde{T}(q) = \mathrm{proj}_{\lfloor u/q \rfloor}^{u} \left((\mathrm{proj}_u^\infty \circ \mathrm{FJ})(f | T(q)) \right).$$

Thus we must check that $\left(\sum_{j=1}^{u} \phi_j\, \xi^{Nj}\right)|\tilde{T}(q) = \mathrm{proj}^{\infty}_{\lfloor u/q \rfloor}(f|T(q))$. By equation (27) the right-hand side is $\sum_{j=1}^{\lfloor u/q \rfloor} \left(q^{k-2}\phi_{j/q}|V_q + \phi_{qj}|W_q\right)\xi^{Nj}$, which is the definition of the left-hand side. □

3.7. *Extending $T(q)_p$ to $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$.* Our goal in this section is to lift the map $T(q)_p : S_k(K(N))^{\epsilon}(\mathbb{F}_p) \to S_k(K(N))^{\epsilon}(\mathbb{F}_p)$ to a map $\mathcal{T}(q) : \mathcal{J}_u^{\epsilon}(\mathbb{F}_p) \to \mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{J}_u^{\epsilon}(\mathbb{F}_p) & \xrightarrow{\ \mathcal{T}(q)\ } & \mathcal{J}_u^{\epsilon}(\mathbb{F}_p) \\
\uparrow{\scriptstyle \mathrm{proj}^{\infty}_{u,p}\circ \mathrm{FJ}_p} & & \uparrow{\scriptstyle \mathrm{proj}^{\infty}_{u,p}\circ \mathrm{FJ}_p} \\
S_k(K(N))^{\epsilon}(\mathbb{F}_p) & \xrightarrow{\ T(q)_p\ } & S_k(K(N))^{\epsilon}(\mathbb{F}_p)\,.
\end{array} \tag{33}$$

Recall the definition (32) of the map $\tilde{T}(q)$. By equations (25) and (26), the action of V_q and W_q is integral for $k \geq 2$; so we may consider the reduction of the map $\tilde{T}(q)$ mod p:

$$\tilde{T}(q)_p : \bigoplus_{j=1}^{u} J^{\mathrm{cusp}}_{k,Nj}(\mathbb{F}_p) \to \bigoplus_{j=1}^{\lfloor u/q \rfloor} J^{\mathrm{cusp}}_{k,Nj}(\mathbb{F}_p),$$

and restrict $\tilde{T}(q)_p$ to $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p) \subseteq \bigoplus_{j=1}^{u} J^{\mathrm{cusp}}_{k,Nj}(\mathbb{F}_p)$ to obtain $\tilde{T}(q)_p : \mathcal{J}_u^{\epsilon}(\mathbb{F}_p) \to \bigoplus_{j=1}^{\lfloor u/q \rfloor} J^{\mathrm{cusp}}_{k,Nj}(\mathbb{F}_p)$. As in the previous section, it is reasonable to impose needed conditions on $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$ that are easy to check. We will require that $\tilde{T}(q)_p$ be relatively stable on $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$ with respect to $\mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}$. This condition is achieved whenever $S_k(K(N))^{\epsilon}[u]_p$ actually equals $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$, which is what the whole set-up aims to prove, so there is no harm in requiring relative stability. If relative stability fails, we should increase u and try again. We will also require that $\mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}$ be injective on $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$. This second condition is achieved when $\lfloor u/q \rfloor \geq u_1^{\epsilon}$, which may be costly.

Proposition 3.7.1. *Let $N, u, k \in \mathbb{N}$ with $k \geq 2$, and $\epsilon \in \{-1, 1\}$. Let p and q be primes with $q \nmid N$. Assume that*:

i) *$\tilde{T}(q)_p$ is relatively stable on $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$ with respect to $\mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}$.*
ii) *The restriction of $\mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}$ to $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$ is injective.*

Then $\mathcal{T}(q) : \mathcal{J}_u^{\epsilon}(\mathbb{F}_p) \to \mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$ is well-defined by: $\mathfrak{f}|\mathcal{T}(q) = \mathfrak{g}$ means $\mathfrak{f}|\tilde{T}(q)_p = \mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}\,\mathfrak{g}$. Under these hypotheses, diagram (33) commutes.

Proof. We show that $\mathcal{T}(q)$ is well-defined and $\mathbb{F}_p$-linear. Take $\mathfrak{f} \in \mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$. Since $\tilde{T}(q)_p$ is relatively stable on $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$ with respect to $\mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}$, there exists a $\mathfrak{g} \in \mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$ such that $\mathfrak{f}|\tilde{T}(q)_p = \mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}\,\mathfrak{g}$. If there were another such $\mathfrak{g}'$, then $\mathfrak{g}' = \mathfrak{g}$ because $\mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}$ is injective on $\mathcal{J}_u^{\epsilon}(\mathbb{F}_p)$. This shows that $\mathcal{T}(q)$ is well-defined. Linearity follows from $\mathfrak{f}|\tilde{T}(q)_p = \mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}\,\mathfrak{g}$ and the uniqueness of $\mathfrak{g}$.

In order to show the commutativity of the diagram, take $\mathfrak{f} \in S_k(K(N))^{\epsilon}(\mathbb{F}_p)$. We must show

$$\left(\mathrm{proj}^{\infty}_{u,p}(\mathrm{FJ}_p(\mathfrak{f}))\right)|\mathcal{T}(q) = (\mathrm{proj}^{\infty}_{u,p}\circ \mathrm{FJ}_p)(\mathfrak{f}|T(q)_p),$$

which, by definition of $\mathcal{T}(q)$, means

$$\left(\mathrm{proj}^{\infty}_{u,p}(\mathrm{FJ}_p(\mathfrak{f}))\right)|\tilde{T}(q)_p = \mathrm{proj}^{u}_{\lfloor u/q \rfloor,p}\left((\mathrm{proj}^{\infty}_{u,p}\circ \mathrm{FJ}_p)(\mathfrak{f}|T(q)_p)\right),$$

or equivalently,

$$\big(\mathrm{proj}^{\infty}_{u,p}(\mathrm{FJ}_p(\mathfrak{f}))\big)|\tilde{T}(q)_p = \mathrm{proj}^{u}_{\lfloor u/q\rfloor,p}\,\mathrm{FJ}_p(\mathfrak{f}|T(q)_p). \tag{34}$$

There is an $f \in S_k(K(N))^{\epsilon}(\mathbb{Z})$ such that $\mathfrak{f} = \mathrm{FJ}(f)_p$, so that (34) would follow by reduction from

$$\big(\mathrm{proj}^{\infty}_{u}(\mathrm{FJ}(f))\big)\,|\tilde{T}(q) = \mathrm{proj}^{u}_{\lfloor u/q\rfloor}\,\mathrm{FJ}(f|T(q)). \tag{35}$$

Writing $\mathrm{FJ}(f) = \sum_{j=1}^{\infty}\phi_j\,\xi^{Nj}$, we verify (35) from the definition of $\tilde{T}(q)$ and equation (27),

$$\bigg(\sum_{j=1}^{u}\phi_j\,\xi^{Nj}\bigg)|\tilde{T}(q) = \sum_{j=1}^{\lfloor u/q\rfloor}\big(q^{k-2}\phi_{j/q}|V_q + \phi_{qj}|W_q\big)\,\xi^{Nj} = \sum_{j=1}^{\lfloor u/q\rfloor}\phi_j(f|T(q))\,\xi^{Nj}. \qquad \square$$

3.8. ***Bootstrapping and lower bounds.*** We now explain the technique of *bootstrapping*, a combination of Jacobi restriction and Hecke spreading, which computes lower bounds for $\dim S_k(K(N))^{\epsilon} = \dim S_k(K(N))^{\epsilon}(\mathbb{F}_p)$. As motivation, we first discuss Borcherds products. The theory of Borcherds products and the theory of Hecke operators bear little relation. A Borcherds product, for example, seems to only be a Hecke eigenform when forced to be by dimensional reasons. In general, if a Borcherds product is written as a linear combination of Hecke eigenforms it seems that the Borcherds product is often supported on every eigenspace with the same Atkin–Lehner signs as the Borcherds product. Thus repeated applications of $T(q)$ on a Borcherds product are likely to span the entire Atkin–Lehner space of paramodular forms that the Borcherds product belongs to. Over $\mathbb{Q}$, many iterations of $T(q)$ on a Borcherds product are much too expensive, but over $\mathbb{F}_p$ many iterations of $\mathcal{T}(q)$ on $\mathcal{J}^{\epsilon}_u(\mathbb{F}_p)$ are feasible.

Let $S \subseteq S_k(K(N))^{\epsilon}(\mathbb{F}_p)$. Define

$$B_p(S;\mathcal{T}(q)) = \mathrm{Span}_{\mathbb{F}_p}\big\{(\mathrm{proj}^{\infty}_{u,p}\circ\mathrm{FJ}_p(\mathfrak{f}))|\mathcal{T}(q)^i \in \mathcal{J}^{\epsilon}_u(\mathbb{F}_p) : i \in \mathbb{Z}_{\geq 0}, \mathfrak{f} \in S\big\}.$$

Lemma 3.8.1. *Let u be large enough so that $\mathrm{proj}^{\infty}_{u,p}\circ\mathrm{FJ}_p$ injects on $S_k(K(N))^{\epsilon}(\mathbb{F}_p)$. Assume the hypotheses of Proposition 3.7.1. Then*

$$\dim B_p(S;\mathcal{T}(q)) \leq \dim S_k(K(N))^{\epsilon}(\mathbb{F}_p).$$

Proof. By the commutative diagram (33), the subspace $B_p(S;\mathcal{T}(q)) \subseteq \mathcal{J}^{\epsilon}_u(\mathbb{F}_p)$ is the injective image under $\mathrm{proj}^{\infty}_{u,p}\circ\mathrm{FJ}_p$ of the span of $\mathfrak{f}|T(q)^i_p \in S_k(K(N))^{\epsilon}(\mathbb{F}_p)$ for $i \in \mathbb{Z}_{\geq 0}$, and $\mathfrak{f} \in S$. $\square$

3.9. ***Specific upper bounds: Jacobi restriction.*** We use the technique of Jacobi restriction to compute upper bounds for $\dim S_k(K(N))^{\epsilon}$. Jacobi restriction over $\mathbb{Q}$ requires a lot of memory. It is better, when sufficient, to run Jacobi restriction modulo p. Table 3 gives u_0 large enough to make projection onto the first u_0 Jacobi coefficients injective. Using the containments in (30), Table 4 reports the resulting upper bound $\dim S_k(K(N))^{\epsilon} = \dim S_k(K(N))^{\epsilon}[u_0] \leq \dim \mathcal{J}^{\epsilon}_{u_0}(\mathbb{F}_p)$ given as output by the Jacobi restriction program, using the same determinant bounds D_0 and prime p as in Section 3.5. In Table 4 we have further refined these upper bounds to apply to the spaces of nonlifts, which is a direct adjustment because the dimensions of the lift spaces are known by [Eichler and Zagier 1985]. Because $\dim S_k(K(4))$ is known and the upper bounds for the three subspaces of $S_k(K(4))$ add up to the known total dimension, the dimensions of the subspaces of $S_k(K(4))$ listed in Table 4 are the actual dimensions without further argument. We will prove that the upper bounds of the dimensions of the nonlift subspaces of $S_k(K(8))$ and $S_k(K(16))$ as listed in Table 4 are in fact the true dimensions in all cases. This illustrates the power

	$K(1)$			$K(2)$			$K(4)$			$K(8)$			$K(16)$		
k	lifts	nonlifts		lifts	nonlifts		lifts	nonlifts		lifts	nonlifts		lifts	nonlifts	
		$+$	$-$		$+$	$-$		$+$	$-$		$+$	$-$		$+$	$-$
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
5	0	0	0	0	0	0	0	0	0	1	0	0	2	0	0
6	0	0	0	0	0	0	0	0	0	1	0	0	3	1	0
7	0	0	0	0	0	0	1	0	0	2	0	0	5	0	2
8	0	0	0	1	0	0	1	0	0	3	0	0	6	5	0
9	0	0	0	0	0	0	1	0	0	3	0	1	7	1	8
10	1	0	0	1	0	0	2	0	0	4	2	0	9	13	2
11	0	0	0	1	0	0	2	0	1	5	1	3	10	4	19
12	1	0	0	2	0	0	3	1	0	6	5	1	12	27	6
13	0	0	0	0	0	0	2	0	1	5	2	6	12	10	34
14	1	0	0	2	0	0	3	2	0	7	9	3	14	46	14

Table 4. Dimensions of cusp forms of weight k. The signs $+$ and $-$ refer to the paramodular Atkin–Lehner sign, which is the same as the Fricke sign in these cases.

of Jacobi restriction. The proof involves constructing enough paramodular forms to show these numbers are also lower bounds.

3.10. ***Specific lower bounds: Borcherds products and bootstrapping.*** In the previous section we computed the upper bounds for $\dim S_k(K(N))$ given in Table 4. This section will compute matching lower bounds, mainly by constructing Gritsenko lifts and Borcherds products, but also via Hecke operators, and oldform theory. The theory of Borcherds products [Borcherds 1998; Gritsenko and Nikulin 1998] creates meromorphic paramodular forms, transforming by a character χ of $K(N)$, in $M_k^{\text{mero}}(K(N))^{\epsilon}(\chi)$ from weakly holomorphic Jacobi forms $\psi \in J_{0,N}^{\text{wh}}$ of weight zero and index N whose Fourier coefficients are integral on singular indices. We will only use Borcherds products that turn out to be holomorphic and cuspidal with trivial character. There is an algorithm [Poor et al. 2018] to find all Borcherds products in a given space $S_k(K(N))$, so we simply post the constructions of the Borcherds products that we use here on the website [Yuen 2018]. Given an appropriate $\psi \in J_{0,N}^{\text{wh}}$, we write $\text{Borch}(\psi) \in M_k^{\text{mero}}(K(N))^{\epsilon}(\chi)$ for the associated Borcherds product. If we write the Fourier expansion of ψ as $\psi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n,r) q^n \zeta^r$, then $\text{Borch}(\psi)$ is defined by analytic continuation of the following infinite product for $\Omega = \left[\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right] \in \mathcal{H}_2$:

$$\text{Borch}(\psi)(\Omega) = q^A \zeta^B \xi^C \prod_{(m,n,r)\geq 0} (1 - q^n \zeta^r \xi^{mN})^{c(nm,r)}.$$

The product is taken over $m, n, r \in \mathbb{Z}$ such that $m \geq 0$, and if $m = 0$ then $n \geq 0$, and if $m = n = 0$ then $r < 0$. Set $\mathbb{N} = \{1, 2, 3, \ldots\}$. The exponents A, B, C are given by $24A = \sum_{r\in\mathbb{Z}} c(0,r)$, $2B = \sum_{r\in\mathbb{N}} rc(0,r)$, and $2C = \sum_{r\in\mathbb{N}} r^2 c(0,r)$. Borcherds products always come with a Fricke sign. The sign ϵ is given by $\epsilon = (-1)^{d_o}$ where $d_o = \sum_{n\in\mathbb{N}} \sigma_0(n)c(-n,0)$, and $\sigma_0(n)$ is the number of positive divisors of n.

Here are our methods for obtaining lower bounds on $\dim S_k(K(N))^\epsilon$. Fix k, N, and $\epsilon = \pm 1$. We search for Borcherds products in $S_k(K(N))^\epsilon$. If we find enough to span a space whose dimension equals that of the upper bound, then we are done. If not, we employ the method of bootstrapping from Section 3.8. We check the hypotheses of Proposition 3.7.1: that $\tilde{T}(3)_p$ is relatively stable on $\mathcal{J}^\epsilon_{u_0}(\mathbb{F}_p)$ with respect to $\mathrm{proj}^{u_0}_{\lfloor u_0/3\rfloor}$, and that $u_0 \geq 3u_1^\epsilon$ so that $\mathrm{proj}^{u_0}_{\lfloor u_0/3\rfloor}$ is injective on $\mathcal{J}^\epsilon_{u_0}(\mathbb{F}_p)$. There are three places in Table 3 where $u_0 < 3u_1^\epsilon$, but these occur for $K(4)$ and weight $k \in \{7, 11, 12\}$ where the dimension is already known. Still using the u_0 from Table 3, we compute a matrix representation for $\mathcal{T}(3)$ on a fixed basis for $\mathcal{J}^\epsilon_{u_0}(\mathbb{F}_p)$. We find a set $\tilde{S} \subseteq S_k(K(N))^\epsilon$ of Borcherds products and take $f \in \tilde{S}$; see [Yuen 2018] for the Borcherds products found. It is feasible to expand a Borcherds product f out far enough to determine $(\mathrm{proj}^\infty_{u_0} \mathrm{FJ}(f))_p$ in this basis. Define $S = (\mathrm{FJ}(\tilde{S}))_p \subseteq S_k(K(N))^\epsilon(\mathbb{F}_p)$. Once we get the coordinates of $(\mathrm{proj}^\infty_{u_0} \mathrm{FJ}(f))_p$ in this basis, it is linear algebra to compute the bootstrapped subspace on S. Then $u_0 \geq 3u_1^\epsilon$ and Lemma 3.8.1 imply that $\dim B_p(S; \mathcal{T}(3)) \leq \dim S_k(K(N))^\epsilon(\mathbb{F}_p)$. It turns out that the dimension of each bootstrapped subspace $B_p(S; \mathcal{T}(3))$ gives the same lower bound as the upper bound $\dim \mathcal{J}^\epsilon_{u_0}(\mathbb{F}_p)$ in every case in Table 4 except in the single case $S_{14}(K(8))^-$. Thus we know $\dim_{\mathbb{C}} S_k(K(N))^\epsilon = \dim_{\mathbb{C}} S_k(K(N))^\epsilon[u_0] = \dim_{\mathbb{F}_p} S_k(K(N))^\epsilon[u_0]_p = \dim_{\mathbb{F}_p} \mathcal{J}^\epsilon_{u_0}(\mathbb{F}_p)$ in all cases in Table 4 except $S_{14}(K(8))^-$. There are no Borcherds products in $S_{14}(K(8))^-$. We now explain the additional argument needed for this exceptional case.

We know that $\dim S_{14}(K(8))^- \leq 3$. We found all the eigenforms in each of $S_{14}(K(N))^\pm$ for $N \in \{1, 2, 4, 8, 16\}$ except $S_{14}(K(8))^-$. We show there is an eigenform in $S_{14}(K(16))^-$ of $T(3)$-eigenvalue $3^{11}\lambda_3 = -1580472$ which is not a $T_{1,0}$-eigenform. The eigenspace of $S_{14}(K(16))^-$ with this $T(3)$-eigenvalue is one-dimensional. Lemma 3.10.1 implies that there exists a newform $f_{\mathrm{new}} \in S_{14}(K(2^j))$ for some $j \in \{0, 1, 2, 3\}$ with the same $T(3)$-eigenvalue. Looking at $T(3)$-eigenvalues for the lifts, we see that f_{new} must be a nonlift. There are no nonlifts in $S_{14}(K(N))$ for $N \in \{1, 2\}$ and there are two nonlift eigenforms in $S_{14}(K(4))$. But the $T(3)$-eigenvalue -1580472 does not show as an eigenvalue in $S_{14}(K(8))^+$ or in $S_{14}(K(4))$. We conclude that f_{new} must be in $S_{14}(K(8))^-$. Together with the two oldforms in $S_{14}(K(8))^-$ coming from the two newforms in $S_{14}(K(4))$, we conclude that $\dim S_{14}(K(8))^- \geq 2 + 1 = 3$.

Lemma 3.10.1. *Let N be a positive integer and p be a prime dividing N. Let $W \subset S_k(K(N))$ be a nonzero eigenspace for a Hecke operator T at some good place $q \nmid N$. Assume that the operators $T_{0,1}(p)$, $T_{1,0}(p)$ and the Atkin–Lehner α_p are not simultaneously diagonalizable on W. Then there exists a new-eigenform $f_{\mathrm{new}} \in S_k(K(M))$ for some $M|N$ with $v_p(M) < v_p(N)$ and with the same T-eigenvalue as the elements of W.*

Proof. Since Hecke operators at good places commute, we can find a basis $f_1, \dots, f_n$ of W consisting of eigenforms for almost all good Hecke operators, including the place q. By Theorem 2.6 i) of [Schmidt 2018], the adelization Φ_i of f_i generates an irreducible, cuspidal, automorphic representation $\pi_i \cong \bigotimes_s \pi_{i,s}$ of $\mathrm{PGSp}(4, \mathbb{A}_{\mathbb{Q}})$, for each i. The automorphic form Φ_i corresponds to a sum of pure tensors $\sum_j \big(\bigotimes_s w_{i,s,j}\big)$, where $w_{i,s,j}$ is in the space of $\pi_{i,s}$. After averaging, we may assume that $w_{i,s,j}$ is a paramodular vector of level $v_s(N)$, for each prime number s. In particular, each $w_{i,q,j}$ is a spherical vector in $\pi_{i,q}$, and hence an eigenvector for the local operator T_q corresponding to T, with the same eigenvalue as T on W.

We claim that there exists an $i \in \{1, \ldots, n\}$ such that the conductor exponent $a(\pi_{i,p})$ is less than $v_p(N)$. Clearly, we must have $a(\pi_{i,p}) \le v_p(N)$ for each i, since $a(\pi_{i,p})$ is the smallest possible level of any paramodular vector in $\pi_{i,p}$ by Corollary 7.5.5 of [Roberts and Schmidt 2007]. Assume that we would have $a(\pi_{i,p}) = v_p(N)$ for all i. Then each $w_{i,p,j}$ would be a local newform in $\pi_{i,p}$, which is unique up to scalars by Theorem 7.5.4 of the same reference. In particular, $T_{0,1}(p)$, $T_{1,0}(p)$ and α_p would be simultaneously diagonalizable on W, contradicting our hypothesis. This proves our claim that there exists an $i_0 \in \{1, \ldots, n\}$ such that $a(\pi_{i_0,p}) < v_p(N)$.

Let Φ_{new} be the automorphic form corresponding to the global holomorphic, paramodular newform in π_{i_0}. De-adelizing Φ_{new}, we obtain a Siegel modular form f_{new} with the desired properties. □

We have now proven that Table 4 gives true dimensions and not just upper bounds. Once we know that the dimension of $S_k(K(N))^\epsilon$ agrees with our upper bound, we have $\mathcal{J}^\epsilon_{u_0}(\mathbb{F}_p) = S_k(K(N))^\epsilon[u_0]_p$ and can use the improved u_1^ϵ in Table 3 for which the projection $\text{proj}^{u_0}_{u_1^\epsilon} : \mathcal{J}^\epsilon_{u_0}(\mathbb{F}_p) \to \mathcal{J}^\epsilon_{u_1^\epsilon}(\mathbb{F}_p)$ injects. It follows that $\text{proj}^{\infty}_{u_1^\epsilon} : S_k(K(N))^\epsilon \to S_k(K(N))^\epsilon[u_1^\epsilon]$ injects. With these improved u_1^ϵ, we run Jacobi restriction over $\mathbb{Q}$ to $u = 3u_1^\epsilon$ Jacobi coefficients and break $S_k(K(N))^\epsilon$ into $T(3)$-eigenspaces by verifying the hypotheses of Proposition 3.6.1 and using $\hat{T}(3)$. We stress that we postpone running Jacobi restriction over $\mathbb{Q}$ until we have the improved u_1^ϵ from Table 3 available for $S_k(K(N))^\epsilon$. We are eventually forced to run Jacobi restriction over $\mathbb{Q}$ however, in order to compute Hecke eigenspaces. Once we have $S_k(K(N))^\epsilon$ broken into one-dimensional eigenspaces, we can revert, if we wish, to using $\mathcal{T}(q)$ to compute further good rational eigenvalues inside $\mathcal{J}^\epsilon_{qu_1^\epsilon}(\mathbb{F}_p)$. The point here is that, for $T(q)f = \lambda_q f$, good eigenvalues have simple archimedean bounds $|\lambda_q| \le (1+q)(1+q^2)$ (see [Freitag 1983], page 269, Hilfsatz 4.8), and $q^{k-3}\lambda_q$ is integral for $k \ge 2$. In the next section, however, we are more interested in computing eigenvalues at the bad primes, as a step toward identifying the local representations.

3.11. *Nonlift newforms.* From Table 4, we can count how many of each dimension of nonlifts are oldforms from lower levels using the global theory of newforms in [Roberts and Schmidt 2006]. Table 5 breaks $S_k(K(16))^\pm$ into the dimension of newforms and oldforms.

By computing the eigenvalue λ_3 for all the nonlift eigenforms, we are able to distinguish the newforms

	$K(16)^+$		$K(16)^-$	
k	new	old	new	old
6	1	0	0	0
7	0	0	2	0
8	5	0	0	0
9	0	1	7	1
10	11	2	0	2
11	1	3	14	5
12	20	7	1	5
13	3	7	25	9
14	32	14	4	10

Table 5. Breakdown into new and old *nonlift* eigenforms for $S_k(K(16))^\pm$.

k	$K(4)$ +	$K(4)$ −	$K(8)$ +	$K(8)$ −
9				-2760
10			-18360 -3672	
11		-13464		$-24(781 \pm 128\sqrt{55})$
12	-88488		-14760 -229032 $-504(-65 \pm 64\sqrt{6})$	
13		-154440	-685224	-271944 $\alpha_{13,8}$ (degree 4)
14	-1422360 -319896		-1176984 199368 $216(1231 \pm 8\sqrt{1129})$ $\alpha_{14,8}$ (degree 3)	-1580472

Table 6. Eigenvalues $3^{k-3}\lambda_3$ of nonlift newforms. Here $\alpha_{13,8}$ represents the four roots of $151059326544225331200 - 2859911841342873 6x - 271045699200x^2 + 463392x^3 + x^4$ and $\alpha_{14,8}$ represents the three roots of $70155550286581248 - 1194997748544x + 186408x^2 + x^3$.

from the oldforms. See Table 6 for the eigenvalues of nonlift newforms for $S_k(K(4))$ and $S_k(K(8))$ for $k \le 14$. Note that there are no nonlifts for $S_k(K(N))$ for $N \in \{1, 2\}$ and $k \le 14$. The eigenvalues of the nonlift newforms for $S_k(K(16))$ with $k \le 14$ are in Table 7 along with other eigenvalues. We were able to easily distinguish the newforms because it turns out that these newforms have different λ_3 eigenvalues than the oldforms of the same level.

3.12. ***Computing $T_{0,1}$ and $T_{1,0}$.*** The global Hecke operators at the bad primes have their origin in the local theory [Roberts and Schmidt 2006]. The global operators $T_{0,1}(p)$ and $T_{1,0}(p)$ at a bad prime p were defined and studied in [PSY 2018], where eigenvalues were computed that required information from Fourier expansions at multiple zero-dimensional cusps. From Proposition 5.2 of [PSY 2018], the two bad Hecke operators $T_{0,1}(2)$ and $T_{1,0}(2)$ may be written on $S_k(K(16))$ as

$$T_{0,1}F = \sum_{x,y,z\in\{0,1\}} F \,\Big|\, \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & y & z/16 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} + \sum_{x,z\in\{0,1\}} F \,\Big|\, \begin{bmatrix} 2 & 0 & 0 & 0 \\ x & 1 & 0 & z/16 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 2 \end{bmatrix} + \sum_{x,y\in\{0,1\}} F \,\Big|\, \begin{bmatrix} 1 & -16y & x & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 16y & 1 \end{bmatrix}$$
$$+ F \,\Big|\, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + F \,\Big|\, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 16 & 7 & -3 \\ -3 & -8 & 1 & -5/16 \\ 0 & 0 & 1 & -3/8 \\ 0 & 0 & 2 & -1 \end{bmatrix},$$

$$T_{1,0}F = \sum_{\substack{x,y\in\{0,1\}\\ z\in\{0,1,2,3\}}} F \,\Big|\begin{bmatrix} 2&0&0&2y\\ x&1&y&-xy+z/16\\ 0&0&2&-2x\\ 0&0&0&4\end{bmatrix} + \sum_{x,y\in\{0,1\}} F\,\Big|\begin{bmatrix} 1&-16y&0&0\\ -x/2&1+8xy&y/2&1/32\\ 0&0&1+8xy&x/2\\ 0&0&16y&1\end{bmatrix}$$
$$+\sum_{y\in\{0,1\}} F\,\Big|\begin{bmatrix} 2&-32y&0&0\\ 0&4&0&0\\ 0&0&2&0\\ 0&0&16y&1\end{bmatrix} + F\,\Big|\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\2&0&0&1\end{bmatrix}\begin{bmatrix}8&32&14&-3\\-3&-16&2&-5/16\\0&0&2&-3/8\\0&0&4&-1\end{bmatrix} + F\,\Big|\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\2&0&0&1\end{bmatrix}\begin{bmatrix}8&32&22&-3\\-3&-16&-1&-5/16\\0&0&2&-3/8\\0&0&4&-1\end{bmatrix}.$$

The zero-dimensional cusps of $K(16)$ are given by the disjoint union

$$\mathrm{GSp}(4,\mathbb{Q})^+ = K(16)\,GP_{2,0}(\mathbb{Q}) \cup K(16)C_0(2)GP_{2,0}(\mathbb{Q}) \cup K(16)C_0(4)GP_{2,0}(\mathbb{Q})$$

(see [Poor and Yuen 2013], Theorem 1.3), where

$$C_0(m)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&m&1&0\\m&0&0&1\end{bmatrix};\quad GP_{2,0}(R)=\begin{bmatrix}*&0&*&*\\ *&*&*&*\\ *&0&*&*\\ 0&0&0&*\end{bmatrix}\cap \mathrm{GSp}(4,R).$$

The difficulty in computing $T_{0,1}F$ and $T_{1,0}F$ is that although most of the coset representatives defining $T_{0,1}$ and $T_{1,0}$ lie in the first cusp, a few lie in the second. As in [PSY 2018], we overcome this difficulty by using the technique of restriction to a modular curve to compute the restrictions $F(s\tau+s')$ and $(T_{0,1}F)(s\tau+s')$ for some serviceable choice of s, s'. The point is that it is straightforward to compute $(F\,|\,u)(s\tau+s')$ when $u\in GP_{2,0}(\mathbb{Q})$, but a trick is required to compute $(F\,|\,C_0(2)u)(s\tau+s')$ for the last coset representative in $T_{0,1}$. The strategy of Section 4.2 in [PSY 2018] is to access the cusp $K(N)C_0(m)GP_{2,0}(\mathbb{Q})$ by finding $\sigma=\left[\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right]\in \mathrm{SL}(2,\mathbb{Z})$ and a positive definite $s_0\in\left[\begin{smallmatrix}\mathbb{Z}&\mathbb{Z}\\ \mathbb{Z}&\frac{1}{N}\mathbb{Z}\end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix}\alpha I&\beta s_0\\ \gamma s_0^{-1}&\delta I\end{smallmatrix}\right]\in K(N)C_0(m)W_0$ for some $W_0\in GP_{2,0}(\mathbb{Q})$. Setting

$$W_1=\left[\begin{smallmatrix}A_1&B_1\\ 0&D_1\end{smallmatrix}\right]=u^{-1}W_0 \quad\text{and}\quad s\tau+s'=W_1\langle s_0\tau\rangle=(A_1s_0\tau+B_1)D_1^{-1},$$

it formally follows that

$$(F|_kC_0(m)u)\,(s\tau+s')=\det(A_1D_1)^{-k/2}\det(D_1)^k\,(g|_{2k}\sigma)(\tau),$$

for $g(\tau)=F(s_0\tau)$. For ℓ with $\ell s_0^{-1}\in\left[\begin{smallmatrix}\mathbb{Z}&N\mathbb{Z}\\ N\mathbb{Z}&N\mathbb{Z}\end{smallmatrix}\right]$, we have $g\in S_{2k}(\Gamma_0(\ell))$, and we have reduced the problem of specializing F at the $C_0(m)$-cusp to transforming an elliptic modular form.

By choosing $\ell=16$ and σ, s_0, W_0, s, s' as

$$\sigma=\left[\begin{smallmatrix}3&1\\8&3\end{smallmatrix}\right],\; s_0=\left[\begin{smallmatrix}4&1\\1&1/2\end{smallmatrix}\right],\; W_0=\begin{bmatrix}-8&8&-1&6\\1/2&0&-2&-33/16\\0&0&0&1/8\\0&0&2&2\end{bmatrix},\; s=\begin{bmatrix}58&-41/2\\-41/2&29/4\end{bmatrix},\; s'=\begin{bmatrix}41/2&-29/4\\-29/4&81/32\end{bmatrix},$$

we get that

$$\left(F\,|_k\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\2&0&0&1\end{bmatrix}\begin{bmatrix}8&16&7&-3\\-3&-8&1&-5/16\\0&0&1&-3/8\\0&0&2&-1\end{bmatrix}\right)(s\tau+s')=\left(\tfrac{1}{4}\right)^{-k/2}(1)^k(g|_{2k}\sigma)(\tau),$$

where $g(\tau)=F(s_0\tau)\in S_{2k}(\Gamma_0(16))$. We therefore need to be able to work with cusp forms in $S_{2k}(\Gamma_0(16))$, namely we need to compute a basis of $S_{2k}(\Gamma_0(16))$ and the action of σ on this basis. We show how to do this in Lemma 3.12.1.

To be able to compute the restrictions $F(s\tau+s')$ and $(T_{1,0}F)(s\tau+s')$, for $F\in S_k(K(16))$ and some choice of s, s', we follow the instructions of Section 4.4 in [PSY 2018]. For $T_{1,0}$, the delicate issue is simultaneously computing $(F\,|\,C_0(2)u)(s\tau+s')$ for the last two coset representatives in $T_{1,0}$. By choosing $\ell=16$ and σ, s_0, s, s', τ_0, W_0 as

$$\sigma=\left[\begin{smallmatrix}3&1\\8&3\end{smallmatrix}\right],\ s_0=\left[\begin{smallmatrix}10&3\\3&1\end{smallmatrix}\right],\ \tau_0=1/2,$$

$$s=\begin{bmatrix}9441370&-2347216\\-2347216&4668325/8\end{bmatrix},\ s'=\begin{bmatrix}3152523&-3134991/4\\-3134991/4&12470225/64\end{bmatrix},\ W_0=\begin{bmatrix}-24&8&-65&0\\-1055/2&176&-1739&-14897/16\\0&0&-44&-1055/8\\0&0&2&6\end{bmatrix},$$

we get the following, for $g(\tau)=F(s_0\tau)\in S_{2k}\,(\Gamma_0(16))$,

$$\left(F\,|_k\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\2&0&0&1\end{bmatrix}\begin{bmatrix}8&32&14&-3\\-3&-16&2&-5/16\\0&0&2&-3/8\\0&0&4&-1\end{bmatrix}\right)(s\tau+s')=\left(\tfrac{1}{4}\right)^{-k/2}(1)^k(g|_{2k}\sigma)(\tau),$$

$$\left(F\,|_k\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\2&0&0&1\end{bmatrix}\begin{bmatrix}8&32&22&-3\\-3&-16&-1&-5/16\\0&0&2&-3/8\\0&0&4&-1\end{bmatrix}\right)(s\tau+s')=\left(\tfrac{1}{4}\right)^{-k/2}(1)^k(g|_{2k}\sigma)(\tau+\tau_0).$$

The last thing we need before using this choice to compute $T_{0,1}F$ is a knowledge of how forms in $M_k(\Gamma_0(16))$ transform by $\sigma=\left[\begin{smallmatrix}3&1\\8&3\end{smallmatrix}\right]$. We discuss the ring generators of $M(\Gamma_0(16))=\bigoplus_{k=0}^{\infty}M_k(\Gamma_0(16))$. Let

$$E_2(\tau)=1-24\sum_{n=1}^{\infty}\sigma(n)q^n=1-24q-72q^2-96q^3-168q^4-144q^5-\cdots$$

be the nearly modular weight two Eisenstein series transforming, for all $\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in\mathrm{SL}(2,\mathbb{Z})$, by

$$\left(E_2|_2\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\right)(\tau)=E_2(\tau)-\frac{3}{\pi^2}\left(\frac{2\pi ic}{c\tau+d}\right). \tag{36}$$

For $d>1$, we define $E^-_{2,d}\in M_2(\Gamma_0(d))$ by $E^-_{2,d}(\tau)=\frac{1}{1-d}\,(E_2(\tau)-dE_2(d\tau))$. We define five elements in $M_2(\Gamma_0(16))$ by

$$a(\tau)=\tfrac{1}{2}E^-_{2,2}(\tau)-3E^-_{2,4}(\tau)+\tfrac{7}{2}E^-_{2,8}(\tau)=1-24q^2+24q^4-96q^6+24q^8-144q^{10}+\cdots$$

$$b(\tau)=-\tfrac{1}{48}E^-_{2,2}(\tau)+\tfrac{7}{48}E^-_{2,8}(\tau)-\tfrac{5}{8}E^-_{2,16}(\tau)+\tfrac{1}{2}\vartheta\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&8&0\\0&0&0&8\end{bmatrix}(\tau)=q-4q^3+6q^5-8q^7+13q^9\cdots$$

$$c(\tau)=-\tfrac{1}{6}E^-_{2,2}(\tau)+\tfrac{7}{6}E^-_{2,8}(\tau)=1+8q^2+24q^4+32q^6+24q^8+48q^{10}+\cdots$$

$$d(\tau)=\tfrac{1}{16}E^-_{2,2}(\tau)-\tfrac{1}{16}E^-_{2,4}(\tau)=q+4q^3+6q^5+8q^7+13q^9+\cdots$$

$$e(\tau)=\tfrac{1}{4}E^-_{2,4}(\tau)-\tfrac{7}{4}E^-_{2,8}(\tau)+\tfrac{5}{2}E^-_{2,16}(\tau)=1-8q^4+24q^8-328q^{12}+\cdots.$$

The theta series $\vartheta[Q]$ of an even m-by-m quadratic form, used above to define basis element b, is defined by $\vartheta[Q](\tau)=\sum_{n\in\mathbb{Z}^m}e\left(\frac{1}{2}Q[n]\tau\right)$. If ℓQ^{-1} is also even then $\vartheta[Q]\in M_{m/2}(\Gamma_0(\ell),\chi)$ for some character χ. The character is trivial when $\det(Q)$ is a square and $4\mid m$, see [Freitag 1983], page 203. Using Satz 0.3 of [Freitag 1983], we also have, for even m,

$$\vartheta[Q]|F_\ell=\ell^{m/4}\det(Q)^{-1/2}(-i)^{m/2}\vartheta[\ell Q^{-1}],\qquad\text{for } F_\ell=\tfrac{1}{\sqrt{\ell}}\left[\begin{smallmatrix}0&-1\\\ell&0\end{smallmatrix}\right]. \tag{37}$$

A D_4-subgroup of the normalizer of $\Gamma_0(16)$ in $\mathrm{SL}(2,\mathbb{Q})$, modulo $\langle \pm I, \Gamma_0(16)\rangle$, acts on $M_k(\Gamma_0(16))$. This representation of D_4 on $M_2(\Gamma_0(16))$ is 5-dimensional and decomposes into a 2-dimensional irreducible representation and three 1-dimensional representations. The basis of $M_2(\Gamma_0(16))$ defined above was selected to decompose this representation into its irreducible components.

Lemma 3.12.1. *The graded ring $M(\Gamma_0(16))$ consists of homogeneous polynomials in the five elements $a, b, c, d, e \in M_2(\Gamma_0(16))$, subject to the six relations*

$$2e^2 = c^2 + ac,\quad 32d^2 = c^2 - ac,\quad c^2 = a^2 + 64b^2,\quad cd = 2be - ad,\quad ce = ae + 32bd,\quad de = bc.$$

Every element in $M_k(\Gamma_0(16))$ can be uniquely written as

$$P_k(a, b) + C_{k-2}(a, b)c + D_{k-2}(a, b)d + E_{k-2}(a, b)e,$$

where P_k is a homogeneous polynomial of degree $k/2$ and the C_{k-2}, D_{k-2}, E_{k-2} are homogeneous of degree $(k-2)/2$. The Fricke involution $F = \left[\begin{smallmatrix} 0 & -1/4 \\ 4 & 0 \end{smallmatrix}\right]$ and the translation $A = \left[\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right]$ normalize $\Gamma_0(16)$ and generate a subgroup isomorphic to the dihedral group D_4, with $T = AF = \left[\begin{smallmatrix} 2 & -1/4 \\ 4 & 0 \end{smallmatrix}\right]$ of order four, and $\sigma = T^3F = \left[\begin{smallmatrix} 3 & 1 \\ 8 & 3 \end{smallmatrix}\right]$ of order two. For the representation $\rho : D_4 \to \mathrm{GL}(5,\mathbb{C})$ defined by $(a, b, c, d, e)|_2 g = (a, b, c, d, e)\rho(g)$, we have

$$\rho(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ \rho(F) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & -1/4 & 0 \end{bmatrix},\ \rho(T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1/4 & 0 \end{bmatrix},\ \rho(\sigma) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Proof. The transformation under A is obvious and the transformation under F may be worked out using (36) and (37). A helpful intermediate step is $(E^-_{2,d}|F)(\tau) = -\frac{16}{d}E^-_{2,d}\left(\frac{16}{d}\tau\right)$. The normalizer in $\mathrm{SL}(2,\mathbb{Q})$ of $\Gamma_0(16)$, modulo $\langle \pm I, \Gamma_0(16)\rangle$, contains a dihedral group of order 8: $\langle A, F\rangle = \langle T, \sigma\rangle$. The index of $\Gamma_0(16)$ in $\mathrm{SL}(2,\mathbb{Z})$ is 24, so, by the Valence Inequality, to prove equality in $M_k(\Gamma_0(16))$ it suffices to check the equality of the first $2k+1$ Fourier coefficients. In this way we verify the six given relations and the images of ρ.

Every modular form in $M_k(\Gamma_0(16))$ that can be written as a polynomial in a, b, c, d, e, may be written in the form $P_k(a, b) + C_{k-2}(a, b)c + D_{k-2}(a, b)d + E_{k-2}(a, b)e$, by appying the given relations in the order given. We will show that no nontrivial relation of the given form can be zero. First, by applying T^2, we would have both $P_k(a, b) + C_{k-2}(a, b)c = 0$ and $D_{k-2}(a, b)d + E_{k-2}(a, b)e = 0$. Second, applying T to the first we obtain $P_k(a, b) - C_{k-2}(a, b)c = 0$ and hence $P_k(a, b) = C_{k-2}(a, b) = 0$. The modular forms a and b have the same weight, and so are algebraically independent because b/a is nonconstant. Hence the polynomials P_k and C_{k-2} are also trivial. Third, applying T to the second we obtain $D_{k-2}(a, b)(4e) - E_{k-2}(a, b)(d/4) = 0$ as well. Over the field of meromorphic functions, we thus have $E_{k-2}(a, b) = \pm 4D_{k-2}(a, b)$ and this is also an equality among holomorphic functions. From $0 = D_{k-2}(a, b)d + E_{k-2}(a, b)e = D_{k-2}(a, b)(d \pm 4e)$, we conclude that D_{k-2} and E_{k-2} are zero as polynomials. The dimension of $\mathbb{C}[a, b, c, d, e] \cap M_k(\Gamma_0(16))$ is then $\left(\frac{k}{2}+1\right) + 3\left(\frac{k-2}{2}+1\right) = 2k+1$. By the Riemann–Roch theorem, $\dim M_k(\Gamma_0(16)) = 2k+1$ for even $k \geq 0$, and thus $M(\Gamma_0(16)) = \mathbb{C}[a, b, c, d, e]$ as graded rings. □

We have all the ingredients to apply the techniques of Section 4.2 and 4.4 of [PSY 2018] to compute the eigenvalues $\lambda_{0,1}$ and $\lambda_{1,0}$. We successfully computed the eigenvalues $\lambda_{0,1}$ and $\lambda_{1,0}$ of the nonlift newforms in $S_k(K(16))^{\pm}$ for $k \leq 14$. The results are in Table 7. By applying the knowledge of these

k	AL	$3^{k-3}\lambda_3$	$\lambda_{0,1}$	$\lambda_{1,0}$	type
6	+	-96	-5	0	X
7	−	-600	-2	-4	XIa
	−	-144	-3	0	I, IIa, or X
8	+	-1992	0	-4	VII, VIIIa or IXa
	+	912	-3	0	X
	+	-168	-2	-4	XIa
	+	$-864\pm 112\sqrt{33}$	$1/8(-7\mp\sqrt{33})$	0	X
9	−	-8136	2	-4	XIa
	−	5856	-5	0	I, IIa, or X
	−	-2280	0	-4	sc(16)
	−	-1920	$1/4$	0	I, IIa, or X
	−	1464	-2	-4	XIa
	−	$\pm 480\sqrt{33}$	$1/4(-3\mp\sqrt{33})$	0	I, IIa, or X
10	+	-12888	2	-4	XIa
	+	5928	-2	-4	XIa
	+	-3768	0	-4	VII, VIIIa or IXa
	+	-1080	0	-4	VII, VIIIa or IXa
	+	$7248\pm 240\sqrt{505}$	$1/8(-19\pm\sqrt{505})$	0	X
	+	$\alpha_{10,16}$ (degree 5)	t_{10}	0	X
11	+	-66096	$-29/8$	0	X
	−	8040	0	-4	sc(16)
	−	$24(-1245\pm 32\sqrt{21})$	2	-4	XIa
	−	$120(111\pm 8\sqrt{69})$	-2	-4	XIa
	−	-73584	$9/2$	0	I, IIa, or X
	−	18768	1	0	I, IIa, or X
	−	35568	$-3/4$	0	I, IIa, or X
	−	$48(425\pm 2\sqrt{3961})$	$1/32(-107\pm\sqrt{3961})$	0	I, IIa, or X
	−	$\alpha_{11,16}$ (degree 4)	t_{11}	0	I, IIa, or X

Table 7. Eigenvalues λ_3, $\lambda_{0,1}$ and $\lambda_{1,0}$ of nonlift newforms in $S_k(K(16))^{\pm}$. The algebraic numbers $\alpha_{10,16}$, $\alpha_{11,16}$ and the corresponding eigenvalues t_{10}, t_{11} are given below. (Table continues on the next page.)

	Symbolic constants in Table 7, for $k = 10, 11$: minimal polynomial of α_* and eigenvalues
$\alpha_{10,16}$:	$-392100597530099712+36717761396736000x-1936322592768x^2-384208896x^3+12000x^4+x^5$
$t_{10}=$	$(20068447042323522728755 2+942556117843692748 80\alpha+2115778851231744\alpha^2-1410266234784\alpha^3-54792385\alpha^4)/4109075318872714688593 92$
$\alpha_{11,16}$:	$332724999250575360-1154234880x^2+x^4$
$t_{11}=$	$(858199620022272+28477875456\alpha-1490544\alpha^2-53\alpha^3)/21539386294272$

k	AL	$3^{k-3}\lambda_3$	$\lambda_{0,1}$	$\lambda_{1,0}$	type
12	$+$	-12456	0	-4	VII, VIIIa or IXa
	$+$	$72(819 \pm 64\sqrt{85})$	0	-4	VII, VIIIa or IXa
	$+$	$72(-521 \pm 128\sqrt{5})$	2	-4	XIa
	$+$	$72(831 \pm 8\sqrt{85})$	-2	-4	XIa
	$+$	$\alpha_{12,16,a}$ (degree 5)	$t_{12,a}$	0	X
	$+$	$\alpha_{12,16,b}$ (degree 8)	$t_{12,b}$	0	X
	$-$	-185616	$-21/8$	0	I, IIa, or X
13	$+$	-183168	$-33/8$	0	X
	$+$	$-144(3879 \pm 41\sqrt{609})$	$(-53 \pm \sqrt{609})/32$	0	X
	$-$	-220968	2	-4	XIa
	$-$	$72(-333 \pm 80\sqrt{609})$	2	-4	XIa
	$-$	$\alpha_{13,16,a}$ (degree 3)	-2	-4	XIa
	$-$	$\alpha_{13,16,b}$ (degree 3)	0	-4	sc(16)
	$-$	0	$3/2$	0	I, IIa, or X
	$-$	725184	-1	0	I, IIa, or X
	$-$	$\alpha_{13,16,c}$ (degree 6)	$t_{13,c}$	0	I, IIa, or X
	$-$	$\alpha_{13,16,d}$ (degree 4)	$t_{13,d}$	0	I, IIa, or X
	$-$	$\alpha_{13,16,e}$ (degree 4)	$t_{13,e}$	0	I, IIa, or X
14	$+$	517320	2	-4	XIa
	$+$	527688	-2	-4	XIa
	$+$	$216(-597 \pm 16\sqrt{51})$	2	-4	XIa
	$+$	$24(40387 \pm 320\sqrt{25561})$	-2	-4	XIa
	$+$	-499608	0	-4	VII, VIIIa or IXa
	$+$	$216(2927 \pm 56\sqrt{3889})$	0	-4	VII, VIIIa or IXa
	$+$	$24(20759 \pm 88\sqrt{8689})$	0	-4	VII, VIIIa or IXa
	$+$	$\alpha_{14,16,a}$ (degree 8)	$t_{14,a}$	0	X
	$+$	$\alpha_{14,16,b}$ (degree 13)	$t_{14,b}$	0	X
	$-$	-2434968	0	-4	sc(16)
	$-$	-927072	$-17/8$	0	I, IIa, or X
	$-$	$-432(1935 \pm 23\sqrt{2377})$	$(-97 \pm \sqrt{2377})/32$	0	I, IIa, or X

(For the minimal polynomials of the algebraic numbers α_* and the corresponding eigenvalues see [Yuen 2018].)

Table 7, continued.

eigenvalues to Table A.14 of [Roberts and Schmidt 2007], we also identify the possibilities for the corresponding local representations at $p = 2$ of the underlying automorphic representations. Further information on the entries of these tables may be found at [Yuen 2018].

3.13. ***Supercuspidal forms found.*** From Table 7, we see that we found supercuspidal forms in weights 9, 11, 13, 14. The website [Yuen 2018] gives formulas for these supercuspidal forms. For the odd weights

$k = 9, 11, 13$, the supercuspidal form is given as a linear combination of Gritsenko lifts and repeated $T(3)$ images of one or more Borcherds products. For the even weight $k = 14$, the supercuspidal form is given as a linear combination of the repeated $T(3)$ images of one Borcherds product. We also give the formula for the weight 14 supercuspidal form here to provide a bridge to the database [Yuen 2018] and to aid any future reproduction of our results. Let Δ be the cusp form in $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ normalized to have leading term q. Theta blocks are the invention of Gritsenko, Skoruppa, and Zagier, and the special case we use here may be defined, for $d_j \in \mathbb{N}$, by

$$\mathrm{TB}_k(d_1, d_2, \ldots, d_\ell)(\tau, z) = \eta(\tau)^{2k-\ell} \prod_{j=1}^{\ell} \vartheta(\tau, d_j z),$$

where η is the Dedekind eta function and $\vartheta(\tau, z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{(n+1/2)^2/2} \zeta^{n+1/2}$ is the odd Jacobi theta function. A basis $\mathbf{B}$ of $J^{\mathrm{cusp}}_{12,16}$ is given in Table 8 in terms of W_2 and W_3 images of theta blocks.

$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4)\vert W_2$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3)\vert W_2$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 8)\vert W_3$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 4, 7)\vert W_3$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 4, 4, 4, 5)\vert W_3$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 3, 3, 3, 4, 4, 4)\vert W_3$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 7)\vert W_3$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 5, 5)\vert W_3$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4)\vert W_3$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3)\vert W_3$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 5)\vert W_3$
$\mathrm{TB}_{12}(1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3)\vert W_3$

Table 8. A basis $\mathbf{B}$ of $J^{\mathrm{cusp}}_{12,16}$.

Define the weight-zero weakly holomorphic form $\psi_{14} \in J^{\mathrm{wh}}_{0,16}(\mathbb{Z})$ by

$$\phi_{14} = \mathrm{TB}_{14}(1, 1, 1, 1, 1, 1, 2, 2, 3, 3); \qquad \psi_{14} = \frac{\phi_{14} \vert V_2}{\phi_{14}} + \frac{\mathbf{b}_{14} \cdot \mathbf{B}}{\Delta},$$

where the vector $\mathbf{b}_{14}$ is given by

$$\mathbf{b}_{14} = \frac{1}{279268001096663167080660} \cdot (-1155865602408281719819 2, -1056598136932746256247 7, -2926740930944006282896, 9167023003084404792024, 9262973271453152666448, 5762211536895867593392, 2926740930944006282896, -5759260672816406314 44, 1918503995959964699328, -1300786643303689051 44, 1584969973757747488 80, 3516352762722050847 68)$$

We have $\mathrm{Borch}(\psi_{14}) \in S_{14}(K(16))^-$. It happens that $\{T(3)^j \,\mathrm{Borch}(\psi_{14}) : j = 0, \ldots, 13\}$ is a basis of the space $S_k(K(16))^-$. We state on the next page the linear combination vector $\mathbf{c}_{14}$ that defines

$$f_{14} = \sum_{j=0}^{13} (\mathbf{c}_{14})_j \, T(3)^j \,\mathrm{Borch}(\psi_{14}) :$$

$$
\mathbf{c}_{14} = \frac{1}{3725738216385042356336483182434882958372211606631219200} \\
\begin{aligned}
(&34824415729731223424619948791663663097413524374159360 0, \\
&-1231050015269758711257977743259890444383052096717455360, \\
&1875755810226739139339249977817168623115122557101015 04, \\
&125498398831599670823396733818930835695798087485646438 4, \\
&-12747907866219085265773767892551614619748795829295513 6, \\
&-48755161139221022969580251268238302688265271154153881 6, \\
&-5141021228456145989487013649851790987626368922445414 4, \\
&6375351289634317218668183191220580480064617628670361 6, \\
&2045229986855668665249903450571345856571047582485708 8, \\
&1123239782891661890888908622454818983032675662143488, \\
&-4291836146958951718615844344886862194096934870835 20, \\
&-865110231933857931075636730023128902729602127075 20, \\
&-6355772893990016890233522775734788662836903493392, \\
&-171792506910670443678820376588540424234035840667)
\end{aligned}
$$

The relevant definitions for other weights are at the website [Yuen 2018].

We stopped at $k = 14$ because we found a supercuspidal paramodular form in an even weight space of the lowest possible level. Also, weight $k = 14$ for $K(16)$ is on the edge of tractability for the method of Jacobi restriction.

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