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PACKET STRUCTURE AND PARAMODULAR FORMS

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ABSTRACT. We explore the consequences of the structure of the discrete automorphic spectrum of the split orthogonal group SO(5) for holomorphic Siegel modular forms of degree 2. In particular, the combination of the local and global packet structure with the local paramodular newform theory for GSp(4) leads to a strong multiplicity one theorem for paramodular cusp forms.

INTRODUCTION

As is well known, Siegel modular forms of degree n can be considered as vectors in automorphic representations of the group $\mathrm{GSp}(2n, \mathbb{A}_\mathbb{Q})$. In many interesting cases the representations involved will have trivial central character, hence descend to representations of the group $\mathrm{PGSp}(2n, \mathbb{A}_\mathbb{Q})$. For $n = 2$, there is an exceptional isomorphism of the algebraic group PGSp(4) with the split orthogonal group SO(5). The latter is among the groups for which Arthur [2] has given a classification of the discrete automorphic spectrum in terms of automorphic representations of general linear groups. The assumptions made in [2] have recently been verified, mostly thanks to the efforts of Moeglin and Waldspurger. In this work we explore the consequences of the structure of the discrete automorphic spectrum of SO(5) for Siegel modular forms of degree 2.

Let F be any number field and $\mathbb{A}$ its ring of adeles. Let G be any group for which the theory of [2] applies; this includes all symplectic and split orthogonal groups. According to [2], there is a certain set $\Psi_2(G)$ of ***parameters***, which are formal objects composed of cuspidal, automorphic data on general linear groups. To each $\psi \in \Psi_2(G)$ is associated a ***packet*** Π_ψ of irreducible, admissible representations $\pi = \otimes\pi_v$ of $G(\mathbb{A})$. The packets are constructed by choosing the local representations π_v from finite, local packets Π_{ψ_v}, for each place v. The global packets may be finite or infinite, depending on whether finitely or infinitely many of the local packets have more than one element. Arthur's main result, Theorem 1.5.2 of [2], gives a precise condition, and multiplicity, for any $\pi = \otimes\pi_v$ in a packet Π_ψ to occur in the discrete automorphic spectrum $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$, and asserts that all of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$ is exhausted by such π.

From now on let G be the split orthogonal group SO(5). We identify representations of $G(\mathbb{A})$ with representations of $\mathrm{GSp}(4, \mathbb{A})$ for which the center acts trivially. The classification of [2] for the discrete automorphic spectrum of $G(\mathbb{A})$ has been extended to $\mathrm{GSp}(4, \mathbb{A})$, and made more explicit, in the work [1]. Whenever we refer to [1], we will restrict ourselves to representations with trivial central character, for which all results are unconditional.

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The parameters $\psi \in \Psi_2(G)$, and with them the corresponding packets Π_ψ and the representations in these packets, fall naturally into six classes. The simplest of these is the class **(F)**, which consists of finite-dimensional (in fact, one-dimensional) representations. There are three classes **(Q)**, **(P)** and **(B)** consisting mostly of CAP representations (cuspidal associated to parabolics); their names come from the three proper parabolic subgroups Q, P and B of $G = \mathrm{SO}(5)$. The representations in these three classes are all non-tempered and non-generic. We further have the Yoshida class **(Y)**, consisting of conjecturally tempered representations. The representations in this class are characterized by their L-functions being the product of two L-functions of cuspidal, automorphic representations of $\mathrm{GL}(2, \mathbb{A})$. Finally, there is the "general" class **(G)**, which contains all the remaining representations. The representations in this class are characterized by admitting a functorial transfer to a cuspidal, automorphic representation on $\mathrm{GL}(4, \mathbb{A})$. In Table 2 below we will give a characterization of the six classes in terms of the analytic properties of their associated degree 4 (spin) and 5 (standard) L-functions.

The Siegel modular forms we will consider are all holomorphic and vector-valued. Let $S_{k,j}(\Gamma)$ be the space of cuspidal Siegel modular forms of weight $\det^k \mathrm{sym}^j$ with respect to the congruence subgroup Γ. As explained in [22], such modular forms originate as vectors in cuspidal, automorphic representations $\pi = \otimes \pi_p$ of $G(\mathbb{A}_\mathbb{Q})$ with a certain archimedean component π_∞. An eigenform $F \in S_{k,j}(\Gamma)$ (with respect to almost all good Hecke operators) will in fact determine a unique parameter $\psi \in \Psi_2(G)$. We can hence talk about F being of type **(G)**, **(Y)**, **(Q)**, **(P)** or **(B)** (the type **(F)** cannot occur). One can tell from a single Euler factor at a good place whether F is of one of the (conjecturally) tempered types **(G)** or **(Y)**, as opposed to one of the CAP types; see Proposition 2.1. Since it is easy to distinguish between the tempered and non-tempered types, our approach in this work is to prove results about eigenforms of types **(G)** or **(Y)**; a more detailed investigation of modular forms that can be found inside the CAP classes will be part of a future work.

One immediate consequence of the parametrization of discrete automorphic forms on $G(\mathbb{A})$ by cuspidal data on general linear groups is that the analytic properties of the spin and standard L-functions are known. This has been a problem for holomorphic Siegel modular forms, since the non-generic nature of the underlying archimedean representation prevents the direct application of standard techniques in automorphic forms. As a consequence of Arthur's work, we can now say that the partial spin L-function of any eigenform F in $S_{k,j}(\Gamma)$ of type **(G)** or **(Y)** can always be completed to a "nice" L-function; see Proposition 2.4 for details. Note that this does not solve the problem of determining the Euler factors at the bad places, given a specific F. For paramodular forms we will come back to this problem in Section 2.3.

The ***paramodular group*** of level N is defined as

$$K(N) = \mathrm{Sp}(4, \mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}. \tag{0.1}$$

Siegel modular forms with respect to $K(N)$ have received much attention in recent years because of their appearance in what has become known as the ***paramodular conjecture***; see [3], [17]. There is also a local theory of paramodular fixed vectors, developed in [19], with properties similar to the familiar local newform theory

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for GL(2). As explained in [18], this local theory results in a global theory of paramodular oldforms and newforms, which we briefly recall in Section 2.2.

In Lemma 2.5 we will prove the important fact that a paramodular eigen-cuspform cannot be of type **(Y)**. The key here is Theorem 1.1, which connects the structure of the local non-archimedean packets Π_{ψ_v} with the existence of paramodular vectors. More precisely, it states that if ψ is of type **(G)** or **(Y)**, then Π_{ψ_v} contains a unique paramodular representation, and it coincides with the unique generic representation. This is closely related to Theorem 7.5.8 of [19], which says that a tempered representation is paramodular if and only if it is generic. Hence a (holomorphic!) paramodular eigen-cuspform generates a representation $\pi = \otimes\pi_p$ which is generic everywhere except at the archimedean place. Such π violate a sign condition imposed on representations of type **(Y)**.

We may therefore concentrate on paramodular eigenforms of type **(G)**. Theorem 2.6 is the expected strong multiplicity one result for paramodular newforms. It follows from the combination of local multiplicity one (Theorem 7.5.1 of [19]) and global multiplicity one (Theorem 1.5.2 of [2]), and again Theorem 1.1 about paramodularity in local packets. As a consequence, one can prove (Corollary 2.8) multiplicity one for the spaces $S_k(\mathrm{Sp}(4,\mathbb{Z}))$, which to the best of our knowledge was an open problem.

In the final Section 2.3 we turn to the problem of determining the Euler factors at places $p|N$ for a given newform F in $S_{k,j}(K(N))$ of type **(G)**. If the local component π_p of the underlying automorphic representation would be known, one could just look up this factor in Table A.8 of [19]. Typically though, only the classical object F is given, maybe in terms of a number of Fourier coefficients, and determining π_p can be a difficult problem. It turns out that one can still calculate the spin Euler factor $L_p(s,F) := L(s,\pi_p)$ from F without knowing π_p. This is accomplished with the help of two paramodular Hecke operators, $T_{0,1}(p)$ and $T_{1,0}(p)$. Every newform is automatically an eigenform for these two operators, and the two resulting eigenvalues encode all the information one needs to write down $L_p(s,F)$. In a local context, this has been observed in Sect. 7.5 of [19]. Here, we rewrite $T_{0,1}(p)$ and $T_{1,0}(p)$ in a form in which they can be applied to classical modular forms; see Proposition 2.10.

1. Packet structure

For any commutative ring R, let

$$\mathrm{GSp}(4,R) = \{g \in \mathrm{GL}(4,R) \mid {}^tgJg = \mu(g)J, \text{ for some } \mu(g) \in R^\times\}, \quad J = \begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix}.$$

The kernel of the multiplier homomorphism $\mu : \mathrm{GSp}(4,R) \to R^\times$ is the group $\mathrm{Sp}(4,R)$. The split orthogonal group $\mathrm{SO}(n)$ is defined by

$$\mathrm{SO}(n,R) = \{g \in \mathrm{SL}(n,R) \mid {}^tgJg = J\}, \qquad J = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}.$$

There is an isomorphism of algebraic groups $\mathrm{PGSp}(4) \cong \mathrm{SO}(5)$. For an explicit realization of this isomorphism in characteristic zero, see Appendix A.7 of [19].

Let B, P, Q be the ***Borel subgroup***, the ***Siegel parabolic subgroup*** and the ***Klingen parabolic subgroup*** of GSp(4), respectively, defined as the matrices of

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the following shapes:

$$B = \begin{bmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}, \qquad P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}, \qquad Q = \begin{bmatrix} * & & * & * \\ * & * & * & * \\ * & & * & * \\ & & & * \end{bmatrix}. \tag{1.1}$$

We denote the images of P, B, Q under the map $\mathrm{GSp}(4) \to \mathrm{SO}(5)$ by the same letters.

1.1. **Global parameters.** Let $\mathbb{A}$ be the ring of adeles of an algebraic number field F. Let G be one of the groups for which the classification theorems of [2] apply; among such are the symplectic groups $\mathrm{Sp}(2n)$ and the split orthogonal groups $\mathrm{SO}(n)$. The central result of [2], Theorem 1.5.2, takes the form

$$L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})) \cong \bigoplus_{\psi \in \Psi_2(G)} \ \bigoplus_{\{\pi \in \Pi_\psi :\, \langle \cdot, \pi\rangle = \varepsilon_\psi\}} m_\psi \pi. \tag{1.2}$$

Here, ψ runs through certain ***Arthur parameters***, which are formal objects composed of cuspidal data on general linear groups; Π_ψ is a global ***Arthur packet***, consisting of certain equivalence classes of global representations of $G(\mathbb{A})$ determined by the parameter ψ; the quantities ε_ψ and $\langle \cdot, \pi\rangle$ are characters of a centralizer group $\mathcal{S}_\psi \cong (\mathbb{Z}/2\mathbb{Z})^t$; and m_ψ is a multiplicity which can only take the values 1 or 2.

Assume from now on that $G = \mathrm{SO}(5)$. In this case the multiplicities m_ψ are all 1 and the groups $\mathcal{S}_\psi$ have at most two elements. The characters ε_ψ and $\langle \cdot, \pi\rangle$ will be explained in Section 1.3. The set $\Psi_2(G)$ consists of formal expressions

$$\psi = (\mu_1 \boxtimes \nu_1) \boxplus \ldots \boxplus (\mu_r \boxtimes \nu_r),$$

where μ_i is a self-dual, unitary, cuspidal automorphic representation of $\mathrm{GL}(m_i, \mathbb{A})$, and ν_i is the irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ of dimension n_i. The following conditions need to be satisfied:

i) $\sum_{i=1}^r m_i n_i = 4$.
ii) $\mu_i \boxtimes \nu_i \neq \mu_j \boxtimes \nu_j$ for $i \neq j$.
iii) If n_i is odd (resp. even), then μ_i is ***symplectic*** (resp. ***orthogonal***), i.e., the exterior (resp. symmetric) square L-function $L(s, \mu_i, \Lambda^2)$ (resp. $L(s, \mu_i, \mathrm{sym}^2)$) has a pole at $s = 1$.

For $\mathrm{GL}(2)$ and $\mathrm{GL}(4)$, there are alternative characterizations of symplectic and orthogonal cuspidal automorphic representations, as explained in Sect. 4 of [1]. The above conditions lead to six different types of parameters:

(G) $\psi = \mu \boxtimes 1$, where μ is a self-dual, symplectic, unitary, cuspidal automorphic representation of $\mathrm{GL}(4, \mathbb{A})$. This is the ***general type*** of Arthur parameter.

(Y) $\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1)$, where μ_1 and μ_2 are distinct, unitary, cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character. These parameters are said to be of ***Yoshida type***.

(Q) $\psi = \mu \boxtimes \nu(2)$, where $\nu(2)$ is the two-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$, and μ is a self-dual, unitary, cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with non-trivial central character. These parameters are said to be of ***Soudry type***.

(P) $\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu(2))$, where μ is a unitary, cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character, and χ is a quadratic

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Hecke character[1]. These parameters are said to be of ***Saito-Kurokawa type***.

(B) $\psi = (\chi_1 \boxtimes \nu(2)) \boxplus (\chi_2 \boxtimes \nu(2))$, where χ_1, χ_2 are distinct, quadratic Hecke characters. These parameters are said to be of ***Howe–Piatetski-Shapiro type***.

(F) $\psi = \chi \boxtimes \nu(4)$, where $\nu(4)$ is the four-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$, and χ is a quadratic Hecke character.

The parameters **(Q)**, **(P)** and **(B)** get their names from the parabolic subgroups Q, P and B. The cusp forms in the Arthur packet corresponding to a parameter in **(Q)** are CAP (cuspidal associated to parabolics) with respect to the Klingen parabolic subgroup Q; similarly for **(P)** and **(B)**. The parameters of type (**F**) correspond to one-dimensional representations.

Each parameter ψ comes with a group $\mathcal{L}_\psi$ and a homomorphism $\tilde{\psi} : \mathcal{L}_\psi \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$; see (1.4.4) and (1.4.5) of [2]. In general the groups $\mathcal{L}_\psi$ are extensions of the absolute Galois group Γ_F by a complex reductive group, but in our case we can neglect the Galois part. We will describe $\mathcal{L}_\psi$ and $\tilde{\psi}$ for each of the six types of parameters above. They depend only on the type of parameter.

For parameters of type **(G)**, we have $\mathcal{L}_\psi = \mathrm{Sp}(4,\mathbb{C})$. The map $\tilde{\psi} : \mathcal{L}_\psi \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$ is the projection onto the first component. For parameters of type **(Y)**, we have $\mathcal{L}_\psi = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. The map $\tilde{\psi} : \mathcal{L}_\psi \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$ is trivial on the second factor, and is given by

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) \longmapsto \begin{bmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{bmatrix} \tag{1.3}$$

on $\mathcal{L}_\psi$. For parameters of type **(Q)**, we have

$$\mathcal{L}_\psi = \mathrm{O}(2,\mathbb{C}) = \{g \in \mathrm{GL}(2,\mathbb{C}) \mid {}^t g \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} g = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\}.$$

The identity component of $\mathrm{O}(2,\mathbb{C})$ consists of all matrices $\begin{bmatrix} x & \\ & x^{-1} \end{bmatrix}$ with $x \in \mathbb{C}^\times$, and the non-identity component is represented by $\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$. The homomorphism $\tilde{\psi} : \mathrm{O}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{Sp}(4,\mathbb{C})$ maps

$$\left(\begin{bmatrix} x & \\ & x^{-1} \end{bmatrix}, 1\right) \longmapsto \begin{bmatrix} x & & & \\ & x^{-1} & & \\ & & x^{-1} & \\ & & & x \end{bmatrix}, \qquad \left(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, 1\right) \longmapsto \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \tag{1.4}$$

and

$$\left(1, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \longmapsto \begin{bmatrix} a & & & b \\ & a & b & \\ & c & d & \\ c & & & d \end{bmatrix}. \tag{1.5}$$

For parameters of type **(P)**, we have $\mathcal{L}_\psi = \mathrm{SL}(2,\mathbb{C}) \times \{\pm 1\}$, and the map $\tilde{\psi} : \mathrm{SL}(2,\mathbb{C}) \times \{\pm 1\} \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$ is given by

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \pm 1, 1\right) \longmapsto \begin{bmatrix} a & & b & \\ & \pm 1 & & \\ c & & d & \\ & & & \pm 1 \end{bmatrix}, \qquad \left(1, 1, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \longmapsto \begin{bmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{bmatrix}. \tag{1.6}$$

For parameters of type **(B)**, we have $\mathcal{L}_\psi = \{\pm 1\} \times \{\pm 1\}$, and $\tilde{\psi} : \{\pm 1\} \times \{\pm 1\} \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$ maps

$$(x, y, 1) \mapsto \begin{bmatrix} x & & & \\ & y & & \\ & & x & \\ & & & y \end{bmatrix}, \qquad \left(1, 1, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \mapsto \begin{bmatrix} a & & b & \\ & a & & b \\ c & & d & \\ & c & & d \end{bmatrix}. \tag{1.7}$$

[1]Meaning χ is a character of $F^\times \backslash \mathbb{A}^\times$ satisfying $\chi^2 = 1$.

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For parameters of type **(F)**, we have $\mathcal{L}_\psi = \{\pm 1\}$. The map $\tilde{\psi} : \{\pm 1\} \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$ identifies $\{\pm 1\}$ with the center of $\mathrm{Sp}(4,\mathbb{C})$. Its restriction to $\mathrm{SL}(2,\mathbb{C})$ is given by the four-dimensional irreducible representation of this group (in some realization that takes values in $\mathrm{Sp}(4,\mathbb{C})$).

In each case, let S_ψ be the centralizer of the image of $\tilde{\psi}$ and S_ψ^0 its identity component. The group $\mathcal{S}_\psi = S_\psi / S_\psi^0 Z$, where $Z \cong \{\pm 1\}$ is the center of $\mathrm{Sp}(4,\mathbb{C})$, has either one or two elements. Easy verifications show that $\mathcal{S}_\psi = 1$ for types **(G)**, **(Q)**, **(F)**, and $\mathcal{S}_\psi \cong \mathbb{Z}/2\mathbb{Z}$ for types **(Y)**, **(P)**, **(B)**. In the latter cases, the non-trivial element of $\mathcal{S}_\psi$ is represented by $\mathrm{diag}(-1, 1, -1, 1)$.

1.2. **Local parameters.** Let ψ be one of the global parameters of the previous section. We shall describe how to localize ψ to a family of local parameters

$$\psi_v : L_{F_v} \times \mathrm{SU}(2) \longrightarrow \mathrm{Sp}(4,\mathbb{C}), \tag{1.8}$$

for each place v of F. Here, L_{F_v} is the Weil group W_{F_v} if v is archimedean, and the Weil-Deligne group $W_{F_v} \times \mathrm{SU}(2)$ if v is non-archimedean. The localizations fit into a commutative diagram:

$$\begin{tikzcd} L_{F_v} \times \mathrm{SU}(2) \arrow[r, "\psi_v"] \arrow[d, "\phi_v \times \mathrm{id}"'] & \mathrm{Sp}(4,\mathbb{C}) \arrow[d, "\sim"', "\mathrm{id}"] \\ \mathcal{L}_\psi \times \mathrm{SL}(2,\mathbb{C}) \arrow[r, "\tilde{\psi}"'] & \mathrm{Sp}(4,\mathbb{C}) \end{tikzcd} \tag{1.9}$$

We will define ψ_v by defining the left vertical map ϕ_v. The general procedure is explained after Theorem 1.4.2 of [2]; we will describe it in our simplified situation. Essentially, ϕ_v is the Langlands parameter of the local component at v of the $\mathrm{GL}(n)$ data in the parameter ψ.

Let $\psi = \mu \boxtimes 1$ be a parameter of type **(G)**. Recall that μ is a self-dual, symplectic, unitary, cuspidal automorphic representation of $\mathrm{GL}(4,\mathbb{A})$. We factor $\mu = \otimes \mu_v$, where μ_v is an irreducible, admissible representation of the local group $\mathrm{GL}(4, F_v)$. Let

$$\phi_v : L_{F_v} \longrightarrow \mathrm{GL}(4,\mathbb{C})$$

be the parameter of μ_v attached to it by the local Langlands correspondence, determined up to conjugation. Since μ is symplectic, Theorem 1.4.2 of [2] asserts that, after a suitable conjugation, the image of ϕ_v is contained in $\mathrm{Sp}(4,\mathbb{C})$. This is the map ϕ_v in (1.9). Observe that the resulting ψ_v is trivial on the factor $\mathrm{SU}(2)$.

We similarly localize parameters $\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1)$ of type **(Y)**. Recall that μ_1 and μ_2 are distinct, unitary, cuspidal automorphic representations of $\mathrm{GL}(2,\mathbb{A})$ with trivial central character. Each μ_i gives rise to local parameters $\phi_{i,v} : L_{F_v} \to \mathrm{GL}(2,\mathbb{C})$. Since μ_i has trivial central character, the image of $\phi_{i,v}$ lies in $\mathrm{SL}(2,\mathbb{C})$. We combine $\phi_{1,v}$ and $\phi_{2,v}$ to a map ϕ_v from L_{F_v} into $\mathcal{L}_\psi = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. Again, the resulting ψ_v is trivial on the factor $\mathrm{SU}(2)$.

Next consider $\psi = \mu \boxtimes \nu(2)$, a parameter of type **(Q)**, where μ is a self-dual, unitary, cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with non-trivial central character ω_μ. Since ω_μ is non-trivial and quadratic, it determines a quadratic extension E of F. We have $\mu = \mu \otimes \omega_\mu$ because μ is self-dual. It follows that there exists a character θ of $\mathbb{A}_E^\times$, not Galois-invariant, such that μ is obtained from θ by automorphic induction. It is then not difficult to see that the L-parameter ϕ_v of each

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local representation μ_v, after a suitable conjugation, takes values in $\mathcal{L}_\psi = \mathrm{O}(2,\mathbb{C})$. This defines the map ϕ_v in the diagram (1.9), and thus the localization ψ_v.

Recall that parameters of type **(P)** are of the form $\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu(2))$, where μ is a unitary, cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with trivial central character, and χ is a quadratic Hecke character. The local components of μ give rise to L-parameters $\phi_{1,v}$ taking values in $\mathrm{SL}(2,\mathbb{C})$, and the local components of χ give rise to L-parameters $\phi_{2,v}$ taking values in $\{\pm 1\}$. We thus obtain the map $\phi_v = \phi_{1,v} \times \phi_{2,v}$ from L_{F_v} into $\mathcal{L}_\psi = \mathrm{SL}(2,\mathbb{C}) \times \{\pm 1\}$.

It is now obvious how to define ϕ_v for a parameter $\psi = (\chi_1 \boxtimes \nu(2)) \boxplus (\chi_2 \boxtimes \nu(2))$ of type **(B)**, where χ_1, χ_2 are distinct, quadratic Hecke characters. Each χ_i gives rise to local parameters $\phi_{i,v}$, and we combine them to a map $\phi_v = \phi_{1,v} \times \phi_{2,v}$ from L_{F_v} into $\mathcal{L}_\psi = \{\pm 1\} \times \{\pm 1\}$.

Finally, for a parameter $\psi = \chi \boxtimes \nu(4)$ of type **(F)**, where χ is a quadratic Hecke character, we let ϕ_v be the L-parameter of the local character χ_v. It takes values in $\{\pm 1\} = \mathcal{L}_\psi$.

Local packets, centralizers and characters. For a global $\psi \in \Psi_2(G)$, recall the centralizer groups $\mathcal{S}_\psi = S_\psi / S_\psi^0 Z$, which have either one or two elements. For each localization ψ_v we make an analogous definition. Let S_{ψ_v} be the centralizer of the image of ψ_v in $\mathrm{Sp}(4,\mathbb{C})$, and $S_{\psi_v}^0$ its identity component. We define the local centralizer group to be

$$\mathcal{S}_{\psi_v} = S_{\psi_v} / S_{\psi_v}^0 Z, \tag{1.10}$$

where Z is the center of $\mathrm{Sp}(4,\mathbb{C})$. It is obvious from (1.9) that $S_\psi \subset S_{\psi_v}$. It follows that there is a natural map $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$.

The theory of [2] attaches to each ψ_v a finite packet of admissible representations Π_{ψ_v} of $G(F_v)$; see Theorem 1.5.1 of [2] and the remarks following it. It is expected, but not known in general, that the elements of Π_{ψ_v} are irreducible and unitary. The packets Π_{ψ_v} come with a canonical mapping

$$\pi_v \longmapsto \langle \cdot, \pi_v \rangle, \qquad \pi_v \in \Pi_{\psi_v}, \tag{1.11}$$

into the group of characters $\widehat{\mathcal{S}}_{\psi_v}$ of the centralizer group $\mathcal{S}_{\psi_v}$. If π_v is unramified, then $\langle \cdot, \pi_v \rangle = 1$. If ψ_v is trivial on $\mathrm{SU}(2)$ and has bounded image, i.e., a tempered L-parameter, then the map $\Pi_{\psi_v} \to \widehat{\mathcal{S}}_{\psi_v}$ is injective; see Theorem 1.5.1 of [2].

As an archimedean example, consider a discrete series parameter, or a limit of discrete series parameter, as in Sects. 1.2, 1.3 of [22]. Then Π_{ψ_v} consists of two elements, a *holomorphic* (limit of) discrete series representation π^{hol}, and a *large*, or *generic*, (limit of) discrete series representation π^{gen}. The centralizer group has two elements, so that the map $\Pi_{\psi_v} \to \widehat{\mathcal{S}}_{\psi_v}$ is a bijection. By Proposition 8.3.2 of [2], the generic representation is assigned the trivial character of $\widehat{\mathcal{S}}_{\psi_v}$. Thus

$$\langle \cdot, \pi^{\mathrm{gen}} \rangle = 1, \qquad \langle \cdot, \pi^{\mathrm{hol}} \rangle = -1, \tag{1.12}$$

where we wrote -1 for the non-trivial character of $\widehat{\mathcal{S}}_{\psi_v}$.

Local packets for types **(G)** *and* **(Y)**. Assume that $\psi \in \Psi_2(G)$ is a global parameter of type **(G)** or **(Y)**. Let v be a place of F, and consider the localization ψ_v. Then ψ_v is trivial on the factor $\mathrm{SU}(2)$. We may thus think of ψ_v as a "traditional" L-parameter $L_{F_v} \to \mathrm{Sp}(4,\mathbb{C})$. The representations in Π_{ψ_v} are irreducible and unitary by the remarks after Conjecture 8.3.1 of [2]. If v is archimedean, then the packets Π_{ψ_v} defined in [2] coincide with the packets defined by the local Langlands

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correspondence. For our group $G = \mathrm{SO}(5)$, the same statement is true for non-archimedean v as well; this follows from the results of [4]. In general, it is not known whether the packets of [2], which are characterized by endoscopic character identities, satisfy the desiderata of the local Langlands correspondence; see the remarks made on pages 43 and 44 of [2].

If v is non-archimedean, we say that an irreducible, admissible representation of $G(F_v)$, viewed as a representation of $\mathrm{GSp}(4, F_v)$ with trivial central character, is ***paramodular***, if it admits a non-zero vector fixed by the paramodular group

$$K(\mathfrak{p}^n) := \{g \in \mathrm{GSp}(4, F_v) \mid \mu(g) \in \mathfrak{o}^\times\} \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{o} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix} \tag{1.13}$$

for some $n \geq 0$; here $\mathfrak{o}$ is the ring of integers of F_v, and $\mathfrak{p}$ is the maximal ideal of $\mathfrak{o}$. The following result is key in connecting paramodular cusp forms with the packet structure on the group $G(\mathbb{A})$.

Theorem 1.1. *Let $\psi \in \Psi_2(G)$ be of type* **(G)** *or* **(Y)**. *For a place v of F, let Π_{ψ_v} be the associated local packet.*

i) *The packet Π_{ψ_v} contains a unique generic representation $\pi^{\mathrm{gen}}_{\psi_v}$. It has the property that the character $\langle\cdot, \pi^{\mathrm{gen}}_{\psi_v}\rangle$ of $\mathcal{S}_{\psi_v}$ is trivial.*

ii) *Assume that v is non-archimedean. Then the packet Π_{ψ_v} contains a unique paramodular representation. It coincides with the unique generic representation $\pi^{\mathrm{gen}}_{\psi_v}$ from i).*

Proof. i) follows from Proposition 8.3.2 of [2] and the Remark 2 following it. For ii) recall from Theorem 7.5.4 of [19] that generic representations are paramodular. Hence Π_{ψ_v} contains at least one paramodular representation, namely $\pi^{\mathrm{gen}}_{\psi_v}$. We have to show that there are no other paramodular representations in Π_{ψ_v}. If Π_{ψ_v} consists entirely of supercuspidal representations, this follows from the fact that non-generic supercuspidals are not paramodular; see Theorem 3.4.3 of [19]. Assume that Π_{ψ_v} contains at least one non-supercuspidal representation. Then, by the requirements of the local Langlands correspondence, Π_{ψ_v} must be one of the L-packets exhibited in Sect. 2.4 of [19] and summarized in Appendix A; it is here that we are using the fact that the packets of [2] coincide with the packets of [9], and thus with those of [19] in the non-supercuspidal cases. We may assume that Π_{ψ_v} has more than one element, since otherwise we have nothing to prove. There are exactly four types of L-packets with more than one element, namely

$$\{\mathrm{VIa}, \mathrm{VIb}\}, \qquad \{\mathrm{VIIIa}, \mathrm{VIIIb}\}, \qquad \{\mathrm{Va}, \mathrm{Va}^*\}, \qquad \{\mathrm{XIa}, \mathrm{XIa}^*\}.$$

Observe that VIa, VIIIa, Va and XIa are the generic members of the packet. The representations Va^* and XIa^* are certain supercuspidals; all we need to know about them is that they are not generic, which follows from the uniqueness statement in i). By Theorem 3.4.3 of [19], the non-generic member in each packet is not paramodular. This concludes the proof. □

Local packets for types **(Q)**, **(P)**, **(B)** *and* **(F)**. Now assume that $\psi \in \Psi_2(G)$ is not of type **(G)** or **(Y)**. Theorem 1.5.1 of [2], or rather a slight variation of it as explained on p. 45 of [2], attaches a finite packet Π_{ψ_v} of admissible representations to each localization ψ_v. In general it is not known whether these representations are irreducible or unitary, and we will not use these assumptions. There is, however,

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always one irreducible representation contained in Π_{ψ_v}, which is obtained as follows. Define $\phi_{\psi_v} : L_{F_v} \to \mathrm{Sp}(4,\mathbb{C})$ by

$$\phi_{\psi_v}(w) = \psi_v(w, \begin{bmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{bmatrix}) \tag{1.14}$$

(where ψ_v has been extended to a map on $L_{F_v} \times \mathrm{SL}(2,\mathbb{C})$). Then ϕ_{ψ_v} is a Langlands parameter whose image is contained in a proper Levi subgroup. Therefore the irreducible, admissible representation π_{ψ_v} corrresponding to ϕ_{ψ_v} via the local Langlands correspondence for GSp(4) is easily identified; one can use [9], or even the explicit description in Sect. 2.4 of [19]. This π_{ψ_v} is an element of the packet Π_{ψ_v} (see Proposition 7.4.1 of [2]).

We will describe π_{ψ_v} more explicitly. Assume that $\psi = \mu \boxtimes \nu(2)$ is of type **(Q)**. Let $\mu = \otimes \mu_v$, and let $\phi_v : L_{F_v} \to \mathrm{O}(2,\mathbb{C})$ be the Langlands parameter of μ_v. It follows from (1.4), (1.5) and (1.9) that

$$\phi_{\psi_v}(w) = \begin{bmatrix} |w|^{1/2}\phi_v(w) & \\ & |w|^{-1/2}\,{}^t\phi_v(w)^{-1} \end{bmatrix}. \tag{1.15}$$

Taking duality for GSp(4) into account, as in (2.40) of [19], we see that π_{ψ_v} is the Langlands quotient of the Klingen induced representation

$$|\cdot|_v\,\omega_{\mu_v} \rtimes |\cdot|_v^{-1/2}\mu_v, \tag{1.16}$$

where ω_{μ_v} is the central character of μ_v. If μ_v is unramified, then π_{ψ_v} is either of type IIIb, Vd or VId in the classification of [19].

Next assume that $\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu(2))$ is of type **(P)**. Let $\mu = \otimes \mu_v$ and $w \mapsto \begin{bmatrix} a(w) & b(w) \\ c(w) & d(w) \end{bmatrix}$ be the Langlands parameter of μ_v. Let $\chi = \otimes \chi_v$, and identify χ_v with a character $L_{F_v} \to \{\pm 1\}$. It follows from (1.6) and (1.9) that

$$\phi_{\psi_v}(w) = \begin{bmatrix} a(w) & & b(w) & \\ & \chi_v(w)|w|^{1/2} & & \\ c(w) & & d(w) & \\ & & & \chi_v(w)|w|^{-1/2} \end{bmatrix}. \tag{1.17}$$

Again taking duality into account, as in (2.46) of [19], we see that π_{ψ_v} is the Langlands quotient of the Siegel induced representation

$$\chi_v|\cdot|_v^{1/2}\mu_v \rtimes \chi_v|\cdot|_v^{-1/2}. \tag{1.18}$$

If μ_v is unramified, then π_{ψ_v} is a representation of type IIb in the classification of [19].

Now assume that $\psi = (\chi_1 \boxtimes \nu(2)) \boxplus (\chi_2 \boxtimes \nu(2))$ is of type **(B)**. We factor $\chi_i = \otimes \chi_{i,v}$ and identify $\chi_{i,v}$ with a character $L_{F_v} \to \{\pm 1\}$. By (1.7) and (1.9),

$$\phi_{\psi_v}(w) = \begin{bmatrix} \chi_{1,v}(w)|w|^{1/2} & & & \\ & \chi_{2,v}(w)|w|^{1/2} & & \\ & & \chi_{1,v}(w)|w|^{-1/2} & \\ & & & \chi_{2,v}(w)|w|^{-1/2} \end{bmatrix}. \tag{1.19}$$

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It follows, similarly to (2.28) of [19], that π_{ψ_v} is the Langlands quotient of the Borel induced representation

$$\chi_{1,v}\chi_{2,v}|\cdot|_v \times \chi_{1,v}\chi_{2,v} \rtimes \chi_{2,v}|\cdot|_v^{-1/2}. \tag{1.20}$$

If $\chi_{1,v}$ and $\chi_{2,v}$ are unramified, π_{ψ_v} is either of type Vd or VId in the classification of [19].

Finally, assume that $\psi = \chi \boxtimes \nu(4)$ is of type **(F)**. We factor χ and identify the local components χ_v with maps $L_{F_v} \to \{\pm 1\}$. In this case

$$\phi_{\psi_v}(w) = \begin{bmatrix} \chi_v(w)|w|^{1/2} &&& \\ & \chi_v(w)|w|^{3/2} && \\ && \chi_v(w)|w|^{-1/2} & \\ &&& \chi_v(w)|w|^{-3/2} \end{bmatrix}. \tag{1.21}$$

We have $\pi_{\psi_v} = \chi_v 1_{\mathrm{GSp}(4)}$, a one-dimensional representation. It is of type IVd in the classification of [19].

1.3. **Global packets and L-functions.** Let $\psi \in \Psi_2(G)$ be a global parameter. In the previous section we defined localizations $\psi_v : L_{F_v} \times \mathrm{SU}(2) \to \mathrm{Sp}(4,\mathbb{C})$, for each place v. To each ψ_v there is associated a packet Π_{ψ_v} of admissible representations of $G(F_v)$. We now define the global packet associated to ψ as

$$\Pi_\psi = \big\{\pi = \otimes \pi_v \mid \pi_v \in \Pi_{\psi_v} \text{ for all } v\big\}. \tag{1.22}$$

Each $\pi \in \Pi_\psi$ defines a character of $\mathcal{S}_\psi$ by

$$\langle x, \pi\rangle = \prod_v \langle x_v, \pi_v\rangle. \tag{1.23}$$

Here, $x \mapsto x_v$ denotes the natural map $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$. The characters (1.23) are the ones appearing in (1.2). Of course, in cases **(G)**, **(Q)**, **(F)**, where $\mathcal{S}_\psi = 1$, we have $\langle\cdot,\pi\rangle = 1$ for any π.

The only ingredient of (1.2) that has not been explained yet are the characters ε_ψ. They are given in [1], and we simply copy the result: They are always trivial, except for parameters $\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu(2))$ of type **(P)** for which $\varepsilon(1/2, \chi \otimes \mu) = -1$, in which case ε_ψ is non-trivial. Writing -1 for the unique non-trivial character of $\mathcal{S}_\psi \cong \mathbb{Z}/2\mathbb{Z}$, we thus have $\varepsilon_\psi = \varepsilon(1/2, \chi \otimes \mu)$. Arthur's main result (1.2) for $G = \mathrm{SO}(5)$ now takes the following form:

$$L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})) \cong \bigoplus_{\psi \in \mathbf{(G)}} \bigoplus_{\pi \in \Pi_\psi} \pi \tag{1.24}$$

$$\oplus \bigoplus_{\psi \in \mathbf{(Y)}} \bigoplus_{\{\pi \in \Pi_\psi :\, \langle\cdot,\pi\rangle = 1\}} \pi \tag{1.25}$$

$$\oplus \bigoplus_{\psi \in \mathbf{(Q)}} \bigoplus_{\pi \in \Pi_\psi} \pi \tag{1.26}$$

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$$\oplus \bigoplus_{\substack{\psi\in(\mathbf{P})\\ \psi=(\mu\boxtimes 1)\boxplus(\chi\boxtimes\nu(2))}} \bigoplus_{\{\pi\in\Pi_\psi:\,\langle\cdot,\pi\rangle=\varepsilon(1/2,\chi\otimes\mu)\}} \pi \tag{1.27}$$

$$\oplus \bigoplus_{\psi\in(\mathbf{B})} \bigoplus_{\{\pi\in\Pi_\psi:\,\langle\cdot,\pi\rangle=1\}} \pi \tag{1.28}$$

$$\oplus \bigoplus_{\substack{\psi\in(\mathbf{F})\\ \psi=\chi\boxtimes\nu(4)}} \chi. \tag{1.29}$$

We see that the global packets of type **(G)**, **(Q)** and **(F)** are ***stable***, meaning we can choose representations from the local packets arbitrarily. Global packets of type **(Y)**, **(P)** and **(B)** are ***unstable***; if $\pi=\otimes\pi_v$ is in such a packet, then the π_v have to satisfy a parity condition in order for π to appear in the discrete spectrum.

For $n\in\{1,4,5\}$, let ρ_n be the irreducible n-dimensional representation of $\mathrm{Sp}(4,\mathbb{C})$. Of course, ρ_1 is the trivial representation, and ρ_4 is given by the natural action on column vectors of length 4. An explicit form of ρ_5 is given in Appendix A.7 of [19]; it can be realized as a map $\mathrm{Sp}(4,\mathbb{C})\to\mathrm{SO}(5,\mathbb{C})$. Note that

$$\Lambda^2\rho_4=\rho_1\oplus\rho_5. \tag{1.30}$$

Suppose that ψ is a parameter of type **(G)** or **(Y)**. If $\pi=\otimes\pi_v$ is any representation in the packet defined by ψ, we define the ***spin L-function*** of π by

$$L(s,\pi,\rho_4):=\prod_v L(s,\psi_v), \tag{1.31}$$

where the product extends over all places, and $\psi_v: L_{F_v}\to\mathrm{Sp}(4,\mathbb{C})$ is the localization of ψ. We further set

$$L(s,\pi,\rho_5):=\prod_v L(s,\rho_5\circ\psi_v), \tag{1.32}$$

which is called the ***standard L-function*** of π.

Recall that an L-function is called ***nice*** if it has analytic continuation to an entire function, satisfies the expected functional equation, and is bounded in vertical strips.

Lemma 1.2. *Let π be a representation in a packet of type* **(G)**. *Then the L-functions $L(s,\pi,\rho_4)$ and $L(s,\pi,\rho_5)$ are nice.*

Proof. This is clear for $L(s,\pi,\rho_4)$, since it coincides with the standard L-function $L(s,\mu)$ of the cuspidal, automorphic representation μ on $\mathrm{GL}(4,\mathbb{A})$ appearing in the parameter ψ of π. It follows from Theorem A of [14] (see also [12]) that

$$L(s,\pi,\Lambda^2)=L(s,\mu,\Lambda^2)=\prod_{i=1}^m L(s,\tau_i)$$

for cuspidal, automorphic representations τ_i of $\mathrm{GL}(n_i,\mathbb{A})$. Since $L(s,\mu,\Lambda^2)$ has a simple pole at $s=1$, exactly one τ_i, say τ_m, is the trivial representation of $\mathrm{GL}(1,\mathbb{A})$. In view of (1.30), it follows that

$$L(s,\pi,\rho_5)=\prod_{i=1}^{m-1} L(s,\tau_i). \tag{1.33}$$

Hence $L(s,\pi,\rho_5)$ is nice. □

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Lemma 1.3. *Let π be a representation in a packet of type* **(Y)**. *Then $L(s,\pi,\rho_4)$ is nice, while $L(s,\pi,\rho_5)$ has a simple pole at $s=1$.*

Proof. The parameter of π is of the form $\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1)$ with distinct, cuspidal, automorphic representations μ_i of $\mathrm{GL}(2,\mathbb{A})$. It follows from (1.3) and (1.9) that $L(s,\pi,\rho_4) = L(s,\mu_1)L(s,\mu_2)$. In particular, this L-function is nice. A straightforward calculation shows that

$$L(s,\pi,\rho_5) = L(s,\mu_1\times\mu_2)Z_F(s), \tag{1.34}$$

where $Z_F(s)$ denotes the L-function of the trivial representation of $\mathrm{GL}(1,\mathbb{A})$ (i.e., the Dedekind zeta function of the number field F). Since μ_1 and μ_2 are self-dual and distinct, the Rankin-Selberg L-function $L(s,\mu_1\times\mu_2)$ is entire. Hence $L(s,\pi,\rho_5)$ has a simple pole at $s=1$. □

The L-functions for representations π in a packet Π_ψ of type **(Q)**, **(P)**, **(B)** or **(F)** will not only depend on ψ, but on the individual element π in the packet (provided there is more than one element in the packet). A "base point" in each global packet is

$$\pi_\psi := \otimes \pi_{\psi_v}, \tag{1.35}$$

where π_{ψ_v} is the representation with L-parameter ϕ_{ψ_v}, defined in (1.14). We set

$$L(s,\pi_\psi,\rho_n) := \prod_v L(s,\rho_n\circ\phi_{\psi_v}), \tag{1.36}$$

for $n\in\{4,5\}$. In the rest of this section we present two tables listing these L-functions and their analytic properties.

Table 1 shows $L(s,\pi_\psi,\rho_n)$ for all types other than **(G)**, disregarding the question of whether π_ψ occurs in the discrete spectrum or not. The L-functions $L(s,\pi_\psi,\rho_4)$ are immediate from (1.15), (1.17), (1.19) and (1.21), while the $L(s,\pi_\psi,\rho_5)$ are easily calculated, for example, by using the explicit form of the map ρ_5 given in A.7 of [19]. For any other π in Π_ψ, the functions $L(s,\pi,\rho_n)$ differ from the ones given in Table 1 only by finitely many Euler factors.

The degree 5 L-function for type **(Q)** appearing in Table 1 contains the adjoint L-function $L(s,\mu,\mathrm{Ad})$, which is defined by $L(s,\mu,\mathrm{Ad})Z_F(s) = L(s,\mu\times\mu^\vee)$. It follows from the calculation on page 488 of [10] that

$$L(s,\mu,\mathrm{Ad}) = L(s,\omega_\mu)L(s,\theta/\theta'), \tag{1.37}$$

where ω_μ is the central character of μ, and θ is the Hecke character for the quadratic extension E (determined by ω_μ) such that μ is obtained from θ by automorphic induction. The character θ' is the Galois conjugate of θ, and $L(s,\theta/\theta')$ is the L-function of the Hecke character θ/θ' over E (hence an L-function of degree 2 over F).

Table 2 summarizes the analytic properties of the L-functions in Table 1. In this table, the twist of an L-function $L(s,\pi)=\prod L(s,\pi_v)$ by a Hecke character χ is to be understood as follows: At each good place v, factor $L(s,\pi_v)=\prod_i L(s,\eta_i)$ with unramified characters η_i, replace $L(s,\eta_i)$ by $L(s,\chi_v\eta_i)$ to obtain new local factors $L(s,\pi_v,\chi_v)$, and then take the product of these factors over unramified places only.

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TABLE 1. The L-functions $L(s,\pi_\psi,\rho_4)$ and $L(s,\pi_\psi,\rho_5)$ for the representation π_ψ in packets parametrized by GL(2) and GL(1) data. For type **(Q)**, the symbol ω_μ denotes the central character of μ, and $L(s,\mu,\mathrm{Ad})$ is the adjoint L-function, defined by $L(s,\mu,\mathrm{Ad})Z_F(s)=L(s,\mu\times\mu^\vee)$.

type	ψ	$L(s,\pi_\psi,\rho_4)$	$L(s,\pi_\psi,\rho_5)$
(Y)	$(\mu_1\boxtimes 1)\boxplus(\mu_2\boxtimes 1)$	$L(s,\mu_1)L(s,\mu_2)$	$L(s,\mu_1\times\mu_2)Z_F(s)$
(Q)	$\mu\boxtimes\nu(2)$	$L(s+\frac12,\mu)L(s-\frac12,\mu)$	$L(s+1,\omega_\mu)L(s-1,\omega_\mu)L(s,\mu,\mathrm{Ad})$
(P)	$(\mu\boxtimes 1)\boxplus(\chi\boxtimes\nu(2))$	$L(s,\mu)L(s+\frac12,\chi)L(s-\frac12,\chi)$	$L(s+\frac12,\chi\mu)L(s-\frac12,\chi\mu)Z_F(s)$
(B)	$(\chi_1\boxtimes\nu(2))\boxplus(\chi_2\boxtimes\nu(2))$	$L(s+\frac12,\chi_1)L(s-\frac12,\chi_1)$ $L(s+\frac12,\chi_2)L(s-\frac12,\chi_2)$	$L(s+1,\chi_1\chi_2)L(s-1,\chi_1\chi_2)$ $L(s,\chi_1\chi_2)^2Z_F(s)$
(F)	$\chi\boxtimes\nu(4)$	$L(s+\frac32,\chi)L(s+\frac12,\chi)$ $L(s-\frac12,\chi)L(s-\frac32,\chi)$	$Z_F(s+2)Z_F(s+1)Z_F(s)$ $Z_F(s-1)Z_F(s-2)$

TABLE 2. Analytic properties of $L(s,\pi_\psi,\rho_4)$ and $L(s,\pi_\psi,\rho_5)$.

type	ψ	$L(s,\pi_\psi,\rho_4)$	$L(s,\pi_\psi,\rho_5)$
(G)	$\mu\boxtimes 1$	nice	nice
(Y)	$(\mu_1\boxtimes 1)\boxplus(\mu_2\boxtimes 1)$	nice	pole at $s=1$
(Q)	$\mu\boxtimes\nu(2)$	nice	pole at $s\in\{1,2\}$ after twist by ω_μ
(P)	$(\mu\boxtimes 1)$ $\boxplus(\chi\boxtimes\nu(2))$	pole at $s=\frac32$ after twist by χ	$L(\frac12,\chi\mu)=0$: nice $L(\frac12,\chi\mu)\neq 0$: pole at $s=1$
(B)	$(\chi_1\boxtimes\nu(2))$ $\boxplus(\chi_2\boxtimes\nu(2))$	pole at $s=\frac32$ after twist by χ_1 or χ_2	pole at $s\in\{1,2\}$ after twist by $\chi_1\chi_2$
(F)	$\chi\boxtimes\nu(4)$	pole at $s\in\{\frac32,\frac52\}$ after twist by χ	pole at $s\in\{1,2,3\}$

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2. Siegel modular forms

In this section we work exclusively over the number field $\mathbb{Q}$. Let $\mathbb{A}$ be its ring of adeles. In Section 2.1 we briefly explain our conventions on Siegel modular forms. We exploit the fact that Siegel modular forms can be understood as special vectors in automorphic representations of the group $\mathrm{GSp}(4,\mathbb{A})$. For more details on the relationship between modular forms and representations, see [22] and [16].

All L-functions attached to either automorphic representations or Siegel modular forms are given in analytic normalization, meaning they satisfy a functional equation relating s and $1-s$.

2.1. Modular forms and parameters.

By a ***congruence subgroup*** Γ of $\mathrm{Sp}(4,\mathbb{Q})$ we mean a group of the form

$$\Gamma=\mathrm{GSp}(4,\mathbb{Q})\cap\mathrm{GSp}(4,\mathbb{R})^+\prod_{p<\infty}K_p,$$

where K_p is an open-compact subgroup of $\mathrm{GSp}(4,\mathbb{Q}_p)$ containing $\mathrm{diag}(a,a,b,b)$ with $a,b\in\mathbb{Z}_p^\times$, and $K_p=\mathrm{GSp}(4,\mathbb{Z}_p)$ for almost all p. The ***good primes*** for Γ are those p for which $K_p=\mathrm{GSp}(4,\mathbb{Z}_p)$.

Let $\mathrm{GSp}(4,\mathbb{R})^+$ be the subgroup of $\mathrm{GSp}(4,\mathbb{R})$ consisting of elements g for which the multiplier $\mu(g)$ is positive. Let $\mathbb{H}_2$ be the Siegel upper half-space of degree 2, i.e., the space of all symmetric complex 2×2-matrices Z whose imaginary part is positive definite. For $g=\left[\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right]\in\mathrm{GSp}(4,\mathbb{R})^+$ and $Z\in\mathbb{H}_2$, let $gZ=(AZ+B)(CZ+D)^{-1}$ and $J(g,Z)=CZ+D$.

Let k be an integer, and let j be a non-negative integer. Let $U_j\simeq\mathrm{sym}^j(\mathbb{C}^2)$ be the space of all complex homogeneous polynomials of degree j in the two variables S and T. For $g\in\mathrm{GL}(2,\mathbb{C})$ and $P(S,T)\in U_j$ define $\eta_{k,j}(g)P(S,T)=\det(g)^kP((S,T)g)$. Then $(\eta_{k,j},U_j)$ gives a concrete realization of the irreducible representation $\det^k\mathrm{sym}^j$ of $\mathrm{GL}(2,\mathbb{C})$. We define a right action of $\mathrm{GSp}(4,\mathbb{R})^+$ on the space of U_j-valued functions on $\mathbb{H}_2$ by

(2.1)

$$(F\big|_{k,j}g)(Z)=\mu(g)^{k+\frac{j}{2}}\,\eta_{k,j}(J(g,Z))^{-1}F(gZ)\qquad\text{for } g\in\mathrm{GSp}(4,\mathbb{R})^+,\ Z\in\mathbb{H}_2.$$

The center of $\mathrm{GSp}(4,\mathbb{R})^+$ acts trivially. A ***Siegel modular form*** of weight $\det^k\mathrm{sym}^j$ (or simply of weight (k,j)) with respect to the congruence subgroup Γ is a holomorphic function $F:\mathbb{H}_2\to U_j$ satisfying $F|_{k,j}\gamma=F$ for all $\gamma\in\Gamma$. In this work we will only consider cusp forms, which can be defined as usual. Let $S_{k,j}(\Gamma)$ be the space of Siegel cusp forms of weight (k,j) with respect to Γ.

Let $F\in S_{k,j}(\Gamma)$ be an ***eigenform***, by which we mean that F is non-zero, and is an eigenfunction for the local Hecke algebra $\mathcal{H}_p$ for almost all good primes p (for Γ). As explained in [22], even though F is in general vector-valued, it can be adelized to a scalar-valued function on $\mathrm{GSp}(4,\mathbb{A})$. Let π be the representation generated by this adelization under right translation. Since F is a cusp form, π decomposes into a direct sum

$$\pi=\pi_1\oplus\ldots\oplus\pi_n$$

of irreducible, cuspidal, automorphic representations of $G(\mathbb{A})$. The assumption that F is an eigenform implies that the π_i are all near-equivalent. Keeping in mind the structure (1.24)–(1.29) and the strong multiplicity one theorem for $\mathrm{GL}(n)$, it follows that there exists a parameter $\psi\in\Psi_2(G)$ such that $\pi_i\in\Pi_\psi$ for all i. We have thus unambigously assigned a parameter ψ to each eigenform $F\in S_{k,j}(\Gamma)$. We say that

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the ***type*** of F is the type of ψ. Hence, F can be of type **(G)**, **(Y)**, etc. Note that F can never be of type **(F)**, since one can prove that one-dimensional representations do not occur in the cuspidal spectrum of GSp(4).

The space $S_{k,j}(\Gamma)$ has a basis consisting of eigenforms. Let $S_{k,j}(\Gamma)_X$ be the subspace spanned by eigenforms of type X, where $X \in \{\mathbf{(G)}, \mathbf{(Y)}, \mathbf{(Q)}, \mathbf{(P)}, \mathbf{(B)}\}$. Then $S_{k,j}(\Gamma)_X$ is well defined, and we have the ***type decomposition***

$$S_{k,j}(\Gamma) = \bigoplus_{X \in \{\mathbf{(G)}, \mathbf{(Y)}, \mathbf{(Q)}, \mathbf{(P)}, \mathbf{(B)}\}} S_{k,j}(\Gamma)_X. \tag{2.2}$$

The space $S_{k,j}(\Gamma)_X$ is spanned by those eigenforms that can be found in cuspidal, automorphic representations in packets Π_ψ, where ψ is of type X. The decomposition (2.2) is orthogonal with respect to the Petersson inner product.

Given an eigenform F, it is desirable to have a practical way of determining the type of F. If the partial L-functions $L_S(s, F, \rho_4)$ and $L_S(s, F, \rho_5)$ are known, this determination can often be made using Table 2. The following result provides alternative criteria using only a single Euler factor at a good place.

Proposition 2.1. *Let $(k, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$. Let $F \in S_{k,j}(\Gamma)$ be an eigenform. Let p be a good prime for which F is an eigenform under the local Hecke algebra at p. Let $Q_p(F) = \prod_{j=1}^4 (1 - \alpha_j X)$ be the degree 4 Hecke polynomial at p.*

i) *If $|\alpha_j| = 1$ for all j, then F is of type* **(G)** *or* **(Y)**.
ii) *If $|\alpha_j| = p^{\pm 1/2}$ for all j, then F is of type* **(Q)** *or* **(B)**.
iii) *If $|\alpha_j| = 1$ for exactly two j's and $|\alpha_j| = p^{\pm 1/2}$ for the other two j's, then F is of type* **(P)**.

Proof. This follows from the shape of the Euler factors of the L-functions in Table 1, together with (weak) general estimates on Satake parameters. □

Remark 2.2. The proposition implies in particular that Gritsenko lifts (see [11]), which are paramodular forms of type **(P)**, can be detected using a single good prime. A similar result for full level is given in Theorem 4.1 of [8].

Given any eigenform $F \in S_{k,j}(\Gamma)$, we can always write down the incomplete L-functions $L_S(s, F, \rho_n)$ for $n \in \{4, 5\}$, where S is a large enough set of places such that any prime $p \notin S$ is good for Γ. The problem of determining the correct Euler factors at the bad places directly from the cusp form F is in general unsolved. If Γ is the paramodular group $K(N)$ of some level, we will provide a method in Section 2.3. For arbitrary Γ, we at least have the existence and uniqueness statement Proposition 2.4 below.

Lemma 2.3. *Let $(k, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$. Let $F \in S_{k,j}(\Gamma)$ be an eigenform of type* **(G)** *or* **(Y)**. *Then $k \geq 2$.*

Proof. The underlying archimedean representation of an eigenform $F \in S_{1,j}(\Gamma)$ is one of the non-tempered lowest weight modules discussed in Sect. 1.4 of [22]. By weak Ramanujan estimates for GL(2) and GL(4), such non-tempered archimedean parameters cannot occur in packets of type **(G)** or **(Y)**. □

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Proposition 2.4. *Let $(k,j)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geq 0}$. Let $F\in S_{k,j}(\Gamma)$ be an eigenform. Let S be a set of finite places such that for each prime $p\notin S$, p is good for Γ and F is an eigenform for the local Hecke algebra at p. Let*

$$L_S(s,F,\rho_4)=\prod_{p\notin S}L_p(s,F,\rho_4),\qquad L_S(s,F,\rho_5)=\prod_{p\notin S}L_p(s,F,\rho_5)$$

be the resulting incomplete spin and standard L-functions of F.

i) *If F is of type* **(G)** *or* **(Y)***, then there exist uniquely determined Euler factors $L_p(s,F,\rho_4)$ for $p\in S$ such that the completed L-function*

$$L(s,F,\rho_4):=\Gamma_{\mathbb{C}}\Big(s+\frac{2k+j-3}{2}\Big)\Gamma_{\mathbb{C}}\Big(s+\frac{j+1}{2}\Big)\Big(\prod_{p\in S}L_p(s,F,\rho_4)\Big)L_S(s,F,\rho_4)$$

is nice.

ii) *If F is of type* **(G)***, then there exist uniquely determined Euler factors $L_p(s,F,\rho_5)$ for $p\in S$ such that the completed L-function*

$$L(s,F,\rho_5):=\Gamma_{\mathbb{C}}(s+k+j-1)\Gamma_{\mathbb{C}}(s+k-2)\Gamma_{\mathbb{R}}(s)\Big(\prod_{p\in S}L_p(s,F,\rho_5)\Big)L_S(s,F,\rho_5)$$

is nice.

Proof. i) By the nature of the parameters, $L_S(s,F,\rho_4)$ is the partial L-function of a cuspidal automorphic representation of $\mathrm{GL}(4,\mathbb{A})$ (in the **(G)** case), or the product of two partial L-functions of cuspidal automorphic representations of $\mathrm{GL}(2,\mathbb{A})$ (in the **(Y)** case). In either case we can complete $L_S(s,F,\rho_4)$ to a nice L-function. The Γ-factors follow from the archimedean local Langlands correspondence; see Proposition 2.5.1 of [22]. The uniqueness statement is a general rigidity property of L-functions; see Proposition 2.1 of [7].

ii) The proof is analogous to that of i), keeping in mind that $L_S(s,F,\rho_5)$ is the product of partial L-functions of cuspidal automorphic representations of certain $\mathrm{GL}(n,\mathbb{A})$'s; see (1.33). The archimedean Euler factor is again given by Proposition 2.5.1 of [22]; observe here that $k\geq 2$ by Lemma 2.3 (the archimedean factor for $k=1$ is slightly different). $\square$

2.2. **Paramodular oldforms and newforms.** For a positive integer N, let $K(N)$ be the paramodular group of level N, defined in (0.1). We consider the spaces $S_{k,j}(K(N))$ of paramodular cusp forms of weight $(k,j)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geq 0}$. These spaces admit a theory of oldforms and newforms, which we now describe. For proofs, see [18].

Let p be any prime. At the heart of the theory are three level raising operators:

$$\theta_p: S_{k,j}(K(N))\longrightarrow S_{k,j}(K(Np)), \tag{2.3}$$

$$\theta'_p: S_{k,j}(K(N))\longrightarrow S_{k,j}(K(Np)), \tag{2.4}$$

$$\eta_p: S_{k,j}(K(N))\longrightarrow S_{k,j}(K(Np^2)). \tag{2.5}$$

(They should really be called $\theta_{N,p}$, etc., but we suppress N to ease the notation.) These three operators commute with each other; operators for different primes commute as well; and they commute with the local Hecke algebras at primes not

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dividing Np. Explicit formulas are as follows:

$$\theta_pF = F\big|_{k,j}\Big(\begin{bmatrix} p&&&\\&p&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix} &&1&\\&1&&\\-1&&&\\&&&1\end{bmatrix}\Big) + \sum_{c\in\mathbb{Z}/p\mathbb{Z}} F\big|_{k,j}\Big(\begin{bmatrix} p&&&\\&p&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix} 1&&&\\&1&&\\c&&1&\\&&&1\end{bmatrix}\Big), \tag{2.6}$$

$$\theta'_pF = F\big|_{k,j}\begin{bmatrix} 1&&&\\&p&&\\&&1&\\&&&p^{-1}\end{bmatrix} + \sum_{c\in\mathbb{Z}/p\mathbb{Z}} F\big|_{k,j}\begin{bmatrix} 1&&&\\&1&&cp^{-1}N^{-1}\\&&1&\\&&&1\end{bmatrix}, \tag{2.7}$$

$$\eta_pF := F\big|_{k,j}\begin{bmatrix} 1&&&\\&p&&\\&&1&\\&&&p^{-1}\end{bmatrix}. \tag{2.8}$$

Here θ'_p is Atkin-Lehner conjugate to θ_p, while η_p commutes with Atkin-Lehner involutions. All three operators are well-behaved with respect to Fourier-Jacobi expansions. While it is obvious that η_p is injective, the same is also true of θ_p and θ'_p; see Theorem 6.2 of [18].

As in the classical Atkin-Lehner theory, we define the space $S_{k,j}^{\rm old}(K(N))$ of oldforms as the space spanned by all cusp forms coming from strictly smaller levels via repeated applications of the three level raising operators for primes dividing N. The space of newforms $S_{k,j}^{\rm new}(K(N))$ is by definition the orthogonal complement of $S_{k,j}^{\rm old}(K(N))$ inside $S_{k,j}(K(N))$.

It is clear that the level raising operators preserve the spaces $S_{k,j}(K(N))_X$ appearing in (2.2). We may thus define oldforms and newforms for each of the spaces $S_{k,j}(K(N))_X$. A moment's consideration shows that

$$S_{k,j}^{\rm old}(K(N)) = \bigoplus_X S_{k,j}^{\rm old}(K(N))_X, \qquad S_{k,j}^{\rm new}(K(N)) = \bigoplus_X S_{k,j}^{\rm new}(K(N))_X. \tag{2.9}$$

Paramodular cusp forms can in fact not be of type $\mathbf{(Y)}$.

Lemma 2.5. *$S_{k,j}(K(N))_{\mathbf{(Y)}} = 0$.*

Proof. Assume that F is a non-zero eigenform in $S_{k,j}(K(N))_{\mathbf{(Y)}}$; we will obtain a contradiction. Let $\pi = \otimes\pi_p$ be an irreducible constituent of the automorphic representation generated by the adelization of F. By definition, π is an element of a global packet Π_ψ, where $\psi \in \Psi_2(G)$ is a parameter of type $\mathbf{(Y)}$. Since F is a paramodular form, each π_p for $p < \infty$ is paramodular. By Theorem 1.1, the local character $\langle\cdot,\pi_p\rangle$ is trivial for $p < \infty$. Since F is holomorphic, the archimedean component π_∞ is one of the lowest weight representations considered in Sect. 2.5 of [22]. More precisely, since $k \geq 2$ by Lemma 2.3, π_∞ is a holomorphic discrete series representation of $G(\mathbb{R})$, or a limit of such. By (1.12), the character $\langle\cdot,\pi_\infty\rangle$ is non-trivial. Since the map $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$ is easily seen to be a bijection, it follows from (1.23) that $\langle\cdot,\pi\rangle$ is non-trivial. In view of (1.25), this is the desired contradiction. $\square$

Theorem 2.6. *Let N, N_1, N_2 and k, k_1, k_2 be positive integers, and j, j_1, j_2 be non-negative integers.*

i) *Assume that $F \in S_{k,j}(K(N))_{\mathbf{(G)}}$ is an eigenform for the unramified local Hecke algebra $\mathcal{H}_p$ for almost all p not dividing N. Then F is an eigenform for $\mathcal{H}_p$ for <u>all</u> $p \nmid N$. The cuspidal, automorphic representation π of $G(\mathbb{A})$ generated by the adelization of F is irreducible and lifts to a cuspidal, automorphic representation of $\mathrm{GL}(4,\mathbb{A})$. The conductor of π divides N, with equality if and only if F is a newform.*

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ii) *Let $F_i \in S^{\mathrm{new}}_{k_i,j_i}(K(N_i))_{\mathbf{(G)}}$, $i=1,2$, be two eigenforms. Assume that for almost all primes p the Hecke eigenvalues of F_1 and F_2 coincide. Then $(k_1,j_1)=(k_2,j_2)$, $N_1=N_2$, and F_1 is a multiple of F_2.*

Proof. i) The adelization of F generates a representation $\pi = \pi_1 \oplus \ldots \oplus \pi_r$, where each π_i is a cuspidal, automorphic representation of $G(\mathbb{A})$. The π_i are all near-equivalent, and thus lie in a packet Π_ψ for some $\psi \in \Psi_2(G)$. Since $F \in S_{k,j}(K(N))_{\mathbf{(G)}}$, the parameter ψ is of type **(G)**. The archimedean component of each π_i is the lowest weight representation denoted by $\mathcal{B}_{k,j}$ in [22]. Since F is a paramodular form, the non-archimedean components of each π_i are paramodular, for each prime p. Theorem 1.1 ii) thus implies that all π_i are isomorphic. Since the representations in (1.24) occur with multiplicity one, it follows that there can be only one π_i, i.e., π is irreducible. By definition of parameters of type **(G)**, π lifts to a cuspidal, automorphic representation μ of $\mathrm{GL}(4,\mathbb{A})$. The non-archimedean components of π outside N must be unramified, implying that F is an eigenform for $\mathcal{H}_p$ for all $p \nmid N$. The last statement follows from the fact that the minimal paramodular level of any irreducible, admissible, generic representation of $G(\mathbb{Q}_p)$ coincides with the conductor of the representation; see Corollary 7.5.5 of [19].

ii) For $i=1,2$, let π_i be the cuspidal, automorphic representation of $G(\mathbb{A})$ generated by the adelization of F_i. By hypothesis, π_1 and π_2 are near-equivalent. Therefore, they are elements of the same packet Π_ψ, for some $\psi \in \Psi_2(G)$. Since π_1 and π_2 are both holomorphic at infinity and paramodular at all finite places, it follows that $\pi_1 \cong \pi_2$. By multiplicity one, $\pi_1 = \pi_2$ (as spaces of automorphic forms). In particular, $(k_1,j_1)=(k_2,j_2)$. The conductor of π_i is N_i by i), and hence $N_1 = N_2$. Since the adelization of F_i is a pure tensor consisting of distinguished vectors in the local components of π_i, it follows from local multiplicity one (see Theorem 7.5.1 of [19] for the non-archimedean places, and the uniqueness of the minimal K-type at the archimedean place) that F_1 and F_2 are multiples of each other. □

Corollary 2.7. *Let $(k,j) \in \mathbb{Z}_{\geq 3} \times \mathbb{Z}_{\geq 0}$. Let $F \in S_{k,j}(K(N))$ be an eigenform. For a good place p let $Q_{F,p}(X) = \prod_{j=1}^4 (1-\alpha_{p,j}X)$ be the degree 4 Hecke polynomial of F at p (so that $Q_{F,p}(p^{-s})^{-1}$ is the spin L-factor of F at p). Then the following are equivalent:*

i) $F \in S_{k,j}(K(N))_{\mathbf{(G)}}$.
ii) $|\alpha_{p,j}| = 1$ *for all* $p \nmid N$ *and all* $j \in \{1,\ldots,4\}$.
iii) $|\alpha_{p,j}| = 1$ *for some* $p \nmid N$ *and all* $j \in \{1,\ldots,4\}$.

Proof. i) $\Rightarrow$ ii) Assume that $F \in S_{k,j}(K(N))_{\mathbf{(G)}}$. Let $\pi = \otimes \pi_p$ be the automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$ generated by the adelization of F. Our hypothesis $k \geq 3$ implies that π_∞ is a discrete series representation. By Theorem 3.3 of [24], the Ramanujan conjecture holds at all good places, implying that the Satake parameters of π_p have absolute value 1 for all $p \nmid N$ (since π can be transferred to a cuspidal representation on $\mathrm{GL}(4,\mathbb{A})$, we can alternatively apply the main result of [5]).

Since ii) $\Rightarrow$ iii) is trivial, it remains to prove iii) $\Rightarrow$ i). Hence assume that $|\alpha_{p,j}| = 1$ for some $p \nmid N$ and all $j \in \{1,\ldots,4\}$. By Lemma 2.5, F cannot be of type **(Y)**. The size of the Satake parameters precludes F from being of type **(Q)**, **(P)** or **(B)**; see Table 1. Hence F must be of type **(G)**. □

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Corollary 2.8. *For any integer k, eigenforms in $S_k(\mathrm{Sp}(4,\mathbb{Z}))$ are determined, up to scalars, by almost all of their Hecke eigenvalues.*

Proof. Let $F_1, F_2 \in S_k(\mathrm{Sp}(4,\mathbb{Z}))$ be two eigenforms whose eigenvalues coincide for almost all primes. From their L-functions it is clear that F_1 and F_2 are either both Saito-Kurokawa liftings, or neither one of them is. Assume that they are both Saito-Kurokawa liftings. Then they come from elliptic modular forms $f_1, f_2 \in S_{2k-2}(\mathrm{SL}(2,\mathbb{Z}))$; see Corollary 1 on page 80 of [6]. Strong multiplicity one for GL(2) implies that f_1 is a multiple of f_2. Consequently F_1 is a multiple of F_2. Now assume that F_1 and F_2 are not Saito-Kurokawa liftings. Then Theorem 5.1.2 of [15] implies that $F_1, F_2 \in S_k(\mathrm{Sp}(4,\mathbb{Z}))_{\mathbf{(G)}}$. We can thus apply Theorem 2.6 ii). $\square$

Remark 2.9. Let F be any number field. For $i = 1, 2$ let $\pi_i \cong \otimes \pi_{i,v}$ be a cuspidal, automorphic representation of $G(\mathbb{A}_F)$ of type **(G)** or **(Y)**. Assume that the following holds: $\pi_{1,v} \cong \pi_{2,v}$ for all archimedean v; π_v is paramodular for all finite v; and $\pi_{1,v} \cong \pi_{2,v}$ for almost all finite v. Then $\pi_1 = \pi_2$ as spaces of automorphic forms. The argument is as in the proof of ii) of Theorem 2.6.

2.3. **Paramodular Hecke operators.** Let $F \in S_{k,j}^{\mathrm{new}}(K(N))_{\mathbf{(G)}}$ be an eigenform. By Theorem 2.6 i), F is an eigenform for the local Hecke algebra $\mathcal{H}_p$ for all primes p outside N. We may thus attach to F a partial L-function, defined as an Euler product over all places $p \nmid N$.

In this section we describe a way to attach Euler factors to F at the places p dividing N. The method is based on two paramodular Hecke operators $T_{0,1}(p)$ and $T_{1,0}(p)$. It follows from the local theory of the paramodular group that F is an eigenform under these operators. In general, knowledge of the resulting eigenvalues is not enough to determine the underlying local representation, but it is sufficient to determine the Euler factor.

There is more than one way to attach a (spin) Euler factor $L(s,\pi)$ to an irreducible, admissible representation (π, V) of $\mathrm{GSp}(4,\mathbb{Q}_p)$ with trivial central character. We always understand $L(s,\pi)$ to be the local factor attached to the L-parameter $\phi : L_{\mathbb{Q}_p} \to \mathrm{Sp}(4,\mathbb{C})$ of π, using the local Langlands correspondence of [9]. Assume that π appears in a global representation of type **(G)**, thus transferring to a representation μ of $\mathrm{GL}(4,\mathbb{Q}_p)$ appearing in a global cuspidal representation of $\mathrm{GL}(4,\mathbb{A})$. Since the results of [4] imply that the L-packets of [9] coincide with the L-packets of [2] for SO(5), we have $L(s,\pi) = L(s,\mu)$, the standard L-factor for the representation μ of $\mathrm{GL}(4,\mathbb{Q}_p)$.

For generic π, there is also the spin Euler factor attached to π via the theory of local zeta integrals, as in [23]. It has been verified in [19] for non-supercuspidal, generic π that this Euler factor coincides with the one defined via the local Langlands correspondence. The same is true for generic, supercuspidal π as well, since in this case both types of Euler factors are 1: See Proposition 3.9 of [23] and Sect. 7 of [9]. Since in the following we will apply Theorem 7.5.3 of [19], which makes a statement about $L(s,\pi)$ defined via zeta integrals, it is important to know that this Euler factor coincides with the one defined via the local Langlands correspondence, thus is the correct factor to fit into a global L-function.

Review of some local theory. Let p be any prime. Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4,\mathbb{Q}_p)$ with trivial central character. For a non-negative integer n, let $V(n)$ be the subspace of vectors fixed by the paramodular

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group

$$K(p^n) = \{g \in \mathrm{GSp}(4,\mathbb{Q}_p) \mid \det(g) \in \mathbb{Z}_p^\times\} \cap \begin{bmatrix} \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-n}\mathbb{Z}_p \\ \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix}. \tag{2.10}$$

We assume that π is paramodular, i.e., $V(n) \neq 0$ for some n. Let n_0 be the minimal n for which $V(n) \neq 0$. Then, by the results of [19], n_0 coincides with the (exponent of the) conductor $a(\pi)$ of the representation. Moreover, $\dim V(n_0) = 1$. Therefore, the paramodular Hecke algebra consisting of locally constant, left and right $K(p^{n_0})$-invariant functions, acts on $V(n_0)$ via a character. Consider in particular the elements

(2.11)

$$T_{0,1} = \mathrm{char}(K(p^{n_0}) \begin{bmatrix} p &&& \\ & p && \\ && 1 & \\ &&& 1 \end{bmatrix} K(p^{n_0})), \qquad T_{1,0} = \mathrm{char}(K(p^{n_0}) \begin{bmatrix} p &&& \\ & p^2 && \\ && p & \\ &&& 1 \end{bmatrix} K(p^{n_0})),$$

where "char" means "characteristic function of". In general, the action of the characteristic function T of a double coset $K(p^n)gK(p^n)$, where $g \in \mathrm{GSp}(4,\mathbb{Q}_p)$, on $V(n)$ is given by

$$Tv = \sum_{i=1}^{r} \pi(g_i)v, \qquad \text{if} \quad K(p^n)gK(p^n) = \bigsqcup_{i=1}^{r} g_iK(p^n).$$

Explicit coset representatives for the double cosets in (2.11) are given in Lemma 6.1.2 of [19], provided that $n_0 \geq 1$. In the unramified case $n_0 = 0$ the coset representatives are well known from the classical theory of Siegel modular forms; see (6.5) and (6.6) of [19].

To each π as above are thus associated the two eigenvalues $\lambda_{0,1}$ and $\lambda_{1,0}$ of the Hecke operators $T_{0,1}$, respectively $T_{1,0}$, on the one-dimensional space $V(n_0)$. These eigenvalues can be calculated explicitly for each π; the results are listed in Table A.14 of [19]. We further have the eigenvalue $\varepsilon_\pi = \pm 1$ of the Atkin-Lehner involution

$$u_{n_0} = \begin{bmatrix} &&& -1 \\ && 1 & \\ & p^{n_0} && \\ -p^{n_0} &&& \end{bmatrix} \tag{2.12}$$

on $V(n_0)$. The relevance of these eigenvalues for us is that they determine the spin L-factor of π. More precisely, by Theorems 7.5.3 and 7.5.9 of [19] we have the following:

i) Assume $n_0 = 0$, so that π is unramified. Then

(2.13)

$$L(s,\pi) = \frac{1}{1 - p^{-3/2}\lambda_{0,1}p^{-s} + (p^{-2}\lambda_{1,0} + 1 + p^{-2})p^{-2s} - p^{-3/2}\lambda_{0,1}p^{-3s} + p^{-4s}}.$$

ii) Assume that $n_0 = 1$, and let $\varepsilon_\pi = \pm 1$ be the Atkin-Lehner eigenvalue on $V(n_0)$. Then

$$L(s,\pi) = \frac{1}{1 - p^{-3/2}(\lambda_{0,1} + \varepsilon_\pi)p^{-s} + (p^{-2}\lambda_{1,0} + 1)p^{-2s} + \varepsilon_\pi p^{-1/2}p^{-3s}}. \tag{2.14}$$

iii) Assume $n_0 \geq 2$. Then

$$L(s,\pi) = \frac{1}{1 - p^{-3/2}\lambda_{0,1}p^{-s} + (p^{-2}\lambda_{1,0} + 1)p^{-2s}}. \tag{2.15}$$

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In the following we will translate the local operators $T_{0,1}$ and $T_{1,0}$ into operators on classical Siegel modular forms.

The operators $T_{0,1}$ and $T_{1,0}$ on Siegel modular forms. Let $F \in S_{k,j}(K(N))$ with adelization Φ. Let p be a prime and assume that p^n is the maximal power of p dividing N; we indicate this by writing $p^n \| N$. The case $n = 0$ is allowed. The Hecke operators $T_{0,1}$ and $T_{1,0}$ at the place p act on Φ by right translation by elements of $\mathrm{GSp}(4, \mathbb{Q}_p)$, producing automorphic forms with the same invariance properties. Translating back to functions on the Siegel upper half-space, we obtain elements $T_{0,1}(p)F$ and $T_{1,0}(p)F$ of $S_{k,j}(K(N))$. We thus have endomorphisms $T_{0,1}(p)$ and $T_{1,0}(p)$ for each $p|N$. If $n = 0$, so that $p \nmid N$, then $T_{0,1}(p)$ and $T_{1,0}(p)$ commute, since the unramified local Hecke algebra $\mathcal{H}_p$ is commutative. If $p|N$, then $T_{0,1}(p)$ and $T_{1,0}(p)$ do not in general commute. However, from the local nature of their definition it is clear that two such endomorphisms for different primes commute with each other.

Proposition 2.10. *Let $(k, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$, and let N be a positive integer. Let p be a prime such that $p^n \| N$ with $n \geq 1$. Let M be any integer such that $M(N/p^n) \equiv 1$* mod p. *Then, for $F \in S_{k,j}(K(N))$,*

$$\begin{aligned} T_{0,1}(p)F = &\sum_{x,y,z \in \mathbb{Z}/p\mathbb{Z}} F|_{k,j} \begin{bmatrix} 1&&&\\ &1&&\\ &&p&\\ &&&p \end{bmatrix} \begin{bmatrix} 1&&x&y\\ &1&y&zp^{-n}\\ &&1&\\ &&&1 \end{bmatrix} \\ &+ \sum_{x,z \in \mathbb{Z}/p\mathbb{Z}} F|_{k,j} \begin{bmatrix} p&&&\\ &1&&\\ &&1&\\ &&&p \end{bmatrix} \begin{bmatrix} 1&&&\\ x&1&&zp^{-n}\\ &&1&-x\\ &&&1 \end{bmatrix} \\ &+ \sum_{x,y \in \mathbb{Z}/p\mathbb{Z}} F|_{k,j} \begin{bmatrix} 1&&&\\ &p&&\\ &&p&\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&-yMN&x&\\ &1&&\\ &&1&\\ &&yMN&1 \end{bmatrix} \\ &+ \sum_{x \in \mathbb{Z}/p\mathbb{Z}} F|_{k,j} \begin{bmatrix} p&&&\\ &p&&\\ &&1&\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&&&\\ &1&&\\ &xMN&1&\\ xMN&&&1 \end{bmatrix} \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} T_{1,0}(p)F = &\sum_{x,y \in \mathbb{Z}/p\mathbb{Z}} \sum_{z \in \mathbb{Z}/p^2\mathbb{Z}} F|_{k,j} \begin{bmatrix} p&&&\\ &1&&\\ &&p&\\ &&&p^2 \end{bmatrix} \begin{bmatrix} 1&&&y\\ &1&y&zp^{-n}\\ &&1&\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&&&\\ x&1&&\\ &&1&-x\\ &&&1 \end{bmatrix} \\ &+ \sum_{x,y \in \mathbb{Z}/p\mathbb{Z}} F|_{k,j} \begin{bmatrix} p&&&\\ &p^2&&\\ &&p&\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&&&\\ &1&&\\ &xMN&1&\\ xMN&&&1 \end{bmatrix} \begin{bmatrix} 1&-yMN&&\\ &1&&\\ &&1&\\ &&yMN&1 \end{bmatrix} \\ &+ \sum_{\substack{x,y \in \mathbb{Z}/p\mathbb{Z} \\ z \in (\mathbb{Z}/p\mathbb{Z})^\times}} F|_{k,j} \begin{bmatrix} 1&&&\\ -xp^{-1}&1&&zp^{-n-1}\\ &&1&xp^{-1}\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&-yMN&&\\ &1&&\\ &&1&\\ &&yMN&1 \end{bmatrix}. \end{aligned} \tag{2.17}$$

Proof. This follows in a straightforward way from the double coset decomposition given in Lemma 5.1 of [16]. See Proposition 5.2 of [16] for details in the scalar-valued case; the proof in the vector-valued case is similar. □

The local Atkin-Lehner element (2.12) can also be globalized to an operator on $S_{k,j}(K(N))$; see Sect. 3.3 of [13]. This operator is an involution on $S_{k,j}(K(N))$ which we denote by u_p. For primes $p \nmid N$ it is trivial, and for $p|N$ it splits $S_{k,j}(K(N))$ into ± 1 eigenspaces.

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Lemma 2.11. *Let $(k,j)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geq 0}$, and let N be a positive integer. Suppose that $F\in S^{\rm new}_{k,j}(K(N))_{\mathbf{(G)}}$ is an eigenform (at almost all good places). Then F is an eigenform for $T_{0,1}(p)$ and $T_{1,0}(p)$, and for the Atkin-Lehner involutions u_p, for all primes p.*

Proof. Let $\pi\cong\otimes\pi_p$ be the automorphic representation of $G(\mathbb{A})$ generated by the adelization of F. As explained in the proof of Theorem 2.6, this adelization corresponds to a pure tensor $\otimes v_p$. For finite primes, v_p is the local newform in π_p, i.e., it spans the one-dimensional space of paramodular vectors of the smallest possible level. The corresponding local paramodular Hecke algebra therefore acts by scalars on this space. In particular, v_p is an eigenvector for the local operators $T_{0,1}$, $T_{1,0}$ and u_p. It follows that F is an eigenvector for the corresponding global operators. $\square$

To any eigenform $F\in S^{\rm new}_{k,j}(K(N))_{\mathbf{(G)}}$, and any prime p, we have thus attached three eigenvalues $\lambda_{0,1}(p)$, $\lambda_{1,0}(p)$ and ε_p, defined by

$$T_{0,1}(p)F=\lambda_{0,1}(p)F,\qquad T_{1,0}(p)F=\lambda_{1,0}(p)F,\qquad u_pF=\varepsilon_pF. \tag{2.18}$$

We define local Euler factors $L_p(s,F)$ using the right-hand sides of the formulas (2.13)–(2.15), where n_0 is the maximal power of p dividing N; in (2.14), ε_π is to be replaced by ε_p. If $p\nmid N$, then $L_p(s,F)$ is the usual spin Euler factor of F (in analytic normalization).

Proposition 2.12. *Let $(k,j)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geq 0}$, and let N be a positive integer. Suppose that $F\in S^{\rm new}_{k,j}(K(N))_{\mathbf{(G)}}$ is an eigenform. Let*

$$L(s,F)=\Gamma_{\mathbb{C}}\Big(s+\frac{2k+j-3}{2}\Big)\Gamma_{\mathbb{C}}\Big(s+\frac{j+1}{2}\Big)\prod_{p<\infty}L_p(s,F), \tag{2.19}$$

where $L_p(s,F)$ are the Euler factors defined above. Then $L(s,F)$ has analytic continuation to an entire function, is bounded in vertical strips, and satisfies the functional equation

$$L(s,F)=(-1)^k\Big(\prod_{p|N}\varepsilon_p\Big)L(1-s,F), \tag{2.20}$$

where ε_p are the Atkin-Lehner eigenvalues of F.

Proof. Let $\pi=\otimes\pi_p$ be the automorphic representation of $G(\mathbb{A})$ generated by F. By definition, $L_p(s,F)=L(s,\pi_p)$ for all primes p. The Γ-factors coincide with the L-factor of π_∞; see Proposition 2.5.1 of [22]. Hence $L(s,F)=L(s,\pi)$. Since F is of type **(G)**, $L(s,\pi)$ is the L-function of a self-dual, cuspidal, automorphic representation of $\mathrm{GL}(4,\mathbb{A})$. This proves most of our claims about the analytic properties. The sign in the functional equation is the product of the root numbers $\varepsilon(1/2,\pi_p)$ over all places (they are all ±1 and independent of the choice of additive characters). We have $\varepsilon(1/2,\pi_\infty)=(-1)^k$ by Proposition 2.5.1 of [22]; note here that j is necessarily even. The fact that $\varepsilon(1/2,\pi_p)$ coincides with ε_p, the Atkin-Lehner eigenvalue on the newform, for each prime p is a feature of the local paramodular theory; see Corollary 7.5.5 of [19]. $\square$

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We remark that the product $\prod_{p|N} \varepsilon_p$ coincides with the eigenvalue of F under the ***Fricke involution***

$$\begin{bmatrix} &&&1\\ &&1&\\ &N&&\\ N&&& \end{bmatrix}, \tag{2.21}$$

which normalizes $K(N)$.

The square-free case. As explained above, the Euler factors for an eigenform $F \in S_{k,j}^{\text{new}}(K(N))_{\mathbf{(G)}}$ can be obtained with the help of the paramodular Hecke operators $T_{0,1}$ and $T_{1,0}$. In practice, if F is given in terms of its Fourier expansion, it may still be difficult to calculate the action of these operators, owing to the presence of lower triangular coset representatives in the decompositions given in Proposition 2.10. For an example of how this difficulty can be overcome in certain situations, see Sect. 5 of [16].

In the square-free case, however, all coset representatives can be brought into block upper triangular form, i.e., taken from the Siegel parabolic subgroup $P(\mathbb{Q}_p)$. This is due to the "Iwasawa decomposition"

$$\mathrm{GSp}(4,\mathbb{Q}_p) = P(\mathbb{Q}_p)K(p); \tag{2.22}$$

see Proposition 5.1.2 of [19]. In fact, the matrix identity

$$\begin{bmatrix} 1&&&\\ &1&&\\ &xp&1&\\ xp&&&1 \end{bmatrix}\begin{bmatrix} 1&&&\\ &1&&\\ &&p&\\ &&&p \end{bmatrix} = \begin{bmatrix} 1&&&x^{-1}p^{-1}\\ &1&x^{-1}p^{-1}&\\ &&1&\\ &&&1 \end{bmatrix}\begin{bmatrix} 0&p&&\\ 1&0&&\\ &&0&1\\ &&p&0 \end{bmatrix}\begin{bmatrix} &&-x^{-1}&0\\ &&0&-x^{-1}p^{-1}\\ x&0&0&1\\ 0&px&p&0 \end{bmatrix} \tag{2.23}$$

is all that is needed to replace the lower triangular representatives appearing in Lemma 4.1.1 of [16] by block upper triangular matrices. This leads to the following formulas for the endomorphisms $T_{0,1}(p)$ and $T_{1,0}(p)$ of the space $S_{k,j}(K(p))$, for any prime p:

$$\begin{aligned} T_{0,1}(p)F = &\sum_{x,y,z\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} 1&&&\\ &1&&\\ &&p&\\ &&&p \end{bmatrix}\begin{bmatrix} 1&&x&y\\ &1&y&zp^{-1}\\ &&1&\\ &&&1 \end{bmatrix}\\ &+\sum_{x,z\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} p&&&\\ &1&&\\ &&1&\\ &&&p \end{bmatrix}\begin{bmatrix} 1&&&\\ x&1&&zp^{-1}\\ &&1&-x\\ &&&1 \end{bmatrix} + \sum_{x,y\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} 1&&&\\ &p&&\\ &&p&\\ &&&1 \end{bmatrix}\begin{bmatrix} 1&-yp&x&\\ &1&&\\ &&1&\\ &&yp&1 \end{bmatrix}\\ &+F\Big|\begin{bmatrix} p&&&\\ &p&&\\ &&1&\\ &&&1 \end{bmatrix} + \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^\times} F\Big|\begin{bmatrix} 0&p&&\\ 1&0&&\\ &&0&1\\ &&p&0 \end{bmatrix}\begin{bmatrix} 1&&&xp^{-1}\\ &1&xp^{-1}&\\ &&1&\\ &&&1 \end{bmatrix}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} T_{1,0}(p)F = &\sum_{x,y\in\mathbb{Z}/p\mathbb{Z}}\ \sum_{z\in\mathbb{Z}/p^2\mathbb{Z}} F\Big|\begin{bmatrix} p&&&\\ &1&&\\ &&p&\\ &&&p^2 \end{bmatrix}\begin{bmatrix} 1&&&y\\ &1&y&zp^{-1}\\ &&1&\\ &&&1 \end{bmatrix}\begin{bmatrix} 1&&&\\ x&1&&\\ &&1&-x\\ &&&1 \end{bmatrix}\\ &+\sum_{y\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} p&&&\\ &p^2&&\\ &&p&\\ &&&1 \end{bmatrix}\begin{bmatrix} 1&-yp&&\\ &1&&\\ &&1&\\ &&yp&1 \end{bmatrix}\\ &+\sum_{\substack{y\in\mathbb{Z}/p\mathbb{Z}\\ x\in(\mathbb{Z}/p\mathbb{Z})^\times}} F\Big|\begin{bmatrix} 0&p^2&&\\ 1&0&&\\ &&0&1\\ &&p^2&0 \end{bmatrix}\begin{bmatrix} 1&&&xp^{-1}\\ &1&xp^{-1}&\\ &&1&\\ &&&1 \end{bmatrix}\begin{bmatrix} 1&-yp&&\\ &1&&\\ &&1&\\ &&yp&1 \end{bmatrix}\\ &+\sum_{\substack{x,y\in\mathbb{Z}/p\mathbb{Z}\\ z\in(\mathbb{Z}/p\mathbb{Z})^\times}} F\Big|\begin{bmatrix} 1&&&\\ -xp^{-1}&1&&zp^{-n-1}\\ &&1&xp^{-1}\\ &&&1 \end{bmatrix}\begin{bmatrix} 1&-yp&&\\ &1&&\\ &&1&\\ &&yp&1 \end{bmatrix}. \end{aligned} \tag{2.25}$$

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In case $p||N$ but N has additional prime factors, one has to be more careful in globalizing the middle matrix on the right-hand side of (2.23). One way to do this is by choosing an integer M such that $M(N/p) \equiv 1 \bmod p$, and observing that

$$\begin{bmatrix} 0 & p & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & p & 0 \end{bmatrix} \in \begin{bmatrix} p & & & \\ & -1 & & \\ & & 1 & \\ & & & -p \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & MN/p & & \\ & 1 & & \\ & & 1 & \\ & & -MN/p & 1 \end{bmatrix} K(p).$$

Substituting appropriately, we find that all local representatives can be globalized, meaning they have the property that, as elements of $\mathrm{GSp}(4,\mathbb{Q}_q)$ for $q \neq p$, they lie in $K(q^n)$, where $q^n\|N$. This way we arrive at the formulas

$$\begin{aligned} T_{0,1}(p)F = F|_{k,j} \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix} &+ \sum_{x,y,z\in\mathbb{Z}/p\mathbb{Z}} F|_{k,j} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & y & zp^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ &+ \sum_{x,z\in\mathbb{Z}/p\mathbb{Z}} F|_{k,j} \begin{bmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{bmatrix} \begin{bmatrix} 1 & & & \\ x & 1 & & zp^{-1} \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \\ &+ \sum_{x,y\in\mathbb{Z}/p\mathbb{Z}} F|_{k,j} \begin{bmatrix} 1 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -yMN & x & \\ & 1 & & \\ & & 1 & \\ & & yMN & 1 \end{bmatrix} \\ &+ \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^\times} F|_{k,j} \begin{bmatrix} 1-MN/p & MN & & \\ 1 & -p & & \\ & & p & 1 \\ & & MN & MN/p-1 \end{bmatrix} \begin{bmatrix} 1 & & & xp^{-1} \\ & 1 & xp^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} T_{1,0}(p)F = \sum_{x,y\in\mathbb{Z}/p\mathbb{Z}} \sum_{z\in\mathbb{Z}/p^2\mathbb{Z}} & F|_{k,j} \begin{bmatrix} p & & & \\ & 1 & & \\ & & p & \\ & & & p^2 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & y & zp^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \\ &+ \sum_{y\in\mathbb{Z}/p\mathbb{Z}} F|_{k,j} \begin{bmatrix} p & & & \\ & p^2 & & \\ & & p & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -yMN & & \\ & 1 & & \\ & & 1 & \\ & & yMN & 1 \end{bmatrix} \\ &+ \sum_{\substack{y\in\mathbb{Z}/p\mathbb{Z} \\ x\in(\mathbb{Z}/p\mathbb{Z})^\times}} F|_{k,j} \begin{bmatrix} 1-MN/p & MNp & & \\ 1 & -p^2 & & \\ & & p^2 & 1 \\ & & MNp & MN/p-1 \end{bmatrix} \begin{bmatrix} 1 & & & xp^{-1} \\ & 1 & xp^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ &\qquad \begin{bmatrix} 1 & -yMN & & \\ & 1 & & \\ & & 1 & \\ & & yMN & 1 \end{bmatrix} \\ &+ \sum_{\substack{x,y\in\mathbb{Z}/p\mathbb{Z} \\ z\in(\mathbb{Z}/p\mathbb{Z})^\times}} F|_{k,j} \begin{bmatrix} 1 & & & \\ -xp^{-1} & 1 & & zp^{-n-1} \\ & & 1 & xp^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -yMN & & \\ & 1 & & \\ & & 1 & \\ & & yMN & 1 \end{bmatrix} \end{aligned} \tag{2.27}$$

for $p||N$. These formulas are designed to be easily applicable to the Fourier expansion of an element in $S_{k,j}(K(N))$.

Proposition 2.13. *Let $(k,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$, and let N be a positive integer. Let p be a prime with $p\|N$. Let F be an eigenform in $S_{k,j}^{\mathrm{new}}(K(N))_{(\mathbf{G})}$. Let $\lambda_{0,1}(p)$, $\lambda_{1,0}(p)$ and ε_p be the eigenvalues defined by* (2.18)*. Then*

$$\lambda_{0,1}(p)\varepsilon_p + \lambda_{1,0}(p) + p + 1 = 0. \tag{2.28}$$

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We have $\lambda_{0,1}(p) \neq 0$, so that two of the three quantities $\lambda_{0,1}(p)$, $\lambda_{1,0}(p)$ and ε_p determine the third. The Euler factor at p is given by

(2.29)
$$L_p(s,F) = \frac{1}{1 - p^{-3/2}(\lambda_{0,1}(p) + \varepsilon_p)p^{-s} + (p^{-2}\lambda_{1,0}(p) + 1)p^{-2s} + \varepsilon_p p^{-1/2}p^{-3s}}.$$

Proof. Let $\pi \cong \otimes \pi_{p'}$ be the cuspidal, automorphic representation generated by F. The local component π_p has conductor $a(\pi_p) = 1$, because F is a newform and $p \| N$. Thus π_p, being Iwahori-spherical, occurs among the representations listed in Table A.15 of [19]. Since π is of type **(G)**, weak estimates on Satake parameters show that the only possibility for π_p is the representation $\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ of type IIa, where χ and σ are unramified characters of $\mathbb{Q}_p^\times$ with $\chi^2\sigma^2 = 1$ and $(\chi\sigma)(p) = -\varepsilon_p$. For these one can readily verify the relation (2.28); see the remark at the end of Sect. 7.2 of [19]. It follows easily from the unitary conditions given in Table A.2 of [19] that $\lambda_{0,1}(p) \neq 0$. □

Remarks. a) The polynomial in the denominator of (2.29) factors as

$$(1 - \alpha p^{-s})(1 - \alpha^{-1}p^{-s})(1 + \varepsilon_p p^{-1/2}p^{-s}), \tag{2.30}$$

where $\alpha + \alpha^{-1} = p^{-3/2}(\lambda_{0,1}(p) + \varepsilon_p(p+1))$. The Ramanujan conjecture predicts that $|\alpha| = 1$.

b) The L-factor (2.29) determines the underlying representation $\pi_p = \chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ completely. Knowing this L-factor, we can thus derive additional quantities. For example, the degree 5 L-factor can be read off Table A.10 of [19]. In non-square-free cases, it is not in general possible to determine the degree 5 from the degree 4 L-factor.

Appendix A. L-parameters for non-supercuspidal representations of $\mathrm{GSp}(4,F)$

Let F be a non-archimedean local field of characteristic zero. In this appendix we reproduce, in a modified form, Table A.7 of [19], which lists the L-parameters of all non-supercuspidal representations of $\mathrm{GSp}(4,F)$.

Let W_F be the Weil group of F, and let $L_F = W_F \times \mathrm{SU}(2)$ be the Weil-Deligne group of F. A representation of L_F is a continuous homomorphism $L_F \to \mathrm{GL}(n,\mathbb{C})$ whose restriction to $\mathrm{SU}(2)$ comes from a holomorphic representation of $\mathrm{SL}(2,\mathbb{C})$. Let ν_i be the irreducible representation of $\mathrm{SU}(2)$ of dimension i. Then the irreducible representations of L_F are precisely those of the form $\sigma \boxtimes \nu_i$, where σ is an irreducible representation of W_F.

The dual group of the algebraic F-group $\mathrm{GSp}(4)$ is $\hat{G} = \mathrm{GSp}(4,\mathbb{C})$. An L-parameter for $\mathrm{GSp}(4)$ is a continuous homomorphism $\phi : L_F \to \hat{G}$ such that $\phi(W_F)$ consists of semisimple elements, and such that the restriction of ϕ to $\mathrm{SU}(2)$ comes from a holomorphic representation of $\mathrm{SL}(2,\mathbb{C})$. Each such ϕ is a semisimple, four-dimensional representation of L_F. Two L-parameters are equivalent if they are conjugate by an element of $\hat{G}$.

Table 3 shows the L-parameters associated to all irreducible, admissible, non-supercuspidal representations of $\mathrm{GSp}(4,F)$. Listed are the parameters as four-dimensional representations; not given is the way that these parameters map into $\hat{G}$. For the latter one should consult Table A.7 of [19], which is the basis for Table 3, and in which the L-parameters are given in their "(ρ, N)" form. To translate

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TABLE 3. L-parameters for non-supercuspidal representations of $\mathrm{GSp}(4,F)$.

		representation	ϕ	$\#\mathcal{S}_\phi$	g
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$\chi_1\chi_2\sigma \oplus \chi_1\sigma \oplus \chi_2\sigma \oplus \sigma$	1	•
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2\sigma \oplus (\nu^{-1/2}\chi\sigma \boxtimes \nu_2) \oplus \sigma$	1	•
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2\sigma \oplus \nu^{1/2}\chi\sigma \oplus \nu^{-1/2}\chi\sigma \oplus \sigma$	1	
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	$(\nu^{-1/2}\chi\sigma \boxtimes \nu_2) \oplus (\nu^{-1/2}\sigma \boxtimes \nu_2)$	1	•
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	$\nu^{1/2}\chi\sigma \oplus \nu^{-1/2}\chi\sigma \oplus \nu^{1/2}\sigma \oplus \nu^{-1/2}\sigma$	1	
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^{-3/2}\sigma \boxtimes \nu_4$	1	•
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	$(\nu^{1/2}\sigma \boxtimes \nu_2) \oplus (\nu^{-3/2}\sigma \boxtimes \nu_2)$	1	
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\nu^{3/2}\sigma \oplus (\nu^{-1/2}\sigma \boxtimes \nu_2) \oplus \nu^{-3/2}\sigma$	1	
	d	$\sigma 1_{\mathrm{GSp}(4)}$	$\nu^{3/2}\sigma \oplus \nu^{1/2}\sigma \oplus \nu^{-1/2}\sigma \oplus \nu^{-3/2}\sigma$	1	
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$(\nu^{-1/2}\sigma \boxtimes \nu_2) \oplus (\nu^{-1/2}\xi\sigma \boxtimes \nu_2)$	2	•
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma \oplus (\nu^{-1/2}\xi\sigma \boxtimes \nu_2) \oplus \nu^{-1/2}\sigma$	1	
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$\nu^{1/2}\xi\sigma \oplus (\nu^{-1/2}\sigma \boxtimes \nu_2) \oplus \nu^{-1/2}\xi\sigma$	1	
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma \oplus \nu^{1/2}\xi\sigma \oplus \nu^{-1/2}\xi\sigma \oplus \nu^{-1/2}\sigma$	1	
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	$(\nu^{-1/2}\sigma \boxtimes \nu_2) \oplus (\nu^{-1/2}\sigma \boxtimes \nu_2)$	2	•
	b	$\tau(T, \nu^{-1/2}\sigma)$			
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma \oplus (\nu^{-1/2}\sigma \boxtimes \nu_2) \oplus \nu^{-1/2}\sigma$	1	
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma \oplus \nu^{1/2}\sigma \oplus \nu^{-1/2}\sigma \oplus \nu^{-1/2}\sigma$	1	
VII		$\chi \rtimes \pi$	$\chi\varphi_\pi \oplus \varphi_\pi$	1	•
VIII	a	$\tau(S, \pi)$	$\varphi_\pi \oplus \varphi_\pi$	2	•
	b	$\tau(T, \pi)$			
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	$\nu^{-1/2}\varphi_\pi \boxtimes \nu_2$	1	•
	b	$L(\nu\xi, \nu^{-1/2}\pi)$	$\nu^{1/2}\varphi_\pi \oplus \nu^{-1/2}\varphi_\pi$	1	
X		$\pi \rtimes \sigma$	$\sigma\omega_\pi \oplus \sigma\varphi_\pi \oplus \sigma$	1	•
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\sigma\varphi_\pi \oplus (\nu^{-1/2}\sigma \boxtimes \nu_2)$	2	•
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma \oplus \sigma\varphi_\pi \oplus \nu^{-1/2}\sigma$	1	
Va*		$\delta^*([\xi, \nu\xi], \nu^{-1/2}\sigma)$	same as Va	2	
XIa*		$\delta^*(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	same as XIa	2	

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between the two ways of writing representations of the Weil-Deligne group, see the proposition in Section 6 of [21].

The symbols χ, χ_1, χ_2, σ and ξ in the table denote characters of $F^\times$, and π denotes an irreducible, admissible, supercuspidal representation of $\mathrm{GL}(2, F)$. Often these have to satisfy additional conditions, for which we refer to Table A.1 of [19]. As usual, we identify characters of W_F and of $F^\times$. We simply write σ for the representation $\sigma \boxtimes \nu_1$ of L_F. The symbol ν (not to be confused with ν_i) stands for the normalized absolute value of $F^\times$. The notation for the representations of $\mathrm{GSp}(4, F)$ is explained in Sect. 2.2 of [19].

For any $\phi : L_F \to \mathrm{GSp}(4, \mathbb{C})$ in the table, let S_ϕ be the centralizer of its image, and S_ϕ^0 the identity component of S_ϕ. Let $\mathcal{S}_\phi = S_\phi / S_\phi^0 Z$, where $Z \cong \mathbb{C}^\times$ is the center of $\hat{G}$. The order of this centralizer group is listed in the next-to-last column of Table 3. It is the size of the L-packet associated to ϕ. The last column in the table indicates the generic representations.

In addition to all non-supercuspidal representations, Table 3 also lists two types of supercuspidals, namely Va* and XIa*. The reason they are included is that Va* constitutes a two-element L-packet with Va, and XIa* constitutes a two-element L-packet with XIa. We refer to Sect. 4 of [20] for a construction of Va* and XIa* in terms of the theta correspondence.

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