On the Number of Local Newforms in a Metaplectic Representation*

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Abstract

The nonarchimedean local analogues of modular forms of half-integral weight with level and character are certain vectors in irreducible, admissible, genuine representations of the metaplectic group over a nonarchimedean local field of characteristic zero. Two natural level raising operators act on such vectors, leading to the concepts of oldforms and newforms. We prove that the number of newforms for a given representation and character is finite and equal to the number of square classes with respect to which the representation admits a Whittaker model.

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Let F be a nonarchimedean local field of characteristic zero with Hilbert symbol $(\,\cdot\,,\,\cdot\,)$ and ring of integers $\mathfrak{o}$, let $\mathfrak{p} \subset \mathfrak{o}$ be the maximal ideal of $\mathfrak{o}$, let ϖ be a generator for $\mathfrak{p}$, and fix a character ψ of F with conductor $\mathfrak{o}$. Let $\widetilde{\mathrm{SL}}(2,F)$ be the two-fold cover of $\mathrm{SL}(2,F)$, as defined below. For m and a in $F^\times$ let $\gamma_m(a)$ be the Weil index of ax^2 with respect to ψ^m, and define $\delta_m(a) = (-1,a)\gamma_m(a)\gamma_m(1)^{-1}$. Let (τ, V) be an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$. The center of $\widetilde{\mathrm{SL}}(2,F)$ consists of the four elements

$$(\begin{bmatrix} \varepsilon & \\ & \varepsilon \end{bmatrix}, \varepsilon')$$

where $\varepsilon, \varepsilon' = \pm 1$. Consider the operator

$$\tau(\begin{bmatrix} -1 & \\ & -1 \end{bmatrix}, 1).$$

*This work is a contribution in honor of Stephen Kudla. Kudla's work spans the theory of modular forms, representation theory, and arithmetic geometry. Though it is difficult to emulate, we have been inspired by his dedication to creating theories and making arithmetic applications.

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By Schur's Lemma, this operator acts by a scalar, and the square of this scalar is the Hilbert symbol $(-1,-1)$. Also, $\delta_1(-1)^2 = (-1,-1)$. It follows that there exists $\varepsilon(\tau,\psi) = \pm 1$ such that

$$\tau(\begin{bmatrix} -1 & \\ & -1 \end{bmatrix}, 1) = \varepsilon(\tau,\psi)\delta_1(-1).$$

We let $F_\psi(\tau)$ be the set of a in $F^\times$ such that τ admits a Whittaker model with respect to ψ^a. The group $F^{\times 2}$ acts on $F_\psi(\tau)$. Let χ be a character of $\mathfrak{o}^\times$. For n an integer, we let $V_\psi(\tau,n,\chi)$ be the subspace of vectors v in V such that

$$\tau(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}, 1)v = v \qquad \text{for all } b \text{ in } \mathfrak{o}, \tag{1}$$

$$\tau(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}, 1)v = \delta_1(a)\chi(a)v \qquad \text{for all } a \text{ in } \mathfrak{o}^\times, \tag{2}$$

$$\tau(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, 1)v = v \qquad \text{for all } c \text{ in } \mathfrak{p}^n. \tag{3}$$

We refer to the vectors in the spaces $V_\psi(\tau,n,\chi)$ as *metaplectic vectors*, and say that the vectors in $V_\psi(\tau,n,\chi)$ have level $\mathfrak{p}^n$. Any metaplectic vector of level $\mathfrak{p}^n$ is a metaplectic vector of level $\mathfrak{p}^{n+1}$. That is, the inclusion of $V_\psi(\tau,n,\chi)$ in $V_\psi(\tau,n+1,\chi)$ is a level raising operator. There is another natural level raising operator that takes metaplectic vectors of level $\mathfrak{p}^n$ to metaplectic vectors of level $\mathfrak{p}^{n+2}$. Define

$$\alpha_2 : V_\psi(\tau,n,\chi) \longrightarrow V_\psi(\tau,n+2,\chi)$$

by

$$\alpha_2 v = \tau(\begin{bmatrix} \varpi^{-1} & \\ & \varpi \end{bmatrix}, 1)v. \tag{4}$$

We note that the definition of α_2 does not depend on n. We define the subspace $V_\psi(\tau,n,\chi)_{\rm old}$ of *oldforms* in $V_\psi(\tau,n,\chi)$ as the subspace spanned by the images of vectors of lower level, i.e., as the subspace generated by $V_\psi(\tau,n-1,\chi)$ and $\alpha_2 V_\psi(\tau,n-2,\chi)$. We define

$$V_\psi(\tau,n,\chi)_{\rm new} = V_\psi(\tau,n,\chi)/V_\psi(\tau,n,\chi)_{\rm old}.$$

In this paper we study the dimensions of the spaces $V_\psi(\tau,n,\chi)_{\rm new}$ and prove the following theorem.

Main Theorem. *Let (τ,V) be an irreducible, admissible, genuine representation of $\widetilde{\rm SL}(2,F)$, and let χ be a character of $\mathfrak{o}^\times$. If $\chi(-1) \neq \varepsilon(\tau,\psi)$, then $V_\psi(\tau,n,\chi)$ is zero for all n. Assume that $\chi(-1) = \varepsilon(\tau,\psi)$. The sum $\sum_n \dim V_\psi(\tau,n,\chi)_{\rm new}$ is finite and*

$$\sum_n \dim V_\psi(\tau,n,\chi)_{\rm new} = \#F_\psi(\tau)/F^{\times 2}. \tag{5}$$

This result has a GL(2) analogue. Let (π, V) be a generic, irreducible, admissible representation of $\mathrm{GL}(2,F)$. For n a non-negative integer, let $V(\pi,n)$ be the subspace of vectors v in V that are stabilized by the subgroup of elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of $\mathrm{GL}(2,\mathfrak{o})$ such that $c \equiv 0 \bmod \mathfrak{p}^n$ and $d \equiv 1 \bmod \mathfrak{p}^n$. In this setting, the inclusion of $V(\pi,n)$ in $V(\pi,n+1)$ is again a level raising operator, and there is another level raising operator from $V(\pi,n,\chi)$ to $V(\pi,n+1,\chi)$ that sends v to

$$\pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v.$$

In this GL(2) case, the sum analogous to the sum in the main theorem has value 1, so that there is an essentially unique newform. This GL(2) result is directly analogous to the result of the main theorem because π admits a Whittaker model with respect to ψ^a for all a in $F^\times$.

The result presented here builds on the works of Waldspurger, but also introduces some new ideas. As far as we know, the spaces $V_\psi(\tau,n,\chi)$ for $F=\mathbb{Q}_p$ were first considered in [W2]; some subsequent works that also used these spaces are [BM] and [M]. For the case $F=\mathbb{Q}_p$ it should be possible to deduce the main theorem from results in Waldspurger. However, our approach is more abstract than the approach in [W2]. To prove the main theorem we introduce the concept of *primitive vectors*. Primitive vectors comprise the kernel of a certain projection μ on the union $V_\psi(\tau,\infty,\chi)$ of the spaces $V_\psi(\tau,n,\chi)$, and the dimension of the subspace of primitive vectors is equal to the sum in the main theorem. Proving the main theorem is thus reduced to computing the dimension of the space of primitive vectors. This is achieved by using various models for τ. This method can be deployed in other settings. For example, an analogous argument proves the above mentioned analogue for GL(2), as we explain at the end of this paper.

Our interest in the spaces $V_\psi(\tau,n,\chi)$ stems from our project to understand the subspaces $W_0(n)$ of vectors in irreducible, admissible representations (π,W) of $\mathrm{GSp}(4,F)$ with trivial central character that are stabilized by the groups $\Gamma_0(\mathfrak{p}^n)$ of elements

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

of $\mathrm{GSp}(4,\mathfrak{o})$ with $B \equiv 0 \bmod \mathfrak{p}^n$ (we use the notation from [RS] for GSp(4)). We refer to the elements of $W_0(n)$ as *Siegel vectors*. If (π,W) is a Saito-Kurokawa representation of $\mathrm{GSp}(4,F)$, then the quotient $W_{Z^J,\psi^{-1}}$ of W by the subspace spanned by the vectors $\pi(g)w-\psi(-x)w$ for w in W and g of the form

$$g = \begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for x in F is isomorphic to $\pi_{SW}^{-1} \otimes \tau$, as a representation of the Jacobi group G^J of $\mathrm{GSp}(4, F)$, for some irreducible, admissible, genuine representation (τ, V) of $\widetilde{\mathrm{SL}}(2, F)$. Here, G^J consists of the elements of $\mathrm{GSp}(4, F)$ of the form

$$\begin{bmatrix} 1 & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & 1 \end{bmatrix},$$

and π_{SW}^{-1} is the Siegel-Weil representation of G^J (see [BS] for the definition of π_{SW}^{-1}). Note that the subgroup of g as above is the center of G^J. It turns out that there is a natural connection between Siegel vectors and metaplectic vectors in τ. If the residual characteristic of F is even, then one must additionally consider certain other subspaces $V_{\psi,j}(\tau, n, 1)$ of V, where j varies between 0 and $\mathrm{val}(2)$; the space $V_\psi(\tau, n, 1)$ from above is $V_{\psi,\mathrm{val}(2)}(\tau, n, 1)$. In particular, in the case of even residual characteristic the consideration of unramified Saito-Kurokawa representations leads to the definition of the Kohnen plus space in $V_\psi(\tau, 2\mathrm{val}(2), 1)$. We plan to return to these topics in subsequent publications.

1 Background

In this section we gather some necessary basic definitions and results about the underlying field, the metaplectic group $\widetilde{\mathrm{SL}}(2, F)$, and representations of $\widetilde{\mathrm{SL}}(2, F)$. Throughout this paper, F is a nonarchimedean local field of characteristic zero with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$ in $\mathfrak{o}$, and Hilbert symbol $(\,\cdot\,,\,\cdot\,)$. Let ϖ be a generator of $\mathfrak{p}$, and let q be the order of $\mathfrak{o}/\mathfrak{p}$. We will use the absolute value $|\cdot|$ on F such that $|\varpi| = 1/q$. Fix a character ψ of F with conductor $\mathfrak{o}$, i.e., $\psi(\mathfrak{o}) = 1$ but $\psi(\mathfrak{p}^{-1}) \neq 1$. We will always use the Haar measure on F that assigns $\mathfrak{o}$ volume 1. If ξ is in $F^\times$, then we define an associated character χ_ξ of $F^\times$ by

$$\chi_\xi(x) = (\xi, x)$$

for x in $F^\times$. If $n = 0$ we take $1 + \mathfrak{p}^n$ to be $\mathfrak{o}^\times$.

Number theory

1.1 Lemma. *Assume that F has even residual characteristic.*

i) The map $\mathfrak{o}/\mathfrak{p} \to \mathfrak{o}/\mathfrak{p}$ sending x to $x^2 + x$ is a group homomorphism and is 2-to-1.

ii) Let a be in $\mathfrak{o}$. The congruence $a \equiv x^2 + x \bmod \mathfrak{p}$ has a solution if and only if the equation $a = x^2 + x$ has a solution in $\mathfrak{o}$.

iii) The group $(1 + 4\mathfrak{o})/(1 + 2\mathfrak{o})^2$ has two elements. By i), there exist a in $\mathfrak{o}$ such that the congruence $a \equiv x^2 + x \bmod \mathfrak{p}$ has no solution, and for any such a the element $1 + 4a$ is a representative for the non-trivial coset of $(1 + 4\mathfrak{o})/(1 + 2\mathfrak{o})^2$.

iv) If a in $\mathfrak{o}$ is such that $a \equiv x^2+x \mod \mathfrak{p}$ has no solution, then $(\varpi, 1+4a) = -1$.

v) The Hilbert symbol satisfies $(\mathfrak{o}^\times, 1+4\mathfrak{o}) = 1$.

Proof. i) It is easy to check that the map is a group homomorphism. Also, it is easy to see that x and $x+1$ have the same image. Assume that $x^2+x = y^2+y$. Then $x^2 - y^2 + x - y = 0$, i.e., $(x-y)(x+y+1) = 0$. It follows that $x = y$ or $x+y+1 = 0$. The latter is equivalent to $y = x+1$.

ii) Assume that $a \equiv c^2 + c \mod \mathfrak{p}$ for some c in $\mathfrak{o}$. Let $f(X) = X^2 + X - a$. Then $|f(c)| < |f'(c)|^2$. By Hensel's Lemma, there exists y in $\mathfrak{o}$ such that $f(y) = 0$.

iii) Let a be any element of $\mathfrak{o}$ such that $a \equiv x^2+x \mod \mathfrak{p}$ has no solution; by i), such an a exists. We need to prove that 1 and $1+4a$ represent all the distinct cosets in $(1+4\mathfrak{o})/(1+2\mathfrak{o})^2$. It is easy to see that they represent distinct cosets. Let b in $\mathfrak{o}$ be such that $1+4b$ is not in $(1+2\mathfrak{o})^2$. Then the identity $(1+2x)^2 = 1+4(x^2+x)$ implies that the equation $b = x^2+x$ has no solution in $\mathfrak{o}$. By ii), the congruence $b \equiv x^2+x \mod \mathfrak{p}$ has no solution. By i), the congruence $b-a \equiv x^2+x \mod \mathfrak{p}$ has a solution. By ii), there exists x in $\mathfrak{o}$ such that $b - a = x^2 + x$. Hence

$$1+4b = (1+4a)\big(1+4\frac{x^2+x}{1+4a}\big).$$

We have

$$\frac{x^2+x}{1+4a} \equiv x^2+x \mod \mathfrak{p}.$$

Therefore, by ii), there exists y in $\mathfrak{o}$ such that

$$\frac{x^2+x}{1+4a} = y^2+y.$$

Hence

$$1+4b = (1+4a)\big(1+4(y^2+y)\big) = (1+4a)(1+2y)^2.$$

This proves iii).

iv) Let a in $\mathfrak{o}$ be such that $a \equiv x^2+x$ has no solution mod $\mathfrak{p}$; clearly, a is in $\mathfrak{o}^\times$. Assume that $(1+4a, \varpi) = 1$; we will obtain a contradiction. By the definition of the Hilbert symbol, there exist x and y in F such that

$$x^2 - (1+4a)y^2 = \varpi.$$

Since the valuation on the right side is odd, x and y must have the same valuation. Write $x = \varpi^k x'$ and $y = \varpi^k y'$ with k in $\mathbb{Z}$ and x' and y' in $\mathfrak{o}^\times$. Then

$$x'^2 - (1+4a)y'^2 = \varpi^{1-2k}.$$

Assume that $1-2k < 2\mathrm{val}(2)$. Then it follows from $(x'-y')(x'+y') = u\varpi^{1-2k} + 4ay'^2$ that $\mathrm{val}(x'-y')+\mathrm{val}(x'+y') = 1-2k$. Now $x'+y' = x'-y'+2y'$. Therefore, if $\mathrm{val}(x'-y') \geq \mathrm{val}(2)$, then $\mathrm{val}(x'+y') \geq \mathrm{val}(2)$, and consequently $1-2k \geq 2\mathrm{val}(2)$, a contradiction. Hence, $\mathrm{val}(x'-y') < \mathrm{val}(2)$. But then $\mathrm{val}(x'+y') = \mathrm{val}(x'-y')$, so that $\mathrm{val}(x'-y') + \mathrm{val}(x'+y')$ is an even number; this is also a contradiction.

Thus, $1-2k \geq 2\mathrm{val}(2)$, and then in fact $1-2k > 2\mathrm{val}(2)$. Again, $(x'-y')(x'+y') = u\varpi^{1-2k} + 4ay'^2$; this now implies that $\mathrm{val}(x'-y') + \mathrm{val}(x'+y') = 2\mathrm{val}(2)$. Using again $x'+y' = x'-y'+2y'$, we see that necessarily $\mathrm{val}(x'+y') \geq \mathrm{val}(2)$ and $\mathrm{val}(x'-y') \geq \mathrm{val}(2)$, and indeed $\mathrm{val}(x'+y') = \mathrm{val}(x'-y') = \mathrm{val}(2)$. Write $x'-y' = 2w$ with $w \in \mathfrak{o}^\times$. Then $2w(2w+2y') = \varpi^{1-2k} + 4ay'^2$, which implies $w(w+y') \equiv ay'^2 \mod \mathfrak{p}$. Hence $a \equiv (\frac{w}{y'})^2 + \frac{w}{y'} \mod \mathfrak{p}$, contradicting the choice of a.

v) Let v be in $\mathfrak{o}^\times$. Let a in $\mathfrak{o}$ be such that $a \equiv x^2 + x \mod \mathfrak{p}$ has no solution. Such an a exists by i). By iii), to prove that $(v, 1+4\mathfrak{o}) = 1$ it suffices to prove that $(v, 1+4a) = 1$. Now by iv) we have $(\varpi, 1+4a) = (v\varpi, 1+4a) = -1$. Therefore, $(v, 1+4a) = (v\varpi^2, 1+4a) = (-1)(-1) = 1$. □

1.2 Lemma. *The following statements hold about the Hilbert symbol of F.*

i) Every element of $1+4\varpi\mathfrak{o}$ is a square, so that $(F^\times, 1+4\varpi\mathfrak{o}) = 1$.

ii) $(\mathfrak{o}^\times, (1+4\mathfrak{o}) \cap \mathfrak{o}^\times) = 1$.

iii) $(\varpi, (1+4\mathfrak{o}) \cap \mathfrak{o}^\times) \neq 1$.

Proof. i) Let a be in $\mathfrak{o}$ and define $f(X) = X^2 - (1+4\varpi a)$. Then $|f(1)| = |4\varpi a| < |2|^2 = |f'(1)|^2$. By Hensel's Lemma, the equation $f(X) = 0$ has a solution in $\mathfrak{o}$.

ii) If the residual characteristic of F is odd, then the assertion is $(\mathfrak{o}^\times, \mathfrak{o}^\times) = 1$, which is well-known. If the residual characteristic of F is even, this is v) of Lemma 1.1.

iii) If the residual characteristic of F is odd, then the assertion is $(\varpi, \mathfrak{o}^\times) \neq 1$, which is well-known. If the residual characteristic of F is even, then this follows from iv) of Lemma 1.1. □

The cocycle

In this paper we define $\widetilde{\mathrm{SL}}(2,F)$ using the same cocycle c as is commonly used in [G], [W1], [W2] and [W3] (though c is denoted by β in these works). The cocycle c is a Borel measurable function

$$c : \mathrm{SL}(2,F) \times \mathrm{SL}(2,F) \to \{\pm 1\}$$

such that

$$c(g_1 g_2, g_3) c(g_1, g_2) = c(g_1, g_2 g_3) c(g_2, g_3) \tag{6}$$

for g_1, g_2 and g_3 in $\mathrm{SL}(2,F)$, and $c(g,1) = c(1,g) = 1$ for g in $\mathrm{SL}(2,F)$. As a set $\widetilde{\mathrm{SL}}(2,F) = \mathrm{SL}(2,F) \times \{\pm 1\}$, and the group law for $\widetilde{\mathrm{SL}}(2,F)$ is

$$(g, \varepsilon)(g', \varepsilon') = (gg', \varepsilon\varepsilon' c(g, g'))$$

for g and g' in $\mathrm{SL}(2,F)$ and ε and ε' equal to ± 1. Explicitly, c is given by the formula

$$c(g, g') = (x(g), x(g'))(-x(g)x(g'), x(gg'))s(g)s(g')s(gg'),$$

where

$$x(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0 \end{cases}$$

and

$$s(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{cases} (c,d) & \text{if } cd \neq 0 \text{ and } \mathrm{val}(c) \text{ is odd}, \\ 1 & \text{otherwise.} \end{cases}$$

It is known that $c(K^4, K^4) = 1$, where

$$K^4 = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathfrak{o}) : a \equiv 1 \ (4\mathfrak{o}), \ c \equiv 0 \ (4\mathfrak{o})\}.$$

Thus, if the residual characteristic of F is odd, then $K^4 = \mathrm{SL}(2,\mathfrak{o})$. The subset $K^4 \times \{1\}$ is a subgroup of $\widetilde{\mathrm{SL}}(2,F)$. Calculations show that the center of $\widetilde{\mathrm{SL}}(2,F)$ consists of the elements

$$(\begin{bmatrix} \varepsilon & \\ & \varepsilon \end{bmatrix}, \varepsilon')$$

for ε and ε' equal to ± 1.

The factor $\delta_m(a)$

For m in $F^\times$ define the character $\psi^m : F \to \mathbb{C}^\times$ by $\psi^m(x) = \psi(mx)$, where ψ is our fixed character of F. If m and a are in $F^\times$ then we let $\gamma_m(a)$ denote the *Weil index* of the quadratic form ax^2 on F with respect to ψ^m, as defined in paragraph 24, page 172 of [Weil]. By paragraph 27, page 175 of [Weil], one has

$$\gamma_m(a) = \frac{\lim\limits_{n\to\infty} \int\limits_{\mathfrak{p}^{-n}} \psi^m(ax^2)\, dx}{|\lim\limits_{n\to\infty} \int\limits_{\mathfrak{p}^{-n}} \psi^m(ax^2)\, dx|}. \tag{7}$$

From this formula it follows that $\gamma_m(a) = \gamma_{ma}(1) = \gamma_1(ma)$, $\gamma_{mb^2}(a) = \gamma_m(a)$ and $\gamma_m(ab^2) = \gamma_m(a)$ for a, b and m in $F^\times$. We define

$$\delta_m(a) = (a,-1)\gamma_m(a)\gamma_m(1)^{-1}.$$

The number $\delta_m(a)$ is written as $\chi_{\psi^m}(a)$ in [W3], page 223 and in [W1], page 4, and is denoted by $(a,-1)\gamma_F(a,\psi^m)$ in [Rao], page 367. It is proven in paragraph 28, page 176 of [Weil] (this is the formula on the bottom of this page if one uses Proposition 3 of [Weil], page 172) that

$$\delta_m(ab) = (a,b)\delta_m(a)\delta_m(b) \tag{8}$$

for a, b and m in $F^\times$. From this, and other properties of the Weil index, one can prove that the following hold for a, c and m in $F^\times$:

$$\begin{aligned}
\delta_m(c^2a) &= \delta_m(a),\\
\delta_{mc}(a) &= (a,c)\delta_m(a),\\
\delta_m(-1) &= (-1,-1)\gamma_m(1)^{-2},\\
\delta_m(a)^{-1} &= (a,-1)\delta_m(a) = \delta_{-m}(a),\\
\delta_m(a)^4 &= 1,\\
\gamma_m(1)^8 &= 1.
\end{aligned}$$

1.3 Lemma. *We have* $\delta_1((1+4\mathfrak{o})\cap\mathfrak{o}^\times)=1$.

Proof. We will first prove that

$$\gamma_1(a) = \frac{\sum\limits_{z\in\mathfrak{o}/\mathfrak{p}^{\mathrm{val}(2a)}} \psi(az^2\varpi^{-2\mathrm{val}(2a)})}{\Big|\sum\limits_{z\in\mathfrak{o}/\mathfrak{p}^{\mathrm{val}(2a)}} \psi(az^2\varpi^{-2\mathrm{val}(2a)})\Big|} \tag{9}$$

for all non-zero a in $\mathfrak{o}$. Fix a non-zero element a of $\mathfrak{o}$. Let n be a positive integer. Using that ψ has conductor $\mathfrak{o}$ we have

$$\begin{aligned}
\int\limits_{\mathfrak{p}^{-n}} \psi(ax^2)\,dx &= \sum_{z\in\mathfrak{p}^{-n}/\mathfrak{o}} \int\limits_{\mathfrak{o}} \psi(a(x+z)^2)\,dx\\
&= \sum_{z\in\mathfrak{o}/\mathfrak{p}^n} \int\limits_{\mathfrak{o}} \psi(a(x+z\varpi^{-n})^2)\,dx\\
&= \sum_{z\in\mathfrak{o}/\mathfrak{p}^n} \int\limits_{\mathfrak{o}} \psi(a(2xz\varpi^{-n}+z^2\varpi^{-2n}))\,dx\\
&= \sum_{z\in\mathfrak{o}/\mathfrak{p}^n} \psi(az^2\varpi^{-2n}) \int\limits_{\mathfrak{o}} \psi(2axz\varpi^{-n})\,dx\\
&= \sum_{\substack{z\in\mathfrak{o}/\mathfrak{p}^n,\\ \mathrm{val}(2az\varpi^{-n})\ge 0}} \psi(az^2\varpi^{-2n})\\
&= \sum_{z\in\mathfrak{p}^{n-\mathrm{val}(2a)}/\mathfrak{p}^n} \psi(az^2\varpi^{-2n})\\
&= \sum_{z\in\mathfrak{o}/\mathfrak{p}^{\mathrm{val}(2a)}} \psi(az^2\varpi^{-2\mathrm{val}(2a)}).
\end{aligned}$$

The statement (9) now follows from (7). Now let a be in $(1+4\mathfrak{o})\cap\mathfrak{o}^\times$. The formula (9) shows that $\gamma_1(a)=\gamma_1(1)$. We now have $\delta_1(a)=(a,-1)\gamma_1(a)\gamma_1(1)^{-1}=(a,-1)=1$ by ii) of Lemma 1.2. □

Representation theory

If (τ, V) is a representation of $\widetilde{\mathrm{SL}}(2,F)$, then we say that τ is *genuine* if $\tau(1,\varepsilon)v = \varepsilon v$ for $\varepsilon = \pm 1$. We say that τ is *smooth* if for every v in V there exists a compact, open subgroup Γ of $\mathrm{SL}(2,F)$ such that $\tau(k,1)v = v$ for k in Γ. We say that τ is *admissible* if τ is smooth and for any compact, open subgroup Γ of $\mathrm{SL}(2,F)$ the subspace of v in V such that $\tau(k,1)v = v$ is finite-dimensional.

Weil representations

We will use the *Weil representation* π_W^m of $\widetilde{\mathrm{SL}}(2,F)$ on the space of locally constant, compactly supported functions $\mathcal{S}(F)$ on F associated to the quadratic form $q(x) = x^2$ and ψ^m. This is as defined as on pages 3–4 of [W1] and page 223 of [W3], though our notation is different. If f is in $\mathcal{S}(F)$, b and x are in F and a is in $F^\times$, then

$$(\pi_W^m(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}, \varepsilon)f)(x) = \varepsilon\psi(mbx^2)f(x), \tag{10}$$

$$(\pi_W^m(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}, \varepsilon)f)(x) = \varepsilon\delta_1(a)(m,a)|a|^{1/2}f(ax), \tag{11}$$

$$\pi_W^m(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \varepsilon)f = \varepsilon\gamma_m(1)\hat{f}. \tag{12}$$

Here, the Fourier transform is given by formula

$$\hat{f}(x) = q^{-\mathrm{val}(2m)/2}\int_F \psi(2mxy)f(y)\,dy,$$

Note, as is the case throughout this paper, we use the Haar measure on F that assigns $\mathfrak{o}$ volume 1. Let $\mathcal{S}(F)^+$ and $\mathcal{S}(F)^-$ be the space of even and odd Schwartz functions, respectively. These are invariant, irreducible subspaces for π_W^m. We denote the representation of $\widetilde{\mathrm{SL}}(2,F)$ on $\mathcal{S}(F)^\pm$ by $\pi_W^{m\pm}$. If m and b are in $F^\times$ then $\pi_W^{mb^2\pm} \cong \pi_W^{m\pm}$.

Principal series representations

The *principal series representations* of $\widetilde{\mathrm{SL}}(2,F)$ are defined as follows. Let α be a character of $F^\times$. We let $\tilde{\mathcal{B}}(\alpha)$ denote the complex vector space of all functions $f : \widetilde{\mathrm{SL}}(2,F) \to \mathbb{C}$ satisfying the following two conditions. First,

$$f(\begin{bmatrix} a & b \\ & a^{-1} \end{bmatrix}, \varepsilon)x) = \varepsilon\delta_1(a)\alpha(a)|a|f(x)$$

for a in $F^\times$, b in F and x in $\widetilde{\mathrm{SL}}(2,F)$; second, there exists a compact, open subgroup Γ of $\mathrm{SL}(2,F)$ such that $f(x(k,1)) = f(x)$ for x in $\widetilde{\mathrm{SL}}(2,F)$ and k in Γ. The group $\widetilde{\mathrm{SL}}(2,F)$ acts on $\tilde{\mathcal{B}}(\alpha)$ by right translation, and defines an admissible, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$. This representation is irreducible if and only if $\alpha^2 \neq |\cdot|^{\pm 1}$. If this representation is irreducible, then we denote the representation by $\tilde{\pi}(\alpha)$.

Twisting

There is a right action of $F^\times$ on $\widetilde{\mathrm{SL}}(2,F)$ that is defined as follows. For each element ξ in $F^\times$ define an automorphism of $\mathrm{SL}(2,F)$ by sending g to g^ξ, where

$$g^\xi = \begin{bmatrix} 1 & \\ & \xi^{-1} \end{bmatrix} g \begin{bmatrix} 1 & \\ & \xi \end{bmatrix}.$$

One can prove that if ξ is in $F^\times$, then there is a unique automorphism of $\widetilde{\mathrm{SL}}(2,F)$, also denoted by $x \mapsto x^\xi$, such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm1\} & \longrightarrow & \widetilde{\mathrm{SL}}(2,F) & \longrightarrow & \mathrm{SL}(2,F) & \longrightarrow & 1 \\ & & {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle x\mapsto x^\xi} & & \big\downarrow{\scriptstyle g\mapsto g^\xi} & & \\ 1 & \longrightarrow & \{\pm1\} & \longrightarrow & \widetilde{\mathrm{SL}}(2,F) & \longrightarrow & \mathrm{SL}(2,F) & \longrightarrow & 1. \end{array}$$

This implies that there exists a function $v : F^\times \times \mathrm{SL}(2,F) \to \{\pm1\}$ such that $(g,\varepsilon)^\xi = (g^\xi, v(\xi,g)\varepsilon)$ for g in $\mathrm{SL}(2,F)$ and ε in $\{\pm1\}$. One can prove that

$$v(\xi,g) = \begin{cases} (\xi,a) & \text{if } g = \begin{bmatrix} a & b \\ & a^{-1} \end{bmatrix}, \\ s(g^\xi)s(g) & \text{if } g \text{ is not of the form } \begin{bmatrix} * & * \\ & * \end{bmatrix} \end{cases} \tag{13}$$

for ξ in $F^\times$ and g in $\mathrm{SL}(2,F)$. For further reference, we note that if ξ is in $F^\times$ and (g,ε) is in $\widetilde{\mathrm{SL}}(2,F)$, then a computation proves that

$$(g,\varepsilon)^{\xi^2} = (\begin{bmatrix} \xi & \\ & \xi^{-1} \end{bmatrix}, 1)(g,\varepsilon)(\begin{bmatrix} \xi & \\ & \xi^{-1} \end{bmatrix}, 1)^{-1}. \tag{14}$$

Using this right action of $F^\times$ on $\widetilde{\mathrm{SL}}(2,F)$ we can define a left action of $F^\times$ on the set of representations of $\widetilde{\mathrm{SL}}(2,F)$. Let ξ be in $F^\times$ and let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$. We define the representation $\xi\cdot\tau$ to have the same space as τ, with action given by $(\xi\cdot\tau)(x) = \tau(x^\xi)$ for ξ in $F^\times$ and x in $\widetilde{\mathrm{SL}}(2,F)$. Computations using (13) show that the following formulas hold for a in $F^\times$ and b and c in F:

$$(\xi\cdot\tau)(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}, 1) = \chi_\xi(a)\tau(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}, 1), \tag{15}$$

$$(\xi\cdot\tau)(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}, 1) = \tau(\begin{bmatrix} 1 & b\xi \\ & 1 \end{bmatrix}, 1), \tag{16}$$

$$(\xi\cdot\tau)(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, 1) = \tau(\begin{bmatrix} 1 & \\ c\xi^{-1} & 1 \end{bmatrix}, 1). \tag{17}$$

Finally, by (14), we have $\xi^2\cdot\tau \cong \tau$: the subgroup $F^{\times 2}$ acts trivially on the equivalence classes of representations of $\widetilde{\mathrm{SL}}(2,F)$.

The Kirillov-type model of Waldspurger

Let π be an infinite-dimensional, unitary, irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character. Let $\tau = \theta(\pi,\psi)$ be the representation of $\widetilde{\mathrm{SL}}(2,F)$ defined in [W3], pages 228–231. This is an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$. The work [W1] proves the existence of a certain model for τ, which is discussed in Assertion 7, page 396, of [W2] and on pages 228–229 of [W3]. The assertion about this model is as follows. Let χ be a character of $F^\times$ such that $\chi(-1) = \varepsilon(\tau,\psi)$. Then there exists a space $\mathcal{M}(\tau)$ of functions $f : F^\times \to \mathbb{C}$ and an action of $\widetilde{\mathrm{SL}}(2,F)$ on $\mathcal{M}(\tau)$ such that, with this action, $\mathcal{M}(\tau)$ is isomorphic to τ. Moreover, $\mathcal{M}(\tau)$ and the action have the following properties:

i) The functions in $\mathcal{M}(\tau)$ are locally constant, have relatively compact support in F, and are supported in $F_\psi(\tau)$. The space $\mathcal{S}(F_\psi(\tau))$ of locally constant, compactly supported functions on $F_\psi(\tau)$ is contained in $\mathcal{M}(\tau)$.

ii) For f in $\mathcal{M}(\tau)$, n in F and x in $F^\times$ we have

$$\tau(\begin{bmatrix} 1 & n \\ & 1 \end{bmatrix}, 1)f(x) = \psi(nx)f(x).$$

iii) For f in $\mathcal{M}(\tau)$, a in $F^\times$ and x in $F^\times$ we have

$$\tau(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}, 1)f(x) = \delta_1(a)|a|^{1/2}\chi(a)f(a^2x).$$

We note that the discussion on pages 228–229 of [W3] mentions the set $F(\pi)$ instead of $F_\psi(\tau)$ as the domain of the elements of $\mathcal{M}(\tau)$; however, these sets are the same by 1) of Lemme 6 of [W3], page 234.

2 Basic observations

Let (τ, V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. In this section we answer two basic questions about the spaces $V_\psi(\tau,n,\chi)$. The first three lemmas determine the general conditions on χ and n that must be satisfied for $V_\psi(\tau,n,\chi)$ to be non-zero. We will prove that if $V_\psi(\tau,n,\chi)$ is non-zero then $n \geq 2\mathrm{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$.

2.1 Lemma. *Let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. Assume that the space $V_\psi(\tau, 2\mathrm{val}(2), \chi)$ is non-zero. Then χ is trivial on $(1+4\mathfrak{o}) \cap \mathfrak{o}^\times$.*

Proof. Let v be a non-zero vector in $V_\psi(\tau, 2\mathrm{val}(2), \chi)$. Let x be in F, let y be in

$F^\times$ and assume that $1+xy$ is in $F^\times$. A computation shows that

$$\begin{aligned}&(\begin{bmatrix}1&\\y&1\end{bmatrix},1)(\begin{bmatrix}1&x\\&1\end{bmatrix},1)\\&=(\begin{bmatrix}(1+xy)^{-1}&\\&1+xy\end{bmatrix},1)(\begin{bmatrix}1&x(1+xy)\\&1\end{bmatrix},1)(\begin{bmatrix}1&\\(1+xy)^{-1}y&1\end{bmatrix},(-y,1+xy)).\end{aligned}\tag{18}$$

Now set $y=4$ and assume that x is in $\mathfrak{o}$ and $1+4x$ is in $\mathfrak{o}^\times$. Applying both sides of (18) to v, we find that $1=(-4,1+4x)\chi(1+4x)^{-1}\delta_1((1+4x)^{-1})=(-1,1+4x)\chi(1+4x)^{-1}\delta_1(1+4x)$. By Lemma 1.3 we have $\delta_1(1+4x)=1$ and by ii) of Lemma 1.2 we have $(-1,1+4x)=1$, so that $\chi(1+4x)=1$ for all $x\in\mathfrak{o}$ such that $1+4x$ is in $\mathfrak{o}^\times$. □

2.2 Lemma. *Let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. The space $V_\psi(\tau,2\mathrm{val}(2)-1,\chi)$ is zero.*

Proof. Assume that $V_\psi(\tau,2\mathrm{val}(2)-1,\chi)$ contains a non-zero vector v; we will obtain a contradiction. Let x in $\mathfrak{p}$ and y in $4\varpi^{-1}\mathfrak{o}$ with y non-zero be such that $1+xy$ is in $\mathfrak{o}^\times$, so that $1+xy$ is in $(1+4\mathfrak{o})\cap\mathfrak{o}^\times$. Applying both sides of (18) to v, we get $1=(-y,1+xy)\chi(1+xy)^{-1}\delta_1(1+xy)$. By Lemma 2.1 we have $\chi(1+xy)=1$; by Lemma 1.3 we have $\delta_1(1+xy)=1$. Therefore, $(-y,1+xy)=1$ for all x in $\mathfrak{p}$ and non-zero y in $4\varpi^{-1}\mathfrak{o}$ such that $1+xy$ is in $\mathfrak{o}^\times$. Letting y be $-4\varpi^{-1}$ and x be $-\varpi b$ where b is in $\mathfrak{o}$, we find that $(\varpi,1+4b)=1$ for all b in $\mathfrak{o}$ such that $1+4b$ is in $\mathfrak{o}^\times$. In other words, $(\varpi,(1+4\mathfrak{o})\cap\mathfrak{o}^\times)=1$. This contradicts iii) of Lemma 1.2. □

2.3 Lemma. *Let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$, let χ be a character of $\mathfrak{o}^\times$ and let n be an integer. Assume that $V_\psi(\tau,n,\chi)$ is non-zero. Then $n\geq 2\mathrm{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$.*

Proof. Let v be a non-zero vector in $V_\psi(\tau,n,\chi)$. By Lemma 2.2 we have $n\geq 2\mathrm{val}(2)$. We may assume $n>2\mathrm{val}(2)$, since the case $n=2\mathrm{val}(2)$ is Lemma 2.1. Let x be in $\mathfrak{o}$ and y in $\mathfrak{p}^n$ with y non-zero. Applying both sides of (18) to v we obtain $1=(-y,1+xy)\chi(1+xy)^{-1}\delta_1((1+xy)^{-1})$. By i) of Lemma 1.2 we have $(-y,1+xy)=1$, and by Lemma 1.3 we have $\delta_1(1+xy)=1$. Hence, $1=\chi(1+xy)$. The lemma follows. □

The second question that we deal with in this section concerns an alternative characterization of the spaces $V_\psi(\tau,n,\chi)$. To formulate the question, assume that $V_\psi(\tau,n,\chi)$ is non-zero. By Lemma 2.3 we know that $n\geq 2\mathrm{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$. Define

$$\tilde{\Gamma}_0(\mathfrak{p}^n)=\Gamma_0(\mathfrak{p}^n)\times\{\pm1\}\tag{19}$$

where $\Gamma_0(\mathfrak{p}^n)$ is the subgroup of $\mathrm{SL}(2,\mathfrak{o})$ of elements with lower left entries in $\mathfrak{p}^n$. The set $\tilde{\Gamma}_0(\mathfrak{p}^n)$ is a subgroup of $\widetilde{\mathrm{SL}}(2,F)$. Moreover, the group $\tilde{\Gamma}_0(\mathfrak{p}^n)$ is generated by $(1,\pm1)$ and the elements of $\widetilde{\mathrm{SL}}(2,F)$ that appear in (1), (2) and (3). It follows that for every element (k,ε) of $\tilde\Gamma_0(\mathfrak{p}^n)$ there exists an element $\tilde\chi(k,\varepsilon)$ of $\mathbb{C}^\times$ such that

$$\tau(k,\varepsilon)v = \tilde\chi(k,\varepsilon)v \tag{20}$$

for all v in $V_\psi(\tau,n,\chi)$. Evidently, the function that sends (k,ε) to $\tilde\chi(k,\varepsilon)$ is a character of $\tilde\Gamma_0(\mathfrak{p}^n)$. The next two results determine the formula for the character $\tilde\chi$ on an arbitrary element of $\tilde\Gamma_0(\mathfrak{p}^n)$. Though we will not need this formula to prove the main theorem, we include it because it may be of some use in other investigations. For example, this formula is essential for determining explicit information about metaplectic vectors in principal series representations if the residual characteristic of F is even.

2.4 Lemma. *Let χ be a character of $\mathfrak{o}^\times$ and let n be an integer such that $n \geq 2\mathrm{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$. Define a function $f : \tilde\Gamma_0(\mathfrak{p}^n) \to \mathbb{C}^\times$ in the following way. If $n = 0$, then define $f(k,\varepsilon) = \varepsilon$. If n is positive, then define*

$$f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \varepsilon) = \varepsilon y(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\chi(d)^{-1}\delta_1(d)$$

where $y : \Gamma_0(\mathfrak{p}) \to \{\pm 1\}$ is given by

$$y(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{cases} 1 & \text{if } c = 0, \\ (d,-1) & \text{if } c \neq 0 \text{ and } \mathrm{val}(c) \text{ is odd}, \\ (d,-c) & \text{if } c \neq 0 \text{ and } \mathrm{val}(c) \text{ is even.} \end{cases} \tag{21}$$

The function f is a character of $\tilde\Gamma_0(\mathfrak{p}^n)$.

Proof. If F has odd residual characteristic, then it is straightforward to verify that f is a character: note that in this case the cocycle c is trivial on $\Gamma_0(\mathfrak{p}^n)$, the function y is constantly 1, and δ_1 is 1 on $\mathfrak{o}^\times$ by Lemma 1.3. Assume that F has even residual characteristic, and let

$$k = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad k' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

be in $\Gamma_0(\mathfrak{p}^n)$. Since we are assuming that F has even residual characteristic, the integer n is positive and a, d, a' and d' are in $\mathfrak{o}^\times$. We have to prove that

$$f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, 1) f(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, 1) = f((\begin{bmatrix} a & b \\ c & d \end{bmatrix}, 1)(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, 1)).$$

Using the definition of f and (8) this is equivalent to

$$y(k)y(k') = y(kk')c(k,k')(d,d'). \tag{22}$$

Using the definitions and the formula for the cocycle, some computations show that (22) is true if $c = 0$ or $c' = 0$. Assume that $c \neq 0$ and $c' \neq 0$. The formulas for y and the cocycle imply that, in general,

$$y(\begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} g) = y(g) \qquad \text{and} \qquad c(\begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} g, g') = c(g,g').$$

We may therefore assume that $b=0$. In other words, we are reduced to proving that

$$y(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix})y(k') = y(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}k')c(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, k')(a, d'). \quad (23)$$

Now (22) has already been verified in general for upper triangular k. Applying this observation to the first term on the left hand side and the first term on the right hand side, using the cocycle property (6), and using the $(\mathfrak{o}^\times, 1+4\mathfrak{o}) = 1$ rule of ii) of Lemma 1.2, we find that (23) reduces to

$$y(k') = y(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}k')c(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, k'). \quad (24)$$

Writing $k' = \begin{bmatrix} a'd' & b'd'^{-1} \\ c'd' & 1 \end{bmatrix}\begin{bmatrix} d'^{-1} & \\ & d' \end{bmatrix}$ and using a similar argument, (24) reduces to

$$1 = y(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & 1 \end{bmatrix})c(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & 1 \end{bmatrix}) = y(\begin{bmatrix} a' & b' \\ c'+ca' & 1+cb' \end{bmatrix})c(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & 1 \end{bmatrix}). \quad (25)$$

Assume that $c'+ca'=0$. Then (25) is equivalent to

$$1 = (c, c')(-cc', 1+cb'). \quad (26)$$

Consider the second Hilbert symbol. Since $c' = -ca'$, $\mathrm{val}(-cc')$ is even. Hence, the second Hilbert symbol is 1 because of the $(\mathfrak{o}^\times, 1+4\mathfrak{o}) = 1$ rule. Using the determinant condition $a' - b'c' = 1$ and $c' + ca' = 0$, we get $c' = -(1+b'c)^{-1}c$. Therefore, $(c, c') = (c, -(1+b'c)^{-1}c) = (c, 1+b'c)$. If $\mathrm{val}(c) = 2\mathrm{val}(2)$, this is of the form $(\mathfrak{o}^\times, 1+4\mathfrak{o}) = 1$. If $\mathrm{val}(c) > 2\mathrm{val}(2)$, then $(c, 1+b'c) = 1$ by the $(F^\times, 1+4\varpi\mathfrak{o}) = 1$ rule of i) of Lemma 1.2, so that $(c, c') = 1$. Hence $(c, c') = 1$ in both cases, and (26) is verified.

Assume that $\mathrm{val}(c'+ca')$ is non-zero. Applying the definitions of y and c shows that (25) is equivalent to

$$1 = (1+cb', -1)(c, c')(-cc', c'+ca')(c'+ca', 1+cb'). \quad (27)$$

The first Hilbert symbol is 1 by the $(\mathfrak{o}^\times, 1+4\mathfrak{o}) = 1$ rule. Using the determinant condition $a' - b'c' = 1$ to eliminate a', we get

$$1 = (c, c')(-cc', c+c'+cc'b')(1+cb', c+c'+cc'b'). \quad (28)$$

Assume that $\mathrm{val}(c') > \mathrm{val}(c)$. Then $c+c'+cc'b' = c(1+c^{-1}c')(1+\frac{c'b'}{1+c^{-1}c'})$. Since $\mathrm{val}(c') > \mathrm{val}(c) \geq 2\mathrm{val}(2)$, we have $(1+cb', 1+\frac{c'b'}{1+c^{-1}c'}) = 1$ by the $(F^\times, 1+4\varpi\mathfrak{o}) = 1$ rule; also, $1+c^{-1}c'$ is in $\mathfrak{o}^\times$. Hence, we have to show

$$1 = (c, c')(-cc', c(1+c^{-1}c'))(1+cb', c(1+c^{-1}c')), \quad (29)$$

which is

$$1 = (-cc', 1+c^{-1}c')(1+cb', c(1+c^{-1}c')). \quad (30)$$

The first Hilbert symbol is 1 since $(x, 1-x) = 1$ for all x in F such that x and $1-x$ are in $F^\times$. Hence we are reduced to

$$1 = (1 + cb', 1 + c^{-1}c')(1 + cb', c). \tag{31}$$

The first Hilbert symbol is 1 by the $(\mathfrak{o}^\times, 1 + 4\mathfrak{o}) = 1$ rule. If $\mathrm{val}(c) > 2\mathrm{val}(2)$, then the second Hilbert symbol is also 1 by the $(F^\times, 1 + 4\varpi\mathfrak{o}) = 1$ rule. If $\mathrm{val}(c) = 2\mathrm{val}(2)$, which is even, then the second Hilbert symbol is 1 by the $(\mathfrak{o}^\times, 1+4\mathfrak{o}) = 1$ rule. Hence (31) is verified.

Now assume that $\mathrm{val}(c) > \mathrm{val}(c')$. Then $c+c'+cc'b' = c'(1+c'^{-1}c)(1+\frac{cb'}{1+c'^{-1}c})$. Again, $1 + c'^{-1}c$ is in $\mathfrak{o}^\times$ and $1 + \frac{cb'}{1+c'^{-1}c}$ is in $1 + 4\varpi\mathfrak{o}$. Hence we have to show

$$1 = (c, c')(-cc', c'(1 + c'^{-1}c))(1 + cb', c'(1 + c'^{-1}c)), \tag{32}$$

which is

$$1 = (-cc', 1 + c'^{-1}c)(1 + cb', c'(1 + c'^{-1}c)). \tag{33}$$

The first Hilbert symbol is 1 by the $(x, 1-x) = 1$ rule. Hence we are reduced to

$$1 = (1 + cb', 1 + c'^{-1}c)(1 + cb', c'). \tag{34}$$

The first Hilbert symbol is 1 by the $(\mathfrak{o}^\times, 1 + 4\mathfrak{o}) = 1$ rule. Since $\mathrm{val}(c) > 2\mathrm{val}(2)$, the element $1 + cb'$ is in $1 + 4\varpi\mathfrak{o}$, and again the second Hilbert symbol is also 1 by the $(F^\times, 1 + 4\varpi\mathfrak{o}) = 1$ rule. Hence (34) is verified.

Finally, assume that $\mathrm{val}(c) = \mathrm{val}(c')$. Write $c = u\varpi^k$ and $c' = v\varpi^k$ with u and v in $\mathfrak{o}^\times$ and $k \geq n \geq 2\mathrm{val}(2)$. Then (28) is equivalent to

$$1 = (u, v)(-uv, u + v + uvb'\varpi^k)(1 + u\varpi^k b', \varpi^k)(1 + u\varpi^k b', u + v + uvb'\varpi^k). \tag{35}$$

If $k > 2\mathrm{val}(2)$, this simplifies to

$$1 = (-uv, 1 + u^{-1}v + vb'\varpi^k). \tag{36}$$

But

$$\begin{aligned}(-uv, 1 + u^{-1}v + vb'\varpi^k) &= (-u^{-1}v, 1 + u^{-1}v + vb'\varpi^k)\\ &\qquad \cdot (1 + ub'\varpi^k, 1 + u^{-1}v + vb'\varpi^k)\\ &= (-u^{-1}v - vb'\varpi^k, 1 + u^{-1}v + vb'\varpi^k)\\ &= 1\end{aligned}$$

by the $(1 - x, x) = 1$ rule. Hence (35) is verified if $k > 2\mathrm{val}(2)$. Assume now that $k = 2\mathrm{val}(2)$, so that in particular k is even. Then (35) is equivalent to

$$1 = (-uv, 1 + u^{-1}v + vb'\varpi^k)(1 + u\varpi^k b', 1 + u^{-1}v + vb'\varpi^k). \tag{37}$$

If $u + v$ is in $\mathfrak{o}^\times$, then this is equivalent to

$$1 = (-uv, 1 + u^{-1}v). \tag{38}$$

This is true by the $(1-x,x)=1$ rule. Assume that $u+v$ is in $\mathfrak{p}$. Write $v=u(-1+w\varpi^t)$ with w in $\mathfrak{o}^\times$ and $t\geq 1$. Substituting $u^{-1}v=-1+w\varpi^t$ and $-uv=u^2(1-w\varpi^t)$ into (37), we get

$$1=(1-w\varpi^t,w\varpi^t+vb'\varpi^k)(1+u\varpi^k b',w\varpi^t+vb'\varpi^k). \tag{39}$$

Since u is in $-v+\mathfrak{p}$, the second Hilbert symbol equals $(1-v\varpi^k b',w\varpi^t+vb'\varpi^k)$ by the $(F^\times,1+4\varpi\mathfrak{o})=1$ rule. Hence (39) is equivalent to

$$1=((1-w\varpi^t)(1-v\varpi^k b'),w\varpi^t+vb'\varpi^k). \tag{40}$$

Multiplying out, we get

$$1=(1-w\varpi^t-vb'\varpi^k+vw\varpi^{t+k}b',w\varpi^t+vb'\varpi^k). \tag{41}$$

The term $vw\varpi^{t+k}b'$ can be omitted by the $(F^\times,1+4\varpi\mathfrak{o})=1$ rule because $t+k>2\mathrm{val}(2)$. Then (41) holds by the $(x,1-x)=1$ rule. This completes the proof. □

2.5 Proposition. *Let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$, let χ be a character of $\mathfrak{o}^\times$, and let n be an integer. If $V_\psi(\tau,n,\chi)$ is non-zero, then $n\geq 2\mathrm{val}(2)$, χ is trivial on $1+\mathfrak{p}^n$, and the character $\tilde\chi$ of $\tilde\Gamma_0(\mathfrak{p}^n)$ defined in (20) is the character f from Lemma 2.4.*

Proof. Assume that $V_\psi(\tau,n,\chi)$ is non-zero. Then $n\geq 2\mathrm{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$ by Lemma 2.3. To prove that $\tilde\chi$ is f it suffices to prove that these two characters agree on the elements in (1), (2) and (3). This follows from the involved formulas. □

3 Proof of the main theorem

In this section we prove the main theorem. We begin with two algebraic reductions. Let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. We define $V_\psi(\tau,\infty,\chi)$ to be the union of all the spaces $V_\psi(\tau,n,\chi)$ as n runs over the integers. The set $V_\psi(\tau,\infty,\chi)$ is a subspace of V because the $V_\psi(\tau,n,\chi)$ are an ascending sequence of vector spaces. Because τ is a smooth representation, a vector v in V is contained in $V_\psi(\tau,\infty,\chi)$ if and only if (1) and (2) hold. We define

$$\alpha_2:V_\psi(\tau,\infty,\chi)\to V_\psi(\tau,\infty,\chi)$$

by the formula (4). For all n, this operator extends the level raising operator α_2 from $V_\psi(\tau,n,\chi)$ to $V_\psi(\tau,n+2,\chi)$. The first reduction proves that the sum from the main theorem can be written in terms of $V_\psi(\tau,\infty,\chi)$.

3.1 Lemma. *Let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. Then*

$$\sum_n \dim V_\psi(\tau,n,\chi)_{\mathrm{new}}=\dim V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi). \tag{42}$$

Proof. It is easy to see that the inclusion maps induce a sequence of inclusions

$$\cdots \hookrightarrow V_\psi(\tau, n-1, \chi)/\alpha_2 V_\psi(\tau, n-3, \chi) \hookrightarrow V_\psi(\tau, n, \chi)/\alpha_2 V_\psi(\tau, n-2, \chi) \hookrightarrow \cdots.$$

If $n \le 2\mathrm{val}(2) - 1$, then the n-th term of the sequence is zero by Lemma 2.2. Also, each of the terms of the sequence is included in $V_\psi(\tau, \infty, \chi)/\alpha_2 V_\psi(\tau, \infty, \chi)$, and the subspace generated by all the images is the entire space $V_\psi(\tau, \infty, \chi)/\alpha_2 V_\psi(\tau, \infty, \chi)$. Since the quotient of the n-th vector space of the sequence by the image of the preceding vector space is $V_\psi(\tau, n, \chi)_{\mathrm{new}}$ we conclude that (42) holds. □

To prove the main theorem we thus need to compute the dimension of the quotient $V_\psi(\tau, \infty, \chi)/\alpha_2 V_\psi(\tau, \infty, \chi)$; the following lemma describes the key property of the elements of $\alpha_2 V_\psi(\tau, \infty, \chi)$ that leads to the second reduction.

3.2 Lemma. *Let (τ, V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2, F)$ and let χ be a character of $\mathfrak{o}^\times$. Let v be in $V_\psi(\tau, \infty, \chi)$. The vector v is in $\alpha_2 V_\psi(\tau, \infty, \chi)$ if and only if v is invariant under the subgroup*

$$(\begin{bmatrix} 1 & \mathfrak{p}^{-2} \\ & 1 \end{bmatrix}, 1) \tag{43}$$

of $\widetilde{\mathrm{SL}}(2, F)$.

Proof. This follows by direct computations. □

The preceding lemma suggests that the subspace $\alpha_2 V_\psi(\tau, \infty, \chi)$ can be characterized as the image of a projection whose kernel would hence be isomorphic to $V_\psi(\tau, \infty, \chi)/\alpha_2 V_\psi(\tau, \infty, \chi)$. Define

$$\mu : V_\psi(\tau, \infty, \chi) \to V_\psi(\tau, \infty, \chi)$$

by

$$\mu v = \frac{1}{q^2} \int\limits_{\mathfrak{p}^{-2}} \tau(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, 1) v \, dx$$

for v in $V_\psi(\tau, \infty, \chi)$. It is straightforward to verify that the operator μ is well-defined. Let $V_{\psi,\mathrm{prim}}(\tau, \infty, \chi)$ be the kernel of μ. We refer to the elements of $V_{\psi,\mathrm{prim}}(\tau, \infty, \chi)$ as *primitive vectors*. The following lemma is the second algebraic reduction.

3.3 Lemma. *Let (τ, V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2, F)$ and let χ be a character of $\mathfrak{o}^\times$. Then $\mu^2 = \mu$, so that $V_\psi(\tau, \infty, \chi) = \ker \mu \oplus \mathrm{im}\, \mu$. Moreover, the image of μ is $\alpha_2 V_\psi(\tau, \infty, \chi)$, so that there is a natural isomorphism*

$$\ker \mu = V_{\psi,\mathrm{prim}}(\tau, \infty, \chi) \xrightarrow{\sim} V_\psi(\tau, \infty, \chi)/\alpha_2 V_\psi(\tau, \infty, \chi).$$

Proof. A direct computation shows that $\mu^2 = \mu$; note that we always use the Haar measure on F that assigns $\mathfrak{o}$ volume 1, and that the volume of $\mathfrak{p}^{-2}$ is q^2. It is clear from the definition of μ that the vectors in the image of μ are invariant under the group (43), so that such vectors are contained in $\alpha_2 V_\psi(\tau, \infty, \chi)$ by Lemma 3.2. Conversely, if v is in $V_\psi(\tau, \infty, \chi)$, then a computation shows that $\mu \alpha_2 v = \alpha_2 v$, so that $\alpha_2 v$ is contained in the image of μ. □

Thanks to Lemma 3.3, the proof of the main theorem has been reduced to the computation of the dimension of the space of primitive vectors, and to make further progress on the proof we will need to use that τ is irreducible. First, however, we need to prove two general technical lemmas. In the following lemma we twist representations of $\widetilde{\mathrm{SL}}(2,F)$ by elements of $F^\times$; see Section 1.

3.4 Lemma. *Let (τ, V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. Let ξ be in $F^\times$. Then*

$$\dim V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi) = \dim V_{\psi,\mathrm{prim}}(\tau,\infty,\chi).$$

Proof. In this proof we will use the projection μ for different representations; the dependence on the representation will be indicated by a subscript. Because $\xi b^2\cdot\tau \cong \xi\cdot\tau$ and $\chi_{\xi b^2} = \chi_\xi$ for ξ and b in $F^\times$, we may assume that $\mathrm{val}(\xi)=0$ or $\mathrm{val}(\xi)=1$. Assume first that $\mathrm{val}(\xi)=0$ so that ξ is in $\mathfrak{o}^\times$. The formulas (15) and (16) show that there is an equality $V_\psi(\xi\cdot\tau,\infty,\chi_\xi\chi) = V_\psi(\tau,\infty,\chi)$ and that $\mu_{\xi\cdot\tau}=\mu_\tau$, so that there is an equality $V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi) = V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$; in particular, these vector spaces have the same dimension. Assume now that $\mathrm{val}(\xi)=1$. To deal with this case we introduce a new operator. Define $\nu_\tau : V_\psi(\tau,\infty,\chi)\to V_\psi(\tau,\infty,\chi)$ by

$$\nu_\tau v = \frac{1}{q}\int\limits_{\mathfrak{p}^{-1}} \tau(\begin{bmatrix}1&x\\&1\end{bmatrix},1)v\,dx.$$

Again, it is straightforward to verify that ν_τ is well-defined. It is evident that μ_τ and ν_τ commute, so that the restriction of ν_τ preserves $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$. The operator ν_τ is also a projection, i.e., $\nu_\tau\nu_\tau=\nu_\tau$. Therefore, if v is in $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$, then $v=(v-\nu_\tau v)+\nu_\tau v$, with $v-\nu_\tau v$ in the space $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)_{\nu_\tau,0}$ and $\nu_\tau v$ in the space $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)_{\nu_\tau,1}$, where $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)_{\nu_\tau,c}$ is the c-eigenspace of ν_τ on $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$. In other words,

$$V_{\psi,\mathrm{prim}}(\tau,\infty,\chi) = V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)_{\nu_\tau,0}\oplus V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)_{\nu_\tau,1}. \tag{44}$$

Similarly,

$$V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi) = V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi)_{\nu_{\xi\cdot\tau},0}\oplus V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi)_{\nu_{\xi\cdot\tau},1}. \tag{45}$$

We claim that there is an equality of vector spaces

$$V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)_{\nu_\tau,0} = V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi)_{\nu_{\xi\cdot\tau},1}. \tag{46}$$

To see this, let v be in the first space. For a in $\mathfrak{o}^\times$ we have by (15)

$$(\xi\cdot\tau)(\begin{bmatrix}a&\\&a^{-1}\end{bmatrix},1)v = \chi_\xi(a)\tau(\begin{bmatrix}a&\\&a^{-1}\end{bmatrix},1)v = \chi_\xi(a)\chi(a)\delta_1(a)v.$$

For b in $\mathfrak{o}$ we have by (16) and $\mathrm{val}(\xi)=1$,

$$(\xi\cdot\tau)(\begin{bmatrix}1&b\\&1\end{bmatrix},1)v = \tau(\begin{bmatrix}1&b\xi\\&1\end{bmatrix},1)v = v.$$

It follows that v is in the space $V_\psi(\xi\cdot\tau,\infty,\chi_\xi\chi)$. Also, by (16),

$$\begin{aligned}\mu_{\xi\cdot\tau}v &= \frac{1}{q^2}\int\limits_{\mathfrak{p}^{-2}}(\xi\cdot\tau)(\begin{bmatrix}1 & x\\ & 1\end{bmatrix},1)v\,dx\\ &= \frac{1}{q^2}\int\limits_{\mathfrak{p}^{-2}}\tau(\begin{bmatrix}1 & x\xi\\ & 1\end{bmatrix},1)v\,dx\\ &= \frac{1}{q}\int\limits_{\mathfrak{p}^{-1}}\tau(\begin{bmatrix}1 & x\\ & 1\end{bmatrix},1)v\,dx\\ &= \nu_\tau v\\ &= 0.\end{aligned}$$

Therefore, v is in $V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi)$. Finally,

$$\begin{aligned}\nu_{\xi\cdot\tau}v &= \frac{1}{q}\int\limits_{\mathfrak{p}^{-1}}(\xi\cdot\tau)(\begin{bmatrix}1 & x\\ & 1\end{bmatrix},1)v\,dx\\ &= \frac{1}{q}\int\limits_{\mathfrak{p}^{-1}}\tau(\begin{bmatrix}1 & x\xi\\ & 1\end{bmatrix},1)v\,dx\\ &= v.\end{aligned}$$

Hence, v is in $V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi)_{\nu_{\xi\cdot\tau},1}$. This shows that the first vector space in (46) is contained in the second vector space. A similar argument proves the opposite inclusion, proving that the two vector spaces are the same. Therefore,

$$\dim V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)_{\nu_\tau,0} = \dim V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi)_{\nu_{\xi\cdot\tau},1}. \tag{47}$$

This equality holds for all τ and χ. Replacing τ by $\xi\cdot\tau$ and χ by $\chi_\xi\chi$ and noting that $\xi^2\cdot\tau\cong\tau$ and $\chi_\xi^2=1$, we also have

$$\dim V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi)_{\nu_{\xi\cdot\tau},0} = \dim V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)_{\nu_\tau,1}. \tag{48}$$

The proof of the lemma is completed by applying (44), (45), (47) and (48). □

The next lemma will be used to transfer information from unitary to non-unitary representations.

3.5 Lemma. *Let (τ_1,V_1) and (τ_2,V_2) be smooth, genuine representations of the group $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. Let $T:V_1\to V_2$ be an isomorphism of vector spaces such that $T(\tau_1(k)v)=\tau_2(k)T(v)$ for v in V_1 and k in the subgroup J of $\widetilde{\mathrm{SL}}(2,F)$ consisting of the elements*

$$(\begin{bmatrix}a & b\varpi^{-2}\\ c\varpi^2 & d\end{bmatrix},\pm1)$$

with a,b,c and d in $\mathfrak{o}$. Then T maps $V_{\psi,\mathrm{prim}}(\tau_1,\infty,\chi)$ onto $V_{\psi,\mathrm{prim}}(\tau_2,\infty,\chi)$ so that these vector spaces have the same dimension.

Proof. The equivariance property of T implies that T maps $V_\psi(\tau_1,\infty,\chi)$ into $V_\psi(\tau_2,\infty,\chi)$. Let μ_{τ_1} and μ_{τ_2} be the μ operators for τ_1 and τ_2, respectively. By the equivariance property of T, we have $T\mu_{\tau_1}v=\mu_{\tau_2}Tv$ for v in $V_\psi(\tau_1,\infty,\chi)$. Therefore, $TV_{\psi,\mathrm{prim}}(\tau_1,\infty,\chi)$ is contained in $V_{\psi,\mathrm{prim}}(\tau_2,\infty,\chi)$. Similarly, the space $T^{-1}V_{\psi,\mathrm{prim}}(\tau_2,\infty,\chi)$ is contained in $V_{\psi,\mathrm{prim}}(\tau_1,\infty,\chi)$. □

We can now begin the proof of the main theorem. The next two lemmas prove the main theorem for two families of representations; by applying the preceding two lemmas, this will lead to a complete proof of the main theorem. In the following lemma we refer to the theta lift $\theta(\pi,\psi)$ of a unitary, generic, irreducible, admissible representation π of $\mathrm{GL}(2,F)$ with trivial central character with respect to our fixed character ψ. This is the representation of $\widetilde{\mathrm{SL}}(2,F)$ defined in [W3], pages 228–231.

3.6 Lemma. *Let π be a unitary, generic, irreducible, admissible representation of* $\mathrm{GL}(2,F)$ *with trivial central character. The main theorem is true for $\tau=\theta(\pi,\psi)$.*

Proof. Assume $\chi(-1)\neq\varepsilon(\tau,\psi)$. Let n be an integer and assume that v is in $V_\psi(\tau,n,\chi)$. Then by the definition of $V_\psi(\tau,n,\chi)$ we have

$$\tau(\begin{bmatrix}-1&\\&-1\end{bmatrix},1)v=\chi(-1)\delta_1(-1)v.$$

On the other hand, by the definition of $\varepsilon(\tau,\psi)$,

$$\tau(\begin{bmatrix}-1&\\&-1\end{bmatrix},1)v=\varepsilon(\tau,\psi)\delta_1(-1)v.$$

Since $\chi(-1)\neq\varepsilon(\tau,\psi)$ we must have $v=0$.

Assume that $\chi(-1)=\varepsilon(\tau,\psi)$. By Lemma 3.1 and Lemma 3.3, it suffices to prove that

$$\dim V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)=\#F_\psi(\tau)/F^{\times 2}.$$

We will use the Kirillov-type model $\mathcal{M}(\tau)$ of τ discovered by Waldspurger; see Section 1. We recall that:

i) The vectors in $\mathcal{M}(\tau)$ are certain functions $f:F^\times\to\mathbb{C}$ that are locally constant, have relatively compact support in F, and are supported in $F_\psi(\tau)$; moreover, the space $\mathcal{S}(F_\psi(\tau))$ of locally constant, compactly supported functions on $F_\psi(\tau)$ is contained in $\mathcal{M}(\tau)$.

ii) For f in $\mathcal{M}(\tau)$, n in F and x in $F^\times$ we have

$$\tau(\begin{bmatrix}1&n\\&1\end{bmatrix},1)f(x)=\psi(nx)f(x).$$

iii) For f in $\mathcal{M}(\tau)$, a in $F^\times$ and x in $F^\times$ we have

$$\tau(\begin{bmatrix}a&\\&a^{-1}\end{bmatrix},1)f(x)=\delta_1(a)|a|^{1/2}\chi(a)f(a^2x).$$

From ii), (1), and the fact that ψ has conductor $\mathfrak{o}$, we see that if f is in $V_\psi(\tau,\infty,\chi)$, then the support of f is contained in $\mathfrak{o}$; from iii) and (2), we see that $f(v^2x)=f(x)$ for all x in $F^\times$ and v in $\mathfrak{o}^\times$. Now let f be in $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$. Then

$$0=\mu f=\frac{1}{q^2}\int\limits_{\mathfrak{p}^{-2}}\tau(\begin{bmatrix}1 & x\\ & 1\end{bmatrix})f\,dx.$$

Hence, for all y in $\mathfrak{o}$,

$$0=(\int\limits_{\mathfrak{p}^{-2}}\psi(xy)\,dx)f(y).$$

Since the conductor of ψ is $\mathfrak{o}$, f is supported on $\mathfrak{o}^\times\sqcup\varpi\mathfrak{o}^\times$. Using this and i), it follows that f is determined by its values on the set

$$\big(F_\psi(\tau)\cap\mathfrak{o}^\times\big)/\mathfrak{o}^{\times 2}\sqcup\big(F_\psi(\tau)\cap\varpi\mathfrak{o}^\times\big)/\mathfrak{o}^{\times 2}. \tag{49}$$

The natural map from this set to $F_\psi(\tau)/F^{\times 2}$ is a bijection. Therefore, the dimension of the vector space $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$ is at most $\#F_\psi(\tau)/F^{\times 2}$. Conversely, suppose that $t\mathfrak{o}^{\times 2}$ is in the set in (49) with t in $F_\psi(\tau)\cap\mathfrak{o}^\times$ or in $F_\psi(\tau)\cap\varpi\mathfrak{o}^\times$. Let $f_{t\mathfrak{o}^{\times 2}}$ be the characteristic function of $t\mathfrak{o}^{\times 2}$. This function lies in the model $\mathcal{M}(\tau)$ by i). Moreover, a calculation shows that $f_{t\mathfrak{o}^{\times 2}}$ is in $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$. The functions $f_{t\mathfrak{o}^{\times 2}}$ as $t\mathfrak{o}^{\times 2}$ varies over the set (49) are linearly independent elements of $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$. Therefore, the dimension of $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$ is at least $\#F_\psi(\tau)/F^{\times 2}$. This completes the proof. □

The next lemma proves the main theorem for the Weil representations π_W^{m+}; see Section 1.

3.7 Lemma. *If m is in $F^\times$, then the main theorem is true for $\tau=\pi_W^{m+}$.*

Proof. The proof that $V_\psi(\pi_W^{m+},n,\chi)$ is zero for all n if $\chi(-1)\neq\varepsilon(\pi_W^{m+},\psi)$ is as in the proof of Lemma 3.6. Assume that $\chi(-1)=\varepsilon(\tau,\psi)$. By (11), this means that $\chi(-1)=(m,-1)$. We may assume that $\mathrm{val}(m)=0$ or $\mathrm{val}(m)=1$ since $\pi_W^{mb^2+}\cong\pi_W^{m+}$ for m and b in $F^\times$. By Proposition 3 of [W1], page 14, $\#F_\psi(\pi_W^{m+})/F^{\times 2}=1$, so that we need to prove that $V_{\psi,\mathrm{prim}}(\pi_W^{m+},\infty,\chi)$ is one-dimensional. Let f be in $V_\psi(\pi_W^{m+},\infty,\chi)$. Then

$$f(y)=(\pi_W^{m+}(\begin{bmatrix}1 & b\\ & 1\end{bmatrix},1)f)(y)=\psi(mby^2)f(y)$$

for all b in $\mathfrak{o}$ and y in F. Since $\mathrm{val}(m)=0$ or $\mathrm{val}(m)=1$ and ψ has conductor $\mathfrak{o}$ we conclude that the support of f lies in $\mathfrak{o}$. Next, for any a in $\mathfrak{o}^\times$ and x in F,

$$\delta_1(a)\chi(a)f(x)=(\pi_W^{m+}(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1)f)(x)=\delta_1(a)(m,a)f(ax),$$

so that

$$f(ax)=(m,a)\chi(a)f(x)$$

for all a in $\mathfrak{o}^\times$ and x in F. Hence f is determined by the values $f(\varpi^k)$ for non-negative integers k. Assume further that f is in $V_{\psi,\mathrm{prim}}(\pi_W^{m+},\infty,\chi)$, i.e., $\mu f = 0$. Then, for all y in F,

$$0 = \Big(\frac{1}{q^2}\int\limits_{\mathfrak{p}^{-2}} (\pi_W^{m+}(\begin{bmatrix} 1 & x \\ & 1\end{bmatrix},1)f)\, dx\Big)(y),$$

$$0 = \Big(\int\limits_{\mathfrak{p}^{-2}} \psi(mxy^2)\, dx\Big) f(y).$$

It follows that $f(y)=0$ for y in $\mathfrak{p}$. Hence f is determined by $f(1)$, and the space $V_{\psi,\mathrm{prim}}(\pi_W^{m+},\infty,\chi)$ is at most one-dimensional. We are thus reduced to proving that $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$ is non-zero. Define an element of $\mathcal{S}(F)$ by

$$f(x) = \begin{cases} (m,x)\chi(x) & \text{if } x \text{ is in } \mathfrak{o}^\times, \\ 0 & \text{if } x \text{ is not in } \mathfrak{o}^\times. \end{cases}$$

Since $\chi(-1) = (m,-1)$, we see that f is an even function, so that f is in the space of π_W^{m+}. Finally, computations using (10) and (11) show that f is contained in $V_{\psi,\mathrm{prim}}(\pi_W^{m+},\infty,\chi)$, completing the proof. $\square$

Finally, we can give the proof of the main theorem.

Proof of the Main Theorem. The proof that $V_\psi(\tau,n,\chi)$ is zero for all n if $\chi(-1)\neq \varepsilon(\tau,\psi)$ is as in the proof of Lemma 3.6. Assume that $\chi(-1)=\varepsilon(\tau,\psi)$. By Lemma 3.1 and Lemma 3.3, it suffices to prove that $\dim V_{\psi,\mathrm{prim}}(\tau,\infty,\chi) = \#F_\psi(\tau)/F^{\times 2}$.

Assume first that τ is an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ that is unitary and is not isomorphic to π_W^{m+} for all m in $F^\times$, that is, τ is in the set $\tilde{P}$ defined on page 225 of [W3]. By Lemme 2 on page 226 of [W3], the set $F_\psi(\tau)$ is not empty, i.e., there is ξ in $F^\times$ such that τ has a ψ^ξ Whittaker model. By Théorème 1 on page 249 of [W3] the theta lift $\pi=\theta(\tau,\psi^\xi)$ to $\mathrm{GL}(2,F)$ is defined; it is a unitary, generic, irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character, and $\theta(\pi,\psi^\xi)=\tau$. Using the definitions and facts about the Weil representation, one can show that for any a in $F^\times$ and non-trivial character ψ' of F one has $a\cdot\theta(\pi,\psi')\cong\theta(\pi,\psi'^a)$. Recalling also that, as mentioned in Section 1, $\xi^2\cdot\tau\cong\tau$ so that $\xi\cdot\tau\cong\xi^{-1}\cdot\tau$, we see that $\xi\cdot\tau\cong\xi^{-1}\cdot\tau\cong\xi^{-1}\cdot\theta(\pi,\psi^\xi)\cong\theta(\pi,\psi)$. A computation using (15) and $\chi(-1)=\varepsilon(\tau,\psi)$ shows that $(\chi_\xi\chi)(-1)=\varepsilon(\xi\cdot\tau,\psi)$. By Lemma 3.6, we have

$$\dim V_{\psi,\mathrm{prim}}(\xi\cdot\tau,\infty,\chi_\xi\chi) = \#F_\psi(\xi\cdot\tau)/F^{\times 2}.$$

Using (16), it is easy to see that $\#F_\psi(\xi\cdot\tau)/F^{\times 2} = \#F_\psi(\tau)/F^{\times 2}$. Applying Lemma 3.4, we now have

$$\dim V_{\psi,\mathrm{prim}}(\tau,\infty,\chi) = \#F_\psi(\tau)/F^{\times 2}, \tag{50}$$

as desired.

By Lemma 3.7, if τ is isomorphic to π_W^{m+} for some m in $F^\times$, then (50) holds.

Finally, assume that τ is not unitary. We will use a "deformation" argument, via Lemma 3.5, to reduce the proof to the unitary case. Since τ is not unitary, τ is isomorphic to an irreducible principal series representation $\tilde{\pi}(\alpha_1)$ for some character α_1 of $F^\times$ such that $\alpha_1^2 \neq |\cdot|^{\pm 1}$. We can find a real number r such that if $\alpha_2 = \alpha_1|\cdot|^r$ then $\alpha_2^2 \neq |\cdot|^{\pm 1}$ and $\tilde{\pi}(\alpha_2)$ is a unitary irreducible principal series representation. We note that α_1 and α_2 agree on $\mathfrak{o}^\times$. Let $\tilde{B}$ be the subgroup of $\widetilde{\mathrm{SL}}(2,F)$ consisting of the elements of the form

$$\tilde{b} = (\begin{bmatrix} a & * \\ & a^{-1} \end{bmatrix}, \varepsilon)$$

for a in $F^\times$ and ε equal to ± 1. If α is a character of F we define a character $\tilde{\alpha}$ of $\tilde{B}$ by

$$\tilde{\alpha}(\tilde{b}) = \varepsilon\delta_1(a)\alpha(a)|a|.$$

This is the character that defines the associated principal series representation $\tilde{\pi}(\alpha)$, i.e., the space of $\tilde{\pi}(\alpha)$ consists of the complex valued functions f on $\widetilde{\mathrm{SL}}(2,F)$ such that $f(\tilde{b}g) = \tilde{\alpha}(\tilde{b})f(g)$ for all $\tilde{b}$ in $\tilde{B}$ and g in $\widetilde{\mathrm{SL}}(2,F)$, and there exists a compact open subgroup Γ of $\mathrm{SL}(2,F)$ such that $f(g(k,1)) = f(g)$ for all g in $\widetilde{\mathrm{SL}}(2,F)$ and all k in Γ. Let J be the subgroup of $\widetilde{\mathrm{SL}}(2,F)$ defined in Lemma 3.5. We have $\widetilde{\mathrm{SL}}(2,F) = \tilde{B}J$. Now define $T : \tilde{\pi}(\alpha_1) \to \tilde{\pi}(\alpha_2)$ by

$$T(f)(g) = T(f)(\tilde{b}\tilde{k}) = \tilde{\alpha}_2(\tilde{b})f(\tilde{k})$$

for g in $\widetilde{\mathrm{SL}}(2,F)$ with $g = \tilde{b}\tilde{k}$ for $\tilde{b}$ in $\tilde{B}$ and $\tilde{k}$ in J. A computation, using that α_1 and α_2 agree on $\mathfrak{o}^\times$, shows that T is well-defined. The map T is an isomorphism because the analogously defined map from $\tilde{\pi}(\alpha_2)$ to $\tilde{\pi}(\alpha_1)$ is the inverse of T. It is also evident that T is a J map. Applying now Lemma 3.5, we have

$$\dim V_{\psi,\mathrm{prim}}(\tilde{\pi}(\alpha_1), \infty, \chi) = \dim V_{\psi,\mathrm{prim}}(\tilde{\pi}(\alpha_2), \infty, \chi).$$

Since $\tilde{\pi}(\alpha_2)$ is unitary, by the first paragraph of this proof,

$$\dim V_{\psi,\mathrm{prim}}(\tilde{\pi}(\alpha_2), \infty, \chi) = \#F_\psi(\tilde{\pi}(\alpha_2)).$$

On the other hand, by Proposition 3 of [W1], page 14,

$$\#F_\psi(\tilde{\pi}(\alpha_2)) = \#F^\times/F^{\times 2} = \#F_\psi(\tilde{\pi}(\alpha_1)).$$

It follows that $\dim V_{\psi,\mathrm{prim}}(\tilde{\pi}(\alpha_1), \infty, \chi) = \#F_\psi(\tilde{\pi}(\alpha_1))$, completing the proof. $\square$

To end this paper we briefly describe how similar reasoning proves the analogous theorem in the GL(2) setting. Let (π, V) be a generic, irreducible, admissible representation of $\mathrm{GL}(2,F)$. For n a non-negative integer, let $V(\pi, n)$ be the subspace of vectors v in V that are stabilized by the subgroup of elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of $\mathrm{GL}(2,\mathfrak{o})$ such that $c \equiv 0 \bmod \mathfrak{p}^n$ and $d \equiv 1 \bmod \mathfrak{p}^n$. Define $\alpha : V(\pi,n) \longrightarrow V(\pi,n+1)$ by

$$\alpha v = \pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v.$$

Define the subspace $V(\pi,n)_{\mathrm{old}}$ of *oldforms* in $V(\pi,n)$ as the subspace spanned by $V(\pi,n-1)$ and $\alpha V(\pi,n-1)$. Our goal is to prove that $\sum_n \dim V(\pi,n)/V(\pi,n)_{\mathrm{old}}$ is one. Define $V(\pi,\infty)$ to be the subspace that is the union of all the spaces $V(\pi,n)$. We have $\sum_n \dim V(\pi,n)/V(\pi,n)_{\mathrm{old}} = \dim V(\pi,\infty)/\alpha V(\pi,\infty)$, as in the $\widetilde{\mathrm{SL}}(2)$ case. Define $\mu_{\mathrm{GL}(2)} : V(\pi,\infty) \to V(\pi,\infty)$ by

$$\mu_{\mathrm{GL}(2)} v = \frac{1}{q} \int\limits_{\mathfrak{p}^{-1}} \pi(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix})v\, dx.$$

The operator $\mu_{\mathrm{GL}(2)}$ is a well-defined projection, and $\ker \mu_{\mathrm{GL}(2)}$ is isomorphic to $V(\pi,\infty)/\alpha V(\pi,\infty)$, so that we are reduced to proving that the space $\ker \mu_{\mathrm{GL}(2)}$ of primitive vectors is one-dimensional. A computation now shows that if the space of π is taken to be the Kirillov model of π with respect to ψ, then the space of primitive vectors is spanned by the characteristic function of $\mathfrak{o}^\times$, which completes the proof. In closing, we note that if π is supercuspidal, then the characteristic function of $\mathfrak{o}^\times$ in the Kirillov model with respect to ψ is the newform of π; the above development shows that this vector is also significant in the non-supercuspidal case.

References

[BM] E.M. Baruch and Z. Mao, *Central value of automorphic L-functions*, Geom. Funct. Anal. **17** (2007), 333–384.

[BS] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Progress in Mathematics **163**, Birkhäuser Verlag, 1998.

[G] S.S. Gelbart, *Weil's representation and the spectrum of the metaplectic group*, Lecture Notes in Mathematics **530**, Springer Verlag, 1967.

[M] Z. Mao, *A generalized Shimura correspondence for newforms*, J. Number Theory **128** (2008), 71–95.

[Rao] R. Ranga Rao, *On some explicit formulas in the theory of Weil representation*, Pacific J. Math. **157** (1993), 335–371.

[RS] B. Roberts and R. Schmidt, *Local newforms for* GSp(4), Lecture Notes in Mathematics **1918**, Springer Verlag, 2007.

[W1] J.-L. Waldspurger, *Correspondance de Shimura*, J. Math. Pures Appl. **59** (1980), 1–132.

[W2] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures Appl. **60** (1981), 375–484.

[W3] J.-L. Waldspurger, *Correspondances de Shimura et quaternions*, Forum Math. **3** (1991), 219–307.

[Weil] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211.