J. Ramanujan Math. Soc. **34,** No.4 (2019) 373–388

On Klingen Eisenstein series with level in degree two

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Communicated by: Prof. Ritabrata Munshi

Received: November 26, 2017

Abstract. We give a representation theoretic approach to the Klingen lift in degree 2, generalizing the classical construction of Klingen Eisenstein series to arbitrary levels for both paramodular and Siegel congruence subgroups.

2000 Mathematics Subject Classification: Primary 11F46.

1. Introduction

Let $M_k^{(n)}$ be the space of holomorphic Siegel modular forms of degree n and weight k with respect to the full modular group $\mathrm{Sp}(2n,\mathbb{Z})$, and let $S_k^{(n)}$ be the subspace of cusp forms. The Siegel Φ operator is a surjective linear map $M_k^{(n)} \to M_k^{(n-1)}$ whose kernel is $S_k^{(n)}$. To prove surjectivity, Klingen [8] defined what is now known as Klingen Eisenstein series. Starting with $f \in S_k^{(r)}$ for $0 \leq r < n$, his construction produces, via a natural summation, an element $E(Z,f) \in M_k^{(n)}$ for which $\Phi^{n-r}E(\cdot,f) = f$. To ensure convergence, one needs to assume that $k > n+r+1$.

A fair amount of work has been done on the Fourier coefficients and Hecke eigenvalues of Klingen Eisenstein series, in particular in degree $n = 2$; see e.g. [1,2,7,10,12,13]. Most of these works consider modular forms with respect to the full modular group. Klingen Eisenstein series with respect to $\Gamma_0^{(n)}(N)$ are employed in Theorem 1.3 of [3] and in the recent work [5].

In this paper we concentrate on the case $n = 2$ and $r = 1$, and investigate Klingen Eisenstein series from a representation theoretic point of view. Assume that $f \in S_k^{(1)}$ is an eigenform. Then there is an associated cuspidal, automorphic representation π of $\mathrm{GL}(2,\mathbb{A})$, which is unramified at all finite places. For a complex parameter s, consider the representation

$$\Pi_s := |\cdot|^s \rtimes |\cdot|^{-s/2}\pi \tag{1.1}$$

of $\mathrm{GSp}(4,\mathbb{A})$. The notation indicates global parabolic induction from the Klingen parabolic subgroup $Q(\mathbb{A})$; see Sect. 2 for details. Given a section ϕ of Π_s, we can construct a corresponding Eisenstein series $E(g,s,\phi)$ in a standard way. Assume that ϕ is a pure tensor $\otimes\phi_p$, where ϕ_p is a spherical vector for each finite p. As we will explain in Sect. 6, if $s = k-2$, then ϕ_∞ can be chosen to be a "holomorphic" vector of weight k. It turns out that the function on the Siegel upper half space $\mathbb{H}_2$ corresponding to $E(g,k-2,\phi)$ is a multiple of the Klingen Eisenstein series $E(\cdot,f)$. See Sect. 5 for the relevant calculation.

The representation theoretic method allows for a generalization to modular forms with arbitrary level and character. Hence, let f be an eigen-newform in $S_k^{(1)}(\Gamma_0(N),\chi)$ for some positive integer N and some Dirichlet character χ mod N. Generalizing (1.1), we consider the induced representation $\Pi_s := \chi^{-1}|\cdot|^s \rtimes |\cdot|^{-s/2}\pi$, where we denote by the same symbol χ the adelization of the Dirichlet character. Even though π has central character χ, the induced representation Π_s always has trivial central character. We may therefore hope to find nice vectors in the local representations $\Pi_{s,p} := \chi_p^{-1}|\cdot|_p^s \rtimes |\cdot|_p^{-s/2}\pi_p$ invariant under the local *paramodular group* of an appropriate level. This level turns out to be the square of the level of π_p. In fact, the local paramodular theory of [19] yields a unique local newform. Descending to the Siegel upper half space, one ends up with the Eisenstein series

$$E(Z,f) = \sum_{\gamma \in \mathrm{L}_N^{-1} Q(\mathbb{Q})\, \mathrm{L}_N \cap \mathrm{K}(N^2)\backslash \mathrm{K}(N^2)} \det(j(\gamma,Z))^{-k} f((\mathrm{L}_N\, \gamma\, \langle Z\rangle)^*), \tag{1.2}$$

where

$$\mathrm{L}_N = \begin{bmatrix} 1 & N & & \\ & 1 & & \\ & & 1 & \\ & & -N & 1 \end{bmatrix}, \tag{1.3}$$

and where

$$Z^* = \tau \quad \text{for an element} \quad Z = \begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix} \in \mathbb{H}_2. \tag{1.4}$$

The local squaring of the level implies that $E(\cdot,f) \in M_k^{(2)}(\mathrm{K}(N^2))$, where $\mathrm{K}(N^2)$ is the paramodular group of level N^2; see (2.2).

The representation theoretic origin of the function $E(\cdot,f)$ implies that it is an eigenfunction for all good Hecke operators. The eigenvalues are most easily encoded in the local Euler factors. In fact, we give the complete L-function of (the underlying representation Π_{k-2} of) $E(\cdot,f)$ in Theorem 6.2. Of course, once the formula (1.2) is established, it applies to all elements of the space of newforms in $S_k^{(1)}(\Gamma_0(N),\chi)$, not only eigenforms.

Starting with a cusp form $f \in S_k^{(1)}(\Gamma_0(N))$, we prove in Sect. 7 that the same automorphic representation Π_{k-2} containing the paramodular form $E(\cdot,f)$ of level N^2 also contains a modular form $E_1(\cdot,f)$ with respect to $\Gamma_0^{(2)}(N)$. This object is less canonical however than the paramodular Klingen lift. For example, in general it is not an Atkin-Lehner eigenfunction, and in particular is not the unique $\Gamma_0^{(2)}(N)$-invariant vector in Π_{k-2}. It is not even clear if there are not $\Gamma_0^{(2)}(M)$-invariant vectors for proper divisors M of N. Also, the construction only works for trivial character χ. This is one of the many instances where paramodular forms are better behaved than $\Gamma_0^{(2)}$ forms. One reason for the different behavior is the more complicated relationship between certain local and global double cosets (cusps) in the $\Gamma_0^{(2)}$ case; see [23] for more information.

Paramodular forms have received a lot of attention in recent years because of their appearance in the *Paramodular Conjecture* formulated in [4] predicting a relationship between abelian surfaces and Siegel modular forms via their L-functions. In Sect. 8 we will review this conjecture and reformulate it in terms of automorphic representations. In this form one may include all abelian surfaces over $\mathbb{Q}$, not only those with trivial endomorphism ring, even though the substance of the conjecture certainly lies with the latter. The connection with Klingen Eisenstein series is that the same type of induced representation (1.1) underlying those Eisenstein series also serves to prove a degenerate case of the Paramodular Conjecture, namely that of abelian surfaces isogenous to a product of two isogenous elliptic curves.

After establishing notations, we explain in Sect. 3 the basic mechanism of obtaining classical Klingen Eisenstein series from certain globally induced representations of the group $\mathrm{GSp}(4,\mathbb{A})$. A lot depends on the choice of local sections in these induced representations, which is the topic of Sect. 4. The main calculation leading from an adelic Eisenstein series to a classical object is carried out in Sect. 5. The main result here is Proposition 5.3, which can in principle be applied to lifting Maass forms as well. We concentrate on the holomorphic case, however, for which we need some additional information at the archimedean place. This information is gathered in Sect. 6, which also contains our main result, Theorem 6.2, on the paramodular Klingen lift. The following section explains the necessary modifications for obtaining a Klingen lift with respect to $\Gamma_0^{(2)}(N)$. Finally, Sect. 8 explains the connection with the Paramodular Conjecture.

2. Notations

We let $\mathrm{GSp}(2n) := \{g \in \mathrm{GL}(2n) \mid {}^tgJg = \lambda(g)J \text{ for some } \lambda(g) \in \mathrm{GL}(1)\}$, with $J = \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix}$. The kernel of the multiplier homomorphism λ is the group $\mathrm{Sp}(2n)$.

Let $K_\infty = \mathrm{GSp}(2n,\mathbb{R}) \cap U(2n)$ and $K'_\infty = \mathrm{Sp}(2n,\mathbb{R}) \cap U(2n)$ be the standard maximal compact subgroups of $\mathrm{GSp}(2n,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{R})$. Then $K'_\infty \cong U(n)$ via $\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \mapsto A + iB$.

Let $\mathrm{GSp}(2n,\mathbb{R})^+$ be the subgroup of $\mathrm{GSp}(2n,\mathbb{R})$ consisting of all elements with positive multiplier. Let $\mathbb{H}_n$ be the Siegel upper half space of degree n, consisting of $n \times n$ complex symmetric matrices whose imaginary part is positive definite. For $m = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GSp}(2n,\mathbb{R})^+$ and $Z \in \mathbb{H}_n$ we use the traditional notations

$$m\langle Z\rangle = (AZ+B)(CZ+D)^{-1}, \quad j(m,Z) = CZ+D.$$

For a function f on $\mathbb{H}_n$, an integer k, and an element $g \in \mathrm{GSp}(2n,\mathbb{R})^+$, let

$$(f\big|_k g)(Z) = \det(g)^{k/2}\det(CZ+D)^{-k} f(g\langle Z\rangle). \tag{2.1}$$

This defines a right action of $\mathrm{GSp}(2n,\mathbb{R})^+$ on functions $f : \mathbb{H}_n \to \mathbb{C}$. The center of $\mathrm{GSp}(2n,\mathbb{R})^+$ acts trivially.

Let $I = iI_n \in \mathbb{H}_n$. Then K'_∞ is the stabilizer of I under the (transitive) action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathbb{H}_n$. Hence $\mathbb{H}_n \cong \mathrm{Sp}(2n,\mathbb{R})/K'_\infty$.

If k is a positive integer and Γ is a congruence subgroup of $\mathrm{Sp}(2n,\mathbb{Q})$, then let $M_k^{(n)}(\Gamma)$ be the space of holomorphic functions $f : \mathbb{H}_n \to \mathbb{C}$ satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma$. If $n = 1$ we impose the additional condition that f is holomorphic at all cusps. Then $M_k^{(n)}(\Gamma)$ is the space of Siegel modular forms of degree n and weight k with respect to Γ. Let $S_k^{(n)}(\Gamma)$ be the subspace of cusp forms.

For a positive integer N, let $\Gamma_0^{(n)}(N) = \left\{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}(2n,\mathbb{Z}) \mid C \equiv 0 \bmod N\right\}$. This is the *Siegel congruence subgroup* of $\mathrm{Sp}(2n,\mathbb{Z})$.

Notations particular to $n = 1$. We write $M_k(\Gamma)$ for $M_k^{(1)}(\Gamma)$ and $S_k(\Gamma)$ for $S_k^{(1)}(\Gamma)$; these are the usual spaces of elliptic modular forms. We write $\Gamma_0(N)$ for $\Gamma_0^{(1)}(N)$. Let $\Gamma_1(N) = \mathrm{SL}(2,\mathbb{Z}) \cap \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & 1+N\mathbb{Z} \end{smallmatrix}\right]$. For a Dirichlet character χ mod N, let $M_k(\Gamma_0(N),\chi)$ (resp. $S_k(\Gamma_0(N),\chi)$) be the space of all f in $M_k(\Gamma_1(N))$ (resp. $S_k(\Gamma_1(N))$) satisfying $f|_k\gamma = \chi(d)f$ for all $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$.

Notations particular to $n = 2$. The Borel subgroup B, the Siegel parabolic P, and the Klingen parabolic Q of $\mathrm{GSp}(4)$ are defined as

$$B = \mathrm{GSp}(4) \cap \begin{bmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}, \quad P = \mathrm{GSp}(4) \cap \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}, \quad Q = \mathrm{GSp}(4) \cap \begin{bmatrix} * & & * & * \\ * & * & * & * \\ * & & * & * \\ & & & * \end{bmatrix}.$$

Let $s_1 := \left[\begin{smallmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{smallmatrix}\right]$ and $s_2 := \left[\begin{smallmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{smallmatrix}\right]$. Then s_1 and s_2 represent elements generating the 8-element Weyl group of $\mathrm{Sp}(4)$.

The *paramodular group* of level N is defined as

$$\mathrm{K}(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}. \tag{2.2}$$

The local version of the paramodular group is

$$\mathrm{K}_p(p^n) = \left\{ g \in \mathrm{GSp}(4,\mathbb{Q}_p) \mid \lambda(g) \in \mathbb{Z}_p^\times,\ g \in \begin{bmatrix} \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-n}\mathbb{Z}_p \\ \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \right\}. \tag{2.3}$$

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. Let $|\cdot|$ be the global absolute value on $\mathbb{A}$. Let δ_Q be the square root of the modulus character of the parabolic $Q(\mathbb{A})$. Explicitly, $\delta_Q(h) = |t^2(ad-bc)^{-1}|$ for

$$h = \begin{bmatrix} a & b & & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} \in Q(\mathbb{A}). \tag{2.4}$$

Using the Iwasawa decomposition, we extend δ_Q to a function on all of $\mathrm{GSp}(4,\mathbb{A})$, by setting

$$\delta_Q(h\kappa) = \delta_Q(h) \quad \text{for} \quad h \in Q(\mathbb{A}) \text{ and } \kappa \in K, \tag{2.5}$$

where K is the standard maximal compact subgroup of $\mathrm{GSp}(4,\mathbb{A})$.

If χ is a character of $\mathbb{A}^\times$, and if (π, V_π) is an admissible representation of $\mathrm{GL}(2,\mathbb{A})$, we denote by $\chi \rtimes \pi$ the representation of $\mathrm{GSp}(4,\mathbb{A})$ obtained via normalized parabolic induction from the representation of $Q(\mathbb{A})$ given by $h \mapsto \chi(t)\pi\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right)$. The standard space of $\chi \rtimes \pi$ consists of smooth functions $\phi : \mathrm{GSp}(4,\mathbb{A}) \to V_\pi$ with the transformation property

$$\phi(hg) = |t^2(ad-bc)|\, \chi(t)\pi\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right)\phi(g) \tag{2.6}$$

for all $g \in \mathrm{GSp}(4,\mathbb{A})$ and $h \in Q(\mathbb{A})$. An analogous definition can be made in the local case.

3. Klingen induction and Eisenstein series

Let π be an irreducible, unitary, cuspidal, automorphic representation of $\mathrm{GL}(2,\mathbb{A})$. Let V_π be the space of cuspidal, automorphic forms on $\mathrm{GL}(2,\mathbb{A})$ realizing π. Abstractly, π factors as a restricted tensor product $\pi \cong \otimes\pi_p$ with irreducible, admissible, unitary representations π_p of $\mathrm{GL}(2,\mathbb{Q}_p)$ for all $p \le \infty$. Let V_p be any model for π_p.

Let χ be the central character of π, and let s be a complex parameter. We consider the global induced representation

$$\Pi_s := \chi^{-1}|\cdot|^s \rtimes |\cdot|^{-s/2}\pi \tag{3.1}$$

of $\mathrm{GSp}(4,\mathbb{A})$. Note that the central character of Π_s is trivial. The space $I(s,\pi)$ of Π_s consists of smooth functions $\phi :\ \mathrm{GSp}(4,\mathbb{A}) \to V_\pi$ with the transformation property

$$\phi(hg) = |t^2\,(ad-bc)^{-1}|^{1+s/2}\,\chi^{-1}(t)\,\pi\left(\begin{bmatrix} a b \\ c d \end{bmatrix}\right)\phi(g) \tag{3.2}$$

for all $g \in \mathrm{GSp}(4,\mathbb{A})$ and all $h \in Q(\mathbb{A})$ as in (2.4). For each place p, a local induced representation $\Pi_{s,p}$, acting on a space $I(s,\pi_p)$, can be defined analogously. If $\pi \cong \otimes\pi_p$ with irreducible, admissible representations π_p of $\mathrm{GL}(2,\mathbb{Q}_p)$, then $\Pi_s \cong \otimes\Pi_{s,p}$ in a canonical way.

For $\phi \in I(s,\pi)$ consider the complex-valued function $\phi_{\mathbb{C}} : \mathrm{GSp}(4,\mathbb{A}) \to \mathbb{C}$ defined by $\phi_{\mathbb{C}}(g) = (\phi(g))(1)$. The map $\phi \mapsto \phi_{\mathbb{C}}$ is injective. Thus, if we denote by $I_{\mathbb{C}}(s,\pi)$ the space of all functions $\phi_{\mathbb{C}}$, where ϕ runs through $I(s,\pi)$, then we obtain another model of Π_s, consisting of a space of complex-valued functions on which $\mathrm{GSp}(4,\mathbb{A})$ acts by right translation.

Let $K = \prod K_p$ be the standard maximal compact subgroup of $\mathrm{GSp}(4,\mathbb{A})$, where $K_\infty = \mathrm{GSp}(4,\mathbb{R}) \cap U(4)$ as above and $K_p = \mathrm{GSp}(4,\mathbb{Z}_p)$ for $p < \infty$. Let ϕ be a fixed K-finite element of $I(0,\pi)$, and let $\phi_s(g) = \delta_Q^{s/2}(g)\phi(g)$; see (2.5), Then $\phi_s \in I(s,\pi)$. We call the familiy ϕ_s a *section*.

For a section $\phi_s \in I(s,\pi)$, originating from $\phi \in I(0,\pi)$, and $g \in \mathrm{GSp}(4,\mathbb{A})$, consider

$$E(g,s,\phi) = \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} \phi_{s,\mathbb{C}}(\gamma g) = \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} (\phi_s(\gamma g))(1). \tag{3.3}$$

By the general theory of Eisenstein series, $E(g,s,\phi)$ converges for $\mathrm{Re}(s)$ large enough and defines an automorphic form on $\mathrm{GSp}(4,\mathbb{A})$.

Suppose that ϕ is chosen in the following way. For each $p < \infty$, let K'_p be a compact-open subgroup of $\mathrm{GSp}(4, \mathbb{Q}_p)$ such that the multiplier map $K'_p \to \mathbb{Z}_p^\times$ is surjective, and such that $K'_p = K_p$ for almost all p. Then

$$\mathrm{GSp}(4, \mathbb{A}) = \mathrm{GSp}(4, \mathbb{Q})\, \mathrm{GSp}(4, \mathbb{R})^+ \prod_{p<\infty} K'_p \tag{3.4}$$

by strong approximation. For $p < \infty$, choose non-zero vectors $\phi_p \in I(0, \pi_p)$ such that ϕ_p is K'_p-invariant for all p. At the archimedean place, let $\phi_\infty \in I(0, \pi_\infty)$ have weight k, i.e.,

$$\pi_\infty(g)\phi_\infty = \eta_k(g)\phi_\infty \quad \text{for} \quad g \in K'_\infty; \tag{3.5}$$

here, $\eta_k(g) = \det(A + iB)^k$ for $g = \left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \in K'_\infty$. Having chosen the vectors ϕ_p for all $p \le \infty$, let ϕ be the element of $I(0, \pi) \cong \otimes I(0, \pi_p)$ corresponding to $\otimes \phi_p$.

For ϕ constructed in this way from local vectors ϕ_p, it follows from (3.4) that the automorphic form $E(\cdot, s, \phi)$ is determined on $\mathrm{GSp}(4, \mathbb{R})^+$. Since the center acts trivially, it is in fact determined on the subgroup $\mathrm{Sp}(4, \mathbb{R})$. It follows from (3.5) that the function

$$\mathrm{Sp}(4, \mathbb{R}) \ni g \mapsto \det(j(g, I))^k E(g, s, \phi)$$

is right-invariant under K'_∞; note here that $\det(j(g, I))^k = \eta_k(g)^{-1}$ for $g \in K'_\infty$. Hence, $E(\cdot, s, \phi)$ is determined by the function on $\mathbb{H}_2 \cong \mathrm{Sp}(4, \mathbb{R})/K'_\infty$ defined by

$$E(Z, s, \phi) := \det(j(g, I))^k E(g, s, \phi), \tag{3.6}$$

where $Z \in \mathbb{H}_2$, and g is any element of $\mathrm{Sp}(4, \mathbb{R})$ for which $g\langle I \rangle = Z$. To be explicit, let

$$Z = \begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix}, \quad \tau = x + iy, \quad z = u + iv, \quad \tau' = x' + iy', \tag{3.7}$$

where x, y, u, v, x', y' are real numbers, $y, y' > 0$, and $yy' - v^2 > 0$. Set

$$b_Z = \begin{bmatrix} 1 & & x & u \\ & 1 & u & x' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ v/y & 1 & & \\ & & 1 & -v/y \\ & & & 1 \end{bmatrix} \begin{bmatrix} b & & & \\ & a & & \\ & & b^{-1} & \\ & & & a^{-1} \end{bmatrix} \tag{3.8}$$

with

$$a = \sqrt{y' - \frac{v^2}{y}} \quad \text{and} \quad b = \sqrt{y}. \tag{3.9}$$

Then $b_Z\langle I \rangle = Z$. Setting $g = b_Z$ in (3.6) gives

$$E(Z, s, \phi) = \det(Y)^{-k/2} E(b_Z, s, \phi), \quad Z = X + iY \in \mathbb{H}_2. \tag{3.10}$$

Let $\Gamma = \mathrm{Sp}(4, \mathbb{Q}) \cap \prod_{p<\infty} K'_p$. It follows easily from (3.6) that

$$E(\gamma\langle Z \rangle, s, \phi) = \det(j(\gamma, Z))^k E(Z, s, \phi) \;\text{ for all }\; \gamma \in \Gamma. \tag{3.11}$$

Hence, in case $E(Z, s, \phi)$ is holomorphic as a function of Z, it is a Siegel modular form of weight k with respect to the congruence subgroup Γ. Our goal is to calculate $E(Z, s, \phi)$ for a specific choice of local vectors ϕ_p.

4. Local sections

We temporarily switch to the local setting, starting with the non-archimedean case. Let p be a prime, and let (π, V) be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2, \mathbb{Q}_p)$. Let $\chi = \omega_\pi$ be the central character of π. Let $a(\pi)$ be the conductor of π, i.e., $a(\pi)$ is the smallest non-negative integer n for which there exists a non-zero vector $v \in V$ with the property

$$\pi(g)v = v \quad \text{for all} \quad g \in \mathrm{GL}(2, \mathbb{Z}_p) \cap \begin{bmatrix} 1 + p^n\mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix}. \tag{4.1}$$

For $n = a(\pi)$ we fix a non-zero vector $v_{\mathrm{new}} \in V$ with the property (4.1); it is called a *local newform*. Consider the induced representation

$$\Pi_s := \chi^{-1}|\cdot|^s \rtimes |\cdot|^{-s/2}\pi \tag{4.2}$$

in its standard model $I(s, \pi)$ consisting of functions $\phi : \mathrm{GSp}(4, \mathbb{Q}_p) \to V$ with the transformation property (3.2). By Theorem 5.4.2 of [19], the minimal paramodular level of Π_0 is $m = 2a(\pi)$; the space of $\mathrm{K}(p^m)$-invariant vectors (see (2.3)) in Π_0 is one-dimensional; a non-zero vector ϕ spanning this space is supported on $Q(\mathbb{Q}_p)\,\mathrm{L}_{a(\pi)}\,\mathrm{K}(p^m)$, and on this double coset is given by

$$\phi(h\,\mathrm{L}_{a(\pi)}\,g) = |t^2\,(ad - bc)^{-1}|\chi^{-1}(t)\,\pi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)v_{\mathrm{new}} \tag{4.3}$$

for $g \in \mathrm{K}(p^m)$ and

$$h = \begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad - bc) \end{bmatrix} \in Q(\mathbb{Q}_p). \tag{4.4}$$

Of course, if $a(\pi) = 0$, then v_{new} is a spherical vector for π, and ϕ is a spherical vector for Π_0. The functions ϕ_s defined by $\phi_s(g) := \delta_Q^{s/2}(g)\phi(g)$ are $\mathrm{GSp}(4, \mathbb{Z}_p)$-finite vectors in $I(s, \pi)$ with the same right-invariance properties.

Remark 4.1. Assume that $\pi = \chi_1 \times \chi_2$ with unitary characters χ_1, χ_2 of $\mathbb{Q}_p^\times$, and that $s \notin \{\pm 1\}$. Then Π_s is irreducible and of type I in the notation of Table A.1 of [19]. Assume that $\pi = \sigma\,\mathrm{St}_{\mathrm{GL}(2)}$ with a unitary character σ of $\mathbb{Q}_p^\times$, and that $\mathrm{Re}(s) \notin \{-2, 0, 2\}$. Then Π_s is irreducible and of type IIIa. Assume that π is supercuspidal and $\mathrm{Re}(s) \notin \{-1, 0, 1\}$. Then Π_s is irreducible and of type VII. In particular, if π is a local component of an automorphic representation attached to a classical cusp form, and if $\mathrm{Re}(s) \notin \{-2, -1, 0, 1, 2\}$, then Π_s is irreducible.

Now we consider the archimedean place. Hence, let π be an irreducible $(\mathfrak{gl}(2, \mathbb{R}), \mathrm{SO}(2))$-module. Assume that the weight $k \in \mathbb{Z}$ appears in π, meaning there exists a non-zero v_∞ in the space of π for which

$$\pi(r(\theta))v_\infty = e^{ik\theta}v_\infty, \quad r(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{4.5}$$

for all $\theta \in \mathbb{R}$.

We choose a globalization of π and let χ be its central character. As before, let $I(s, \pi)$ be the standard space of $\Pi_s := \chi^{-1}|\cdot|^s \rtimes |\cdot|^{-s/2}\pi$.

By the Iwasawa decomposition, we can write any element of $\mathrm{GSp}(4, \mathbb{R})$ as $h\kappa$ with $h \in Q(\mathbb{R})$ and $\kappa \in K'_\infty$. We define $\phi \in I(0, \pi)$ by

$$\phi(h\kappa) = \eta_k(\kappa)|t^2(ad - bc)^{-1}|\chi^{-1}(t)\pi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)v_\infty, \tag{4.6}$$

where $h \in Q(\mathbb{R})$ is written in the form (4.4) and $\kappa \in K'_\infty$. It is straightforward to verify that ϕ is well-defined. By definition, ϕ has weight k in the sense of (3.5). The functions $\phi_s(g) := \delta_Q^{s/2}(g)\phi(g)$ are weight k vectors in $I(s, \pi)$.

5. Calculation of the Eisenstein series

Let $\pi \cong \otimes\pi_p$ be an irreducible, unitary, cuspidal, automorphic representation of GL(2, $\mathbb{A}$) with central character $\chi \cong \otimes\chi_p$. Let $n_p = a(\pi_p)$ be the local conductor at p, and set $N = \prod_p p^{n_p}$. Hence, N is the global conductor of π. Let V_π be the space of automorphic forms realizing π. We fix an isomorphism $\iota : \otimes V_p \to V_\pi$, where V_p is a fixed model of π_p.

We define $I(s, \pi)$ and the local spaces $I(s, \pi_p)$ as before. For a finite prime p, define $\phi_p \in I(0, \pi_p)$ as in (4.3). At the archimedean place, define $\phi_\infty \in I(0, \pi_\infty)$ as in (4.6); here, we assume that π_∞ contains a weight k vector v_∞. Let $\phi = \otimes\phi_p \in I(0, \pi)$, and consider the corresponding Eisenstein series $E(g, s, \phi)$ defined in (3.3).

As explained in Sect. 3, the automorphic form $E(g, s, \phi)$ is determined by the function $E(Z, s, \phi) : \mathbb{H}_2 \to \mathbb{C}$ defined in (3.6). By (3.10),

$$\begin{aligned} E(Z, s, \phi) &= \det(Y)^{-k/2} E(b_Z, s, \phi) \\ &= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} (\phi_s(\gamma\, b_Z))(1). \end{aligned} \tag{5.1}$$

We need the following double coset decompositions. Recall the matrices L_N defined in (1.3).

Lemma 5.1.

(1) *Let p be a prime and m be a non-negative integer. Then*

$$\mathrm{GSp}(4, \mathbb{Q}_p) = \bigsqcup_{j=0}^{m} Q(\mathbb{Q}_p)\, \mathrm{L}_{p^j}\, \mathrm{K}_p(p^m). \tag{5.2}$$

For $x, y \in \mathbb{Z}_p$, we have

$$Q(\mathbb{Q}_p)\, \mathrm{L}_x\, \mathrm{K}_p(p^m) = Q(\mathbb{Q}_p)\, \mathrm{L}_y\, \mathrm{K}_p(p^m) \tag{5.3}$$

if and only if $\min(v_p(x), m) = \min(v_p(y), m)$.

(2) *Let $M = \prod_p p^{m_p}$ be a positive integer. Then,*

$$\mathrm{GSp}(4, \mathbb{Q}) = \bigsqcup_{\substack{\delta \in \mathbb{Z}_{>0} \\ \delta | M}} Q(\mathbb{Q})\, \mathrm{L}_\delta\, \mathrm{K}(M). \tag{5.4}$$

For integers x, y, we have

$$Q(\mathbb{Q})\, \mathrm{L}_x\, \mathrm{K}(M) = Q(\mathbb{Q})\, \mathrm{L}_y\, \mathrm{K}(M) \tag{5.5}$$

if and only if $\min(v_p(x), m_p) = \min(v_p(y), m_p)$ *for all* $p|M$.

Proof. For (1), see Sect. 5.1 of [19]. The global result (2) is due to Reefschläger; see Theorem 1.2 of [16]. □

By (2) of this lemma, each γ in the summation (5.1) is of the form $q\, \mathrm{L}_\delta\, g$ with $q \in Q(\mathbb{Q})$, $\delta | N^2$ and $g \in K(N^2)$. We have

$$\phi_s(\gamma\, b_Z) = \iota\big(\phi_{\infty,s}(\gamma\, b_Z) \otimes \big(\otimes_{p<\infty}\, \phi_{p,s}(\gamma)\big)\big) \in V_\pi.$$

By the definition (4.3) of the local non-archimedean sections, $\phi_{p,s}$ is non-zero if and only if $\mathrm{L}_\delta \in Q(\mathbb{Q}_p)\, \mathrm{L}_{p^{n_p}}\, \mathrm{K}_p(p^{2n_p})$. Using Lemma 5.1 it follows that $\otimes_{p<\infty}\phi_{p,s}(\gamma)$ is non-zero if and only if $\delta = N$. Hence

$$\begin{aligned} E(Z, s, \phi) &= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash Q(\mathbb{Q})\, \mathrm{L}_N\, \mathrm{K}(N^2)} (\phi_s(\gamma\, b_Z))(1) \\ &= \det(Y)^{-k/2} \sum_{\gamma \in D(N)} (\phi_s(\mathrm{L}_N\, \gamma\, b_Z))(1). \end{aligned} \tag{5.6}$$

where $D(N)$ is a set of representatives for $\mathrm{L}_N^{-1}\, Q(\mathbb{Q})\, \mathrm{L}_N \cap \mathrm{K}(N^2)\backslash \mathrm{K}(N^2)$. For $\gamma \in K(N^2)$, we have

$$\begin{aligned}\phi_s(\mathrm{L}_N\,\gamma\, b_Z) &= \iota\big(\phi_{\infty,s}(\mathrm{L}_N\,\gamma\, b_Z) \otimes \big(\otimes_{p<\infty}\,\phi_{p,s}(\mathrm{L}_N\,\gamma)\big)\big)\\ &= \iota\big(\phi_{\infty,s}(\mathrm{L}_N\,\gamma\, b_Z) \otimes \big(\otimes_{p<\infty}\, v_{p,\mathrm{new}}\big)\big)\end{aligned} \tag{5.7}$$

by (4.3). We calculate the archimedean component with the help of the following lemma.

Lemma 5.2. *Let $\delta \in \mathrm{Sp}(2n,\mathbb{R})$ and $Z \in \mathbb{H}_2$. Then $\delta b_Z = b_{\tilde{Z}}\kappa$, where $\tilde{Z} = \delta\langle Z\rangle$ and $\kappa \in K'_\infty$ is such that*

$$\det(j(\kappa, I)) = \sqrt{\frac{\det(\tilde{Y})}{\det(Y)}}\,\det(j(\delta, Z)). \tag{5.8}$$

Proof. The relation $\tilde{Z} = \delta\langle Z\rangle$ follows from applying $\delta b_Z = b_{\tilde{Z}}\kappa$ to I. From $j(\delta b_Z, I) = j(b_{\tilde{Z}}\kappa, I)$ we get

$$j(\delta, Z)\, j(b_Z, I) = j(b_{\tilde{Z}}, I)\, j(\kappa, I).$$

Now (5.8) follows by taking determinants on both sides, observing from (3.8) and (3.9) that $\det(j(b_Z, I)) = \det(Y)^{-1/2}$. □

Applying Lemma 5.2 to $\delta = \mathrm{L}_N\,\gamma$, we get $\mathrm{L}_N\,\gamma\, b_Z = b_{\tilde{Z}}\kappa$ with $\kappa \in K'_\infty$ such that

$$\det(j(\kappa, I)) = \sqrt{\frac{\det(\tilde{Y})}{\det(Y)}}\,\det(j(\mathrm{L}_N\,\gamma, Z)) = \sqrt{\frac{\det(\tilde{Y})}{\det(Y)}}\,\det(j(\gamma, Z)). \tag{5.9}$$

Let

$$b_{\tilde{Z}} = \begin{bmatrix}1 & & \tilde{x} & \tilde{u}\\ & 1 & \tilde{u} & \tilde{x}'\\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ \tilde{v}/\tilde{y} & 1 & & \\ & & 1 & -\tilde{v}/\tilde{y}\\ & & & 1\end{bmatrix}\begin{bmatrix}\tilde{b} & & & \\ & \tilde{a} & & \\ & & \tilde{b}^{-1} & \\ & & & \tilde{a}^{-1}\end{bmatrix}$$

with $\tilde{a}, \tilde{b}$ defined as in (3.9). Then, by (4.6),

$$\begin{aligned}\phi_{\infty,s}(\mathrm{L}_N\,\gamma\, b_Z) &= \phi_{\infty,s}(b_{\tilde{Z}}\kappa)\\ &= \eta_k(\kappa)|\tilde{a}^2|^{1+s/2}\,\pi_\infty\left(\begin{bmatrix}1 & \tilde{x}\\ & 1\end{bmatrix}\begin{bmatrix}\tilde{b} & \\ & \tilde{b}^{-1}\end{bmatrix}\right)v_\infty\\ &= j(\kappa, I)^{-k}\left|\tilde{y}' - \frac{\tilde{v}^2}{\tilde{y}}\right|^{1+s/2}\pi_\infty\left(\begin{bmatrix}1 & \tilde{x}\\ & 1\end{bmatrix}\begin{bmatrix}\tilde{y}^{1/2} & \\ & \tilde{y}^{-1/2}\end{bmatrix}\right)v_\infty\\ &= \left(\frac{\det(\tilde{Y})}{\det(Y)}\right)^{-k/2}\det(j(\gamma, Z))^{-k}\,\frac{\det(\tilde{Y})^{1+s/2}}{\tilde{y}^{1+s/2}}\,\pi_\infty\left(\begin{bmatrix}1 & \tilde{x}\\ & 1\end{bmatrix}\begin{bmatrix}\tilde{y}^{1/2} & \\ & \tilde{y}^{-1/2}\end{bmatrix}\right)v_\infty.\end{aligned}$$

Substituting into (5.7), we get

$$\phi_s(\mathrm{L}_N\,\gamma\, b_Z) = \frac{\det(\tilde{Y})^{1+s/2-k/2}}{\det(Y)^{-k/2}}\,\det(j(\gamma, Z))^{-k}\,\tilde{y}^{-1-s/2}\,\iota\left(\pi_\infty\left(\begin{bmatrix}1 & \tilde{x}\\ & 1\end{bmatrix}\begin{bmatrix}\tilde{y}^{1/2} & \\ & \tilde{y}^{-1/2}\end{bmatrix}\right)v_\infty \otimes (\otimes_{p<\infty} v_{p,\mathrm{new}})\right). \tag{5.10}$$

Let $\varphi = \iota(\otimes v_p)$ be the automorphic form in the space of π corresponding to the pure tensor $\otimes_{p\le\infty} v_p$. Then

$$(\phi_s(\mathrm{L}_N\,\gamma\, b_Z))(1) = \frac{\det(\tilde{Y})^{1+s/2-k/2}}{\det(Y)^{-k/2}}\,\det(j(\gamma, Z))^{-k}\,\tilde{y}^{-1-s/2}\,\varphi\left(\begin{bmatrix}1 & \tilde{x}\\ & 1\end{bmatrix}\begin{bmatrix}\tilde{y}^{1/2} & \\ & \tilde{y}^{-1/2}\end{bmatrix}\right). \tag{5.11}$$

Going back to (5.6), it follows that

$$E(Z,s,\phi)=\sum_{\gamma\in D(N)}\det(\tilde{Y})^{1+s/2-k/2}\det(j(\gamma,Z))^{-k}\tilde{y}^{-1-s/2}\varphi\left(\begin{bmatrix}1&\tilde{x}\\&1\end{bmatrix}\begin{bmatrix}\tilde{y}^{1/2}&\\&\tilde{y}^{-1/2}\end{bmatrix}\right). \tag{5.12}$$

Analogous to (3.6), the automorphic form φ is determined by the function $f:\mathbb{H}_1\to\mathbb{C}$ defined by

$$f(\tau):=j(g,i)^k\varphi(g), \tag{5.13}$$

where g is any element of $\mathrm{SL}(2,\mathbb{R})$ for which $g\langle i\rangle=\tau$. Explicitly,

$$f(x+iy)=y^{-k/2}\varphi\left(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}y^{1/2}&\\&y^{-1/2}\end{bmatrix}\right)\quad\text{for}\quad x+iy\in\mathbb{H}_1. \tag{5.14}$$

We may thus rewrite (5.12) as

$$E(Z,s,\phi)=\sum_{\gamma\in D(N)}\left(\frac{\det(\tilde{Y})}{\tilde{y}}\right)^{1+s/2-k/2}\det(j(\gamma,Z))^{-k}f(\tilde{\tau}), \tag{5.15}$$

where $\tilde{\tau}=\tilde{x}+i\tilde{y}$ is the upper left entry of $\tilde{Z}=\mathrm{L}_N\,\gamma\,\langle Z\rangle$. We summarize:

Proposition 5.3. *Let $\pi\cong\otimes\pi_p$ be an irreducible, unitary, cuspidal, automorphic representation of* $\mathrm{GL}(2,\mathbb{A})$ *with central character $\chi\cong\otimes\chi_p$. Let N be the global conductor of π. Let k be a weight occurring in π_∞. Let φ be the automorphic form in π corresponding to $\otimes v_p$, where v_∞ is a weight k vector in π_∞, and v_p is a local newform in π_p for each $p<\infty$. Let f be the function on the upper half plane defined by* (5.14)*. Let ϕ be the element of the global induced representation $\chi^{-1}\rtimes\pi$ defined above, and let $E(g,s,\phi)$ be the corresponding Eisenstein series* (3.3)*. Then, using the notation* (1.4)*, the function $E(Z,s,\phi)$ on the Siegel upper half space defined in* (3.6) *is given by*

$$E(Z,s,\phi)=\sum_{\gamma\in D(N)}\left(\frac{\det(\tilde{Y})}{\tilde{y}}\right)^{1+s/2-k/2}\det(j(\gamma,Z))^{-k}f((\mathrm{L}_N\,\gamma\,\langle Z\rangle)^*), \tag{5.16}$$

where $\tilde{Y}=\begin{bmatrix}\tilde{y}&\tilde{v}\\\tilde{v}&\tilde{y}'\end{bmatrix}$ is the imaginary part of $\mathrm{L}_N\,\gamma\,\langle Z\rangle$, and where $D(N)$ is a set of representatives for $\mathrm{L}_N^{-1}\,Q(\mathbb{Q})\,\mathrm{L}_N\cap\mathrm{K}(N^2)\backslash\,\mathrm{K}(N^2)$. We have

$$E(\gamma\,\langle Z\rangle,s,\phi)=\det(j(\gamma,Z))^kE(Z,s,\phi)\ \textit{for all}\ \ \gamma\in K(N^2). \tag{5.17}$$

Proof. The transformation property (5.17) follows from (3.11). □

6. The holomorphic case

We fix the following notations for archimedean representations. For a positive integer p, let $\mathcal{D}(p)$ be the discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with Harish-Chandra parameter p (i.e., minimal weight $p+1$) and central character sgn^{p+1}. For the Harish-Chandra parameters (i.e., infinitesimal characters) of discrete series representations of $\mathrm{GSp}(4,\mathbb{R})$ we use the same conventions as in [15]. In particular, for integers $p>t>0$, let $\mathcal{D}^{\mathrm{hol}}(p,t)$ be the holomorphic discrete series representation of $\mathrm{GSp}(4,\mathbb{R})$ with Harish-Chandra parameter (p,t) (i.e., minimal K_∞-type $(p+1,t+2)$) and central character sgn^{p+t+1}, and let $\mathcal{D}^{\mathrm{large}}(p,t)$ be the "large" (generic) discrete series representation of $\mathrm{GSp}(4,\mathbb{R})$ with Harish-Chandra parameter (p,t) (i.e., minimal K_∞-type $(p,-t+1)$) and central character sgn^{p+t+1}. The two representations $\mathcal{D}^{\mathrm{hol}}(p,t)$ and $\mathcal{D}^{\mathrm{large}}(p,t)$ constitute an L-packet. (Note that [15] works with $\mathrm{Sp}(4,\mathbb{R})$ instead of $\mathrm{GSp}(4,\mathbb{R})$. The restriction of $\mathcal{D}^{\mathrm{hol}}(p,t)$ to $\mathrm{Sp}(4,\mathbb{R})$ is $X(p,t)\oplus X(-t,-p)$ in the notation of [15], and the restriction of $\mathcal{D}^{\mathrm{large}}(p,t)$ to $\mathrm{Sp}(4,\mathbb{R})$ is $X(p,-t)\oplus X(t,-p)$.)

For an integer $k\geq 2$, the discrete series representation $\mathcal{D}(k-1)$ appears as the archimedean component of the automorphic representations of $\mathrm{GL}(2,\mathbb{A})$ generated by newforms in $S_k(\Gamma_0(N))$. Similarly, $\mathcal{D}^{\mathrm{hol}}(k-1,k-2)$ appears as the archimedean component of the automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$ generated by holomorphic Siegel modular forms of weight k.

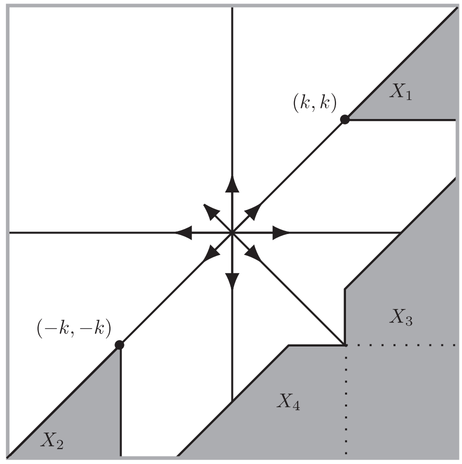

Figure 1. $\mathcal{D}^{\rm hol}(p,t) = X_1 \oplus X_2$ and $\mathcal{D}^{\rm large}(p,t) = X_3 \oplus X_4$ for $p = k-1$ and $t = k-2$.

Lemma 6.1. *For an integer $k \geq 2$, consider the Klingen induced representation*

$$|\cdot|^{k-2}{\rm sgn}^k \rtimes |\cdot|^{-\frac{k-2}{2}}\mathcal{D}(k-1) \tag{6.1}$$

of ${\rm GSp}(4,\mathbb{R})$.

(1) *If $k = 2$, then* (6.1) *decomposes as the direct sum $\mathcal{D}^{\rm hol}(1,0) \oplus \mathcal{D}^{\rm large}(1,0)$. The two components constitute an L-packet of limits of discrete series representations.*
(2) *If $k > 2$, then* (6.1) *has length* 3. *It contains $\mathcal{D}^{\rm hol}(k-1,k-2) \oplus \mathcal{D}^{\rm large}(k-1,k-2)$ as a subrepresentation. The two components of this subrepresentation constitute an L-packet of discrete series representations.*

Proof. (1) follows by setting $p = 1$ in Lemma 8.1 of [15]. (2) follows by setting $p = k-1$ and $t = k-2$ in (10.2) in Theorem 10.1 of [15]. □

We now continue in the setting of Proposition 5.3. Recall that $\chi = \otimes\chi_p$ is the central character of π. We denote by the same symbol χ also the corresponding Dirichlet character mod N; it is characterized by

$$\chi(d) = \prod_{p|N} \chi_p(d) \quad \text{for } d \in \mathbb{Z} \text{ with } \gcd(d,N) = 1. \tag{6.2}$$

Assume that the cuspidal, automorphic representation $\pi \cong \otimes\pi_p$ of ${\rm GL}(2,\mathbb{A})$ is such that $\pi_\infty \cong \mathcal{D}(k-1)$ for some integer $k \geq 2$. Then the function f defined in (5.14) is a newform and eigenform in $S_k(\Gamma_0(N),\chi)$. Conversely, we could have started with an eigen-newform f in $S_k(\Gamma_0(N),\chi)$ and defined π as the automorphic representation generated by (the adelization of) f.

Theorem 6.2. *Let $k \geq 5$ and $N \geq 1$ be integers. Let χ be a Dirichlet character* mod N. *Assume that $f \in S_k(\Gamma_0(N),\chi)$ is an elliptic eigenform and a newform. For $Z \in \mathbb{H}_2$, set*

$$E(Z,f) = \sum_{\gamma \in D(N)} \det(j(\gamma,Z))^{-k} f(({\rm L}_N\,\gamma\,\langle Z\rangle)^*), \tag{6.3}$$

where $D(N)$ is a set of representatives for ${\rm L}_N^{-1}\,Q(\mathbb{Q})\,{\rm L}_N \cap {\rm K}(N^2)\backslash\,{\rm K}(N^2)$, and ${\rm L}_N$ is as in (1.3).

(1) *The summation* (6.3) *is absolutely and locally uniformly convergent and defines a holomorphic Siegel modular form $E(\cdot,f) \in M_k^{(2)}(K(N^2))$.*
(2) *We have $\Phi(E(\cdot,f)) = f$.*
(3) *Let $N = \prod p^{n_p}$. The modular form $E(\cdot,f)$ is an eigenfunction for the Atkin-Lehner involution at p, for all primes p. For a prime $p|N$, the Atkin-Lehner eigenvalue of $E(\cdot,f)$ at p equals $\chi(m)$, where m is an integer such that $m \equiv -1$ mod p^{n_p} and $m \equiv 1$ mod q^{n_q} for $q|N$, $q \neq p$. In particular, $E(\cdot,f)$ is an eigenfunction for the Fricke involution with eigenvalue $\chi(-1)$.*

(4) *The adelization of $E(\cdot, f)$ generates an irreducible, automorphic representation $\Pi \cong \otimes\Pi_p$ of* $\mathrm{GSp}(4, \mathbb{A})$*, where* $\Pi_\infty = \mathcal{D}^{\mathrm{hol}}(k-1, k-2)$ *and* $\Pi_p = |\cdot|^{k-2}\chi_p^{-1} \rtimes |\cdot|^{-\frac{k-2}{2}}\pi_p$*; here, $\pi \cong \otimes\pi_p$ is the cuspidal, automorphic representation of* $\mathrm{GL}(2, \mathbb{A})$ *generated by the adelization of f, and $\chi = \otimes\chi_p$ is the adelization of the Dirichlet character χ.*

(5) *The modular form $E(\cdot, f)$ is an eigenfunction for the local Hecke algebra at all places $p \nmid N$. The (complete) L-function of the underlying automorphic representation Π is given by*

$$L(s, \Pi) = L\left(s + \frac{k-2}{2}, \pi^\vee\right) L\left(s - \frac{k-2}{2}, \pi\right), \tag{6.4}$$

where $\pi^\vee$ is the contragredient of π (and all L-functions are normalized so that they satisfy a functional equation relating s and $1-s$). It admits analytic continuation to an entire function and satisfies the functional equation

$$L(s, \Pi) = \varepsilon(s, \Pi)L(1-s, \Pi), \tag{6.5}$$

where $\varepsilon(s, \Pi) = \varepsilon\big(s + \frac{k-2}{2}, \pi^\vee\big)L\big(s - \frac{k-2}{2}, \pi\big)$.

Proof.

(1) The convergence statement can be proven along the classical lines; see [8], or Sect. 5 of [9]. For the second claim we apply Proposition 5.3 with $s = k-2$. Hence $E(Z, f)$ is the function appearing in (5.16). It follows from (5.17) that it is an element of $M_k^{(2)}(K(N^2))$.

(2) From the first part above, as the sum is absolutely convergent, the Φ operator could be applied term by term. Then a straightforward calculation shows that for $\gamma \in \mathrm{L}_N^{-1}\, Q(\mathbb{Q})\, \mathrm{L}_N \cap \mathrm{K}(N^2)$,

$$\det(j(\gamma, Z))^{-k} f((\mathrm{L}_N\, \gamma\, \langle Z\rangle)^*) = f(\tau), \quad Z = \begin{bmatrix} \tau & \\ & i\lambda \end{bmatrix}.$$

One can show, by using an argument similar to Prop. 5, Sect. 5 in [9], that on application of the Φ operator all the other terms vanish.

(3) Recall from (4.6) that the archimedean sections $\phi_{\infty,s} \in I(s, \pi_\infty)$ transform according to the character η_k of K'_∞. Hence $\phi_{\infty,s}$ spans the scalar K'_∞-type (k, k), which appears exactly once in $I(s, \pi_\infty)$. Now set $s = k-2$. By Lemma 6.1, the function $\phi_{\infty,k-2}$ generates an irreducible subrepresentation of $I(k-2, \pi_\infty)$ isomorphic to $\mathcal{D}^{\mathrm{hol}}(k-1, k-2)$. Hence the Eisenstein series $E(g, s, \phi)$ is a lowest weight vector in an automorphic representation $\Pi \cong \prod_{p \le \infty} \Pi_p$, where $\Pi_\infty \cong \mathcal{D}^{\mathrm{hol}}(k-1, k-2)$. For finite p, we have $\Pi_p \cong I(k-2, \pi_p)$, which is irreducible by Remark 4.1. Hence we obtain the asserted irreducibility.

(4) The modular form $E(\cdot, f)$ is an eigenfunction for the Atkin-Lehner involution at p, because the local vectors $\phi_{p,s}$ are eigenvectors for the local Atkin-Lehner elements. By Corollary 7.5.5 of [19], the eigenvalues coincide with the value of the ε-factor at $1/2$ of the representation Π_p. These ε-factors are listed in Table A.9 of [19]. Taking into account Remark 4.1, we see that the eigenvalue at p is $\chi_p(-1)$, where we recall that χ_p is the central character of π_p. Our assertion about this eigenvalue follows in view of (6.2). The last assertion follows since the Fricke eigenvalue is the product of all Atkin-Lehner eigenvalues.

(5) Since the section ϕ is composed of spherical vectors outside of N, it follows that ϕ is an eigenvector for the local (unramified) Hecke algebras for all $p \nmid N$. The same is then true for the corresponding function $E(\cdot, f)$ on the Siegel upper half space. From the L-factors listed in Table A.8 of [19], and taking into account Remark 4.1, we see that

$$L(s, \Pi_p) = L\left(s + \frac{k-2}{2}, \pi_p^\vee\right) L\left(s - \frac{k-2}{2}, \pi_p\right) \tag{6.6}$$

for all finite places p. Using [21], the same identity can be verified at the archimedean place. Hence we obtain (6.4). The functional equation (6.5) follows from the functional equation $L(s, \pi) = \varepsilon(s, \pi)L(1-s, \pi^\vee)$ for π.

□

We remark that, using classical calculations as in Proposition 6.3 of [14], one can prove that $E(\cdot, f)$ is an eigenfunction at all good places without recourse to the underlying representation.

The Siegel modular form $E(Z, f)$ in Theorem 6.2 is a newform in the sense that the automorphic representation Π does not contain a modular form of level M with M being a proper divisor of N^2. This is because the local vectors chosen in (4.3) are of minimal paramodular level.

If $N = 1$, then (6.3) reduces to

$$E(Z, f) = \sum_{\gamma \in C_{2,1} \backslash \mathrm{Sp}(4,\mathbb{Z})} \det(j(\gamma, Z))^{-k} f((\gamma \langle Z \rangle)^*), \tag{6.7}$$

where $C_{2,1} = \mathrm{Sp}(4, \mathbb{Z}) \cap Q(\mathbb{Q})$. This is the Klingen lift without level, as it is defined in [8].

7. Klingen Eisenstein series with respect to $\Gamma_0^{(2)}(N)$

Starting from an elliptic cusp form $f \in S_k(\Gamma_0(N))$, we constructed in Theorem 6.2 a Klingen Eisenstein series $E(Z, f)$ with respect to the paramodular group $K(N^2)$. The question arises whether we can also construct an Eisenstein series with respect to the Siegel congruence subgroup $\Gamma_0^{(2)}(M)$ for some level M.

To distinguish local and global congruence subgroups, we let $\mathrm{Si}(p^n)$ be the subgroup of $\mathrm{GSp}(4, \mathbb{Z}_p)$ consisting of all matrices of the form $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ where each entry of C is in $p^n\mathbb{Z}_p$. If we can choose $\mathrm{Si}(p^{m_p})$-invariant vectors in each local representation $I(0, \pi_p)$ (see (4.2)), then a similar construction as in the paramodular case will yield an Eisenstein series with respect to $\Gamma_0^{(2)}(M)$, where $M = \prod p^{m_p}$.

The $\mathrm{Si}(p^m)$-invariant vectors in $I(0, \pi_p)$ are less well understood than the paramodular vectors. But at least if the central character of π_p is trivial, we can identify two natural double cosets $Q(\mathbb{Q}_p)r\mathrm{Si}(p^n)$ supporting such vectors, where $n = a(\pi_p)$. These are the double cosets represented by $r = 1$ and by the Atkin-Lehner element η_{p^n}, where

$$\eta_N = \begin{bmatrix} &&&-1 \\ &&1& \\ &N&& \\ -N&&& \end{bmatrix} \tag{7.1}$$

for a positive integer N. It is easy to see that the double cosets $Q(\mathbb{Q}_p)\eta_{p^n}\mathrm{Si}(p^n)$ and $Q(\mathbb{Q}_p)\mathrm{Si}(p^n)$ are different, provided that $n \geq 1$.

Lemma 7.1. *Let (π, V) be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ with trivial central character. Let $n = a(\pi)$, and let $v_{\mathrm{new}} \in V$ be a local newform; hence v_{new} is invariant under $\mathrm{GL}(2, \mathbb{Z}_p) \cap \left[\begin{smallmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p \end{smallmatrix}\right]$. Let $\Pi_s = |\cdot|^s \rtimes |\cdot|^{-s/2}\pi$ given in its standard model $I(s, \pi)$.*

(1) *There exists an $\mathrm{Si}(p^n)$-invariant vector ϕ_1 in $I(s, \pi)$ given by*

$$\phi_1(h\kappa) = |t^2(ad - bc)^{-1}|^{1+s/2}\pi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)v_{\mathrm{new}} \tag{7.2}$$

if $h \in Q(\mathbb{Q}_p)$ is written in the form (4.4) and $\kappa \in \mathrm{Si}(p^n)$, and $\phi(g) = 0$ if g is not in the double coset $Q(\mathbb{Q}_p)\mathrm{Si}(p^n)$.

(2) *There exists an $\mathrm{Si}(p^n)$-invariant vector ϕ_2 in $I(s, \pi)$ given by*

$$\phi_2(h\eta_{p^n}\kappa) = |t^2(ad - bc)^{-1}|^{1+s/2}\pi_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)v_{\mathrm{new}} \tag{7.3}$$

if $h \in Q(\mathbb{Q}_p)$ is written in the form (4.4) and $\kappa \in \mathrm{Si}(p^n)$, and $\phi(g) = 0$ if g is not in the double coset $Q(\mathbb{Q}_p)\eta_{p^n}\mathrm{Si}(p^n)$.

(3) *We have $\Pi_s(\eta_{p^n})\phi_1 = \phi_2$ and $\Pi_s(\eta_{p^n})\phi_2 = \phi_1$.*

Proof. It is straightforward to verify that ϕ_1 and ϕ_2 are well-defined. Evidently, these functions have the asserted transformation and invariance properties. Assertion (3) is obvious. □

Lemma 7.2. *Let $N = \prod p^{n_p}$ be a positive integer, and let $g \in \mathrm{GSp}(4, \mathbb{Q})$.*

(1) *Assume that $g \in Q(\mathbb{Q}_p)\mathrm{Si}(p^{n_p})$ for all $p|N$. Then $g \in Q(\mathbb{Q})\Gamma_0^{(2)}(N)$.*
(2) *Assume that $g \in Q(\mathbb{Q}_p)\eta_{p^n}\mathrm{Si}(p^{n_p})$ for all $p|N$. Then $g \in Q(\mathbb{Q})\eta_N\Gamma_0^{(2)}(N)$.*

Proof.

(1) By the results of [23], we can write $g = qr\gamma$ with $q \in Q(\mathbb{Q})$, $\gamma \in \Gamma_0^{(2)}(N)$, and r of the form

$$r = \begin{bmatrix} 1 &&& \\ & 1 && \\ & v & 1 & \\ v & u && 1 \end{bmatrix} \quad \text{with} \quad u, v \in \mathbb{Z}.$$

A straightforward calculation verifies that the condition $g \in Q(\mathbb{Q}_p)\mathrm{Si}(p^{n_p})$ implies $p^{n_p}|u$ and $p^{n_p}|v$. Hence $r \in \Gamma_0^{(2)}(N)$, proving our claim.
(2) is an easy consequence of (1).

□

Starting from a unitary, cuspidal, automorphic representation $\pi \cong \otimes\pi_p$ of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character and global conductor N, we can now construct an Eisenstein series $E(Z, s, \phi)$ with respect to $\Gamma_0^{(2)}(N)$. The global section $\phi \in I(0, \pi)$ is composed of local sections $\phi_{1,p}$ as in (7.2). At the archimedean place we choose the local section as before; see (4.6). Using Lemma 7.2 (1), we see that

$$\begin{aligned} E(Z, s, \phi) &= \det(Y)^{-k/2} E(b_Z, s, \phi) \\ &= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q}) \backslash \mathrm{GSp}(4,\mathbb{Q})} (\phi_s(\gamma b_Z))(1) \\ &= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q}) \backslash Q(\mathbb{Q})\Gamma_0^{(2)}(N)} (\phi_s(\gamma b_Z))(1) \\ &= \det(Y)^{-k/2} \sum_{\gamma \in D_1(N)} (\phi_s(\gamma b_Z))(1), \end{aligned} \tag{7.4}$$

where $D_1(N)$ is a set of representatives for $Q(\mathbb{Q}) \cap \Gamma_0^{(2)}(N) \backslash \Gamma_0^{(2)}(N)$. A similar calculation as the one leading to (5.16) now shows that

$$E(Z, s, \phi) = \sum_{\gamma \in D_1(N)} \left(\frac{\det(\tilde{Y})}{\tilde{y}} \right)^{1+s/2-k/2} \det(j(\gamma, Z))^{-k} f((\gamma\langle Z\rangle)^*), \tag{7.5}$$

where $\tilde{Y} = \begin{bmatrix} \tilde{y} & \tilde{v} \\ \tilde{v} & \tilde{y}' \end{bmatrix}$ is the imaginary part of $\gamma\langle Z\rangle$. Just as in the paramodular case, we now obtain the following result.

Theorem 7.3. *Let $k \geq 5$ and $N \geq 1$ be integers. Assume that $f \in S_k(\Gamma_0(N))$ is an elliptic eigenform and a newform. For $Z \in \mathbb{H}_2$, set*

$$E_1(Z, f) = \sum_{\gamma \in D_1(N)} \det(j(\gamma, Z))^{-k} f((\gamma\langle Z\rangle)^*), \tag{7.6}$$

where $D_1(N)$ is a set of representatives for $Q(\mathbb{Q}) \cap \Gamma_0^{(2)}(N) \backslash \Gamma_0^{(2)}(N)$.

(1) *The summation* (7.6) *is absolutely and locally uniformly convergent and defines a holomorphic Siegel modular form* $E_1(\cdot, f) \in M_k^{(2)}(\Gamma_0^{(2)}(N))$.
(2) *We have* $\Phi(E_1(\cdot, f)) = f$.
(3) *The adelization of* $E_1(\cdot, f)$ *generates an irreducible, automorphic representation* $\Pi \cong \otimes\Pi_p$ *of* $\mathrm{GSp}(4, \mathbb{A})$, *which is identical to the one from Theorem 6.2 (3).*

If in our construction we replace the local sections $\phi_{1,p}$ in (7.2) by $\phi_{2,p}$ in (7.3), then we obtain another Klingen Eisenstein series $E_2(\cdot, f) \in S_k(\Gamma_0^{(2)}(N))$. It is the Fricke image of $E_1(\cdot, f)$, i.e.,

$$E_2(\cdot, f) = E_1(\cdot, f)\,|_k\, \eta_N. \tag{7.7}$$

8. A connection with the Paramodular conjecture

The famous *Modularity Theorem* states that, given an elliptic curve E over $\mathbb{Q}$ with conductor N, there exists an eigen-newform $f \in S_2(\Gamma_0(N))$ such that the L-functions of E and f coincide: $L(s, E) = L(s, f)$. The *Paramodular Conjecture* of Brumer and Kramer, formulated in [4], is an analogue in degree 2. It states that to every abelian surface A over $\mathbb{Q}$ with conductor N and $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ there exists an eigen-newform $F \in S_k(K(N))$ such that the degree-4 L-functions of A and F coincide: $L(s, A) = L(s, F)$. In fact, Conjecture 1.1 of [4] predicts a one-one correspondence of isogeny classes of such abelian surfaces and lines of *non-lift* eigen-newforms $F \in S_k(K(N))$ with rational Hecke eigenvalues. Here, non-lift refers to Siegel modular forms that are orthogonal to the space of *Gritsenko lifts*, which are Saito-Kurokawa lifts with level.

The mechanism leading from a geometric object to a modular form is the same for the elliptic curve and the abelian surface case. In degree 2, the following diagram illustrates the procedure:

$$\boxed{\begin{array}{c}\text{abelian}\\ \text{surface } A\end{array}} \xrightarrow{1} \boxed{\begin{array}{c}\text{local repre-}\\ \text{sentations } \pi_p\end{array}} \xrightarrow{2} \boxed{\begin{array}{c}\text{automorphic}\\ \pi = \otimes\pi_p\end{array}} \xrightarrow{3} \boxed{\begin{array}{c}\text{paramodular}\\ \text{form } F\end{array}} \tag{8.1}$$

Given any abelian surface over $\mathbb{Q}$, the step 1 is to find local representations π_p for each place $p \le \infty$. For finite primes, one considers an auxiliary prime ℓ different than p and the action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on the Tate module $T_\ell(A)$. This results in an ℓ-adic representation $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}(4, \mathbb{Q}_\ell)$. Then following the procedure described in §4 of [20], this ℓ-adic representation can be converted to a complex representation σ_p of the Weil-Deligne group $W'(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, which is independent of ℓ. After a twist, and using the symplectic structure on the Tate module coming from the Weil pairing, σ_p can be assumed to be a map

$$\sigma_p : W'(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathrm{Sp}(4, \mathbb{C}). \tag{8.2}$$

Note that $\mathrm{Sp}(4, \mathbb{C})$ is the dual group of the split orthogonal group $\mathrm{SO}(5) \cong \mathrm{PGSp}(4)$. We can thus invoke the local Langlands correspondence [6] and obtain from σ_p an irreducible, admissible representation π_p of $\mathrm{GSp}(4, \mathbb{Q}_p)$ with trivial central character. In some cases there may be a choice for π_p among two members of a local L-packet. In this case we choose π_p to be the unique *generic* member of the packet. This is because we would like π_p to a) have paramodular vectors, and b) have these vectors appear at the "correct" level, namely the conductor $a(\pi_p)$. See Theorem 7.5.4 and Corollary 7.5.5 of [19].

There is an analogous procedure at the archimedean place. The Hodge decomposition of the first cohomology group $H^1(A, \mathbb{C}) \cong H^{1,0} \oplus H^{0,1}$ leads to a 4-dimensional representation of the real Weil group $W_{\mathbb{R}} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$; see §20 of [20]. Again we may view this representation as a homomorphism $\sigma_\infty : W_{\mathbb{R}} \to \mathrm{Sp}(4, \mathbb{C})$ and consider the corresponding L-packet of representations of $\mathrm{GSp}(4, \mathbb{R})$. As explained in [21], this L-packet consists of the holomorphic limit of discrete series representation $\mathcal{D}^{\mathrm{hol}}(1, 0)$ (with minimal K'_∞-type $(2, 2)$), and a large (generic) limit of discrete series representation $\mathcal{D}^{\mathrm{large}}(1, 0)$. We let $\pi_\infty = \mathcal{D}^{\mathrm{hol}}(1, 0)$, i.e., we choose the *non-generic* member of the L-packet at the archimedean place.

Having defined local representations π_p at all places, we consider the admissible global representation $\pi := \otimes\pi_p$ of $\mathrm{GSp}(4, \mathbb{A})$. The decisive step in (8.1) is to prove that π is an *automorphic* representation.

In the elliptic curve case, this is essentially the content of the Modularity Theorem. In the abelian surface case it is largely conjectural, although there has been recent progress and there is convincing evidence. In fact, the Paramodular Conjecture claims that π is a *cuspidal* automorphic representation if $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$.

The final step 3 in (8.1) is to extract from π a classical modular form F. This can be done in a standard way: For $p < \infty$, let v_p be a paramodular newform in the space of π_p. At the archimedean place, let v_∞ be a vector of weight $(2, 2)$. Assuming π has been proven to be cuspidal automorphic, let Φ be the automorphic form corresponding to $\otimes v_p$. From the adelic function Φ we obtain a function F on $\mathbb{H}_2$ in the same way that $E(Z, s, \phi)$ was obtained from $E(g, s, \phi)$; see (3.6). The function F thus obtained is an element of $S_2(K(N))$, where $N = \prod p^{n_p}$ is the conductor of the abelian surface A. This is because $a(\pi_p) = n_p$ by the definition of the conductor of an abelian surface, and the paramodular level of v_p is $a(\pi_p)$ by the properties of the local paramodular newform theory of [19].

We have $L(s, F) = L(s, \pi)$ by definition of $L(s, F)$, and $L(s, \pi) = L(s, A)$ since the Euler factors of both L-functions are defined as the factors attached to the L-parameters σ_p. In practice, of course, evidence for the Paramodular Conjecture comes from constructing paramodular forms rather than automorphic representations; see [17].

We now offer the following alternative version of one direction of the Paramodular Conjecture.

Conjecture 8.1. *Let A be an abelian variety over $\mathbb{Q}$. Then there exists an automorphic representation $\Pi \cong \otimes\Pi_p$ of $\mathrm{GSp}(4, \mathbb{A})$ with trivial central character such that $L(s, A) = L(s, \Pi)$ and such that Π_p is generic for all places p.*

Assume that $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$, and that we can find a paramodular form $F \in S_2(K(N))$ such that $L(s, F) = L(s, A)$, as in the original Paramodular Conjecture. Let $\Pi \cong \otimes\Pi_p$ be the cuspidal, automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ generated by F. By Arthur's classification of automorphic representations, we may exchange $\Pi_\infty \cong \mathcal{D}^{\mathrm{hol}}(1, 0)$ by $\mathcal{D}^{\mathrm{large}}(1, 0)$ and obtain a globally generic, cuspidal, automorphic representation Π' of $\mathrm{GSp}(4, \mathbb{A})$; see [22]. Hence the original Paramodular Conjecture implies Conjecture 8.1 for abelian varieties with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$.

A case not covered by the original conjecture is that of abelian surfaces A isogeneous to a direct product $E_1 \times E_2$ of two non-isogenous elliptic curves E_1 and E_2. For $i = 1, 2$ let π_i be the cuspidal, automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ attached to E_i by the Modularity Theorem. Since $\pi_1 \not\cong \pi_2$, the Yoshida lifting construction from [18] produces a cuspidal automorphic representation $\Pi = \Pi(\pi_1, \pi_2) \cong \otimes\Pi_p$ of $\mathrm{GSp}(4, \mathbb{A})$ such that all Π_p are generic and such that $L(s, \Pi) = L(s, \pi_1)L(s, \pi_2)$. Hence Conjecture 8.1 is true for such A.

We finally consider the degenerate case that A is isogenoues to a product of two isogeneous elliptic curves. We may assume that $A = E \times E$ for an elliptic curve E over $\mathbb{Q}$. Let $\pi \cong \otimes\pi_p$ be the cuspidal, automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ attached to E by the Modularity Theorem. The L-parameters σ_p arising from A are of the form $\sigma_p(w) = \begin{bmatrix} \varphi_p(w) & \\ & {}^t\varphi_p(w)^{-1} \end{bmatrix}$, where φ_p is the L-parameter of π_p. If π_p is not square-integrable, the corresponding L-packet consists of the irreducible representation $1 \rtimes \pi_p$. If π_p is square-integrable, then $1 \rtimes \pi_p$ decomposes as a direct sum of two irreducible representations Π^{g} and Π^{n}, and $\{\Pi^{\mathrm{g}}, \Pi^{\mathrm{n}}\}$ is the L-packet parameterized by σ_p. One of these representations, Π^{g}, is generic, and the other is non-generic. In the archimedean case, see Lemma 6.1 (1). In the non-archimedean case, using the classification of Sect. 2.4 of [19], $\Pi^{\mathrm{g/n}}$ are either of type VIa/b or of type VIIIa/b.

Now consider the global representation $1 \rtimes \pi \cong \otimes(1 \rtimes \pi_p)$. Note that this is the case $s = 0$ of the representation defined in (3.1). By [11], every irreducible subquotient of $1 \rtimes \pi$ is automorphic. For finite places p such that π_p is not square-integrable, let Π_p be the irreducible $1 \rtimes \pi_p$. For all other places, let Π_p be the generic constituent Π^{g} mentioned in the previous paragraph. Then $\Pi := \otimes\Pi_p$ is a non-cuspidal, automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ such that $L(s, \Pi) = L(s, \pi)^2$ and Π_p is generic for all p.

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