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Paramodular forms of level 8 and weights 10 and 12

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We study degree 2 paramodular eigenforms of level 8 and weights 10 and 12, and determine all their local representations. We prove dimensions by the technique of Jacobi restriction. A level divisible by a cube permits a wide variety of local representations, but also complicates the Hecke theory by involving Fourier expansions at more than one zero-dimensional cusp. We overcome this difficulty by the technique of restriction to modular curves. An application of our determination of the local representations is that we obtain the Euler 2-factor of each newform.

Keywords: Siegel modular form; paramodular; automorphic representation.

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1. Introduction

Let $\mathcal{H} = \{Z \in M_2(\mathbb{C}) \mid {}^tZ = Z, \operatorname{Im}(Z) > 0\}$ be the Siegel upper half space of degree 2. Siegel modular forms of weight k are holomorphic functions $\mathcal{H} \to \mathbb{C}$ satisfying a transformation property with respect to a congruence subgroup Γ of $\operatorname{Sp}(4, \mathbb{Q})$. In recent years Siegel modular forms with respect to $\Gamma = K(N)$, the *paramodular group* of level N, have received much attention, partly due to the *paramodular conjecture* formulated in [6], and partly due to the theory of newforms on paramodular groups, see [12, 23].

The space $S_k(K(N))$ of paramodular cusp forms of weight k and level N contains the subspace $S_k^*(K(N))$ spanned by *Gritsenko lifts* $\mathrm{Grit}(\phi)$, where ϕ runs through Jacobi cusp forms of weight k and index N, see [9]. The same subspace can also be obtained as generalized Saito–Kurokawa liftings from the space of elliptic newforms in $S_{2k-2}(\Gamma_0(N))$, as in [23, Theorem 6.1]. We will refer to the non-zero elements of $S_k^*(K(N))$ simply as *lifts*. Any non-zero element of $S_k(K(N))$ in the orthogonal complement $S_k'(K(N))$ of $S_k^*(K(N))$ is a *non-lift*.

The spaces $S_k(K(N))$ also admit a theory of oldforms and newforms, analogous to the familiar Atkin–Lehner theory for elliptic modular forms. In particular, there is an orthogonal decomposition

$$S_k(K(N)) = S_k(K(N))^{\mathrm{new}} \oplus S_k(K(N))^{\mathrm{old}},$$

where $S_k(K(N))^{\mathrm{old}}$ consists of forms arising from lower paramodular level via a fixed set of three level raising operators θ, θ' and η (for each prime). The level raising operators take lifts to lifts and non-lifts to non-lifts, so that if we define

$$S_k^*(K(N))^{\mathrm{old/new}} = S_k^*(K(N)) \cap S_k(K(N))^{\mathrm{old/new}},$$

and similarly define $S_k'(K(N))^{\mathrm{old/new}}$, then we have orthogonal decompositions

$$S_k^*(K(N)) = S_k^*(K(N))^{\mathrm{old}} \oplus S_k^*(K(N))^{\mathrm{new}},$$

$$S_k'(K(N)) = S_k'(K(N))^{\mathrm{old}} \oplus S_k'(K(N))^{\mathrm{new}}.$$

Oldforms are old in the sense of arising from a discrete group of smaller *level*, and lifts are old in the sense of arising from a Lie group of smaller *rank*. Each of the spaces mentioned so far has a basis consisting of eigenforms with respect to the local Hecke algebras for all p not dividing N. At least conjecturally, the eigenforms in $S_k(K(N))^{\mathrm{new}}$ are in one-to-one correspondence with a set of cuspidal, automorphic representations of conductor N of the adelic group $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$. More precisely, each eigen-newform should adelize to a distinguished vector in such an automorphic representation. The oldforms adelize to give non-distinguished vectors in automorphic representations of strictly smaller conductor.

Consider the set of *all* cuspidal, automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ contributing to $S_k(K(N))$. Some of these automorphic representations will be “lifts” $\Lambda_1, \ldots, \Lambda_m$ of automorphic representations of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$, in the sense of [29, Theorem 3.1]; these correspond to the eigenforms in $S_k^*(K(N))$. The rest, $\Pi_1, \ldots, \Pi_n$, will be cuspidal, automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ that are not lifts; they correspond to the eigenforms in $S_k'(K(N))$.

Our goal in this paper is to analyze the spaces $S_{10}(K(8))$ and $S_{12}(K(8))$ with respect to these structures, and to discover, by examples, which local representations can actually be hit by global automorphic forms. We will determine the dimension of the spaces of old/new lifts and old/new non-lifts. The starting point is to determine the dimension of the full spaces $S_{10}(K(8))$ and $S_{12}(K(8))$. By Theorems 4.3 and 6.4, these dimensions are 6 and 12, respectively. Our method for proving dimensions is to find an upper bound on the dimension using "Jacobi restriction" [5, 13], and then to find a lower bound by constructing lifts of theta blocks, and products of such lifts. This method gives not only the dimensions, but also the initial Fourier expansions of a $\mathbb{Q}$-basis.

We then consider the automorphic representations generated by these eigenforms. For both weights $k = 10$ and $k = 12$ we determine the automorphic lifts $\Lambda_1, \ldots, \Lambda_m$ precisely in terms of their GL(2) data; we have $m = 3$ for $k = 10$ and $m = 4$ for $k = 12$. Some of the automorphic representations $\Pi_1, \ldots, \Pi_n$ contain cusp forms with respect to other congruence subgroups that have previously appeared in the literature, in particular in [11]. The rest of $\Pi_1, \ldots, \Pi_n$ are newly discovered automorphic representations, generated by certain eigen-newforms in $S_k(K(8))$. We have two non-lift newforms for $k = 10$ and 4 for $k = 12$.

Let Π be one of the cuspidal, automorphic representations generated by a non-lift eigen-newform F in $S_k(K(N))$. We can decompose Π as a restricted tensor product $\Pi = \otimes\Pi_p$, where Π_p is an irreducible, admissible representation of the local group $\mathrm{GSp}(4, \mathbb{Q}_p)$. Since we are working with level $N = 8$, the Π_p are unramified representations for each finite $p > 2$. It is an interesting problem to determine the local representation at $p = 2$ of this global automorphic form in terms of the classification of [24, Table A.1]. Even for elliptic modular forms the analogous problem is generally not easy; see [17]. In degree 2 it does not seem to have been addressed beyond the Iwahori-spherical cases in [28]. In our case, we are getting some help from the fact that there are no characters of $\mathbb{Q}_2^\times$ of conductor exponent 1. This limits the possibilities for Π_2 to a small number of families; see Table 2.

To determine Π_2 precisely requires additional information. We extract this information from F with the help of two paramodular Hecke operators $T_{0,1}$ and $T_{1,0}$, which are the topic of Sec. 5. These operators have their origin in the local theory of the paramodular group; their local counterparts appear in [24, Sec. 6.1]. The local newform theory implies that eigenforms (at all good places) in $S_k(K(N))^{\mathrm{new}}$, provided they generate an irreducible automorphic representation, are also eigenforms (at all bad places) for the operators $T_{0,1}$ and $T_{1,0}$. The calculation of these operators can be challenging however, since some of their double coset representatives consist of lower triangular matrices. In other words, these Hecke operators can mix the Fourier expansions at different zero-dimensional cusps. *The difficulty of simultaneously accessing Fourier expansions at multiple cusps is one reason that the computations here have not been previously attempted.* We explain in Sec. 5.2 how we overcome this difficulty by using the method of *restriction to a modular curve*. Here again we receive some help from the fact that our only ramified place is $p = 2$, since in this case the number of problematic double coset representatives is small. The results of our eigenvalue calculations are contained in Table 7 (for $k = 10$) and 17 (for $k = 12$). With this information we can determine the local components Π_2 precisely; see Proposition 6.2, which contains the arguments in full detail for the $k = 10$ case.

Our main results are Table 11 (for $k = 10$) and 16 (for $k = 12$). These tables show precisely how the eigenforms in $S_k(K(8))$ are distributed among the various automorphic representations. We also include the spaces $S_k(B(2))$ and $S_k(\Gamma_0(2))$, since there is significant overlap of their automorphic representations with those of $S_k(K(8))$. As an application, we obtain the "correct" Euler factors at $p = 2$ for all eigenforms considered; see Tables 12 and 18.

2. Notation

For any commutative ring R, let

$$\mathrm{GSp}(4,R) = \{g \in \mathrm{GL}(4,R) \mid {}^t gJg = \lambda(g)J, \text{ for some } \lambda \in R^\times\}, \quad J = \begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix}.$$

The kernel of the multiplier homomorphism $\lambda : \mathrm{GSp}(4,R) \to R^\times$ is the group $\mathrm{Sp}(4,R)$.

Let $G = \mathrm{GL}(2)$ or $G = \mathrm{GSp}(4)$. Let $G(\mathbb{R})^\circ$ be the identity component of $G(\mathbb{R})$. Let $\mathcal{H}$ be the usual upper half plane if $G = \mathrm{GL}(2)$, or the Siegel upper half space of degree 2 if $G = \mathrm{GSp}(4)$. Hence, in the latter case, $\mathcal{H}$ consists of all symmetric complex (2×2)-matrices Z whose imaginary part is positive definite. In either case $G(\mathbb{R})^\circ$ acts on $\mathcal{H}$ by

$$g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}, \quad g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G(\mathbb{R})^\circ.$$

For a function f on $\mathcal{H}$, an integer k, and an element $g \in G(\mathbb{R})^\circ$, let

$$(f|_k g)(Z) = \det(CZ+D)^{-k} \det(g)^{k/2} f(g\langle Z\rangle). \tag{2.1}$$

This defines a right action of $G(\mathbb{R})^\circ$ on functions $f : \mathcal{H} \to \mathbb{C}$. The center of $G(\mathbb{R})^\circ$ acts trivially, both in the GL(2) and the GSp(4) cases. (This would not have been the case with the "classical" normalization of $|_k$, which uses $\lambda(g)^{nk-n(n+1)/2}$ instead of $\det(g)^{k/2}$.)

Let N be a positive integer. The only congruence subgroup we need in the GL(2) case is

$$\Gamma_0(N) = \mathrm{SL}(2,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

In the GSp(4) case we consider both the *Borel* and the *Siegel congruence subgroups*, defined respectively by

$$B(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix},$$

$$\Gamma_0(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}. \tag{2.2}$$

It will be clear from the context whether $\Gamma_0(N)$ stands for a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ or of $\mathrm{Sp}(4,\mathbb{Z})$. In addition, we consider the *paramodular group*

$$K(N) = \mathrm{Sp}(4,\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}. \tag{2.3}$$

Let Γ be one of these congruence subgroups, and let k be a non-negative integer. A *modular form* of weight k with respect to Γ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma$, and being holomorphic at the cusps of Γ. If f vanishes at the cusps, then it is called a *cusp form*. The space of cusp forms of weight k with respect to Γ is denoted by $S_k(\Gamma)$, both in the GL(2) and the GSp(4) cases. For $g \in G(\mathbb{Q})$, a double coset $\Gamma g \Gamma$ acts as a Hecke operator on $M_k(\Gamma)$ via $f|_k\Gamma g\Gamma = \sum_j f|_k g_j$, for any finite disjoint union $\bigcup_j \Gamma g_j = \Gamma g \Gamma$.

3. Modular Forms and Representations

3.1. *Obtaining modular forms from automorphic representations*

In this section, we explain the mechanism of constructing modular forms from special vectors inside the space of an automorphic representation of a reductive algebraic group. We only consider the groups relevant for this work, namely GL(2) and GSp(4). These lead to elliptic modular forms and Siegel modular forms of degree 2, respectively. We also limit ourselves to cusp forms and trivial central character, thus avoiding a number of technical issues irrelevant for this paper.

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. Let G be either GL(2) or GSp(4). Recall that automorphic forms are complex-valued functions on $G(\mathbb{A})$, left-invariant under the diagonally embedded $G(\mathbb{Q})$, and satisfying certain regularity conditions; we refer to [4] for details. The group $G(\mathbb{A})$ acts on the space of automorphic forms, and on the subspace of *cuspidal* automorphic forms, by right translation. Strictly speaking, at the Archimedean place we have to consider the action of a $(\mathfrak{g}, K)$-module, but we will allow ourselves the usual simplification and speak of "representations of $G(\mathbb{A})$".

Let π be a cuspidal, automorphic representation of $G(\mathbb{A})$. Hence, π is an irreducible representation which can be realized on a space V consisting of cuspidal, automorphic forms on $G(\mathbb{A})$. By the tensor product theorem, π is isomorphic to a restricted tensor product $\bigotimes_{p\leq\infty}\pi_p$, where π_p is an irreducible, admissible representation of the local group $G(\mathbb{Q}_p)$. (Again, when $p=\infty$, we really mean a $(\mathfrak{g},K)$-module.)

We assume that π has trivial central character. The same is then also true for all the local representations π_p. The restriction to trivial central character means that the modular forms we can construct from π will have trivial character (i.e. no nebentypus).

Since our goal is to construct holomorphic modular forms, we will require that the Archimedean component π_∞ is an infinite-dimensional lowest weight module. Hence, if $G=\mathrm{GL}(2)$, we will assume that π_∞ is the unique representation of $\mathrm{GL}(2,\mathbb{R})$ with trivial central character and a lowest weight vector of weight $k\geq 1$; see [15, §5]. It is a discrete series representation if $k\geq 2$, and a limit of discrete series if $k=1$. If $G=\mathrm{GSp}(4)$, then we will assume that π_∞ is the unique representation of $\mathrm{GSp}(4,\mathbb{R})$ with trivial central character and a lowest weight vector of weight (k,k), where $k\geq 1$ (see [18] for more details). It is a holomorphic discrete series representation if $k\geq 3$, a limit of such if $k=2$, and a certain non-tempered representation if $k=1$. In each case, let v_∞ be the lowest weight vector; it is unique up to scalars.

We will construct a vector in $\pi\cong\bigotimes\pi_p$ by choosing local distinguished vectors v_p in each π_p and piecing them together to a "pure tensor" $\bigotimes v_p$. At the Archimedean place we have the lowest weight vector v_∞. For almost all primes p the representation π_p is *unramified*, meaning it has a non-zero vector fixed under the maximal compact subgroup $G(\mathbb{Z}_p)$ of $G(\mathbb{Q}_p)$; we let v_p be such a fixed vector. (In fact, the restricted tensor product $\bigotimes\pi_p$ is constructed with respect to a choice of such fixed vectors at almost all places, and pure tensors in $\bigotimes\pi_p$ are forced to have v_p be this fixed vector almost everywhere.)

Let p be a prime for which π_p is ramified. Let V_p be the space of π_p; which model we take for π_p is irrelevant. Consider the case $G=\mathrm{GL}(2)$ first. By the local newform theory of [8], the space V_p contains a non-zero vector fixed by the local congruence subgroup

$$\Gamma_0(p^n)=\mathrm{GL}(2,\mathfrak{o})\cap\begin{bmatrix}\mathbb{Z}_p & \mathbb{Z}_p\\ p^n\mathbb{Z}_p & \mathbb{Z}_p,\end{bmatrix}, \tag{3.1}$$

for some n. Let n_p be the minimal n such that the space of fixed vectors $V_p(n):=V_p^{\Gamma_0(p^n)}$ is non-zero; then it is known that $V_p(n_p)$ is one-dimensional. We let v_p be any non-zero vector in this one-dimensional space. It is known that p^{n_p} coincides with the *conductor* of the representation π_p. This implies that the integer $N=\prod p^{n_p}$ appears in the global functional equation of the L-function $L(s,\pi)$; see [15].

Now assume that $G=\mathrm{GSp}(4)$ and that π is not of "Yoshida type" or "CAP type", notions that are explained in [31]. In this case, by the results of [24], the space V_p contains a non-zero vector fixed under the local paramodular group

$$K(p^n)=\{g\in\mathrm{GSp}(4,\mathbb{Q}_p)\,|\,\det(g)\in\mathbb{Z}_p^\times\}\cap\begin{bmatrix}\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p\\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-n}\mathbb{Z}_p\\ \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p\\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p\end{bmatrix},\qquad(3.2)$$

for some n. Let n_p be the minimal n such that the space of fixed vectors $V_p(n):=V_p^{K(p^n)}$ is non-zero; then we know from [24] that $V_p(n_p)$ is one-dimensional. We let v_p be any non-zero vector in this one-dimensional space. As in the GL(2) case, the number p^{n_p} coincides with the conductor of the representation π_p.

For either group $G=\mathrm{GL}(2)$ or $G=\mathrm{GSp}(4)$, we have now chosen local vectors v_p at each place, canonical up to normalization. To have a unified notation for $p<\infty$, let us write C_p for the compact group under which v_p is invariant, i.e. $C_p=\Gamma_0(p^{n_p})$ in the GL(2) case, and $C_p=K(p^{n_p})$ in the GSp(4) case. Via $\pi\cong\otimes\pi_p$, the pure tensor $\bigotimes v_p$ corresponds to an automorphic form Φ on $G(\mathbb{A})$. Among other properties, Φ satisfies, for all $g\in G(\mathbb{A})$,

$$\Phi(\rho g)=\Phi(g),\quad \rho\in G(\mathbb{Q}),\qquad(3.3)$$

$$\Phi(gh)=\Phi(g),\quad h\in\prod_{p<\infty}C_p,\qquad(3.4)$$

$$\Phi(g\kappa)=\rho_k(\kappa)\Phi(g),\quad \kappa\in K_\infty.\qquad(3.5)$$

Property (3.3) holds simply because Φ is an automorphic form. Property (3.4) follows from our choice of local vectors v_p at all non-Archimedean places. Property (3.5) follows from our choice of v_∞. The group K_∞ is the identity component of the standard maximal compact subgroup of $G(\mathbb{R})$, and ρ_k is its weight k representation. Explicitly, in the GL(2) case, $K_\infty=\mathrm{SO}(2)$ and $\rho_k\big(\big[\begin{smallmatrix}\cos(\theta)&\sin(\theta)\\-\sin(\theta)&\cos(\theta)\end{smallmatrix}\big]\big)=e^{ik\theta}$. In the GSp(4) case, $K_\infty=\mathrm{Sp}(4,\mathbb{R})\cap O(4,\mathbb{R})$ consists of all matrices in $\mathrm{Sp}(4,\mathbb{R})$ of the form $\kappa=\big[\begin{smallmatrix}A&B\\-B&A\end{smallmatrix}\big]$, and $\rho_k(\kappa)=\det(A+iB)^k$.

The strong approximation theorem implies that $G(\mathbb{A})=G(\mathbb{Q})G(\mathbb{R})^\circ\prod_{p<\infty}C_p$, where $G(\mathbb{R})^\circ$ is the identity component of $G(\mathbb{R})$. In view of the above transformation properties, Φ is determined by its values on $G(\mathbb{R})^\circ$. Let $\mathcal{H}$ be the usual upper half plane if $G=\mathrm{GL}(2)$, and the Siegel upper half space of degree 2 if $G=\mathrm{GSp}(4)$. Using the property (3.5), it is easy to verify that there exists a unique function f on $\mathcal{H}$ for which

$$(f|_kg)(i1_n)=\Phi(g)\quad\text{for all } g\in G(\mathbb{R})^\circ,\qquad(3.6)$$

where $n=1$ in the GL(2) case and $n=2$ in the GSp(4) case. Since v_∞ is a lowest weight vector, the function f is holomorphic; see [1, Sec. 4.2]. One verifies immediately that

$$f|_k\gamma=f\quad\text{for } \gamma\in\Gamma:=G(\mathbb{Q})\cap G(\mathbb{R})^\circ\prod_{p<\infty}C_p.\qquad(3.7)$$

Evidently, $\Gamma = \Gamma_0(N)$ in the GL(2) case, and $\Gamma = K(N)$ in the GSp(4) case, where $N = \prod p^{n_p}$. The function f is holomorphic at the cusps of Γ, since Φ is of moderate growth. In fact, f vanishes at the cusps, since Φ is a cuspidal automorphic form, hence $f \in S_k(\Gamma)$.

We have thus extracted a cusp form f from π of the same level N as the representation. Since the numbers n_p above were chosen to be minimal, this f will be a *newform*. Here, for $G = \mathrm{GL}(2)$, we mean a newform in the traditional sense of Atkin–Lehner, and for $G = \mathrm{GSp}(4)$ we mean a paramodular newform as defined in [23]. In each case, a consequence of being new is that the level lowering operators annihilate f at each finite place, just as a consequence of being holomorphic is that the weight lowering operators annihilate f.

Especially in the GSp(4) case, there are other important choices for the local congruence subgroups instead of the $K(p^n)$ defined in (3.2). For example, we could have taken vectors v_p fixed under

$$\Gamma_0(p^n) = \mathrm{GSp}(4, \mathbb{Z}_p) \cap \begin{bmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix}, \tag{3.8}$$

for some minimal $n = n_p$. Any resulting f would then be a cusp form with respect to some $\Gamma_0(N) \subset \mathrm{Sp}(4, \mathbb{Z})$, but only rarely will $N = \prod p^{n_p}$ coincide with the conductor of π. Another choice of local congruence subgroup, important for this paper, is the *Borel congruence subgroup*

$$B(p^n) = \mathrm{GSp}(4, \mathbb{Z}_p) \cap \begin{bmatrix} \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix}. \tag{3.9}$$

If $n = 1$, this is also called an *Iwahori subgroup*. The resulting global congruence subgroups are the $B(N)$ defined in (2.2).

We note that the cusp forms f constructed from automorphic representations by the above procedure are automatically eigenforms for the local Hecke algebras $\mathcal{H}_p$ at all places p where $p \nmid N$. Conversely, the adelization Φ of any eigenform f (meaning eigenform for $\mathcal{H}_p$ for almost all good places p) can be used to generate a representation π. Automorphic representations generated by paramodular eigenforms with respect to $K(N)$ will always have local representations with a fixed vector for some $K(p^n)$, for $p^n \mid N$. Here, a technical issue arises in the GSp(4) case in that π need not be irreducible; this is due to the failure of strong multiplicity one[a] for GSp(4). It is still true though that the eigenforms constructed from all

[a]Examples for the failure of strong multiplicity one for GSp(4) are provided by the Yoshida liftings; see [2, 3], and the exposition in [26, Sec. 3].

cuspidal, automorphic representations π (with the correct Archimedean type and the correct choice of local congruence subgroups) span the entire space $S_k(\Gamma)$.

3.2. *Local representations*

In this section, we will take a closer look at the local non-Archimedean representations π_p occurring in an automorphic representation $\pi = \otimes\pi_p$ of either $G = \mathrm{GL}(2)$ or $G = \mathrm{GL}(4)$. Recall that π_p is an irreducible, admissible representation of $G(\mathbb{Q}_p)$. The only ramification that will occur in our examples is at the place $p = 2$. For all other primes π_p will always be an unramified principal series representation. Only representations with trivial central character will be relevant for us.

First, we consider characters χ of $\mathbb{Q}_2^\times$, meaning continuous homomorphisms $\chi : \mathbb{Q}_2^\times \to \mathbb{C}^\times$. If χ is trivial on $\mathbb{Z}_2^\times$, then we say that χ is unramified and write $a(\chi) = 0$. Otherwise let $a(\chi)$ be the smallest positive integer a such that χ is trivial on $1 + p^a\mathbb{Z}_2$, but not on $(1 + p^{a-1}\mathbb{Z}_2) \cap \mathbb{Z}_2^\times$. Note that $a(\chi) = 1$ is impossible, since $1 + 2\mathbb{Z}_2 = \mathbb{Z}_2^\times$.

The GL(2) *case*

We first recall some general facts for irreducible, admissible representations of $\mathrm{GL}(2, \mathbb{Q}_p)$ that hold for any p. Since we will be considering local representations only, we change notation and write π instead of π_p. We will assume throughout that π is infinite-dimensional and has trivial central character.

Recall that the (exponent of the) *conductor* $a(\pi)$ of π is characterized as the smallest integer n such that the space V of π contains a non-zero vector fixed under the congruence subgroup $\Gamma_0(p^n)$ defined in (3.1). If $V(n) = V^{\Gamma_0(p^n)}$ is the space of fixed vectors, then $\dim V(a(\pi) + i) = i + 1$ for $i \geq 0$. In other words, starting at level $a(\pi)$, the dimensions grow like $1, 2, 3, \ldots$. The essentially unique vector at level $a(\pi)$ is called a *local newform*; the spaces $V(a(\pi) + i)$ for $i > 0$ consist of *local oldforms*.

The *Atkin–Lehner element*

$$u_n = \begin{bmatrix} & 1 \\ p^n & \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Q}_p) \tag{3.10}$$

normalizes the group $\Gamma_0(p^n)$, and hence acts on the space $V(n)$. In particular, the Atkin–Lehner action on the one-dimensional space $V(a(\pi))$ defines a sign ± 1 canonically attached to the representation. It follows from the local functional equation for zeta integrals that this sign coincides with the value at $1/2$ of the ε-factor, so we will denote these signs by $\varepsilon(1/2, \pi)$. In case that a newform $f \in S_k(\Gamma_0(N))$ corresponds to an automorphic representation $\otimes\pi_p$, as in the previous section, the sign $\varepsilon(1/2, \pi_p)$ coincides with the classical Atkin–Lehner eigenvalue at p of the modular form f, for each prime p.

In a standard notation, as in [27], the *principal series representations* (with trivial central character) of $\mathrm{GL}(2, \mathbb{Q}_p)$ are written in the form $\pi = \chi \times \chi^{-1}$, where

χ is a character of $\mathbb{Q}_p^\times$ not of the form $|\cdot|^{\pm 1/2}$. The conductor of π can be calculated as $a(\pi) = 2a(\chi)$, and the Atkin–Lehner eigenvalue of the newform as $\varepsilon(1/2,\pi) = \chi(-1)$. The simplest case occurs if χ is unramified, i.e. $a(\chi) = 0$. Then π is called *unramified*, or *spherical*. In a global representation $\otimes\pi_p$, almost every π_p is unramified.

There are exactly two representations with conductor $a(\pi) = 1$, the *Steinberg representation* $\mathrm{St}_{\mathrm{GL}(2)}$, and its twist $\xi\mathrm{St}_{\mathrm{GL}(2)}$ by the unique non-trivial, unramified, quadratic character ξ of $\mathbb{Q}_2^\times$. The two representations can be distinguished by their Atkin–Lehner eigenvalue, as $\varepsilon(1/2, \mathrm{St}_{\mathrm{GL}(2)}) = -1$ and $\varepsilon(1/2, \xi\mathrm{St}_{\mathrm{GL}(2)}) = 1$.

From now on we consider only $p = 2$. Since there are no characters χ of $\mathbb{Q}_2^\times$ with $a(\chi) = 1$, any representation π (always assumed to have trivial central character) with $a(\pi) = 2$ must be *supercuspidal*, i.e. not accessible via parabolic induction. Using [34, Proposition 3.5] one can show that there is a unique such supercuspidal. We denote it by $\mathsf{sc}(4)$, since it contributes a factor $4 = 2^2$ to the conductor in a global situation. It is not difficult to show that $\varepsilon(1/2, \mathsf{sc}(4)) = -1$.

By the remark after [34, Theorem 3.9], or alternatively [7, Theorem 5], applied to the field $\mathbb{Q}_2$, there are exactly two supercuspidal representations π of $\mathrm{GL}(2,\mathbb{Q}_2)$ with trivial central character and $a(\pi) = 3$. We denote these two supercuspidals by $\mathsf{sc}(8)^+$ and $\mathsf{sc}(8)^-$. They are unramified twists of each other and can be distinguished by their Atkin–Lehner eigenvalue; we fix the notation such that $\varepsilon(1/2, \mathsf{sc}(8)^\pm) = \pm 1$.

It follows from the conductor formulas for principal series representations and for twists of the Steinberg representation that there are no other π with $a(\pi) = 3$. Table 1 summarizes all the representations of $\mathrm{GL}(2,\mathbb{Q}_2)$ with trivial central character and conductor up to 3.

The GSp(4) *case*

We next consider several irreducible, admissible representations of $\mathrm{GSp}(4,\mathbb{Q}_p)$ relevant for our analysis of spaces of Siegel modular forms. Even though we will only

Table 1. The irreducible, admissible, infinite-dimensional representations π of $\mathrm{GL}(2,\mathbb{Q}_2)$ with trivial central character and $a(\pi) \le 3$.

$a(\pi)$	π	$\varepsilon(1/2,\pi)$	$V(0)$	$V(1)$	$V(2)$	$V(3)$
0	Unramified	1	1	2	3	4
1	$\mathrm{St}_{\mathrm{GL}(2)}$	−1	0	1	2	3
	$\xi\mathrm{St}_{\mathrm{GL}(2)}$	1	0	1	2	3
2	$\mathsf{sc}(4)$	−1	0	0	1	2
3	$\mathsf{sc}(8)^+$	1	0	0	0	1
	$\mathsf{sc}(8)^-$	−1	0	0	0	1

Table 2. Some irreducible, admissible, infinite-dimensional representations Π of $\mathrm{GSp}(4,\mathbb{Q}_p)$ with trivial central character and $a(\Pi) \le 3$. For each types X and XIb, the supercuspidal representation π of $\mathrm{GL}(2,\mathbb{Q}_p)$ is assumed to have $a(\pi) = 3$, and the character σ is unramified. For type XIa, the supercuspidal representation π of $\mathrm{GL}(2,\mathbb{Q}_p)$ is assumed to have $a(\pi) = 2$, and the character σ is unramified. The number α for types X abbreviates $\sigma(p)$. The quantity $\sigma(p)$ for type XIa and XIb is ± 1.

$a(\Pi)$	Π	Type	$\varepsilon(1/2,\Pi)$	$V(0)$	$V(1)$	$V(2)$	$V(3)$	V^I	$T_{0,1}$	$T_{1,0}$
0	$\chi_1 \times \chi_2 \rtimes \sigma$	I	1	1	2	4	6	8	Irrelevant	
	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	IIb	1	1	1	2	2	4	Irrelevant	
2	$\tau(T,\nu^{-1/2}\sigma)$	VIb	1	0	0	0	0	1	—	—
3	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	IVa	$-\sigma(p)$	0	0	0	1	1	$\sigma(p)$	$-p^2$
	$\pi \rtimes \sigma$	X	$\varepsilon(1/2,\sigma\pi)$	0	0	0	1	0	$p^{\frac{3}{2}}(\alpha+\alpha^{-1})$	0
	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	XIa	$\sigma(p)$	0	0	0	1	0	$\sigma(p)p$	$-p^2$
	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	XIb	$\varepsilon(1/2,\sigma\pi)$	0	0	0	1	0	$\sigma(p)p(p+1)$	0
	Supercuspidal			0	0	0	1	0	0	$-p^2$

need the case $p = 2$, it is not more difficult to work with general p. Just as in the GL(2) case, all representations are assumed to be infinite-dimensional and to have trivial central character.

Table 2 lists all the representations Π that are important for our purposes. The precise meaning of the notation in the "Π" column need not concern us; it is taken from [27]. We shall mostly refer to these representations by their "type", which is simply a label. The symbols χ, χ_i, σ stand for characters of $\mathbb{Q}_p^\times$, which are all assumed to be unramified. The symbol π stands for a supercuspidal representation of $\mathrm{GL}(2,\mathbb{Q}_p)$. For types X and XIb, we assume $a(\pi) = 3$, and for type XIa we assume $a(\pi) = 2$. We make these assumptions so that the conductor $a(\Pi)$ is as listed in the first column. See [24, Table A.9], where the conductors for all non-supercuspidal representations of $\mathrm{GSp}(4,\mathbb{Q}_p)$ are listed.

Let V be the space of one of these representations Π. For $n \ge 0$, let $V(n)$ be the subspace of vectors fixed under the local paramodular group $K(p^n)$ defined in (3.2). We note that for all representations in Table 2, except for VIb, the conductor $a(\Pi)$ coincides with the minimal n such that $V(n) \ne 0$. This is a general feature of the paramodular theory. The VIb representation does not admit any paramodular vectors at all, but it shares an L-packet with a representation of type VIa, for which $a(\Pi)$ coincides with the minimal paramodular level.

Another feature of the paramodular theory, proven in [24], is that if n is minimal such that $V(n) \ne 0$, then $V(n)$ is one-dimensional. Any non-zero vector in this one-dimensional space is called a *local newform*. As in the GL(2) case, there is an

Atkin–Lehner element

$$u_n = \begin{bmatrix} & & 1 & \\ & & & -1 \\ p^n & & & \\ & -p^n & & \end{bmatrix} \in \mathrm{GSp}(4, \mathbb{Q}_p) \tag{3.11}$$

normalizing $K(p^n)$. The action of u_n on the one-dimensional $V(n)$ thus defines a sign canonically attached to each paramodular representation. Since this sign coincides with the value of the ε-factor at 1/2, we denote it by $\varepsilon(1/2, \Pi)$. Table 2 lists these ε-factors, except for supercuspidal representations, for which we make no general statement.

Also listed in Table 2 are the dimensions of the space of fixed vectors V^I under the *Iwahori subgroup*

$$I = \mathrm{GSp}(4, \mathbb{Q}_p) \cap \begin{bmatrix} \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix}. \tag{3.12}$$

Representations for which V^I is non-zero are called *Iwahori-spherical*. In a global setting the group I corresponds to the Borel congruence subgroup $B(p)$; see (2.2).

It remains to explain the last two columns in Table 2. The $T_{0,1}$ and $T_{1,0}$ are certain paramodular Hecke operators, which we consider in more detail in Sec. 5.1 below. They act on the one-dimensional space $V(n)$, where n is minimal such that $V(n) \neq 0$, and thus produce two eigenvalues. It is these eigenvalues that are listed in Table 2. The source of this information is [24, Tables A.9 and A.14]. The representations of types I and IIb also define $T_{0,1}$ and $T_{1,0}$ eigenvalues, given by slightly more complicated expressions; since they are irrelevant for our purposes, we refrain from listing them.

4. $S_{10}(K(8))$

The main goal of this section is to prove Theorem 4.3, which says that $\dim S_{10}(K(8)) = 6$.

4.1. *Cusp structure of* $K(8)$

In this section, we reduce the task of computing the Fourier expansion of a paramodular form slashed by an arbitrary element of $\mathrm{Sp}(4, \mathbb{Q})$ to a finite number of cases, one for each zero-dimensional cusp. The cusp structure of $K(8)$ is as follows: Define

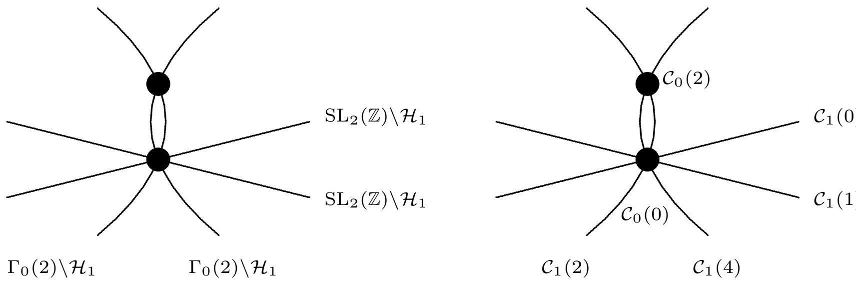

Fig. 1. The cusp structure of $K(8)$ and double coset representatives.

$\mathcal{C}_0(m)$ and $\mathcal{C}_1(m)$ by

$$\mathcal{C}_0(m) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & m & 1 & 0 \\ m & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{C}_1(m) = \begin{bmatrix} 1 & m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m & 1 \end{bmatrix},$$

$$P_{2,0}(\mathbb{Q}) = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \quad P_{2,1}(\mathbb{Q}) = \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \cap \mathrm{Sp}_2(\mathbb{Q});$$

the $\mathcal{C}_i(m)$ are the standard double coset representatives corresponding to the zero-dimensional and one-dimensional paramodular cusps given by $P_{2,i}(\mathbb{Q})$. Applying [21, Theorems 1.2 and 1.3], we have the following double coset decompositions:

$$\begin{aligned} \mathrm{Sp}(4,\mathbb{Q}) &= K(8)\mathcal{C}_0(0)P_{2,0}(\mathbb{Q}) \cup K(8)\mathcal{C}_0(2)P_{2,0}(\mathbb{Q}), \\ \mathrm{Sp}(4,\mathbb{Q}) &= K(8)\mathcal{C}_1(0)P_{2,1}(\mathbb{Q}) \cup K(8)\mathcal{C}_1(1)P_{2,1}(\mathbb{Q}) \\ &\quad \cup K(8)\mathcal{C}_1(2)P_{2,1}(\mathbb{Q}) \cup K(8)\mathcal{C}_1(4)P_{2,1}(\mathbb{Q}). \end{aligned}$$

So there are two zero-dimensional cusps and four one-dimensional cusps. When we slash a form $f \in S_k(K(8))$ and take the Fourier expansion, we may need the Fourier expansion of f at either of these two zero-dimensional cusps. This will come up when we apply certain Hecke operators later. Figure 1 shows how the cusps intersect each other.

4.2. *Upper bound on the dimension*

Denote

$$\mathcal{X}_2(N) = \left\{ \begin{bmatrix} a & b/2 \\ b/2 & cN \end{bmatrix} \,\middle|\, a,b,c \in \mathbb{Z}, a,c > 0, 4acN - b^2 > 0 \right\}.$$

These are the indices that occur in the Fourier expansion of a form in $S_k(K(N))$. Let $\langle A, B\rangle = \mathrm{tr}(AB)$. For $t \in \mathcal{X}_2(N)$, define

$$m_N^*(t) = \min\left\{ \frac{1}{N} \left\langle gt^tg, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \,\middle|\, g \in \Gamma_0^*(N) \right\},$$

where $\Gamma_0^*(N)$ is generated by $\Gamma_0(N)$ and all the Atkin–Lehner involutions. Let $\mathcal{P}_n(\mathbb{R})$ be positive definite symmetric real matrices. Let $\phi : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^+$ be *type one*, which means

(i) $\phi(\alpha t) = \alpha\phi(t)$, for all $\alpha > 0$ and $t \in \mathcal{P}_n(\mathbb{R})$,
(ii) $\phi(s + t) \geq \phi(s) + \phi(t)$ for all $s, t \in \mathcal{P}_n(\mathbb{R})$.

For $\lambda > 0$, define

$$J_N^*(\phi, \lambda) = \max\{m_N^*(t) : t \in \mathcal{X}_2(N), \phi(t) \leq \lambda\}.$$

The following is [5, Theorem 7.3].

Theorem 4.1. *Let ϕ be a type one function that is a $\mathrm{GL}(2, \mathbb{Z})$-class function. Let $f \in S_k(K(N))$ be an eigenform under all paramodular Atkin–Lehner involutions. Let*

$$f\left(\begin{bmatrix} \tau & z \\ z & w \end{bmatrix}\right) = \sum_{j=1}^{\infty} \phi_{Nj}(\tau, z)(\exp(2\pi i w))^{Nj}$$

be its Fourier–Jacobi expansion, where $\phi_{Nj} \in J_{k,Nj}^{\mathrm{cusp}}$ are Jacobi cusp forms. Let

$$\lambda = \phi\left(\frac{1}{30}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\right) kN \prod_{q^r \| N} \frac{q^r + q^{r-2}}{q^r + 1}.$$

If $\phi_{Nj} = 0$ for all $j \leq J_N^*(\phi, \lambda)$, then $f = 0$.

Applying this to $S_{10}(K(8))$ and using the reduced trace function $\phi = \tilde{\mathrm{tr}}$, where $\tilde{\mathrm{tr}}(t) = \min\{\mathrm{tr}(gt\,{}^tg) \,|\, g \in \mathrm{GL}(2, \mathbb{Z})\}$, we calculate that $\lambda = \frac{160}{9}$ and $J_8^*(\tilde{\mathrm{tr}}, \frac{160}{9}) = 9$. We conclude that nine Fourier–Jacobi coefficients determine a paramodular Atkin–Lehner eigenform in $S_{10}(K(8))$. Note that because $8 = 2^3$, there is only one Atkin–Lehner involution. We run the "Jacobi restriction" method with a chosen determinant bound of $B = 800$. This value of B was just a choice. Here is a short summary of the Jacobi restriction method; see [5, 13].

Fix an Atkin–Lehner sign $\epsilon = 1$ or $\epsilon = -1$.

(i) Find bases of Jacobi cusp forms $J_{10,8m}^{\mathrm{cusp}}$ for $m = 1, \ldots, 9$. Call such a basis $\{g_{mj}\}_{j=1}^{d_m}$, where $d_m = \dim J_{10,8m}^{\mathrm{cusp}}$. The dimensions of these spaces are 4, 9, 13, 19, 24, 28, 34, 40, 43, respectively. These bases were found by searching for theta blocks of the shape 26 thetas over 6 etas, and possibly using a down operator from Jacobi forms of higher index on a theta block of this same shape.

(ii) Find the subspace $(\phi_1, \dots, \phi_9) \in \prod_{m=1}^{9} J_{10,8m}^{\mathrm{cusp}}$ such that:

- (Involution condition) For all $(n, r, m) \in \mathbb{Z}^3$ with $1 \le m \le 9$, $1 \le n \le 9$, and with the determinant bound $0 < 4nm \cdot 8 - r^2 \le B$, we have

$$c(n, r; \phi_m) = \epsilon c(m, -r; \phi_n).$$

- (Siegel modular form consistency condition) Let $\hat{\Gamma}^0(8) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathbb{Z}) \mid b \in 8\mathbb{Z} \right\}$. For all (n_i, r_i, m_i), $i = 1, 2$, where $\left[\begin{smallmatrix} n_1 & r_1/2 \\ r_1/2 & 8m_1 \end{smallmatrix}\right] = {}^t g \left[\begin{smallmatrix} n_2 & r_2/2 \\ r_2/2 & 8m_2 \end{smallmatrix}\right] g$ for some $g \in \hat{\Gamma}^0(8)$, we have, whenever $1 \le m_1, m_2 \le 9$,

$$c(n_1, r_1; \phi_{m_1}) = \det(g)^k c(n_2, r_2; \phi_{m_2}).$$

(iii) Here, we are really solving for linear conditions on α_{mj}, $1 \le m \le 9$, $1 \le j \le d_m$, such that the Jacobi forms $\phi_m = \sum_{j=1}^{d_m} \alpha_{mj} g_{mj}$ satisfy these two conditions. The dimension of the null space of these relations is an upper bound on the dimension of $S_{10}(K(8))^\epsilon$, which is the subspace of $S_{10}(K(8))$ where the paramodular Atkin–Lehner sign is ϵ.

With our choice of $B = 800$, the above instructions, when run with $\epsilon = 1, -1$, return the results

$$\dim S_{10}(K(8))^+ \le 6, \quad \dim S_{10}(K(8))^- = 0. \tag{4.1}$$

So the total dimension of $S_{10}(K(8))$ is at most 6.

4.3. *Lower bound on the dimension*

The theory of theta blocks is due to Gritsenko, Skoruppa, and Zagier [10], see also [5, 22] for applications. We review a simpler version of theta blocks that fits our needs. Let ϑ be Jacobi's odd theta function and η be the Dedekind eta function, letting $e(z) = e^{2\pi i z}$, $q = e(\tau)$, and $\zeta = e(z)$,

$$\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{(2n+1)^2}{8}} \zeta^{\frac{2n+1}{2}}, \quad \eta(\tau) = \sum_{n \in \mathbb{Z}^+} \left(\frac{12}{n}\right) q^{n^2/24}.$$

For positive integers $k, d_1, \dots, d_\ell$, define a theta block to be

$$\mathrm{TB}_k(d_1, \dots, d_\ell) = \eta(z)^{2k-\ell} \prod_{i=1}^{\ell} \vartheta(\tau, d_i z).$$

Theorem 4.2 (Gritsenko, Skoruppa, Zagier). *Define* $\bar{B}_2(x) = (x - \lfloor x \rfloor)^2 - (x - \lfloor x \rfloor) + \frac{1}{6}$. *Let* $k, d_1, \dots, d_\ell$ *be positive integers. If*

(i) $\frac{k}{12} + \frac{1}{2} \sum_{i=1}^{\ell} \bar{B}_2(d_i x) > 0$ *for all* $x \in [0, 1]$,

(ii) $12 | (k + \ell)$,

(iii) $\sum_{i=1}^{\ell} d_i^2 = 2m$ *for some* $m \in \mathbb{Z}^+$,

then $\mathrm{TB}_k(d_1, \dots, d_\ell) \in J_{k,m}^{\mathrm{cusp}}$.

We will use the down operator, see [16], $W_\ell : J^{\mathrm{cusp}}_{k,m\ell} \to J^{\mathrm{cusp}}_{k,m}$ defined for prime ℓ by

$$(\phi \,|\, W_\ell)(\tau, z) = \ell^{-2} \sum_{\beta,\gamma \bmod \ell} \phi\left(\frac{\tau+\gamma}{\ell}, \frac{z+\beta}{\ell}\right) + \ell^{k-2} \sum_{\alpha \bmod \ell} \phi(\ell\tau, z+\alpha\tau) e(m(2\alpha z + \alpha^2 \tau)).$$

We now explain how to use the above theorem and the down operators to span spaces of Jacobi forms. By using the formula for dimensions of Jacobi forms from [33], we know $\dim J^{\mathrm{cusp}}_{10,8} = 4$. We find a basis by searching for theta blocks in $J^{\mathrm{cusp}}_{10,8}$, and if there are not enough then we search for theta blocks in $J^{\mathrm{cusp}}_{10,8p}$ for $p = 2, 3, 5, \ldots$ and apply the down operator W_p to get back to $J^{\mathrm{cusp}}_{10,8}$. Using this method, here is one basis of $J^{\mathrm{cusp}}_{10,8}$:

$$\Xi_1 = \mathrm{TB}_{10}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,4,5) \,|\, W_5,$$

$$\Xi_2 = \mathrm{TB}_{10}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,3,3,3,4) \,|\, W_5,$$

$$\Xi_3 = \mathrm{TB}_{10}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,5) \,|\, W_5,$$

$$\Xi_4 = \mathrm{TB}_{10}(1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,3,3,3) \,|\, W_5.$$

We construct paramodular forms via $\mathrm{Grit}(\Xi_i) \in S_{10}(K(8))$ for $i = 1, 2, 3, 4$, where

$$\mathrm{Grit} : J^{\mathrm{cusp}}_{k,N} \to S_k(K(N))$$

is the Gritsenko lift; see [9]. Let $\Xi_5 = \mathrm{TB}_5(1,1,1,1,2,2,2) \in J^{\mathrm{cusp}}_{5,8}$, and consider the lift $\mathrm{Grit}(\Xi_5) \in S_5(K(8))$. Let

$$h_i = \mathrm{Grit}(\Xi_i) \quad \text{for } i = 1,2,3,4; \quad h_5 = \mathrm{Grit}(\Xi_5)^2; \quad h_6 = \mathrm{Grit}(\Xi_5)^2 | T(3).$$

Computing several Fourier coefficients, these six forms are seen to be linearly independent, and therefore, combined with the upper bound $\dim S_{10}(K(8)) \le 6$, we have the following theorem.

Theorem 4.3. *$\dim S_{10}(K(8)) = 6$.*

Now that we know the relations generated by the Jacobi restriction method up to nine Jacobi coefficients with determinant bound 800 actually specify the space $S_{10}(K(8))$, we can use these relations to determine all Fourier coefficients within the first nine Jacobi coefficients whose indices have determinant bounded by 800. It turns out that, up to $\hat{\Gamma}^0(8)$ equivalence, there are 7320 Fourier coefficient indices $\left[\begin{smallmatrix} n & r/2 \\ r/2 & 8m \end{smallmatrix}\right] \in \mathcal{X}_2(8)$ satisfying $32nm - r^2 \le 3200$ and $m \le 9$. If we needed to, we could try to prolong these expansions further (either by going farther with the Jacobi expansion method or by expanding $h_1, \ldots, h_6$ further). But it turns out that the set of coefficients that we already have is sufficient for the calculations in this paper.

Table 3. The eigenforms in $S_{10}(K(8))$ expressed as a linear combination $\sum_{i=1}^{6} c_i h_i$.

f	c_1	c_2	c_3	c_4	c_5	c_6
$N(8)^{\mathrm{a}}$	$\frac{31030273487}{37479316069125}$	$\frac{673571370572}{37479316069125}$	$\frac{649935283}{2498621071275}$	$\frac{8584961865272}{112437948207375}$	40	$\frac{5}{459}$
$N(8)^{\mathrm{b}}$	$-\frac{2848884919}{62465526781875}$	$\frac{252938075876}{62465526781875}$	$-\frac{1011851}{4164368452125}$	$-\frac{12288827502424}{187396580345625}$	-330	$-\frac{11}{612}$
L_1	$\frac{40223180409 + 68555687\sqrt{4449}}{19597028010000}$	$\frac{1115262352089 + 405818327\sqrt{4449}}{19597028010000}$	$\frac{454514743 + 223249\sqrt{4449}}{653234267000}$	$\frac{1010469321033 + 211320919\sqrt{4449}}{7348885503750}$	0	0
L_2	$\frac{40223180409 - 68555687\sqrt{4449}}{19597028010000}$	$\frac{1115262352089 - 405818327\sqrt{4449}}{19597028010000}$	$\frac{454514743 - 223249\sqrt{4449}}{653234267000}$	$\frac{1010469321033 - 211320919\sqrt{4449}}{7348885503750}$	0	0
$L(8)^{\mathrm{a}}$	$\frac{-38856656771 - 2975285952\sqrt{114}}{408271416875}$	$\frac{-16(57368515151 + 4568874012\sqrt{114})}{408271416875}$	$\frac{-2265348177 - 207937024\sqrt{114}}{81654283375}$	$\frac{-8(920781343327 + 70177771824\sqrt{114})}{1224814250625}$	0	0
$L(8)^{\mathrm{b}}$	$\frac{-38856656771 + 2975285952\sqrt{114}}{408271416875}$	$\frac{-16(57368515151 - 4568874012\sqrt{114})}{408271416875}$	$\frac{-2265348177 + 207937024\sqrt{114}}{81654283375}$	$\frac{-8(920781343327 - 70177771824\sqrt{114})}{1224814250625}$	0	0

Computing the action of the Hecke operator $T(3)$ on this basis, we get eigenforms $N(8)^{\mathrm{a}}$, $N(8)^{\mathrm{b}}$ and $L_1, L_2, L(8)^{\mathrm{a}}, L(8)^{\mathrm{b}}$. To aid the presentation, we write "L" for "lift" and "N" for "non-lift". The number 8 in parentheses indicates that the forms live in an automorphic representation of conductor 8, as we will see later. The forms L_1 and L_2 turn out to be oldforms, living in an automorphic representation of conductor 1.

Table 3 expresses each eigenform as a linear combination $\sum_{i=1}^{6} c_i h_i$ of the basis $h_1, \ldots h_6$. Note that $L_1, L_2, L(8)^{\mathrm{a}}, L(8)^{\mathrm{b}}$ are Gritsenko lifts because they are each a linear combination of $h_1, \ldots, h_4$, which are Gritsenko lifts. The eigenforms $N(8)^{\mathrm{a}}$ and $N(8)^{\mathrm{b}}$ are non-lifts. The test for being a lift is simple: $f \in S_k(K(N))$ is a Gritsenko lift if and only if $f = \mathrm{Grit}(\phi_1)$, where $\phi_1 \in J_{k,N}^{\mathrm{cusp}}$ is the first Fourier–Jacobi coefficient of f. Table 4 shows some Fourier coefficients of these six eigenforms; note that L_2 is a conjugate of L_1 and $L(8)^{\mathrm{b}}$ is a conjugate of $L(8)^{\mathrm{a}}$.

Table 5 shows the eigenvalues of the Hecke operators $T(3)$, $T(5)$, $T(7)$ and $T(9)$ on the eigenforms in $S_{10}(K(8))$. It is possible to compute the action of these Hecke operators because we have expansions up to nine Jacobi coefficients. Letting

Table 4. Fourier coefficients of the eigenforms in $S_{10}(K(8))$. The index $t = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is denoted by $[a, b, c]$.

t	$a(t; N(8)^{\mathrm{a}})$	$a(t; N(8)^{\mathrm{b}})$	$a(t; L_1)$	$a(t; L(8)^{\mathrm{a}})$
$[1, 5/2, 8]$	1	1	1	$-9 + \sqrt{114}$
$[1, 7/2, 16]$	15	15	15	$281 - 17\sqrt{114}$
$[2, 7/2, 8]$	15	15	15	$281 - 17\sqrt{114}$
$[1, 2, 8]$	64	0	$-49 + \sqrt{4449}$	0
$[1, 3/2, 8]$	469	-715	-171	$-157 + 117\sqrt{114}$
$[3, 13/2, 16]$	-171	341	-171	$-157 + 117\sqrt{114}$
$[2, 13/2, 24]$	-171	341	-171	$-157 + 117\sqrt{114}$
$[1, 1, 8]$	896	-800	$8(-81 + \sqrt{4449})$	$256(-9 + \sqrt{114})$
$[1, 3, 16]$	128	800	$8(-17 + \sqrt{4449})$	$-256(-9 + \sqrt{114})$
$[2, 3, 8]$	128	800	$8(-17 + \sqrt{4449})$	$-256(-9 + \sqrt{114})$
$[2, 5, 16]$	-640	-800	$8(47 + \sqrt{4449})$	$256(-9 + \sqrt{114})$
$[1, 1/2, 8]$	-214	970	426	$4518 - 374\sqrt{114}$
$[4, 15/2, 16]$	426	-86	426	$4518 - 374\sqrt{114}$
$[2, 15/2, 32]$	426	-86	426	$4518 - 374\sqrt{114}$
$[1, 0, 8]$	-2432	1088	$-18(-49 + \sqrt{4449})$	-4096
$[1, 4, 24]$	128	-1088	$-18(-49 + \sqrt{4449})$	4096
$[3, 4, 8]$	128	-1088	$-18(-49 + \sqrt{4449})$	4096
$[3, 8, 24]$	2688	1088	$-18(-49 + \sqrt{4449})$	-4096

Table 5. Hecke eigenvalues and eigenforms, $f \,|\, T(q) = \lambda_q f$, for $f \in S_{10}(K(8))$.

f	$3^7\lambda_3$	$5^7\lambda_5$	$7^7\lambda_7$	$9^7\lambda_9$
$N(8)^{\mathrm{a}}$	-18360	741900	-2990960	-2973591
$N(8)^{\mathrm{b}}$	-3672	-253300	13196624	-167855895
L_1	21960	1317900	49344400	293343849
L_2	21960	1317900	49344400	293343849
$L(8)^{\mathrm{a}}$	$72(445+16\sqrt{114})$	$1947788 - 78336\sqrt{114}$	$36652112 - 1822464\sqrt{114}$	$81(8943385 + 538112\sqrt{114})$
$L(8)^{\mathrm{b}}$	$72(445-16\sqrt{114})$	$1947788 + 78336\sqrt{114}$	$36652112 + 1822464\sqrt{114}$	$81(8943385 - 538112\sqrt{114})$

Table 6. The 3-Euler factors of the eigenforms in $S_{10}(K(8))$. Arithmetic (respectively, analytic) normalization indicates that the factors fit into an L-function with a functional equation relating s and $2k-2-s$ (respectively, $1-s$). If the arithmetic normalization is $Q_p(x,f)$, then the analytic normalization is $Q_p(p^{\frac{3}{2}-k}x, f)$. The actual Euler factor is $Q_3(3^{-s}, f)^{-1}$.

f	$Q_3(x,f)$	
	Arithmetic normalization	Analytic normalization
$N(8)^{\mathrm{a}}$	$1+18360x+297016470x^2 + 2371013392680x^3+3^{34}x^4$	$1+\frac{680}{3^{11/2}}x+\frac{5030}{3^7}x^2+\frac{680}{3^{11/2}}x^3+x^4$
$N(8)^{\mathrm{b}}$	$1+3672x+138292758x^2 + 474202678536x^3+3^{34}x^4$	$1+\frac{136}{3^{11/2}}x+\frac{2342}{3^7}x^2+\frac{136}{3^{11/2}}x^3+x^4$
L_1	$(1-3^8x)(1-3^9x)(1+4284x+3^{17}x^2)$	$\left(1-\frac{1}{\sqrt{3}}x\right)(1-\sqrt{3}x)\left(1+\frac{476}{3^{13/2}}x+x^2\right)$
L_2	$(1-3^8x)(1-3^9x)(1+4284x+3^{17}x^2)$	$\left(1-\frac{1}{\sqrt{3}}x\right)(1-\sqrt{3}x)\left(1+\frac{476}{3^{13/2}}x+x^2\right)$
$L(8)^{\mathrm{a}}$	$(1-3^8x)(1-3^9x)(1-(5796 + 1152\sqrt{114})x+3^{17}x^2)$	$\left(1-\frac{1}{\sqrt{3}}x\right)(1-\sqrt{3}x) \times\left(1-\left(\frac{644}{3^{13/2}}+\frac{128\sqrt{38}}{3^6}\right)x+x^2\right)$
$L(8)^{\mathrm{b}}$	$(1-3^8x)(1-3^9x)(1-(5796 - 1152\sqrt{114})x+3^{17}x^2)$	$\left(1-\frac{1}{\sqrt{3}}x\right)(1-\sqrt{3}x) \times\left(1-\left(\frac{644}{3^{13/2}}-\frac{128\sqrt{38}}{3^6}\right)x+x^2\right)$

$\Gamma = K(8)$, the definitions of $T(q^2)$ and $T(q)$, for a good prime q, are

$$T(q^2) = \Gamma\begin{bmatrix}1&&&\\&1&&\\&&q^2&\\&&&q^2\end{bmatrix}\Gamma + \Gamma\begin{bmatrix}1&&&\\&q&&\\&&q^2&\\&&&q\end{bmatrix}\Gamma + \Gamma\begin{bmatrix}q&&&\\&q&&\\&&q&\\&&&q\end{bmatrix}\Gamma;$$

$$T(q) = \Gamma \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & q & \\ & & & q \end{bmatrix} \Gamma.$$

Table 6 shows the 3-Euler factors of the eigenforms in $S_{10}(K(8))$. We note that $(1-\sqrt{3}^{-1}x)(1-\sqrt{3}x)$ is the 3-Euler factor of $\zeta(s-1/2)\zeta(s+1/2)$, which must be a factor of the L-function of a Gritsenko lift. All other polynomials in this table have roots of absolute value 1. The spin q-Euler factor $Q_q(x,f)$, for a good prime q, is given, in the arithmetic normalization, by

$$Q_q(x,f) = 1 - q^{k-3}\lambda_q x + q^{2k-6}(\lambda_q^2 - \lambda_{q^2} - q^2)x^2 - q^{3k-6}\lambda_q x^3 + q^{4k-6}x^4.$$

We note from Table 6 that L_1 and L_2 have the same Euler factor at $p=3$. We will see later that this is explained by the fact that L_1 and L_2 are vectors in the same automorphic representation. In fact, they are oldforms originating from Igusa's X_{10}, which also has this same 3-Euler factor. Similarly, the factor for $N(8)^{\mathrm{a}}$ given in Table 6 is the same as the factor for the cusp form $F_{10} \in S_{10}(B(2))$ given by Ibukiyama in [11, Theorem 3.3]. This is also explained by the fact that both modular forms lie in the same automorphic representation; see Table 11.

5. Paramodular Hecke Operators

We introduce two Hecke operators $T_{0,1}(p)$ and $T_{1,0}(p)$ acting on $S_k(K(N))$ for $p \mid N$. We calculate the eigenvalues of these operators on the eigenforms in $S_{10}(K(8))$ constructed in the previous section. Knowledge of these eigenvalues is key to determining the local components at $p=2$ of the underlying automorphic representations.

5.1. *Classical and adelic Hecke operators*

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4,\mathbb{Q}_p)$ with trivial central character. As before, let $V(n)$ be the subspace of vectors fixed by the paramodular group $K(p^n)$ defined in (3.2). Any double coset $T = K(p^n)gK(p^n)$, where $g \in \mathrm{GSp}(4,\mathbb{Q}_p)$, defines an endomorphism of $V(n)$ by

$$Tv = \sum_{i=1}^{r} \pi(g_i)v, \quad \text{if } T = \bigsqcup_{i=1}^{r} g_i K(p^n).$$

Of particular interest are the double cosets

$$T_{0,1} = K(p^n) \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K(p^n), \quad T_{1,0} = K(p^n) \begin{bmatrix} p & & & \\ & p^2 & & \\ & & p & \\ & & & 1 \end{bmatrix} K(p^n). \quad (5.1)$$

If $n = 0$, then $T_{0,1}$ and $T_{1,0}$, together with a central element, generate the local Hecke algebra. This is no longer the case if $n > 0$, but the action of these two elements on the minimal paramodular level still reveals interesting information about the representation π. Recall that if n is minimal such that $V(n) \neq 0$, then $\dim V(n) = 1$. The action of $T_{0,1}$ and $T_{1,0}$ thus gives two eigenvalues. These can be calculated for any π that admits non-zero paramodular vectors; see [24, Table A.14]. For some of the representations of interest to us, we have listed these eigenvalues in Table 2. The main goal of this section is to rewrite the local operators $T_{0,1}$ and $T_{1,0}$ in terms of operators on Siegel paramodular forms.

Lemma 5.1. *We have the following coset decompositions in* $\mathrm{GSp}(4, \mathbb{Q}_p)$.

(i) *For any* $n \geq 1$,

$$
\begin{aligned}
K(p^n) \begin{bmatrix} p &&& \\ & p && \\ && 1 & \\ &&& 1 \end{bmatrix} K(p^n)
&= \bigsqcup_{x,y,z \in \mathbb{Z}/p\mathbb{Z}} \begin{bmatrix} 1 && x & y \\ & 1 & y & zp^{-n} \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} p &&& \\ & p && \\ && 1 & \\ &&& 1 \end{bmatrix} K(p^n) \\
&\sqcup \bigsqcup_{x,z \in \mathbb{Z}/p\mathbb{Z}} \begin{bmatrix} 1 &&& \\ x & 1 && zp^{-n} \\ && 1 & -x \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & p && \\ && p & \\ &&& 1 \end{bmatrix} K(p^n) \\
&\sqcup \bigsqcup_{x,y \in \mathbb{Z}/p\mathbb{Z}} \begin{bmatrix} 1 & -yp^n & x & \\ & 1 && \\ && 1 & \\ && yp^n & 1 \end{bmatrix} \begin{bmatrix} p &&& \\ & 1 && \\ && 1 & \\ &&& p \end{bmatrix} K(p^n) \\
&\sqcup \bigsqcup_{x \in \mathbb{Z}/p\mathbb{Z}} \begin{bmatrix} 1 &&& \\ & 1 && \\ & xp^n & 1 & \\ xp^n &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && p & \\ &&& p \end{bmatrix} K(p^n).
\end{aligned}
\tag{5.2}
$$

In particular, the number of cosets for $T_{0,1}$ *is* $p(p+1)^2$.

(ii) *For any* $n \geq 1$,

$$K(p^n)\begin{bmatrix} p &&& \\ & p^2 && \\ && p & \\ &&& 1 \end{bmatrix}K(p^n)$$

$$= \bigsqcup_{x,y\in\mathbb{Z}/p\mathbb{Z}} \bigsqcup_{z\in\mathbb{Z}/p^2\mathbb{Z}} \begin{bmatrix} 1 &&& \\ x & 1 && \\ && 1 & -x \\ &&& 1 \end{bmatrix}\begin{bmatrix} 1 &&& y \\ & 1 & y & zp^{-n} \\ && 1 & \\ &&& 1 \end{bmatrix}$$

$$\times \begin{bmatrix} p &&& \\ & p^2 && \\ && p & \\ &&& 1 \end{bmatrix}K(p^n)$$

$$\sqcup \bigsqcup_{x,y,z\in\mathbb{Z}/p\mathbb{Z}} \begin{bmatrix} 1 & -yp^n && \\ & 1 && \\ && 1 & \\ && yp^n & 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ & xp^n & 1 & \\ xp^n & zp^{n+1} && 1 \end{bmatrix}$$

$$\begin{bmatrix} p &&& \\ & 1 && \\ && p & \\ &&& p^2 \end{bmatrix}K(p^n). \tag{5.3}$$

Alternatively,

$$K(p^n)\begin{bmatrix} p &&& \\ & p^2 && \\ && p & \\ &&& 1 \end{bmatrix}K(p^n)$$

$$= \bigsqcup_{x,y\in\mathbb{Z}/p\mathbb{Z}} \bigsqcup_{z\in\mathbb{Z}/p^2\mathbb{Z}} \begin{bmatrix} 1 &&& \\ x & 1 && \\ && 1 & -x \\ &&& 1 \end{bmatrix}\begin{bmatrix} 1 &&& y \\ & 1 & y & zp^{-n} \\ && 1 & \\ &&& 1 \end{bmatrix}$$

$$\times \begin{bmatrix} p & & & \\ & p^2 & & \\ & & p & \\ & & & 1 \end{bmatrix} K(p^n)$$

$$\sqcup \bigsqcup_{x,y \in \mathbb{Z}/p\mathbb{Z}} \begin{bmatrix} 1 & -yp^n & & \\ & 1 & & \\ & & 1 & \\ & & yp^n & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & xp^n & 1 & \\ xp^n & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} p & & & \\ & 1 & & \\ & & p & \\ & & & p^2 \end{bmatrix} K(p^n)$$

$$\sqcup \bigsqcup_{\substack{x,y \in \mathbb{Z}/p\mathbb{Z} \\ z \in (\mathbb{Z}/p\mathbb{Z})^\times}} p \begin{bmatrix} 1 & -yp^n & & \\ & 1 & & \\ & & 1 & \\ & & yp^n & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ -xp^{-1} & 1 & & zp^{-n-1} \\ & & 1 & xp^{-1} \\ & & & 1 \end{bmatrix} K(p^n). \tag{5.4}$$

In particular, the number of cosets for $T_{1,0}$ *is* $p^3(p+1)$.

Proof. Single coset representatives for $T_{0,1}$ and $T_{1,0}$ are given in [24, Lemma 6.1.2]. We modify these representatives slightly by moving the element t_n appearing in these representatives to the right and absorbing t_n into $K(p^n)$; this gives us (5.2) and (5.3). To obtain the alternative formula (5.4) for $T_{1,0}$, we use the matrix identity

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & zp^{n+1} & & 1 \end{bmatrix} \begin{bmatrix} p & & & \\ & 1 & & \\ & & p & \\ & & & p^2 \end{bmatrix}$$

$$= p\begin{bmatrix}1&&&\\&1&&z^{-1}p^{-n-1}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&&&-p^{-n}\\&&1&\\&p^n&&\end{bmatrix}$$

$$\times\begin{bmatrix}1&&&\\&1&&zp^{-n+1}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&z&&\\&&1&\\&&&z^{-1}\end{bmatrix},$$

which holds for $z \in (\mathbb{Z}/p\mathbb{Z})^\times$. Since the three right-most matrices are in $K(p^n)$, we can rewrite the terms in the second line of (5.3) for which $z \in (\mathbb{Z}/p\mathbb{Z})^\times$ as follows:

$$\bigsqcup_{\substack{x,y\in\mathbb{Z}/p\mathbb{Z}\\ z\in(\mathbb{Z}/p\mathbb{Z})^\times}} p\begin{bmatrix}1&-yp^n&&\\&1&&\\&&1&\\&&yp^n&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&xp^n&1&\\xp^n&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&z^{-1}p^{-n-1}\\&&1&\\&&&1\end{bmatrix}K(p^n).$$

Since

$$\begin{bmatrix}1&&&\\&1&&\\&xp^n&1&\\xp^n&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&z^{-1}p^{-n-1}\\&&1&\\&&&1\end{bmatrix}$$

$$=\begin{bmatrix}1&&&\\-xz^{-1}p^{-1}&1&&z^{-1}p^{-n-1}\\&&1&xz^{-1}p^{-1}\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\-x^2z^{-1}p^{n-1}&xp^n&1&\\xp^n&&&1\end{bmatrix},$$

the same terms also equal the last line in (5.4); note that we have replaced x by xz and then z by z^{-1}. □

Now let $F \in S_k(K(N))$ for some positive integer N. We recall the definition of the associated adelic function Φ on $G(\mathbb{A})$, where $G = \mathrm{GSp}(4)$. Let $N = \prod p^{n_p}$ be the prime factorization of N. The corresponding global compact subgroup is

$$K_N := K_\infty \times \prod_{p<\infty} K(p^{n_p});$$

if $n_p = 0$, then $K(p^{n_p}) = G(\mathbb{Z}_p)$. It follows from strong approximation for $\mathrm{Sp}(4)$ that $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^\circ K_N$. There is then a unique function $\Phi : G(\mathbb{A}) \to \mathbb{C}$ that is left invariant under $G(\mathbb{Q})$, right invariant under K_N, and satisfies

$$\Phi(g) = (F \,|\, g)(i1_2), \quad \text{for } g \in G(\mathbb{R})^{\circ}. \tag{5.5}$$

We say that Φ is the automorphic form corresponding to F.

The double cosets $T_{0,1}$ and $T_{1,0}$, defined with respect to p^n, act not only on local representations, but also on automorphic forms that are right invariant under $K(p^n)$. In particular, we may apply them to the function Φ. The action is given by right translation, so that $(T_{0,1}\Phi)(g) = \sum_i \Phi(gh_i)$, where h_i runs through the representatives given in (5.2) for $n = n_p$; similarly for $T_{1,0}$. Here, g is any element in $G(\mathbb{A})$, but the h_i, which are rational matrices, are embedded at the place p only.

Proposition 5.2. *Let N be a positive integer, and let $F \in S_k(K(N))$. Let p be a prime such that $p^n \| N$ with $n \geq 1$. Let M be any integer such that $M(N/p^n) \equiv 1$ mod p.*

(i) *Define*

$$\begin{aligned}
T_{0,1}(p)F = {} & \sum_{x,y,z \in \mathbb{Z}/p\mathbb{Z}} F | \begin{bmatrix} 1 &&& \\ & 1 && \\ && p & \\ &&& p \end{bmatrix} \begin{bmatrix} 1 && x & y \\ & 1 & y & zp^{-n} \\ && 1 & \\ &&& 1 \end{bmatrix} \\
& + \sum_{x,z \in \mathbb{Z}/p\mathbb{Z}} F | \begin{bmatrix} p &&& \\ & 1 && \\ && 1 & \\ &&& p \end{bmatrix} \begin{bmatrix} 1 &&& \\ x & 1 && zp^{-n} \\ && 1 & -x \\ &&& 1 \end{bmatrix} \\
& + \sum_{x,y \in \mathbb{Z}/p\mathbb{Z}} F | \begin{bmatrix} 1 &&& \\ & p && \\ && p & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 & -yMN & x & \\ & 1 && \\ && 1 & \\ && yMN & 1 \end{bmatrix} \\
& + \sum_{x \in \mathbb{Z}/p\mathbb{Z}} F | \begin{bmatrix} p &&& \\ & p && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ & xMN & 1 & \\ xMN &&& 1 \end{bmatrix}.
\end{aligned} \tag{5.6}$$

Then $T_{0,1}(p)F \in S_k(K(N))$. If Φ corresponds to F in the sense of (5.5)*, then $T_{0,1}(p)F$ corresponds to $T_{0,1}\Phi$.*

(ii) *Define*

$$
\begin{aligned}
T_{1,0}(p)F = {} & \sum_{x,y\in\mathbb{Z}/p\mathbb{Z}} \sum_{z\in\mathbb{Z}/p^2\mathbb{Z}} F\left| \begin{bmatrix} p & & & \\ & 1 & & \\ & & p & \\ & & & p^2 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & y & zp^{-n} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right. \\
& \times \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \\
& + \sum_{x,y,z\in\mathbb{Z}/p\mathbb{Z}} F\left| \begin{bmatrix} p & & & \\ & p^2 & & \\ & & p & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & xMN & 1 & \\ xMN & zpMN & & 1 \end{bmatrix} \right. \\
& \times \begin{bmatrix} 1 & -yMN & & \\ & 1 & & \\ & & 1 & \\ & & yMN & 1 \end{bmatrix},
\end{aligned}
\tag{5.7}
$$

or alternatively,

$$
\begin{aligned}
T_{1,0}(p)F = {} & \sum_{x,y\in\mathbb{Z}/p\mathbb{Z}} \sum_{z\in\mathbb{Z}/p^2\mathbb{Z}} F\left| \begin{bmatrix} p & & & \\ & 1 & & \\ & & p & \\ & & & p^2 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & y & zp^{-n} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right. \\
& \times \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \\
& + \sum_{x,y\in\mathbb{Z}/p\mathbb{Z}} F\left| \begin{bmatrix} p & & & \\ & p^2 & & \\ & & p & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & xMN & 1 & \\ xMN & & & 1 \end{bmatrix} \right.
\end{aligned}
$$

$$\times \begin{bmatrix} 1 & -yMN & & \\ & 1 & & \\ & & 1 & \\ & & yMN & 1 \end{bmatrix}$$

$$+ \sum_{\substack{x,y\in\mathbb{Z}/p\mathbb{Z} \\ z\in(\mathbb{Z}/p\mathbb{Z})^\times}} F|\begin{bmatrix} 1 & & & \\ -xp^{-1} & 1 & & zp^{-n-1} \\ & & 1 & xp^{-1} \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & -yMN & & \\ & 1 & & \\ & & 1 & \\ & & yMN & 1 \end{bmatrix}. \tag{5.8}$$

Then $T_{1,0}(p)F \in S_k(K(N))$*. If* Φ *corresponds to* F *in the sense of* (5.5)*, then* $T_{1,0}(p)F$ *corresponds to* $T_{1,0}\Phi$*.*

Proof. We will only prove (i), since the proof for (ii) is analogous. For $h \in G(\mathbb{Q})$, write $h_\mathbb{Q}$ for h diagonally embedded into $G(\mathbb{A})$, and h_v for h embedded at the place v only (meaning $h_v \in G(\mathbb{A})$ is such that the v-component equals h and all other components equal 1).

We first note that in (5.2) we may

$$\text{replace} \begin{bmatrix} 1 & -yp^n & x & \\ & 1 & & \\ & & 1 & \\ & & yp^n & 1 \end{bmatrix}\begin{bmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{bmatrix} \quad \text{by}$$

$$\begin{bmatrix} 1 & -yMN & x & \\ & 1 & & \\ & & 1 & \\ & & yMN & 1 \end{bmatrix}\begin{bmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{bmatrix}$$

and

$$\text{replace} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & xp^n & 1 & \\ xp^n & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \quad \text{by}$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & xMN & 1 & \\ xMN & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix},$$

since MNp^{-n} is a p-adic unit and x, y run over $\mathbb{Z}/p\mathbb{Z}$. By these replacements, all representatives h in (5.6) have the property that $h_q \in K_N$ for all $q \neq p$. As a consequence, for any $g \in G(\mathbb{R})^\circ$,

$$G(\mathbb{Q})gh_pK_N = G(\mathbb{Q})h_{\mathbb{Q}}^{-1}gh_pK_N = G(\mathbb{Q})h_\infty^{-1}gK_N.$$

The representatives h for $T_{0,1}$ acting on Φ correspond to the representatives $\lambda(h)h^{-1}$ for $T_{0,1}$ acting on F. For example,

$$h = \begin{bmatrix} 1 & & x & y \\ & 1 & y & zp^{-n} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix};$$

$$\lambda(h)h^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \begin{bmatrix} 1 & & -x & -y \\ & 1 & -y & -zp^{-n} \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Since we may sum over $-x, -y, -z$ as well as x, y, z, and since F is scalar invariant, we have

$$\begin{aligned}(T_{0,1}\Phi)(g) &= \sum_i \Phi(g(h_i)_p) = \sum_i \Phi((h_i)_\infty^{-1}g) = \sum_i (F \,|\, h_i^{-1}g)(iI_2) \\ &= \sum_i (F \,|\, \lambda(h_i)h_i^{-1}g)(iI_2) = ((T_{0,1}F) \,|\, g)(iI_2).\end{aligned}$$

Thus $T_{0,1}F$ corresponds to $T_{0,1}\Phi$ in the sense of (5.5). □

5.2. *A method to compute* $T_{0,1}$

In order to get more information about $S_{10}(K(8))$, we will apply the Hecke operators from Proposition 5.2 to the eigenforms in this space. Since only one prime is involved, we will denote them simply by $T_{0,1}$ and $T_{1,0}$. From (5.6) we get the formula

$$T_{0,1}F = \sum_{x,y,z\in\{0,1\}} F \left| \begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & y & z/8 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right. + \sum_{x,z\in\{0,1\}} F \left| \begin{bmatrix} 2 & 0 & 0 & 0 \\ x & 1 & 0 & z/8 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 2 \end{bmatrix} \right.$$

$$+\sum_{x,y\in\{0,1\}} F|\begin{bmatrix}1 & -8y & x & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 8y & 1\end{bmatrix} + F|\begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$+F|\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 2 & 1 & 0\\ 2 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}4 & 8 & 3 & -1\\ -1 & -4 & 1 & -3/8\\ 0 & 0 & 1 & -1/4\\ 0 & 0 & 2 & -1\end{bmatrix} \tag{5.9}$$

for $F \in S_k(K(8))$. Note that we have replaced the last representative in (5.6) by an equivalent one, which is more convenient for our computations. Let us write (5.9) as

$$T_{0,1}F = \sum_{i=1}^{17} F|U_i + F|\mathcal{C}_0(2)U_{18}, \tag{5.10}$$

where $U_1, \ldots, U_{17}$ are defined as the first 17 upper triangular matrices in some order, and U_{18} is the last block upper triangular matrix. It will be straightforward to apply $|U_i$ for $1 \le i \le 17$. The difficulty will be in applying $|\mathcal{C}_0(2)U_{18}$, because it seems we would need the expansion $F|\mathcal{C}_0(2)$, namely the expansion of F at the other cusp. In this section, we will present a method to calculate $T_{0,1}F$ that appears to avoid the Fourier expansion of $F|\mathcal{C}_0(2)$, but really does access information about it, albeit in a targeted manner. This method can potentially be applied to more general situations as well.

The technique we use is called *restriction to a modular curve*, compare [20]. Let s be a symmetric positive definite 2×2 matrix with rational entries and let s' be a symmetric matrix. We will evaluate F at $\Omega = s\tau + s'$ to get a one-variable power series in $q = e^{2\pi i\tau}$. If we can compute $(T_{0,1}F)(s\tau + s')$, then the eigenvalue is the ratio of these two series, assuming of course that $F(s\tau + s')$ is non-zero. We now derive formulas for this purpose. Recall $\langle t, \Omega\rangle = \mathrm{tr}(t\Omega)$. Let F have Fourier series expansion

$$F(\Omega) = \sum_{t\in\mathcal{X}_2(N)} a(t;F)e(\langle t,\Omega\rangle);$$

we understand $N = 8$ in what follows. Then we have

$$F(s\tau + s') = \sum_{n\in\mathbb{Q}^+}\left(\sum_{t\in\mathcal{X}_2(N):\langle s,t\rangle = n} a(t;F)e(\langle s',t\rangle)\right)q^n,$$

where $q = e^{2\pi i\tau}$. Let $U = \left[\begin{smallmatrix}A & B\\ 0 & D\end{smallmatrix}\right] \in \mathrm{GSp}(4,\mathbb{Q})$. Then

$$(F|_kU)(\Omega) = \det(AD)^{k/2}(\det D)^{-k}F(A\Omega D^{-1} + BD^{-1}),$$

and so

$$(F|_k U)(s\tau + s') = \det(AD)^{k/2}(\det D)^{-k} F(AsD^{-1}\tau + As'D^{-1} + BD^{-1}), \quad (5.11)$$

which can be calculated as another restriction. This restriction formula applies well to $U = U_1, \ldots, U_{17}$, which takes care of 17 of the 18 coset representatives of $T_{0,1}$. But the 18th coset representative will require a roundabout technique, to be described presently. We have the following useful proposition, whose proof is a modification of [19, Proposition 2.3].

Proposition 5.3. *Let $s \in \mathcal{P}_2(\mathbb{Q})$ have the form $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}/N \end{smallmatrix}\right]$. Let $F \in S_k(K(N))$ and set $g(\tau) = F(s\tau)$. Let $\sigma = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z})$. If $M := \left[\begin{smallmatrix} \alpha I & \beta s \\ \gamma s^{-1} & \delta I \end{smallmatrix}\right] \in \mathrm{Sp}(4,\mathbb{Q})$, then*

$$(g|_{2k}\sigma)(\tau) = (F|_k M)(s\tau), \quad (5.12)$$

where on the left-hand side we have the slash operator for functions on the upper half plane. In particular, if $\ell \in \mathbb{Z}^+$ is such that $\ell s^{-1} \in \left[\begin{smallmatrix} \mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} \end{smallmatrix}\right]$, then $g \in S_{2k}(\Gamma_0(\ell))$. If there are $m \in \mathbb{Z}$, $K \in K(N)$, and $\left[\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right] \in \mathrm{GSp}_4(\mathbb{Q})$ such that $M = K\mathcal{C}_0(m)\left[\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right]$, then

$$(g|_{2k}\sigma)(\tau) = \det(AD)^{k/2}(\det D)^{-k}(F|_k\mathcal{C}_0(m))(AsD^{-1}\tau + BD^{-1}). \quad (5.13)$$

Here is our method to deal with the 18th coset representative of $T_{0,1}$. We pick an s_0 so that Proposition 5.3 applies, with the following additional three conditions.

(1) There exists $\sigma = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z})$ such that

$$\begin{bmatrix} \alpha I & \beta s_0 \\ \gamma s_0^{-1} & \delta I \end{bmatrix} = K\mathcal{C}_0(2)W_0$$

for some $K \in K(8)$ and some $W_0 = \left[\begin{smallmatrix} A_0 & B_0 \\ 0 & D_0 \end{smallmatrix}\right] \in P_{2,0}(\mathbb{Q})$.

(2) There is an $\ell \in \mathbb{Z}^+$ such that $\ell s_0^{-1} \in \left[\begin{smallmatrix} \mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} \end{smallmatrix}\right]$, and we can effectively compute $g|\sigma$ when we are given a q-expansion for $g \in S_{20}(\Gamma_0(\ell))$.

Suppose for the moment that both (1) and (2) are feasible. Note that

$$F \,|\, \mathcal{C}_0(2)W_0 = F|\mathcal{C}_0(2)U_{18}|W_1, \quad \text{where } W_1 = U_{18}^{-1}W_0 = \begin{bmatrix} A_1 & B_1 \\ 0 & D_1 \end{bmatrix}.$$

The key is now to choose

$$s = A_1 s_0 D_1^{-1} \quad \text{and} \quad s' = B_1 D_1^{-1},$$

and compute the restriction $(T_{0,1}F)(s\tau + s')$ with this choice. As stated before, $(F \,|\, U_i)(s\tau + s')$, for $i = 1, \ldots, 17$, will be straightforward using (5.11). To compute $(F \,|\, \mathcal{C}_0(2)U_{18})(s\tau + s')$, note that with $g(\tau) = F(s_0\tau)$ we have

$$\begin{aligned} &(\det A_1D_1)^{10/2}(\det D_1)^{-10}(F \,|\, \mathcal{C}_0(2)U_{18})(s\tau + s') \\ &\quad = (\det A_1D_1)^{10/2}(\det D_1)^{-10}(F \,|\, \mathcal{C}_0(2)U_{18})(A_1 s_0 D_1^{-1}\tau + B_1 D_1^{-1}) \\ &\quad = (F \,|\, \mathcal{C}_0(2)U_{18}W_1)(s_0\tau) = (F \,|\, \mathcal{C}_0(2)W_0)(s_0\tau) = (g|_{20}\sigma)(\tau). \end{aligned}$$

In the last step we applied (5.12), with s_0 instead of s. By assumption (2), we can compute $(g|_{20}\sigma)(\tau)$ by using the presumably known action on $g(\tau) \in S_{20}(\Gamma_0(\ell))$. Hence we can compute

$$(F \mid \mathcal{C}_0(2)U_{18})(s\tau + s') = (\det A_1 D_1)^{-10/2}(\det D_1)^{10}(g|_{20}\sigma)(\tau). \tag{5.14}$$

This is where we access some targeted information about $F|\mathcal{C}_0(2)$. Thus we would be able to calculate the series $(T_{0,1}F)(s\tau + s')$. The last item required for this method to succeed is that

(3) The restriction $F(s\tau + s')$ must be non-zero.

5.3. *Carrying out the computation for* $T_{0,1}$

We will show that the conditions (1)–(3) from the previous section are satisfied with the following choice of s_0, ℓ, and $\sigma = \sigma_4 = \left[\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right]$:

$$s_0 = \begin{bmatrix} 4 & 1 \\ 1 & 1/2 \end{bmatrix}, \quad \ell = 8, \quad \sigma_4 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}.$$

Following the above instructions, we compute that $s_0^{-1} = \left[\begin{smallmatrix}1/2 & -1\\ -1 & 4\end{smallmatrix}\right]$ and

$$\begin{bmatrix} \alpha I & \beta s_0 \\ \gamma s_0^{-1} & \delta I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 \\ -4 & 16 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 8 & -3 & -1 \\ 2 & 5 & -2 & -5/8 \\ -5 & 0 & 0 & 2 \\ 24 & 40 & -16 & -8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} W_0,$$

$$W_0 = \begin{bmatrix} -2 & 4 & 3 & 2 \\ 3/2 & -2 & 1 & 5/8 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

So condition (1) holds. Since $\ell s_0^{-1} = \left[\begin{smallmatrix}4 & -8\\ -8 & 32\end{smallmatrix}\right]$, the first part of condition (2) also holds. Next,

$$W_1 = U_{18}^{-1} W_0 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From this, we compute that $s = \left[\begin{smallmatrix}2 & -3/2\\ -3/2 & 5/4\end{smallmatrix}\right]$, $s' = \left[\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right]$.

The last thing we need before using this choice to compute $T_{0,1}F$ is a knowledge of how forms in $S_{20}(\Gamma_0(8))$ transform by σ_4, which is the second part of condition (2). We discuss the ring generators of $M(\Gamma_0(8)) = \bigoplus_{k=0}^{\infty} M_k(\Gamma_0(8))$. Let

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma(n) q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 - \cdots$$

be the nearly modular weight two Eisenstein series transforming, for all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z})$, by

$$\left(E_2|_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)(\tau) = E_2(\tau) - \frac{3}{\pi^2}\left(\frac{2\pi i c}{c\tau + d}\right). \tag{5.15}$$

For $d > 1$, we define $E^-_{2,d} \in M_2(\Gamma_0(d))$ by $E^-_{2,d}(\tau) = \frac{1}{1-d}(E_2(\tau) - dE_2(d\tau))$. We define three elements in $M_2(\Gamma_0(8))$ by

$$\begin{aligned}
a(\tau) &= E^-_{2,2}(2\tau) = 1 + 24q^2 + 24q^4 + 96q^6 + 24q^8 + 144q^{10} + \cdots, \\
b(\tau) &= -\frac{1}{3}(E^-_{2,2}(\tau) - 4E^-_{2,2}(4\tau)) = 1 - 8q - 8q^2 - 32q^3 + 24q^4 - 48q^5 - \cdots, \\
c(\tau) &= \frac{1}{3}(E^-_{2,2}(\tau) - 2E^-_{2,2}(2\tau) + 4E^-_{2,2}(4\tau)) \\
&= 1 + 8q - 8q^2 + 32q^3 + 24q^4 + 48q^5 - \cdots.
\end{aligned}$$

Lemma 5.4. *The graded ring $M(\Gamma_0(8))$ consists of homogeneous polynomials in the three elements a, b, $c \in M_2(\Gamma_0(8))$, subject to the relation $c^2 = 2a^2 - b^2$. Every element in $M_k(\Gamma_0(8))$ can be uniquely written as $P_k(a,b) + cQ_{k-2}(a,b)$, where P_k and Q_{k-2} are homogeneous polynomials of degrees $k/2$ and $(k-2)/2$. The ideal of cusp forms is principal, and a generator is $d = (\eta(2\tau)\eta(4\tau))^4 \in S_4(\Gamma_0(8))$. Furthermore, we have*

$$(a|_2\sigma_4)(\tau) = +a\left(\tau - \frac{1}{2}\right), \quad (b|_2\sigma_4)(\tau) = -b\left(\tau - \frac{1}{2}\right), \quad (c|_2\sigma_4)(\tau) = -c\left(\tau - \frac{1}{2}\right).$$

Proof. The transformation under $\mathrm{SL}(2,\mathbb{Z})$ of a, b, c may be worked out using (5.15). The normalizer in $\mathrm{GL}(2,\mathbb{Q})$ of $\Gamma_0(8)$ modulo $\langle \mathbb{Q}I, \Gamma_0(8)\rangle$ is a dihedral group of order 8; we have $T^4 \equiv I$ and $STS \equiv T^{-1}$, for $T = \left[\begin{smallmatrix} 0 & 1 \\ -8 & 4 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right]$. The index of $\Gamma_0(8)$ in $\mathrm{SL}(2,\mathbb{Z})$ is 12, so, by the Valence Inequality, to prove equality in $\dim M_k(\Gamma_0(8))$ it suffices to check the equality of the first $k+1$ Fourier coefficients. In this way we verify $(a,b,c)|_2T = (-a, c, -b)$, $(a,b,c)|_2S = (a, -c, -b)$, and $c^2 = 2a^2 - b^2$. By the Riemann–Roch theorem, $\dim M_k(\Gamma_0(8)) = k+1$ for even $k \ge 0$, and $\dim S_k(\Gamma_0(8)) = k - 3$ for even $k > 2$. It follows that the ideal of cusp forms is principal, and a non-trivial cusp form of weight 4 is $d = \frac{1}{16}(a^2 - b^2) = (\eta(2\tau)\eta(4\tau))^4$.

Every modular form in $M_k(\Gamma_0(8))$ that can be written as a polynomial in a, b, c, may be written in the form $P_k(a,b) + cQ_{k-2}(a,b)$, where P_k and Q_{k-2} are homogeneous polynomials of degrees $k/2$ and $(k-2)/2$, respectively. The modular forms a and b have the same weight, and so are algebraically independent because b/a is non-constant. No non-trivial relation of the form $P_k(a,b) + cQ_{k-2}(a,b) = 0$ exists because slashing $P_k(1, b/a) + \frac{c}{a}Q_{k-2}(1, b/a) = 0$ by T^3S implies $P_k(1, b/a) - \frac{c}{a}Q_{k-2}(1, b/a) = 0$, so that P_k and Q_{k-2} are trivial. The dimension of $\mathbb{C}[a,b,c] \cap M_k(\Gamma_0(8))$ is then $(\frac{k}{2} + 1) + (\frac{k-2}{2} + 1) = k + 1$, and thus $M(\Gamma_0(8)) = \mathbb{C}[a,b,c]$ as graded rings. Noting that $T^2 \equiv \left[\begin{smallmatrix} -2 & -1 \\ 8 & 2 \end{smallmatrix}\right]$, the result of slashing a, b, c by σ_4 follows from $\sigma_4 = \left[\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right] = -\frac{1}{4}\left[\begin{smallmatrix} 1 & 0 \\ 8 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} -2 & -1 \\ 8 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 & -1 \\ 0 & 2 \end{smallmatrix}\right]$. ∎

We have $\dim S_{20}(\Gamma_0(8)) = 17$, and a basis is

$$\{a^i b^{8-i} d \,|\, 0 \le i \le 8\} \cup \{ca^i b^{7-i} d \,|\, 0 \le i \le 7\}.$$

Using sufficiently many terms of the expansion of $g(\tau) := F(s_0\tau)$, we determine g as a linear combination of this basis, and can then compute $g|_{20}\sigma_4$. For $F = N(8)^{\mathrm{b}}$, we compute the first 17 terms of $g^b(\tau) := N(8)^{\mathrm{b}}(s_0\tau)$ to be

$$\begin{aligned} g^b(\tau) &:= q^3 - 684q^5 + 17802q^7 - 91144q^9 + 208107q^{11} \\ &\qquad - 152172q^{13} - 3426194q^{15} + 9701496q^{17} + O(q^{18}) \\ &= \frac{-a^8 d}{128} + \frac{5a^6b^2d}{128} - \frac{17a^4b^4d}{256} + \frac{3a^2b^6d}{64} - \frac{3b^8d}{256}. \end{aligned}$$

Using the representation in a, b, c, d, we compute $g^b(\tau)|\sigma_4 = -q^3 + O(q^4)$. By (5.14), we have

$$(N(8)^{\mathrm{b}} \,|\, \mathcal{C}_0(2)U_{18})(s\tau + s') = -1024q^3 + O(q^4).$$

Here, we decided to truncate power series at q^3. Note that exponents of q may increase by $1/2$ by looking at the entries of s. By contrast, it is fairly straightforward to apply (5.11) to compute

$$\sum_{j=1}^{17} (N(8)^{\mathrm{b}} \,|\, U_j)(s\tau + s') = -\frac{9}{4}q^{\frac{3}{2}} + 1539q^{\frac{5}{2}} + 1024q^3 + O(q^{7/2}).$$

Combining these two equations, we have that

$$(T_{0,1}N(8)^{\mathrm{b}})(s\tau + s') = -\frac{9}{4}q^{\frac{3}{2}} + 1539q^{\frac{5}{2}} + O(q^{7/2}).$$

We also compute that

$$N(8)^{\mathrm{b}}(s\tau + s') = q^{\frac{3}{2}} - 684q^{\frac{5}{2}} + O(q^{7/2}).$$

We conclude that the eigenvalue is

$$\lambda_{0,1}(N(8)^{\mathrm{b}}) = \frac{-9/4}{1} = \frac{1539}{-684} = -\frac{9}{4}.$$

The fact that we get the same answer from the coefficient of $q^{3/2}$ as well as from $q^{5/2}$ provides a check on this calculation.

We apply the same choices of s_0, ℓ, σ_4, s, s' and compute that

$$\begin{aligned} N(8)^{\mathrm{a}}(s\tau + s') &= q^{\frac{3}{2}} + 64q^2 + 500q^{\frac{5}{2}} + 512q^3 + O(q^{7/2}), \\ (T_{0,1}(N(8)^{\mathrm{a}}))(s\tau + s') &= -q^{\frac{3}{2}} - 64q^2 - 500q^{\frac{5}{2}} - 512q^3 + O(q^{7/2}). \end{aligned}$$

We conclude that the eigenvalue is $\lambda_{0,1}(N(8)^{\mathrm{a}}) = -1$. It turns out that this choice of s_0, ℓ, σ_4, s does calculate the action of $T_{0,1}$ on the subspace spanned by L_1, L_2, because the restriction applied to this two-dimensional space has an image of full dimension 2. In fact, L_1, L_2 each span the one-dimensional eigenspaces of $T_{0,1}$; they were chosen with hindsight to have this property. (There are other choices for which the restriction yields an image of dimension less than 2, and in those cases, such restrictions do not tell us what the map $T_{0,1}$ is on the subspace spanned by L_1, L_2, and those choices would not give us the eigenspaces under $T_{0,1}$.) The results for the Gritsenko lift eigenforms are

$$\lambda_{0,1}(L_1) = \frac{1}{16}(111 + \sqrt{4449}); \quad \lambda_{0,1}(L(8)^{\mathrm{a}}) = 6,$$
$$\lambda_{0,1}(L_2) = \frac{1}{16}(111 - \sqrt{4449}); \quad \lambda_{0,1}(L(8)^{\mathrm{b}}) = 6.$$

5.4. *Calculating* $T_{1,0}$

The computation of $T_{1,0}$ will be trickier because, as is evident from (5.8), there are two coset representatives that are in the $K(8)\mathcal{C}_0(2)P_{2,0}(\mathbb{Q})$ double coset. We want to modify the technique to take care of these two coset representatives simultaneously. First, similar to the process for $T_{0,1}$, we write the coset representatives of $T_{1,0}$ as follows. For $F \in S_k(K(8))$, we have

$$\begin{aligned} T_{1,0}F &= \sum_{i=1}^{22} F|U_i + F|\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&2&1&0\\2&0&0&1 \end{bmatrix}\begin{bmatrix} 4&16&6&-1\\-1&-8&2&-3/8\\0&0&2&-1/4\\0&0&4&-1 \end{bmatrix} \\ &\quad + F|\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&2&1&0\\2&0&0&1 \end{bmatrix}\begin{bmatrix} 4&16&10&-1\\-1&-8&1&-3/8\\0&0&2&-1/4\\0&0&4&-1 \end{bmatrix} \\ &= \sum_{i=1}^{22} F|U_i + F|\mathcal{C}_0(2)U_{23} + F|\mathcal{C}_0(2)U_{24}, \end{aligned}$$

where $U_1, \ldots, U_{22}$ are defined as the 22 obviously upper triangular matrices in some order, and U_{23}, U_{24} are the upper triangular matrices to the right of $\mathcal{C}_0(2)$ in the last two coset representatives. It is straightforward to apply $|U_i$ for $1 \le i \le 22$. For $\mathcal{C}_0(2)U_{23}$ and $\mathcal{C}_0(2)U_{24}$, we want to apply the trick from the previous section, but simultaneously to both representatives.

It will turn out that with the following choices of s_0, ℓ, and σ, we will be able to calculate the restriction to some $(s\tau + s')$ of $F \,|\, \mathcal{C}_0(2)U_{23}$ and $F \,|\, \mathcal{C}_0(2)U_{24}$ simultaneously. Let

$$s_0 = \begin{bmatrix} 2&1\\1&1 \end{bmatrix}, \quad \ell = 8, \quad \sigma_4 = \begin{bmatrix} 1&0\\4&1 \end{bmatrix}.$$

Following the instructions for $T_{0,1}$, we compute that $s_0^{-1} = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 2 \end{smallmatrix}\right]$ and

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & -4 & 1 & 0 \\ -4 & 8 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 8 & -3 & -1 \\ 3 & 10 & -4 & -7/8 \\ -5 & 0 & 0 & 2 \\ 16 & 40 & -16 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} W_0 \in K(8)\mathcal{C}_0(2)W_0,$$

$$W_0 = \begin{bmatrix} -4 & 4 & 3 & 4 \\ 5/2 & -2 & 1 & 7/8 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 2 & 2 \end{bmatrix}.$$

Let

$$W_{23} = U_{23}^{-1} W_0 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -3/8 & 1/4 & 0 & 1/32 \\ 0 & 0 & 1/2 & 3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$W_{24} = U_{24}^{-1} W_0 = \begin{bmatrix} 1/2 & 0 & -1/2 & -3/4 \\ -3/8 & 1/4 & 0 & 1/32 \\ 0 & 0 & 1/2 & 3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Setting $g(\tau) := F(s_0\tau)$, we get that

$$(F \,|\, \mathcal{C}_0(2)U_{23})(s_{23}\tau + s'_{23}) = 1024(g \,|\, \sigma_4)(\tau),$$
$$(F \,|\, \mathcal{C}_0(2)U_{24})(s_{24}\tau + s'_{24}) = 1024(g \,|\, \sigma_4)(\tau),$$

where

$$s_{23} = \begin{bmatrix} 2 & -1 \\ -1 & 5/8 \end{bmatrix}, \quad s'_{23} = \begin{bmatrix} 0 & 0 \\ 0 & 1/32 \end{bmatrix}, \quad s_{24} = \begin{bmatrix} 2 & -1 \\ -1 & 5/8 \end{bmatrix}, \quad s'_{24} = \begin{bmatrix} -1 & 0 \\ 0 & 1/32 \end{bmatrix}.$$

We choose $s = s_{23} = s_{24}$, $s' = s'_{23}$. With this choice, we can compute $(F \,|\, \mathcal{C}_0(2)U_{23})(s\tau + s')$ as $1024(g \,|\, \sigma_4)(\tau)$. But the issue is to handle $(F \,|\, \mathcal{C}_0(2)U_{24})(s\tau + s')$ as well. Towards this end, define

$$\tau_0 = 1/2, \quad B = s\tau_0 + s'_{24} - s' = \begin{bmatrix} 0 & -1/2 \\ -1/2 & 5/16 \end{bmatrix}.$$

We have that

$$\mathcal{C}_0(2)U_{24} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} U_{24}^{-1}\mathcal{C}_0(2)^{-1} = \begin{bmatrix} 9 & -8 & 4 & -4 \\ -6 & 9 & -4 & 3 \\ -12 & 16 & -7 & 6 \\ 16 & -16 & 8 & -7 \end{bmatrix} \in K(8),$$

and thus

$$F \,|\, \mathcal{C}_0(2)U_{24} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} = F \,|\, \mathcal{C}_0(2)U_{24}.$$

We conclude

$$\begin{aligned}(F \,|\, \mathcal{C}_0(2)U_{24})(s\tau + s') &= (F \,|\, \mathcal{C}_0(2)U_{24})(s\tau + s' + B) \\ &= (F \,|\, \mathcal{C}_0(2)U_{24})(s(\tau + \tau_0) + s'_{24}) = 1024(g|\sigma_4)(\tau + \tau_0).\end{aligned}$$

We now show how this works out in the case of $F = N(8)^{\mathrm{a}}$. As an element of $S_{20}(\Gamma_0(8))$ in terms of the ring generators a, b, c, we compute

$$\begin{aligned}g^a(\tau) := N(8)^{\mathrm{a}}(s_0\tau) = \frac{9a^8d}{4096} - \frac{11a^6b^2d}{1024} + \frac{35a^4b^4d}{2048} - \frac{11a^2b^6d}{1024} \\ + \frac{9b^8d}{4096} - \frac{a^7cd}{1024} + \frac{5a^5b^2cd}{1024} - \frac{7a^3b^4cd}{1024} + \frac{3ab^6cd}{1024}.\end{aligned}$$

We truncate our computations to q^3. Using the action of σ_4 on a, b, c as before, we get

$$\begin{aligned}(N(8)^{\mathrm{a}} \,|\, \mathcal{C}_0(2)U_{23})(s\tau + s') &= 1024(g^a \,|\, \sigma_4)(\tau) = 1024q^3 + O(q^4), \\ (N(8)^{\mathrm{a}} \,|\, \mathcal{C}_0(2)U_{24})(s\tau + s') &= 1024(g^a \,|\, \sigma_4)(\tau + 1/2) = -1024q^3 + O(q^4).\end{aligned}$$

Along with

$$\sum_{i=1}^{22} (N(8)^{\mathrm{a}}|U_i)(s\tau + s') = -128iq^2 + (2048 - 2048i)q^3 + O(q^4),$$

we get

$$(T_{1,0}N(8)^{\mathrm{a}})(s\tau + s') = -128iq^2 + (2048 - 2048i)q^3 + O(q^4).$$

We also compute

$$N(8)^{\mathrm{a}}(s\tau + s') = 32iq^2 + (-512 + 512i)q^3 + O(q^4).$$

Table 7. The eigenvalues of $T_{0,1}(2)$ and $T_{1,0}(2)$ on the eigenforms in $S_{10}(K(8))$.

f	$\lambda_{0,1}(f)$	$\lambda_{1,0}(f)$
$N(8)^{\mathrm{a}}$	-1	-4
$N(8)^{\mathrm{b}}$	$-\frac{9}{4}$	0
L_1	$\frac{111 + \sqrt{4449}}{16}$	$\frac{15 + \sqrt{4449}}{8}$
L_2	$\frac{111 - \sqrt{4449}}{16}$	$\frac{15 - \sqrt{4449}}{8}$
$L(8)^{\mathrm{a}}$	6	0
$L(8)^{\mathrm{b}}$	6	0

We conclude that

$$\lambda_{1,0}(N(8)^{\mathrm{a}}) = \frac{-128}{32} = \frac{2048 - 2048i}{-512 + 512i} = -4.$$

Again, the fact that we see the same ratio from two different coefficients provides a check on the calculation. We perform this calculation on the rest of the eigenforms in $S_{10}(K(8))$ and summarize the results, as well as those for $\lambda_{0,1}$ from the previous section, in Table 7.

6. Distributing Cusp Forms Among Automorphic Representations

Recall from Sec. 4.3 the eigenforms $N(8)^{\mathrm{a}}$, $N(8)^{\mathrm{b}}$ and $L_1, L_2, L(8)^{\mathrm{a}}, L(8)^{\mathrm{b}}$ spanning the space $S_{10}(K(8))$. We now identify the automorphic representations generated by these eigenforms.

6.1. *Some elliptic cusp forms and their lifts*

In this section, we consider the cuspidal automorphic representations π of $\mathrm{GL}(2,\mathbb{A})$ generated by the eigenforms in $S_{18}(\Gamma_0(8))$. It turns out that there are eight such π. The reason we consider this weight and level is that these π lift to cuspidal automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$, some of which correspond to elements of $S_{10}(K(8))$, our space of interest. Here, by "lift" we mean the representation-theoretic Saito–Kurokawa lifting constructed in [29, 30]. Only three of the eight π's lift to paramodular representations. Inside the lifts of these three, called $\Lambda(1)$, $\Lambda(8)^{\mathrm{a}-}$ and $\Lambda(8)^{\mathrm{b}-}$ we can find the Gritsenko lifts $L_1, L_2, L(8)^{\mathrm{a}}, L(8)^{\mathrm{b}}$ considered earlier.

Our notation will be as follows. Automorphic representations of $\mathrm{GL}(2,\mathbb{A})$ will be called $\pi(m)$, where m is the global conductor. Only $m \in \{1,2,4,8\}$ will appear. If there is more than one π with the same conductor, we will write $\pi(m)^{\mathrm{a}}$, $\pi(m)^{\mathrm{b}}$, etc. We may decompose a $\pi(m)$ as $\otimes\pi(m)_p$, a restricted tensor product of irreducible,

Table 8. Dimensions of spaces of elliptic modular forms of weight 18 and Jacobi forms of weight 10. The "new, −" row gives the dimension of the space spanned by eigen-newforms that have a minus sign in the functional equation of their L-function.

		$m=1$	$m=2$	$m=4$	$m=8$
$S_{18}(\Gamma_0(m))$ (elliptic)	Total	1	3	7	15
	New	1	1	2	4
	New,−	1	0	0	2
$J_{10,m}^{\mathrm{cusp}}$		1	1	2	4

(6.1)

admissible representations $\pi(m)_p$ of $\mathrm{GL}(2,\mathbb{Q}_p)$. For $\mathrm{GSp}(4)$ we use similar notations, but with Λ or Π instead of π. We choose Λ if the $\mathrm{GSp}(4)$ representation is a lift from $\mathrm{GL}(2)$, otherwise Π.

We start with some dimension data for the spaces of elliptic cusp forms $S_{18}(\Gamma_0(m))$ for $m \in \{1,2,4,8\}$. Table 8 lists these dimensions, together with the dimensions of the spaces of newforms, and the space spanned by newforms whose L-function has sign -1 in its functional equation. The reason we are looking at weight 18 is that these forms lift to weight 10 Siegel modular forms; in general, weight $2k-2$ lifts to weight k. Also given in Table 8 are the dimensions of spaces of Jacobi cusp forms of weight 10 and index m for $m \in \{1,2,4,8\}$; see [33].

The newforms in Table 8 generate eight automorphic representations of $\mathrm{GL}(2,\mathbb{A})$:

- $\pi(1)$: The representation generated by the eigenform in $S_{18}(\Gamma_0(1))$. Its 2-component $\pi(1)_2$ has conductor $a(\pi(1)_2)=0$, i.e. it is an unramified principal series representation of $\mathrm{GL}(2,\mathbb{Q}_2)$. The sign in the functional equation is the product of all local ε-factors at $1/2$. Since all p-adic components are unramified, the only contribution comes from the Archimedean place. In general the Archimedean contribution for weight k is $(-1)^{k/2}$. Hence, in this case, the Archimedean contribution is -1. This is the sign in the functional equation for $L(s,\pi(1))$.
- $\pi(2)$: The representation generated by the eigen-newform in $S_{18}(\Gamma_0(2))$. Its 2-component $\pi(2)_2$ has conductor $a(\pi(2)_2)=1$. Using the notation of Table 1, we see that either $\pi(2)_2=\mathrm{St}_{\mathrm{GL}(2)}$ or $\pi(2)_2=\xi\mathrm{St}_{\mathrm{GL}(2)}$. According to the table above, the global sign is $+1$. Consequently, $\varepsilon(1/2,\pi(2)_2)=-1$, and it follows that $\pi(2)_2=\mathrm{St}_{\mathrm{GL}(2)}$.
- $\pi(4)^{\mathrm{a}}$ and $\pi(4)^{\mathrm{b}}$: The representations generated by the two eigen-newforms in $S_{18}(\Gamma_0(4))$. Their 2-components $\pi(4)_2^{\mathrm{ab}}$ have conductor $a(\pi(4)_2^{\mathrm{ab}})=2$. By Table 1, $\pi(4)_2^{\mathrm{ab}}=\mathsf{sc}(4)$, the unique representation with conductor 2. Since $\varepsilon(1/2,\mathsf{sc}(4))=-1$, it follows that $L(s,\pi(4)^{\mathrm{a}})$ and $L(s,\pi(4)^{\mathrm{b}})$ both satisfy a functional equation with sign $+1$. This is consistent with the data in Table 8.
- $\pi(8)^{\mathrm{a}-}$ and $\pi(8)^{\mathrm{b}-}$: The representations generated by the two eigen-newforms in $S_{18}(\Gamma_0(8))$ for which the L-function satisfies a functional equation with sign -1. Their 2-components $\pi(8)_2^{\mathrm{ab}-}$ have conductor $a(\pi(8)_2^{\mathrm{ab}-})=3$ and sign $\varepsilon(1/2,\pi(8)_2^{\mathrm{ab}-})=1$. By Table 1, this identifies them uniquely as $\pi(8)_2^{\mathrm{ab}-}=\mathsf{sc}(8)^+$.
- $\pi(8)^{\mathrm{a}+}$ and $\pi(8)^{\mathrm{b}+}$: The representations generated by the two eigen-newforms in $S_{18}(\Gamma_0(8))$ for which the L-function satisfies a functional equation with sign $+1$. The same argument shows that $\pi(8)_2^{\mathrm{ab}+}=\mathsf{sc}(8)^-$.

Table 9 summarizes the eight automorphic representations. Next we are going to lift all these representations to $\mathrm{GSp}(4)$ using [29, Theorem 3.1]. The liftings will be denoted by $\Lambda(M)$, where M is the conductor of the lift. To lift any $\pi=\otimes\pi_p$, [29, Theorem 3.1] requires the choice of a set of places S where π_p is square

Table 9. The automorphic representations $\pi = \otimes\pi_p$ of $\mathrm{GL}(2,\mathbb{A})$ generated by the newforms in $S_{18}(\Gamma_0(m))$ for $m \in \{1,2,4,8\}$. The notation for the 2-components π_2 is the same as in Table 1. The last four columns show the dimensions of the spaces of fixed vectors in π_2 under the local groups $\Gamma_0(2^n)$ for $n = 0,1,2,3$; this data is taken from Table 1. The "certain space" is the subspace of fixed vectors that have the same Atkin–Lehner sign as the newform; it is the local version of the "certain space" of Skoruppa and Zagier; see [32].

π	$\varepsilon(1/2,\pi)$	π_2	$\varepsilon(1/2,\pi_2)$		$V(0)$	$V(1)$	$V(2)$	$V(3)$
$\pi(1)$	-1	Unramified	1	Total dim	1	2	3	4
				"Certain space"	1	1	2	2
$\pi(2)$	1	$\mathrm{St}_{\mathrm{GL}(2)}$	-1	Total dim	0	1	2	3
$\pi(4)^{\mathrm{ab}}$	1	$\mathsf{sc}(4)$	-1	Total dim	0	0	1	2
$\pi(8)^{\mathrm{ab}-}$	-1	$\mathsf{sc}(8)^+$	1	Total dim	0	0	0	1
				"Certain space"	0	0	0	1
$\pi(8)^{\mathrm{ab}+}$	1	$\mathsf{sc}(8)^-$	-1	Total dim	0	0	0	1

integrable. This set needs to satisfy the parity condition $(-1)^{\#S} = \varepsilon(1/2,\pi)$. If this condition is satisfied, the lift exists. Moreover, the lift is cuspidal as long as S is non-empty.

In our situation, we always want S to contain ∞, since we want to produce holomorphic Siegel modular forms. By [29, Sec. 4], these Siegel modular forms will all have weight 10.

The only freedom then is whether S contains the place 2 or not. For $\pi(1)$ there is no choice, since $\pi(1)_2$ is not square-integrable. Hence, for $\pi(1)$ we are forced to choose $S = \{\infty\}$. But then the parity condition is satisfied, and we get a cuspidal lifting $\Lambda(1)$.

In all other cases the 2-component of π is square-integrable, so we have a choice for S. There is exactly one choice that satisfies the parity condition. Hence *all* our π's can be lifted in a unique way to a cuspidal, automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$. Table 10 summarizes the lifts. Note here that $\pi(2)$ lifts to a representation which we call $\Lambda(4)$, since it has conductor $4 = 2^2$. Similarly, $\pi(4)^{\mathrm{ab}}$ lifts to $\Lambda(8)^{\mathrm{ab}}$, and $\pi(8)^{\mathrm{ab}+}$ lifts to $\Lambda(16)^{\mathrm{ab}+}$. In general, we know from [29] that whenever π lifts to Λ, the rule for the global conductor is

$$a(\Lambda) = a(\pi) \prod_{\substack{p<\infty \\ p\in S}} p. \tag{6.2}$$

The last three columns in Table 10 summarize the 2-component of the lift. The information in the "Λ_2" column is taken from [29, Table 2]. The "Type" column refers to the classification from [24, Table A.1]; we have already used this classification in Sec. 3.2. As indicated in the last column, some of these 2-components have no paramodular vectors of any level; see [24, Theorem 3.4.3]. In the first row, χ is an unramified character of $\mathbb{Q}_2^\times$, and $\chi \times \chi^{-1}$ is an unramified principal series representation of $\mathrm{GL}(2,\mathbb{Q}_2)$ in standard notation.

Table 10. The lifts $\Lambda(M)$ of the automorphic representations $\pi(m)$. We have $M = m$ if $2 \notin S$ and $M = 2m$ if $2 \in S$. Here S is the set of places ν with π_ν square integrable. For the notations in the Λ_2 column, see [24, Table A.1], except for the supercuspidal representations $\delta^*(\ldots)$, which are explained in [25].

π	$\varepsilon(1/2,\pi)$	π_2	S	Λ	Λ_2	Type	Para
$\pi(1)$	-1	$\chi \times \chi^{-1}$	$\{\infty\}$	$\Lambda(1)$	$\chi 1_{\mathrm{GL}(2)} \rtimes \chi^{-1}$	IIb	Yes
$\pi(2)$	1	$\mathrm{St}_{\mathrm{GL}(2)}$	$\{\infty, 2\}$	$\Lambda(4)$	$\tau(T, \nu^{-1/2})$	VIb	No
$\pi(4)^{\mathrm{ab}}$	1	$\mathsf{sc}(4)$	$\{\infty, 2\}$	$\Lambda(8)^{\mathrm{ab}}$	$\delta^*(\nu^{1/2}\mathsf{sc}(4), \nu^{-1/2})$	XIa*	No
$\pi(8)^{\mathrm{ab}-}$	-1	$\mathsf{sc}(8)^+$	$\{\infty\}$	$\Lambda(8)^{\mathrm{ab}-}$	$L(\nu^{1/2}\mathsf{sc}(8)^+, \nu^{-1/2})$	XIb	Yes
$\pi(8)^{\mathrm{ab}+}$	1	$\mathsf{sc}(8)^-$	$\{\infty, 2\}$	$\Lambda(16)^{\mathrm{ab}+}$	$\delta^*(\nu^{1/2}\mathsf{sc}(8)^-, \nu^{-1/2})$	XIa*	No

We focus on those π that can be lifted to paramodular representations, namely, $\pi(1)$ and $\pi(8)^{\mathrm{ab}-}$. Given that the sequence of dimensions for the IIb type representation in Table 2 is $1, 1, 2, 2$, we see that inside $\Lambda(1)$ we can find the following modular forms:

- A full-level cusp form of weight 10. Up to multiples, there is only one such cusp form, namely Igusa's X_{10}; see [14]. It is the Saito–Kurokawa lifting of the unique eigenform in $S_{18}(\mathrm{SL}(2,\mathbb{Z}))$.
- Two linearly independent oldforms in $S_{10}(K(8))$. Since every oldform originates from the newform via level raising operators, we may assume that these two oldforms are $\theta^3 X_{10}$ and $\theta\eta X_{10}$. Here, θ and η are the paramodular level raising operators introduced in [23].

Since X_{10}, $\theta^3 X_{10}$ and $\theta\eta X_{10}$ originate from the same automorphic representation, they have the same Euler factors at all places. We also know that $\theta^3 X_{10}$ and $\theta\eta X_{10}$ must be in the span of the four Gritsenko-liftings $L_1, L_2, L(8)^{\mathrm{a}}, L(8)^{\mathrm{b}}$ identified in Sec. 4.3. It therefore follows from Table 6 that

$$\mathrm{span}(\theta^3 X_{10}, \theta\eta X_{10}) = \mathrm{span}(L_1, L_2). \tag{6.3}$$

In fact, the Fourier coefficients in Table 4 show that

$$\theta^3 X_{10} = -8(L_1 + L_2) - 8\frac{37\sqrt{3}}{\sqrt{1483}}(L_1 - L_2), \quad \theta\eta X_{10} = \frac{-1}{\sqrt{3 \cdot 1483}}(L_1 - L_2). \tag{6.4}$$

As a Saito–Kurokawa lift, the 3-Euler factor for X_{10} is known, and is the same as the quadratic factor of the 3-Euler factor for L_1 and L_2 given in Table 6.

Next consider $\Lambda(8)^{\mathrm{a}-}$ and $\Lambda(8)^{\mathrm{b}-}$. Since the dimensions for the XIb representation in Table 2 are $0, 0, 0, 1$, it follows that we can find inside $\Lambda(8)^{\mathrm{ab}-}$ a newform in $S_{10}(K(8))$, unique up to multiples. These two newforms must match up with $L(8)^{\mathrm{a}}$ and $L(8)^{\mathrm{b}}$. In fact, the quadratic factors of the Hecke polynomials for $L(8)^{\mathrm{a}}$ and $L(8)^{\mathrm{b}}$ in Table 6 are the 3-Hecke polynomials for the two newforms in $S_{18}(\Gamma_0(8))$ for which the L-function satisfies a functional equation with sign -1; compare label

8.18.1b in the LMFDB. These independent consistency checks are reassuring when a good deal of coding has been required. Since we have not specified the order of these two newforms, we might as well assume that $L(8)^{\mathrm{a}}$ lies in $\Lambda(8)^{\mathrm{a}-}$ and $L(8)^{\mathrm{b}}$ lies in $\Lambda(8)^{\mathrm{b}-}$.

6.2. *The non-lifts in $S_{10}(K(8))$*

In the previous section we identified the automorphic representation $\Lambda(1)$ that contains the eigenforms L_1 and L_2, and the representations $\Lambda(8)^{\mathrm{a}-}$ and $\Lambda(8)^{\mathrm{b}-}$ that contain the eigenforms $L(8)^{\mathrm{a}}$ and $L(8)^{\mathrm{b}}$. This covers all the lifts in $S_{10}(K(8))$. We now proceed to identify the representations generated by (the adelizations of) the non-lifts $N(8)^{\mathrm{a}}$ and $N(8)^{\mathrm{b}}$.

Lemma 6.1. *Let $\Pi(8)^{\mathrm{a}}$ and $\Pi(8)^{\mathrm{b}}$ be the automorphic representations generated by $N(8)^{\mathrm{a}}$ and $N(8)^{\mathrm{b}}$, respectively.*

(i) *$\Pi(8)^{\mathrm{a}}$ and $\Pi(8)^{\mathrm{b}}$ are irreducible, cuspidal automorphic representations of $G(\mathbb{A})$.*
(ii) *Let Π_2 be the 2-component of either $\Pi(8)^{\mathrm{a}}$ or $\Pi(8)^{\mathrm{b}}$. Then its conductor (exponent) is $a(\Pi_2) = 3$. If Π_2 is not generic, then $\Pi_2 = L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ (type XIb) with unramified σ and a supercuspidal representation π of $\mathrm{GL}(2, \mathbb{Q}_2)$ with trivial central character and conductor $a(\pi) = 3$.*

Proof. (i) Certainly, $\Pi(8)^{\mathrm{a}}$ and $\Pi(8)^{\mathrm{b}}$ are cuspidal, since $N(8)^{\mathrm{a}}$ and $N(8)^{\mathrm{b}}$ are cusp forms; we will prove irreducibility. Since we are within the space of cuspidal automorphic forms, we can write

$$\Pi(8)^{\mathrm{a}} = \Pi_1 \oplus \cdots \oplus \Pi_m, \qquad \Pi(8)^{\mathrm{b}} = \Pi'_1 \oplus \cdots \oplus \Pi'_n, \tag{6.5}$$

with irreducible, cuspidal representations Π_i and Π'_i. None of the Π_i or Π'_i can be equal to one of the lifts $\Lambda(1)$, $\Lambda(8)^{\mathrm{a}-}$ and $\Lambda(8)^{\mathrm{b}-}$; one way to see this is to look at the 3-Euler factors in Table 6. We write the adelization Φ of $N(8)^{\mathrm{a}}$ (respectively, Φ' of $N(8)^{\mathrm{b}}$) as $\Phi_1 + \cdots + \Phi_m$ (respectively, $\Phi'_1 + \cdots + \Phi'_n$) according to (6.5). The automorphic forms Φ_i and Φ'_i have the same invariance properties as Φ and Φ'. Each one of them can therefore be de-adelized to an element of $S_{10}(K(8))$. It follows that each Π_i and Π'_i contributes at least 1 to the dimension of $S_{10}(K(8))$. Since $\dim S_{10}(K(8)) = 6$ by Theorem 4.3, and four of these dimensions are contributed by $L_1, L_2, L(8)^{\mathrm{a}}, L(8)^{\mathrm{b}}$, it follows that $m = n = 1$.

(ii) Let V_2 be the space of Π_2, and let $V_2(n)$ be the subspace of vectors invariant under the local paramodular group $K(2^n)$; see (3.2). Then $\dim V_2(3) \geq 1$, since Φ and Φ' are invariant under $K(2^3)$. For reasons of dimension, similar to the argument in (i), we cannot have $\dim V_2(3) > 1$. Hence $\dim V_2(3) = 1$.

Assume that Π_2 is generic. Then the dimensions of the spaces $V_2(n)$, starting with $n = a(\Pi_2)$, grow like $1, 2, 4, 6, \ldots$; see [24, Theorem 7.5.6]. Since $\dim V_2(3) = 1$, it follows that $a(\Pi_2) = 3$.

Assume that Π_2 is non-generic. Then Π_2 cannot be supercuspidal, since non-generic supercuspidals do not admit paramodular vectors of any level by [24, Theorem 3.4.3]. Since we are within the space of cusp forms, all local representations must be unitary. Going through [24, Table A.12], we find that the only irreducible, non-supercuspidal, unitary representation satisfying $\dim V_2(3) = 1$ is of type XIb with π and σ as indicated, and for this representation we have $a(\Pi_2) = 3$. □

By (ii) of this lemma, the 2-components of $\Pi(8)^{\mathrm{a}}$ and $\Pi(8)^{\mathrm{b}}$ contribute 2^3 to the global conductor. Of course, the p-components for $p \geq 3$ contribute nothing, since everything is unramified outside 2. It follows that the global conductor is 8, justifying our notation. Using the Hecke eigenvalue information from Table 7, we can now completely determine the 2-components.

Proposition 6.2. *For* $* = \mathrm{a}$ *or* $* = \mathrm{b}$, *let* $\Pi(8)^* = \otimes \Pi(8)^*_p$ *be the factorization of* $\Pi(8)^*$ *into irreducible, admissible representations of* $\mathrm{GSp}(4, \mathbb{Q}_p)$.

(i) *The* 2*-component of* $\Pi(8)^{\mathrm{a}}$ *is*

$$\Pi(8)^{\mathrm{a}}_2 = \xi \mathrm{St}_{\mathrm{GSp}(4)} \quad (\textit{type IVa}), \tag{6.6}$$

where ξ *is the non-trivial, unramified quadratic character of* $\mathbb{Q}_2^\times$.

(ii) *The* 2*-component of* $\Pi(8)^{\mathrm{b}}$ *is*

$$\Pi(8)^{\mathrm{b}}_2 = \pi \rtimes \sigma \quad (\textit{type X}), \tag{6.7}$$

where $\pi = \sigma^{-1}\mathsf{sc}(8)^+$ *and* σ *is the unramified character of* $\mathbb{Q}_2^\times$ *with*

$$\sigma(2) = \frac{-1}{16\sqrt{2}}(9 + i\sqrt{431}). \tag{6.8}$$

Proof. (i) According to Table 7, the $T_{0,1}$ and $T_{1,0}$ eigenvalues of $N(8)^{\mathrm{a}}$ are $\lambda_{0,1} = -1$ and $\lambda_{1,0} = -4$. The Hecke operators $T_{0,1}$ and $T_{1,0}$ are compatible with the local operators at the place 2 of the same name; see Proposition 5.2. We are thus looking for a representation Π_2 where the local operators act on the newform with eigenvalues -1 and -4. Moreover, Π_2 must be unitary, and must be one of the representations admitted by Lemma 6.1(ii). Going through [24, Table A.14], we see that the only possibility is $\Pi_2 = \xi \mathrm{St}_{\mathrm{GSp}(4)}$, where ξ is the non-trivial, unramified quadratic character of $\mathbb{Q}_2^\times$.

(ii) The argument in this case is similar. According to Table 7, the $T_{0,1}$ and $T_{1,0}$ eigenvalues of $N(8)^{\mathrm{b}}$ are $\lambda_{0,1} = -9/4$ and $\lambda_{1,0} = 0$. The only unitary representation with these eigenvalues and satisfying the conditions of Lemma 6.1(ii) is the type X representation $\pi \rtimes \sigma$, where σ is unramified and $a(\pi) = 3$. By [24, Table A.14], we must have

$$-\frac{9}{4} = 2^{3/2}(\sigma(2) + \sigma(2)^{-1}). \tag{6.9}$$

Solving the quadratic equation for $\sigma(2)$ gives (6.8) as one of its roots (which root we take is irrelevant). The central character of $\pi \rtimes \sigma$ must be trivial, i.e. $\omega_\pi \sigma^2 = 1$, where ω_π is the central character of π. The twist $\sigma\pi$ therefore has trivial central character. By Table 1, it follows that $\sigma\pi = \mathsf{sc}(8)^\pm$. To determine the sign, we note from (4.1) that each eigenform in $S_{10}(K(8))$ has Atkin–Lehner eigenvalue $+1$. By [24, Table A.9], $\varepsilon(1/2, \sigma\pi) = 1$; note here that ε-factor values at $1/2$ coincide with Atkin–Lehner eigenvalues by [24, Corollary 7.5.5]. From Table 1 we thus see $\sigma\pi = \mathsf{sc}(8)^+$. □

We note that the number $\sigma(2)$ in (6.8) has absolute value 1. This implies that $\Pi(8)^{\rm b}_2$ is a tempered representation, as it should be according to the Ramanujan conjecture.

6.3. *Eigenforms in $S_{10}(K(8))$ and $S_{10}(B(2))$ and their representations*

In the previous two sections we identified a total of five automorphic representations that contribute to the six-dimensional space $S_{10}(K(8))$. Recall that these were $\Lambda(1)$, $\Lambda(8)^{\rm a-}$ and $\Lambda(8)^{\rm b-}$, which contain the lifts $L_1, L_2, L(8)^{\rm a}, L(8)^{\rm b}$, and $\Pi(8)^{\rm a}$, $\Pi(8)^{\rm b}$, which contain the non-lifts $N(8)^{\rm a}$ and $N(8)^{\rm b}$. Note that L_1 and L_2 are oldforms, since they lie in $\Lambda(1)$, a representation of conductor smaller than 8. The forms $L(8)^{\rm a}$, $L(8)^{\rm b}$, $N(8)^{\rm a}$ and $N(8)^{\rm b}$ are new, since they lie in automorphic representations of conductor 8. The notion of oldforms and newforms used here is the one defined in [23].

We will now consider the automorphic representations that contribute to the space $S_{10}(B(2))$. First note that $\dim S_{10}(B(2)) = 6$ by [11, Theorem 3.3]. The congruence subgroup $B(2)$ corresponds to the Iwahori subgroup I in $\mathrm{GSp}(4, \mathbb{Q}_2)$. By Table 10, the automorphic representation $\Lambda(1)$ has an unramified representation of type IIb as its 2-component. By Table 2, the space of I-invariant vectors in this IIb representation is four-dimensional. It follows that $\Lambda(1)$ contributes four of the six dimensions of $S_{10}(B(2))$.

Next consider $\Lambda(4)$, which is the lift of $\pi(2)$. By Table 10, $\Lambda(4)$ has a representation of type VIb as its 2-component. By Table 2, the space of I-invariant vectors in VIb is one-dimensional. Hence, $\Lambda(4)$ contributes one of the six dimensions of $S_{10}(B(2))$.

The last of the six dimensions comes from $\Pi(8)^{\rm a}$, the automorphic representation that also contains $N(8)^{\rm a}$. By Proposition 6.2(i), its 2-component is an unramified twist of the Steinberg representation, $\xi\mathrm{St}_{\mathrm{GSp}(4)}$. By Table 2, the space of I-invariant vectors in this local representation is one-dimensional. Hence, $\Pi(8)^{\rm a}$ contributes one dimension to $S_{10}(B(2))$.

Lemma 6.3. *The cusp form in $S_{10}(B(2))$ coming from $\Pi(8)^{\rm a}$ is Ibukiyama's F_{10}.*

Proof. The Steinberg representation $\mathrm{St}_{\mathrm{GSp}(4)}$ and its unramified twist are the only representations of $\mathrm{GSp}(4, \mathbb{Q}_2)$ that contain an I-invariant vector that is not invariant under any bigger parahoric subgroup; see [24, Table A.15]. This means that the cusp form F constructed from $\Pi(8)^{\rm a}$ is not invariant under any congruence subgroup bigger than $B(2)$, like $\Gamma_0(2)$. In other words, F is a newform in $S_{10}(B(2))$ in the

Table 11. The eigenforms in $S_{10}(K(8))$, $S_{10}(B(2))$ and $S_{10}(\Gamma_0(2))$ and their automorphic representations. The $K(n)$ column in the top half of the table gives the dimensions of the spaces $S_{10}(K(n))$, and their subspaces of newform/oldforms and lifts/non-lifts; similarly for $B(2)$ and $\Gamma_0(2)$, except that we give no concept of newform/oldform for $\Gamma_0(2)$. Specific eigenforms in these spaces are indicated in small print under the dimension. The lower half of the table shows the automorphic representations contributing to $S_{10}(K(8))$, $S_{10}(B(2))$ and $S_{10}(\Gamma_0(2))$. The entries in the $K(n)$, $B(2)$ and $\Gamma_0(2)$ columns are now the dimensions of the spaces of fixed vectors under the corresponding local groups in the 2-components of the automorphic representations. The local type of the 2-component is indicated next to the name of the automorphic representation.

		$K(1)$	$K(2)$	$K(4)$	$K(8)$	$B(2)$	$\Gamma_0(2)$	
S_{10}	Lifts	1 X_{10}	1	2	4 $L_1, L_2, L(8)^{\mathrm{a}}, L(8)^{\mathrm{b}}$	5	4	
	Non-lifts	0	0	0	2 $N(8)^{\mathrm{a}}, N(8)^{\mathrm{b}}$	1 F_{10}	0	
	Total	1	1	2	6	6	4	
S_{10}^{old}	Lifts	0	1	2	2 L_1, L_2	5		
	Non-lifts	0	0	0	0	0		
	Total	0	1	2	2	5		
S_{10}^{new}	Lifts	1	0	0	2 $L(8)^{\mathrm{a}}, L(8)^{\mathrm{b}}$	0		
	Non-lifts	0	0	0	2 $N(8)^{\mathrm{a}}, N(8)^{\mathrm{b}}$	1 F_{10}		
	Total	1	0	0	4	1		
Contributions from automorphic representations								
$\Lambda(1)$	IIb	1 X_{10}	1	2	2 L_1, L_2	4	3	(lift)
$\Lambda(8)^{\mathrm{a}-}$	XIb	0	0	0	1 $L(8)^{\mathrm{a}}$	0	0	(lift)
$\Lambda(8)^{\mathrm{b}-}$	XIb	0	0	0	1 $L(8)^{\mathrm{b}}$	0	0	(lift)
$\Pi(8)^{\mathrm{a}}$	IVa	0	0	0	1 $N(8)^{\mathrm{a}}$	1 F_{10}	0	
$\Pi(8)^{\mathrm{b}}$	X	0	0	0	1 $N(8)^{\mathrm{b}}$	0	0	
$\Lambda(4)$	VIb	0	0	0	0	1	1	(lift)

sense of [11]. Since the space of newforms is one-dimensional and spanned by F_{10} according to the theorem in [11, §1], it follows that $F = F_{10}$, up to multiples. □

To illustrate the newform concept for $B(2)$ further, consider the essentially unique I-invariant vector in VIb, the 2-component of $\Lambda(4)$. According to [24, Table A.15], this vector is invariant under the bigger parahoric subgroup $\Gamma_0(2)$.

Table 12. The Euler factors at $p = 2$ for the automorphic representations contributing to $S_{10}(K(8))$ and $S_{10}(B(2))$. In the $\Lambda(1)$ Euler factor, f is the newform in $S_{18}(\mathrm{SL}(2,\mathbb{Z}))$. All factors are normalized to fit into a functional equation relating s and $1-s$.

Π	$L(s, \Pi_2)^{-1}$
$\Lambda(1)$	$L_2(s,f)^{-1}(1-p^{-s-1/2})(1-p^{-s+1/2})$
$\Lambda(8)^{\mathrm{a}-}$	$(1-p^{-s-1/2})(1-p^{-s+1/2})$
$\Lambda(8)^{\mathrm{b}-}$	$(1-p^{-s-1/2})(1-p^{-s+1/2})$
$\Pi(8)^{\mathrm{a}}$	$1+2^{-s-3/2}$
$\Pi(8)^{\mathrm{b}}$	$1+9\cdot 2^{-7/2-s}+2^{-2s}$
$\Lambda(4)$	$(1-p^{-s-1/2})^2$

Thus the cusp form in $S_{10}(B(2))$ constructed from $\Lambda(4)$ lies in fact in $S_{10}(\Gamma_0(2))$, i.e. it is an oldform in $S_{10}(B(2))$ in the sense of [11].

Table 11 summarizes our findings. The dimension data for $S_{10}(K(2))$ and $S_{10}(B(2))$ is taken from [11, Theorem 3.3], and the one for $S_{10}(K(4))$ is taken from [21]. We have also included the space $S_{10}(\Gamma_0(2))$ since its eigenforms are covered by the automorphic representations in our list. Note however that we do not define oldforms or newforms for this space.

Using [24, Table A.8], it is easy to determine the 2-Euler factors of the automorphic representations in Table 11. The results are summarized in Table 12. The factors for $\Pi(8)^{\mathrm{a}}$ and $\Pi(8)^{\mathrm{b}}$ follow from Proposition 6.2. Note that, by (6.9), the factor for $\Pi(8)^{\mathrm{b}}$ equals $L(s,\sigma)L(s,\sigma^{-1})$, with σ as in Proposition 6.2(ii).

6.4. *Weight* 12

With the methods explained in the previous sections we can also analyze the space $S_{12}(K(8))$. We list the results without giving all details. The starting point is the following theorem.

Theorem 6.4. $\dim S_{12}(K(8)) = 12$.

Proof. The proof of this result is analogous to that of Theorem 4.3. Using Theorem 4.1 and 11 Fourier-Jacobi coefficients, running the Jacobi restriction method yields that the dimension is at most 11 for $S_{12}(K(8))^+$ and at most 1 for $S_{12}(K(8))^-$. Because the non-lift newform in $S_{12}(K(4))$ yields an old form in $S_{12}(K(8))^-$, it follows that $\dim S_{12}(K(8))^- = 1$. We can generate 11 linearly independent forms in $S_{12}(K(8))^+$ as follows: six weight 12 Gritsenko lifts, the square of the weight 6 Gritsenko lift, the two products obtained by multiplying the one weight 5 Gritsenko lift with the two weight 7 Gritsenko lifts, and $T(3)$ applied to these two products. □

Table 13. Dimensions of spaces of elliptic modular forms of weight 22 and Jacobi forms of weight 12. The "new, −" row gives the dimension of the space spanned by eigen-newforms that have a minus sign in the functional equation of their L-function.

		$m = 1$	$m = 2$	$m = 4$	$m = 8$
$S_{22}(\Gamma_0(m))$ (elliptic)	Total	1	4	9	19
	New	1	2	2	5
	New, −	1	1	0	2
$J_{12,m}^{\text{cusp}}$		1	2	3	6

Table 14. The automorphic representations $\pi = \otimes\pi_p$ of $\mathrm{GL}(2,\mathbb{A})$ generated by the newforms in $S_{22}(\Gamma_0(m))$ for $m \in \{1, 2, 4, 8\}$. The notation for the 2-components π_2 is the same as in Table 1. The last four columns show the dimensions of the spaces of fixed vectors in π_2 under the local groups $\Gamma_0(2^n)$ for $n = 0, 1, 2, 3$; this data is taken from Table 1. The "certain space" is the subspace of fixed vectors that have the same Atkin–Lehner sign as the newform; it is the local version of the "certain space" of Skoruppa and Zagier; see [32].

π	$\varepsilon(1/2,\pi)$	π_2	$\varepsilon(1/2,\pi_2)$		$V(0)$	$V(1)$	$V(2)$	$V(3)$
$\pi(1)$	-1	Unramified	1	Total dim	1	2	3	4
				"Certain space"	1	1	2	2
$\pi(2)^+$	1	$\mathrm{St}_{\mathrm{GL}(2)}$	-1	Total dim	0	1	2	3
$\pi(2)^-$	-1	$\xi\mathrm{St}_{\mathrm{GL}(2)}$	1	Total dim	0	1	2	3
$\pi(4)^{\mathrm{ab}}$	1	$\mathsf{sc}(4)$	-1	Total dim	0	0	1	2
$\pi(8)^{\mathrm{ab}-}$	-1	$\mathsf{sc}(8)^+$	1	Total dim	0	0	0	1
				"Certain space"	0	0	0	1
$\pi(8)^{\mathrm{abc}+}$	1	$\mathsf{sc}(8)^-$	-1	Total dim	0	0	0	1

Table 15. The lifts $\Lambda(M)$ of the automorphic representations $\pi(m)$. We have $M = m$ if $2 \notin S$ and $M = 2m$ if $2 \in S$.

π	$\varepsilon(1/2,\pi)$	π_2	S	Π	Π_2	Type	Para
$\pi(1)$	-1	$\chi \times \chi^{-1}$	$\{\infty\}$	$\Lambda(1)$	$\chi 1_{\mathrm{GL}(2)} \rtimes \chi^{-1}$	IIb	Yes
$\pi(2)^+$	1	$\mathrm{St}_{\mathrm{GL}(2)}$	$\{\infty, 2\}$	$\Lambda(4)$	$\tau(T, \nu^{-1/2})$	VIb	No
$\pi(2)^-$	-1	$\xi\mathrm{St}_{\mathrm{GL}(2)}$	$\{\infty\}$	$\Lambda(2)$	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2})$	Vb	Yes
$\pi(4)^{\mathrm{ab}}$	1	$\mathsf{sc}(4)$	$\{\infty, 2\}$	$\Lambda(8)^{\mathrm{ab}}$	$\delta^*(\nu^{1/2}\mathsf{sc}(4), \nu^{-1/2})$	XIa*	No
$\pi(8)^{\mathrm{ab}-}$	-1	$\mathsf{sc}(8)^+$	$\{\infty\}$	$\Lambda(8)^{\mathrm{ab}-}$	$L(\nu^{1/2}\mathsf{sc}(8)^+, \nu^{-1/2})$	XIb	Yes
$\pi(8)^{\mathrm{abc}+}$	1	$\mathsf{sc}(8)^-$	$\{\infty, 2\}$	$\Lambda(16)^{\mathrm{abc}+}$	$\delta^*(\nu^{1/2}\mathsf{sc}(8)^-, \nu^{-1/2})$	XIa*	No

Table 16. The eigenforms in $S_{12}(K(8))$ and $S_{12}(B(2))$ and their automorphic representations.

		$K(1)$	$K(2)$	$K(4)$	$K(8)$	$B(2)$	$\Gamma_0(2)$	
S_{12}	Lifts	1 X_{12}	2	3	6	11	5	
	Non-lifts	0	0	1 $N(4)$	6	1 F_{12}	2	
	Total	1	2	4	12	12	7	
S_{12}^{old}	Lifts	0	1	3	4	7		
	Non-lifts	0	0	0	2	4		
	Total	0	1	3	6	11		
S_{12}^{new}	Lifts	1 X_{12}	1 $L(2)$	0	2 $L(8)^{\mathrm{ab}}$	0		
	Non-lifts	0	0	1 $N(4)$	4 $N(8)^*$	1 F_{12}		
	Total	1	1	1	6	1		
	Contributions from automorphic representations							
$\Lambda(1)$	IIb	1 X_{12}	1	2	2	4	3	(lift)
$\Lambda(2)$	Vb	0	1 $L(2)$	1	2	2	1	(lift)
$\Pi(4)$	IIIa	0	0	1 $N(4)$	2	4	2	
$\Lambda(8)^{\mathrm{a}-}$	XIb	0	0	0	1 $L(8)^{\mathrm{a}}$	0	0	(lift)
$\Lambda(8)^{\mathrm{b}-}$	XIb	0	0	0	1 $L(8)^{\mathrm{b}}$	0	0	(lift)
$\Pi(8)^{\mathrm{a}}$	IVa	0	0	0	1 $N(8)^{\mathrm{a}}$	1 F_{12}	0	
$\Pi(8)^{\mathrm{b}}$	XIa	0	0	0	1 $N(8)^{\mathrm{b}}$	0	0	
$\Pi(8)^{\mathrm{c}}$	X	0	0	0	1 $N(8)^{\mathrm{c}}$	0	0	
$\Pi(8)^{\mathrm{d}}$	X	0	0	0	1 $N(8)^{\mathrm{d}}$	0	0	
$\Lambda(4)$	VIb	0	0	0	0	1	1	(lift)

The method also shows that the subspace of $S_{12}(K(8))$ spanned by Gritsenko lifts is six-dimensional. To understand the representation-theoretic lifts, we start from Table 13, which shows the dimensions of some spaces of elliptic modular forms of weight 22. Note that weight 22 on GL(2) lifts to weight 12 on GSp(4); in general, weight $2k-2$ lifts to weight k. Table 14 shows the automorphic representations of $\mathrm{GL}(2,\mathbb{A})$ generated by these eigenforms. Just as before, each of these representations

Table 17. The newforms in the automorphic representations contributing to $S_{12}(K(8))$. The second and third columns show the eigenvalues of $T_{0,1}(2)$ and $T_{1,0}(2)$ on some of these newforms. The last column shows the Hecke eigenvalue of $T(3)$, normalized to facilitate comparison with the eigenvalues given in [11, Theorem 3.4].

Form	$\lambda_{0,1}$	$\lambda_{1,0}$	Representation	Type	$3^9\lambda_3$
X_{12}			$\chi 1_{\mathrm{GL}(2)} \rtimes \chi^{-1}$	IIb	107352
$L(2)$			$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2})$	Vb	307800
$N(4)$	$-\frac{7}{2}$		$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	IIIa	-88488
$L(8)^{\mathrm{a}}$	6	0	$L(\nu^{1/2}\mathsf{sc}(8)^+, \nu^{-1/2})$	XIb	$24(7645 + 8\sqrt{358549})$
$L(8)^{\mathrm{b}}$	6	0	$L(\nu^{1/2}\mathsf{sc}(8)^+, \nu^{-1/2})$	XIb	$24(7645 - 8\sqrt{358549})$
$N(8)^{\mathrm{a}}$	-1	-4	$\xi\mathrm{St}_{\mathrm{GSp}(4)}$	IVa	-14760
$N(8)^{\mathrm{b}}$	2	-4	$\delta(\nu^{1/2}\mathsf{sc}(4), \nu^{-1/2})$	XIa	-229032
$N(8)^{\mathrm{c}}$	$\frac{1}{8}(-12 - 5\sqrt{6})$	0	$(\sigma^{-1}\mathsf{sc}(8)^+) \rtimes \sigma$	X	$504(65 + 64\sqrt{6})$
$N(8)^{\mathrm{d}}$	$\frac{1}{8}(-12 + 5\sqrt{6})$	0	$(\sigma^{-1}\mathsf{sc}(8)^+) \rtimes \sigma$	X	$504(65 - 64\sqrt{6})$

Table 18. The Euler factors at $p = 2$ for the automorphic representations contributing to $S_{12}(K(8))$ and $S_{12}(B(2))$. In the $\Lambda(1)$ Euler factor, f is the newform in $S_{22}(\mathrm{SL}(2, \mathbb{Z}))$. All factors are normalized to fit into a functional equation relating s and $1 - s$.

Π	$L(s, \Pi_2)^{-1}$
$\Lambda(1)$	$L(s, f)^{-1}(1 - p^{-s-1/2})(1 - p^{-s+1/2})$
$\Lambda(2)$	$(1 + p^{-s-1/2})(1 - p^{-s-1/2})(1 - p^{-s+1/2})$
$\Pi(4)$	$1 + \frac{7}{4}p^{-s-1/2} + p^{-2s-1}$
$\Lambda(8)^{\mathrm{a}-}$	$(1 - p^{-s-1/2})(1 - p^{-s+1/2})$
$\Lambda(8)^{\mathrm{b}-}$	$(1 - p^{-s-1/2})(1 - p^{-s+1/2})$
$\Pi(8)^{\mathrm{a}}$	$1 + p^{-s-3/2}$
$\Pi(8)^{\mathrm{b}}$	$1 - p^{-s-1/2}$
$\Pi(8)^{\mathrm{c}}$	$1 + \frac{1}{16}(6\sqrt{2} + 5\sqrt{3})p^{-s} + p^{-2s}$
$\Pi(8)^{\mathrm{d}}$	$1 + \frac{1}{16}(6\sqrt{2} - 5\sqrt{3})p^{-s} + p^{-2s}$
$\Lambda(4)$	$(1 - p^{-s-1/2})^2$

admits a unique lift to a cuspidal, automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$; see Table 15.

Table 16 is the main result for weight 12. There are nine automorphic representations that contribute to the 12-dimensional $S_{12}(K(8))$. There is a tenth representation $\Lambda(4)$ which does not contribute to $S_{12}(K(8))$, but to $S_{12}(B(2))$ and $S_{12}(\Gamma_0(2))$. We have $\dim S_{12}(B(2)) = 12$ and $\dim S_{12}(\Gamma_0(2)) = 7$ by [11, Theorem 3.4], showing that no other automorphic representations contribute to these spaces besides the ones in Table 16. Our $F_{12} \in S_{12}(B(2))$ is the same F_{12} as in [11, Theorem 3.4]. Our form $L(2)$ is the same as the $F_{12}^{(2)}$ from [11].

To determine the local representations at two of the non-lifts, we again need to calculate the $T_{0,1}$ and $T_{1,0}$ eigenvalues on the non-lift newforms. Because the technique of choosing s_0, σ, etc. is independent of the weight, the same technique used to calculate $T_{0,1}$ and $T_{1,0}$ in weight 10 can be used for other weights. The results are listed in Table 17. The characters of some of the local components in Table 17 are determined as follows:

- For X_{12}: χ is unramified such that $(1-\chi(2)2^{-s})(1-\chi(2)^{-1}2^{-s})$ is the reciprocal of the Euler factor at $p = 2$ (in the analytic normalization) of the elliptic cusp form spanning the space $S_{22}(\mathrm{SL}(2,\mathbb{Z}))$.
- For $L(2)$ and $N(8)^{\mathrm{a}}$: ξ is unramified with $\xi(2) = -1$.
- For $N(4)$: σ is unramified with $2(\sigma(2)+\sigma(2)^{-1}) = \lambda_{0,1} = -7/2$. The character χ is unramified with $\chi\sigma^2 = 1$.
- For $N(8)^{\mathrm{c}}$: σ is unramified with $2^{3/2}(\sigma(2)+\sigma(2)^{-1}) = \lambda_{0,1} = \frac{1}{8}(-12-5\sqrt{6})$.
- For $N(8)^{\mathrm{d}}$: σ is unramified with $2^{3/2}(\sigma(2)+\sigma(2)^{-1}) = \lambda_{0,1} = \frac{1}{8}(-12+5\sqrt{6})$.

As an application we obtain the L-factors of all automorphic representations involved; see Table 18. Observe that the Atkin–Lehner eigenvalue of all newforms is $+1$. Hence $\varepsilon(1/2, \Pi) = +1$ for all representations Π in Table 16.

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