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Integral representation for L-functions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$

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ABSTRACT

Let π be a cuspidal, automorphic representation of GSp_4 attached to a Siegel modular form of degree 2. We refine the method of Furusawa [M. Furusawa, On L-functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ and their special values, J. Reine Angew. Math. 438 (1993) 187–218] to obtain an integral representation for the degree-8 L-function $L(s, \pi \times \tau)$, where τ runs through certain cuspidal, automorphic representation of GL_2. Our calculations include the case of any representation with unramified central character for the p-adic components of τ, and a wide class of archimedean types including Maaß forms. As an application we obtain a special value result for $L(s, \pi \times \tau)$.

1. Introduction

Let $\pi = \bigotimes \pi_v$ and $\tau = \bigotimes \tau_v$ be irreducible, cuspidal, automorphic representations of $\mathrm{GSp}_4(\mathbb{A})$ and $\mathrm{GL}_2(\mathbb{A})$, respectively. Here, $\mathbb{A}$ is the ring of adeles of a number field F. We want to investigate the degree eight twisted L-functions $L(s, \pi \times \tau)$ of π and τ, which are important for a number of reasons. For example, when π and τ are obtained from holomorphic modular forms, then Deligne [8] has conjectured that a finite set of special values of $L(s, \pi \times \tau)$ are algebraic up to certain period integrals. Another very important application is the conjectured Langlands functorial transfer of π to an automorphic representation of $\mathrm{GL}_4(\mathbb{A})$. One approach to obtain the transfer to $\mathrm{GL}_4(\mathbb{A})$ is to use the converse theorem due to Cogdell and Piatetski-Shapiro [6], which requires precise information about the L-functions $L(s, \pi \times \tau)$.

In the special case that π is generic, Asgari and Shahidi [2] have been successful in obtaining the above transfer using the converse theorem. They analyze the twisted L-functions using the Langlands–Shahidi method. In this method, one has to consider a larger group in which GSp_4 is embedded and

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then use the representation π to construct an Eisenstein series on the larger group. Then the L-functions are obtained in the constant and non-constant terms of the Eisenstein series. Unfortunately, this method only works when π is generic. It is known that if π is obtained from a holomorphic Siegel modular form then it is not generic.

Another method to understand L-functions is via integral representations. For this method one constructs an integral that is Eulerian, i.e., one that can be written as an infinite product of local integrals, $Z(s) = \prod_v Z_v(s)$. Then the local integrals are computed to obtain the local L-functions. In many of the constructions, the local calculations are done only when all the local data is unramified. This gives information about the partial L-functions, which already leads to remarkable applications. The calculations for the ramified data are unfortunately often very involved and not available in the literature. (For more on integral representations of L-functions, see [11,12,21].)

In the GSp_4 and $\mathrm{GSp}_4 \times \mathrm{GL}_2$ case, Novodvorsky, Piatetski-Shapiro and Soudry (see [18,20,22]) were the first ones to construct integral representations for $L(s, \pi \times \tau)$. Their constructions were for the special case when π is either generic or has a special Bessel model. Examples of Siegel modular forms which do not have a special Bessel model have been constructed by Schulze-Pillot [29]. The first construction of an integral representation for $L(s, \pi \times \tau)$ with no restriction on the Bessel model of π is the work of Furusawa [9]. In this remarkable paper, Furusawa embeds GSp_4 in a unitary group $\mathrm{GU}(2,2)$ and constructs an Eisenstein series on $\mathrm{GU}(2,2)$ using the GL_2 representation τ. He then integrates the Eisenstein series against a vector in π. He shows that this integral is Eulerian and, when the local data is unramified, he computes the local integral to obtain the local L-function $L(s, \pi_v \times \tau_v)$ up to a normalizing factor. He also calculates the archimedean integral for the case that both π and τ are holomorphic of the same weight. Thus, Furusawa obtains an integral representation for the completed L-function $L(s, \pi \times \tau)$ in the case when π and τ are obtained from holomorphic modular forms of full level and same weight. He uses this to obtain a special value result, which fits into the context of Deligne's conjectures, and to prove meromorphic continuation and functional equation for the L-function. The main limitation of [9] is that, if we fix a Siegel modular form, then the results allow us to obtain information on a very small family of twists only, namely those coming from elliptic modular forms of full level and the same weight as the Siegel modular form, which is a finite dimensional vector space.

For the applications that we discussed above, we need twists of π by all representations τ of GL_2, i.e., twists by all GL_2 modular forms, holomorphic or non-holomorphic, of arbitrary weight and level. For this purpose, one needs to compute the non-archimedean local integral obtained in [9] when the local representation τ_v is ramified. Also, one needs to extend Furusawa's archimedean calculation to include more general archimedean representations.

In this paper, we will compute the local non-archimedean integral from [9] in the case when τ_v is any irreducible, admissible representation with unramified central character. We will also compute the archimedean integral for a larger family of archimedean representations τ_∞.

Before we state the results of this paper, let us recall the integral representation of [9] in some more detail. Let L be a quadratic extension of the number field F, and let $\mathrm{GU}(2,2)$ be the unitary group defined using the field L. Let P be the standard maximal parabolic subgroup of the unitary group $\mathrm{GU}(2,2)$ with a non-abelian radical. Given an irreducible, admissible representation τ of $\mathrm{GL}_2(\mathbb{A})$ and suitable characters χ and χ_0 of $\mathbb{A}_L^\times$, one considers an induced representation $I(s, \chi, \chi_0, \tau)$ from P to $\mathrm{GU}(2,2)$, where s is a complex parameter. Let $f(g,s)$ be an analytic family in $I(s, \chi, \chi_0, \tau)$. Define an Eisenstein series on $\mathrm{GU}(2,2)$ by the formula

$$E(g,s) = E(g,s;f) = \sum_{\gamma \in P(F)\backslash \mathrm{GU}(2,2)(F)} f(\gamma g, s), \quad g \in \mathrm{GU}(2,2)(\mathbb{A}).$$

For an automorphic form ϕ in the space of π, consider the integral

$$Z(s) = Z(s, f, \bar{\phi}) = \int\limits_{Z(\mathbb{A})\,\mathrm{GSp}_4(F)\backslash \mathrm{GSp}_4(\mathbb{A})} E(h,s;f)\bar{\phi}(h)\,dh. \tag{1}$$

In [9], Furusawa has shown that these integrals have the following two important properties.

(i) There is a "basic identity"

$$Z(s)=\int\limits_{R(\mathbb{A})\backslash \mathrm{GSp}_4(\mathbb{A})} W_f(\eta h,s)B_{\bar{\phi}}(h)\,dh, \tag{2}$$

where $R\subset \mathrm{GSp}(4)$ is a Bessel subgroup of the Siegel parabolic subgroup, η is a certain fixed element, $B_{\bar{\phi}}$ corresponds to $\bar{\phi}$ in the Bessel model for π, and W_f is a function on $\mathrm{GU}(2,2)$ obtained from the Whittaker model of τ and depending on the section f used to define the Eisenstein series.

(2) $Z(s)$ is Eulerian, i.e.,

$$Z(s)=\prod_\nu Z_\nu(s)=\prod_\nu \int\limits_{R(F_\nu)\backslash \mathrm{GSp}_4(F_\nu)} W_\nu(\eta h,s)B_\nu(h)\,dh. \tag{3}$$

In Theorems 3.8.1 and 3.8.2 below we show that the local integral can be computed to give $L(3s+\frac{1}{2},\tilde{\pi}_\nu\times\tilde{\tau}_\nu)$ up to a normalizing factor.

Theorem 1. *Let F_ν be a non-archimedean local field with characteristic zero. Let π_ν be an unramified, irreducible, admissible representation of $\mathrm{GSp}_4(F_\nu)$. Let τ_ν be an irreducible, admissible, generic representation of $\mathrm{GL}_2(F_\nu)$ with unramified central character and conductor $\mathfrak{p}^n$, $n\geqslant 1$. Then we can make a choice of vectors W_ν and B_ν such that the local integral in (3) is given by*

$$Z_\nu(s)=\begin{cases}\dfrac{L(3s+\frac{1}{2},\tilde{\pi}_\nu\times\tilde{\tau}_\nu)}{L(3s+1,\tau_\nu\times\mathcal{AI}(\Lambda_\nu)\times(\chi_\nu|_{F_\nu^\times}))} & \text{if } n=1;\\ 1 & \text{if } n\geqslant 2.\end{cases}$$

Here, Λ_ν is the Bessel character on $L_\nu^\times$ used to define the Bessel model B_ν, and $\mathcal{AI}(\Lambda_\nu)$ is the representation of $\mathrm{GL}_2(F_\nu)$ obtained from Λ_ν by automorphic induction.

Note that, for $n\geqslant 2$ in the above theorem, we have $L(s,\pi_\nu\times\tau_\nu)=1$, and hence the integral $Z_\nu(s)$ indeed computes the L-function. We point out that the ramified calculation is not a trivial generalization of the unramified calculation in [9]. There are two main steps. First is the choice of the vector W_ν and B_ν – making the "correct" choice of local vectors to be used to compute the local integral is delicate and, probably, is the main contribution of this paper. For example, we will have to make a choice of local compact subgroup $K^\#(\mathfrak{P}^n)$, for which the Borel congruence subgroup turns out to be too small, while the Klingen congruence subgroup is too large; the group we will work with lies in between these two natural congruence subgroups. Secondly, the actual computation of the local integral is complicated and depends heavily on the structure theory of the groups involved. We will explain this in detail in Section 3.

In Theorem 4.4.1, we compute the local archimedean integral in the following cases:

(i) π_∞ is the holomorphic discrete series representation of $\mathrm{GSp}_4(\mathbb{R})$ with trivial central character and Harish-Chandra parameter $(l-1,l-2)$. This is the archimedean component of the automorphic representations generated by Siegel modular forms of weight l.

(ii) τ_∞ is either a principal series representation of $\mathrm{GL}_2(\mathbb{R})$ whose K-types have the same parity as l or is a holomorphic discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with lowest weight l_2 satisfying $l_2\leqslant l$ and $l_2\equiv l \pmod 2$.

This extends the calculations in [9], where τ_∞ is only allowed to be a holomorphic discrete series representation with lowest weight l.

Putting together the local computations we get the following global result (see Theorem 5.3.1).

Theorem 2. *Let Φ be a cuspidal Siegel eigenform of weight l with respect to $\mathrm{Sp}_4(\mathbb{Z})$ (satisfying the two mild assumptions formulated in Section 5.1). Let N be any positive integer. Let f be a cuspidal Maaß eigenform of weight $l_1 \in \mathbb{Z}$ with respect to $\Gamma_0(N)$. If (the adelic function corresponding to) f lies in a holomorphic discrete series representation with lowest weight l_2, then assume that $l_2 \leqslant l$. Let π_Φ and τ_f be the corresponding cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, respectively. Then a choice of local vectors can be made such that the global integral $Z(s)$ defined in (1) is given by*

$$Z(s) = \kappa_\infty(s)\kappa_N(s)\frac{L(3s+\frac{1}{2}, \pi_\Phi \times \tau_f)}{\zeta(6s+1)L(3s+1, \tau_f \times \mathcal{AI}(\Lambda))}, \tag{4}$$

where $\kappa_\infty(s)$ and $\kappa_N(s)$ are explicitly known factors obtained from the local computations.

Using (4), we get the following special value result (see Theorem 5.4.4).

Theorem 3. *Let Φ be a cuspidal Siegel eigenform of weight l with respect to $\mathrm{Sp}_4(\mathbb{Z})$ (satisfying the two mild assumptions formulated in Section 5.1). Let N be any positive integer. Let Ψ be a holomorphic, cuspidal Hecke eigenform of weight l with respect to $\Gamma_0(N)$. Then*

$$\frac{L(\frac{l}{2}-1, \pi_\Phi \times \tau_\Psi)}{\pi^{5l-8}(\Phi,\Phi)_2(\Psi,\Psi)_1} \in \overline{\mathbb{Q}}.$$

Note that in [3], using a completely different method, special value results in the spirit of Deligne's conjectures were proven under the assumption that Ψ is a cusp form with respect to $\mathrm{SL}_2(\mathbb{Z})$ with weight $k \leqslant 2l-2$, where l is the weight of the Siegel modular form Φ. Since the results of [3] cannot be applied to modular forms with respect to congruence subgroups, there is no overlap of [3] with this paper.

This paper is organized as follows. In Section 2 we make the basic definitions and describe the setup for the local integrals from (3) for a non-archimedean local field F of characteristic zero or $F=\mathbb{R}$. We use the fact that the basic local setup is uniform and can be stated in full generality. The main input of the local integrals are the choices of the functions W and B from (3). In Sections 3 and 4 we consider the non-archimedean and archimedean case, respectively. We make the choice of the appropriate functions W and B and compute the local integrals. In Section 5, we consider the global situation corresponding to modular forms on GSp_4 and GL_2. We use the local calculations from Sections 3 and 4 to obtain an integral representation for the global L-function. Finally, in Section 5.4, we use the global theorem to obtain a special values result.

After the completion of this work it has been brought to our attention that there is some overlap with the doctoral thesis [26] of Abhishek Saha. Amongst the differences, Saha has obtained an interpretation of the integral representation for the L-function due to Furusawa using pullbacks of Eisenstein series on GU(3, 3), and can also include GSp_4 Steinberg representations under certain conditions.

Finally, we would like to thank A. Raghuram for many helpful discussions and for pointing out a gap in an earlier draft of the paper.

2. General setup

In this section, we give the basic definitions and set up the data required to compute the local integrals. Let F be a non-archimedean local field of characteristic zero, or $F=\mathbb{R}$. We fix three elements $a, b, c \in F$ such that $d := b^2-4ac \neq 0$. Let

$$L = \begin{cases} F(\sqrt{d}) & \text{if } d \notin F^{\times 2}, \\ F \oplus F & \text{if } d \in F^{\times 2}. \end{cases} \tag{5}$$

In case $L = F \oplus F$, we consider F diagonally embedded. If L is a field, we denote by $\bar{x}$ the Galois conjugate of $x \in L$ over F. If $L = F \oplus F$, let $\overline{(x, y)} = (y, x)$. In any case we let $N(x) = x\bar{x}$ and $\mathrm{tr}(x) = x + \bar{x}$.

2.1. The unitary group

We define the symplectic and unitary similitude groups by

$$H(F) = \mathrm{GSp}_4(F) := \left\{g \in \mathrm{GL}_4(F):\ {}^t g J g = \mu(g) J,\ \mu(g) \in F^\times\right\},$$
$$G(F) = \mathrm{GU}(2,2; L) := \left\{g \in \mathrm{GL}_4(L):\ {}^t \bar{g} J g = \mu(g) J,\ \mu(g) \in F^\times\right\},$$

where $J = \left[\begin{smallmatrix} & 1_2 \\ -1_2 & \end{smallmatrix}\right]$. Note that $H(F) = G(F) \cap \mathrm{GL}_4(F)$. As a minimal parabolic subgroup we choose the subgroup of all matrices that become upper triangular after switching the last two rows and last two columns. Let P be the standard maximal parabolic subgroup of $G(F)$ with a non-abelian unipotent radical. Let $P = MN$ be the Levi decomposition of P. We have $M = M^{(1)} M^{(2)}$, where

$$M^{(1)}(F) = \left\{ \begin{bmatrix} \zeta &&& \\ & 1 && \\ && \bar{\zeta}^{-1} & \\ &&& 1 \end{bmatrix} :\ \zeta \in L^\times \right\}, \tag{6}$$

$$M^{(2)}(F) = \left\{ \begin{bmatrix} 1 &&& \\ & \alpha && \beta \\ && \mu & \\ & \gamma && \delta \end{bmatrix} \in G(F) \right\}, \tag{7}$$

$$N(F) = \left\{ \begin{bmatrix} 1 & z && \\ & 1 && \\ && 1 & \\ && -\bar{z} & 1 \end{bmatrix} \begin{bmatrix} 1 && w & y \\ & 1 & \bar{y} & \\ && 1 & \\ &&& 1 \end{bmatrix} :\ w \in F,\ y, z \in L \right\}. \tag{8}$$

For a matrix in $M^{(2)}(F)$ as the one above, the unitary conditions are equivalent to $\mu = \bar{\mu}$ (i.e., $\mu \in F^\times$), $\mu = \bar{\alpha}\delta - \beta\bar{\gamma}$, $\bar{\alpha}\gamma = \bar{\gamma}\alpha$ and $\bar{\delta}\beta = \bar{\beta}\delta$. In addition, we have $\bar{\alpha}\beta = \bar{\beta}\alpha$, $\bar{\delta}\gamma = \bar{\gamma}\delta$, $\bar{\alpha}\delta = \bar{\delta}\alpha$, $\bar{\gamma}\beta = \bar{\beta}\gamma$. Hence the following holds.

Lemma 2.1.1. *Let*

$$\begin{bmatrix} 1 &&& \\ & \alpha && \beta \\ && \mu & \\ & \gamma && \delta \end{bmatrix}$$

be an element of $M^{(2)}(F)$, as above. Then the quotient of any two entries of the matrix $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$, if defined, lies in F. Hence, if λ is any invertible entry of $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$, then

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} \alpha/\lambda & \beta/\lambda \\ \gamma/\lambda & \delta/\lambda \end{bmatrix}}_{\in \mathrm{GL}_2(F)}.$$

Consequently, the map

$$\begin{aligned} &L^\times \times \mathrm{GL}_2(F) \longrightarrow M^{(2)}(F),\\ &\left(\lambda, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) \longmapsto \begin{bmatrix} 1 &&& \\ & \lambda\alpha && \lambda\beta \\ && N(\lambda)(\alpha\delta-\beta\gamma) & \\ & \lambda\gamma && \lambda\delta \end{bmatrix}, \end{aligned} \tag{9}$$

is surjective with kernel $\{(\lambda, \lambda^{-1})\colon\ \lambda \in F^\times\}$.

The modular factor of the parabolic P is given by

$$\delta_P\left(\begin{bmatrix} \zeta &&& \\ & 1 && \\ && \bar\zeta^{-1} & \\ &&& 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & \alpha && \beta \\ && \mu & \\ & \gamma && \delta \end{bmatrix}\right) = \left|N(\zeta)\mu^{-1}\right|^3 \quad (\mu = \bar\alpha\delta - \beta\bar\gamma), \tag{10}$$

where $|\cdot|$ is the normalized absolute value on F.

2.2. *The Bessel subgroup*

Recall that we fixed three elements $a, b, c \in F$ such that $d = b^2 - 4ac \neq 0$. Let

$$S = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}, \qquad \xi = \begin{bmatrix} \frac{b}{2} & c \\ -a & \frac{-b}{2} \end{bmatrix}.$$

Then $F(\xi) = F + F\xi$ is a two-dimensional F-algebra isomorphic to L. If $L = F(\sqrt{d})$ is a field, then an isomorphism is given by $x + y\xi \mapsto x + y\frac{\sqrt{d}}{2}$. If $L = F \oplus F$, then an isomorphism is given by $x + y\xi \mapsto (x + y\frac{\sqrt{d}}{2}, x - y\frac{\sqrt{d}}{2})$. The determinant map on $F(\xi)$ corresponds to the norm map on L. Let

$$T(F) = \left\{g \in \mathrm{GL}_2(F)\colon\ {}^tgSg = \det(g)S\right\}.$$

One can check that $T(F) = F(\xi)^\times$. Note that $T(F) \cong L^\times$ via the isomorphism $F(\xi) \cong L$. We consider $T(F)$ a subgroup of $H(F) = \mathrm{GSp}_4(F)$ via

$$T(F) \ni g \longmapsto \begin{bmatrix} g & \\ & \det(g)\,{}^tg^{-1} \end{bmatrix} \in H(F).$$

Let

$$U(F) = \left\{\begin{bmatrix} 1_2 & X \\ & 1_2 \end{bmatrix} \in \mathrm{GSp}_4(F)\colon\ {}^tX = X\right\}$$

and $R(F) = T(F)U(F)$. We call $R(F)$ the *Bessel subgroup* of $\mathrm{GSp}_4(F)$ (with respect to the given data a, b, c). Let ψ be any non-trivial character $F \to \mathbb{C}^\times$. Let $\theta : U(F) \to \mathbb{C}^\times$ be the character given by

$$\theta\left(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}\right) = \psi\big(\mathrm{tr}(SX)\big). \tag{11}$$

Explicitly,

$$\theta\left(\begin{bmatrix} 1 && x & y \\ & 1 & y & z \\ && 1 & \\ &&& 1 \end{bmatrix}\right) = \psi(ax + by + cz). \tag{12}$$

We have $\theta(t^{-1}ut)=\theta(u)$ for all $u\in U(F)$ and $t\in T(F)$. Hence, if Λ is any character of $T(F)$, then the map $tu\mapsto\Lambda(t)\theta(u)$ defines a character of $R(F)$. We denote this character by $\Lambda\otimes\theta$.

2.3. Parabolic induction from $P(F)$ to $G(F)$

Let (τ, V_τ) be an irreducible, admissible representation of $\mathrm{GL}_2(F)$, and let χ_0 be a character of $L^\times$ such that $\chi_0|_{F^\times}$ coincides with ω_τ, the central character of τ. Then the representation $(\lambda, g)\mapsto \chi_0(\lambda)\tau(g)$ of $L^\times\times\mathrm{GL}_2(F)$ factors through $\{(\lambda,\lambda^{-1})\colon \lambda\in F^\times\}$, and consequently, by Lemma 2.1.1, defines a representation of $M^{(2)}(F)$ on the same space V_τ. Let us denote this representation by $\chi_0\times\tau$. Every irreducible, admissible representation of $M^{(2)}(F)$ is of this form. If V_τ is a space of functions on $\mathrm{GL}_2(F)$ on which $\mathrm{GL}_2(F)$ acts by right translation, then $\chi_0\times\tau$ can be realized as a space of functions on $M^{(2)}(F)$ on which $M^{(2)}(F)$ acts by right translation. This is accomplished by extending every $W\in V_\tau$ to a function on $M^{(2)}(F)$ via

$$W(\lambda g)=\chi_0(\lambda)W(g),\quad \lambda\in L^\times,\ g\in\mathrm{GL}_2(F). \tag{13}$$

If V_τ is the Whittaker model of τ with respect to the character ψ, then the extended functions W satisfy the transformation property

$$W\left(\begin{bmatrix}1&&&\\&1&&x\\&&1&\\&&&1\end{bmatrix}g\right)=\psi(x)W(g),\quad x\in F,\ g\in M^{(2)}(F). \tag{14}$$

If s is a complex parameter, χ is any character of $L^\times$, and $\chi_0\times\tau$ is a representation of $M^{(2)}(F)$ as above, we denote by $I(s,\chi,\chi_0,\tau)$ the representation of $G(F)$ obtained by parabolic induction from the representation of $P(F)=M(F)N(F)$ given on the Levi part by

$$\begin{bmatrix}\zeta&&&\\&1&&\\&&\bar\zeta^{-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&\lambda\alpha&&\lambda\beta\\&&N(\lambda)(\alpha\delta-\beta\gamma)&\\&\lambda\gamma&&\lambda\delta\end{bmatrix}$$
$$\longmapsto \left|N\big(\zeta\lambda^{-1}\big)(\alpha\delta-\beta\gamma)^{-1}\right|^{3s}\chi(\zeta)\chi_0(\lambda)\tau\left(\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}\right).$$

Explicitly, the space of $I(s,\chi,\chi_0,\tau)$ consists of functions $f: G(F)\to V_\tau$ with the transformation property

$$f\left(\begin{bmatrix}\zeta&&&\\&1&&\\&&\bar\zeta^{-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&\lambda\alpha&&\lambda\beta\\&&N(\lambda)(\alpha\delta-\beta\gamma)&\\&\lambda\gamma&&\lambda\delta\end{bmatrix}g\right)$$
$$=\left|N\big(\zeta\lambda^{-1}\big)(\alpha\delta-\beta\gamma)^{-1}\right|^{3(s+\frac12)}\chi(\zeta)\chi_0(\lambda)\tau\left(\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}\right)f(g). \tag{15}$$

Now assume that V_τ is the Whittaker model of τ with respect to the character ψ of F. If we associate to each f as above the function on $G(F)$ given by $W^\#_f(g)=f(g)(1)$, then we obtain another model of $I(s,\chi,\chi_0,\tau)$ consisting of functions $W^\#: G(F)\to\mathbb{C}$. These functions satisfy

$$W^\#\left(\begin{bmatrix}\zeta&&&\\&1&&\\&&\bar\zeta^{-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&\lambda&&\\&&N(\lambda)&\\&&&\lambda\end{bmatrix}g\right)$$
$$=\left|N\big(\zeta\lambda^{-1}\big)\right|^{3(s+\frac12)}\chi(\zeta)\chi_0(\lambda)W^\#(g),\quad \zeta,\lambda\in L^\times, \tag{16}$$

and

$$W^{\#}\left(\begin{bmatrix}1&z&&\\&1&&\\&&1&\\&&-\bar z&1\end{bmatrix}\begin{bmatrix}1&&w&y\\&1&\bar y&x\\&&1&\\&&&1\end{bmatrix}g\right)=\psi(x)W^{\#}(g),\quad w,x\in F,\ y,z\in L. \tag{17}$$

The following lemma gives a transformation property of $W^{\#}$ under the action of the elements of the Bessel subgroup $R(F)$.

Lemma 2.3.1. *Let (τ, V_τ) be a generic, irreducible, admissible representation of $\mathrm{GL}_2(F)$. We assume that V_τ is the Whittaker model of τ with respect to the non-trivial character $\psi^{-c}(x)=\psi(-cx)$ of F. Let χ and χ_0 be characters of $L^\times$ such that $\chi_0|_{F^\times}=\omega_\tau$. Let $W^{\#}(\cdot,s): G(F)\to\mathbb{C}$ be a function in the above model of the induced representation $I(s,\chi,\chi_0,\tau)$, where s is a complex parameter. Let θ be the character of $U(F)$ defined in (11). Let Λ be the character of $L^\times\cong T(F)$ given by*

$$\Lambda(\zeta)=\chi(\bar\zeta)^{-1}\chi_0(\zeta)^{-1}. \tag{18}$$

Let

$$\eta=\begin{bmatrix}1&0&&\\\alpha&1&&\\&&1&-\bar\alpha\\&&0&1\end{bmatrix},\quad \text{where } \alpha:=\begin{cases}\frac{b+\sqrt{d}}{2c} & \text{if } L \text{ is a field},\\ \left(\frac{b+\sqrt{d}}{2c},\frac{b-\sqrt{d}}{2c}\right) & \text{if } L=F\oplus F.\end{cases} \tag{19}$$

Then

$$W^{\#}(\eta tuh,s)=\Lambda(t)^{-1}\theta(u)^{-1}W^{\#}(\eta h,s) \tag{20}$$

for $t\in T(F)$, $u\in U(F)$ and $h\in G(F)$.

Proof. If L is a field, then the proof is word for word the same as on pp. 197/198 of [9]. The case $L=F\oplus F$ requires the only modification that the element $\zeta=x+\frac{y}{2}\sqrt{d}$ is to be replaced by $\zeta=x+\frac{y}{2}(\sqrt{d},-\sqrt{d})$. $\square$

2.4. The local integral

Let (π,V_π) be an irreducible, admissible representation of $H(F)=\mathrm{GSp}_4(F)$. Let the Bessel subgroup $R(F)$ be as defined in Section 2.2; it depends on the given data $a,b,c\in F$. We assume that V_π is a Bessel model for π with respect to the character $\Lambda\otimes\theta$ of $R(F)$. Hence, V_π consists of functions $B: H(F)\to\mathbb{C}$ satisfying the Bessel transformation property

$$B(tuh)=\Lambda(t)\theta(u)B(h)\quad \text{for } t\in T(F),\ u\in U(F),\ h\in H(F).$$

Let (τ,V_τ) be a generic, irreducible, admissible representation of $\mathrm{GL}_2(F)$ such that V_τ is the ψ^{-c}-Whittaker model of τ (we assume $c\neq 0$). Let χ_0 be a character of $L^\times$ such that $\chi_0|_{F^\times}=\omega_\tau$. Let χ be the character of $L^\times$ for which (18) holds. Let $W^{\#}(\cdot,s)$ be an element of $I(s,\chi,\chi_0,\tau)$ for which the restriction of $W^{\#}(\cdot,s)$ to the standard maximal compact subgroup of $G(F)$ (see below for more details) is independent of s, i.e., $W^{\#}(\cdot,s)$ is a "flat section" of the family of induced representations $I(s,\chi,\chi_0,\tau)$. By Lemma 2.3.1 it is meaningful to consider the integral

$$Z(s)=\int\limits_{R(F)\backslash H(F)} W^{\#}(\eta h,s)B(h)\,dh. \tag{21}$$

In the following we shall compute these integrals for certain choices of $W^\#$ and B. We shall only consider $\mathrm{GSp}_4(F)$ representations π that are relevant for the global application to Siegel modular forms we have in mind. In the real case we shall assume that π is a holomorphic discrete series representation and that B corresponds to the highest weight vector. In the p-adic case we shall assume that π is an unramified representation and that B corresponds to the spherical vector.

The generic $\mathrm{GL}_2(F)$ representation τ, however, will be only mildly restricted in the real case, and, in the p-adic case, will be any representation with an unramified central character. In the real case, the function $W^\#$ will be constructed from a certain vector of the "correct" weight in V_τ. In the p-adic case, the function $W^\#$ will be constructed from the local newform in V_τ. In each case our calculations will show that the integral (21) converges absolutely for $\mathrm{Re}(s)$ large enough and has meromorphic continuation to all of $\mathbb{C}$. Our choice of $W^\#$ will be such that $Z(s)$ is closely related to the local L-factor $L(s,\pi\times\tau)$. Note that the integral (21) has been calculated in [9] for π and τ both holomorphic discrete series representations with related lowest weights in the real case and π and τ both unramified representations in the p-adic case.

3. Local non-archimedean theory

In this section, we evaluate (21) in the non-archimedean setting. The key steps are the choices of the vector $W^\#$ and the actual computation of the integral $Z(s)$.

3.1. Setup

Let F be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$, $\mathfrak{p}$, ϖ, q be the ring of integers, prime ideal, uniformizer and cardinality of the residue class field $\mathfrak{o}/\mathfrak{p}$, respectively. Recall that we fix three elements $a,b,c\in F$ such that $d:=b^2-4ac\neq 0$. Let L be as in (5). We shall make the following *assumptions*:

(A1) $a,b\in\mathfrak{o}$ and $c\in\mathfrak{o}^\times$.
(A2) If $d\notin F^{\times 2}$, then d is the generator of the discriminant of L/F. If $d\in F^{\times 2}$, then $d\in\mathfrak{o}^\times$.

Remark. In [9, p. 198], Furusawa makes a stronger assumption on a,b,c, namely, $\left[\begin{smallmatrix} a & b/2\\ b/2 & c\end{smallmatrix}\right]\in M_2(\mathfrak{o})$. However, it is necessary to make the weaker assumption $a,b,c\in\mathfrak{o}$ for the global integral calculation (4.5) in [9, p. 210] to be valid for $D\equiv 3\pmod 4$. (This is because the matrix $S(-D)$ on p. 208 is not in $M_2(\mathfrak{o}_2)$ for $D\equiv 3\pmod 4$.) One can check that the non-archimedean unramified calculation in [9] is valid with the weaker assumption $a,b,c\in\mathfrak{o}$. Hence, the global result of [9] is still valid but the assumptions (A1) and (A2) above are the correct ones.

We set the Legendre symbol as follows,

$$\left(\frac{L}{\mathfrak{p}}\right):=\begin{cases}-1, & \text{if } d\notin F^{\times 2},\ d\notin\mathfrak{p}\quad\text{(the inert case)},\\ 0, & \text{if } d\notin F^{\times 2},\ d\in\mathfrak{p}\quad\text{(the ramified case)},\\ 1, & \text{if } d\in F^{\times 2}\quad\text{(the split case)}.\end{cases}\tag{22}$$

If L is a field, then let $\mathfrak{o}_L$ be its ring of integers. If $L=F\oplus F$, then let $\mathfrak{o}_L=\mathfrak{o}\oplus\mathfrak{o}$. Note that $x\in\mathfrak{o}_L$ if and only if $N(x)$, $\mathrm{tr}(x)\in\mathfrak{o}$. If L is a field then we have $x\in\mathfrak{o}_L^\times$ if and only if $N(x)\in\mathfrak{o}^\times$. If L is not a field then $x\in\mathfrak{o}_L$, $N(x)\in\mathfrak{o}^\times$ implies that $x\in\mathfrak{o}_L^\times=\mathfrak{o}^\times\oplus\mathfrak{o}^\times$. Let ϖ_L be the uniformizer of $\mathfrak{o}_L$ if L is a field and set $\varpi_L=(\varpi,1)$ if L is not a field. Note that, if $(\frac{L}{\mathfrak{p}})\neq -1$, then $N(\varpi_L)\in\varpi\mathfrak{o}^\times$. Let

$$\xi_0:=\begin{cases}\frac{-b+\sqrt{d}}{2} & \text{if } L \text{ is a field},\\ \left(\frac{-b+\sqrt{d}}{2},\frac{-b-\sqrt{d}}{2}\right) & \text{if } L=F\oplus F,\end{cases}\tag{23}$$

and

$$\alpha := \begin{cases} \frac{b+\sqrt{d}}{2c} & \text{if } L \text{ is a field,} \\ \left(\frac{b+\sqrt{d}}{2c}, \frac{b-\sqrt{d}}{2c}\right) & \text{if } L = F \oplus F. \end{cases} \tag{24}$$

We fix the following ideal in $\mathfrak{o}_L$,

$$\mathfrak{P} := \mathfrak{p}\mathfrak{o}_L = \begin{cases} \mathfrak{p}_L & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \mathfrak{p}_L^2 & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \mathfrak{p} \oplus \mathfrak{p} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases} \tag{25}$$

Here, $\mathfrak{p}_L$ is the maximal ideal of $\mathfrak{o}_L$ when L is a field extension. Note that $\mathfrak{P}$ is prime only if $(\frac{L}{\mathfrak{p}}) = -1$. We have $\mathfrak{P}^n \cap \mathfrak{o} = \mathfrak{p}^n$ for all $n \geqslant 0$. We now state a number-theoretic lemma which will be crucial in Section 3.6.

Lemma 3.1.1. *Let notations be as above.*

(i) *The elements* 1 *and* ξ_0 *constitute an integral basis of* L/F *(i.e., a basis of the free* $\mathfrak{o}$*-module* $\mathfrak{o}_L$*). The elements* 1 *and* α *also constitute an integral basis of* L/F.
(ii) *There exists no* $x \in \mathfrak{o}$ *such that* $\alpha + x \in \mathfrak{P}$.

Proof. (i) Since $c \in \mathfrak{o}^\times$ and $b \in \mathfrak{o}$, the second assertion of (i) follows from the first one. To prove the first assertion, first note that ξ_0 satisfies $\xi_0^2 + \xi_0 b + ac = 0$, and therefore belongs to $\mathfrak{o}_L$. Since the claim is easily verified if $L = F \oplus F$, we will assume that L is a field. Let $A, B \in F$ be such that 1 and $\xi_1 := A + B\sqrt{d}$ is an integral basis of L/F. Then

$$\det\left(\begin{bmatrix} 1 & \xi_1 \\ 1 & \bar{\xi}_1 \end{bmatrix}\right)^2 = 4B^2 d$$

generates the discriminant of L/F. Since d also generates the discriminant by assumption (A2), it follows that $2B \in \mathfrak{o}_F^\times$. Dividing ξ_1 by this unit, we may assume $\xi_1 = A + \frac{1}{2}\sqrt{d}$ for some $A \in F$. Now let us represent ξ_0 in this integral basis,

$$\xi_0 = x + y\xi_1, \quad x, y \in \mathfrak{o}_F,$$

i.e.,

$$\frac{-b + \sqrt{d}}{2} = x + y\left(A + \frac{1}{2}\sqrt{d}\right).$$

Comparing coefficients, we get $y = 1$ and $A = -\frac{b}{2} - x$. We may modify ξ_1 by adding the integral element x and still obtain an integral basis. But $\xi_1 + x = \xi_0$, and the assertion follows.

(ii) Let $X \subset \mathfrak{o}_L/\mathfrak{P}$ be the image of the injection

$$\mathfrak{o}/\mathfrak{p} \longrightarrow \mathfrak{o}_L/\mathfrak{P}.$$

Note that the field on the left-hand side has q elements, and the ring on the right-hand side has q^2 elements, for any value of $(\frac{L}{\mathfrak{p}})$. Our claim is equivalent to the statement that $\bar{\alpha}$, the image of α in

$\mathfrak{o}_L/\mathfrak{P}$, does not lie in the subring X of $\mathfrak{o}_L/\mathfrak{P}$. Assume that $\bar{\alpha} \in X$. By (i), any element $z \in \mathfrak{o}_L$ can be (uniquely) written as

$$z = x\alpha + y, \quad x, y \in \mathfrak{o}.$$

Applying the projection to $\mathfrak{o}_L/\mathfrak{P}$, it follows that $\bar{z} = \bar{x}\bar{\alpha} + \bar{y} \in X$. This is a contradiction, since $\bar{z}$ runs through all elements of $\mathfrak{o}_L/\mathfrak{P}$, but X is a proper subset. $\square$

Note that, via the identification $T(F) = L^\times$ described in Section 2.2, the element ξ_0 corresponds to the matrix $\left[\begin{smallmatrix} 0 & c \\ -a & -b \end{smallmatrix}\right]$. Therefore, by Lemma 3.1.1(i),

$$\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o}\xi_0 = \left\{ \begin{bmatrix} x & yc \\ -ya & x - yb \end{bmatrix} :\ x, y \in \mathfrak{o} \right\}. \tag{26}$$

Since c is assumed to be a unit, it follows that

$$\mathfrak{o}_L = T(F) \cap M_2(\mathfrak{o}) \quad \text{and} \quad \mathfrak{o}_L^\times = T(F) \cap \mathrm{GL}_2(\mathfrak{o}). \tag{27}$$

3.2. The spherical Bessel function

Let (π, V_π) be an unramified, irreducible, admissible representation of $\mathrm{GSp}_4(F)$. Then π can be realized as the unramified constituent of an induced representation of the form $\chi_1 \times \chi_2 \rtimes \sigma$, where χ_1, χ_2 and σ are unramified characters of $F^\times$; here, we used the notation of [27] for parabolic induction. Let

$$\gamma^{(1)} = \chi_1\chi_2\sigma, \qquad \gamma^{(2)} = \chi_1\sigma, \qquad \gamma^{(3)} = \sigma, \qquad \gamma^{(4)} = \chi_2\sigma.$$

Then $\gamma^{(1)}\gamma^{(3)} = \gamma^{(2)}\gamma^{(4)}$ is the central character of π. The numbers $\gamma^{(1)}(\varpi), \dots, \gamma^{(4)}(\varpi)$ are the Satake parameters of π. The degree-4 L-factor of π is given by $\prod_{i=1}^4 (1 - \gamma^{(i)}(\varpi)q^{-s})^{-1}$.

Let Λ be any character of $T(F) \cong L^\times$. We assume that V_π is the Bessel model with respect to the character $\Lambda \otimes \theta$ of $R(F)$; see Section 2.2. Let $B \in V_\pi$ be a spherical vector. By [32, Propositions 2–5], we have $B(1) \neq 0$. It follows from $B(1) \neq 0$ and (27) that necessarily $\Lambda|_{\mathfrak{o}_L^\times} = 1$. For $l, m \in \mathbb{Z}$ let

$$h(l,m) = \begin{bmatrix} \varpi^{2m+l} & & & \\ & \varpi^{m+l} & & \\ & & 1 & \\ & & & \varpi^m \end{bmatrix}. \tag{28}$$

Then, as in (3.4.2) of [9],

$$H(F) = \bigsqcup_{l \in \mathbb{Z}} \bigsqcup_{m \geqslant 0} R(F)h(l,m)K^H, \quad K^H = \mathrm{GSp}_4(\mathfrak{o}). \tag{29}$$

The double cosets on the right-hand side are pairwise disjoint. Since B transforms on the left under $R(F)$ by the character $\Lambda \otimes \theta$ and is right K^H-invariant, it follows that B is determined by the values $B(h(l,m))$. By Lemma 3.4.4 of [9] we have $B(h(l,m)) = 0$ for $l < 0$, so that B is determined by the values $B(h(l,m))$ for $l, m \geqslant 0$.

In [32, 2–4], Sugano has given a formula for $B(h(l,m))$ in terms of a generating function. It turns out that for our purposes we only require the values $B(h(l,0))$. In this special case Sugano's formula reads

$$\sum_{l \geqslant 0} B\big(h(l,0)\big)y^l = \frac{1 - A_5 y - A_2 A_4 y^2}{Q(y)}, \tag{30}$$

Table 1

	$\left(\frac{L}{\mathfrak{p}}\right)=-1$	$\left(\frac{L}{\mathfrak{p}}\right)=0$	$\left(\frac{L}{\mathfrak{p}}\right)=1$
A_2	$q^{-2}\Lambda(\varpi)$	$q^{-2}\Lambda(\varpi)$	$q^{-2}\Lambda(\varpi)$
A_4	q^{-2}	0	$-q^{-2}$
A_5	0	$q^{-2}\Lambda(\varpi_L)$	$q^{-2}(\Lambda(\varpi_L)+\Lambda(\varpi\varpi_L^{-1}))$
$H(y)$	$1-q^{-4}\Lambda(\varpi)y^2$	$1-q^{-2}\Lambda(\varpi_L)y$	$1-q^{-2}(\Lambda(\varpi_L)+\Lambda(\varpi\varpi_L^{-1}))y+q^{-4}\Lambda(\varpi)y^2$

where

$$Q(y)=\prod_{i=1}^{4}\left(1-\gamma^{(i)}(\varpi)q^{-3/2}y\right), \tag{31}$$

and where A_2, A_4, A_5 are given in Table 1. Set $H(y)=1-A_5y-A_2A_4y^2$.

3.3. *The local compact subgroup*

We define congruence subgroups of $\mathrm{GL}_2(F)$, as follows. For $n=0$ let $K^{(1)}(\mathfrak{p}^0)=\mathrm{GL}_2(\mathfrak{o})$. For $n>0$ let

$$K^{(1)}(\mathfrak{p}^n)=\mathrm{GL}_2(F)\cap\begin{bmatrix}\mathfrak{o}^\times & \mathfrak{o}\\ \mathfrak{p}^n & \mathfrak{o}^\times\end{bmatrix}. \tag{32}$$

The following result is well known (see [5,7]).

Theorem 3.3.1. *Let (τ, V) be a generic, irreducible, admissible representation of* $\mathrm{GL}_2(F)$ *with unramified central character. Then the spaces*

$$V(n)=\left\{v\in V\colon\ \tau(g)v=v \text{ for all } g\in K^{(1)}(\mathfrak{p}^n)\right\}$$

are non-zero for n large enough. If n is minimal with $V(n)\neq 0$, then $\dim(V(n))=1$.

If n is minimal such that $V(n)\neq 0$, then $\mathfrak{p}^n$ is called the *conductor* of τ. In this section we shall define a family $K^\#(\mathfrak{P}^n)$, $n\geqslant 0$, of compact-open subgroups of $G(F)$, the relevance of which is as follows. Recall that our goal is to evaluate integrals of the form

$$Z(s)=\int\limits_{R(F)\backslash H(F)} W^\#(\eta h,s)B(h)\,dh, \tag{33}$$

where $W^\#(\cdot,s)$ is a section in a family of induced representations $I(s,\chi,\chi_0,\tau)$. The choice of the function $W^\#(\cdot,s)$ is crucial for our purposes. We will define it in such a way that $W^\#(\cdot,s)$ is supported on $M(F)N(F)K^\#(\mathfrak{P}^n)$, where $\mathfrak{p}^n$ is the conductor of the $\mathrm{GL}_2(F)$ representation τ.

Recall that $\mathfrak{P}=\mathfrak{p}\mathfrak{o}_L$. Let

$$I:=\left\{g\in\mathrm{GU}(2,2;\mathfrak{o}_L)\colon\ g\equiv\begin{bmatrix}*&0&*&*\\ *&*&*&*\\ 0&0&*&*\\ 0&0&0&*\end{bmatrix}\ (\mathrm{mod}\ \mathfrak{P})\right\} \tag{34}$$

be the Iwahori subgroup and

$$\mathrm{Kl}(\mathfrak{P}^n) := \left\{ g \in \mathrm{GU}(2,2;\mathfrak{o}_L):\ g \equiv \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \pmod{\mathfrak{P}^n} \right\} \tag{35}$$

be the Klingen congruence subgroup. We define $K^{\#}(\mathfrak{P}^0) := \mathrm{GU}(2,2;\mathfrak{o}_L)$, and for $n \geqslant 1$

$$K^{\#}(\mathfrak{P}^n) := I \cap \mathrm{Kl}(\mathfrak{P}^n) = \mathrm{GU}(2,2;\mathfrak{o}_L) \cap \begin{bmatrix} \mathfrak{o}_L^{\times} & \mathfrak{P}^n & \mathfrak{o}_L & \mathfrak{o}_L \\ \mathfrak{o}_L & \mathfrak{o}_L^{\times} & \mathfrak{o}_L & \mathfrak{o}_L \\ \mathfrak{P} & \mathfrak{P}^n & \mathfrak{o}_L^{\times} & \mathfrak{o}_L \\ \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{o}_L^{\times} \end{bmatrix}. \tag{36}$$

Furthermore, let

$$K^{\#}(\mathfrak{p}^n) := K^{\#}(\mathfrak{P}^n) \cap \mathrm{GSp}_4(F) = \mathrm{GSp}_4(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o}^{\times} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o}^{\times} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p}^n & \mathfrak{o}^{\times} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o}^{\times} \end{bmatrix}. \tag{37}$$

Note that $K^{\#}(\mathfrak{P}) = I$. The GL_2 congruence subgroup $K^{(1)}(\mathfrak{p}^n)$ defined above can be embedded into $K^{\#}(\mathfrak{P}^n)$ in the following way,

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \longmapsto \begin{bmatrix} 1 & & & \\ & \alpha & & \beta \\ & & \mu & \\ & \gamma & & \delta \end{bmatrix}, \quad \text{where } \mu = \alpha\delta - \beta\gamma. \tag{38}$$

It follows from Lemma 2.1.1 that the map

$$\begin{gathered} \mathfrak{o}_L^{\times} \times \mathrm{GL}_2(\mathfrak{o}) \longrightarrow M^{(2)}(F) \cap \mathrm{GL}_4(\mathfrak{o}_L), \\ \left(\lambda, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) \longmapsto \begin{bmatrix} 1 & & & \\ & \lambda\alpha & & \lambda\beta \\ & & N(\lambda)(\alpha\delta - \beta\gamma) & \\ & \lambda\gamma & & \lambda\delta \end{bmatrix}, \end{gathered} \tag{39}$$

is surjective with kernel $\{(\lambda, \lambda^{-1}):\ \lambda \in \mathfrak{o}_F^{\times}\}$.

3.4. The function $W^{\#}$

We shall now define the specific function $W^{\#}(\cdot, s)$ for which we shall evaluate the integral (33). Let (τ, V_τ) be a generic, irreducible, admissible representation of $\mathrm{GL}_2(F)$ with unramified central character. We assume that V_τ is the Whittaker model of τ with respect to the character of F given by $\psi^{-c}(x) = \psi(-cx)$. Let $\mathfrak{p}^n$ be the conductor of τ. Let $W^{(0)} \in V(n)$ be the local newform, i.e., the essentially unique non-zero $K^{(1)}(\mathfrak{p}^n)$ invariant vector in V_τ. We can make it unique by requiring that $W^{(0)}(1) = 1$, since this value is known to be non-zero.

We choose any character χ_0 of $L^{\times}$ such that

$$\chi_0|_{F^{\times}} = \omega_\tau \quad \text{and} \quad \chi_0|_{\mathfrak{o}_L^{\times}} = 1. \tag{40}$$

If $(\frac{L}{\mathfrak{p}}) = -1$, there is only one such character, but in the other cases the choice of χ_0 is not unique. We extend $W^{(0)}$ to a function on $M^{(2)}(F)$ via

$$W^{(0)}(ag) = \chi_0(a) W^{(0)}(g), \quad a \in L^\times,\ g \in \mathrm{GL}_2(F) \tag{41}$$

(see (9)). It follows from (39) that

$$W^{(0)}(g\kappa) = W^{(0)}(g), \quad \text{for } g \in M^{(2)}(F) \text{ and } \kappa \in M^{(2)}(F) \cap K^{\#}\big(\mathfrak{P}^n\big). \tag{42}$$

As in Section 3.2, let (π, V_π) be an unramified, irreducible, admissible representation of $\mathrm{GSp}_4(F)$, where V_π is the Bessel model for π with respect to the character $\Lambda \otimes \theta$ of $R(F) = T(F)U(F)$. As was pointed out in Section 3.2, the character Λ is necessarily unramified. Let χ be the character of $L^\times$ given by

$$\chi(\zeta) = \Lambda(\bar\zeta)^{-1} \chi_0(\bar\zeta)^{-1}, \tag{43}$$

so that (18) holds.

Given a complex number s, there exists a unique function $W^{\#}(\cdot, s) : G(F) \to \mathbb{C}$ with the following properties.

(i) If $g \notin M(F)N(F)K^{\#}(\mathfrak{P}^n)$, then $W^{\#}(g, s) = 0$.
(ii) If $g = mnk$ with $m \in M(F)$, $n \in N(F)$, $k \in K^{\#}(\mathfrak{P}^n)$, then $W^{\#}(g, s) = W^{\#}(m, s)$.
(iii) For $\zeta \in L^\times$ and $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in M^{(2)}(F)$,

$$W^{\#}\left(\begin{bmatrix} \zeta &&& \\ & 1 && \\ && \bar\zeta^{-1} & \\ &&& 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & \alpha && \beta \\ && \mu & \\ & \gamma && \delta \end{bmatrix}, s\right) = \big|N(\zeta) \cdot \mu^{-1}\big|^{3(s+1/2)} \chi(\zeta) W^{(0)}\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right). \tag{44}$$

Here $\mu = \bar\alpha\delta - \beta\bar\gamma$.

To verify that such a function exists, use (42) and

$$\big(M(F)N(F)\big) \cap K^{\#}\big(\mathfrak{P}^n\big) = \big(M(F) \cap K^{\#}\big(\mathfrak{P}^n\big)\big)\big(N(F) \cap K^{\#}\big(\mathfrak{P}^n\big)\big).$$

Also, one has to use the fact that $\chi|_{\mathfrak{o}_L^\times} = 1$. Note that $W^{\#}(\cdot, s)$ is an element of the induced representation $I(s, \chi, \chi_0, \tau)$ discussed in Section 2.3. In particular, Lemma 2.3.1 applies. Note that if $n = 0$, i.e., if τ is unramified, then $W^{\#}(\cdot, s)$ coincides with the function $W_\nu(\cdot, s)$ defined on p. 200 of [9].

3.5. Basic local integral computation

Let $W^{\#}(\cdot, s)$ be the element of $I(s, \chi, \chi_0, \tau)$ defined in the previous section. Let B be the spherical vector in the $\Lambda \otimes \theta$ Bessel model of the unramified representation π of $\mathrm{GSp}_4(F)$, as in Section 3.2. We shall compute the integral

$$Z(s) = \int\limits_{R(F)\backslash H(F)} W^{\#}(\eta h, s) B(h)\, dh. \tag{45}$$

By Lemma 2.3.1, the integral (45) is well-defined. By (29) and the fact that $B(h(l, m)) = 0$ for $l < 0$ [9, Lemma 3.4.4], we have

$$\begin{aligned} Z(s) &= \sum_{l,m\geqslant 0} \int\limits_{R(F)\backslash R(F)h(l,m)K^H} W^{\#}(\eta h, s) B(h)\, dh \\ &= \sum_{l,m\geqslant 0} \int\limits_{h(l,m)^{-1}R(F)h(l,m)\cap K^H\backslash K^H} W^{\#}\big(\eta h(l,m)h, s\big) B\big(h(l,m)h\big)\, dh \\ &= \sum_{l,m\geqslant 0} B\big(h(l,m)\big) \int\limits_{h(l,m)^{-1}R(F)h(l,m)\cap K^H\backslash K^H} W^{\#}\big(\eta h(l,m)h, s\big)\, dh. \end{aligned} \tag{46}$$

The function $W^{\#}$ is only invariant under $K^{\#}(\mathfrak{P}^n)$. Since our integral (46) is over elements of $H(F)$, all that is relevant is that $W^{\#}$ is invariant under the group $K^{\#}(\mathfrak{p}^n)$ defined in (37). Let us abbreviate $K_{l,m} := h(l,m)^{-1}R(F)h(l,m) \cap K^H$. Suppose we had a system of representatives $\{s_i\}$ for the double coset space $K_{l,m} \setminus K^H/K^{\#}(\mathfrak{p}^n)$ (it will depend on l and m, of course). Then, from (46),

$$\begin{aligned} Z(s) &= \sum_{l,m\geqslant 0} \sum_i B\big(h(l,m)\big) \int\limits_{K_{l,m}\backslash K_{l,m}s_iK^{\#}(\mathfrak{p}^n)} W^{\#}\big(\eta h(l,m)h, s\big)\, dh \\ &= \sum_{l,m\geqslant 0} \sum_i B\big(h(l,m)\big)\, W^{\#}\big(\eta h(l,m)s_i, s\big) \int\limits_{K_{l,m}\backslash K_{l,m}s_iK^{\#}(\mathfrak{p}^n)} dh. \end{aligned} \tag{47}$$

In practice it will be difficult to obtain the system $\{s_i\}$. However, we can save some work by exploiting the fact that $W^{\#}$ is supported on the small subset $M(F)N(F)K^{\#}(\mathfrak{P}^n)$ of $G(F)$. Hence, we shall proceed as follows.

Step 1. First we determine a *preliminary* decomposition

$$K^H = \bigcup_j K_{l,m}s'_j K^{\#}\big(\mathfrak{p}^n\big), \tag{48}$$

which is not necessarily disjoint. We may assume that the s'_j are taken from the system of representatives for $K^H/K^{\#}(\mathfrak{p}^n)$ to be determined in the next section (but some of these will be absorbed in $K_{l,m}$, so that we get an initial reduction).

Step 2. Then we consider the values $W^{\#}(\eta h(l,m)s'_j, s)$. If $\eta h(l,m)s'_j \notin M(F)N(F)K^{\#}(\mathfrak{P}^n)$, then s'_j makes no contribution to the integral (46). Therefore, all that is relevant is the subset $\{s''_j\} \subset \{s'_j\}$ of representatives for which $\eta h(l,m)s''_j \in M(F)N(F)K^{\#}(\mathfrak{P}^n)$. Hence we consider the set

$$S := \bigcup_j K_{l,m}s''_j K^{\#}\big(\mathfrak{p}^n\big).$$

Step 3. Now, from this much smaller set of representatives $\{s''_j\}$ we determine a subset $\{s'''_j\}$ such that this union becomes disjoint:

$$S = \bigsqcup_j K_{l,m}s'''_j K^{\#}\big(\mathfrak{p}^n\big).$$

The integral (46) is then given by

$$Z(s) = \sum_{l,m\geqslant 0} \sum_j B\big(h(l,m)\big)\, W^{\#}\big(\eta h(l,m)s'''_j, s\big) \int\limits_{K_{l,m}\backslash K_{l,m}s'''_jK^{\#}(\mathfrak{p}^n)} dh. \tag{49}$$

Finally, we have to compute the volumes, evaluate $W^{\#}$, and carry out the summations with the help of Sugano's formula (30).

3.6. Double coset decomposition

3.6.1. The cosets $K^{\#}(\mathfrak{p}^0)/K^{\#}(\mathfrak{p}^n)$

We need to determine representatives for the coset space

$$K^{\#}(\mathfrak{p}^0)/K^{\#}(\mathfrak{p}^n), \quad \text{where } K^{\#}(\mathfrak{p}^0) = K^H = \mathrm{GSp}_4(\mathfrak{o}). \tag{50}$$

Note that this coset space is isomorphic to

$$K_1^{\#}(\mathfrak{p}^0)\big/K_1^{\#}(\mathfrak{p}^n), \quad \text{where } K_1^{\#}(\mathfrak{p}^n) = K^{\#}(\mathfrak{p}^n) \cap \big\{g \in H(F)\colon\ \mu(g) = 1\big\}. \tag{51}$$

Let

$$s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \qquad s_2 = \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}. \tag{52}$$

It follows from the Bruhat decomposition for $\mathrm{Sp}(4, \mathfrak{o}/\mathfrak{p})$ that

$$\begin{aligned} K^{\#}(\mathfrak{p}^0) = K^{\#}(\mathfrak{p}^1) \sqcup \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} s_1 K^{\#}(\mathfrak{p}^1) \sqcup \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & x & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 K^{\#}(\mathfrak{p}^1) && (53)\\ \sqcup \bigsqcup_{x,y \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & & \\ x & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix} s_1 s_2 K^{\#}(\mathfrak{p}^1) \sqcup \bigsqcup_{x,y \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & x & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 K^{\#}(\mathfrak{p}^1) && (54)\\ \sqcup \bigsqcup_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & & y \\ x & 1 & y & xy+z \\ & & 1 & -x \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 K^{\#}(\mathfrak{p}^1) \sqcup \bigsqcup_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 s_2 K^{\#}(\mathfrak{p}^1) && (55)\\ \sqcup \bigsqcup_{w,x,y,z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & x & y \\ w & 1 & wx+y & wy+z \\ & & 1 & -w \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 s_2 K^{\#}(\mathfrak{p}^1). && (56) \end{aligned}$$

Let $n \geqslant 1$. It is easy to see that

$$K^{\#}(\mathfrak{p}^1) = \bigsqcup_{w,y,z \in \mathfrak{o}/\mathfrak{p}^{n-1}} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y\varpi & 1 & \\ y\varpi & z\varpi & & 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n). \tag{57}$$

Let $\{r_i\}$ be the system of representatives for $K^{\#}(\mathfrak{p}^0)/K^{\#}(\mathfrak{p}^1)$ determined in (53)–(56). Combining these with (57) we get

$$K^H = \bigsqcup_i \bigsqcup_{w,y,z \in \mathfrak{o}/\mathfrak{p}^{n-1}} r_i \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y\varpi & 1 & \\ y\varpi & z\varpi & & 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n). \tag{58}$$

Recall that we are interested in the double cosets $K_{l,m} \setminus K^H/K^{\#}(\mathfrak{p}^n)$, where $K_{l,m} = h(l,m)^{-1}R(F)h(l,m) \cap K^H$.

3.6.2. Step 1: Preliminary decomposition

Observe that $K_{l,m}$ contains all elements $\begin{bmatrix} 1 & & \mathfrak{o} & \mathfrak{o} \\ & 1 & \mathfrak{o} & \mathfrak{o} \\ & & 1 & \\ & & & 1 \end{bmatrix}$. From (58) we therefore get the following preliminary decomposition, which is not disjoint:

$$K^H = \bigcup_{y,z,w\in\mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y\varpi & 1 & \\ y\varpi & z\varpi & & 1 \end{bmatrix} K^{\#}\big(\mathfrak{p}^n\big) \tag{59}$$

$$\cup \bigcup_{w\in\mathfrak{o}/\mathfrak{p}^n} \bigcup_{y,z\in\mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ z\varpi & y\varpi & 1 & \\ y\varpi & & & 1 \end{bmatrix} s_1 K^{\#}\big(\mathfrak{p}^n\big) \tag{60}$$

$$\cup \bigcup_{w,y,z\in\mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & w\varpi & 1 & \\ w\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y\varpi & & \\ & 1 & & \\ & & 1 & \\ & z\varpi & -y\varpi & 1 \end{bmatrix} s_2 K^{\#}\big(\mathfrak{p}^n\big) \tag{61}$$

$$\cup \bigcup_{w\in\mathfrak{o}/\mathfrak{p}^n} \bigcup_{y,z\in\mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ y\varpi & z\varpi & 1 & \\ z\varpi & & & 1 \end{bmatrix} s_1 s_2 K^{\#}\big(\mathfrak{p}^n\big) \tag{62}$$

$$\cup \bigcup_{w\in\mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} s_2 s_1 K^{\#}\big(\mathfrak{p}^n\big) \tag{63}$$

$$\cup \bigcup_{w\in\mathfrak{o}/\mathfrak{p}^n} K_{l,m} \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 K^{\#}\big(\mathfrak{p}^n\big) \tag{64}$$

$$\cup \bigcup_{w\in\mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} s_2 s_1 s_2 K^{\#}\big(\mathfrak{p}^n\big) \tag{65}$$

$$\cup \bigcup_{w\in\mathfrak{o}/\mathfrak{p}^n} K_{l,m} \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 s_2 K^{\#}\big(\mathfrak{p}^n\big). \tag{66}$$

3.6.3. Step 2: Support of $W^{\#}$

We assumed that $c \in \mathfrak{o}^\times$, so that $\alpha \in \mathfrak{o}_L$. We have $\eta h(l,m) = h(l,m)\eta_m$, where for $m \geqslant 0$ we let

$$\eta_m = \begin{bmatrix} 1 & & & \\ \alpha\varpi^m & 1 & & \\ & & 1 & -\bar{\alpha}\varpi^m \\ & & & 1 \end{bmatrix}. \tag{67}$$

Fix $l, m \geqslant 0$, and let r run through the representatives for $K_{l,m} \setminus K^H/K^{\#}(\mathfrak{p}^n)$ from (59)–(66). In view of (49) we want to find out for which r is $\eta h(l,m)r \in M(F)N(F)K^{\#}(\mathfrak{P}^n)$, since this set is the support of $W^\#$. Since $h(l,m) \in M(F)$, this is equivalent to $\eta_m r \in M(F)N(F)K^\#(\mathfrak{P}^n)$. Hence, this condition depends only on $m \geqslant 0$ and not on the integer l.

(i) Let $r = \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y\varpi & 1 & \\ y\varpi & z\varpi & & 1 \end{bmatrix}$ with $w, y, z \in \mathfrak{o}/\mathfrak{p}^{n-1}$. Suppose $\eta_m r = \tilde{m}\tilde{n}k$ with $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$ and $k \in K^\#(\mathfrak{P}^n)$. Let $A = (\tilde{m}\tilde{n})^{-1}\eta_m r$. Looking at the $(3,2)$ and $(3,3)$ coefficient of A we get

$$y + \varpi^{m+1}\alpha wy - \varpi^m \alpha z \in \mathfrak{P}^{n-1}, \quad \text{and hence} \quad \alpha\varpi^m(\varpi wy - z) + y \in \mathfrak{P}^{n-1}.$$

If $\nu(\varpi wy - z) < n-m-1$ then $\alpha + y/(\varpi^m(\varpi wy - z)) \in \mathfrak{P}$, which contradicts Lemma 3.1.1(ii). Hence, $\nu(\varpi wy - z) \geqslant n-m-1$, which implies $\varpi^m(\varpi wy - z) \in \mathfrak{p}^{n-1}$. It follows that $y \in \mathfrak{p}^{n-1}$. To summarize, necessary conditions for $A \in K^\#(\mathfrak{P}^n)$ are $y = 0$ and $z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$. The following matrix identity shows that these are also sufficient conditions:

$$\eta_m r = \begin{bmatrix} a^{-1} & & & \\ & a & & \\ & & \bar{a} & \\ & \varpi z\bar{a}^{-1} & & \bar{a}^{-1} \end{bmatrix} \begin{bmatrix} 1 & \varpi wa & & \\ & 1 & & \\ & & 1 & \\ & & -\varpi w\bar{a} & 1 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & & & \\ \varpi^m\alpha a^{-1} & 1 & & \\ & -\varpi^{m+1}\bar{\alpha}z\bar{a}^{-1} & 1 & -\varpi^m\bar{\alpha}\bar{a}^{-1} \\ -\varpi^{m+1}\alpha za^{-1} & & & 1 \end{bmatrix} \in M(F)N(F)K^\#\bigl(\mathfrak{P}^n\bigr), \qquad (68)$$

where $a = 1 + \varpi^{m+1}\alpha w \in \mathfrak{o}_L^\times$. Hence, the values of w, y, z for which $\eta_m r \in M(F)N(F)K^\#(\mathfrak{P}^n)$ are

$$w \in \mathfrak{o}/\mathfrak{p}^{n-1}, \qquad y = 0, \qquad z \in \bigl(\mathfrak{p}^{n-m-1} \cap \mathfrak{o}\bigr)/\mathfrak{p}^{n-1}.$$

(ii) Let $r = \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ z\varpi & y\varpi & 1 & \\ y\varpi & & & 1 \end{bmatrix} s_1$ with $w \in \mathfrak{o}/\mathfrak{p}^n$ and $y, z \in \mathfrak{o}/\mathfrak{p}^{n-1}$. Suppose $\eta_m r = \tilde{m}\tilde{n}k$ with $\tilde{m} \in M(F), \tilde{n} \in N(F)$ and $k \in K^\#(\mathfrak{P}^n)$. Let $A = (\tilde{m}\tilde{n})^{-1}\eta_m r$. Looking at the $(3,2)$ and $(3,3)$ coefficients of A we get

$$\beta := \varpi^m\alpha + w \in \mathfrak{o}_L^\times \quad \text{and} \quad \varpi^m\alpha y + wy - z \in \mathfrak{P}^{n-1}.$$

If $\nu(y) < n-m-1$, then $\alpha + (wy - z)/(\varpi^m y) \in \mathfrak{P}$, which contradicts Lemma 3.1.1(ii). Hence, $\nu(y) \geqslant n-m-1$, which implies $wy - z \in \mathfrak{P}^{n-1}$. We may therefore assume that $z = wy$. To summarize, necessary conditions for $A \in K^\#(\mathfrak{P}^n)$ are $\varpi^m\alpha + w \in \mathfrak{o}_L^\times$, $y \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$ and $z = wy$. The following matrix identity shows that these are also sufficient conditions:

$$\eta_m r = \begin{bmatrix} -\beta^{-1} & & & \\ & \beta & & \\ & & -\bar{\beta} & \\ & \varpi wy\bar{\beta}^{-1} & & \bar{\beta}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\beta & & \\ & 1 & & \\ & & 1 & \\ & & \bar{\beta} & 1 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & & & \\ \beta^{-1} & 1 & & \\ -\varpi y\bar{\beta}^{-1} & \varpi^{m+1}\bar{\alpha}y\bar{\beta}^{-1} & 1 & -\bar{\beta}^{-1} \\ \varpi^{m+1}\alpha y\beta^{-1} & & & 1 \end{bmatrix} \in M(F)N(F)K^\#\bigl(\mathfrak{P}^n\bigr). \qquad (69)$$

Hence, the values of w, y, z for which $\eta_m r \in M(F)N(F)K^\#(\mathfrak{P}^n)$ are as follows.
(a) If $m = 0$, then all $w \in \mathfrak{o}/\mathfrak{p}^n$ such that $\alpha + w \in \mathfrak{o}_L^\times$ and $y = z = 0$.
(b) If $m > 0$, then all $w \in \mathfrak{o}^\times$, $y \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$ and $z = wy$.

(iii) Let $r = \begin{bmatrix} 1&&& \\ &1&& \\ &w\varpi&1& \\ w\varpi&&&1 \end{bmatrix} \begin{bmatrix} 1&y\varpi&& \\ &1&& \\ &&1& \\ &z\varpi&-y\varpi&1 \end{bmatrix} s_2$ with $w, y, z \in \mathfrak{o}/\mathfrak{p}^{n-1}$.
Then $\eta_m r \notin M(F)N(F)K^\#(\mathfrak{P}^n)$, since the $(3,3)$-coefficient divided by the $(3,1)$-coefficient of any matrix product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_m r$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is in $\mathfrak{o}_L$.

(iv) Let $r = \begin{bmatrix} 1&&& \\ w&1&& \\ &&1&-w \\ &&&1 \end{bmatrix} \begin{bmatrix} 1&&& \\ &1&& \\ y\varpi&z\varpi&1& \\ z\varpi&&&1 \end{bmatrix} s_1 s_2$ with $w \in \mathfrak{o}/\mathfrak{p}^n$ and $y, z \in \mathfrak{o}/\mathfrak{p}^{n-1}$.
Then $\eta_m r \notin M(F)N(F)K^\#(\mathfrak{P}^n)$, since the $(3,3)$-coefficient of any product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_m r$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is in $\mathfrak{P}$.

(v) Let $r = \begin{bmatrix} 1&w\varpi&& \\ &1&& \\ &&1& \\ &&-w\varpi&1 \end{bmatrix} s_2 s_1$ with $w \in \mathfrak{o}/\mathfrak{p}^{n-1}$.
Then $\eta_m r \notin M(F)N(F)K^\#(\mathfrak{P}^n)$, since the $(4,1)$-coefficient of any product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_m r$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is in $\mathfrak{o}_L^\times$.

(vi) Let $r = \begin{bmatrix} 1&&& \\ w&1&& \\ &&1&-w \\ &&&1 \end{bmatrix} s_1 s_2 s_1$ with $w \in \mathfrak{o}/\mathfrak{p}^n$. Suppose $\eta_m r = \tilde{m}\tilde{n}k$ with $\tilde{m} \in M(F), \tilde{n} \in N(F)$ and $k \in K^\#(\mathfrak{P}^n)$. Let $A = (\tilde{m}\tilde{n})^{-1}\eta_m r$. Looking at the $(3,2)$ and $(3,3)$ coefficients of A we get $\varpi^m\alpha + w \in \mathfrak{P}^n$. If $m < n$, then we get $\alpha + w/\varpi^m \in \mathfrak{P}$ which contradicts Lemma 3.1.1(ii). Hence $m \geqslant n$, which implies that $w \in \mathfrak{P}^n$. We may therefore assume that $w = 0$. To summarize, necessary conditions for $A \in K^\#(\mathfrak{P}^n)$ are $m \geqslant n$ and $w = 0$. The following matrix identity shows that these are also sufficient conditions:

$$\eta_m r = \begin{bmatrix} 1&&& \\ &&&1 \\ &&1& \\ &-1&& \end{bmatrix} \begin{bmatrix} 1&&& \\ &1&& \\ &\varpi^m\bar{\alpha}&1& \\ \varpi^m\alpha&&&1 \end{bmatrix} \in M(F)N(F)K^\#\bigl(\mathfrak{P}^n\bigr). \tag{70}$$

(vii) Let $r = \begin{bmatrix} 1&w\varpi&& \\ &1&& \\ &&1& \\ &&-w\varpi&1 \end{bmatrix} s_2 s_1 s_2$ with $w \in \mathfrak{o}/\mathfrak{p}^{n-1}$.
Then $\eta_m r \notin M(F)N(F)K^\#(\mathfrak{P}^n)$, since the $(3,3)$-coefficient of any product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_m r$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is zero.

(viii) Let $r = \begin{bmatrix} 1&&& \\ w&1&& \\ &&1&-w \\ &&&1 \end{bmatrix} s_1 s_2 s_1 s_2$ with $w \in \mathfrak{o}/\mathfrak{p}^n$.
Then $\eta_m r \notin M(F)N(F)K^\#(\mathfrak{P}^n)$, since the $(3,3)$-coefficient of any product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_m r$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is zero.

Let us summarize the double cosets that can possibly make a non-trivial contribution to the integral (49).

$$\bigcup_{\substack{w \in \mathfrak{o}/\mathfrak{p}^{n-1} \\ z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}}} K_{l,m} \begin{bmatrix} 1&w\varpi&& \\ &1&& \\ &&1& \\ &&-w\varpi&1 \end{bmatrix} \begin{bmatrix} 1&&& \\ &1&& \\ &&1& \\ &z\varpi&&1 \end{bmatrix} K^\#\bigl(\mathfrak{p}^n\bigr) \quad \text{for } l, m \geqslant 0, \tag{71}$$

$$\bigcup_{\substack{w\in\mathfrak{o}/\mathfrak{p}^n\\ \varpi^m\alpha+w\in\mathfrak{o}_L^\times\\ y\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}}} K_{l,m}\begin{bmatrix}1&&&\\w&1&&\\&&1&-w\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&&1&\\yw\varpi&y\varpi&1&\\y\varpi&&&1\end{bmatrix}s_1K^\#(\mathfrak{p}^n)\quad\text{for } l,m\geqslant 0, \tag{72}$$

$$K_{l,m}s_1s_2s_1K^\#(\mathfrak{p}^n)\quad\text{for } l\geqslant 0,\ m\geqslant n. \tag{73}$$

3.6.4. Step 3: Disjointness of double cosets

We will now investigate the overlap between double cosets in (71), (72) and (73). First we will consider the case $m=0$.

Equivalences among double cosets from (71) with $m=0$

For $w\in\mathfrak{o}/\mathfrak{p}^{n-1}$, set $\beta=c+b(\varpi w)+a(\varpi w)^2\in\mathfrak{o}^\times$. Let $g=\left[\begin{smallmatrix}x+yb/2&yc\\-ya&x-yb/2\end{smallmatrix}\right]$ with $y=\varpi w$ and $x=c+yb/2$. Then we have the matrix identity

$$h(l,0)^{-1}\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}h(l,0)=\begin{bmatrix}1&w\varpi&&\\&1&&\\&&1&\\&&-w\varpi&1\end{bmatrix}\begin{bmatrix}\beta&&&\\-a\varpi w&c&&\\&&c&a\varpi w\\&&&\beta\end{bmatrix}.$$

The rightmost matrix above is in $K^\#(\mathfrak{p}^n)$, so that

$$\bigcup_{w\in\mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,0}\begin{bmatrix}1&w\varpi&&\\&1&&\\&&1&\\&&-w\varpi&1\end{bmatrix}K^\#(\mathfrak{p}^n)=K_{l,0}K^\#(\mathfrak{p}^n)\quad\text{for all } l\geqslant 0. \tag{74}$$

Equivalences among double cosets from (74) and (72) with $m=0$

Let $w\in\mathfrak{o}/\mathfrak{p}^n$ be such that $\alpha+w\in\mathfrak{o}_L^\times$. Set $\beta=a+bw+cw^2$. Let $g=\left[\begin{smallmatrix}x+yb/2&yc\\-ya&x-yb/2\end{smallmatrix}\right]$ with $y=1$ and $x=-(cw+b/2)$. Then we have the matrix identity

$$h(l,0)^{-1}\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}h(l,0)\begin{bmatrix}1&&&\\w&1&&\\&&1&-w\\&&&1\end{bmatrix}s_1=\begin{bmatrix}c&&&\\-(b+cw)&-\beta&&\\&&\beta&-(b+cw)\\&&&-c\end{bmatrix}.$$

The matrix on the right-hand side is in $K^\#(\mathfrak{p}^n)$ if $\beta\in\mathfrak{o}^\times$. We will now show that the condition $\alpha+w\in\mathfrak{o}_L^\times$ forces $\beta\in\mathfrak{o}^\times$. First observe the identity

$$a+bw+cw^2=-c(\alpha+w)\big(\alpha-\big(w+bc^{-1}\big)\big).$$

If $\beta\in\mathfrak{p}$, then it would follow that $\alpha-(w+bc^{-1})\in\mathfrak{p}\mathfrak{o}_L=\mathfrak{P}$. By Lemma 3.1.1(ii), this is impossible. It follows that indeed $\beta\in\mathfrak{o}^\times$, so that *all* double cosets in (72) with $m=0$ are equivalent to the double coset in (74).

Equivalence among double cosets from (71) or (72) and (73) with $m>0$

Let h_1 be a double coset representative from either (71) or (72), and let h_2 be a double coset representative from (73). Then, in either case, the double cosets are not equivalent, since, for any $r\in R(F)$ the $(2,2)$ coordinate of the matrix $h_2^{-1}h(l,m)^{-1}rh(l,m)h_1$ is in $\mathfrak{p}$.

Equivalence among double cosets from (71) and (72) with $m>0$

For $m>0$ the condition $\varpi^m\alpha+w\in\mathfrak{o}_L^\times$ in (72) is equivalent to $w\in\mathfrak{o}^\times$. Hence let $w\in\mathfrak{o}^\times$ and $z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}$. Let $\beta_1=a\varpi^{2m}+b\varpi^m+c$, $\beta_2=a\varpi^{2m}+b\varpi^m+cw$ and $\beta_3=a\varpi^{2m}+bw\varpi^m+cw^2$. We have $\beta_1,\beta_2,\beta_3\in\mathfrak{o}^\times$. Let $g=\left[\begin{smallmatrix}x+yb/2&yc\\-ya&x-yb/2\end{smallmatrix}\right]$ with $y=\varpi^m(1-w)/\beta_3$ and $x=\beta_2/\beta_3-by/2$. Then we have the matrix identity

$$h(l,m)^{-1}\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}h(l,m)\begin{bmatrix}1&&&\\1&1&&\\&&1&-1\\&&&1\end{bmatrix}\begin{bmatrix}&1&&\\&&1&\\z\varpi/w&z\varpi/w&1&\\z\varpi/w&&&1\end{bmatrix}s_1$$
$$=\begin{bmatrix}1&&&\\w&1&&\\&&1&-w\\&&&1\end{bmatrix}\begin{bmatrix}&1&&\\&&1&\\zw\varpi&z\varpi&1&\\z\varpi&&&1\end{bmatrix}s_1\kappa,$$

where

$$\kappa=\begin{bmatrix}1&0&0&0\\\frac{c(1-w)}{\beta_3}&\frac{\beta_1}{\beta_3}&0&0\\\frac{cz\varpi(w^2-1)}{w\beta_3}&-\frac{\varpi^{m+1}z(w-1)(b+a\varpi^m)}{w\beta_3}&\frac{\beta_1}{\beta_3}&\frac{c(w-1)}{\beta_3}\\-\frac{\varpi^{m+1}z(w-1)(bw+a\varpi^m)}{w\beta_3}&-\frac{\varpi^{m+1}z(w-1)(bw+a\varpi^m(1+w))}{w\beta_3}&0&1\end{bmatrix}\in K^\#(\mathfrak{p}^n).$$

Hence

$$\bigcup_{\substack{w\in\mathfrak{o}/\mathfrak{p}^n\\w\in\mathfrak{o}^\times\\z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}}}K_{l,m}\begin{bmatrix}1&&&\\w&1&&\\&&1&-w\\&&&1\end{bmatrix}\begin{bmatrix}&1&&\\&&1&\\zw\varpi&z\varpi&1&\\z\varpi&&&1\end{bmatrix}s_1K^\#(\mathfrak{p}^n)$$
$$=\bigcup_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}}K_{l,m}\begin{bmatrix}1&&&\\1&1&&\\&&1&-1\\&&&1\end{bmatrix}\begin{bmatrix}&1&&\\&&1&\\z\varpi&z\varpi&1&\\z\varpi&&&1\end{bmatrix}s_1K^\#(\mathfrak{p}^n).\tag{75}$$

Now let $w\in\mathfrak{o}/\mathfrak{p}^{n-1}$ and $z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}$. Set $\beta=c+(\varpi^{m+1}w)b+(\varpi^{m+1}w)^2a\in\mathfrak{o}^\times$. Let $g_1=\left[\begin{smallmatrix}x_1+y_1b/2&y_1c\\-y_1a&x_1-y_1b/2\end{smallmatrix}\right]$ with $y_1=\varpi^{m+1}w/\beta$ and $x_1=1-by_1/2-a\varpi^{m+1}wy_1$. Then we have the matrix identity

$$h(l,m)^{-1}\begin{bmatrix}g_1&\\&\det(g_1)\,{}^t(g_1)^{-1}\end{bmatrix}h(l,m)\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z\varpi&&1\end{bmatrix}$$
$$=\begin{bmatrix}1&w\varpi&&\\&1&&\\&&1&\\&&-w\varpi&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z\varpi&&1\end{bmatrix}\kappa_1,$$

where

$$\kappa_1=\begin{bmatrix}1&&&\\-a\varpi^{2m+1}w/\beta&c/\beta&&\\&a\varpi^{2+2m}wz/\beta&c/\beta&a\varpi^{2m+1}w/\beta\\a\varpi^{2+2m}wz/\beta&\varpi^{2+m}w(b+a\varpi^{m+1}w)z/\beta&&1\end{bmatrix}\in K^{\#}(\mathfrak{p}^n).$$

Hence

$$\bigcup_{\substack{w\in\mathfrak{o}/\mathfrak{p}^{n-1}\\ z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}}}K_{l,m}\begin{bmatrix}1&w\varpi&&\\&1&&\\&&1&\\&&-w\varpi&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z\varpi&&1\end{bmatrix}K^{\#}(\mathfrak{p}^n)$$
$$=\bigcup_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}}K_{l,m}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z\varpi&&1\end{bmatrix}K^{\#}(\mathfrak{p}^n). \tag{76}$$

We will now show that the double cosets in (75) are all equivalent to double cosets in (76). Given $z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}$, let $g_2=\left[\begin{smallmatrix}x_2+y_2b/2&y_2c\\-y_2a&x_2-y_2b/2\end{smallmatrix}\right]$ with $y_2=\varpi^m$ and $x_2=-(c+by_2/2)$. Then we have the matrix identity

$$h(l,m)^{-1}\begin{bmatrix}g_1&\\&\det(g_1)\,{}^t(g_1)^{-1}\end{bmatrix}h(l,m)\begin{bmatrix}1&&&\\1&1&&\\&&1&-1\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\z\varpi&z\varpi&1&\\z\varpi&&&1\end{bmatrix}s_1$$
$$=\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z\varpi&&1\end{bmatrix}\kappa_2,$$

where

$$\kappa_2=\begin{bmatrix}c&&&\\-c-b\varpi^m&-c-b\varpi^m-a\varpi^{2m}&&\\-\varpi(c+b\varpi^m)z&a\varpi^{1+2m}z&c+b\varpi^m+a\varpi^{2m}&-c-b\varpi^m\\b\varpi^{m+1}z&\varpi^{m+1}(b+a\varpi^m)z&0&-c\end{bmatrix}\in K^{\#}(\mathfrak{p}^n).$$

We conclude that, for $m>0$ and any $l\geqslant 0$, the double cosets in (71) and (72) are all contained in the union

$$\bigcup_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}}K_{l,m}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z\varpi&&1\end{bmatrix}K^{\#}(\mathfrak{p}^n). \tag{77}$$

Equivalence among double cosets from (77) with $m>0$

Finally, we have to determine any equivalences amongst the double cosets in (77). Fix $l\geqslant 0$ and $m>0$, and let

$$h_1=\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z_1\varpi&&1\end{bmatrix},\qquad h_2=\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z_2\varpi&&1\end{bmatrix}$$

with $z_1, z_2 \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$. We want to see if we can find $r = \left[\begin{smallmatrix} g & gX \\ & \det(g)\,{}^tg^{-1} \end{smallmatrix}\right] \in R(F)$ such that

$$A = h_1^{-1} h(l,m)^{-1} r h(l,m) h_2 \in K^\#\big(\mathfrak{p}^n\big);$$

here, $g = \left[\begin{smallmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{smallmatrix}\right] \in T(F)$ and $X = \left[\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right]$ with $e, f, g \in F$. Suppose such an r exists. Looking at the $(1,3)$, $(1,4)$, $(2,3)$ and $(2,4)$ coefficient of A we get

$$\begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \in \begin{bmatrix} \mathfrak{p}^{l+2m} & \mathfrak{p}^{l+m} \\ \mathfrak{p}^{l+m} & \mathfrak{p}^{l} \end{bmatrix}.$$

Looking at the $(1,1)$, $(1,2)$, $(1,4)$ and $(3,3)$ coefficient of A, we see that

$$x \pm by/2 \in \mathfrak{o}^\times,\ y \in \mathfrak{p}^m \quad \text{and hence} \quad \begin{bmatrix} e & f \\ f & g \end{bmatrix} \in \begin{bmatrix} \mathfrak{p}^{l+2m} & \mathfrak{p}^{l+m} \\ \mathfrak{p}^{l+m} & \mathfrak{p}^{l} \end{bmatrix}.$$

Looking at the $(4,2)$ coefficient of A, we get

$$(x - by/2)z_1 + \varpi^{1-l}\big(g(x-by/2) - afy\big)z_1 z_2 - (x+by/2)z_2 \in \mathfrak{p}^{n-1}. \tag{78}$$

From this it follows that $\nu(z_1) = \nu(z_2)$. Using $y \in \mathfrak{p}^m$, it further follows that

$$(z_1 - z_2) + \varpi^{-l} g(\varpi z_1 z_2) \in \mathfrak{p}^{n-1} \tag{79}$$

(first add byz_2 to both sides of (78), then divide by the unit $x - by/2$). Let $\nu(z_1) = \nu(z_2) = j$. Write $z_i = \varpi^j u_i$ for $i = 1, 2$, where $u_i \in \mathfrak{o}^\times$. If $2j + 1 \geqslant n - 1$, then (79) implies that $z_1 = z_2$; hence we get disjoint double cosets in this case. If $2j + 1 < n - 1$, then (79) implies that $u_1 - u_2 \in \mathfrak{p}^{j+1}$. This is a necessary condition for the coincidence of double cosets. We will now show that it is sufficient. So, suppose that $u_1 - u_2 \in \mathfrak{p}^{j+1}$. Set $g = \varpi^l (z_2 - z_1)/(\varpi z_1 z_2) \in \mathfrak{p}^l$ and $e = f = 0$. Then there is a matrix identity

$$h(l,m)^{-1} \begin{bmatrix} I_2 & X \\ & I_2 \end{bmatrix} h(l,m) \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z_2\varpi && 1 \end{bmatrix} = \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z_1\varpi && 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \frac{u_2}{u_1} && \frac{z_2 - z_1}{\varpi z_1 z_2} \\ && 1 & \\ &&& \frac{u_1}{u_2} \end{bmatrix},$$

where the rightmost matrix lies in $K^\#(\mathfrak{p}^n)$. We therefore get the disjoint union

$$\begin{aligned} &\bigcup_{z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} K^\#\big(\mathfrak{p}^n\big) \\ &= \bigsqcup_{z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o} \cap \mathfrak{p}^{[\frac{n-1}{2}]})/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} K^\#\big(\mathfrak{p}^n\big) \\ &\sqcup \bigsqcup_{j=\max(n-m-1,0)}^{[\frac{n-3}{2}]} \bigsqcup_{u \in \mathfrak{o}^\times/(1+\mathfrak{p}^{j+1})} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & u\varpi^{j+1} && 1 \end{bmatrix} K^\#\big(\mathfrak{p}^n\big). \end{aligned}$$

The following proposition summarizes our results in this section.

Proposition 3.6.1. *Let $l, m \geqslant 0$. The following are the disjoint double cosets in $K_{l,m} \setminus K^H/K^{\#}(\mathfrak{p}^n)$ that can possibly make a non-trivial contribution to the integral* (49).

$$\bigsqcup_{z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o} \cap \mathfrak{p}^{[\frac{n-1}{2}]})/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} K^{\#}\big(\mathfrak{p}^n\big) \quad \text{for } l, m \geqslant 0, \tag{80}$$

$$\bigsqcup_{j=\max(n-m-1,0)}^{[\frac{n-3}{2}]} \bigsqcup_{u \in \mathfrak{o}^\times/(1+\mathfrak{p}^{j+1})} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & u\varpi^{j+1} && 1 \end{bmatrix} K^{\#}\big(\mathfrak{p}^n\big) \quad \text{for } l, m \geqslant 0, \tag{81}$$

$$K_{l,m} s_1 s_2 s_1 K^{\#}\big(\mathfrak{p}^n\big) \quad \text{for } l \geqslant 0,\ m \geqslant n. \tag{82}$$

For $n = 1$ this reduces to

$$K_{l,m} K^{\#}(\mathfrak{p}) \quad \text{for } l, m \geqslant 0, \tag{83}$$

$$K_{l,m} s_1 s_2 s_1 K^{\#}(\mathfrak{p}) \quad \text{for } l \geqslant 0,\ m \geqslant 1. \tag{84}$$

3.7. Volume computations

With a view towards the integral (49), we will now compute the volumes of the sets $K_{l,m} \setminus K_{l,m} A K^{\#}(\mathfrak{p}^n)$, where A is one of the representatives of the disjoint double cosets in (80), (81) or (82). Let $\chi_1 : K_{l,m} \setminus K^H \to \mathbb{C}$ be the characteristic function of $K_{l,m} \setminus K_{l,m} A K^{\#}(\mathfrak{p}^n)$, and let $\delta_1 : K^H \to \mathbb{C}$ be the characteristic function of $AK^{\#}(\mathfrak{p}^n)$.

Lemma 3.7.1. *For all $g \in K^H$ we have*

$$\int_{K_{l,m}} \delta_1(tg)\, dt = \chi_1(\dot{g}) \int_{K_{l,m} \cap (AK^{\#}(\mathfrak{p}^n)A^{-1})} dt, \tag{85}$$

where $\dot{g}$ denotes the image of g in $K_{l,m} \setminus K^H$.

Proof. First assume that $g \notin K_{l,m} A K^{\#}(\mathfrak{p}^n)$. Then $tg \notin AK^{\#}(\mathfrak{p}^n)$ for all $t \in K_{l,m}$, and hence the left-hand side is zero. The right-hand side is also zero by definition of χ_1. Thus the equality holds under our assumption. Now assume that $g \in K_{l,m} A K^{\#}(\mathfrak{p}^n)$. In this case $\chi_1(\dot{g}) = 1$. Write $g = kA\kappa$ with $k \in K_{l,m}$ and $\kappa \in K^{\#}(\mathfrak{p}^n)$. We have

$$\begin{aligned} tg \in AK^{\#}\big(\mathfrak{p}^n\big) \quad &\Longleftrightarrow \quad tkA\kappa \in AK^{\#}\big(\mathfrak{p}^n\big) \quad \Longleftrightarrow \quad tkA \in AK^{\#}\big(\mathfrak{p}^n\big) \\ &\Longleftrightarrow \quad tk \in AK^{\#}\big(\mathfrak{p}^n\big)A^{-1} \quad \Longleftrightarrow \quad t \in \big(AK^{\#}\big(\mathfrak{p}^n\big)A^{-1}\big)k^{-1}. \end{aligned}$$

Hence the left-hand side equals

$$\int_{K_{l,m} \cap (AK^{\#}(\mathfrak{p}^n)A^{-1})k^{-1}} dt.$$

But since $k \in K_{l,m}$, this integral equals $\int_{K_{l,m} \cap (AK^{\#}(\mathfrak{p}^n)A^{-1})} dt$. This proves the lemma. $\square$

Integrating both sides of (85) over $K_{l,m} \setminus K^H$, we obtain

$$\int_{K^H} \delta_1(g)\,dg = \Bigl(\int_{K_{l,m}\setminus K_{l,m}AK^\#(\mathfrak{p})} dh\Bigr)\Bigl(\int_{K_{l,m}\cap(AK^\#(\mathfrak{p})A^{-1})} dt\Bigr), \tag{86}$$

so that

$$\int_{K_{l,m}\setminus K_{l,m}AK^\#(\mathfrak{p}^n)} dh = \operatorname{vol}\bigl(K^\#(\mathfrak{p}^n)\bigr)\Bigl(\int_{K_{l,m}\cap(AK^\#(\mathfrak{p}^n)A^{-1})} dt\Bigr)^{-1}. \tag{87}$$

Note that

$$\operatorname{vol}\bigl(K^\#(\mathfrak{p}^n)\bigr) = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)} \tag{88}$$

from (58) and the fact that $\operatorname{vol}(K^H) = 1$. Hence we are reduced to computing

$$V(l,m,A) := \int_{K_{l,m}\cap(AK^\#(\mathfrak{p}^n)A^{-1})} dt. \tag{89}$$

3.7.1. Volume corresponding to double cosets (80) and (81)

In this case $A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & z\varpi & & 1 \end{bmatrix}$ for $z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$. We need to calculate the volume of the set $h(l,m)^{-1}R(F)h(l,m) \cap AK^\#(\mathfrak{p}^n)A^{-1}$. Let $\nu(z) = j$ with $j \leqslant n-1$. Conjugation of $h(l,m)^{-1}R(F)h(l,m) \cap AK^\#(\mathfrak{p}^n)A^{-1}$ with an element of the form $\operatorname{diag}(1,1,u,u)$, where $u \in \mathfrak{o}^\times$, leaves $R(F)$ and $K^\#(\mathfrak{p}^n)$ unchanged and results in replacing z by uz without any change in the volume. We may therefore assume that $z = \varpi^j$. Since $j \leqslant n-1$, it is clear that

$$AK^\#\bigl(\mathfrak{p}^n\bigr)A^{-1} \subset K^\#\bigl(\mathfrak{p}^{j+1}\bigr). \tag{90}$$

If we write an element of $R(F)$ as tn with $t = \bigl[\begin{smallmatrix} x+by/2 & yc \\ -ya & x-by/2 \end{smallmatrix}\bigr] \in T(F)$ and $n = \bigl[\begin{smallmatrix} 1_2 & X \\ & 1_2 \end{smallmatrix}\bigr]$, $X = \bigl[\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\bigr]$, then (90) gives the following necessary condition for $h(l,m)^{-1}tnh(l,m) \in AK^\#(\mathfrak{p}^n)A^{-1}$,

$$\begin{bmatrix} x+by/2 & yc\varpi^{-m} \\ -ya\varpi^m & x-by/2 \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{j+1} \\ \mathfrak{o} & \mathfrak{o}^\times \end{bmatrix} \subset \mathrm{GL}_2(\mathfrak{o}) \quad \text{and} \quad X \in \begin{bmatrix} \mathfrak{p}^{2m+l} & \mathfrak{p}^{m+l} \\ \mathfrak{p}^{m+l} & \mathfrak{p}^l \end{bmatrix}. \tag{91}$$

Set $B = A^{-1}h(l,m)^{-1}tnh(l,m)A$. We want to find further necessary conditions for $B \in K^\#(\mathfrak{p}^n)$. Looking at the $(4,2)$ coefficient of B, we get

$$\varpi^{-l}g(x+by/2)\varpi^{2+2j} \in \mathfrak{p}^n, \quad \text{and hence} \quad g \in \mathfrak{p}^{n-2-2j+l}. \tag{92}$$

Using the $(4,3)$ coefficient of B, we get

$$\varpi^l cy + \varpi^{j+1} f(x \pm by/2) \in \mathfrak{p}^{n+m+l}. \tag{93}$$

A direct computation shows that the conditions (91), (92) and (93) are also sufficient to conclude that $B \in K^\#(\mathfrak{p}^n)$. Note that $\varpi^l cy + \varpi^{j+1} f(x+by/2) \in \mathfrak{p}^{n+m+l}$ and $y \in \mathfrak{p}^{m+j+1}$ implies that $f \in \mathfrak{p}^{m+l}$ and $\varpi^l cy + \varpi^{j+1} f(x-by/2) \in \mathfrak{p}^{n+m+l}$. To summarize, the following are the necessary and sufficient conditions on t and n for $h(l,m)^{-1}tnh(l,m) \in AK^\#(\mathfrak{p}^n)A^{-1}$.

$$
\begin{aligned}
& y \in \mathfrak{p}^{m+j+1}, \qquad x \pm by/2 \in \mathfrak{o}^\times, \\
e \in \mathfrak{p}^{2m+l}, \qquad & g \in \mathfrak{p}^{n-2-2j+l} \cap \mathfrak{p}^l, \qquad \varpi^l cy + \varpi^{j+1} f(x+by/2) \in \mathfrak{p}^{n+m+l}.
\end{aligned} \tag{94}
$$

For fixed values of x, y satisfying the first two conditions, we are interested in

$$
\begin{aligned}
V_X &:= \mathrm{vol}\big(\big\{(e,f,g) \in F^3\colon\ e \in \mathfrak{p}^{2m+l},\ g \in \mathfrak{p}^{n-2-2j+l} \cap \mathfrak{p}^l,\ \varpi^l cy + \varpi^{j+1} f(x+by/2) \in \mathfrak{p}^{n+m+l}\big\}\big) \\
&= \mathrm{vol}\big(\big\{(e,f,g) \in F^3\colon\ e \in \mathfrak{p}^{2m+l},\ g \in \mathfrak{p}^{n-2-2j+l} \cap \mathfrak{p}^l,\ f \in \mathfrak{p}^{n+m+l-j-1} - \varpi^{l-j-1} cy(x+by/2)^{-1}\big\}\big) \\
&= \mathrm{vol}\big(\big\{(e,f,g) \in F^3\colon\ e \in \mathfrak{p}^{2m+l},\ g \in \mathfrak{p}^{n-2-2j+l} \cap \mathfrak{p}^l,\ f \in \mathfrak{p}^{n+m+l-j-1}\big\}\big).
\end{aligned}
$$

Note that if $j \leqslant [\frac{n-3}{2}]$, then $n-2-2j \geqslant 0$, and if $j \geqslant [\frac{n-1}{2}]$, then $n-2-2j \leqslant 0$. Hence, the above volume is

$$
V_X = \begin{cases} q^{-2n-3m-3l+3j+3}, & \text{if } j \leqslant [\frac{n-3}{2}]; \\ q^{-n-3m-3l+j+1}, & \text{if } j \geqslant [\frac{n-1}{2}]. \end{cases} \tag{95}
$$

So far we have $V(l,m,A) = V_X^{-1}\,\mathrm{vol}(T_{m,j})^{-1}$, where $T_{m,j} := T(F) \cap \left[\begin{smallmatrix} \varpi^{m+j} & \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o}^\times \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{-m-j} & \\ & 1 \end{smallmatrix}\right]$.

Lemma 3.7.2. *For any $m \geqslant 0$ and any j we have*

$$
\mathrm{vol}(T_{m,j})^{-1} = \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^{m+j+1}.
$$

Proof. Note that the group $T(F) \cap \left[\begin{smallmatrix} \varpi^{m+j} & \\ & 1 \end{smallmatrix}\right] \mathrm{GL}_2(\mathfrak{o}) \left[\begin{smallmatrix} \varpi^{-m-j} & \\ & 1 \end{smallmatrix}\right]$ lies in $\mathfrak{o}_L^\times$, since the determinants of these matrices lie in $\mathfrak{o}^\times$ and the trace lies in $\mathfrak{o}$. As in [9, p. 202], we define a subring $\mathfrak{o}_{m+j}$ of $\mathfrak{o}_L$ by

$$
\mathfrak{o}_{m+j} := \mathfrak{o}_L \cap \begin{bmatrix} \varpi^{m+j} & \\ & 1 \end{bmatrix} M_2(\mathfrak{o}) \begin{bmatrix} \varpi^{-m-j} & \\ & 1 \end{bmatrix}.
$$

In addition, we define a smaller subring

$$
\mathfrak{o}'_{m+j} := \mathfrak{o}_L \cap \begin{bmatrix} \varpi^{m+j} & \\ & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} \end{bmatrix} \begin{bmatrix} \varpi^{-m-j} & \\ & 1 \end{bmatrix}.
$$

We normalize the measure so that $\mathrm{vol}(\mathfrak{o}_L^\times) = 1$. Hence, we have

$$
\Bigg(\int\limits_{T(F) \cap \left[\begin{smallmatrix} \varpi^{m+j} & \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o}^\times \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{-m-j} & \\ & 1 \end{smallmatrix}\right]} dt \Bigg)^{-1} = \big(\mathfrak{o}_L^\times : \big(\mathfrak{o}'_{m+j}\big)^\times\big).
$$

From (26) we have the integral basis $\mathfrak{o}_L = \mathfrak{o} + \mathfrak{o}\xi_0 = \{\left[\begin{smallmatrix} x & yc \\ -ya & x-yb \end{smallmatrix}\right] x, y \in \mathfrak{o}\}$, where $\xi_0 = \left[\begin{smallmatrix} 0 & c \\ -a & -b \end{smallmatrix}\right]$. Such an element lies in $\left[\begin{smallmatrix} \varpi^{m+j} & \\ & 1 \end{smallmatrix}\right] M_2(\mathfrak{o}) \left[\begin{smallmatrix} \varpi^{-m-j} & \\ & 1 \end{smallmatrix}\right]$ if and only if $y \in \mathfrak{p}^{m+j}$. Similarly, such an element lies in $\left[\begin{smallmatrix} \varpi^{m+j} & \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{-m-j} & \\ & 1 \end{smallmatrix}\right]$ if and only if $y \in \mathfrak{p}^{m+j+1}$. Therefore,

$$
\mathfrak{o}_{m+j} = \big\{x + \varpi^{m+j} y \xi_0\colon\ x, y \in \mathfrak{o}\big\}
$$

and

$$\mathfrak{o}'_m = \{x + \varpi^{m+j+1} y\xi_0\colon\ x, y \in \mathfrak{o}\}.$$

Hence $\mathfrak{o}'_{m+j} = \mathfrak{o}_{m+j+1}$, so that $(\mathfrak{o}_L^\times : (\mathfrak{o}'_{m+j})^\times) = (\mathfrak{o}_L^\times : (\mathfrak{o}_{m+j+1})^\times)$. By Lemma 3.5.3 of [9],

$$\left(\mathfrak{o}_L^\times : \left(\mathfrak{o}'_{m+j}\right)^\times\right) = \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^{m+j+1}.$$

This concludes the proof. □

3.7.2. Volume corresponding to double coset (82)

In this case, we have $A = s_1 s_2 s_1$ and $m \geqslant n$. Note that

$$V(l, m, s_1 s_2 s_1) = \int\limits_{(h(l,m)^{-1} R(F) h(l,m)) \cap (s_1 s_2 s_1 K^{\#}(\mathfrak{p}^n)(s_1 s_2 s_1)^{-1})} dt.$$

We have

$$s_1 s_2 s_1 K^{\#}\left(\mathfrak{p}^n\right)(s_1 s_2 s_1)^{-1} = K^H \cap \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathfrak{o}^\times & \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o}^\times & \mathfrak{p}^n \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o}^\times \end{bmatrix}. \tag{96}$$

We have to find the intersection of this compact group with $h(l, m)^{-1} R(F) h(l, m)$. Set

$$L_1 := \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o}^\times \end{bmatrix} \subset \mathrm{GL}_2(\mathfrak{o}), \qquad N_1 := \left\{ X \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathfrak{p}^n \end{bmatrix}\colon\ {}^t X = X \right\} \subset F^3.$$

Then L_1 and N_1 are the upper left and upper right blocks of (96), respectively. Write a given element of $R(F)$ as tn with $t \in T(F)$ and $n \in U(F)$. If $n = \left[\begin{smallmatrix} 1_2 & X \\ & 1_2 \end{smallmatrix}\right]$, then a direct computation shows that tn lies in $s_1 s_2 s_1 K^{\#}(\mathfrak{p}^n)(s_1 s_2 s_1)^{-1}$ if and only if

$$\begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} t \begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} \in L_1 \tag{97}$$

and

$$\begin{bmatrix} \varpi^{-2m-l} & \\ & \varpi^{-m-l} \end{bmatrix} X \begin{bmatrix} 1 & \\ & \varpi^m \end{bmatrix} \in N_1. \tag{98}$$

It follows that

$$\begin{aligned} &\mathrm{vol}\left(\left\{X \in F^3\colon \begin{bmatrix} \varpi^{-2m-l} & \\ & \varpi^{-m-l} \end{bmatrix} X \begin{bmatrix} 1 & \\ & \varpi^m \end{bmatrix} \in N_1\right\}\right) \\ &\quad = \mathrm{vol}\left(\left\{X \in F^3\colon\ X \in \begin{bmatrix} \varpi^{2m+l} & \\ & \varpi^{m+l} \end{bmatrix} N_1 \begin{bmatrix} 1 & \\ & \varpi^{-m} \end{bmatrix}\right\}\right) \\ &\quad = q^{-3m-3l} \mathrm{vol}(N_1) = q^{-3m-3l-2n}. \end{aligned}$$

Let

$$\hat{T}_m = \left\{ t \in T(F)\colon \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} t \begin{bmatrix} \varpi^{m} & \\ & 1 \end{bmatrix} \in L_1 \right\} = T(F) \cap \begin{bmatrix} \varpi^{m} & \\ & 1 \end{bmatrix} L_1 \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix}. \tag{99}$$

So far, we have $V(l, m, s_1 s_2 s_1)^{-1} = q^{3m+3l+2n} \operatorname{vol}(\hat{T}_m)^{-1}$.

Lemma 3.7.3. *For any $m \geqslant n$ we have*

$$\operatorname{vol}(\hat{T}_m)^{-1} = \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^m.$$

Proof. Let $\mathfrak{o}_m$ be the subring of $\mathfrak{o}_L$ as defined in the proof of Lemma 3.7.2. In addition, we define another subring

$$\mathfrak{o}''_m := \mathfrak{o}_L \cap \begin{bmatrix} \varpi^{m} & \\ & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{bmatrix} \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix}.$$

Since $\operatorname{vol}(\mathfrak{o}_L^\times) = 1$, we have

$$\left(\int\limits_{T(F) \cap \left[\begin{smallmatrix} \varpi^{m} & \\ & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o}^\times \end{smallmatrix}\right] \left[\begin{smallmatrix} \varpi^{-m} & \\ & 1 \end{smallmatrix}\right]} dt \right)^{-1} = \left(\mathfrak{o}_L^\times : \left(\mathfrak{o}''_m\right)^\times\right).$$

As above we have the integral basis $\mathfrak{o}_L = \mathfrak{o} + \mathfrak{o}\xi_0 = \{\left[\begin{smallmatrix} x & yc \\ -ya & x-yb \end{smallmatrix}\right] x, y \in \mathfrak{o}\}$. Such an element lies in $\left[\begin{smallmatrix} \varpi^{m} & \\ & 1 \end{smallmatrix}\right] M_2(\mathfrak{o}) \left[\begin{smallmatrix} \varpi^{-m} & \\ & 1 \end{smallmatrix}\right]$ if and only if $y \in \mathfrak{p}^m$. Similarly, such an element lies in $\left[\begin{smallmatrix} \varpi^{m} & \\ & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{smallmatrix}\right] \left[\begin{smallmatrix} \varpi^{-m} & \\ & 1 \end{smallmatrix}\right]$ if and only if $y \in \mathfrak{p}^m$, since $m \geqslant n$. Therefore,

$$\mathfrak{o}_m = \left\{ x + \varpi_F^m y \xi_0\colon\ x, y \in \mathfrak{o} \right\}$$

and

$$\mathfrak{o}''_m = \left\{ x + \varpi_F^m y \xi_0\colon\ x, y \in \mathfrak{o} \right\},$$

so that actually $\mathfrak{o}_m = \mathfrak{o}''_m$. Hence $(\mathfrak{o}_L^\times : (\mathfrak{o}''_m)^\times) = (\mathfrak{o}_L^\times : (\mathfrak{o}_m)^\times)$. By Lemma 3.5.3 of [9],

$$\left(\mathfrak{o}_L^\times : \left(\mathfrak{o}''_m\right)^\times\right) = \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^m.$$

This concludes the proof. $\square$

The following proposition summarizes the volume computations in this section.

Proposition 3.7.4.

(i) *Let* $m \geqslant 0$. *Let* $A = \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix}$ *for* $z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$ *and set* $\nu(z) = j$. *If* $j \leqslant [\frac{n-3}{2}]$, *then*

$$V_j^{l,m} := \int\limits_{K_{l,m} \backslash K_{l,m} A K^{\#}(\mathfrak{p}^n)} dh = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)} \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^{2n+4m+3l-2j-2}, \quad (100)$$

and if $j \geqslant [\frac{n-1}{2}]$, *then*

$$V^{l,m} := \int\limits_{K_{l,m} \backslash K_{l,m} A K^{\#}(\mathfrak{p}^n)} dh = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)} \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^{n+4m+3l}. \quad (101)$$

(ii) *For all* $m \geqslant n$,

$$V_{s_1 s_2 s_1}^{l,m} := \int\limits_{K_{l,m} \backslash K_{l,m} s_1 s_2 s_1 K^{\#}(\mathfrak{p}^n)} dh = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)} \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^{4m+3l+2n}. \quad (102)$$

(iii) *In particular, for* $n = 1$,

$$\int\limits_{K_{l,m} \backslash K_{l,m} K^{\#}(\mathfrak{p})} dh = \frac{q-1}{(q+1)(q^4-1)} \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^{4m+3l+1} \quad (m \geqslant 0),$$

$$\int\limits_{K_{l,m} \backslash K_{l,m} s_1 s_2 s_1 K^{\#}(\mathfrak{p})} dh = \frac{q-1}{(q+1)(q^4-1)} \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^{4m+3l+2} \quad (m > 0).$$

Note that the right-hand side of (101) is independent of j. This will play an important role in the evaluation of the zeta integral.

3.8. Main local theorem

In this section we will calculate the integral (49). From Proposition 3.6.1, we have

$$\begin{aligned} Z(s) = \sum_{l,m \geqslant 0} B\big(h(l,m)\big) \Bigg(& \sum_{z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o} \cap \mathfrak{p}^{[\frac{n-1}{2}]})/\mathfrak{p}^{n-1}} W^{\#}\big(\eta h(l,m) A(z), s\big) V^{l,m} \\ & + \sum_{j=\max(n-m-1,0)}^{[\frac{n-3}{2}]} \sum_{u \in \mathfrak{o}^\times/(1+\mathfrak{p}^{j+1})} W^{\#}\big(\eta h(l,m) A\big(\varpi^{j+1} u\big), s\big) V_j^{l,m} \Bigg) \\ & + \sum_{l \geqslant 0, m \geqslant n} B\big(h(l,m)\big) W^{\#}\big(\eta h(l,m) s_1 s_2 s_1, s\big) V_{s_1 s_2 s_1}^{l,m} \end{aligned} \quad (103)$$

where $A(z) = \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix}$. By (44), (68) and (70) we get

$$W^{\#}\big(\eta h(l,m)A(z),s\big)=\big|\varpi^{2m+l}\big|^{3(s+\frac{1}{2})}\omega_\pi\big(\varpi^{-2m-l}\big)\omega_\tau\big(\varpi^{-m-l}\big)W^{(0)}\left(\begin{bmatrix}\varpi^l & 0\\ \varpi z & 1\end{bmatrix}\right), \tag{104}$$

$$W^{\#}\big(\eta h(l,m)s_1s_2s_1,s\big)=\big|\varpi^{2m+l}\big|^{3(s+\frac{1}{2})}\omega_\pi\big(\varpi^{-2m-l}\big)\omega_\tau\big(\varpi^{-m-l}\big)W^{(0)}\left(\begin{bmatrix} & \varpi^l\\ -1 & \end{bmatrix}\right). \tag{105}$$

Set $C_{l,m}:=|\varpi^{2m+l}|^{3(s+\frac{1}{2})}\omega_\pi(\varpi^{-2m-l})\omega_\tau(\varpi^{-m-l})$. Substituting (104) and (105) into (103), we get

$$\begin{aligned}Z(s)={}&\sum_{l\geqslant 0}B\big(h(l,0)\big)C_{l,0}W^{(0)}\left(\begin{bmatrix}\varpi^l & 0\\ 0 & 1\end{bmatrix}\right)V^{l,0}\\
&+\sum_{l\geqslant 0,m\geqslant 1}B\big(h(l,m)\big)C_{l,m}V^{l,m}\Bigg(\sum_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o}\cap\mathfrak{p}^{[\frac{n-1}{2}]})/\mathfrak{p}^{n-1}}W^{(0)}\left(\begin{bmatrix}\varpi^l & 0\\ \varpi z & 1\end{bmatrix}\right)\Bigg)\\
&+\sum_{l\geqslant 0,m\geqslant 1}B\big(h(l,m)\big)C_{l,m}\Bigg(\sum_{j=\max(n-m-1,0)}^{[\frac{n-3}{2}]}\ \sum_{u\in\mathfrak{o}^\times/(1+\mathfrak{p}^{j+1})}W^{(0)}\left(\begin{bmatrix}\varpi^l & 0\\ \varpi^{j+1}u & 1\end{bmatrix}\right)V_j^{l,m}\Bigg)\\
&+\sum_{l\geqslant 0,m\geqslant n}B\big(h(l,m)\big)C_{l,m}W^{(0)}\left(\begin{bmatrix} & \varpi^l\\ -1 & \end{bmatrix}\right)V^{l,m}_{s_1s_2s_1}.\end{aligned} \tag{106}$$

The calculation of (106) for $n=1$ is different from the case $n>1$.

3.8.1. The case $n=1$

We will now assume that $\tau=\Omega\,\mathrm{St}_{\mathrm{GL}(2)}$, where Ω is an unramified character of $F^\times$, and $\mathrm{St}_{\mathrm{GL}(2)}$ is the Steinberg representation of $\mathrm{GL}(2,F)$. Then τ has conductor $\mathfrak{p}$, and the central character of τ is $\omega_\tau=\Omega^2$. We work in the ψ^{-c} Whittaker model for τ. In this model, the newform $W^{(0)}$ has the properties

$$W^{(0)}\left(\begin{bmatrix}a & \\ & 1\end{bmatrix}\right)=\begin{cases}|a|\Omega(a) & \text{if } a\in\mathfrak{o},\\ 0 & \text{otherwise,}\end{cases} \tag{107}$$

and

$$W^{(0)}\left(g\begin{bmatrix} & 1\\ \varpi & \end{bmatrix}\right)=-\Omega(\varpi)W^{(0)}(g)\quad\text{for all } g\in\mathrm{GL}_2(F). \tag{108}$$

We refer to [28] for details. If $n=1$, then the inner sum over z in the second term of (106) above reduces to just $z=0$, and the third term is not present. We have

$$\begin{aligned}Z(s)={}&\sum_{\substack{l\geqslant 0\\ m>0}}B\big(h(l,m)\big)C_{l,m}\left(W^{(0)}\left(\begin{bmatrix}\varpi^l & 0\\ 0 & 1\end{bmatrix}\right)V^{l,m}+W^{(0)}\left(\begin{bmatrix} & \varpi^l\\ -1 & \end{bmatrix}\right)V^{l,m}_{s_1s_2s_1}\right)\\
&+\sum_{l\geqslant 0}B\big(h(l,0)\big)C_{l,0}W^{(0)}\left(\begin{bmatrix}\varpi^l & 0\\ 0 & 1\end{bmatrix}\right)V^{l,0}.\end{aligned} \tag{109}$$

It follows from (107) and (108) that $W^{(0)}(\begin{bmatrix} & \varpi^l\\ -1 & \end{bmatrix})=W^{(0)}(\begin{bmatrix} & \varpi^l\\ 1 & \end{bmatrix})=-\Omega(\varpi^l)|\varpi|^{l+1}$ for all $l\geqslant 0$. Hence, from Proposition 3.7.4, we get

$$W^{(0)}\left(\begin{bmatrix}\varpi^l & 0\\ 0 & 1\end{bmatrix}\right)V^{l,m}+W^{(0)}\left(\begin{bmatrix} & \varpi^l\\ -1 & \end{bmatrix}\right)V^{l,m}_{s_1s_2s_1}=|\varpi|^l\Omega\big(\varpi^l\big)\big(V^{l,m}-q^{-1}V^{l,m}_{s_1s_2s_1}\big)=0.$$

Therefore,

$$\begin{aligned} Z(s) &= \sum_{l \geqslant 0} B\big(h(l,0)\big) C_{l,0} W^{(0)}\left(\begin{bmatrix} \varpi^l & 0 \\ 0 & 1 \end{bmatrix}\right) V^{l,0} \\ &= \frac{q-1}{(q+1)(q^4-1)} \sum_{l \geqslant 0} B\big(h(l,0)\big) q^{-l(3s+5/2)} (\omega_\pi \Omega)(\varpi)^{-l} \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) q^{3l+1} \\ &= \frac{q(q-1)}{(q+1)(q^4-1)} \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) \sum_{l \geqslant 0} B\big(h(l,0)\big) \big(q^{-3s+1/2} (\omega_\pi \Omega)(\varpi)^{-1}\big)^l. \end{aligned} \tag{110}$$

Let $\pi = \chi_1 \times \chi_2 \rtimes \sigma$ be an unramified principal series representation of $\mathrm{GSp}_4(F)$; in case $\chi_1 \times \chi_2 \rtimes \sigma$ is not irreducible, take its unramified constituent. Recall the characters $\gamma^{(1)}, \dots, \gamma^{(4)}$ defined in Section 3.2. Let ν be the absolute value in F normalized by $\nu(\varpi) = q^{-1}$. Set

$$L(s, \tilde{\pi} \times \tilde{\tau}) = \prod_{i=1}^{4} \big(1 - \big((\gamma^{(i)})^{-1} \Omega^{-1} \nu^{1/2}\big)(\varpi_F) q^{-s}\big)^{-1}. \tag{111}$$

Then $L(s, \tilde{\pi} \times \tilde{\tau})$ is the standard L-factor attached to the representation $\tilde{\pi} \times \tilde{\tau}$ of $\mathrm{GSp}_4(F) \times \mathrm{GL}_2(F)$ by the local Langlands correspondence. Here, $\tilde{\pi}$ (resp. $\tilde{\tau}$) denotes the contragredient representation of π (resp. τ). Denote by $\mathcal{AI}(\Lambda)$ the irreducible, admissible representation of $\mathrm{GL}_2(F)$ obtained by automorphic induction from the character Λ of $L^\times$. Set

$$L\big(s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times}\big) = \begin{cases} (1 - \chi(\varpi) q^{-1} q^{-2s})^{-1}, & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ (1 - \Lambda(\varpi_L)(\chi\Omega)(\varpi) q^{-1/2} q^{-3s-1})^{-1}, & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ (1 - \Lambda(\varpi_L)(\chi\Omega)(\varpi) q^{-1/2} q^{-3s-1})^{-1} & \\ \quad \times (1 - \Lambda(\varpi \varpi_L^{-1})(\chi\Omega)(\varpi) q^{-1/2} q^{-3s-1})^{-1}, & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases} \tag{112}$$

Then $L(s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times})$ is the standard L-factor attached to the representation $\tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times}$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_1(F)$ by the local Langlands correspondence. We now state the main theorem of the local non-archimedean theory for $n = 1$.

Theorem 3.8.1. *Let π be an unramified, irreducible, admissible representation of* $\mathrm{GSp}_4(F)$ (*not necessarily with trivial central character*), *and let $\tau = \Omega\,\mathrm{St}_{\mathrm{GL}(2)}$ with an unramified character Ω of $F^\times$. Let $Z(s)$ be the integral* (45), *where $W^\#$ is the function defined in Section* 3.4, *and B is the spherical Bessel function defined in Section* 3.2. *Then*

$$Z(s) = \frac{q(q-1)}{(q+1)(q^4-1)} \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) \frac{L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times})}. \tag{113}$$

Proof. By (30) and (110),

$$Z(s) = \frac{q(q-1)}{(q+1)(q^4-1)} \left(1 - \left(\frac{L}{\mathfrak{p}}\right) q^{-1}\right) \frac{H(q^{-3s+1/2}(\omega_\pi \Omega)(\varpi_F)^{-1})}{Q(q^{-3s+1/2}(\omega_\pi \Omega)(\varpi_F)^{-1})}. \tag{114}$$

By (31),

$$\begin{aligned} Q\left(q^{-3s+1/2}(\omega_\pi \Omega)(\varpi_F)^{-1}\right) &= \prod_{i=1}^{4} \big(1 - \gamma^{(i)}(\varpi_F) q^{-3s-1} (\omega_\pi \Omega)(\varpi_F)^{-1}\big) \\ &= \prod_{i=1}^{4} \big(1 - \big(\gamma^{(i)} (\omega_\pi \Omega)^{-1} \nu^{1/2}\big)(\varpi_F) q^{-3s-1/2}\big) \end{aligned}$$

$$= \prod_{i=1}^{4}\big(1-\big(\big(\gamma^{(i)}\big)^{-1}\Omega^{-1}\nu^{1/2}\big)(\varpi_F)q^{-3s-1/2}\big)$$
$$\overset{(111)}{=} L(3s+1/2,\tilde\pi\times\tilde\tau)^{-1}.$$

To compute the numerator of (114), we distinguish cases. If $(\frac{L}{\mathfrak p})=-1$, then $H(y)=1-q^{-4}\Lambda(\varpi_F)y^2$, and hence

$$\begin{aligned}
H\big(q^{-3s+1/2}(\omega_\pi\Omega)(\varpi_F)^{-1}\big) &= 1-q^{-4}\Lambda(\varpi_F)\big(q^{-3s+1/2}(\omega_\pi\Omega)(\varpi_F)^{-1}\big)^2\\
&= 1-\big(\Lambda\omega_\pi^{-2}\Omega^{-2}\big)(\varpi_F)q^{-6s-3}\\
&= 1-\big(\omega_\pi^{-1}\omega_\tau^{-1}\big)(\varpi_F)q^{-6s-3}\\
&= 1-\chi(\varpi_F)q^{-1}q^{-6s-2}\\
&\overset{(112)}{=} L\big(3s+1,\tau\times\mathcal{AI}(\Lambda)\times\chi|_{F^\times}\big)^{-1}.
\end{aligned}$$

If $(\frac{L}{\mathfrak p})=0$, then $H(y)=1-q^{-2}\Lambda(\varpi_L)y$, and hence

$$\begin{aligned}
H\big(q^{-3s+1/2}(\omega_\pi\Omega)(\varpi_F)^{-1}\big) &= 1-q^{-2}\Lambda(\varpi_L)q^{-3s+1/2}(\omega_\pi\Omega)(\varpi_F)^{-1}\\
&= 1-\Lambda(\varpi_L)\big(\omega_\pi\omega_\tau\Omega^{-1}\big)(\varpi_F)^{-1}q^{-3s-3/2}\\
&= 1-\Lambda(\varpi_L)(\chi\Omega)(\varpi_F)q^{-1/2}q^{-3s-1}\\
&\overset{(112)}{=} L\big(3s+1,\tau\times\mathcal{AI}(\Lambda)\times\chi|_{F^\times}\big)^{-1}.
\end{aligned}$$

If $(\frac{L}{\mathfrak p})=1$, then $H(y)=(1-q^{-2}\Lambda(\varpi_L)y)(1-q^{-2}\Lambda(\varpi_F\varpi_L^{-1})y)$, and hence

$$\begin{aligned}
H\big(q^{-3s+1/2}(\omega_\pi\Omega)(\varpi_F)^{-1}\big) &= \big(1-q^{-2}\Lambda(\varpi_L)q^{-3s+1/2}(\omega_\pi\Omega)(\varpi_F)^{-1}\big)\\
&\quad\times\big(1-q^{-2}\Lambda\big(\varpi_F\varpi_L^{-1}\big)q^{-3s+1/2}(\omega_\pi\Omega)(\varpi_F)^{-1}\big)\\
&= \big(1-\Lambda(\varpi_L)\big(\omega_\pi\omega_\tau\Omega^{-1}\big)(\varpi_F)^{-1}q^{-3s-3/2}\big)\\
&\quad\times\big(1-\Lambda\big(\varpi_F\varpi_L^{-1}\big)\big(\omega_\pi\omega_\tau\Omega^{-1}\big)(\varpi_F)^{-1}q^{-3s-3/2}\big)\\
&= \big(1-\Lambda(\varpi_L)(\chi\Omega)(\varpi_F)q^{-1/2}q^{-3s-1}\big)\\
&\quad\times\big(1-\Lambda\big(\varpi_F\varpi_L^{-1}\big)(\chi\Omega)(\varpi_F)q^{-1/2}q^{-3s-1}\big)\\
&\overset{(112)}{=} L\big(3s+1,\tau\times\mathcal{AI}(\Lambda)\times\chi|_{F^\times}\big)^{-1}.
\end{aligned}$$

Hence $H(q^{-3s+1/2}(\omega_\pi\Omega)(\varpi_F)^{-1})=L(3s+1,\tau\times\mathcal{AI}(\Lambda)\times\chi|_{F^\times})^{-1}$ in all cases. This concludes the proof of the theorem. $\square$

3.8.2. The case $n\geqslant 2$

From now on we will assume that $n\geqslant 2$. As the following lemma shows, the fact that the representation τ has conductor $\mathfrak p^n$ implies that the middle two expressions in formula (106) are zero.

Lemma 3.8.1. *Let $m \geqslant 1$ and $n \geqslant 2$.*

(i) *For any $g \in \mathrm{GL}_2(F)$,*

$$\sum_{z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o} \cap \mathfrak{p}^{[\frac{n-1}{2}]})/\mathfrak{p}^{n-1}} W^{(0)}\left(g\begin{bmatrix} 1 & 0 \\ \varpi z & 1 \end{bmatrix}\right) = 0.$$

(ii) *For $2j+2 < n$ and any z with $\nu(z) = j$,*

$$W^{(0)}\left(\begin{bmatrix} \varpi^l & 0 \\ \varpi z & 1 \end{bmatrix}\right) = 0.$$

Proof. (i) Let $t = \max(n-m-1, 0, [\frac{n-1}{2}])$. We have $\mathfrak{p}^{n-m-1} \cap \mathfrak{o} \cap \mathfrak{p}^{[\frac{n-1}{2}]} = \mathfrak{p}^t$ and, since $m \geqslant 1$ and $n \geqslant 2$, we see that $t+1 < n$. Define $\hat{W}(g) = \sum_{z \in \mathfrak{p}^{t+1}/\mathfrak{p}^n} W^{(0)}(g\left[\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right]) \in V_\tau$. A calculation verifies that $\hat{W}$ is invariant under $K^{(1)}(\mathfrak{p}^{t+1})$. Since τ has level $\mathfrak{p}^n$ and $t+1 < n$, this implies $\hat{W} = 0$, as claimed.

(ii) Let z_1, z_2 be such that $\nu(z_1) = \nu(z_2) = j$ and $z_1/z_2 \in 1 + \mathfrak{p}^{j+1}$. Consider the matrix identity

$$\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi z_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\varpi^l(z_2-z_1)}{\varpi z_1 z_2} \\ & 1 \end{bmatrix}\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi z_2 & 1 \end{bmatrix}\begin{bmatrix} \frac{z_1}{z_2} & \frac{(z_2-z_1)}{\varpi z_1 z_2} \\ & \frac{z_2}{z_1} \end{bmatrix}.$$

Since the additive character ψ is trivial on $\mathfrak{o}$ and the rightmost matrix is in $K^{(1)}(\mathfrak{p}^n)$, it implies that

$$W^{(0)}\left(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi z & 1 \end{bmatrix}\right) = W^{(0)}\left(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi zu & 1 \end{bmatrix}\right) \tag{115}$$

for every $u \in 1 + \mathfrak{p}^{j+1}$ and $z \in \mathfrak{o}$ with $\nu(z) = j$ (we have essentially derived the well-definedness of the third sum in (106)). Writing $u = 1 + b\varpi^{j+1}$ with $b \in \mathfrak{o}$ and integrating both sides of (115) with respect to b, we get

$$\begin{aligned} W^{(0)}\left(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi z & 1 \end{bmatrix}\right) &= \int_{\mathfrak{o}} W^{(0)}\left(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi z & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi z b \varpi^{j+1} & 1 \end{bmatrix}\right) db \\ &= \int_{\mathfrak{o}} W^{(0)}\left(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi z & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ b\varpi^{2j+2} & 1 \end{bmatrix}\right) db. \end{aligned}$$

This last expression is zero, since $2j+2 < n$ and $\tilde{W}(g) := \int_{\mathfrak{o}} W^{(0)}(g\left[\begin{smallmatrix} 1 & \\ b\varpi^{2j+2} & 1 \end{smallmatrix}\right])\, db \in V_\tau$ is right invariant under $K^{(1)}(\mathfrak{p}^{2j+2})$. This concludes the proof. □

Using this lemma, (106) now becomes

$$\begin{aligned} Z(s) = {} & \sum_{l \geqslant 0} B\big(h(l,0)\big)\left|\varpi^l\right|^{3(s+\frac{1}{2})}\omega_\pi\left(\varpi^{-l}\right)\omega_\tau\left(\varpi^{-l}\right)W^{(0)}\left(\begin{bmatrix} \varpi^l & 0 \\ 0 & 1 \end{bmatrix}\right)V^{l,0} \\ & + \sum_{l \geqslant 0, m \geqslant n} B\big(h(l,m)\big)\left|\varpi^{2m+l}\right|^{3(s+\frac{1}{2})}\omega_\pi\left(\varpi^{-2m-l}\right)\omega_\tau\left(\varpi^{-m-l}\right)W^{(0)} \\ & \times \left(\begin{bmatrix} & \varpi^l \\ -1 & \end{bmatrix}\right)V^{l,m}_{s_1 s_2 s_1}. \end{aligned} \tag{116}$$

Since $\left[\begin{smallmatrix} & 1 \\ \varpi^n & \end{smallmatrix}\right]$ normalizes $K^{(1)}(\mathfrak{p}^n)$, the vector $W'(g) := W^{(0)}(g\left[\begin{smallmatrix} & 1 \\ \varpi^n & \end{smallmatrix}\right])$ is another element of V_τ that is right invariant under $K^{(1)}(\mathfrak{p}^n)$. Since the space of vectors in V_τ right invariant under $K^{(1)}(\mathfrak{p}^n)$ is one-dimensional, there is a constant $c \in \mathbb{C}$ such that $W^{(0)} = cW'$ (one can check that $c^{-2} = \omega_\tau(\varpi^n)$). Hence,

$$W^{(0)}\left(\begin{bmatrix} & \varpi^l \\ -1 & \end{bmatrix}\right) = cW^{(0)}\left(\begin{bmatrix} \varpi^{l+n} & \\ & -1 \end{bmatrix}\right) = cW^{(0)}\left(\begin{bmatrix} \varpi^{l+n} & \\ & 1 \end{bmatrix}\right). \tag{117}$$

This shows that in order to evaluate (116) we need the formula for the new-vector of τ in the Kirillov model. The possibilities for our generic, irreducible, admissible representation τ of $\mathrm{GL}_2(F)$ with unramified central character and conductor $\mathfrak{p}^n$, $n \geqslant 2$, are as follows. Either τ is a principal series representation $\chi_1 \times \chi_2$, where χ_1 and χ_2 are ramified characters of $F^\times$ (with $\chi_1\chi_2$ unramified); or $\tau = \chi\,\mathrm{St}_{\mathrm{GL}(2)}$, a twist of the Steinberg representation by a ramified character χ (such that χ^2 is unramified); or τ is supercuspidal. In each case the newform in the Kirillov model is given by the characteristic function of $\mathfrak{o}^\times$; see, e.g., [28]. It follows that all the terms in (117) are zero. The integral (116) reduces to

$$Z(s) = V^{0,0} = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)}\left(1 - \left(\frac{L}{\mathfrak{p}}\right)q^{-1}\right)q^n. \tag{118}$$

Thus, we have proved the following result.

Theorem 3.8.2. *Let π be an unramified, irreducible, admissible representation of $\mathrm{GSp}_4(F)$ (not necessarily with trivial central character), and let τ be an irreducible, admissible, generic representation of $\mathrm{GL}_2(F)$ with unramified central character and conductor $\mathfrak{p}^n$ with $n \geqslant 2$. Let $Z(s)$ be the integral (45), where $W^\#$ is the function defined in Section 3.4, and B is the spherical Bessel function defined in Section 3.2. Then*

$$Z(s) = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)}\left(1 - \left(\frac{L}{\mathfrak{p}}\right)q^{-1}\right)q^n. \tag{119}$$

Remark. For any unramified, irreducible, admissible representation π of $\mathrm{GSp}_4(F)$ and any of the representations τ of $\mathrm{GL}_2(F)$ mentioned in the theorem we have $L(s, \pi \times \tau) = 1$. Hence, up to a constant, the integral $Z(s)$ represents the L-factor $L(s, \pi \times \tau)$.

4. Local archimedean theory

In this section we compute the local archimedean integral. As in Section 3, the key step is the choice of vectors B and $W^\#$.

4.1. Real groups

Consider the symmetric domains $\mathbb{H}_2 := \{Z \in M_2(\mathbb{C})\colon\ i({}^t\bar{Z} - Z) \text{ is positive definite}\}$ and $\mathfrak{h}_2 := \{Z \in \mathbb{H}_2\colon\ {}^tZ = Z\}$. The group $G^+(\mathbb{R}) := \{g \in G(\mathbb{R})\colon\ \mu_2(g) > 0\}$ acts on $\mathbb{H}_2$ via $(g, Z) \mapsto g\langle Z\rangle$, where

$$g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}, \quad \text{for } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G^+(\mathbb{R}),\ Z \in \mathbb{H}_2.$$

Under this action, $\mathfrak{h}_2$ is stable by $H^+(\mathbb{R}) = \mathrm{GSp}_4^+(\mathbb{R})$. The group $K_\infty = \{g \in G^+(\mathbb{R})\colon\ \mu_2(g) = 1,\ g\langle I\rangle = I\}$ is a maximal compact subgroup of $G^+(\mathbb{R})$. Here, $I = \left[\begin{smallmatrix} i & \\ & i \end{smallmatrix}\right] \in \mathbb{H}_2$. Explicitly,

$$K_\infty = \left\{\begin{bmatrix} A & B \\ -B & A \end{bmatrix}\colon\ A, B \in M(2, \mathbb{C}),\ {}^t\bar{A}B = {}^t\bar{B}A,\ {}^t\bar{A}A + {}^t\bar{B}B = 1\right\}.$$

By the Iwasawa decomposition

$$G(\mathbb{R}) = M^{(1)}(\mathbb{R})M^{(2)}(\mathbb{R})N(\mathbb{R})K_\infty, \tag{120}$$

where $M^{(1)}(\mathbb{R})$, $M^{(2)}(\mathbb{R})$ and $N(\mathbb{R})$ are as defined in (6), (7), (8). A calculation shows that

$$\begin{aligned} &M^{(1)}(\mathbb{R})M^{(2)}(\mathbb{R})N(\mathbb{R}) \cap K_\infty \\ &\quad = \left\{ \begin{bmatrix} \zeta &&& \\ & \alpha && \beta \\ && \zeta & \\ & -\beta && \alpha \end{bmatrix} :\ \zeta, \alpha, \beta \in \mathbb{C},\ |\zeta| = 1,\ |\alpha|^2 + |\beta|^2 = 1,\ \alpha\bar{\beta} = \beta\bar{\alpha} \right\}. \end{aligned} \tag{121}$$

Note also that

$$M^{(2)}(\mathbb{R}) \cap K_\infty = \left\{ \begin{bmatrix} 1 &&& \\ & \alpha && \beta \\ && 1 & \\ & -\beta && \alpha \end{bmatrix} :\ \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1,\ \alpha\bar{\beta} = \beta\bar{\alpha} \right\}, \tag{122}$$

and that there is an isomorphism

$$\begin{aligned} \big(S^1 \times \mathrm{SO}(2)\big) \Big/ \left\{ \left(\lambda, \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix} \right) :\ \lambda = \pm 1 \right\} &\xrightarrow{\sim} M^{(2)}(\mathbb{R}) \cap K_\infty, \\ \left(\lambda, \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \right) &\longmapsto \begin{bmatrix} 1 &&& \\ & \lambda\alpha && \lambda\beta \\ && 1 & \\ & -\lambda\beta && \lambda\alpha \end{bmatrix}. \end{aligned} \tag{123}$$

For $g \in G^+(\mathbb{R})$ and $Z \in \mathbb{H}_2$, let $J(g, Z) = CZ + D$ be the automorphy factor. Then, for any integer l, the map

$$k \longmapsto \det\big(J(k, I)\big)^l \tag{124}$$

defines a character $K_\infty \to \mathbb{C}^\times$. If $k \in M^{(2)}(\mathbb{R}) \cap K_\infty$ is written in the form (123), then $\det(J(k, I))^l = \lambda^l e^{-il\theta}$, where $\alpha = \cos(\theta)$, $\beta = \sin(\theta)$. Let $K_\infty^H = K_\infty \cap H^+(\mathbb{R})$. Then K_∞^H is a maximal compact subgroup, explicitly given by

$$K_\infty^H = \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} :\ {}^tAB = {}^tBA,\ {}^tAA + {}^tBB = 1 \right\}.$$

Sending $\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right]$ to $A - iB$ gives an isomorphism $K_\infty^H \cong \mathrm{U}(2)$. Recall that we have chosen $a, b, c \in \mathbb{R}$ such that $d = b^2 - 4ac \neq 0$. In the archimedean case we shall assume that $d < 0$ and let $D = -d$. Then $\mathbb{R}(\sqrt{-D}) = \mathbb{C}$. As in Section 2.2 we have

$$T(\mathbb{R}) = \left\{ \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix} :\ x, y \in \mathbb{R},\ x^2 + y^2D/4 > 0 \right\}. \tag{125}$$

Let

$$T_\infty^1 = T(\mathbb{R}) \cap \mathrm{SL}(2, \mathbb{R}) = \left\{ \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix} :\ x, y \in \mathbb{R},\ x^2 + y^2D/4 = 1 \right\}. \tag{126}$$

We have $T(\mathbb{R}) \cong \mathbb{C}^\times$ via $\left[\begin{smallmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{smallmatrix}\right] \mapsto x + y\sqrt{-D}/2$. Under this isomorphism T^1_∞ corresponds to the unit circle. We have

$$T(\mathbb{R}) = T^1_\infty \cdot \left\{ \begin{bmatrix} \zeta & \\ & \zeta \end{bmatrix} : \zeta > 0 \right\}. \tag{127}$$

As in [9, p. 211], let $t_0 \in \mathrm{GL}_2(\mathbb{R})^+$ be such that $T^1_\infty = t_0\,\mathrm{SO}(2)t_0^{-1}$. We will make a specific choice of t_0 when we choose the matrix $S = \left[\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right]$ below. By the Cartan decomposition,

$$\mathrm{GL}_2^+(\mathbb{R}) = \mathrm{SO}(2) \cdot \left\{ \begin{bmatrix} \zeta_1 & \\ & \zeta_2 \end{bmatrix} : \zeta_1, \zeta_2 > 0,\ \zeta_1 \geqslant \zeta_2 \right\} \cdot \mathrm{SO}(2). \tag{128}$$

Therefore,

$$\begin{aligned} \mathrm{GL}_2^+(\mathbb{R}) &= t_0\,\mathrm{SO}(2) \cdot \left\{ \begin{bmatrix} \zeta_1 & \\ & \zeta_2 \end{bmatrix} : \zeta_1, \zeta_2 > 0,\ \zeta_1 \geqslant \zeta_2 \right\} \cdot \mathrm{SO}(2) \\ &= T^1_\infty t_0 \cdot \left\{ \begin{bmatrix} \sqrt{\zeta_1\zeta_2} & \\ & \sqrt{\zeta_1\zeta_2} \end{bmatrix} \begin{bmatrix} \sqrt{\zeta_1/\zeta_2} & \\ & \sqrt{\zeta_2/\zeta_1} \end{bmatrix} : \zeta_1, \zeta_2 > 0,\ \zeta_1 \geqslant \zeta_2 \right\} \cdot \mathrm{SO}(2) \\ &= T(\mathbb{R}) t_0 \cdot \left\{ \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} : \zeta \geqslant 1 \right\} \cdot \mathrm{SO}(2). \end{aligned} \tag{129}$$

Using this, it is not hard to see that

$$H(\mathbb{R}) = R(\mathbb{R}) \cdot \left\{ \begin{bmatrix} \lambda t_0 \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} & \\ & {}^t t_0^{-1} \begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} \end{bmatrix} : \lambda \in \mathbb{R}^\times,\ \zeta \geqslant 1 \right\} \cdot K^H_\infty. \tag{130}$$

Here, $R(\mathbb{R}) = T(\mathbb{R})U(\mathbb{R})$ is the Bessel subgroup defined in Section 2.2. One can check that all the double cosets in (130) are disjoint.

4.2. The Bessel function

Recall that we have chosen three elements $a, b, c \in \mathbb{R}$ such that $d = b^2 - 4ac \neq 0$. We will now make the stronger assumption that $S = \left[\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right] \in M_2(\mathbb{R})$ is a positive definite matrix. Set $D = 4ac - b^2 > 0$, as above. Given a positive integer $l \geqslant 2$, consider the function $B : H(\mathbb{R}) \to \mathbb{C}$ defined by

$$B(h) := \begin{cases} \mu_2(h)^l \overline{\det(J(h, I))^{-l}} e^{-2\pi i\,\mathrm{tr}(S\overline{h\langle I\rangle})} & \text{if } h \in H^+(\mathbb{R}), \\ 0 & \text{if } h \notin H^+(\mathbb{R}), \end{cases} \tag{131}$$

where $I = \left[\begin{smallmatrix} i & \\ & i \end{smallmatrix}\right]$. Note that the function B only depends on the choice of S and l. Recall the character θ of $U(\mathbb{R})$ defined in (12). It depends on the choice of additive character ψ, and throughout we choose $\psi(x) = e^{-2\pi i x}$. Then the function B satisfies

$$B(tuh) = \theta(u)B(h) \quad \text{for } h \in H(\mathbb{R}),\ t \in T(\mathbb{R}),\ u \in U(\mathbb{R}), \tag{132}$$

and

$$B(hk) = \det\bigl(J(k, I)\bigr)^l B(h) \quad \text{for } h \in H(\mathbb{R}),\ k \in K^H_\infty. \tag{133}$$

Property (132) means that B satisfies the Bessel transformation property with the character $\Lambda \otimes \theta$ of $R(\mathbb{R})$, where Λ is trivial. In fact, by the considerations in [32, 1–3], or by [24, Theorem 3.4], B is the highest weight vector (weight $(-l, -l)$) in a holomorphic discrete series representation (or limit of such if $l = 2$) of $\mathrm{PGSp}_4(\mathbb{R})$ corresponding to Siegel modular forms of degree 2 and weight l. By (132) and (133), the function B is determined by its values on a set of representatives for $R(\mathbb{R}) \setminus H(\mathbb{R})/K_\infty^H$. Such a set is given in (130).

4.3. The function $W^\#$

Let (τ, V_τ) be a generic, irreducible, admissible representation of $\mathrm{GL}_2(\mathbb{R})$ with central character ω_τ. We assume that $V_\tau = \mathcal{W}(\tau, \psi_{-c})$ is the Whittaker model of τ with respect to the non-trivial additive character $x \mapsto \psi(-cx)$. Note that S positive definite implies $c > 0$. Let $W^{(0)} \in V_\tau$ have weight l_1. Then $W^{(0)}$ has the following properties.

(i) $$W^{(0)}\big(gr(\theta)\big) = e^{il_1\theta} W^{(0)}(g) \quad \text{for } g \in \mathrm{GL}_2(\mathbb{R}), \qquad r(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \in \mathrm{SO}(2).$$

(ii) $$W^{(0)}\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g\right) = \psi(-cx) W^{(0)}(g) \quad \text{for } g \in \mathrm{GL}_2(\mathbb{R}),\ x \in \mathbb{R}.$$

Let χ_0 be the character of $\mathbb{C}^\times$ with the properties

$$\chi_0|_{\mathbb{R}^\times} = \omega_\tau, \quad \chi_0(\zeta) = \zeta^{-l_1} \quad \text{for } \zeta \in \mathbb{C}^\times,\ |\zeta| = 1. \tag{134}$$

Such a character exists since $\omega_\tau(-1) = (-1)^{l_1}$. We extend $W^{(0)}$ to a function on $M^{(2)}(\mathbb{R})$ via

$$W^{(0)}(\zeta g) = \chi_0(\zeta) W^{(0)}(g), \quad \zeta \in \mathbb{C}^\times,\ g \in \mathrm{GL}_2(\mathbb{R}) \tag{135}$$

(see Lemma 2.1.1). Then it is easy to check that

$$W^{(0)}(gk) = \det\big(J(k, I)\big)^{-l_1} W^{(0)}(g) \quad \text{for } g \in M^{(2)}(\mathbb{R}) \text{ and } k \in M^{(2)}(\mathbb{R}) \cap K_\infty. \tag{136}$$

We will need values of $W^{(0)}$ evaluated at $\left[\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right]$ for $t \neq 0$. For this we consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$ and its elements

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In the universal enveloping algebra $U(\mathfrak{g})$ let

$$\Delta = \frac{1}{4}\big(H^2 + 2RL + 2LR\big). \tag{137}$$

Then Δ lies in the center of $U(\mathfrak{g})$ and acts on V_τ by a scalar, which we write in the form $-(\frac{1}{4} + (\frac{r}{2})^2)$ with $r \in \mathbb{C}$. In particular,

$$\Delta W^{(0)} = -\left(\frac{1}{4} + \left(\frac{r}{2}\right)^2\right) W^{(0)}. \tag{138}$$

If one restricts the function $W^{(0)}$ to $\left[\begin{smallmatrix} t^{1/2} & \\ & t^{-1/2}\end{smallmatrix}\right]$, $t>0$, then (138) reduces to the differential equation satisfied by the classical Whittaker functions. Hence, there exist constants $a^+, a^- \in \mathbb{C}$ such that

$$W^{(0)}\left(\begin{bmatrix} t & 0 \\ 0 & 1\end{bmatrix}\right) = \begin{cases} a^+\omega_\tau((4\pi ct)^{1/2})W_{\frac{l_1}{2},\frac{ir}{2}}(4\pi ct) & \text{if } t>0, \\ a^-\omega_\tau((-4\pi ct)^{1/2})W_{-\frac{l_1}{2},\frac{ir}{2}}(-4\pi ct) & \text{if } t<0. \end{cases} \tag{139}$$

Here, $W_{\pm\frac{l_1}{2},\frac{ir}{2}}$ denotes a classical Whittaker function; see [4, p. 244], [16]. Let χ be the character of $\mathbb{C}^\times$ given by

$$\chi(\zeta) = \chi_0(\bar{\zeta})^{-1}. \tag{140}$$

We interpret χ as a character of $M^{(1)}(\mathbb{R})$; see (6). Given a complex number s, we define a function $W^\#(\cdot,s) : G(\mathbb{R}) \to \mathbb{C}$ as follows. Given $g \in G(\mathbb{R})$, write $g = m_1m_2nk$ according to (120). Then set

$$W^\#(g,s) = \delta_P^{s+1/2}(m_1m_2)\det\big(J(k,I)\big)^{-l_1}\chi(m_1)W^{(0)}(m_2). \tag{141}$$

Property (136) shows that this is well-defined. Explicitly, for $\zeta \in \mathbb{C}^\times$ and $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right] \in M^{(2)}(\mathbb{R})$,

$$W^\#\left(\begin{bmatrix} \zeta &&& \\ & 1 && \\ && \bar{\zeta}^{-1} & \\ &&& 1\end{bmatrix}\begin{bmatrix} 1 &&& \\ & \alpha && \beta \\ && \mu & \\ & \gamma && \delta\end{bmatrix}, s\right) = \left|\zeta^2\mu^{-1}\right|^{3(s+1/2)}\chi(\zeta)W^{(0)}\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta\end{bmatrix}\right). \tag{142}$$

Here $\mu = \bar{\alpha}\delta - \beta\bar{\gamma}$. It is clear that $W^\#(\cdot,s)$ satisfies

$$W^\#(gk,s) = \det\big(J(k,I)\big)^{-l_1}W^\#(g,s) \quad \text{for } g \in G(\mathbb{R}),\ k \in K_\infty. \tag{143}$$

By Lemma 2.3.1, we have

$$W^\#(\eta tuh,s) = \theta(u)^{-1}W^\#(\eta h,s) \tag{144}$$

for $t \in T(\mathbb{R})$, $u \in U(\mathbb{R})$, $h \in G(\mathbb{R})$ and

$$\eta = \begin{bmatrix} 1 &&& \\ \alpha & 1 && \\ && 1 & -\bar{\alpha} \\ &&& 1\end{bmatrix}, \qquad \alpha = \frac{b+\sqrt{d}}{2c}, \quad d = b^2 - 4ac.$$

4.4. The local archimedean integral

Let B and $W^\#$ be as defined in Sections 4.2 and 4.3. By (132) and (144), it makes sense to consider the integral

$$Z_\infty(s) = \int\limits_{R(\mathbb{R})\backslash H(\mathbb{R})} W^\#(\eta h,s)B(h)\,dh. \tag{145}$$

Our goal in the following is to evaluate this integral. It follows from (133) and (143) that it is zero if $l_1 \neq l$. We shall therefore assume that $\underline{l_1 = l}$. Then the function $W^\#(\eta h,s)B(h)$ is right invariant under K_∞^H. From the disjoint double coset decomposition (130) and the fact that $W^\#(\eta h,s)B(h)$ is right invariant under K_∞^H we obtain

$$Z_\infty(s)=\pi\int_{\mathbb{R}^\times}\int_1^\infty W^\#\left(\eta\begin{bmatrix}\lambda t_0\begin{bmatrix}\zeta&\\&\zeta^{-1}\end{bmatrix}&\\&{}^tt_0^{-1}\begin{bmatrix}\zeta^{-1}&\\&\zeta\end{bmatrix}\end{bmatrix},s\right)$$
$$\times B\left(\begin{bmatrix}\lambda t_0\begin{bmatrix}\zeta&\\&\zeta^{-1}\end{bmatrix}&\\&{}^tt_0^{-1}\begin{bmatrix}\zeta^{-1}&\\&\zeta\end{bmatrix}\end{bmatrix}\right)\left(\zeta-\zeta^{-3}\right)\lambda^{-4}\,d\zeta\,d\lambda; \tag{146}$$

see [9, (4.6)] for the relevant integration formulas. The above calculations are valid for any choice of a,b,c as long as $S=\left[\begin{smallmatrix}a&b/2\\b/2&c\end{smallmatrix}\right]$ is positive definite. To compute (146), we will fix $D=4ac-b^2$ and make special choices for a,b,c. First assume that $\underline{D\equiv 0 \pmod 4}$. In this case let $S(-D):=\left[\begin{smallmatrix}\frac{D}{4}&0\\0&1\end{smallmatrix}\right]$. Then $\eta=\begin{bmatrix}1&&&\\\frac{\sqrt{-D}}{2}&1&&\\&&1&\frac{\sqrt{-D}}{2}\\&&&1\end{bmatrix}$, and we can choose $t_0=\left[\begin{smallmatrix}2^{1/2}D^{-1/4}&\\&2^{-1/2}D^{1/4}\end{smallmatrix}\right]$. From (131) we have

$$B\left(\begin{bmatrix}\lambda t_0\begin{bmatrix}\zeta&\\&\zeta^{-1}\end{bmatrix}&\\&{}^tt_0^{-1}\begin{bmatrix}\zeta^{-1}&\\&\zeta\end{bmatrix}\end{bmatrix}\right)=\begin{cases}\lambda^l e^{-2\pi\lambda D^{1/2}\frac{\zeta^2+\zeta^{-2}}{2}}&\text{if }\lambda>0,\\0&\text{if }\lambda<0.\end{cases} \tag{147}$$

Next we rewrite the argument of $W^\#$ as an element of MNK_∞,

$$\eta\begin{bmatrix}\lambda t_0\begin{bmatrix}\zeta&\\&\zeta^{-1}\end{bmatrix}&\\&{}^tt_0^{-1}\begin{bmatrix}\zeta^{-1}&\\&\zeta\end{bmatrix}\end{bmatrix}$$
$$=\begin{bmatrix}\lambda\begin{bmatrix}D^{-\frac14}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{-\frac12}&\\&D^{\frac14}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{\frac12}\end{bmatrix}&\\&\begin{bmatrix}D^{\frac14}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{\frac12}&\\&D^{-\frac14}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{-\frac12}\end{bmatrix}\end{bmatrix}$$
$$\times\begin{bmatrix}1&-i\zeta^2&&\\0&1&&\\&&1&0\\&&-i\zeta^2&1\end{bmatrix}\begin{bmatrix}k_0&0\\0&k_0\end{bmatrix},$$

where $k_0\in\mathrm{SU}(2)=\{g\in\mathrm{SL}_2(\mathbb{C}):\ {}^t\bar{g}g=I_2\}$. Hence, using (142) and (143), we get

$$W^\#\left(\eta\begin{bmatrix}\lambda t_0\begin{bmatrix}\zeta&\\&\zeta^{-1}\end{bmatrix}&\\&t_0^{-1}\begin{bmatrix}\zeta^{-1}&\\&\zeta\end{bmatrix}\end{bmatrix},s\right)$$
$$=\left|\lambda D^{-\frac12}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{-1}\right|^{3(s+\frac12)}\omega_\tau(\lambda)^{-1}W^{(0)}\left(\begin{bmatrix}\lambda D^{\frac12}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)&0\\0&1\end{bmatrix}\right). \tag{148}$$

Let $q\in\mathbb{C}$ be such that $\omega_\tau(y)=y^q$ for $y>0$. It follows from (139), (147) and (148) that

$$Z_\infty(s) = a^+ \pi D^{-\frac{3s}{2}-\frac{3}{4}+\frac{q}{4}} (4\pi)^{\frac{q}{2}} \int_0^\infty \int_1^\infty \lambda^{3s+\frac{3}{2}+l-\frac{q}{2}} \left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{-3s-\frac{3}{2}+\frac{q}{2}} W_{\frac{l}{2},\frac{ir}{2}}\left(4\pi\lambda D^{1/2}\frac{\zeta^2+\zeta^{-2}}{2}\right)$$
$$\times e^{-2\pi\lambda D^{1/2}\frac{\zeta^2+\zeta^{-2}}{2}} \left(\zeta-\zeta^{-3}\right)\lambda^{-4}\, d\zeta\, d\lambda. \tag{149}$$

Substituting $u = (\zeta^2+\zeta^{-2})/2$ we get

$$Z_\infty(s) = a^+ \pi D^{-\frac{3s}{2}-\frac{3}{4}+\frac{q}{4}} (4\pi)^{\frac{q}{2}} \int_1^\infty \int_0^\infty \lambda^{3s-\frac{3}{2}+l-\frac{q}{2}} u^{-3s-\frac{3}{2}+\frac{q}{2}} W_{\frac{l}{2},\frac{ir}{2}}\left(4\pi\lambda D^{1/2}u\right) e^{-2\pi\lambda D^{1/2}u} \frac{d\lambda}{\lambda}\, du.$$

We will first compute the integral with respect to λ. For a fixed u substitute $x = 4\pi\lambda D^{1/2}u$ to get

$$Z_\infty(s) = a^+ \pi D^{-3s-\frac{l}{2}+\frac{q}{2}} (4\pi)^{-3s+\frac{3}{2}-l+q} \int_1^\infty u^{-6s-l+q} \int_0^\infty W_{\frac{l}{2},\frac{ir}{2}}(x) e^{-\frac{x}{2}} x^{3s-\frac{3}{2}+l-\frac{q}{2}} \frac{dx}{x}\, du.$$

Using the integral formula for the Whittaker function from [16, p. 316], we get

$$\begin{aligned}
Z_\infty(s) &= a^+ \pi D^{-3s-\frac{l}{2}+\frac{q}{2}} (4\pi)^{-3s+\frac{3}{2}-l+q} \frac{\Gamma(3s+l-1+\frac{ir}{2}-\frac{q}{2})\Gamma(3s+l-1-\frac{ir}{2}-\frac{q}{2})}{\Gamma(3s+\frac{l}{2}-\frac{1}{2}-\frac{q}{2})} \int_1^\infty u^{-6s-l+q}\, du \\
&= a^+ \pi D^{-3s-\frac{l}{2}+\frac{q}{2}} \frac{(4\pi)^{-3s+\frac{3}{2}-l+q}}{6s+l-q-1} \frac{\Gamma(3s+l-1+\frac{ir}{2}-\frac{q}{2})\Gamma(3s+l-1-\frac{ir}{2}-\frac{q}{2})}{\Gamma(3s+\frac{l}{2}-\frac{1}{2}-\frac{q}{2})} \\
&= \frac{a^+}{2} \pi D^{-3s-\frac{l}{2}+\frac{q}{2}} (4\pi)^{-3s+\frac{3}{2}-l+q} \frac{\Gamma(3s+l-1+\frac{ir}{2}-\frac{q}{2})\Gamma(3s+l-1-\frac{ir}{2}-\frac{q}{2})}{\Gamma(3s+\frac{l+1-q}{2})}.
\end{aligned} \tag{150}$$

Here, for the calculation of the u-integral, we have assumed that $\mathrm{Re}(6s+l-q) > 0$.—Now assume that $\underline{D \equiv 3 \pmod 4}$. In this case we choose

$$S(-D) = \begin{bmatrix} \frac{1+D}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{D}{4} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Let $T(\mathbb{R})$, $R(\mathbb{R})$, η, B be the objects defined with this $\left[\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right] = \left[\begin{smallmatrix} \frac{1+D}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{smallmatrix}\right]$, and let $\tilde{T}(\mathbb{R})$, $\tilde{R}(\mathbb{R})$, $\tilde{\eta}$, $\tilde{B}$ be the corresponding objects defined with $\left[\begin{smallmatrix} \tilde{a} & \tilde{b}/2 \\ \tilde{b}/2 & \tilde{c} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \frac{D}{4} & \\ & 1 \end{smallmatrix}\right]$. Let

$$h_0 = \begin{bmatrix} 1 & & & \\ -\frac{1}{2} & 1 & & \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{bmatrix}.$$

Then

$$T^1(\mathbb{R}) = h_0 \tilde{T}^1(\mathbb{R}) h_0^{-1}, \qquad T(\mathbb{R}) = h_0 \tilde{T}(\mathbb{R}) h_0^{-1}, \qquad R(\mathbb{R}) = h_0 \tilde{R}(\mathbb{R}) h_0^{-1}.$$

Furthermore, $\eta = \tilde{\eta} h_0^{-1}$. The integral (145) becomes

$$
\begin{aligned}
Z_\infty(s) &= \int\limits_{R(\mathbb{R})\backslash H(\mathbb{R})} W^{\#}(\eta h, s) B(h)\, dh \\
&= \int\limits_{h_0 \tilde{R}(\mathbb{R}) h_0^{-1}\backslash H(\mathbb{R})} W^{\#}\big(\tilde{\eta} h_0^{-1} h, s\big) B\big(h_0 h_0^{-1} h\big)\, dh \\
&= \int\limits_{h_0 \tilde{R}(\mathbb{R}) h_0^{-1}\backslash H(\mathbb{R})} W^{\#}\big(\tilde{\eta} h_0^{-1} h h_0, s\big) B\big(h_0 h_0^{-1} h h_0\big)\, dh \\
&= \int\limits_{\tilde{R}(\mathbb{R})\backslash H(\mathbb{R})} W^{\#}(\tilde{\eta} h, s) B(h_0 h)\, dh \\
&= \int\limits_{\tilde{R}(\mathbb{R})\backslash H(\mathbb{R})} W^{\#}(\tilde{\eta} h, s) \tilde{B}(h)\, dh.
\end{aligned}
$$

This integral can be computed just like the one in the case $D \equiv 0 \bmod 4$, and we get the exactly same answer as in (150). We proved the following.

Theorem 4.4.1. *Let l and D be positive integers such that $D \equiv 0, 3 \bmod 4$. Let $S(-D) = \left[\begin{smallmatrix} D/4 & \\ & 1 \end{smallmatrix}\right]$ if $D \equiv 0 \bmod 4$ and $S(-D) = \left[\begin{smallmatrix} (1+D)/4 & 1/2 \\ 1/2 & 1 \end{smallmatrix}\right]$ if $D \equiv 3 \bmod 4$. Let $B : \mathrm{GSp}_4(\mathbb{R}) \to \mathbb{C}$ be the function defined in* (131)*, and let $W^{\#}(\cdot, s)$ be the function defined in* (141)*. Then, for $\mathrm{Re}(6s + l - q) > 0$,*

$$
\begin{aligned}
Z_\infty(s) &:= \int\limits_{R(\mathbb{R})\backslash H(\mathbb{R})} W^{\#}(\eta h, s) B(h)\, dh \\
&= \frac{a^+}{2} \pi D^{-3s-\frac{l}{2}+\frac{q}{2}} (4\pi)^{-3s+\frac{3}{2}-l+q} \frac{\Gamma(3s+l-1+\frac{ir}{2}-\frac{q}{2})\Gamma(3s+l-1-\frac{ir}{2}-\frac{q}{2})}{\Gamma(3s+\frac{l+1-q}{2})}.
\end{aligned} \tag{151}
$$

Here, $q \in \mathbb{C}$ is related to the central character of τ via $\omega_\tau(y) = y^q$ for $y > 0$. The number $r \in \mathbb{C}$ is such that (138) *holds.*

We will state two special cases of formula (151). First assume that $\tau = \chi_1 \times \chi_2$, an irreducible principal series representation of $\mathrm{GL}_2(\mathbb{R})$, where χ_1 and χ_2 are characters of $\mathbb{R}^\times$. Let $\varepsilon_i \in \{0, 1\}$ and $s_i \in \mathbb{C}$ be such that $\chi_i(x) = \mathrm{sgn}(x)^{\varepsilon_i} |x|^{s_i}$, for $i = 1, 2$. Then Δ acts on τ by multiplication with $-\frac{1}{4}(1 - (s_1 - s_2)^2)$. Comparing with (138), we get $(s_1 - s_2)^2 = -r^2$, so that $ir = \pm(s_1 - s_2)$. Furthermore, $q = s_1 + s_2$. Therefore,

$$
Z_\infty(s) = \frac{a^+}{2} \pi D^{-3s-\frac{l}{2}+\frac{s_1+s_2}{2}} (4\pi)^{-3s+\frac{3}{2}-l+s_1+s_2} \frac{\Gamma(3s+l-1-s_1)\Gamma(3s+l-1-s_2)}{\Gamma(3s+\frac{l+1-s_1-s_2}{2})}. \tag{152}
$$

Now assume that l_1 is a positive integer, that $q \in \mathbb{C}$, and that $\tau = \mathcal{D}_q(l_1)$, the discrete series (or limit of discrete series) representation of $\mathrm{GL}_2(\mathbb{R})$ with a lowest weight vector of weight l_1 for which the central element $Z = \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right]$ acts by multiplication with q. Then $ir = \pm(l_1 - 1)$, so that, from (151),

$$
Z_\infty(s) = \frac{a^+}{2} \pi D^{-3s-\frac{l}{2}+\frac{q}{2}} (4\pi)^{-3s+\frac{3}{2}-l+q} \frac{\Gamma(3s+l-1+\frac{l_1-1}{2}-\frac{q}{2})\Gamma(3s+l-1-\frac{l_1-1}{2}-\frac{q}{2})}{\Gamma(3s+\frac{l+1-q}{2})}. \tag{153}
$$

5. Modular forms

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. In this section we will consider a cuspidal, automorphic representation π of $\mathrm{GSp}_4(\mathbb{A})$, obtained from a Siegel cusp form, and a cuspidal, automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$, obtained from a Maaß form. Using the local calculations from the previous sections, we will obtain an integral formula for the L-function $L(s,\pi\times\tau)$.

Given a quadratic field extension $L/\mathbb{Q}$, we define the groups $G=\mathrm{GU}(2,2)$, $H=\mathrm{GSp}_4$, $P=MN$ and $R=TU$ as in Sections 2.1 and 2.2, but now considered as algebraic groups over $\mathbb{Q}$.

5.1. Siegel modular forms and Bessel models

Let $\Gamma_2=\mathrm{Sp}_4(\mathbb{Z})$. For a positive integer l we denote by $S_l(\Gamma_2)$ the space of Siegel cusp forms of degree 2 and weight l with respect to Γ_2. If $\Phi\in S_l(\Gamma_2)$ then Φ satisfies

$$\Phi\big(\gamma\langle Z\rangle\big)=\det\big(J(\gamma,Z)\big)^l\Phi(Z),\quad \gamma\in\Gamma_2,\ Z\in\mathfrak{h}_2.$$

Let us assume that $\Phi\in S_l(\Gamma_2)$ is a Hecke eigenform. It has a Fourier expansion

$$\Phi(Z)=\sum_{S>0}a(S,\Phi)e^{2\pi i\,\mathrm{tr}(SZ)},$$

where S runs through all symmetric semi-integral positive definite matrices of size two. We shall make the following two assumptions about the function Φ.

Assumption 1. $a(S,\Phi)\neq 0$ for some $S=\left[\begin{smallmatrix}a&b/2\\b/2&c\end{smallmatrix}\right]$ such that $b^2-4ac=-D<0$, where $-D$ is the discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$.

Assumption 2. The weight l is a multiple of $w(-D)$, the number of roots of unity in $\mathbb{Q}(\sqrt{-D})$. We have

$$w(-D)=\begin{cases}4 & \text{if } D=4,\\ 6 & \text{if } D=3,\\ 2 & \text{otherwise.}\end{cases}$$

We define a function $\phi=\phi_\Phi$ on $H(\mathbb{A})=\mathrm{GSp}_4(\mathbb{A})$ by

$$\phi(\gamma h_\infty k_0)=\mu_2(h_\infty)^l\det\big(J(h_\infty,I)\big)^{-l}\Phi\big(h_\infty\langle I\rangle\big),\tag{154}$$

where $\gamma\in H(\mathbb{Q})$, $h_\infty\in H^+(\mathbb{R})$, $k_0\in\prod_{p<\infty}H(\mathbb{Z}_p)$. Here $I=\left[\begin{smallmatrix}i&\\&i\end{smallmatrix}\right]$. Note that ϕ is invariant under the center $Z_H(\mathbb{A})$ of $H(\mathbb{A})$. It can be shown (see [1, p. 186]) that the function ϕ_Φ is a cuspidal automorphic form. Let V_Φ be the automorphic representation generated by ϕ_Φ. This representation may not be irreducible, but decomposes into a direct sum of finitely many irreducible, cuspidal, automorphic representations of $H(\mathbb{A})$. Let π_Φ be one of these irreducible components, and write π_Φ as a restricted tensor product $\pi_\Phi\cong\bigotimes_p'\pi_p$, where π_p is an irreducible, admissible, unitarizable representation of $H(\mathbb{Q}_p)$. Since ϕ_Φ is $H(\mathbb{Z}_p)$-invariant for all finite primes p, the representation π_p has a non-zero, essentially unique $H(\mathbb{Z}_p)$-invariant vector. The same calculations as in [1] show that the equivalence class of π_p depends only on Φ and not on the chosen global irreducible component π_Φ.

Let $\psi=\prod_p\psi_p$ be a character of $\mathbb{Q}\setminus\mathbb{A}$ which is unramified at every finite prime and such that $\psi_\infty(x)=e^{-2\pi ix}$ for $x\in\mathbb{R}$. Let

$$S(-D)=\begin{cases}\begin{bmatrix}\frac{D}{4}&0\\0&1\end{bmatrix} & \text{if } D\equiv 0 \pmod 4,\\ \begin{bmatrix}\frac{1+D}{4}&\frac12\\\frac12&1\end{bmatrix} & \text{if } D\equiv 3 \pmod 4.\end{cases}\tag{155}$$

Our quadratic extension is $L=\mathbb{Q}(\sqrt{-D})$. We have $T(\mathbb{Q})\simeq\mathbb{Q}(\sqrt{-D})^{\times}$. Let Λ be an ideal class character of $\mathbb{Q}(\sqrt{-D})$, i.e., a character of

$$T(\mathbb{A})/T(\mathbb{Q})T(\mathbb{R})\prod_{p<\infty}\bigl(T(\mathbb{Q}_p)\cap\mathrm{GL}_2(\mathbb{Z}_p)\bigr),$$

to be specified further below. We define the global Bessel function of type (Λ,θ) associated to $\bar{\phi}$ by

$$B_{\bar{\phi}}(h)=\int\limits_{Z_H(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})}(\Lambda\otimes\theta)(r)^{-1}\bar{\phi}(rh)\,dr, \tag{156}$$

where $\theta(\left[\begin{smallmatrix}1&X\\&1\end{smallmatrix}\right])=\psi(\mathrm{tr}(S(-D)X))$ and $\bar{\phi}(h)=\overline{\phi(h)}$. For a finite prime p, the function $B_p(h_p):=B_{\bar{\phi}}(h_p)$, with $h_p\in H(\mathbb{Q}_p)$, is in the Bessel model for the contragradient representation $\tilde{\pi}_p$ with respect to the character $\Lambda_p\otimes\theta_p$ of $R(\mathbb{Q}_p)$. From uniqueness of Bessel models for GSp_4 (see [19]) we conclude

$$B_{\bar{\phi}}(h)=B_{\bar{\phi}}(h_\infty)\prod_{p<\infty}B_p(h_p), \tag{157}$$

where $h=\bigotimes h_p$. From [32, (1-17), (1-19), (1-26)], we have, for $h_\infty\in H^+(\mathbb{R})$,

$$B_{\bar{\phi}}(h_\infty)=\bigl|\mu_2(h_\infty)\bigr|^l\,\overline{\det\bigl(J(h_\infty,I)\bigr)^{-l}e^{-2\pi i\,\mathrm{tr}(S(-D)\overline{h_\infty\langle I\rangle})}}\sum_{j=1}^{h(-D)}\Lambda(t_j)^{-1}\overline{a(S_j,\Phi)}, \tag{158}$$

and $B_{\bar{\phi}}(h_\infty)=0$ for $h_\infty\notin H^+(\mathbb{R})$. Here, $h(-D)$ is the class number of $\mathbb{Q}(\sqrt{-D})$, the elements t_j, $j=1,\dots,h(-D)$, are representatives of the ideal classes of $\mathbb{Q}(\sqrt{-D})$ and S_j, $j=1,\dots,h(-D)$, are the representatives of $\mathrm{SL}_2(\mathbb{Z})$ equivalent classes of primitive semi-integral positive definite matrices of discriminant $-D$ corresponding to t_j. Thus, by Assumption 1, there exists a Λ such that $B_{\bar{\phi}}(I_4)\neq 0$. We fix such a Λ. Note that $B_{\bar{\phi}}(h_\infty)$ is a non-zero constant multiple of (131). Let us abbreviate $a(\Lambda)=\sum_{j=1}^{h(-D)}\Lambda(t_j)a(S_j,\Phi)$.

5.2. Maaß forms and Eisenstein series

Let $\mathfrak{h}_1=\{z=x+iy\in\mathbb{C}:\ y>0\}$ be the complex upper half-plane. Let $N=\prod_{p\mid N}p^{n_p}$ be a positive integer, and $\Gamma_0(N)=\{\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in\mathrm{SL}_2(\mathbb{Z}):\ N\mid c\}$. A smooth function $f:\mathfrak{h}_1\to\mathbb{C}$ is called a Maaß cusp form of weight l_1 with respect to $\Gamma_0(N)$ if

(i) For every $\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in\Gamma_0(N)$ and $z\in\mathfrak{h}_1$ we have

$$f\left(\frac{az+b}{cz+d}\right)=\left(\frac{cz+d}{|cz+d|}\right)^{l_1}f(z).$$

(ii) f is an eigenfunction of Δ_{l_1}, where

$$\Delta_{l_1}=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-il_1y\frac{\partial}{\partial x}.$$

(iii) f vanishes at the cusps of $\Gamma_0(N)$.

We denote the space of Maaß cusp forms of weight l_1 with respect to $\Gamma_0(N)$ by $S^M_{l_1}(N)$. A function $f \in S^M_{l_1}(N)$ has the Fourier expansion

$$f(x+iy) = \sum_{n\neq 0} a_n W_{\operatorname{sgn}(n)\frac{l_1}{2},\frac{ir}{2}}(4\pi|n|y)e^{2\pi inx}, \tag{159}$$

where $W_{\nu,\mu}$ is a classical Whittaker function (the same function as in (139)) and $(\Delta_{l_1}+\lambda)f = 0$ with $\lambda = 1/4+(r/2)^2$. Let $f \in S^M_{l_1}(N)$ be a Hecke eigenform.

If $ir/2 = (l_2-1)/2$ for some integer $l_2 > 0$, then the cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ constructed below is holomorphic at infinity of lowest weight l_2. In this case we make the additional assumptions that $l_2 \leqslant l$ and $l_2 \leqslant l_1$, where l is the weight of the Siegel cusp form Φ from the previous section.

Starting from f, we obtain another Maaß form $f_l \in S^M_l(N)$ by applying the raising and lowering operators. The raising operator R_* maps $S^M_*(N)$ to $S^M_{*+2}(N)$ and the lowering operator L_* maps $S^M_*(N)$ to $S^M_{*-2}(N)$; for more details on these operators, see [23, p. 3925]. In particular, we have

$$f_l = \begin{cases} R_{l-2}R_{l-4}\cdots R_{l_1+2}R_{l_1}f & \text{if } l_1 < l,\\ f & \text{if } l_1 = l,\\ L_{l+2}L_{l+4}\cdots L_{l_1-2}L_{l_1}f & \text{if } l_1 > l. \end{cases} \tag{160}$$

Note that, by Assumption 2 on the Siegel cusp form Φ, the weight l is always even. Also, $S^M_{l_1}(N)$ is empty if l_1 is odd. Hence (160) makes sense. If $ir/2 = (l_2-1)/2$, then the assumption $l_2 \leqslant l$ guarantees that $f_l \neq 0$. Suppose $\{c(n)\}$ are the Fourier coefficients of f_l. In later calculations we will need $c(1)$. By [23, Lemma 2.5],

$$c(1) = \begin{cases} a_1 & \text{if } l_1 \leqslant l,\\ \prod^{l_1}_{\substack{t=l+2\\ t\equiv l \pmod 2}} \left(\frac{ir}{2}+\frac{1}{2}-\frac{t}{2}\right)\left(\frac{ir}{2}-\frac{1}{2}+\frac{t}{2}\right)a_1 & \text{if } l_1 > l. \end{cases} \tag{161}$$

Define a function $\hat f$ on $\mathrm{GL}_2(\mathbb{A})$ by

$$\hat f(\gamma_0 m k_0) = \left(\frac{\gamma i+\delta}{|\gamma i+\delta|}\right)^{-l} f_l\left(\frac{\alpha i+\beta}{\gamma i+\delta}\right), \tag{162}$$

where $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$, $m = \left[\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$, $k_0 \in \prod_{p|N} K^{(1)}(\mathfrak{p}^{n_p})\prod_{p\nmid N}\mathrm{GL}_2(\mathbb{Z}_p)$. Here, $N = \prod_{p|N} p^{n_p}$ and $K^{(1)}(\mathfrak{p}^m) = \mathrm{GL}_2(\mathbb{Q}_p) \cap \left[\begin{smallmatrix}\mathbb{Z}_p^\times & \mathbb{Z}_p\\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times\end{smallmatrix}\right]$, as in (32). Then $\hat f$ satisfies

$$\hat f\big(gr(\theta)\big) = e^{il\theta}\hat f(g), \qquad g \in \mathrm{GL}_2(\mathbb{A}), \quad r(\theta) = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}. \tag{163}$$

Let (τ_f, V_f) be the cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by $\hat f$. By strong multiplicity one, τ_f is irreducible. Note that τ_f has trivial central character. Write τ_f as a restricted tensor product $\tau_f = \bigotimes_p' \tau_p$. If $p \nmid N$ is a finite prime, then τ_p is an irreducible, admissible, unramified representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. If $p|N$, then τ_p is an irreducible, admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with conductor $\mathfrak{p}^{n_p}$, where $\mathfrak{p} = p\mathbb{Z}_p$ and $n_p = \nu_p(N)$. Let

$$W^{(0)}(g) := \int\limits_{\mathbb{Q}\backslash\mathbb{A}} \hat f\left(\begin{bmatrix}1 & x\\ & 1\end{bmatrix}g\right)\psi(x)\,dx,$$

where ψ is the additive character fixed in the previous section. Then $W^{(0)}$ is in the Whittaker model of τ_f with respect to the character ψ^{-1}. By (163),

$$W^{(0)}\big(gr(\theta)\big) = e^{il\theta} W^{(0)}(g), \qquad g \in \mathrm{GL}_2(\mathbb{A}), \quad r(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{164}$$

For any finite prime p, the function $W_p(g_p) := W^{(0)}(g_p)$, for $g_p \in \mathrm{GL}_2(\mathbb{Q}_p)$, is in the Whittaker model of τ_p. By the uniqueness of Whittaker models for GL_2, we get

$$W^{(0)}(g) = W^{(0)}(g_\infty) \prod_{p<\infty} W_p(g_p)$$

for $g = \bigotimes g_p$. Using the definition (162) for $\hat{f}$ we get, for $t \in \mathbb{R}^\times$,

$$W^{(0)}\left(\begin{bmatrix} t & \\ & 1 \end{bmatrix}\right) = \begin{cases} c(1) W_{\frac{l}{2}, \frac{ir}{2}}(4\pi t) & \text{if } t > 0, \\ c(-1) W_{-\frac{l}{2}, \frac{ir}{2}}(-4\pi t) & \text{if } t < 0. \end{cases} \tag{165}$$

We want to extend $\hat{f}$ to a function on $\mathrm{GU}(1,1;L)(\mathbb{A})$. For this, we need to construct a suitable character χ_0 on $L^\times \setminus \mathbb{A}_L^\times$.

Lemma 5.2.1. *Let S be a divisible group, i.e., a group with the property that $S = \{s^n\colon s \in S\}$ for all positive integers n. Let A and B be abelian groups, and assume that B is finite. Then every exact sequence*

$$1 \longrightarrow S \longrightarrow A \longrightarrow B \longrightarrow 1$$

splits.

Proof. Write B as a product of cyclic groups $\langle b_i\rangle$. Choose pre-images a_i of b_i in A. Modifying a_i by suitable elements of S, we may assume that a_i has the same order as b_i. Then the group generated by all a_i is isomorphic to B. $\square$

Lemma 5.2.2. *Let $L = \mathbb{Q}(\sqrt{-D})$ with $D > 0$ be an imaginary quadratic number field. Let $\mathbb{A}_L^\times$ be the group of ideles of L. Let K_f be the subgroup given by $\prod_{v<\infty} \mathfrak{o}_{L,v}^\times$, where v runs over all finite places of L, and $\mathfrak{o}_{L,v}$ is the ring of integers in the completion of L at v. The archimedean component of $\mathbb{A}_L^\times$ is isomorphic to $\mathbb{C}^\times = \mathbb{R}_{>0} \times S^1$, where S^1 is the unit circle. Let $l \in \mathbb{Z}$ be a multiple of $w(-D)$, the number of roots of unity in L. Then there exists a character χ_0 of $\mathbb{A}_L^\times$ with the properties*

(i) *χ_0 is trivial on $\mathbb{A}_\mathbb{Q}^\times K_f L^\times$; and*
(ii) *$\chi_0(\zeta) = \zeta^{-l}$ for all $\zeta \in S^1$.*

Proof. First note that $\mathbb{A}_\mathbb{Q}^\times K_f L^\times = \mathbb{R}_{>0} K_f L^\times$. There is an exact sequence

$$1 \longrightarrow W \setminus S^1 \longrightarrow \mathbb{R}_{>0} K_f L^\times \setminus \mathbb{A}_L^\times \longrightarrow \mathbb{C}^\times K_f L^\times \setminus \mathbb{A}_L^\times \longrightarrow 1,$$

where W is the group of roots of unity in L. The group on the right is the ideal class group of L. By Lemma 5.2.1,

$$\mathbb{R}_{>0} K_f L^\times \setminus \mathbb{A}_L^\times \cong \big(W \setminus S^1\big) \times \big(\mathbb{C}^\times K_f L^\times \setminus \mathbb{A}_L^\times\big).$$

By hypothesis, the map $S^1 \ni \zeta \mapsto \zeta^l$ factors through $W \setminus S^1$. The assertion follows. $\square$

Let χ_0 be a character of $\mathbb{A}_L^\times$ as in Lemma 5.2.2 (observe our Assumption 2 above). We extend $\hat{f}$ to $\mathrm{GU}(1,1;L)(\mathbb{A})$ by

$$\hat{f}(\zeta g) = \chi_0(\zeta)\hat{f}(g) \quad \text{for } \zeta \in \mathbb{A}_L^\times,\ g \in \mathrm{GL}_2(\mathbb{A}). \tag{166}$$

Since l is even, this is well-defined; see (123) and (163). Let χ be the character of $L^\times \setminus \mathbb{A}_L^\times$ given by $\chi(\zeta) = \Lambda(\bar{\zeta})^{-1}\chi_0(\bar{\zeta})^{-1}$. Let $K_G^\#(N)$ be the compact subgroup $\prod_{p|N} K^\#(\mathfrak{P}^{n_p})\prod_{p\nmid N} K^\#(\mathfrak{P}^0)$ of $\mathrm{GU}(2,2;L)(\mathbb{A})$, where $K^\#(\mathfrak{P}^n)$ is as defined in (36). For a complex variable s, let us define a function $f_\Lambda(\cdot,s)$ on $\mathrm{GU}(2,2;L)(\mathbb{A})$ by

(i) $f_\Lambda(g,s) = 0$ if $g \notin M(\mathbb{A})N(\mathbb{A})K_\infty K_G^\#(N)$.
(ii) If $m = m_1m_2$, $m_i \in M^{(i)}(\mathbb{A})$, $n \in N(\mathbb{A})$, $k = k_0k_\infty$, $k_0 \in K_G^\#(N)$, $k_\infty \in K_\infty$, then

$$f_\Lambda(mnk,s) = \delta_P^{\frac{1}{2}+s}(m)\chi(m_1)\hat{f}(m_2)\det\bigl(J(k_\infty, I)\bigr)^{-l}. \tag{167}$$

Recall from (10) that $\delta_P(m_1m_2) = |N_{L/\mathbb{Q}}(m_1)\mu_1(m_2)^{-1}|^3$.

Here, $M^{(1)}(\mathbb{A})$, $M^{(2)}(\mathbb{A})$, $N(\mathbb{A})$ are the adelic points of the algebraic groups defined by (6), (7) and (8) and K_∞ is as defined in Section 4.1. In fact, f_Λ is a section in the representation $I(s,\chi,\chi_0,\tau)$ of $\mathrm{GU}(2,2;L)(\mathbb{A})$ obtained by parabolic induction from P; see Section 2.3.

Let us define the Eisenstein series on $\mathrm{GU}(2,2;L)(\mathbb{A})$ by

$$E_\Lambda(g,s) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_\Lambda(\gamma g, s). \tag{168}$$

This series is absolutely convergent for $\mathrm{Re}(s) > 1/2$, uniformly convergent in compact subdomains and has a meromorphic continuation to the whole complex plane; see [15].

Remark. Note that our definition (167) differs from the formula for f_Λ given on p. 209 of [9]. In fact, the function f_Λ in [9] is not well-defined, since there is a non-trivial overlap between $M^{(2)}(\mathbb{R})$ and K_∞. It is necessary to extend the function $\hat{f}$ to $\mathrm{GU}(1,1;L)(\mathbb{A})$ using the character χ_0 as in (166), not the trivial character.

5.3. *Global integral and L-functions*

Let ϕ be as in (154). Let $f_\Lambda(\cdot,s)$ and $E_\Lambda(\cdot,s)$ be as in the previous section. We shall evaluate the global integral

$$Z(s,\Lambda) = \int\limits_{Z_H(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} E_\Lambda(h,s)\bar{\phi}(h)\,dh. \tag{169}$$

In Theorem 2.4 of [9], the following basic identity has been proved.

$$Z(s,\Lambda) = \int\limits_{R(\mathbb{A})\backslash H(\mathbb{A})} W_\Lambda(\eta h,s)B_{\bar{\phi}}(h)\,dh, \tag{170}$$

where

$$W_\Lambda(g,s)=\int\limits_{\mathbb{Q}\backslash\mathbb{A}} f_\Lambda\left(\begin{bmatrix}1&&&\\&1&&x\\&&1&\\&&&1\end{bmatrix}g,s\right)\psi(x)\,dx,\qquad \eta=\begin{bmatrix}1&0&&\\ \alpha&1&&\\&&1&-\bar\alpha\\&&0&1\end{bmatrix},$$

$$\alpha=\frac{b+\sqrt{-D}}{2},\tag{171}$$

and $B_{\bar\phi}$ is as defined in (156). Note that the value of b above depends on the choice of $S(-D)$ in (155). For the choice of f_Λ in the previous section, we get

$$W_\Lambda(g,s)=W_\infty(g_\infty,s)\prod_{p<\infty}W_p(g_p,s),$$

where W_p is the function $W^\#$ defined in Section 3.4. For $g_\infty\in G(\mathbb{R})$, the function $W_\infty(g_\infty,s)$ is exactly the function $W^\#$ from (141). Note that, in this case, the values of a^+,a^- in (139) are given by $a^+=c(1)$ and $a^-=c(-1)$. From the basic identity (170) we therefore have

$$Z(s,\Lambda)=\prod_{p\leqslant\infty}Z_p(s),\qquad Z_p(s)=\int\limits_{R(\mathbb{Q}_p)\backslash H(\mathbb{Q}_p)}W_p(\eta h_p,s)B_p(h_p)\,dh_p.$$

Here, B_∞ is the function given in (158). If p is a finite prime such that $p\nmid N$, then all the local data satisfies the hypothesis of Theorem 3.7 from [9], where the corresponding local integral is computed. For $p\mid N$, we apply Theorems 3.8.1, 3.8.2, and for the archimedean integral we apply Theorem 4.4.1. We obtain the following integral representation.

Theorem 5.3.1. *Let $\Phi\in S_l(\Gamma_2)$ be a cuspidal Siegel eigenform of degree 2 and even weight l satisfying the two assumptions from Section 5.1. Let $L=\mathbb{Q}(\sqrt{-D})$, where D is as in Assumption 1. Let $N=\prod p^{n_p}$ be a positive integer. Let f be a Maaß Hecke eigenform of weight $l_1\in\mathbb{Z}$ with respect to $\Gamma_0(N)$. If f lies in a holomorphic discrete series with lowest weight l_2, then assume that $l_2\leqslant l$. Then the integral (169) is given by*

$$Z(s,\Lambda)=\kappa_\infty(s)\kappa_N(s)\frac{L(3s+\frac12,\pi_\Phi\times\tau_f)}{\zeta(6s+1)L(3s+1,\tau_f\times\mathcal{AI}(\Lambda))},\tag{172}$$

where ζ is the Riemann zeta function, $\kappa_N(s)=\prod_{p|N}\kappa_p(s)$ with

$$\kappa_p(s)=\begin{cases}\frac{p(p-1)}{(p+1)(p^4-1)}\big(1-\big(\frac{L}{p}\big)p^{-1}\big)(1-p^{-6s-1})^{-1} & \text{if } n_p=1,\\ \frac{p^{n_p}(p-1)}{p^{3(n_p-1)}(p+1)(p^4-1)}\big(1-\big(\frac{L}{p}\big)p^{-1}\big)\frac{L_p(3s+1,\tau_p\times\mathcal{AI}(\Lambda_p))}{1-p^{-6s-1}} & \text{if } n_p\geqslant2,\end{cases}$$

and

$$\kappa_\infty(s)=\frac12\overline{a(\Lambda)}c(1)\pi D^{-3s-\frac{l}{2}}(4\pi)^{-3s+\frac32-l}\frac{\Gamma(3s+l-1+\frac{ir}{2})\Gamma(3s+l-1-\frac{ir}{2})}{\Gamma(3s+\frac{l+1}{2})}.$$

Here, the non-zero constant $c(1)$ is given by (161), the non-zero constant $a(\Lambda)$ is defined at the end of Section 5.1, and

$$\left(\frac{L}{p}\right)=\begin{cases}-1 & \text{if } p \text{ is inert in } L,\\ 0 & \text{if } p \text{ ramifies in } L,\\ 1 & \text{if } p \text{ splits in } L.\end{cases}$$

The quantity $\frac{ir}{2}$ is as in (159).

5.4. The special value

In this section, we will apply Theorem 5.3.1 to a special case—when f, from the previous section, is a holomorphic cusp form of the same weight l as the Siegel cusp form Φ—to obtain a special L-value result. This result fits into the general conjecture of Deligne on special values of L-functions.

Let $\Psi(z) = \sum_{n>0} b_n e^{2\pi i n z}$ be a holomorphic cuspidal eigenform of weight l with respect to $\Gamma_0(N)$. Here, l is the same as the weight of the Siegel modular form Φ from Section 5.1 and $N = \prod_{p|N} p^{n_p}$ is a positive integer. Let us normalize Ψ so that $b_1 = 1$. The function f_Ψ defined by $f_\Psi(z) = y^{l/2}\Psi(z)$ is a Maaß form in $S_l^M(N)$. Let $\{c(n)\}$ be its Fourier coefficients; see (159). It follows from the formula $W_{\mu+1/2,\mu}(z) = e^{-z/2} z^{\mu+1/2}$ for the Whittaker function that

$$c(n) = \begin{cases} b_n(4\pi n)^{-l/2} & \text{if } n > 0, \\ 0 & \text{if } n < 0. \end{cases} \tag{173}$$

From (162), we have

$$\hat{f}_\Psi(\gamma_0 m k_0) = \left(\frac{\gamma i + \delta}{|\gamma i + \delta|}\right)^{-l} f_\Psi\left(\frac{\alpha i + \beta}{\gamma i + \delta}\right) = \frac{\det(m)^{l/2}}{(\gamma i + \delta)^l} \Psi\left(\frac{\alpha i + \beta}{\gamma i + \delta}\right),$$

where $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$, $m = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$, $k_0 \in \prod_{p|N} K^{(1)}(\mathfrak{p}^{n_p}) \prod_{p\nmid N} \mathrm{GL}_2(\mathbb{Z}_p)$. Let us denote z_{22} by Z^* for $Z = \left[\begin{smallmatrix} * & * \\ * & z_{22} \end{smallmatrix}\right] \in \mathbb{H}_2$. Let us set $\hat{Z} = \frac{i}{2}({}^t\overline{Z} - Z)$ for $Z \in \mathbb{H}_2$. Let $\mathrm{Im}(z)$ denote the imaginary part of a complex number z. Let f_Λ be as defined in (167) and $I = \left[\begin{smallmatrix} i & \\ & i \end{smallmatrix}\right] \in \mathbb{H}_2$.

Lemma 5.4.1. *For $g \in G^+(\mathbb{R})$, we have*

$$f_\Lambda(g, s) = \mu_2(g)^l \det\bigl(J(g, I)\bigr)^{-l} \left(\frac{\det \widehat{g\langle I\rangle}}{\mathrm{Im}(g\langle I\rangle)^*}\right)^{3s+\frac{3}{2}-\frac{l}{2}} \Psi\bigl(\bigl(g\langle I\rangle\bigr)^*\bigr). \tag{174}$$

Proof. For $g \in G^+(\mathbb{R})$ and $Z \in \mathbb{H}_2$ we have $\widehat{g\langle Z\rangle} = \mu_2(g)\,{}^t\overline{J(g, Z)}^{-1} \hat{Z} J(g, Z)^{-1}$. This implies that $\det(\widehat{g\langle I\rangle}) = \mu_2(g)^2 |\det(J(g, I))|^{-2} \det(\hat{I}) = \mu_2(g)^2 |\det(J(g, I))|^{-2}$. It follows from (121) that we can write the element $g \in G^+(\mathbb{R})$ as

$$g = \begin{bmatrix} \zeta & & & \\ & 1 & & \\ & & \zeta^{-1} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \alpha & & \beta \\ & & \mu & \\ & \gamma & & \delta \end{bmatrix} \begin{bmatrix} 1 & x & x\bar{y} + w & y \\ & 1 & \bar{y} & \\ & & 1 & \\ & & -\bar{x} & 1 \end{bmatrix} k,$$

where $\zeta \in \mathbb{R}^\times$, $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$, $x, y \in \mathbb{C}$, $w \in \mathbb{R}$ and $k \in K_\infty$. Then we have

$$\det\bigl(J(g, I)\bigr) = \zeta^{-1}\mu(\gamma i + \delta)\det\bigl(J(k, I)\bigr) \quad \text{and} \quad \bigl(g\langle I\rangle\bigr)^* = \frac{\alpha i + \beta}{\gamma i + \delta}.$$

Hence, the right-hand side of (174) is equal to

$$\mu^l\bigl(\zeta^{-1}\mu(\gamma i + \delta)\det\bigl(J(k, I)\bigr)\bigr)^{-l} \left(\frac{\mu^2|\zeta^{-1}\mu(\gamma i + \delta)\det(J(k, I))|^{-2}}{\mu|\gamma i + \delta|^{-2}}\right)^{3s+\frac{3}{2}-\frac{l}{2}} \Psi\left(\frac{\alpha i + \beta}{\gamma i + \delta}\right).$$

Using the fact that $|\det(J(k, I))|^{-2} = \det(\widehat{k\langle I\rangle}) = 1$, we get the lemma. $\square$

Remark. Eq. (4.4.2) of [9] claims that, for $g \in G^+(\mathbb{R})$, the function $f_\Lambda(g,s)$ satisfies a formula different from (174). In this formula, the term $\det(\mathrm{Im}(g\langle I\rangle))$ replaces the term $\det(\widehat{g\langle I\rangle})$ from (174). Note that $\mathrm{Im}(Z) = \frac{i}{2}(\overline{Z} - Z)$ for $Z \in \mathbb{H}_2$. One easily checks that the resulting function is not invariant under $N(\mathbb{R})$ and hence cannot equal f_Λ, as defined in (167). If one replaces Eq. (4.4.2) in [9] by (174), the subsequent arguments in [9] remain valid.

Let E_Λ be the Eisenstein series defined in (168). From the above lemma, we see that, for $g \in G^+(\mathbb{R})$, $\mu_2(g)^{-l}\det(J(g,I))^l E_\Lambda(g,s)$ only depends on $Z = g\langle I\rangle$. Hence, we can define a function $\mathcal{E}_\Lambda$ on $\mathbb{H}_2$ by the formula

$$\mathcal{E}_\Lambda(Z,s) = \mu_2(g)^{-l}\det\big(J(g,I)\big)^l E_\Lambda\left(g, \frac{s}{3} + \frac{l}{6} - \frac{1}{2}\right), \tag{175}$$

where $g \in G^+(\mathbb{R})$ is such that $g\langle I\rangle = Z$. The series that defines $\mathcal{E}_\Lambda(Z,s)$ is absolutely convergent for $\mathrm{Re}(s) > 3 - l/2$ (see [14]). Since $l \geqslant 12$, we can set $s = 0$ and obtain a holomorphic Eisenstein series $\mathcal{E}_\Lambda(Z,0)$ on $\mathbb{H}_2$. For a finite place p of $\mathbb{Q}$ recall the local congruence subgroups $K^\#(\mathfrak{P}^n) \subset G(\mathbb{Z}_p)$ and $K^\#(\mathfrak{p}^n) = K^\#(\mathfrak{P}^n) \cap H(\mathbb{Z}_p)$ defined in (36) resp. (37). For $N = \prod p^{n_p}$, we let

$$\Gamma_G^\#(N) = G(\mathbb{Q}) \cap G(\mathbb{R})^+ K_G^\#(N), \qquad K_G^\#(N) = \prod_{p|N} K^\#\big(\mathfrak{P}^{n_p}\big) \prod_{p\nmid N} K^\#\big(\mathfrak{P}^0\big),$$

and

$$\Gamma^\#(N) = H(\mathbb{Q}) \cap H(\mathbb{R})^+ K^\#(N), \qquad K^\#(N) = \prod_{p|N} K^\#\big(\mathfrak{p}^{n_p}\big) \prod_{p\nmid N} K^\#\big(\mathfrak{p}^0\big).$$

Explicitly,

$$\Gamma^\#(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N'\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}, \quad \text{where } N' \text{ is the square-free part of } N.$$

Then $\mathcal{E}_\Lambda(Z,0)$ is a modular form of weight l with respect to $\Gamma_G^\#(N)$. Its restriction to $\mathfrak{h}_2$ is a modular form of weight l with respect to $\Gamma^\#(N)$. We see that $\mathcal{E}_\Lambda(Z,0)$ has a Fourier expansion

$$\mathcal{E}_\Lambda(Z,0) = \sum_{\mathcal{S} \geqslant 0} b(\mathcal{S},\mathcal{E}_\Lambda) e^{2\pi i\,\mathrm{tr}(\mathcal{S}Z)},$$

where $\mathcal{S}$ runs through all hermitian half-integral (i.e., $\mathcal{S} = \left[\begin{smallmatrix} t_1 & \bar{t}_2 \\ t_2 & t_3 \end{smallmatrix}\right]$, $t_1, t_3 \in \mathbb{Z}$, $\sqrt{-D}t_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{-D})}$) positive semi-definite matrices of size 2. By [13],

$$b(\mathcal{S},\mathcal{E}_\Lambda) \in \overline{\mathbb{Q}} \quad \text{for any } \mathcal{S}. \tag{176}$$

Here $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. The relation between the global integral $Z(s,\Lambda)$ defined in (169) and the Eisenstein series $\mathcal{E}_\Lambda$ is given in the following lemma.

Lemma 5.4.2. *We have*

$$Z\left(\frac{l}{6} - \frac{1}{2}, \Lambda\right) = \frac{1}{2} V_N \int\limits_{\Gamma^\#(N)\backslash\mathfrak{h}_2} \mathcal{E}_\Lambda(Z,0)\overline{\Phi}(Z)\big(\det(Y)\big)^{l-3}\, dX\, dY,$$

where $V_N = \prod_{p|N} \frac{p-1}{p^{3(n_p-1)}(p+1)(p^4-1)}$ *and* $Z = X + iY$.

Proof. By definition,

$$Z\left(\frac{l}{6} - \frac{1}{2}, \Lambda\right) = \int\limits_{Z_H(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} E_\Lambda\left(h, \frac{l}{6} - \frac{1}{2}\right)\bar{\phi}_\Phi(h)\, dh.$$

Note that the integrand is right invariant under $K_\infty^H K^\#(N)$. Since the volume of $K_\infty^H K^\#(N)$ equals $\prod_{p|N} \frac{p-1}{p^{3(n_p-1)}(p+1)(p^4-1)} = V_N$, it follows that

$$Z\left(\frac{l}{6} - \frac{1}{2}, \Lambda\right) = V_N \int\limits_{Z_H(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})/K_\infty^H K^\#(N)} E_\Lambda\left(h, \frac{l}{6} - \frac{1}{2}\right)\bar{\phi}_\Phi(h)\, dh.$$

Note that

$$Z_H(\mathbb{A})H(\mathbb{Q}) \setminus H(\mathbb{A})/K_\infty^H K^\#(N) = Z_H(\mathbb{R})\Gamma^\#(N) \setminus H(\mathbb{R})^+/K_\infty^H = \Gamma^\#(N) \setminus \mathfrak{h}_2. \tag{177}$$

The $H(\mathbb{R})^+$-invariant measure on $\mathfrak{h}_2$ and dh are related by $dh = \frac{1}{2}\det(Y)^{-3}\, dX\, dY$. From (154) and (175) we get, for $h \in H(\mathbb{R})^+$,

$$\begin{aligned} E_\Lambda\left(h, \frac{l}{6} - \frac{1}{2}\right)\bar{\phi}_\Phi(h) &= \mu_2(h)^l \det\bigl(J(h, I)\bigr)^{-l} \mathcal{E}_\Lambda\bigl(h\langle I\rangle, 0\bigr)\mu_2(h)^l \overline{\det\bigl(J(h, I)\bigr)}^{-l}\overline{\Phi}\bigl(h\langle I\rangle\bigr) \\ &= \det(Y)^l \mathcal{E}_\Lambda(Z, 0)\overline{\Phi}(Z), \end{aligned}$$

where $Z = h\langle I\rangle = X + iY$. We get the last equality because, for $Z \in \mathfrak{h}_2$ and $h \in H(\mathbb{R})^+$,

$$\operatorname{Im}\bigl(h\langle Z\rangle\bigr) = \mu_2(h)\, {}^tJ(h, Z)^{-1} \operatorname{Im}(Z)\, \overline{J(h, Z)}^{-1}.$$

This completes the proof of the lemma. □

Let $\Gamma^{(2)}(N) := \{g \in \mathrm{Sp}_4(\mathbb{Z}):\ g \equiv 1 \pmod N\}$ be the principal congruence subgroup of $\mathrm{Sp}_4(\mathbb{Z})$. Let us denote the space of all Siegel modular forms of weight l with respect to $\Gamma^{(2)}(N)$ by $M_l(\Gamma^{(2)}(N))$ and its subspace of cusp forms by $S_l(\Gamma^{(2)}(N))$. For Φ_1, Φ_2 in $M_l(\Gamma^{(2)}(N))$ with one of the Φ_i a cusp form, one can define the Petersson inner product $\langle \Phi_1, \Phi_2\rangle$ by

$$\langle \Phi_1, \Phi_2\rangle = \bigl[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma^{(2)}(N)\bigr]^{-1} \int\limits_{\Gamma^{(2)}(N)\backslash\mathfrak{h}_2} \Phi_1(Z)\overline{\Phi}_2(Z)\bigl(\det(Y)\bigr)^{l-3}\, dX\, dY. \tag{178}$$

For a Hecke eigenform $\Phi \in S_l(\Gamma^{(2)}(N))$, let $\mathbb{Q}(\Phi)$ be the subfield of $\mathbb{C}$ generated by all the Hecke eigenvalues of Φ. From [10, p. 460], we see that $\mathbb{Q}(\Phi)$ is a totally real number field. Let $S_l(\Gamma^{(2)}(N), \mathbb{Q}(\Phi))$ be the subspace of $S_l(\Gamma^{(2)}(N))$ consisting of cusp forms whose Fourier coefficients lie in $\mathbb{Q}(\Phi)$. Again by [10, p. 460], $S_l(\Gamma^{(2)}(N))$ has an orthogonal basis $\{\Phi_i\}$ of Hecke eigenforms $\Phi_i \in S_l(\Gamma^{(2)}(N), \mathbb{Q}(\Phi_i))$. In addition, if Φ is a Hecke eigenform such that $\Phi \in S_l(\Gamma^{(2)}(N), \mathbb{Q}(\Phi))$, then one can take $\Phi_1 = \Phi$ in the above basis. Hence, let us assume that the Siegel eigenform Φ of weight l with respect to $\mathrm{Sp}_4(\mathbb{Z})$ considered in the previous section satisfies $\Phi \in S_l(\Gamma^{(2)}(N), \mathbb{Q}(\Phi))$. (Also, see [17] for the $N = 1$ case.)

Lemma 5.4.3. *With notations as above, we have*

$$\frac{Z(\frac{l}{6}-\frac{1}{2},\Lambda)}{\langle\Phi,\Phi\rangle}\in\bar{\mathbb{Q}}. \tag{179}$$

Proof. Since $\Gamma^{(2)}(N)\subset\Gamma^{\#}(N)$ we know that $\mathcal{E}_\Lambda|_{\mathfrak{h}_2}$ is a holomorphic Siegel modular form of weight l with respect to $\Gamma^{(2)}(N)$. Let V be the orthogonal complement of $S_l(\Gamma^{(2)}(N))$ in $M_l(\Gamma^{(2)}(N))$ with respect to the Petersson inner product (178). In Corollary 2.4.6 of [13], it is shown, using the Siegel operator, that V is generated by Eisenstein series. By Theorem 3.2.1 of [13], one can choose a basis $\{E_j\}$ such that all the Fourier coefficients of the E_j are algebraic. Let $\{\Phi_i\}$ be the orthogonal basis of $S_l(\Gamma^{(2)}(N))$, with $\Phi_1=\Phi$, as in the remark above. Let us write

$$\mathcal{E}_\Lambda|_{\mathfrak{h}_2}=\sum_i\alpha_i\Phi_i+\sum_j\beta_jE_j. \tag{180}$$

Given a $F\in M_l(\Gamma^{(2)}(N))$ and $\sigma\in\mathrm{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$, let F^σ be defined by applying the automorphism σ to the Fourier coefficients of F. From [31], we know that $F^\sigma\in M_l(\Gamma^{(2)}(N))$. Applying σ to (180) we get

$$\mathcal{E}_\Lambda|_{\mathfrak{h}_2}=\sum_i\sigma(\alpha_i)\Phi_i+\sum_j\sigma(\beta_j)E_j. \tag{181}$$

This follows from the construction of the bases $\{\Phi_i\},\{E_j\}$ and the property (176). From (180) and (181) we now get

$$\sigma\left(\frac{\langle\mathcal{E}_\Lambda|_{\mathfrak{h}_2},\Phi_1\rangle}{\langle\Phi_1,\Phi_1\rangle}\right)=\sigma(\alpha_1)=\frac{\langle\mathcal{E}_\Lambda|_{\mathfrak{h}_2},\Phi_1\rangle}{\langle\Phi_1,\Phi_1\rangle}\quad\text{for all }\sigma\in\mathrm{Aut}(\mathbb{C}/\bar{\mathbb{Q}}),$$

and hence

$$\frac{\langle\mathcal{E}_\Lambda|_{\mathfrak{h}_2},\Phi_1\rangle}{\langle\Phi_1,\Phi_1\rangle}\in\bar{\mathbb{Q}}.$$

Now, using Lemma 5.4.2, we get the result. □

Let $\langle\Psi,\Psi\rangle_1=(\mathrm{SL}_2(\mathbb{Z}):\Gamma_1(N))^{-1}\int_{\Gamma_1(N)\backslash\mathfrak{h}_1}|\Psi(z)|^2y^{l-2}\,dx\,dy$, where $\Gamma_1(N):=\{\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in\Gamma_0(N)\colon\ a,d\equiv 1\pmod N\}$. We have the following generalization of Theorem 4.8.3 of [9].

Theorem 5.4.4. *Let Φ be a cuspidal Siegel eigenform of weight l with respect to Γ_2 satisfying the two assumptions from Section* 5.1 *and $\Phi\in S_l(\Gamma^{(2)}(N),\mathbb{Q}(\Phi))$. Let Ψ be a normalized, holomorphic, cuspidal eigenform of weight l with respect to $\Gamma_0(N)$, with $N=\prod p^{n_p}$ a positive integer. Then*

$$\frac{L(\frac{l}{2}-1,\pi_\Phi\times\tau_\Psi)}{\pi^{5l-8}\langle\Phi,\Phi\rangle\langle\Psi,\Psi\rangle_1}\in\bar{\mathbb{Q}}. \tag{182}$$

Proof. By Theorem 5.3.1, we have

$$Z\left(\frac{l}{6}-\frac{1}{2},\Lambda\right)=C\pi^{4-2l}\frac{L(\frac{l}{2}-1,\pi_\Phi\times\tau_\Psi)}{\zeta(l-2)L(\frac{l-1}{2},\tau_\Psi\times\mathcal{AI}(\Lambda))},$$

where

$$C = \overline{a(\Lambda)}D^{-l+\frac{3}{2}}2^{-4l+6}(2l-5)!$$
$$\times \prod_{p|N} \frac{p^{n_p}(p-1)}{p^{3(n_p-1)}(p+1)(p^4-1)}\left(1-\left(\frac{\mathbb{Q}(\sqrt{-D})}{p}\right)p^{-1}\right)\frac{L_p((l-1)/2, \tau_p \times \mathcal{AI}(\Lambda_p))^{\epsilon_p}}{1-p^{-l+2}} \in \bar{\mathbb{Q}}.$$

Here $\epsilon_p = 1$ if $n_p \geqslant 2$ and 0 otherwise. Observe that $\frac{ir}{2} = \frac{l-1}{2}$, and that $c(1) = (4\pi)^{-l/2}$ by (173). We have used the fact that $L_p((l-1)/2, \tau_p \times \mathcal{AI}(\Lambda_p)) \in \bar{\mathbb{Q}}$, which follows from an argument as in the proof of Proposition 3.17 of [25]. It is well known that $\zeta(l-2)\pi^{2-l} \in \mathbb{Q}$. Using [30], by the same argument as in the proof of Theorem 4.8.3 in [9], we get

$$\frac{L(\frac{l-1}{2}, \tau_\Psi \times \mathcal{AI}(\Lambda))}{\pi^{2l-2}\langle \Psi, \Psi\rangle_1} \in \bar{\mathbb{Q}}.$$

Together with (179), this implies the theorem. □

We remark that it would be interesting to know the behavior of the quantity $\frac{L(\frac{l}{2}-1, \pi_\Phi \times \tau_\Psi)}{\pi^{5l-8}\langle \Phi, \Phi\rangle\langle \Psi, \Psi\rangle_1}$ under the action of $\mathrm{Aut}(\mathbb{C})$. This subject will be considered in a future work.

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