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ARCHIMEDEAN ASPECTS OF SIEGEL MODULAR FORMS OF DEGREE 2

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ABSTRACT. We survey the archimedean representations and Langlands parameters corresponding to holomorphic Siegel modular forms of degree 2. This leads to a determination of archimedean local factors for various L-functions and all vector-valued weights. We determine the Hodge structures that correspond to holomorphic Siegel modular forms and clarify the relationship with four-dimensional symplectic artin representations.

1. Introduction. As is well known, Siegel modular forms of degree 2 are related to automorphic representations of the adelic group $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$. In this note, we concentrate on the archimedean aspects of this relationship. We survey the relevant representations of $\mathrm{GSp}(4, \mathbb{R})$ and their Langlands parameters and use the latter to calculate the Γ- and ε-factors for the "first three" L-functions of (scalar- or vector-valued) Siegel eigenforms. We give some examples of Siegel modular form parameters arising from geometric or motivic situations. We also clarify which parameters can arise from four-dimensional symplectic artin representations which are not related to holomorphic Siegel modular forms.

In the first part of this note we survey the part of the representation theory of $\mathrm{Sp}(4, \mathbb{R})$ that is related to holomorphic Siegel modular forms of degree 2. We recall the parametrization of the (limits of) discrete series representations and of certain non-tempered lowest weight modules and describe their K-type structure. Since this material is well known and far from new, our emphasis is on being explicit.

In Section 2, we describe, again in a very explicit way, the Langlands parameters

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$$W_{\mathbb{R}} \longrightarrow \mathrm{GSp}(4, \mathbb{C})$$

corresponding to the representations exhibited earlier. Knowledge of these parameters makes it simple to calculate the archimedean L- and ε-factors for the spin, standard and adjoint L-functions of holomorphic Siegel modular forms. To the extent that these L-functions have already appeared in the literature, unsurprisingly, the representation-theoretic factors coincide with those obtained by classical methods. The only somewhat unexpected result is that the formula for the standard (degree 5) Γ-factor for weight $\det^k \mathrm{sym}^j$ requires a slight modification for $k = 1$, while the formulae for the spin (degree 4) and adjoint (degree 10) factors admit a uniform description for all weights.

At least conjecturally, Siegel modular forms of degree 2 are related to other arithmetic objects, such as algebraic varieties, Galois representations or motives, via their L-functions. The most prominent of such situations is the "paramodular conjecture" expounded in **[12]**, a rather precise conjectural relationship between abelian surfaces and Siegel modular forms with respect to the paramodular group. Considering Hodge numbers and Langlands parameters, it is easy to see why such a relationship is expected. In subsection 4.4, we describe more generally which Hodge vectors would give rise to Siegel modular forms and give references to the literature where motives with such Hodge vectors have been constructed.

There is a well-known correspondence between odd two-dimensional artin representations and elliptic modular forms of weight 1, see **[14, 20, 40]**. It could be suspected that four-dimensional symplectic artin representations are similarly related to Siegel modular forms of some low weight. This turns out not to be the case, at least not if only holomorphic Siegel modular forms are admitted into such a correspondence. In subsection 4.5, we describe the representations of $\mathrm{GSp}(4, \mathbb{R})$ that can arise from a four-dimensional symplectic artin representation. There are four, none of which is a lowest weight module. Consequently, such artin representations may still be related to Siegel modular forms, but the latter will not be holomorphic. Such an approach was taken in **[19]**.

Most of the material in these notes is well known to experts. The goal was merely to clarify a few details and create a possible reference for some topics related to the archimedean Langlands parameters of (holomorphic) Siegel modular forms of degree 2.

2. Discrete series and lowest weight representations. Depending on their weight, holomorphic Siegel modular forms generate one of three types of representations of $\mathrm{Sp}(4,\mathbb{R})$: holomorphic discrete series, limits of such, or certain non-tempered lowest weight modules. In this section, we recall the parametrization and basic properties of these representations.

2.1. Notation. Let $\mathrm{GSp}(4)$ be the algebraic $\mathbb{Q}$-group whose $\mathbb{R}$-points are given by

$$\mathrm{GSp}(4,\mathbb{R}) = \{g \in \mathrm{GL}(4,\mathbb{R}) \mid {}^tgJg = \mu(g)J,\ \mu(g) \in \mathbb{R}^\times\}, \tag{2.1}$$

$$J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}.$$

Let $\mathrm{Sp}(4,\mathbb{R})$ be the subgroup where $\mu(g) = 1$. Its Lie algebra is

$$\mathfrak{sp}(4,\mathbb{R}) = \{X \in \mathfrak{gl}(4,\mathbb{R}) \mid {}^tXJ + JX = 0\}.$$

Let K be the subgroup of $\mathrm{Sp}(4,\mathbb{R})$ consisting of all matrices of the form $\left[AB{-}BA\right]$. Then, K is a maximal compact subgroup of $\mathrm{Sp}(4,\mathbb{R})$. Mapping

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

to $A + iB$ provides an isomorphism of K with $\mathrm{U}(2)$. Let $\mathfrak{k} \subset \mathfrak{sp}(4,\mathbb{R})$ be the Lie algebra of K. A basis of $\mathfrak{k}$ is given by

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathfrak{h}$ be the Cartan subalgebra spanned by the first two elements. The corresponding analytic subgroup H consists of all elements of the form

$$H = \left\{ \begin{bmatrix} \cos(\theta) & & \sin(\theta) & \\ & \cos(\theta') & & \sin(\theta') \\ -\sin(\theta) & & \cos(\theta) & \\ & -\sin(\theta') & & \cos(\theta') \end{bmatrix} \;\middle|\; \theta,\ \theta' \in \mathbb{R}/2\pi i\mathbb{Z} \right\}. \tag{2.2}$$

We introduce the following basis for the complexification $\mathfrak{k}^{\mathbb{C}}$:

$$Z = -i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Z' = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$N_+ = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{bmatrix}, \qquad N_- = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{bmatrix}.$$

Then $[Z, N_\pm] = \pm N_\pm$ and $[Z', N_\pm] = \mp N_\pm$. The Cartan subgroup $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ is spanned by Z and Z'. The roots are elements of the space

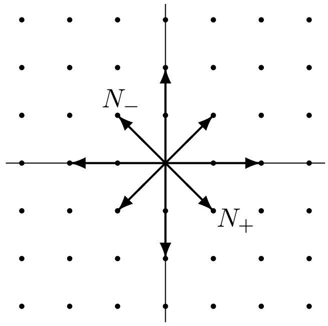
$$(\mathfrak{h}^{\mathbb{C}})' := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C}).$$

We identify an element $\lambda \in (\mathfrak{h}^{\mathbb{C}})'$ with the pair of complex numbers $(\lambda(Z), \lambda(Z'))$. If this pair lies in $\mathbb{R}^2$, we may visualize it as a point in a plane. For example, the compact roots are $\pm(1, -1)$. The non-compact roots are $\pm(0, 2)$, $\pm(2, 0)$ and $\pm(1, 1)$. Let Δ be the set of all roots. The *analytically integral* elements of $(\mathfrak{h}^{\mathbb{C}})'$ are those of the form (n, n') with $n, n' \in \mathbb{Z}$, see [**21,** Proposition 4.13]. The linear map (n, n'), restricted to $\mathfrak{h}$, is the derivative of the character

$$\begin{bmatrix} \cos(\theta) & & \sin(\theta) & \\ & \cos(\theta') & & \sin(\theta') \\ -\sin(\theta) & & \cos(\theta) & \\ & -\sin(\theta') & & \cos(\theta') \end{bmatrix} \longmapsto e^{\mathrm{in}\theta + \mathrm{in}'\theta'}$$

of the group H defined in (2.2). The following picture shows the roots and the analytically integral elements.

(2.3)



Let $E \cong \mathbb{R}^2$ be this plane, i.e., E is the $\mathbb{R}$-subspace of $(\mathfrak{h}^{\mathbb{C}})'$ spanned by the root vectors. We endow E with the inner product

$$\langle (x,y),(x',y')\rangle = xx' + yy'.$$

The roots form a root system of type B_2 in E. The walls are the hyperplanes orthogonal to the roots; in this special case, the walls are the lines spanned by the roots. Let W be the *Weyl group* of this root system, meaning the group generated by reflections in the walls. Let W_K be the two-element subgroup generated by the reflection in the hyperplane perpendicular to the root $(1,-1)$; this is the *compact Weyl group*. We may extend the elements of W to $\mathbb{C}$-linear automorphisms of $(\mathfrak{h}^{\mathbb{C}})'$.

K-types, representations, infinitesimal character. To each analytically integral $\lambda = (\lambda_1, \lambda_2) \in E$ with $\lambda_1 \geq \lambda_2$ corresponds a *K-type* $V_{(\lambda_1,\lambda_2)}$, i.e., an equivalence class of irreducible representations of $K \cong U(2)$. The weights occurring in $V_{(\lambda_1,\lambda_2)}$ are

$$(\lambda_1 - j, \lambda_2 + j) \quad \text{for } j \in \{0, 1, \ldots, \lambda_1 - \lambda_2\},$$

and each weight occurs with multiplicity 1. In particular, the dimension of $V_{(\lambda_1,\lambda_2)}$ is $\lambda_1 - \lambda_2 + 1$. We refer to (λ_1, λ_2) as the highest weight of $V_{(\lambda_1,\lambda_2)}$, and to any non-zero $v_0 \in V_{(\lambda_1,\lambda_2)}$ with this weight as a highest weight vector. Evidently, $N_+ v_0 = 0$. The vector $N_-^j v_0$ has weight $(\lambda_1 - j, \lambda_2 + j)$ for $j \in \{0, 1, \ldots, \lambda_1 - \lambda_2\}$.

Let $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ be the subgroup of $\mathrm{GSp}(4,\mathbb{R})$ where $\mu(g) \in \{\pm 1\}$. It contains $\mathrm{Sp}(4,\mathbb{R})$ with index 2. Its standard maximal compact subgroup is

$$K^{\pm} := \{\mathrm{diag}(1,1,-1,-1)\} \ltimes K.$$

Any K-type $V_{(\lambda_1,\lambda_2)}$ with $\lambda_2 \neq -\lambda_1$ irreducibly induces to $K^{\pm}$. The K-types of the form $V_{(\lambda,-\lambda)}$ extend in two different ways to $K^{\pm}$-types $V^{+}_{(\lambda,-\lambda)}$ and $V^{-}_{(\lambda,-\lambda)}$.

Whenever we say "representation of $\mathrm{Sp}(4,\mathbb{R})$," we mean a Harish-Chandra module, i.e., a $(\mathfrak{g},K)$-module, where $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{R})$. A representation of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ is a $(\mathfrak{g},K^{\pm})$-module, and a representation of $\mathrm{GSp}(4,\mathbb{R})$ is a $(\mathfrak{g}',K^{\pm})$-module, where $\mathfrak{g}' \cong \mathbb{R} \oplus \mathfrak{g}$ is the Lie algebra of $\mathrm{GSp}(4,\mathbb{R})$. A representation of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ can be extended in a trivial way to a representation of $\mathrm{GSp}(4,\mathbb{R})$. Most of the representations of $\mathrm{GSp}(4,\mathbb{R})$ we will consider are such trivial extensions. A representation π of one of the groups $\mathrm{Sp}(4,\mathbb{R})$, $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ or $\mathrm{GSp}(4,\mathbb{R})$ is *admissible*, if each of its K-types occurs with finite multiplicity. In this case, we may write

$$\pi = \bigoplus m_\lambda V_\lambda,$$

where $\lambda = (\lambda_1,\lambda_2)$ runs over analytically integral elements of E with $\lambda_1 \geq \lambda_2$, and m_λ is the multiplicity with which V_λ occurs in π. If $m_\lambda \neq 0$ and λ is closer to the origin than any other λ' with $m_{\lambda'} \neq 0$, then we say that V_λ (or λ) is a *minimal K-type* of π.

Let $\mathcal{Z}$ be the center of the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$; it is a polynomial ring in two variables. For every $\lambda \in (\mathfrak{h}^{\mathbb{C}})'$, an algebra homomorphism

$$\chi_\lambda : \mathcal{Z} \longrightarrow \mathbb{C}$$

may be constructed as in [**21,** subsection VIII.6]. Every algebra homomorphism $\mathcal{Z} \to \mathbb{C}$ is of the form χ_λ, and we have $\chi_\lambda = \chi_{\lambda'}$ if and only if $\lambda = w\lambda'$ for some $w \in W$, see [**21,** Propositions 8.20, 8.21]. If π is an admissible representation of $\mathrm{Sp}(4,\mathbb{R})$ or $\mathrm{GSp}(4,\mathbb{R})$ for which $\mathcal{Z}$ acts via χ_λ, then we say that π has *infinitesimal character* χ_λ (or sometimes just λ). The trivial representation has infinitesimal character $(2,1)$.

Parabolic induction. Let B, P, Q be the *Borel subgroup*, the *Siegel parabolic subgroup* and the *Klingen parabolic subgroup*, respectively

defined as the matrices in $\mathrm{GSp}(4,\mathbb{R})$ of the following shapes:
(2.4)

$$B=\begin{bmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix},\qquad P=\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix},\qquad Q=\begin{bmatrix} * & & * & * \\ * & * & * & * \\ * & & * & * \\ & & & * \end{bmatrix}.$$

Let χ_1, χ_2 and σ be characters of $\mathbb{R}^\times$. Then, we denote by $\chi_1 \times \chi_2 \rtimes \sigma$ the representation of $\mathrm{GSp}(4,\mathbb{R})$ obtained by normalized parabolic induction from the character

$$\begin{bmatrix} b & & * & * \\ * & a & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix} \longmapsto \chi_1(a)\chi_2(b)\sigma(c) \tag{2.5}$$

of B. If π is an admissible representation of $\mathrm{GL}(2,\mathbb{R})$ and σ is a character of $\mathbb{R}^\times$, we denote by $\pi \rtimes \sigma$ the representation of $\mathrm{GSp}(4,\mathbb{R})$ obtained by normalized parabolic induction from the representation

$$\begin{bmatrix} A & * \\ & c\,{}^tA^{-1} \end{bmatrix} \longmapsto \sigma(c)\pi(A) \tag{2.6}$$

of P. If χ is a character of $\mathbb{R}^\times$ and π is an admissible representation of $\mathrm{GSp}(2,\mathbb{R}) = \mathrm{GL}(2,\mathbb{R})$, we denote by $\chi \rtimes \pi$ the representation of $\mathrm{GSp}(4,\mathbb{R})$ obtained by normalized parabolic induction from the representation

$$\begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} \longmapsto \chi(t)\pi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \tag{2.7}$$

of Q. Note that, in (2.6) and (2.7), we take any globalization of π to construct the induced representation and then take K-finite vectors of the result. (Recall that we are working in the category of Harish-Chandra modules. Definitions (2.6) and (2.7), however, require π to be evaluated at any element of $\mathrm{GL}(2,\mathbb{R})$. By globalization, we mean any such representation of $\mathrm{GL}(2,\mathbb{R})$ whose underlying Harish-Chandra module is π.)

We make similar definitions for $\mathrm{Sp}(4,\mathbb{R})$. The relevant parabolic subgroups are again those of the shape (2.4), and we use the same symbols for these subgroups. The notation for the induced representations is:

(2.8) $\chi_1 \times \chi_2 \rtimes 1$ (Borel induction),

(2.9) $\pi \rtimes 1$ (Siegel induction),

(2.10) $\chi \rtimes \pi$ (Klingen induction).

Here, the π in (2.9) is a representation of $\mathrm{GL}(2,\mathbb{R})$, and the π in (2.10) is a representation of $\mathrm{SL}(2,\mathbb{R})$. There should be no danger of confusing the notation between induction on $\mathrm{GSp}(4,\mathbb{R})$ and on $\mathrm{Sp}(4,\mathbb{R})$. The K-types of the representations (2.8), (2.9) and (2.10) are given in [**25,** Lemma 6.1].

Let sgn be the sign character of $\mathbb{R}^\times$. If $\chi_i = |\cdot|^{s_i}\mathrm{sgn}^{\epsilon_i}$ with $s_i \in \mathbb{C}$ and $\epsilon_i \in \{0,1\}$ for $i = 1,2$, then the representation $\chi_1 \times \chi_2 \rtimes 1$ of $\mathrm{Sp}(4,\mathbb{R})$ has infinitesimal character (s_1, s_2). Similarly, for any character σ of $\mathbb{R}^\times$, the representation $\chi_1 \times \chi_2 \rtimes \sigma$ of $\mathrm{GSp}(4,\mathbb{R})$ has infinitesimal character (s_1, s_2).

2.2. Discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$**.** We explain the parametrization of discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$ according to Harish-Chandra. Our main reference is Theorem 9.20 in connection with [**21,** Theorem 12.21].

The space $(i\mathfrak{b})'$ appearing in [**21,** Theorem 9.20] is our Euclidean space E. An element $\lambda \in E$ is *non-singular* if $\langle \lambda, \alpha \rangle \neq 0$ for all roots α, i.e., if λ does not lie on a wall. Every non-singular λ determines a system of positive roots $\Delta_\lambda^+ = \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle > 0\}$. Let

$$\delta_\lambda^{\mathrm{nc}} = \frac{1}{2} \text{ the sum of non-compact roots in } \Delta_\lambda^+$$

and

$$\delta_\lambda^{\mathrm{c}} = \frac{1}{2} \text{ the sum of compact roots in } \Delta_\lambda^+.$$

Consider, in particular, λ from one of the following regions:

(2.11)

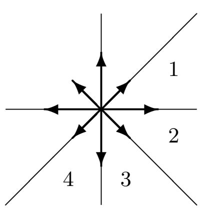


TABLE 1.

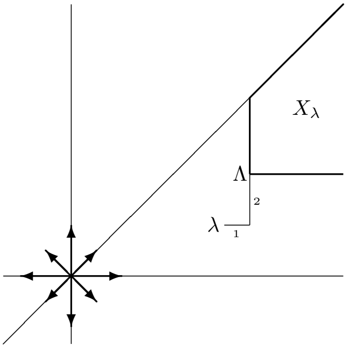
region	$\delta_\lambda^{\rm nc}$	$\delta_\lambda^{\rm c}$	$\delta_\lambda^{\rm nc} - \delta_\lambda^{\rm c}$
1	(3/2, 3/2)	(1/2, −1/2)	(1, 2)
2	(3/2, −1/2)	(1/2, −1/2)	(1, 0)
3	(1/2, −3/2)	(1/2, −1/2)	(0, −1)
4	(−3/2, −3/2)	(1/2, −1/2)	(−2, −1)

Table 1 shows the corresponding quantities $\delta_\lambda^{\rm nc}$ and $\delta_\lambda^{\rm c}$ and also the difference $\delta_\lambda^{\rm nc} - \delta_\lambda^{\rm c}$. Theorem 9.20 of [**21**] states that, for each analytically integral and non-singular $\lambda \in E$, there exists a discrete series representation X_λ of $\mathrm{Sp}(4, \mathbb{R})$ with infinitesimal character λ. The representations X_λ and $X_{\lambda'}$ are equivalent if and only if $\lambda = w\lambda'$ for some $w \in W_K$; consequently, we need only consider λ in one of the regions 1, 2, 3 or 4. By [**21,** Theorem 12.21], every discrete series representation of $\mathrm{Sp}(4, \mathbb{R})$ is of the form X_λ.

The element λ is called the *Harish-Chandra parameter* of X_λ. There is a minimal K-type, occurring in X_λ with multiplicity 1, given by $\Lambda = \lambda + \delta_\lambda^{\rm nc} - \delta_\lambda^{\rm c}$. It is called the *Blattner parameter* of X_λ. One way to determine the multiplicities of each K-type is via the *Blattner formula*, see [**18**]. Some additional details are provided here:

(1) Assume that λ is in region 1 (and non-singular, and analytically integral). Then, X_λ has minimal K-type $\Lambda = \lambda + (1, 2)$. All of the other K-types lie in a region as indicated in the next diagram:

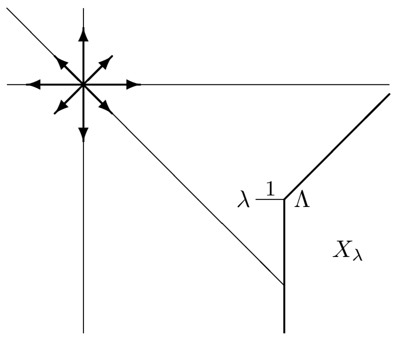
(2.12)



We say that such X_λ are in the *holomorphic discrete series.* If $\lambda = (k-1, k-2)$ with $k \geq 3$, then $\Lambda = (k,k)$, a one-dimensional K-type. These are the discrete series representations generated by holomorphic Siegel modular forms of weight k.

(2) Assume that λ is in region 2 (and non-singular, and analytically integral). Then X_λ has minimal K-type $\Lambda = \lambda + (1,0)$. All of the other K-types lie in a region as indicated in this diagram:

(2.13)



We say that such X_λ are in the *large discrete series.* These are generic representations, meaning they admit a Whittaker model.

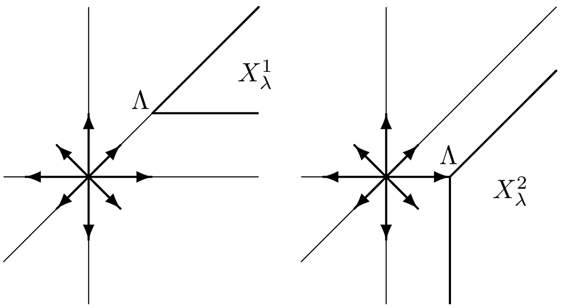
If λ is in region 3, then we obtain a picture symmetric to that for region 2 (with respect to the diagonal running from northwest to southeast). If λ is in region 4, then we obtain a picture symmetric to that for region 1; these X_λ are said to be in the *antiholomorphic discrete series.*

We note that X_λ is a representation for which the center of $\mathrm{Sp}(4,\mathbb{R})$, consisting of ± 1, acts trivially, if and only if $\lambda = (\lambda_1, \lambda_2)$ with λ_1 and λ_2 having different parity.

2.3. Limits of discrete series. The limits of discrete series representations have a *singular* infinitesimal character λ. The λ considered are of one of the forms $(p,0)$, $(0,-p)$ or $(p,-p)$ with integral $p > 0$. There are *two* limits of discrete series for each such λ. We use [**21,** subsection XII.7] as our main reference. By [**21,** Corollary 12.27], the limits of discrete series representations are irreducible, unitary and tempered.

First, consider $\lambda = (p,0)$ with integral $p > 0$. We may consider λ as a limit case of the Harish-Chandra parameters in region 1 or in region 2. Thus, there will be a *holomorphic* limit of discrete series representation X^1_λ, and a *large* (or generic) limit of discrete series representation X^2_λ. The description of their K-types is the same as that for the corresponding discrete series if we formally allow λ to be singular. Hence, X^1_λ has a minimal K-type $\Lambda=(p+1,2)$, and X^2_λ has a minimal K-type $\Lambda=(p+1,0)$. The next diagram illustrates the "first" such limits of discrete series for $\lambda=(1,0)$.

(2.14)



For $\lambda=(0,-p)$ with integral $p>0$, we obtain an anti-holomorphic limit of discrete series X^4_λ and a large limit of discrete series X^3_λ. The K-type structure of $X^4_{(0,-p)}$, respectively $X^3_{(0,-p)}$, is symmetric to that of $X^1_{(p,0)}$, respectively $X^2_{(p,0)}$, with respect to reflection in the diagonal.

By [**25,** Lemma 8.1], these limits of discrete series appear in representations induced from the Klingen parabolic subgroup of $\mathrm{Sp}(4,\mathbb{R})$. More precisely,

$$1\rtimes\mathcal{D}^+_p\cong X^1_\lambda\oplus X^2_\lambda,\qquad \lambda=(p,0), \tag{2.15}$$

and

$$1\rtimes\mathcal{D}^-_p\cong X^3_\lambda\oplus X^4_\lambda,\qquad \lambda=(0,-p), \tag{2.16}$$

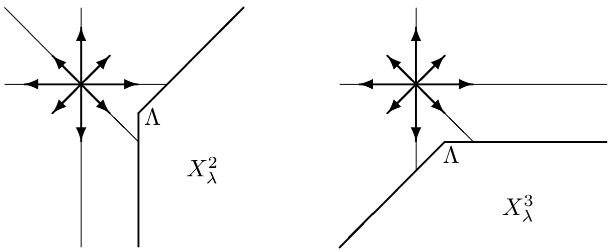
where $\mathcal{D}^+_p$, respectively, $\mathcal{D}^-_p$, is the discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ with lowest weight $p+1$, respectively, highest weight $-p-1$.

It should be noted that, for $\lambda=(1,0)$, the representation X^1_λ is the Θ_{10} of [**1**], which is characterized there in many different ways. The representation X^2_λ is called γ in [**1**].

Now, consider $\lambda=(p,-p)$ with integral $p>0$. For such a λ, there are two large discrete series representations X^2_λ and X^3_λ. Their K-

type structures look similar to that of discrete series representations in region 2, respectively 3, if we allow λ to be singular. In particular, X^2_λ has a minimal K-type $\Lambda = (p+1, -p)$, and X^3_λ has a minimal K-type $\Lambda = (p, -p-1)$. The next diagram illustrates the case $\lambda = (1, -1)$.

(2.17)



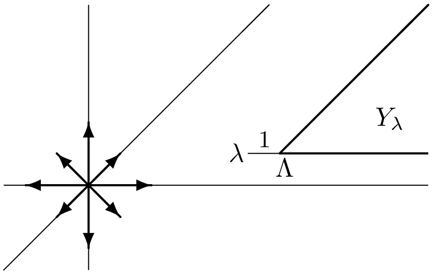
From [**25,** Lemma 8.1], X^2_λ and X^3_λ appear in representations induced from the Siegel parabolic subgroup of $\mathrm{Sp}(4,\mathbb{R})$. More precisely,

$$\mathcal{D}_{2p} \rtimes 1 \cong X^2_\lambda \oplus X^3_\lambda, \quad \lambda = (p, -p), \tag{2.18}$$

where $\mathcal{D}_{2p}$ is the discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with lowest weight $2p+1$ and central character trivial on $\mathbb{R}_{>0}$.

2.4. Non-tempered lowest weight representations. Assume that $\lambda = (p, 1)$ with an integer $p \geq 0$. From [**29,** Proposition 2.8], there exist a lowest weight module Y_λ with infinitesimal character λ, a minimal K-type $\Lambda = (p+1, 1)$, and all other K-types contained in a "wedge" region as indicated in the next picture:

(2.19)



For $\lambda = (-1, -p)$, there exists a similar module whose K-type structure is symmetric. These modules are multiplicity-free, meaning each K-type occurs at most once.

Lemma 2.1. *Let p be a non-negative integer. The representations $Y_{(p,1)}$ and $Y_{(-1,-p)}$ are Langlands quotients,*

$$Y_{(p,1)} = L(|\cdot|\,\mathrm{sgn} \rtimes \mathcal{D}_p^+), \qquad Y_{(-1,-p)} = L(|\cdot|\,\mathrm{sgn} \rtimes \mathcal{D}_p^-),$$

where $\mathcal{D}_p^+$, respectively $\mathcal{D}_p^-$, is the (limit of) discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ *with lowest weight $p+1$, respectively, highest weight $-p-1$.*

Proof. We prove the assertion for $Y_{(p,1)}$, the other case being analogous. First, assume that $p \geq 2$. By [**25,** Theorem 10.1, equation (10.2))], there is an exact sequence

(2.20)
$$0 \longrightarrow X_{(p,1)} \oplus X_{(p,-1)} \longrightarrow |\cdot|\,\mathrm{sgn} \rtimes \mathcal{D}_p^+ \longrightarrow L(|\cdot|\,\mathrm{sgn} \rtimes \mathcal{D}_p^+) \longrightarrow 0.$$

From [**25,** Lemma 6.1], we can determine the K-types of $L(|\cdot|\,\mathrm{sgn}\rtimes\mathcal{D}_p^+)$ and see that this representation coincides with $Y_{(p,1)}$.

The proof for $p = 1$ is similar, starting from the exact sequence

$$0 \longrightarrow X^2_{(1,-1)} \longrightarrow |\cdot|\,\mathrm{sgn} \rtimes \mathcal{D}_1^+ \longrightarrow L(|\cdot|\,\mathrm{sgn} \rtimes \mathcal{D}_1^+) \longrightarrow 0 \tag{2.21}$$

from [**25,** Theorem 10.4 ii)].

Finally, assume $p = 0$. In this case, we first determine the K-types of $L(\delta(|\cdot|^{1/2}, p) \rtimes 1)$ from [**25,** Lemma 6.1, equation (10.25)] (using the notation of this paper). With this knowledge, we can determine the K-types of $L(|\cdot|\,\mathrm{sgn} \rtimes \mathcal{D}_0^+)$ from [**25,** Theorem 11.2 i)]. We see that $L(|\cdot|\,\mathrm{sgn} \rtimes \mathcal{D}_0^+) = Y_{(0,1)}$. □

Using the classification of unitary highest weight modules, it can be proven that the $Y_{(p,1)}$ and $Y_{(-1,-p)}$ are unitary. They are not tempered; see [**25,** Section 8].

2.5. Representations of $\mathrm{GSp}(4,\mathbb{R})$**.** Let $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ be the subgroup of $\mathrm{GSp}(4,\mathbb{R})$ consisting of elements g with multiplier $\mu(g) = \pm 1$. Let $\lambda = (\lambda_1, \lambda_2)$ be a non-singular, analytically integral element of E in one of the regions 1 or 2. Then $\lambda' = (-\lambda_2, -\lambda_1)$ is also non-singular and analytically integral. If λ is in region 1, then λ' is in region 4, and if λ is in region 2, then λ' is in region 3. Let X_λ and $X_{\lambda'}$ be the corresponding discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$. These two representations are conjugate via $\mathrm{diag}(1,1,-1,-1)$. Consequently,

$$\mathrm{ind}_{\mathrm{Sp}(4,\mathbb{R})}^{\mathrm{Sp}(4,\mathbb{R})^{\pm}}(X_\lambda) = \mathrm{ind}_{\mathrm{Sp}(4,\mathbb{R})}^{\mathrm{Sp}(4,\mathbb{R})^{\pm}}(X_{\lambda'}). \tag{2.22}$$

Upon restriction to $\mathrm{Sp}(4,\mathbb{R})$, this induced representation decomposes into a direct sum $X_\lambda \oplus X_{\lambda'}$. In particular, its K-types combine those of X_λ and $X_{\lambda'}$.

We extend the representation (2.22) to

$$\mathrm{GSp}(4,\mathbb{R}) \cong \mathbb{R}_{>0} \times \mathrm{Sp}(4,\mathbb{R})^{\pm}$$

by letting $\mathbb{R}_{>0}$ act trivially. Let the resulting representation again be denoted by X_λ. If λ is in region 1, we refer to X_λ as a *holomorphic* discrete series representation, and if λ is in region 2, as a *large* discrete series representation.

Similarly, we can combine two limits of discrete series representations to one representation of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ and then extend it to $\mathrm{GSp}(4,\mathbb{R})$. More precisely, let $\lambda = (p,0)$ and $\lambda' = (0,-p)$ with $p > 0$. Then, the representations X_λ^1 and $X_{\lambda'}^4$ combine and extend to a representation of $\mathrm{GSp}(4,\mathbb{R})$, which we denote by X_λ^1 and refer to as a *holomorphic limit* of discrete series representation. Similarly, X_λ^2 and $X_{\lambda'}^3$ combine and extend to a representation of $\mathrm{GSp}(4,\mathbb{R})$, which we denote by X_λ^2 and call a *large limit* of discrete series representation. From (2.15) and (2.16), we conclude that, as representations of $\mathrm{GSp}(4,\mathbb{R})$,

$$1 \rtimes \mathcal{D}_p \cong X_\lambda^1 \oplus X_\lambda^2, \quad \lambda = (p,0), \tag{2.23}$$

where $\mathcal{D}_p$ is the discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ obtained by inducing $\mathcal{D}_p^+$ or $\mathcal{D}_p^-$ to $\mathrm{SL}(2,\mathbb{R})^{\pm}$, and extending the result in a trivial way to $\mathrm{GL}(2,\mathbb{R}) \cong \mathbb{R}_{>0} \times \mathrm{SL}(2,\mathbb{R})^{\pm}$.

If $\lambda = (p,-p)$ with $p > 0$, then the representations X_λ^2 and X_λ^3 combine and extend to a representation of $\mathrm{GSp}(4,\mathbb{R})$, which we denote by $X_\lambda^\times$ and also call a *large limit* of discrete series representation. From (2.18) we conclude that, as representations of $\mathrm{GSp}(4,\mathbb{R})$,

$$\mathcal{D}_{2p} \rtimes 1 \cong X_\lambda^\times, \quad \lambda = (p,-p). \tag{2.24}$$

Finally, let $\lambda = (p,1)$ and $\lambda' = (-1,-p)$ with $p \geq 0$. Then, the non-tempered highest weight modules Y_λ and $Y_{\lambda'}$ combine and extend to a representation of $\mathrm{GSp}(4,\mathbb{R})$, which we denote by Y_λ. It follows from Lemma 2.1 that

$$Y_\lambda = L(|\cdot|\,\mathrm{sgn} \rtimes (|\cdot|^{-1/2}\mathcal{D}_p)), \quad \lambda = (p,1). \tag{2.25}$$

Note here that the central character of a Klingen induced representation $\chi \rtimes \pi$ is $\chi\omega_\pi$, where ω_π is the central character of π. Since the central character of Y_λ is, by definition, trivial on $\mathbb{R}_{>0}$, we need to twist $\mathcal{D}_p$ by $|\cdot|^{-1/2}$.

3. Langlands parameters. Let G be a linear reductive group over $\mathbb{R}$ and ${}^LG$ its L-group, as defined in **[10]**. The *local Langlands correspondence* is a bijection between admissible homomorphisms

$$W_\mathbb{R} \longrightarrow {}^LG,$$

where $\mathbb{R}$ is the real Weil group, and L-packets of irreducible, admissible representations of $G(\mathbb{R})$. In this section, we explicate the portion of the correspondence for $\mathrm{GSp}(4,\mathbb{R})$ which involves the previously considered representations.

3.1. The real Weil group. The *real Weil group* is defined as

$$W_\mathbb{R} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times,$$

where the multiplication on $\mathbb{C}^\times$ is standard, and where j is an element satisfying $j^2 = -1$ and $jzj^{-1} = \bar{z}$ (complex conjugation) for $z \in \mathbb{C}^\times$. Hence, $W_\mathbb{R}$ sits in an exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_\mathbb{R} \longrightarrow \{\pm 1\} \longrightarrow 1, \tag{3.1}$$

where the third map is determined by

$$\mathbb{C}^\times \longmapsto 1 \quad \text{and} \quad j \longmapsto -1.$$

There is a homomorphism to the Galois group of $\mathbb{C}/\mathbb{R}$,

$$W_\mathbb{R} \longrightarrow G(\mathbb{C}/\mathbb{R}), \quad \mathbb{C}^\times \mapsto 1,\ j\mathbb{C}^\times \mapsto -1. \tag{3.2}$$

When referring to representations of $W_\mathbb{R}$, we always mean continuous homomorphisms

$$W_\mathbb{R} \longrightarrow \mathrm{GL}(n,\mathbb{C}) \quad \text{for some } n,$$

for which the image consists of semisimple elements.

Basic facts regarding $W_\mathbb{R}$ are explained in **[22]**. For example, every representation of $W_\mathbb{R}$ is completely reducible, and every irreducible representation is either one- or two-dimensional. The complete list of one-dimensional representations is as follows:

$$\varphi_{+,t} : \mathrm{r}e^{i\theta} \longmapsto r^{2t}, \quad j \mapsto 1, \tag{3.3}$$

$$\varphi_{-,t} : \mathrm{r}e^{i\theta} \longmapsto r^{2t}, \quad j \mapsto -1, \tag{3.4}$$

where $t \in \mathbb{C}$, and we write a non-zero complex number as $\mathrm{r}e^{i\theta}$ with $r \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The two-dimensional representations are precisely

$$\varphi_{\ell,t} : \ \mathrm{r}e^{i\theta} \longmapsto \begin{bmatrix} r^{2t}e^{i\ell\theta} & \\ & r^{2t}e^{-i\ell\theta} \end{bmatrix}, \qquad j \longmapsto \begin{bmatrix} & (-1)^\ell \\ 1 & \end{bmatrix}, \tag{3.5}$$

where $\ell \in \mathbb{Z}_{>0}$ and $t \in \mathbb{C}$. Often, we will only consider the case $t = 0$; in this case, we write $\varphi_\pm$ instead of $\varphi_{\pm,0}$ and φ_ℓ instead of $\varphi_{\ell,0}$.

In the local Langlands correspondence for $\mathrm{GL}(n)$ over $\mathbb{R}$ we may replace the L-group by the dual group $\mathrm{GL}(n,\mathbb{C})$. The correspondence is then a bijection between n-dimensional representations of $W_\mathbb{R}$ and irreducible, admissible representations of $\mathrm{GL}(n,\mathbb{R})$. For $\mathrm{GL}(1)$, the local Langlands correspondence is the assignment

$$\varphi_{+,t} \longleftrightarrow \big(\mathbb{R}^\times \ni a \longmapsto |a|^t\big), \tag{3.6}$$

$$\varphi_{-,t} \longleftrightarrow \big(\mathbb{R}^\times \ni a \longmapsto \mathrm{sgn}(a)|a|^t\big). \tag{3.7}$$

The local Langlands correspondence for $\mathrm{GL}(2)$ over $\mathbb{R}$ is such that

$$\varphi_{\ell,t} \longleftrightarrow |\det(\cdot)|^t \otimes \mathcal{D}_\ell, \tag{3.8}$$

where $|\det(\cdot)|^t \otimes \mathcal{D}_\ell$ is the irreducible representation of $\mathrm{GL}(2,\mathbb{R})$ with lowest weight ℓ and central character determined by $a \mapsto a^{2t}$, $a > 0$.

There are L- and ε-factors attached to representations of $W_\mathbb{R}$. A given representation can be decomposed into irreducibles and the product taken of the factors of the irreducible pieces. For those which are irreducible, the factors are defined as in Table 2.

Here,

$$\Gamma_\mathbb{R}(s) := \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right), \qquad \Gamma_\mathbb{C}(s) := 2(2\pi)^{-s}\,\Gamma(s), \tag{3.9}$$

where Γ is the usual gamma function. Note that the ε-factors depend on the choice of a non-trivial character of $\mathbb{R}$. Here, we choose $\psi(x) = e^{2\pi i x}$, see [**39,** (3.4.4)] for the change of additive character.

TABLE 2.

representation φ	$L(s,\varphi)$	$\varepsilon(s,\varphi,\psi)$	$\varepsilon(s,\varphi,\psi^{-1})$
$\varphi_{+,t}$	$\Gamma_{\mathbb{R}}(s+t)$	1	1
$\varphi_{-,t}$	$\Gamma_{\mathbb{R}}(s+t+1)$	i	$-i$
$\varphi_{\ell,t}$	$\Gamma_{\mathbb{C}}\big(s+t+\frac{\ell}{2}\big)$	$i^{\ell+1}$	$(-i)^{\ell+1}$

3.2. Discrete series parameters. Here, we present the Langlands parameters for the discrete series representations X_λ of $\mathrm{GSp}(4,\mathbb{R})$ defined in subsection 2.5. For the generalities, see [**10,** subsection 10.5]. Note that the dual group of $\mathrm{GSp}(4)$ is $\mathrm{GSp}(4,\mathbb{R})$; for details on how to establish the duality, see [**33,** subsection 2.3]. Hence, the Langlands parameters will be continuous homomorphisms

$$W_{\mathbb{R}} \longrightarrow \mathrm{GSp}(4,\mathbb{C}).$$

Let $\lambda = (\lambda_1,\lambda_2)$ be a non-singular, analytically integral element of E contained in region 1, i.e., λ_1 and λ_2 are integers with $\lambda_1 > \lambda_2 > 0$. Let $\lambda' = (\lambda_1,-\lambda_2)$, which is in region 2. Then X_λ is a holomorphic discrete series representation, and $X_{\lambda'}$ is large. The representations $\{X_\lambda, X_{\lambda'}\}$ form a 2-element L-packet. Their common L-parameter is the homomorphism $W_{\mathbb{R}} \to \mathrm{GSp}(4,\mathbb{C})$, given by

$$\mathrm{r}e^{i\theta} \longmapsto \begin{bmatrix} e^{i(\lambda_1+\lambda_2)\theta} &&& \\ & e^{i(\lambda_1-\lambda_2)\theta} && \\ && e^{-i(\lambda_1+\lambda_2)\theta} & \\ &&& e^{-i(\lambda_1-\lambda_2)\theta} \end{bmatrix}, \tag{3.10}$$

$$j \longmapsto \begin{bmatrix} && (-1)^\epsilon & \\ &&& (-1)^\epsilon \\ 1 &&& \\ & 1 && \end{bmatrix},$$

where $\epsilon = \lambda_1 + \lambda_2$. In order to see this, the duality must be established similarly to [**33,** subsection 2.3]. Note that, as a representation, (3.10) equals

$$\varphi_{\lambda_1+\lambda_2} \oplus \varphi_{\lambda_1-\lambda_2}.$$

Composing the parameter (3.10) with the multiplier homomorphism, we obtain the character of $W_{\mathbb{R}}$ given by $\mathrm{re}^{i\theta} \mapsto 1$ and $j \mapsto (-1)^{\epsilon+1}$. Hence, the image of (3.10) lies in $\mathrm{Sp}(4,\mathbb{C})$ if and only if λ_1 and λ_2 have different parity, which is exactly the case if X_λ and $X_{\lambda'}$ have trivial central character. It is a feature of the local Langlands correspondence that the multiplier of the parameter corresponds to the central character of the representations in the L-packet via the local Langlands correspondence for GL(1).

The *component group* $\mathcal{C}(\varphi)$ of an L-parameter

$$\varphi : W_{\mathbb{R}} \longrightarrow \mathrm{GSp}(4,\mathbb{C})$$

is defined as the quotient $\mathrm{Cent}(\varphi)/\mathrm{Cent}(\varphi)^0\mathbb{C}^\times$, where $\mathrm{Cent}(\varphi)$ is the centralizer in $\mathrm{GSp}(4,\mathbb{C})$ of the image of φ, $\mathrm{Cent}(\varphi)^0$ is its identity component and $\mathbb{C}^\times$ is the center of $\mathrm{GSp}(4,\mathbb{C})$. A calculation shows that the component group of the parameter in (3.10) has two elements, represented by the identity and $\mathrm{diag}(1,-1,1,-1)$. The size of the component group is always the size of the L-packet.

3.3. Parameters for limits of discrete series. Assume that $\lambda = (p,0)$ with integral $p > 0$. Associated to this element of E are two limits of discrete series representations of $\mathrm{GSp}(4,\mathbb{R})$, the holomorphic X_λ^1, and the large X_λ^2. By (2.23), their direct sum equals the Klingen induced representation $1 \rtimes \mathcal{D}_p$. From duality, it follows that the Langlands parameters take values in the Siegel parabolic subgroup of $\mathrm{GSp}(4,\mathbb{C})$. The situation is, in fact, analogous to those of type VIII representations in the non-archimedean case. Similar to [**33,** subsection 2.4], in particular, equation (2.41), we thus see that the common L-parameter of X_λ^1 and X_λ^2 is the map

$$W_{\mathbb{R}} \longrightarrow \mathrm{GSp}(4,\mathbb{C}),$$

given by

$$\mathrm{re}^{i\theta} \longmapsto \begin{bmatrix} e^{-ip\theta} & & & \\ & e^{ip\theta} & & \\ & & e^{ip\theta} & \\ & & & e^{-ip\theta} \end{bmatrix}, \tag{3.11}$$

$$j \longmapsto \begin{bmatrix} & -1 & & \\ (-1)^{p+1} & & & \\ & & & (-1)^p \\ & & 1 & \end{bmatrix}.$$

As a representation of $W_{\mathbb{R}}$, this is $\varphi_p \oplus \varphi_p$. The component group of the parameter has two elements, corresponding to the fact that we have a two-element L-packet $\{X_\lambda^1, X_\lambda^2\}$. The matrix

$$g = \frac{1}{\sqrt{2}} \begin{bmatrix} & & i & 1 \\ i & 1 & & \\ i & -1 & & \\ & & -i & 1 \end{bmatrix} \tag{3.12}$$

lies in $\mathrm{Sp}(4, \mathbb{C})$ and has the property that

$$g \begin{bmatrix} a & & b & \\ & a & & b \\ c & & d & \\ & c & & d \end{bmatrix} g^{-1} = \begin{bmatrix} d & c & & \\ b & a & & \\ & & a & -b \\ & & -c & d \end{bmatrix}. \tag{3.13}$$

With its aid, we can conjugate parameter (3.11) into

$$\mathrm{r}e^{i\theta} \longmapsto \begin{bmatrix} e^{ip\theta} & & & \\ & e^{ip\theta} & & \\ & & e^{-ip\theta} & \\ & & & e^{-ip\theta} \end{bmatrix}, \tag{3.14}$$

$$j \longmapsto \begin{bmatrix} & & (-1)^p & \\ & & & (-1)^p \\ 1 & & & \\ & 1 & & \end{bmatrix}.$$

This map looks precisely like the discrete series parameter (3.10), if we allow (λ_1, λ_2) to be the singular $(p, 0)$.

Now, let $\lambda = (p, -p)$ with integral $p > 0$, and consider the corresponding limit of discrete series representation $X_\lambda^\times$ of $\mathrm{GSp}(4, \mathbb{R})$. Recall from (2.24) that $X_\lambda^\times \cong \mathcal{D}_{2p} \rtimes 1$, a representation induced from the Siegel parabolic subgroup. Consequently, the parameter of $X_\lambda^\times$ lies in the Klingen parabolic subgroup of $\mathrm{GSp}(4, \mathbb{C})$. Arguing as in [**33,** subsection 2.4], in particular, equation (2.46), we find that this parameter is the map

$$W_{\mathbb{R}} \longrightarrow \mathrm{GSp}(4,\mathbb{C}),$$

given by

$$\text{(3.15)}\qquad \mathrm{r}e^{i\theta} \longmapsto \begin{bmatrix} e^{2ip\theta} &&& \\ & 1 && \\ && e^{-2ip\theta} & \\ &&& 1 \end{bmatrix},$$

$$j \longmapsto \begin{bmatrix} && 1 & \\ & -1 && \\ 1 &&& \\ &&& 1 \end{bmatrix}.$$

As a representation of $W_{\mathbb{R}}$, (3.15) equals $\varphi_- \oplus \varphi_{2p} \oplus \varphi_+$. Note that the image of j has multiplier -1, corresponding to the fact that the central character of $X_\lambda^\times$ is the sign character of $\mathbb{R}^\times$. This time, the component group is trivial, meaning $X_\lambda^\times$ is the only element in the L-packet. The map (3.15) is conjugate to

$$\text{(3.16)}\qquad \mathrm{r}e^{i\theta} \longmapsto \begin{bmatrix} 1 &&& \\ & e^{2ip\theta} && \\ && 1 & \\ &&& e^{-2ip\theta} \end{bmatrix}, \qquad j \longmapsto \begin{bmatrix} && 1 & \\ &&& 1 \\ 1 &&& \\ & 1 && \end{bmatrix}$$

by an element of $\mathrm{Sp}(4,\mathbb{C})$. We see that (3.16) looks like a discrete series parameter (3.10), if we allow (λ_1,λ_2) to be the singular $(p,-p)$.

3.4. Parameters for non-tempered lowest weight modules. Assume that $\lambda = (p,1)$ with integral $p \geq 0$. Let Y_λ be the corresponding non-tempered lowest weight module of $\mathrm{GSp}(4,\mathbb{R})$; recall that Y_λ has infinitesimal character λ and minimal K-type $(p+1,1)$. Recall from (2.25) that

$$Y_\lambda = L(|\cdot|\,\mathrm{sgn} \rtimes (|\cdot|^{-1/2}\mathcal{D}_p)),$$

a Langlands quotient of a Klingen-induced representation. As a consequence, its L-parameter will take values in the Siegel parabolic subgroup of $\mathrm{GSp}(4,\mathbb{C})$. In fact, Y_λ is the archimedean analogue of a representation of type IXb in the non-archimedean theory. As in [**33,** subsection 2.4], in particular, equation (2.42), we see that the parameter of Y_λ is the homomorphism $W_\mathbb{R} \to \mathrm{GSp}(4,\mathbb{C})$, given by

$$(3.17)\qquad \mathrm{re}^{i\theta} \longmapsto \begin{bmatrix} \mathrm{re}^{-ip\theta} &&& \\ & \mathrm{re}^{ip\theta} && \\ && r^{-1}e^{ip\theta} & \\ &&& r^{-1}e^{-ip\theta} \end{bmatrix},$$

$$j \longmapsto \begin{bmatrix} & 1 && \\ (-1)^p &&& \\ &&& (-1)^p \\ && 1 & \end{bmatrix}.$$

This parameter is unbounded, corresponding to the fact that Y_λ is not tempered. As a representation of $W_\mathbb{R}$, it is equal to

$$\varphi_{p,1/2} \oplus \varphi_{p,-1/2}$$

if $p > 0$, and to

$$\varphi_{+,1/2} \oplus \varphi_{-,1/2} \oplus \varphi_{+,-1/2} \oplus \varphi_{-,-1/2}$$

if $p = 0$. The component group of (3.17) is trivial, meaning that Y_λ is the only element in the L-packet.

3.5. Local factors. Equations (3.10), (3.11), (3.15) and (3.17) show the Langlands parameters of the discrete series representations, the limits of discrete series and certain non-tempered lowest weight modules of $\mathrm{GSp}(4,\mathbb{R})$. From this information and Table 2, it is easy to calculate the L- and ε-factors of these representations. Table 3 summarizes the results. Note that these are the degree 4 factors $L(s,\pi,\rho_4)$ and $\varepsilon(s,\pi,\rho_4,\psi)$, where ρ_4 denotes the natural four-dimensional representation of the dual group $\mathrm{GSp}(4,\mathbb{C})$. For simplicity, ρ_4 is often omitted from the notation. The formulae for the ε-factors are valid for both additive characters $\psi(x) = e^{2\pi ix}$ and $\psi(x) = e^{-2\pi ix}$.

We say a representation is a *lowest weight representation* if it admits a non-zero vector v annihilated by the roots $(-2,0)$, $(-1,-1)$ and $(0,-2)$. (The highest weight vector v_0 in any K-type contributing to v is then annihilated by the same roots, as well as the compact root $(1,-1)$.) Among the ones considered, these are the holomorphic (limits of) discrete series and the non-tempered lowest weight modules. It turns out that we obtain a uniform description of the L- and ε-factors of these representations if we parameterize them by weight rather than by their infinitesimal character. Here, we define the *weight* of one of these representations to be the pair of non-negative integers (k, j) such that $(k+j, k)$ is the minimal K-type. Table 4 shows the representations in question and their weights.

TABLE 3. L- and ε-factors for certain representations of $\mathrm{GSp}(4,\mathbb{R})$. The holomorphic discrete series representation X_λ, where $\lambda = (\lambda_1, \lambda_2)$, forms an L-packet with the large discrete series representation $X_{\lambda'}$, where $\lambda' = (\lambda_1, -\lambda_2)$. The holomorphic and large limits of discrete series X_λ^1, X_λ^2 also form a two-element L-packet. The $X_\lambda^\times$ are a different type of large limit of discrete series and form singleton L-packets. The non-tempered lowest weight modules Y_λ also form singleton L-packets.

π	λ	par.	$L(s,\pi)$	$\varepsilon(s,\pi,\psi)$
$X_\lambda, X_{\lambda'}$	non-sing.	(3.10)	$\Gamma_{\mathbb{C}}\big(s+\frac{\lambda_1+\lambda_2}{2}\big)\Gamma_{\mathbb{C}}\big(s+\frac{\lambda_1-\lambda_2}{2}\big)$	$(-1)^{\lambda_1+1}$
X_λ^1, X_λ^2	$(p,0)$, $p>0$	(3.11)	$\Gamma_{\mathbb{C}}\big(s+\frac{p}{2}\big)\Gamma_{\mathbb{C}}\big(s+\frac{p}{2}\big)$	$(-1)^{p+1}$
$X_\lambda^\times$	$(p,-p)$, $p>0$	(3.15)	$\Gamma_{\mathbb{C}}(s+p)\Gamma_{\mathbb{C}}(s)$	$(-1)^{p+1}$
Y_λ	$(p,1)$, $p\geq 0$	(3.17)	$\Gamma_{\mathbb{C}}\big(s+\frac{p+1}{2}\big)\Gamma_{\mathbb{C}}\big(s+\frac{p-1}{2}\big)$	$(-1)^{p+1}$

For each $(k, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$, there exists exactly one representation of type X_λ, X_λ^1 or Y_λ with weight (k, j). We denote this unique representation by $\mathcal{B}_{k,j}$. It is a discrete series representation if $k \geq 3$, a limit of discrete series if $k = 2$, and non-tempered if $k = 1$. The dimension of the minimal K-type is $j+1$; in particular, $\mathcal{B}_{k,j}$ has scalar minimal K-type if and only if $j = 0$.

Proposition 3.1 gives a unified formula for the L- and ε-factors of the $\mathcal{B}_{k,j}$. In addition to the degree 4 "spin" factors, we may calculate the degree 5 "standard" factors. These are obtained by composing the parameters (3.10), (3.11), (3.17) with the homomorphism

$$\rho_5 : \mathrm{GSp}(4,\mathbb{C}) \longrightarrow \mathrm{SO}(5,\mathbb{C})$$

given in [**33,** Appendix A.7]. Recall that ρ_5 induces an isomorphism

$$\mathrm{PGSp}(4,\mathbb{C}) \cong \mathrm{SO}(5,\mathbb{C}),$$

and that its restriction to $\mathrm{Sp}(4,\mathbb{C})$ is the unique five-dimensional irreducible representation of this group. We denote the resulting factors by $L(s,\pi,\rho_5)$ and $\varepsilon(s,\pi,\rho_5,\psi)$.

TABLE 4.

π	λ	k	j
X_λ	$(\lambda_1,\lambda_2),\ \lambda_1>\lambda_2>0$	λ_2+2	$\lambda_1-\lambda_2-1$
X_λ^1	$(p,0),\ p>0$	2	$p-1$
Y_λ	$(p,1),\ p\geq 0$	1	p

The next proposition also includes the degree 10 "adjoint" factors. In this case, we compose the parameters with the representation

$$\rho_{10}:\ \mathrm{GSp}(4,\mathbb{C})\longrightarrow \mathrm{GL}(10,\mathbb{C}),$$

defined via the conjugation action of $\mathrm{GSp}(4,\mathbb{C})$ on the ten-dimensional Lie algebra $\mathfrak{sp}(4,\mathbb{C})$. The restriction of ρ_{10} to $\mathrm{Sp}(4,\mathbb{C})$ is the unique ten-dimensional irreducible representation of this group. The resulting factors are denoted by $L(s,\pi,\rho_{10})$ and $\varepsilon(s,\pi,\rho_{10},\psi)$.

Proposition 3.1. *Let $(k,j)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geq 0}$ and $\pi=\mathcal{B}_{k,j}$ be the representation with lowest weight (k,j). Then the spin, standard and adjoint L- and ε-factors of π are given in Table* 5. *The formulae for the ε-factors are valid for both additive characters $\psi(x)=e^{2\pi ix}$ and $\psi(x)=e^{-2\pi ix}$.*

Proof. The formulae for ρ_4 are easily verified using Table 5. For ρ_5 and ρ_{10}, if φ is one of the parameters (3.10), (3.11) or (3.17), we first calculate $\rho_5\circ\varphi$ and $\rho_{10}\circ\varphi$, decompose the result into irreducible representations of $W_\mathbb{R}$, and use Table 2. We omit the details of the simple calculations. □

It is easily seen that, while the degree 4 and degree 10 factors have a uniform description for all k and j, this is not quite true for the degree 5 factors. The factor $L(s,\pi,\rho_5)$ for $k=1$ is

$$\frac{\pi^2}{4}s(s-1)$$

times the factor that we would obtain if we set $k = 1$ in the formula for $k \geq 2$.

TABLE 5.

ρ		$L(s,\pi,\rho)$	$\varepsilon(s,\pi,\rho,\psi)$
ρ_4		$\Gamma_{\mathbb{C}}\big(s+\frac{2k+j-3}{2}\big)\Gamma_{\mathbb{C}}\big(s+\frac{j+1}{2}\big)$	$(-1)^{k+j}$
ρ_5	$k \geq 2$	$\Gamma_{\mathbb{C}}(s+k+j-1)\Gamma_{\mathbb{C}}(s+k-2)\Gamma_{\mathbb{R}}(s)$	$(-1)^j$
	$k = 1$	$\Gamma_{\mathbb{C}}(s+j)\Gamma_{\mathbb{R}}(s+2)\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s)$	$(-1)^j$
ρ_{10}		$\Gamma_{\mathbb{C}}(s+2k+j-3)\Gamma_{\mathbb{C}}(s+k+j-1)$ $\Gamma_{\mathbb{C}}(s+k-2)\Gamma_{\mathbb{C}}(s+j+1)\Gamma_{\mathbb{R}}(s+1)^2$	$(-1)^j$

4. Siegel modular forms.

In this section, we define vector-valued Siegel modular forms and explain their connection with automorphic representations of the group $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$. The archimedean components of these automorphic representations are the $\mathcal{B}_{k,j}$ defined earlier. Considerations with Hodge numbers show that Siegel modular forms should correspond to certain motives. In the last section, we explain why four-dimensional symplectic artin representations do *not* correspond to holomorphic Siegel modular forms.

4.1. Vector-valued Siegel modular forms.

Let k be an integer and j a non-negative integer. Let

$$U_j \simeq \mathrm{sym}^j(\mathbb{C}^2)$$

be the space of all complex homogeneous polynomials of total degree j in the two variables S and T. For any $g \in \mathrm{GL}(2, \mathbb{C})$ and $P(S,T) \in U_j$, define

$$\eta_{k,j}(g)P(S,T) = \det(g)^k P((S,T)g).$$

Then $(\eta_{k,j}, U_j)$ gives a concrete realization of the irreducible representation $\det^k \mathrm{sym}^j$ of $\mathrm{GL}(2, \mathbb{C})$. The $\eta_{k,j}$ may be related to our previously introduced K-types as follows:

$$\text{the restriction of } \eta_{k,j} \text{ to } K \cong U(2) \text{ is } V_{(k+j,k)}. \tag{4.1}$$

Here, it is important that we fix the isomorphism $K \cong \mathrm{U}(2)$ given by

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \longmapsto A + iB.$$

See [**29,** Section 3] for details.

Let $\mathcal{H}_2$ be the Siegel upper half space of degree 2. Hence, $\mathcal{H}_2$ consists of all symmetric, complex 2×2 matrices whose imaginary part is positive definite. The group $\mathrm{GSp}(4, \mathbb{R})^+$, consisting of all elements of $\mathrm{GSp}(4, \mathbb{R})$ with positive multiplier, acts on $\mathcal{H}_2$ by

$$gZ = (AZ + B)(CZ + D)^{-1}, \quad Z \in \mathcal{H}_2,$$
$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GSp}(4, \mathbb{R})^+.$$

We set

$$J(g, Z) = CZ + D, \quad Z \in \mathcal{H}_2, \qquad g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GSp}(4, \mathbb{R})^+. \tag{4.2}$$

Then, $J(g_1 g_2, Z) = J(g_1, g_2 Z) J(g_2, Z)$. Let $C^\infty_{k,j}(\mathcal{H}_2)$ be the space of smooth U_j-valued functions on $\mathcal{H}_2$. We define a right action of $\mathrm{GSp}(4, \mathbb{R})^+$ on $C^\infty_{k,j}(\mathcal{H}_2)$ by

$$\big(F\big|_{k,j} g\big)(Z) = \mu(g)^{k+j/2}\, \eta_{k,j}(J(g, Z))^{-1} F(gZ) \quad \text{for } g \in \mathrm{GSp}(4, \mathbb{R})^+, \quad Z \in \mathcal{H}_2. \tag{4.3}$$

The normalization factor $\mu(g)^{k+j/2}$ has the effect that the center of $\mathrm{GSp}(4, \mathbb{R})^+$ acts trivially.

In the following, we fix a congruence subgroup Γ of $\mathrm{Sp}(4, \mathbb{Q})$. Let $C^\infty_{k,j}(\Gamma)$ be the space of smooth functions $F : \mathcal{H}_2 \to U_j$ satisfying

$$F\big|_{k,j} \gamma = F \quad \text{for all } \gamma \in \Gamma. \tag{4.4}$$

Let $M_{k,j}(\Gamma)$ be the subspace of holomorphic functions in $C^\infty_{k,j}(\Gamma)$. The elements of $M_{k,j}(\Gamma)$ are called *Siegel modular forms* of degree 2 and weight (k, j) with respect to Γ. An element $F \in M_{k,j}(\Gamma)$ is called a *cusp form*, if

$$\lim_{\lambda \to \infty} \big(F\big|_{k,j} g\big)\left(\begin{bmatrix} i\lambda & \\ & \tau \end{bmatrix}\right) = 0 \tag{4.5}$$

for all $g \in \mathrm{Sp}(4,\mathbb{Q})$ and all τ in the usual upper half plane $\mathcal{H}_1$. Let $S_{k,j}(\Gamma)$ be the space of cusp forms.

We abbreviate $M_{k,0}(\Gamma)$ as $M_k(\Gamma)$ and $S_{k,0}(\Gamma)$ as $S_k(\Gamma)$. These are the spaces of complex-valued Siegel modular forms, respectively, cusp forms.

4.2. Modular forms and automorphic representations. In this section, we abbreviate $G = \mathrm{GSp}(4)$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$, and let $\mathbb{A}_f$ be the finite part of $\mathbb{A}$. Recall from **[11]** that an *automorphic form* on $G(\mathbb{A})$ is a continuous function

$$\Phi : G(\mathbb{A}) \longrightarrow \mathbb{C},$$

which is left-invariant under $G(\mathbb{Q})$; right-invariant under a compact, open subgroup of $G(\mathbb{A}_f)$; smooth and K-finite as a function of the archimedean component; $\mathcal{Z}$-finite, where $\mathcal{Z}$ is the center of the universal enveloping algebra of $\mathfrak{g}$; and of moderate growth. A *cusp form* is an automorphic form Φ satisfying

$$\int\limits_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \Phi(ng)\,dn = 0 \tag{4.6}$$

and

$$\int\limits_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} \Phi(ng)\,dn = 0 \tag{4.7}$$

for all $g \in G(\mathbb{A})$, where N_P and N_Q are the unipotent radicals of the parabolics P and Q defined in (2.4). Let π be an automorphic representation of $G(\mathbb{A})$, assumed to be irreducible, but not necessarily cuspidal. For simplicity, let us assume that π can be realized as a subspace V of the space of automorphic forms. (This assumption is satisfied in most cases, and, in particular, if π is cuspidal. In general, however, automorphic representations are defined as subquotients of the space of automorphic forms, see **[11]**, and not every subquotient can be realized as a subspace.) Recall that π is not actually a representation of $G(\mathbb{A})$; rather, V carries an action of $G(\mathbb{A}_f)$ and, simultaneously, a $(\mathfrak{g}', K)$-module structure. Here,

$$\mathfrak{g}' \cong \mathbb{R} \oplus \mathfrak{g}$$

is the Lie algebra of $G(\mathbb{R})$. For our purposes, it will be adequate to assume that the center $\mathbb{R}$ of $\mathfrak{g}'$ acts trivially.

Since π is assumed to be irreducible, we may decompose π into a restricted tensor product $\otimes\pi_p$, where p runs over the places of $\mathbb{Q}$. Each π_p is an irreducible, admissible representation of $G(\mathbb{Q}_p)$, and π_p is unramified for almost all finite p. Let V_p be any model for π_p, so that $V \cong \otimes V_p$, a restricted tensor product.

We will make an assumption on π_∞, namely, that there exists a

$$(k,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$$

such that π_∞ is isomorphic to the lowest weight representation $\mathcal{B}_{k,j}$ defined in subsection 3.5. In particular, π_∞ contains the K-type $V_{(k+j,k)}$ with multiplicity 1. Let $v_\infty \in V_\infty$ be a non-zero vector of weight $(k+j,k)$ in this K-type. Then, v_∞ is unique up to scalars, and $\pi_\infty(N_+)v_\infty = 0$. In addition, v_∞ is annihilated by $\mathfrak{n}$, the three-dimensional subspace of $\mathfrak{sp}(4,\mathbb{R})$, spanned by the root vectors for the roots $(-2,0)$, $(-1,-1)$ and $(0,-2)$.

For each finite prime p, let v_p be a non-zero vector in V_p. For almost all p, we assume that v_p is the distinguished unramified vector that has been used to construct the restricted tensor product $\otimes V_p$. Let C_p be an open-compact subgroup of $G(\mathbb{Q}_p)$ stabilizing v_p. We make the following assumptions:

- $C_p = G(\mathbb{Z}_p)$ for almost all p.
- The multiplier map $\mu:\ C_p \to \mathbb{Z}_p^\times$ is surjective for all p. This is certainly satisfied if C_p is one of the standard congruence subgroups, like $\Gamma_0(p^n)$. In general, it follows from **[31,** Corollary 7.2.4] that, as long as π is not one-dimensional, one can always find a v_p stabilized by a C_p for which

$$\mu:\ C_p \longrightarrow \mathbb{Z}_p^\times$$

is surjective.

The surjectivity assumption, together with strong approximation for Sp(4), implies that

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+ \prod_{p<\infty} C_p. \tag{4.8}$$

Here, $G(\mathbb{Q})$ is diagonally embedded into $G(\mathbb{A})$. Note that the product is not direct. In fact,

$$\Gamma := G(\mathbb{Q}) \cap G(\mathbb{R})^+ \prod_{p<\infty} C_p \tag{4.9}$$

is a congruence subgroup of $\mathrm{Sp}(4, \mathbb{Q})$.

From our choices, the pure tensor $\otimes v_p$ is a legitimate element of $\otimes V_p$. Let $\Phi \in V$ be the automorphic form corresponding to $\otimes v_p$ under the isomorphism $V \cong \otimes V_p$. By construction, Φ is right invariant under the open-compact subgroup

$$\prod_{p<\infty} C_p \quad \text{of } G(\mathbb{A}_f).$$

It follows from (4.8) that Φ is determined by its values on $G(\mathbb{R})^+$. Since we assumed that the center of $\mathfrak{g}'$ acts trivially, Φ is in fact determined by its values on $\mathrm{Sp}(4, \mathbb{R})$.

We now define a function $\vec{\Phi}$ on $\mathrm{Sp}(4, \mathbb{R})$ taking values in the polynomial ring $\mathbb{C}[S, T]$ by

$$\vec{\Phi}(g) = \sum_{m=0}^{j} \frac{(-1)^m}{m!} (N_-^m \Phi)(g) S^{j-m} T^m, \quad g \in \mathrm{Sp}(4, \mathbb{R}). \tag{4.10}$$

Here, the action of N_- on Φ is by right translation. Evidently, $\vec{\Phi}$ takes values in the space

$$U_j \subset \mathbb{C}[S, T]$$

of the representation $\eta_{k,j}$. Hence, an expression like $\eta_{k,j}(h)(\vec{\Phi}(g))$ makes sense, for any $h \in \mathrm{GL}(2, \mathbb{C})$. If $j = 0$, then $\vec{\Phi} = \Phi$ is $\mathbb{C}$-valued, and $\eta_{k,j}(h) = \det(h)^k$. We note that, since N_- normalizes $\mathfrak{n}$, the vector-valued $\vec{\Phi}$ is annihilated by $\mathfrak{n}$, similarly to Φ.

As in **[29,** Section 3], it may be verified that the U_j-valued function

$$g \longmapsto \eta_{k,j}(J(g, I))\vec{\Phi}(g)$$

is right K-invariant. Hence, this function descends to a function on $\mathcal{H}_2 \cong \mathrm{Sp}(4, \mathbb{R})/K$. We introduce an additional normalization factor, and define $F : \mathcal{H}_2 \to U_j$ by

$$F(Z) = \mu(g)^{-k-(j/2)}\, \eta_{k,j}(J(g, I))\vec{\Phi}(g), \tag{4.11}$$

where g is any element of $\mathrm{GSp}(4,\mathbb{R})^+$ satisfying $gI = Z$. Since $\Phi(\gamma g) = \Phi(g)$ for all $g \in \mathrm{Sp}(4,\mathbb{R})$ and $\gamma \in \Gamma$, the transformation property (4.4) is satisfied. Furthermore, $\mathfrak{n} \cdot \vec{\Phi} = 0$ implies that F is holomorphic, see [**29,** Corollary 3.4]. It follows that $F \in M_{k,j}(\Gamma)$.

The F thus constructed from π is automatically an eigenform for the Hecke algebra at p for all good places p, i.e., for all places where v_p is $G(\mathbb{Z}_p)$ invariant. It is a cusp form if and only if π is a *cuspidal* automorphic representation. This process may be reversed; one can start with an eigenform and use it to generate an automorphic representation. Although this automorphic representation may not, in general, be irreducible, it is still true that $S_{k,j}(\Gamma)$ is spanned by eigenforms originating from automorphic representations via the above process.

4.3. *L*-functions. Assume for the moment that $\Gamma = \mathrm{Sp}(4,\mathbb{Z})$, the full Siegel modular group. Here is a brief overview of results concerning the analytic properties of L-functions attached to Siegel eigenforms with respect to Γ.

• Let $k \geq 1$ be an integer, and let $F \in M_k(\Gamma)$ be an eigenform for all Hecke operators. The *spin L-function* $Z_F(s)$ is a degree 4 Euler product attached to F. It is proved in [**3**] that the completed function

$$\Psi_F(s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+2)Z_F(s) \tag{4.12}$$

has meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Psi_F(2k-2-s) = (-1)^k\Psi_F(s). \tag{4.13}$$

(The definition of $\Psi_F(s)$ in [**3**] differs from that given in (4.12) by an irrelevant constant.) If F is a cusp form, then $\Psi_F(s)$ has at most two simple poles at $s = k-2$ and $s = k$. The poles occur if and only if F is in the *Maass space*, i.e., F is a *Saito-Kurokawa lifting*, see [**17, 24, 27**].

• Assume that $j > 0$, and let $F \in S_{k,j}(\Gamma)$ be an eigenform. Note that $S_{k,j}(\Gamma) = 0$ if j is odd; thus, we assume j to be even. Attached to F is the Euler product $Z_F(s)$, defined exactly as in the scalar-valued case via Hecke eigenvalues. It is completed to a function $\Psi_F(s)$ using the same formula (4.12). It is proved in [**4**] (under a mild condition) and also in [**36**], that $\Psi_F(s)$ can be analytically continued to an entire function of s satisfying the functional equation

$$\Psi_F(2k+j-2-s) = (-1)^k\Psi_F(s). \tag{4.14}$$

Note that there are no Saito-Kurokawa type liftings for $j > 0$.

• Let $F \in M_k(\Gamma)$ be a scalar-valued eigenform. The *standard L-function* $D_F(s)$ is a degree 5 Euler product attached to F. It is proved in [**8**] that the completed function

$$\Lambda_F(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s+k-1)\Gamma_{\mathbb{C}}(s+k-2)D_F(s) \tag{4.15}$$

has meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda_F(1-s) = \Lambda_F(s). \tag{4.16}$$

The results of [**8**] hold in fact for Siegel modular forms of any degree.

• Let $F \in S_{k,j}(\Gamma)$ be an eigenform, with both k and j even. The standard L-function $D_F(s)$ is defined exactly as in the scalar-valued case via Hecke operators. Generalizing (4.15), the completed function is

$$\Lambda_F(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s+k+j-1)\Gamma_{\mathbb{C}}(s+k-2)D_F(s). \tag{4.17}$$

It was proven in [**38**] that $\Lambda_F(s)$ has analytic continuation to an entire function and satisfies the functional equation (4.16).

• Let $F \in S_k(\Gamma)$ with $k \geq 3$. It is shown in [**29,** Theorem 5.2.1] that the spin, standard and adjoint (degree 10) L-functions attached to F have analytic continuation to entire functions, satisfy the expected functional equation, and are bounded in vertical strips. These L-functions come from the n-dimensional irreducible representation ρ_n of the dual group $\mathrm{Sp}(4,\mathbb{C})$ for $n = 4$, 5 and 10, respectively. The next largest representations are ρ_{14} and ρ_{16}. As proven in [**29**], the resulting L-functions of degrees 14 and 16 attached to F have meromorphic continuation and satisfy a functional equation. For more details on the finite-dimensional representations of $\mathrm{Sp}(4,\mathbb{C})$, see [**13**].

If we replace s by $s + k - (3/2)$ in (4.12), we obtain the completion factor

$$\Gamma_{\mathbb{C}}\Big(s+k-\frac{3}{2}\Big)\Gamma_{\mathbb{C}}\Big(s+\frac{1}{2}\Big),$$

which is the degree 4 L-factor for the representation $\mathcal{B}_{k,0}$, see Proposition 3.1. The shift by $k - (3/2)$ is consistent with the fact that the factors in Proposition 3.1 are all normalized to fit into a functional equation relating s and $1 - s$. The sign $(-1)^k$ in (4.13) is the ε-factor for $\mathcal{B}_{k,0}$. There are no contributions to the global sign coming from the finite places since we are in an everywhere unramified situation.

In the vector-valued case, if we replace s by $s+k+(j-3)/2$, then the completion factor in (4.12) turns into the L-factor for the lowest weight representation $\mathcal{B}_{k,j}$, and the resulting functional equation relates s and $1-s$. The sign $(-1)^k$ in (4.14) coincides with the ε-factor of $\mathcal{B}_{k,j}$ since we assume j to be even.

In the degree 5 case, the completion factors in (4.15) and (4.17) are the archimedean Euler factors $L(s,\pi,\rho_5)$ for $\pi = \mathcal{B}_{k,0}$ and $\pi = \mathcal{B}_{k,j}$, respectively, see Proposition 3.1. The sign $+1$ in the functional equation (4.16) is the ε-factor of π since j is assumed to be even.

In either the scalar- or the vector-valued case, let π be the cuspidal automorphic representation of $G(\mathbb{A})$ generated by the adelization of the eigenform $F \in S_{k,j}(\Gamma)$. Then π is irreducible, see [**26,** Corollary 3.4]. If we factor $\pi = \otimes\pi_p$, then π_∞ is the lowest weight representation $\mathcal{B}_{k,j}$; this eventually follows from (4.1). Comparing classical and adelic Hecke operators, as in [**6**], it is not difficult to see that

$$\Psi_F\left(s+k+\frac{j-3}{2}\right) = L(s,\pi,\rho_4) \tag{4.18}$$

and

$$\Lambda_F(s) = L(s,\pi,\rho_5), \tag{4.19}$$

where

$$L(s,\pi,\rho_n) = \prod_{p\le\infty} L(s,\pi_p,\rho_n) \tag{4.20}$$

are the Langlands L-functions attached to the representation π (and normalized such that the functional equations relate s and $1-s$). The functional equations (4.14) and (4.16) are those expected since the finite places do not contribute to the signs. Note that π is self-contragredient, since it has trivial central character.

General congruence subgroups. Now assume that Γ is an arbitrary congruence subgroup. The most prominent are the *Siegel congruence subgroup* and the *paramodular group* of level N, respectively, given by

$$\Gamma_0(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}, \tag{4.21}$$

$$K(N) = \mathrm{Sp}(4,\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

Let $F \in S_{k,j}(\Gamma)$ be an eigenform with respect to all *good* Hecke operators. We may then define, for all good primes p, local Euler factors $L_p(s,F,\rho_n)$, for $n = 4, 5, 10, \dots$, via the eigenvalues of these Hecke operators; for $n = 4$, see [**3**], and, for $n = 5$, see [**8**]. The problem of defining Euler factors at the *bad* places in terms of data derived from F, and of proving the desired analytic properties of the resulting L-function, is unsolved. It is closely related to the problem of defining old- and new-forms, which is also unsolved except for the paramodular case, see [**32**]. Whatever the correct Euler factors are, the *archimedean* factor used to complete the L-function depends only upon the weight (k, j) and is given in Proposition 3.1.

We may use the adelization of F to generate a cuspidal automorphic representation π. In general, π will not be irreducible. However, we can always write F as a sum of eigenforms for which the associated π's *are* irreducible. Thus, we assume this is the case for F itself. Then, the "correct" manner of assigning local factors to F at all places p is

$$L_p(s,F,\rho_n) := L(s,\pi_p,\rho_n),$$

where, on the right hand side, we have the factors attached to the local representations π_p via the local Langlands correspondence. For all non-supercuspidal π_p, these factors are listed in [**33,** Table A.8] for the $n = 4$ case, [**33,** Table A.10] for the $n = 5$ case and in [**7**] for the $n = 10$ case. For supercuspidal π_p and $n = 4$, the factors are 1.

The problem with this approach is that it is not clear how to determine the π_p from F at a bad place p. Even for elliptic modular forms, the analogous problem is not simple, see [**23**]. For the paramodular case, $L(s,\pi_p,\rho_4)$ may be determined by using two paramodular Hecke operators acting on a newform, although the action of these operators may be difficult to compute. This has been carried out in [**30**] for a particular situation.

Even if all Euler factors can be defined, the problem remains of proving the analytic properties of the resulting L-function. Recall that the archimedean component π_∞ of the automorphic representation $\pi = \otimes\pi_p$ corresponding to an eigenform F in $S_{k,j}(\Gamma)$ is the lowest weight representation $\mathcal{B}_{k,j}$. The $\mathcal{B}_{k,j}$ are all non-generic, meaning they do not admit a Whittaker model. This precludes the use of the Langlands-Shahidi method for π.

It can be shown that a cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$ which is not globally generic admits a global *Bessel model.* The approach of [**28**] was to use local and global Bessel models to define local factors and prove (some of) the expected analytic properties of $L(s,\pi,\rho_4)$. More work must be done, however, both locally and globally, before the theory sketched in [**28**] can be considered on a solid enough basis in order to be applied to Siegel modular forms.

4.4. Hodge numbers. Let X be a compact, complex manifold and $H^i(X,\mathbb{C})$ its ith cohomology group. Then, $H^i(X,\mathbb{C})$ admits a *Hodge decomposition*

$$H^i(X,\mathbb{C}) = \bigoplus_{p+q=i} H^{p,q},$$

where elements of $H^{p,q}$ are represented by closed forms of type (p,q). Complex conjugation induces an involution F_∞ on $H^i(X,\mathbb{C})$ mapping $H^{p,q}$ onto $H^{q,p}$. The *Hodge numbers* $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ thus satisfy $h^{p,q} = h^{q,p}$. The sequence of numbers $(h^{i,0},\ldots,h^{0,i})$ is called the *Hodge vector* for i. If i is even, then there is a space $H^{p,p}$ $(p = i/2)$, on which F_∞ acts as an involution; let $h^{p,\pm}$ be the dimension of the ±1-eigenspace. We attach Γ-factors to X by

(4.22)

$$L(X,s) = \Gamma_{\mathbb{R}}\left(s-\frac{i}{2}\right)^{h^{i/2,+}} \Gamma_{\mathbb{R}}\left(s-\frac{i}{2}+1\right)^{h^{i/2,-}} \prod_{\substack{p+q=i\\ p<q}} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}}$$

(with the first two factors appearing only if i is even), see [**35,** equation (25)]. For example, for an abelian variety of dimension n, the Hodge vector for $i = 1$ is (n,n), and the resulting Γ-factor is $\Gamma_{\mathbb{C}}(s)^n$. See [**15**] for an interpretation of the factor (4.22) as the inverse of a suitably defined characteristic polynomial of an endomorphism of a certain infinite-dimensional vector space.

Assuming that X originates as a localization of an algebraic variety defined over a number field, the factor in (4.22) is designed to fit into an L-function whose functional equation (conjecturally) relates s and $i+1-s$. Replacing s by $s+i/2$ in (4.22), this factor turns into

$$L(X,s+i/2)=\Gamma_{\mathbb{R}}(s)^{h^{i/2,+}}\Gamma_{\mathbb{R}}(s+1)^{h^{i/2,-}}\prod_{\substack{p+q=i\\ p<q}}\Gamma_{\mathbb{C}}\Big(s+\frac{q-p}{2}\Big)^{h^{p,q}}. \tag{4.23}$$

We note that this is precisely the factor attached to the representation

$$h^{i/2,+}\cdot\varphi_+\;\oplus\;h^{i/2,-}\cdot\varphi_-\;\oplus\;\sum_{\substack{p+q=i\\ p<q}}h^{p,q}\cdot\varphi_{q-p} \tag{4.24}$$

of $W_{\mathbb{R}}$, see Table 1. The factor in (4.22) is designed to fit into an L-function whose functional equation relates s and $1-s$.

Sometimes the Weil group representation in (4.24) coincides with the archimedean parameter associated to a lowest weight representation $\mathcal{B}_{k,j}$. For this to occur, we need $k\geq 2$ since, for $k=1$, we have the non-tempered lowest weight modules, whose parameters (3.17) are never of the form (4.23). The parameter for $\mathcal{B}_{k,j}$ with $k\geq 2$, as a representation of $W_{\mathbb{R}}$, is

$$\varphi_{2k+j-3}\oplus\varphi_{j+1}; \tag{4.25}$$

see (3.10), (3.11) and Table 4. Hodge vectors for $i=2k+j-3$ producing the $W_{\mathbb{R}}$ representation in (4.25) are

$$\Big(1,\underbrace{0,\dots,0}_{k-3},1,\underbrace{0,\dots,0}_{j},1,\underbrace{0,\dots,0}_{k-3},1\Big) \tag{4.26}$$

for $k\geq 3$, and

$$\Big(2,\underbrace{0,\dots,0}_{j},2\Big) \tag{4.27}$$

for $k=2$. Note that the *width* $i=2k+j-3$ leads to an L-function with functional equation relating s and $2k+j-2-s$; comparison with (4.14) shows that the classical normalization of the L-function coincides with the motivic normalization. Sometimes this is referred to as the *arithmetic normalization*, as opposed to the *analytic normalization*, which relates s to $1 - s$ in the functional equation.

Here are some examples of Siegel modular forms of degree 2 attached to geometric (more generally, motivic) objects:

• Hodge vectors $(2, 2)$ arise from abelian surfaces, and the corresponding $\mathcal{B}_{2,0}$ is the underlying archimedean representation of a Siegel modular form of weight 2. This explains the appearance of weight 2 Siegel modular forms in the *paramodular conjecture* formulated in [**12**].

• The theory of *hypergeometric motives* is explained in [**34**]. Such motives can lead to many different Hodge vectors, but only a few are of the form (4.26) or (4.27). In fact, most hypergeometric Hodge vectors of the form (4.26) or (4.27) are either $(1, 1, 1, 1)$ or $(2, 2)$. *Special* hypergeometric motives can also have Hodge vectors $(1, 1, 0, 0, 1, 1)$ or $(2, 0, 0, 2)$. There are a few sporadic cases where hypergeometric motives reduce to give additional Hodge vectors of the form (4.26).

• The Hodge vector $(1, 1, 1, 1)$ arises from various families of motives. There are 14 hypergeometric families with these Hodge numbers, see [**16,** Table 1]. The work [**2**] lists these 14 and many others which are not hypergeometric. Yet other examples are those in [**9,** Table 2]. The Hodge numbers $(1, 1, 1, 1)$ give rise to the same $W_{\mathbb{R}}$ representation as $\mathcal{B}_{3,0}$, the underlying archimedean representation of a Siegel modular form of weight 3. The motives listed in the cited works are thus expected to correspond to Siegel modular forms of weight 3 via their L-functions.

• Hodge vectors of the form (4.26) appear in [**41,** (3.1)]. Section 4 of [**41**] considers the hypothetical motive attached to a scalar-valued Siegel modular form of weight k. The Hodge vector in [**41,** (4.2)] is precisely that in (4.26) for $j = 0$.

4.5. artin representations. Let L/F be a finite Galois extension of number fields with Galois group $G(L/F)$. An *artin representation* is a homomorphism

$$\sigma : G(L/F) \longrightarrow \mathrm{GL}(n, \mathbb{C}),$$

for some $n \geq 1$. artin [**5**] associated an Euler product $L(s, \sigma)$ to σ and proved several fundamental properties. Let v be a place of F and w a place of L lying above v. The localization

$$\sigma_v : G(L_w/F_v) \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

determines an irreducible, admissible representation π_v of $\mathrm{GL}(n, F_v)$ via the local Langlands correspondence. It is conjectured that, if σ is irreducible and non-trivial, then $\pi = \otimes \pi_v$ is a cuspidal automorphic representation of $\mathrm{GL}(n, \mathbb{A}_F)$.

Assume, for simplicity, that $F = \mathbb{Q}$. The possibilities for the Archimedean representation π_∞ arising from σ are very limited since its parameter

$$W_{\mathbb{R}} \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

must factor through the homomorphism (3.2). Hence, $W_{\mathbb{R}} \to \mathrm{GL}(n, \mathbb{C})$ is determined by the image of j, which must either be the identity or an element of order 2. Up to conjugation, there are exactly $n+1$ elements of order at most 2 in $\mathrm{GL}(n, \mathbb{C})$, given by

$$\mathrm{diag}\Big(\underbrace{1, \ldots, 1}_{r}, \underbrace{-1, \ldots, -1}_{n-r}\Big), \quad r = 0, \ldots, n.$$

The corresponding irreducible, admissible representation of $\mathrm{GL}(n, \mathbb{R})$ is

$$\underbrace{1_{\mathbb{R}^\times} \times \ldots \times 1_{\mathbb{R}^\times}}_{r} \times \underbrace{\mathrm{sgn} \times \ldots \times \mathrm{sgn}}_{n-r}; \tag{4.28}$$

here, $1_{\mathbb{R}^\times}$, respectively, sgn, denotes the trivial, respectively, sign, character of $\mathbb{R}^\times$, and we use a standard notation for parabolic induction from the Borel subgroup. Note that the induced representation (4.28) is irreducible and unitary, see **[37]** for much more general results.

Assume that $n = 2$ and σ is an *odd* artin representation, i.e., $\det(\sigma_\infty)$ is non-trivial. Then, the only possibility for π_∞ is $1_{\mathbb{R}^\times} \times \mathrm{sgn}$, which is a limit of discrete series representation of $\mathrm{GL}(2, \mathbb{R})$ with lowest weight 1. The expected cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ hence corresponds to an elliptic cusp form of weight 1. The work **[14]** shows that *every* cusp form of weight 1 with odd Dirichlet character arises from an irreducible two-dimensional artin representation.

We now consider the case of four-dimensional *symplectic* artin representations, i.e., homomorphisms

$$\sigma : G(L/\mathbb{Q}) \longrightarrow \mathrm{GSp}(4,\mathbb{C}).$$

Since $\mathrm{GSp}(4,\mathbb{C})$ is the dual group of $\mathrm{GSp}(4)$, it is reasonable to ask whether such artin representations, at least conjecturally, give rise to holomorphic Siegel modular forms of some weight. In order to answer this question, we consider the archimedean parameters

$$W_{\mathbb{R}} \longrightarrow \mathrm{GSp}(4,\mathbb{C}),$$

which, as above, must factor through the homomorphism (3.2). There are exactly four conjugacy classes of order at most 2 in $\mathrm{GSp}(4,\mathbb{C})$, represented by the elements

$$\begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{bmatrix}, \quad \begin{bmatrix}-1&&&\\&1&&\\&&-1&\\&&&1\end{bmatrix}, \quad \begin{bmatrix}-1&&&\\&-1&&\\&&1&\\&&&1\end{bmatrix}, \quad \begin{bmatrix}-1&&&\\&-1&&\\&&-1&\\&&&-1\end{bmatrix}. \tag{4.29}$$

Using the notation introduced in subsection 3.1, the resulting representations of $W_{\mathbb{R}}$ are one of the following:

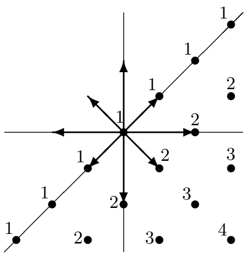
$$\begin{gathered}\varphi_+ \oplus \varphi_+ \oplus \varphi_+ \oplus \varphi_+\,,\\ \varphi_+ \oplus \varphi_+ \oplus \varphi_- \oplus \varphi_-\,,\\ \varphi_- \oplus \varphi_- \oplus \varphi_- \oplus \varphi_-\,.\end{gathered}$$

None is a parameter for a discrete series, limit of discrete series or non-tempered lowest weight representation of $\mathrm{GSp}(4,\mathbb{R})$. It follows that four-dimensional symplectic artin representations do not correspond to holomorphic Siegel modular forms.

In order to see which irreducible, admissible representations of $\mathrm{GSp}(4,\mathbb{R})$ correspond to the parameters determined by the elements in (4.29), we must consider the duality for $\mathrm{GSp}(4)$, as in [**33,** subsection 2.4]. Since the parameters have an image on the diagonal subgroup, the corresponding representations are induced from the Borel subgroup. More precisely, employing the notation (2.5), the representations corresponding to the four elements in (4.29) are

$$\begin{aligned} &1_{\mathbb{R}^\times} \times 1_{\mathbb{R}^\times} \rtimes 1_{\mathbb{R}^\times},\\ &\mathrm{sgn} \times \mathrm{sgn} \rtimes 1_{\mathbb{R}^\times},\\ &\mathrm{sgn} \times 1_{\mathbb{R}^\times} \rtimes 1_{\mathbb{R}^\times},\\ &1_{\mathbb{R}^\times} \times 1_{\mathbb{R}^\times} \rtimes \mathrm{sgn}; \end{aligned} \tag{4.30}$$

see [**33,** equation (2.28)]. In order to understand the K-types of these representations, we consider their restrictions to $\mathrm{Sp}(4,\mathbb{R})$. The first and last representation in (4.30) restrict to $1_{\mathbb{R}^\times} \times 1_{\mathbb{R}^\times} \rtimes 1$ (using notation (2.8), the second restricts to $\mathrm{sgn} \times \mathrm{sgn} \rtimes 1$ and the third to $\mathrm{sgn} \times 1_{\mathbb{R}^\times} \rtimes 1$). From [**25,** Lemmas 5.1 and 6.1], $1_{\mathbb{R}^\times} \times 1_{\mathbb{R}^\times} \rtimes 1$ is irreducible with the multiplicities of its K-types as follows:

(4.31) 

This representation of $\mathrm{Sp}(4,\mathbb{R})$ is invariant under conjugation by $\mathrm{diag}(1,1,-1,-1)$ and can thus be extended in two different ways to a representation of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$. Extending further to

$$\mathrm{GSp}(4,\mathbb{R}) \cong \mathbb{R}_{>0} \times \mathrm{Sp}(4,\mathbb{R})^{\pm}$$

by letting $\mathbb{R}_{>0}$ act trivially, we obtain

$$1_{\mathbb{R}^\times} \times 1_{\mathbb{R}^\times} \rtimes 1_{\mathbb{R}^\times} \quad \text{and} \quad 1_{\mathbb{R}^\times} \times 1_{\mathbb{R}^\times} \rtimes \mathrm{sgn}.$$

Evidently, these two representations of $\mathrm{GSp}(4,\mathbb{R})$ are twists of each other by the sign character.

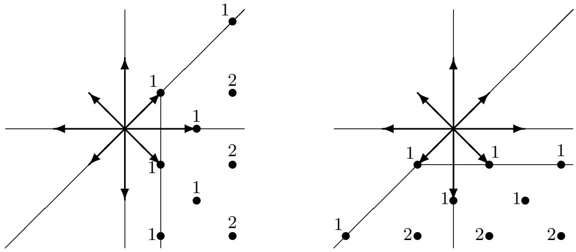
Next, consider $\mathrm{sgn} \times \mathrm{sgn} \rtimes 1$. From [**25,** Corollary 5.2], this representation of $\mathrm{Sp}(4,\mathbb{R})$ is not irreducible, but decomposes as

$$\mathrm{sgn} \rtimes \mathcal{D}_0^+ \oplus \mathrm{sgn} \rtimes \mathcal{D}_0^-,$$

where $\mathcal{D}_0^+$, respectively, $\mathcal{D}_0^-$, is the limit of discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ with lowest weight 1, respectively, highest weight -1.

According to [**25,** Lemma 6.1], the K-types of these two representations are as follows:

(4.32)



$\mathrm{sgn} \rtimes \mathcal{D}_0^+$ $\qquad$ $\mathrm{sgn} \rtimes \mathcal{D}_0^-$

Induction to $\mathrm{Sp}(4,\mathbb{R})^\pm$ of either one of $\mathrm{sgn} \rtimes \mathcal{D}_0^\pm$ results in an irreducible representation combining the K-types of both. Extending trivially to $\mathrm{GSp}(4,\mathbb{R})$ gives the representation $\mathrm{sgn} \times \mathrm{sgn} \rtimes 1_{\mathbb{R}^\times}$, corresponding to the second matrix in (4.29). Note that this representation is invariant under twisting by sgn, i.e.,

$$\mathrm{sgn} \times \mathrm{sgn} \rtimes 1_{\mathbb{R}^\times} \cong \mathrm{sgn} \times \mathrm{sgn} \rtimes \mathrm{sgn},$$

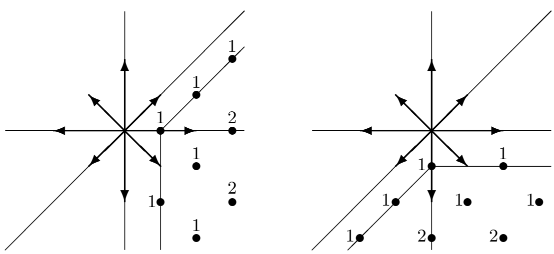
corresponding to the fact that $\mathrm{diag}(-1,1,-1,1)$ is conjugate to $\mathrm{diag}(1, -1,1,-1)$ by an element of $\mathrm{Sp}(4,\mathbb{R})$.

Finally, consider $\mathrm{sgn} \times 1_{\mathbb{R}^\times} \rtimes 1$. From [**25,** Corollary 5.2], this representation of $\mathrm{Sp}(4,\mathbb{R})$ decomposes as

$$1_{\mathbb{R}^\times} \rtimes \mathcal{D}_0^+ \oplus 1_{\mathbb{R}^\times} \rtimes \mathcal{D}_0^-.$$

The K-types of the two are as follows:

(4.33)



$1_{\mathbb{R}^\times} \rtimes \mathcal{D}_0^+$ $\qquad$ $1_{\mathbb{R}^\times} \rtimes \mathcal{D}_0^-$

Upon induction to $\mathrm{Sp}(4,\mathbb{R})^\pm$ and extension to $\mathrm{GSp}(4,\mathbb{R})$, we obtain the irreducible representation $\mathrm{sgn} \times 1_{\mathbb{R}^\times} \rtimes 1_{\mathbb{R}^\times}$, corresponding to the third matrix in (4.29). It is isomorphic to $\mathrm{sgn} \times 1_{\mathbb{R}^\times} \rtimes \mathrm{sgn}$ and combines the K-types of the two representations in (4.33).

We note that the five representations of $\mathrm{Sp}(4, \mathbb{R})$ in (4.31), (4.32) and (4.33) are precisely those which have infinitesimal character $(0, 0)$.

The representation $\mathrm{sgn} \times 1_{\mathbb{R}^\times} \rtimes 1_{\mathbb{R}^\times}$ is the only one among those in (4.30) which has non-trivial central character. It is thus the unique archimedean component attached to any "symplectically odd" four-dimensional artin representation. Such Galois representations appear in **[19]**. One can still relate them to certain types of non-holomorphic Siegel modular forms by singling out a K-type in $\mathrm{sgn} \times 1_{\mathbb{R}^\times} \rtimes 1_{\mathbb{R}^\times}$ and considering the corresponding vector-valued functions on the Siegel upper half space $\mathcal{H}_2$. This is carried out in **[19]**, where the K-type $(2, 1)$ is chosen and an appropriate theory of non-holomorphic Siegel modular forms is developed.

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