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Research Article

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On counting cuspidal automorphic representations for GSp(4)

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Abstract: We find the number $s_k(p, \Omega)$ of cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ with trivial central character such that the archimedean component is a holomorphic discrete series representation of weight $k \geq 3$, and the non-archimedean component at p is an Iwahori-spherical representation of type Ω and unramified otherwise. Using the automorphic Plancherel density theorem, we show how a limit version of our formula for $s_k(p, \Omega)$ generalizes to the vector-valued case and a finite number of ramified places.

Keywords: Plancherel measure, cuspidal automorphic representations, Siegel cusp forms, dimension formula, Arthur packets

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1 Introduction

As is well known, classical modular forms are associated to automorphic representations of the adelic group $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$. Similarly, Siegel modular forms of degree 2 are related to automorphic representations of the adelic group $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$; see [4]. The latter have trivial central character and can hence be regarded as automorphic representations of $\mathrm{PGSp}(4, \mathbb{A}_\mathbb{Q})$. The split orthogonal group SO(5), which is isomorphic to PGSp(4) as algebraic groups, is one of the groups for which Arthur [3] has provided a classification of the discrete automorphic spectrum in terms of the automorphic representations of general linear groups. (This classification has been extended to $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ by Gee and Taïbi in [8].) In the following, let $G = \mathrm{GSp}(4)$.

Definition 1.1. Let k be a positive integer, and let p be a prime. Let $S_k(p, \Omega)$ be the set of cuspidal automorphic representations $\pi \cong \bigotimes_{v \leq \infty} \pi_v$ of $G(\mathbb{A}_\mathbb{Q})$ with trivial central character satisfying the following properties:
(i) π_∞ is the lowest weight module with minimal K-type (k, k); it is a holomorphic discrete series representation if $k \geq 3$, a holomorphic limit of discrete series representation if $k = 2$, and a non-tempered representation if $k = 1$. (It was denoted by $\mathcal{B}_{k,0}$ in [23, Section 3.5].)
(ii) π_v is unramified for each $v \neq p, \infty$.
(iii) π_p is an Iwahori-spherical representation of $G(\mathbb{Q}_p)$ of type Ω.

Here the representation type Ω is one of the types listed in Table 1: I, IIa, IIb, …. This table lists all the irreducible, admissible representations of $G(F)$ supported in the minimal parabolic subgroup, where F is a non-archimedean local field of characteristic zero. By general principles, it is known that $S_k(p, \Omega)$ is finite.

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Let

$$s_k(p,\Omega) := \#S_k(p,\Omega) \tag{1.1}$$

be its cardinality. In this paper, we will find an explicit formula for $s_k(p,\Omega)$ except for $k=2$. Moreover, we give some partial results for $k=2$ in Theorem 3.1, Theorems 3.3–3.4 and Proposition 3.12.

In order to do so, we explore the relationship between Siegel cusp forms of degree 2 and cuspidal automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ in Section 2. Using local representation theory of $G(\mathbb{Q}_p)$ (see [19]), we get a system of equations involving the quantities $s_k(p,\Omega)$ and the global dimensions for the spaces of Siegel cusp forms $S_k(\Gamma_p)$ of degree 2 under the various congruence subgroups Γ_p defined in (1.3) further below. We partition the set $S_k(p,\Omega)$ according to the six Arthur types for GSp(4) and define the Arthur type versions of the quantities $s_k(p,\Omega)$ in Section 2.2. We show that these quantities are all zero for the Arthur packets of types (**B**) and (**Q**) in Section 2.3.

Many mathematicians have studied the dimension formulas for the spaces of Siegel modular forms of degree 2; see for example the references in Table 3. We consider scalar-valued Siegel cusp forms $S_k(\Gamma_p)$ of degree 2, weight k and level p. Using dimension formulas for these spaces, we compute a general formula for $s_k(p,\Omega)$ and a rational expression for the generating series $\sum_{k\geq 3} s_k(p,\Omega)t^k$ in Sections 3.1 and 3.2. Hence we find the numbers $s_k(p,\Omega)$ in a uniform way, where Ω varies over the representation types described in Table 1. Wakatsuki studied these cuspidal automorphic representations for the square-integrable representation types $\Omega=$ IVa, Va in [30].

Furthermore, using the global newform theory for various congruence subgroups of $G(\mathbb{Q})$ defined in [21, Section 3.3], we find the dimensions of the spaces of newforms $S_k^{\mathrm{new}}(\Gamma_p)$ in terms of the quantities $s_k(p,\Omega)$ in Section 3.4.

In the final section, we study a connection between the quantities $s_k(p,\Omega)$ and the total Plancherel measure m_Ω, defined in (4.2), of the tempered Iwahori-spherical representations of $\mathrm{PGSp}(4,\mathbb{Q}_p)$ of type Ω. In Section 4.1, we compute the m_Ω, working over any non-archimedean local field F of characteristic zero. It turns out (see (3.13) and (4.5)) that the leading terms in the *global* formulas for $s_k(p,\Omega)$ are proportional to the purely *local* quantities m_Ω. We will show how this is a consequence of the automorphic Plancherel density theorem from [27], specialized to PGSp(4), even though [27] works on the level of real L-packets, while we require a holomorphic discrete series representation at infinity. Finally, we will use the results of [27] to generalize the limit version of our formula for $s_k(p,\Omega)$ to the vector-valued case and to more than one ramified place; see Theorem 4.5.

Notation

We let

$$G=\mathrm{GSp}(4) := \{g\in \mathrm{GL}(4) : {}^tgJg=\lambda(g)J,\ \lambda(g)\in\mathrm{GL}(1)\},\qquad J=\begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}.$$

The function λ is called the multiplier homomorphism. The kernel of this function is the symplectic group Sp(4). Let Z be the center of GSp(4) and $\mathrm{PGSp}(4)=\mathrm{GSp}(4)/Z$.

Congruence subgroups of GSp(4, *F*). Let F be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$ be the ring of integers of F, and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. We fix a generator ϖ of $\mathfrak{p}$. Let q be the cardinality of $\mathfrak{o}/\mathfrak{p}$, and let ν be the normalized absolute value on F; thus $\nu(\varpi)=q^{-1}$. We consider the following congruence subgroups of $\mathrm{GSp}(4,F)$: the *paramodular group* $\mathrm{K}(\mathfrak{p})$ of level $\mathfrak{p}$, the *Siegel congruence subgroup* $\mathrm{Si}(\mathfrak{p})$ of level $\mathfrak{p}$, the *Klingen congruence subgroup* $\mathrm{Kl}(\mathfrak{p})$ of level $\mathfrak{p}$ and the *Iwahori subgroup* I, which are defined as follows.

$$\begin{aligned}
K &:= \mathrm{GSp}(4,\mathfrak{o}),\\
\mathrm{K}(\mathfrak{p}) &:= \{g \in \mathrm{GSp}(4,F) : \det(g) \in \mathfrak{o}^\times\} \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix},\\
\mathrm{Si}(\mathfrak{p}) &:= \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{bmatrix},\\
\mathrm{Kl}(\mathfrak{p}) &:= \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix},\\
I &:= \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}.
\end{aligned} \tag{1.2}$$

An admissible representation of $G(F)$ is called *Iwahori-spherical* if it has non-zero I-invariant vectors. These are exactly the constituents of the representations parabolically induced from an unramified character of the Borel subgroup. The complete list of all irreducible, admissible representations of $G(F)$ that are constituents of Borel-induced representations is given in Table 1.

Let (π, V) be an Iwahori-spherical representation of $G(F)$, and let H be one of the subgroups defined in (1.2). We denote by V^H, and sometimes by π^H, the space of H-fixed vectors. The quantity $\dim(V^H)$ is the same across all Iwahori-spherical representations of type Ω; here $\Omega \in \{\mathrm{I}, \mathrm{IIa}, \dots, \mathrm{VId}\}$ is one of the types in Table 1. We denote this common dimension by $d_{H,\Omega}$. These numbers are given explicitly in [21, Table 3].

Congruence subgroups for Siegel modular forms. Let $S_k(\Gamma)$ be the space of Siegel cusp forms of degree 2 and weight k with respect to a congruence subgroup Γ of $\mathrm{Sp}(4,\mathbb{Q})$. (When speaking about Siegel modular forms, it is more convenient to realize symplectic groups using the symplectic form $J = \left[\begin{smallmatrix} 0 & 1_2 \\ -1_2 & 0 \end{smallmatrix}\right]$.) The following congruence subgroups, which correspond to the local groups in (1.2), will be of particular interest:

the full modular group $\mathrm{Sp}(4,\mathbb{Z})$,

$$\text{the paramodular group of level } N: \quad \mathrm{K}(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4,\mathbb{Q}),$$

$$\text{the Siegel congruence subgroup of level } N: \quad \Gamma_0(N) = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4,\mathbb{Z}), \tag{1.3}$$

$$\text{the Klingen congruence subgroup of level } N: \quad \Gamma_0'(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4,\mathbb{Z}),$$

$$\text{the Borel congruence subgroup of level } N: \quad B(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4,\mathbb{Z}).$$

2 Iwahori-spherical representations and Arthur packets

In this section, we first discuss the relationship between $s_k(p,\Omega)$ defined as in (1.1) and the dimensions $\dim_{\mathbb{C}} S_k(\Gamma)$, where Γ is one of the congruence subgroups in (1.3) of prime level p. We introduce the refined quantities $s_k^{(*)}(p,\Omega)$, where $(*)$ is one of five types of Arthur packets.

2.1 Siegel modular forms associated to the representations in $S_k(p,\Omega)$

We review the connection between Siegel modular forms of degree 2 and automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$; for more details, see [4] and [23, Section 3.2].

Let $\pi \cong \bigotimes_{p\le\infty}\pi_p$ be a cuspidal automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ with trivial central character, where π_p is an irreducible admissible representation of $G(\mathbb{Q}_p)$. Let V_p be a model for π_p so that $V \cong \bigotimes_p V_p$, a restricted tensor product. In order to get a holomorphic scalar-valued Siegel modular form of weight k, we need to make an assumption that π_∞ is a holomorphic discrete series representation with minimal K-type (k,k). Let $v_\infty \in V_\infty$ be a non-zero vector of weight (k,k) in this K-type. For each finite prime p, let v_p be a non-zero vector in V_p, and let C_p be an open-compact subgroup of $G(\mathbb{Q}_p)$ stabilizing v_p. For almost all primes p, we assume that v_p is the distinguished unramified vector and $C_p = G(\mathbb{Z}_p)$. By our choices, $\bigotimes_p v_p$ is a legitimate element in $\bigotimes_p V_p$, and it corresponds to a cusp form $\Phi \in V$ via the isomorphism $V \cong \bigotimes_p V_p$. Using strong approximation for Sp(4), the automorphic form Φ gives rise to a cuspidal Siegel eigenform f of degree 2 and weight k with respect to the congruence subgroup $\Gamma = G(\mathbb{Q}) \cap G(\mathbb{R})^+ \prod_{p<\infty} C_p$ of $\mathrm{Sp}(4,\mathbb{Q})$. Every eigenform in $S_k(\Gamma)$ arises in this way.

In particular, consider $\pi \cong \bigotimes \pi_v \in S_k(p,\Omega)$. Recall that π_p is an Iwahori-spherical representation of type Ω. Let C_p be one of the compact open subgroups in (1.2), and let Γ_p be the corresponding congruence subgroup of $\mathrm{Sp}(4,\mathbb{Q})$, as in (1.3). Then every eigenform $f \in S_k(\Gamma_p)$ arises from a vector in $\pi_p^{C_p}$ by the above procedure. We thus obtain the formula

$$\dim_{\mathbb{C}} S_k(\Gamma_p) = \sum_{\Omega}\sum_{\pi\in S_k(p,\Omega)} \dim \pi_p^{C_p} = \sum_{\Omega} s_k(p,\Omega)\, d_{C_p,\Omega}, \tag{2.1}$$

which will be the basis for our determination of the quantities $s_k(p,\Omega)$. Here the quantities $d_{C_p,\Omega}$ are defined in the notation section.

Note that the representations of type IVb and IVc are never unitary and hence cannot occur as components of cuspidal, automorphic representations. Thus $s_k(p,\mathrm{IVb}) = s_k(p,\mathrm{IVc}) = 0$. The one-dimensional representations of type IVd can also not occur as components of cuspidal, automorphic representations, so that $s_k(p,\mathrm{IVd}) = 0$. The family Vc is the same as the family Vb due to the fact that a representation of type Vc is just a representation of type Vb twisted by some suitable character; see [19, (2.10)]. So, in the following, we will ignore Vc, as it is subsumed under the Vb case.

2.2 The quantities $s_k(p,\Omega)$ and Arthur packets

Recall from [3] that every $\pi \in S_k(p,\Omega)$ appears in the discrete spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}}))$ inside a certain global Arthur packet. Using [3, Theorem 1.5.2] for SO(5), we have

$$L^2_{\mathrm{disc}}(\mathrm{SO}(5,\mathbb{Q})\backslash \mathrm{SO}(5,\mathbb{A}_{\mathbb{Q}})) \cong \bigoplus_{\psi\in\Psi_2(G)} \bigoplus_{\{\pi\in\Pi_\psi:\langle\cdot,\pi\rangle=\varepsilon_\psi\}} \pi, \tag{2.2}$$

where ψ is an Arthur parameter, and Π_ψ is the corresponding global Arthur packet, consisting of certain equivalence classes of representations of $\mathrm{SO}(5,\mathbb{A}_{\mathbb{Q}})$. The quantities $\langle\cdot,\pi\rangle$ and ε_ψ are characters of a centralizer group $\mathcal{S}_\psi \cong (\mathbb{Z}/2\mathbb{Z})^t$. Since we identify the representations of $\mathrm{SO}(5,\mathbb{A}_{\mathbb{Q}})$ with the representations of $G(\mathbb{A}_{\mathbb{Q}})$ having trivial central character, a representation $\pi \in S_k(p,\Omega)$ appears in some global Arthur packet Π_ψ on the right-hand side of (2.2). The Arthur packets fall into six classes: the finite type (**F**), the

Ω		Representation	Tempered	(G)	(Y)	(P)	(Q)	(B)
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irred.)	•	•	∘			
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	•	•	∘			
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$				•		
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	•	•				
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$					∘	
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	•	•				
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$		never unitary				
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$		never unitary				
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(4)}$						
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	•	•	∘			
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$				•		
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$				∘		
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$					∘	∘
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	•	•	∘			
	b	$\tau(T, \nu^{-1/2}\sigma)$	•	•	•	•		
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$				•	∘	∘
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$					∘	∘

Table 1: Irreducible, admissible representations of $G(F)$ supported on the Borel subgroup.

general type (**G**), the Yoshida type (**Y**), and the types (**P**), (**B**) and (**Q**) consisting mostly of CAP representations (cuspidal associated to parabolics). Cuspidal representations cannot be finite-dimensional, so we will ignore the type (**F**). See [2, 24] for more details about the Arthur packets for GSp(4).

Let $S_k^{(*)}(p, \Omega)$ be the set of those $\pi \in S_k(p, \Omega)$ that lie in an Arthur packet of type $(*)$. Evidently,

$$S_k(p, \Omega) = S_k^{(\mathbf{G})}(p, \Omega) \sqcup S_k^{(\mathbf{Y})}(p, \Omega) \sqcup S_k^{(\mathbf{P})}(p, \Omega) \sqcup S_k^{(\mathbf{Q})}(p, \Omega) \sqcup S_k^{(\mathbf{B})}(p, \Omega), \tag{2.3}$$

so that

$$s_k(p, \Omega) = s_k^{(\mathbf{G})}(p, \Omega) + s_k^{(\mathbf{Y})}(p, \Omega) + s_k^{(\mathbf{P})}(p, \Omega) + s_k^{(\mathbf{Q})}(p, \Omega) + s_k^{(\mathbf{B})}(p, \Omega), \tag{2.4}$$

where $s_k^{(*)}(p, \Omega) = \# S_k^{(*)}(p, \Omega)$.

Not every representation can occur as a local component in every type of packet. In the last five columns of Table 1, we indicate by • or ∘ the possible Arthur packet types in which a given local representation can occur. This information is based on the explicit determination of local Arthur packets given in [25].

A symbol "∘" in the (**Y**) column means that this representation can occur as a local component in a cuspidal automorphic representation π inside an Arthur packet of Yoshida type; however, such π cannot be in $S_k(p, \Omega)$. The reason is that, for discretely appearing $\pi \cong \bigotimes \pi_v$ in packets of type (**Y**), the number of π_v's that are non-generic has to be even; this is the concrete meaning of the multiplicity formula in (2.2) for Yoshida packets. Since, for $\pi \in S_k(p, \Omega)$, the archimedean component is non-generic, the component π_p must also be non-generic. Hence $s_k^{(\mathbf{Y})}(p, \Omega) = 0$ for the generic types $\Omega \in \{\mathrm{I}, \mathrm{IIa}, \mathrm{Va}, \mathrm{VIa}\}$.

We will prove in the next section that $s_k^{(\mathbf{Q})}(p, \Omega) = s_k^{(\mathbf{B})}(p, \Omega) = 0$ for all k and all Ω, hence the ∘ in these columns of Table 1. By (2.4), for all $k \geq 1$,

$$s_k(p, \Omega) = s_k^{(\mathbf{G})}(p, \Omega) + s_k^{(\mathbf{P})}(p, \Omega) + s_k^{(\mathbf{Y})}(p, \Omega),$$

where $s_k^{(\mathbf{Y})}(p, \Omega) = 0$ unless $\Omega = \mathrm{VIb}$.

Note that Arthur packets of type (**G**) are *stable*, meaning one can switch within local L-packets and still retain the automorphic property. Most representations in Table 1 constitute singleton L-packets, except VIa and VIb, which constitute a 2-element L-packet, and Va, which shares an L-packet with a non-generic supercuspidal. Hence $s_k^{(\mathbf{G})}(p, \mathrm{VIa}) = s_k^{(\mathbf{G})}(p, \mathrm{VIb})$, and we denote this common number by $s_k^{(\mathbf{G})}(p, \mathrm{VIa/b})$,

$$s_k^{(\mathbf{G})}(p, \mathrm{VIa/b}) = s_k^{(\mathbf{G})}(p, \mathrm{VIa}) = s_k^{(\mathbf{G})}(p, \mathrm{VIb}). \tag{2.5}$$

A look at [21, Table 3] shows that, for each parahoric subgroup H, the dimensions of H-invariant vectors in a type IIIa representation is the same as the dimensions of H-invariant vectors for the L-packet VIa/b combined. (The reason for this is that both IIIa and VIa ⊕ VIb are of the form $\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$ for unramified characters χ and σ. The dimension of the space of H-invariant vectors in such a parabolically induced representation does not depend on χ and σ.) Hence our methods will not be able to determine the numbers $s_k^{(\mathbf{G})}(p, \mathrm{IIIa})$ and $s_k^{(\mathbf{G})}(p, \mathrm{VIa/b})$ separately, but we will be able to determine

$$s_k(p, \mathrm{IIIa} + \mathrm{VIa/b}) := s_k^{(\mathbf{G})}(p, \mathrm{IIIa}) + s_k^{(\mathbf{G})}(p, \mathrm{VIa/b}). \tag{2.6}$$

Summarizing, we will compute the $s_k(p, \Omega)$ given in Table 2.

Tempered representations	Saito–Kurokawa type representations
$s_k(p, \mathrm{I}) = s_k^{(\mathbf{G})}(p, \mathrm{I})$	$s_k(p, \mathrm{IIb}) = s_k^{(\mathbf{P})}(p, \mathrm{IIb})$
$s_k(p, \mathrm{IIa}) = s_k^{(\mathbf{G})}(p, \mathrm{IIa})$	$s_k(p, \mathrm{Vb}) = s_k^{(\mathbf{P})}(p, \mathrm{Vb})$
$s_k(p, \mathrm{IIIa} + \mathrm{VIa/b}) = s_k^{(\mathbf{G})}(p, \mathrm{IIIa}) + s_k^{(\mathbf{G})}(p, \mathrm{VIa/b})$	$s_k^{(\mathbf{P})}(p, \mathrm{VIb})$
$s_k(p, \mathrm{IVa}) = s_k^{(\mathbf{G})}(p, \mathrm{IVa})$	$s_k(p, \mathrm{VIc}) = s_k^{(\mathbf{P})}(p, \mathrm{VIc})$
$s_k(p, \mathrm{Va}) = s_k^{(\mathbf{G})}(p, \mathrm{Va})$	
$s_k^{(\mathbf{Y})}(p, \mathrm{VIb})$	

Table 2

Representations of type (**P**) and (**Y**) are parametrized by automorphic representations of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. Hence it is not difficult to express the s_k's for the Saito–Kurokawa type representations as well as $s_k^{(\mathbf{Y})}(p, \mathrm{VIb})$ in terms of dimension formulas for certain elliptic modular forms; see Theorem 3.1, Theorem 3.3 and Theorem 3.4. It then remains to determine the quantities $s_k(p, \Omega)$ for the five generic types I, IIa, IIIa + VIa/b, IVa, Va. On the other hand, we have the five congruence subgroups Γ_p appearing in (1.3) (for $N = p$). It turns out that (2.1) is a linear system with an invertible matrix, allowing us to calculate the $s_k(p, \Omega)$ from the $\dim_{\mathbb{C}} S_k(\Gamma_p)$. We do not have a good theoretical explanation for the non-singularity of this linear system.

2.3 Packets of type (B) and (Q) are not Iwahori-spherical

In this section, we will prove that $s_k^{(\mathbf{Q})}(p, \Omega) = s_k^{(\mathbf{B})}(p, \Omega) = 0$ for all k and all Ω. This is a consequence of a more general result about Arthur packets of type (**Q**) or (**B**). This more general result also implies that certain spaces of Siegel modular forms of weight 1 are zero.

Proposition 2.1. *Let $\pi \cong \bigotimes \pi_v$ be an automorphic representation of $\mathrm{PGSp}(4, \mathbb{A}_{\mathbb{Q}})$ in an Arthur packet of type* (**Q**) *or* (**B**)*. Then π_p is not Iwahori-spherical for at least one prime p.*

Proof. Assume first that π lies in an Arthur packet of type (**Q**). Recall from [25] that one of the parameters entering into the definition of such a packet is a non-trivial quadratic character $\xi = \bigotimes \xi_v$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$. There exists a prime p for which the local character ξ_p is ramified. Looking at [25, Table 3], the description of the local packets of type (**Q**), we see that none of the possibilities for π_p (corresponding to a ramified ξ_p) is an Iwahori-spherical representation.

The proof for π being in a packet of type (**B**) is similar. Such packets are parametrized by pairs (χ_1, χ_2) of distinct, quadratic characters of $\chi_i = \bigotimes \chi_{i,v}$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$. There exists a prime p for which $\chi_{1,p}\chi_{2,p}$ is ramified. Then a look at [25, Table 1], the description of the local packets of type (**B**), shows that none of the corresponding possibilities for π_p is Iwahori-spherical. □

Recall, for example from [23, Section 4.1], that vector-valued Siegel cusp forms of degree 2 form spaces $S_{k,j}(\Gamma)$, where k is a positive integer and j is a non-negative integer. We have $S_{k,0}(\Gamma) = S_k(\Gamma)$. The following generalizes [13, Theorem 6.1].

Corollary 2.2. *Let N be a square-free positive integer. Then $S_{1,j}(B(N)) = 0$ for any $j \geq 0$.*

Proof. Eigenforms in $S_{1,j}(B(N))$ would have to originate from cuspidal, automorphic representations

$$\pi \cong \bigotimes \pi_v \quad \text{of } \mathrm{PGSp}(4, \mathbb{A}_\mathbb{Q})$$

for which π_∞ is one of the lowest weight modules described in [23, Section 2.4]. Such lowest weight modules are non-tempered and hence π cannot be of type (**G**). (More precisely, weak archimedean estimates preclude the local parameter given in [23, (3.17)] to be that of the archimedean component of a cusp form on $\mathrm{GL}(4, \mathbb{A})$.) For a similar reason, π cannot be of type (**Y**). It can also not be of type (**P**), by the description of the local packets of type (**P**) given in [25, Table 2]. It follows that π must be of type (**Q**) or (**B**). By Proposition 2.1, π_p is not Iwahori-spherical for at least one prime p. But *all* π_p would have to be Iwahori-spherical in order to extract from π an element of $S_{1,j}(B(N))$. □

Corollary 2.3. $s_k^{(\mathbf{Q})}(p, \Omega) = s_k^{(\mathbf{B})}(p, \Omega) = 0$ *for all k and all Ω.*

Proof. Since the elements of $S_k(p, \Omega)$ are, by definition, Iwahori-spherical at all finite places, this is an immediate consequence of Proposition 2.1. □

3 Counting certain automorphic representations

In Section 3.1, we give formulas for all $s_k^{(\mathbf{P})}(p, \Omega)$ and $s_k^{(\mathbf{Y})}(p, \Omega)$ (i.e., all the lifts), and in Section 3.2 for all the $s_k^{(\mathbf{G})}(p, \Omega)$ (the non-lifts). In Section 3.4, we express the dimensions of the spaces of newforms for some congruence subgroups of prime level in terms of the quantities we found.

For the results in this section, we introduce some notation. The symbol $t = [t_0, t_1, \dots, t_{n-1}; n]_k$ means that $t = t_i$ if $k \equiv i \pmod n$. Then we define

$$\begin{aligned} f_4(k) &= [k-2, -k+1, -k+2, k-1; 4]_k, & c_3(k) &= [1, -1, 0; 3]_k, \\ f_6(k) &= [k-3, -2k+2, -2k+4, k, k-1, k-2; 6]_k, & \hat{c}_3(k) &= [0, 1, -1; 3]_k, \\ c_{12}(k) &= [1, 0, 0, -1, -1, -1, -1, 0, 0, 1, 1, 1; 12]_k, & c_4(k) &= [1, 0, 0, -1; 4]_k, \\ c_6(k) &= [1, 0, 0, -1, 0, 0; 6]_k, & c_4'(k) &= [1, -1, -1, 1; 4]_k, \\ c_6'(k) &= [0, 1, 0, 0, -1, 0; 6]_k, & c_5(k) &= [1, 0, 0, -1, 0; 5]_k, \\ \hat{c}_6(k) &= [0, 1, 1, 0, -1, -1; 6]_k. \end{aligned} \tag{3.1}$$

Let $\delta_{k,n}$ be the Kronecker symbol, i.e., $\delta_{k,n} = 1$ if $k = n$ and $\delta_{k,n} = 0$ otherwise. For any prime p, we define

$$\left(\frac{-1}{p}\right) = \begin{cases} 0 & \text{if } p = 2, \\ 1 & \text{if } p \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv 3 \pmod 4, \end{cases} \qquad \left(\frac{-3}{p}\right) = \begin{cases} 0 & \text{if } p = 3, \\ 1 & \text{if } p \equiv 1 \pmod 3, \\ -1 & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{3.2}$$

Note that $\left(\frac{\cdot}{p}\right)$ is not the usual Legendre symbol as we define $\left(\frac{-1}{2}\right) = 0$.

3.1 The representation types I, IIb, Vb, VIb, VIc

In this section, we compute the numbers $s_k(p, \Omega)$ in Table 2 when the representations in $S_k(p, \Omega)$ are of Saito–Kurokawa type (**P**) or of Yoshida type (**Y**); these are two kinds of liftings from elliptic cuspidal automorphic representations.

We denote by $S_k^{\pm}(\mathrm{SL}(2, \mathbb{Z}))$ the space spanned by the eigenforms in $S_k(\mathrm{SL}(2, \mathbb{Z}))$ which have the sign ± 1 in the functional equation of their L-functions. For the theorems in this section, we use the following standard formula:

$$\dim_\mathbb{C} S_k(\mathrm{SL}_2(\mathbb{Z})) = \frac{k-1}{2^2 3} + \frac{1}{2^2}(-1)^{\frac{k}{2}} + \frac{1}{3}(c_3(k) + \hat{c}_3(k)) - \frac{1}{2} + \delta_{k,2}. \tag{3.3}$$

Theorem 3.1. *For $k \geq 1$, we have*

$$
\begin{aligned}
s_k(p,\mathrm{I}) = {} & \frac{1}{2^7 3^3 5}(k-2)(k-1)(2k-3) + \frac{7(-1)^k}{2^7 3^2}(k-2)(k-1) + \frac{5}{2^4 3} + \delta_{k,3} - \delta_{k,2} - \frac{47}{2^7 3^3}(2k-3) \\
& + \frac{61}{2^7}(-1)^k - \frac{13}{2^2 3^3}\hat{c}_3(k) - \frac{1}{2\cdot 3}c_3(k) + \frac{1}{2^5 3}f_4(k) - \frac{1}{2^3}c_4'(k) + \frac{1}{2^3}c_4(k) + \frac{1}{5}c_5(k) \\
& + \frac{1}{2^2 3^3}f_6(k) + \frac{1}{2^2 3}\hat{c}_6(k) + \frac{1}{3^2}c_6(k) + \frac{1}{2^2 3}c_{12}(k) - \frac{2k-3}{2^4}(-1)^k - \frac{1}{2\cdot 3}\hat{c}_3(k)(-1)^k,
\end{aligned} \tag{3.4}
$$

$$
s_k(p,\mathrm{IIb}) = \begin{cases} \dfrac{2k-3}{2^2 3} - \dfrac{3}{2^2} + \dfrac{1}{3}\hat{c}_3(k) + \delta_{k,2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases} \tag{3.5}
$$

Equivalently,

$$
\begin{aligned}
\sum_{k\geq 1} s_k(p,\mathrm{I})t^k &= \frac{t^{35}+1}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} - \frac{1}{(1-t^4)(1-t^6)} - \frac{t^{10}}{(1-t^2)(1-t^6)}, \\
\sum_{k\geq 1} s_k(p,\mathrm{IIb})t^k &= \frac{t^{10}}{(1-t^2)(1-t^6)}.
\end{aligned} \tag{3.6}
$$

Proof. By [21, Table 3], the spherical representations in Table 1 are I, IIb, IIIb, IVd, Vd and VId. Hence, by (2.1), and observing that one-dimensionals are irrelevant, we have

$$\dim_{\mathbb{C}} S_k(\mathrm{Sp}(4,\mathbb{Z})) = s_k(p,\mathrm{I}) + s_k(p,\mathrm{IIb}) + s_k(p,\mathrm{IIIb}) + s_k(p,\mathrm{Vd}) + s_k(p,\mathrm{VId}).$$

By Corollary 2.3, $s_k(p,\mathrm{IIIb}) = s_k^{(\mathbf{Q})}(p,\mathrm{IIIb}) = 0$, and similarly, $s_k(p,\mathrm{Vd}) = s_k(p,\mathrm{VId}) = 0$. Thus

$$\dim_{\mathbb{C}} S_k(\mathrm{Sp}(4,\mathbb{Z})) = s_k(p,\mathrm{I}) + s_k(p,\mathrm{IIb}). \tag{3.7}$$

The eigenforms constructed from the representations in $S_k(p,\mathrm{IIb})$ are precisely the full-level Saito–Kurokawa liftings. By [7], the space spanned by these is the image of an injective map

$$S^{-}_{2k-2}(\mathrm{SL}(2,\mathbb{Z})) \hookrightarrow S_k(\mathrm{Sp}(4,\mathbb{Z})).$$

Since the sign in the functional equation of the eigenforms in $S_{2k-2}(\mathrm{SL}(2,\mathbb{Z}))$ is $(-1)^{k-1}$, we get

$$s_k(p,\mathrm{IIb}) = \dim_{\mathbb{C}} S^{-}_{2k-2}(\mathrm{SL}(2,\mathbb{Z})) = \begin{cases} \dim_{\mathbb{C}} S_{2k-2}(\mathrm{SL}(2,\mathbb{Z})) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Then (3.5) and (3.6) follow from (3.3). Furthermore, using (3.7) and $\dim_{\mathbb{C}} S_k(\mathrm{Sp}(4,\mathbb{Z}))$ from [9, Theorem 6-2] (which is also implicit in [15, Section 3]), we obtain (3.4). Note that the dimension formula for Sp(4, ℤ)) in [9, Theorem 6-2] does not work for $k = 3$; we add the term $\delta_{k,3}$ so that the formula works for all k. □

The following lemma is useful for finding the quantities $s_k(p,\Omega)$ for the representations of Saito–Kurokawa type and Yoshida type. This result can be derived from the work of Yamauchi [32], and it is explicitly given in [16, Theorem 2.2]. Here $S_k^{\mathrm{new}}(\Gamma_0^{(1)}(p))$ is the new subspace of weight k elliptic cusp forms on the congruence subgroup $\Gamma_0^{(1)}(p)$ of $\mathrm{SL}(2,\mathbb{Z})$. The plus and minus spaces $S_k^{\pm,\mathrm{new}}(\Gamma_0^{(1)}(p))$ are the space spanned by the eigenforms in $S_k^{\mathrm{new}}(\Gamma_0^{(1)}(p))$ which have the sign ± 1 in the functional equation of their L-functions.

Lemma 3.2. *For any even integer $k \geq 2$ and $p \geq 5$,*

$$\dim_{\mathbb{C}} S_k^{\pm,\mathrm{new}}(\Gamma_0^{(1)}(p)) = \frac{1}{2}\dim_{\mathbb{C}} S_k^{\mathrm{new}}(\Gamma_0^{(1)}(p)) \pm \frac{1}{2}\Big(\frac{1}{2}hb - \delta_{k,2}\Big),$$

where h is the class number of $\mathbb{Q}(\sqrt{-p})$ and

$$b = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ 2 & \text{if } p \equiv 7 \pmod 8, \\ 4 & \text{if } p \equiv 3 \pmod 8, \end{cases} \qquad \text{and} \qquad \delta_{k,2} = \begin{cases} 1 & \text{if } k = 2, \\ 0 & \text{if } k \neq 2. \end{cases} \tag{3.8}$$

For $k > 2$,

$$\dim_{\mathbb{C}} S_k^{\pm,\mathrm{new}}(\Gamma_0^{(1)}(2)) = \frac{1}{2}\dim_{\mathbb{C}} S_k^{\mathrm{new}}(\Gamma_0^{(1)}(2)) \pm \begin{cases} \frac{1}{2} & \text{if } k \equiv 0, 2 \pmod 8, \\ 0 & \text{else}, \end{cases}$$

$$\dim_{\mathbb{C}} S_k^{\pm,\mathrm{new}}(\Gamma_0^{(1)}(3)) = \frac{1}{2}\dim_{\mathbb{C}} S_k^{\mathrm{new}}(\Gamma_0^{(1)}(3)) \pm \begin{cases} \frac{1}{2} & \text{if } k \equiv 0, 2, 6, 8 \pmod{12}, \\ 0 & \text{else}. \end{cases}$$

For any even integer $k \geq 2$, we have the following well-known results (for details, see [6, Section 3.5]):

$$\begin{aligned} \dim_{\mathbb{C}} S_k(\Gamma_0^{(1)}(p)) &= \frac{k-1}{2^2 3}(p+1) + \frac{(-1)^{\frac{k}{2}}}{2^2}\Big(1 + \Big(\frac{-1}{p}\Big)\Big) + \frac{c_3(k) + \hat{c}_3(k)}{3}\Big(1 + \Big(\frac{-3}{p}\Big)\Big) - 1 + \delta_{k,2}, \\ \dim_{\mathbb{C}} S_k^{\mathrm{new}}(\Gamma_0^{(1)}(p)) &= \dim_{\mathbb{C}} S_k(\Gamma_0^{(1)}(p)) - 2\dim_{\mathbb{C}} S_k(\mathrm{SL}(2,\mathbb{Z})). \end{aligned} \tag{3.9}$$

Theorem 3.3. *Let* $p \geq 5$ *be a prime. Suppose* h *is the class number of* $\mathbb{Q}(\sqrt{-p})$ *and* b *is defined as in* (3.8).

(i) *For* $k \geq 2$,

$$s_k(p, \mathrm{Vb}) = \begin{cases} \frac{2k-3}{2^3 3}(p-1) + \frac{1-(\frac{-1}{p})}{2^3} - \frac{\hat{c}_3(k)(1-(\frac{-3}{p}))}{2\cdot 3} - \frac{bh}{2^2} & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}. \end{cases}$$

Equivalently,

$$\sum_{k\geq 2} s_k(p, \mathrm{Vb})t^k = \Big[\frac{(1+3t^2)(p-1)}{2^3 3(1-t^2)^2} + \frac{1-(\frac{-1}{p})}{2^3(1-t^2)} + \frac{1-(\frac{-3}{p})}{2\cdot 3(1+t^2+t^4)} - \frac{bh}{2^2(1-t^2)}\Big]t^2.$$

(ii) *For* $k \geq 2$,

$$s_k^{(\mathbf{P})}(p, \mathrm{VIb}) = \begin{cases} \frac{2k-3}{2^3 3}(p-1) + \frac{1-(\frac{-1}{p})}{2^3} - \frac{\hat{c}_3(k)(1-(\frac{-3}{p}))}{2\cdot 3} + \frac{bh}{2^2} - \delta_{k,2} & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}. \end{cases}$$

Equivalently,

$$\sum_{k\geq 2} s_k^{(\mathbf{P})}(p, \mathrm{VIb})t^k = \Big[\frac{(1+3t^2)(p-1)}{2^3 3(1-t^2)^2} + \frac{1-(\frac{-1}{p})}{2^3(1-t^2)} + \frac{1-(\frac{-3}{p})}{2\cdot 3(1+t^2+t^4)} + \frac{bh}{2^2(1-t^2)} - 1\Big]t^2.$$

(iii) *For* $k \geq 2$,

$$s_k(p, \mathrm{VIc}) = \begin{cases} 0 & \text{if } k \text{ is even}, \\ \frac{2k-3}{2^3 3}(p-1) - \frac{1-(\frac{-1}{p})}{2^3} - \frac{\hat{c}_3(k)(1-(\frac{-3}{p}))}{2\cdot 3} - \frac{bh}{2^2} & \text{if } k \text{ is odd}. \end{cases}$$

Equivalently,

$$\sum_{k\geq 2} s_k(p, \mathrm{VIc})t^k = \Big[\frac{(3+t^2)(p-1)}{2^3 3(1-t^2)^2} - \frac{1-(\frac{-1}{p})}{2^3(1-t^2)} + \frac{(1-(\frac{-3}{p}))t^2}{2\cdot 3(1+t^2+t^4)} - \frac{bh}{2^2(1-t^2)}\Big]t^3.$$

Proof. It follows from [22, Section 1] that

$$\begin{aligned} s_k(p, \mathrm{Vb}) &= \begin{cases} \dim_{\mathbb{C}} S_{2k-2}^{-,\mathrm{new}}(\Gamma_0^{(1)}(p)) & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}, \end{cases} \\ s_k^{(\mathbf{P})}(p, \mathrm{VIb}) &= \begin{cases} \dim_{\mathbb{C}} S_{2k-2}^{+,\mathrm{new}}(\Gamma_0^{(1)}(p)) & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}, \end{cases} \\ s_k(p, \mathrm{VIc}) &= \begin{cases} 0 & \text{if } k \text{ is even}, \\ \dim_{\mathbb{C}} S_{2k-2}^{-,\mathrm{new}}(\Gamma_0^{(1)}(p)) & \text{if } k \text{ is odd}. \end{cases} \end{aligned} \tag{3.10}$$

Then, using Lemma 3.2, (3.3) and (3.9), the formulas for $s_k^{(\mathbf{P})}(p, \mathrm{VIb})$, $s_k(p, \mathrm{VIc})$ and $s_k(p, \mathrm{Vb})$ follow from straightforward calculations. □

For the next theorem, we define

$$C(p)=\frac{p-1}{2^3 3}+\frac{1-\left(\frac{-1}{p}\right)}{2^3}+\frac{1-\left(\frac{-3}{p}\right)}{2\cdot 3}-\frac{1}{2}$$

for any prime $p\geq 5$.

Theorem 3.4. *Let $p\geq 5$ be a prime. Suppose h is the class number of $\mathbb{Q}(\sqrt{-p})$ and b is defined as in* (3.8). *For $k\geq 2$,*

$$\begin{aligned}s_k^{(\mathbf{Y})}(p,\mathrm{VIb})&=C(p)\cdot\left(\frac{2k-3}{2^2 3}(p-1)+(-1)^k\frac{1-\left(\frac{-1}{p}\right)}{2^2}-\frac{\hat{c}_3(k)(1-\left(\frac{-3}{p}\right))}{3}-\delta_{k,2}\right)\\&\quad+\frac{2-bh}{2^2}\delta_{k,2}+(-1)^k\frac{b^2h^2-2bh}{2^3}.\end{aligned}$$

Equivalently,

$$\begin{aligned}\sum_{k\geq 2}s_k^{(\mathbf{Y})}(p,\mathrm{VIb})t^k&=\left[C(p)\left(\frac{1+t}{2^2 3(1-t)^2}(p-1)+\frac{1-\left(\frac{-1}{p}\right)}{2^2(1+t)}+\frac{(1+t)(1-\left(\frac{-3}{p}\right))}{3(1+t+t^2)}-1\right)\right.\\&\quad\left.+\frac{2-bh}{2^2}+\frac{b^2h^2-2bh}{2^3(1+t)}\right]t^2.\end{aligned}$$

Proof. In order to compute $s_k^{(\mathbf{Y})}(p,\mathrm{VIb})$, we look at the Yoshida lifting. This lifting associates a holomorphic Siegel modular form $F\in S_k^{\mathrm{new}}(\Gamma_0(p))$ to two eigenforms

$$f\in S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))\quad\text{and}\quad g\in S_2^{\mathrm{new}}(\Gamma_0^{(1)}(p));$$

see [20, Proposition 3.1]. Let $\pi_f=\bigotimes_{v\leq\infty}\pi_{f,v}$ and $\pi_g=\bigotimes_{v\leq\infty}\pi_{g,v}$ be the automorphic representations of $\mathrm{PGL}(2,\mathbb{A}_\mathbb{Q})$ attached to f and g, respectively. Both local representations $\pi_{f,p}$ and $\pi_{g,p}$ are either the Steinberg representation $\mathrm{St}_{\mathrm{GL}(2)}$ or its non-trivial unramified twist $\xi\mathrm{St}_{\mathrm{GL}(2)}$; here ξ is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^\times$. In order to produce a Yoshida lifting $\pi_F=\bigotimes_{p\leq\infty}\pi_{F,p}$ with Iwahori-fixed vectors at p, we must have $\pi_{f,p}=\pi_{g,p}$ by [20, (16)]. In this case, $\pi_{F,p}$ is of type VIb. More precisely, if $\pi_{f,p}=\pi_{g,p}=\mathrm{St}_{\mathrm{GL}(2)}$ (and hence the local root numbers at p are -1), then $\pi_{F,p}=\tau(S,\nu^{-1/2})$, and if $\pi_{f,p}=\pi_{g,p}=\xi\mathrm{St}_{\mathrm{GL}(2)}$ (and hence the local root numbers at p are $+1$), then $\pi_{F,p}=\tau(S,\nu^{-1/2}\xi)$. Since the archimedean signs are $(-1)^{k-1}$ for eigenforms in $S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))$, and -1 for eigenforms in $S_2^{\mathrm{new}}(\Gamma_0^{(1)}(p))$, it follows that

$$s_k^{(\mathbf{Y})}(p,\mathrm{VIb})=\begin{cases}\dim_\mathbb{C}S_{2k-2}^{+,\mathrm{new}}(\Gamma_0^{(1)}(p))\times\dim_\mathbb{C}S_2^{+,\mathrm{new}}(\Gamma_0^{(1)}(p))\\\qquad+\dim_\mathbb{C}S_{2k-2}^{-,\mathrm{new}}(\Gamma_0^{(1)}(p))\times\dim_\mathbb{C}S_2^{-,\mathrm{new}}(\Gamma_0^{(1)}(p))&\text{if }k\text{ is even},\\\dim_\mathbb{C}S_{2k-2}^{-,\mathrm{new}}(\Gamma_0^{(1)}(p))\times\dim_\mathbb{C}S_2^{+,\mathrm{new}}(\Gamma_0^{(1)}(p))\\\qquad+\dim_\mathbb{C}S_{2k-2}^{+,\mathrm{new}}(\Gamma_0^{(1)}(p))\times\dim_\mathbb{C}S_2^{-,\mathrm{new}}(\Gamma_0^{(1)}(p))&\text{if }k\text{ is odd}.\end{cases}\tag{3.11}$$

Then, using (3.3), (3.9), (3.10), (3.11) and Lemma 3.2, we obtain the general formula and hence the generating function for $s_k^{(\mathbf{Y})}(p,\mathrm{VIb})$. □

3.2 The representation types IIa, IIIa + VIa/b, IVa, Va

In this section, we compute the generating functions $\sum_{k\geq 3}s_k(p,\Omega)t^k$ for the representation types

$$\Omega\in\{\mathrm{IIa},\mathrm{IIIa}+\mathrm{VIa/b},\mathrm{IVa},\mathrm{Va}\}.$$

Note that the Arthur packets contributing to such $S_k(p,\Omega)$ are necessarily of type (**G**).

We will use dimension formulas for the spaces $S_k(\Gamma)$, where Γ is one of $\mathrm{K}(p)$, $\Gamma_0(p)$, $\Gamma_0'(p)$ or $B(p)$. Many authors have contributed such dimension formulas. We summarize the sources of the formulas we will use in Table 3, without claim to historical completeness.

In the following theorems, let b and h be as in Lemma 3.2. We will also use the quantities defined in (3.1) and (3.2).

		$p=2$	$p=3$	$p\geq 5$
K(p)	$k=1$	[13, Theorem 6.1]	[13, Theorem 6.1]	[13, Theorem 6.1]
	$k=2$	[11, Section 1]	[14, Section 5.3]	[17] ($p<600$)
	$k=3$	[13, Theorem 2.1]	[13, Theorem 2.1]	[13, Theorem 2.1]
	$k=4$	[13, Section 2.4]	[13, Section 2.4]	[13, Section 2.4]
	$k\geq 5$	[12, Theorem 4]	[12, Theorem 4]	[12, Theorem 4]
$\Gamma_0(p)$	$k=1$	[13, Theorem 6.1]	[13, Theorem 6.1]	[13, Theorem 6.1]
	$k=2$	[11, Section 1]	[14, Section 5.3]	
	$k=3$	[13, Theorem 2.2]	[13, Theorem 2.2]	[13, Theorem 2.2]
	$k=4$	[28, Corollary 4.12]	[28, Corollary 4.12]	[28, Corollary 4.12]
		[13, Section 2.4]	[13, Section 2.4]	[13, Section 2.4]
	$k\geq 5$	[28, Corollary 4.12]	[9, Theorem 7-1]	[9, Theorem 7-1]
		[29, Theorem 7.4]	[28, Corollary 4.12]	[28, Corollary 4.12]
			[29, Theorem 7.4]	[29, Theorem 7.4]
$\Gamma_0'(p)$	$k=1$	[13, Theorem 6.1]	[13, Theorem 6.1]	[13, Theorem 6.1]
	$k=2$	[11, Section 1]	[14, Section 5.3]	
	$k=3$	[13, Theorem 2.4]	[13, Theorem 2.4]	[13, Theorem 2.4]
	$k=4$	[13, Section 2.4]	[13, Section 2.4]	[13, Section 2.4]
	$k\geq 5$	[30, Theorem A.1]	[30, Theorem A.1]	[10, Theorem 3.3]
				[30, Theorem A.1]
$B(p)$	$k=1$	[13, Theorem 6.1]	[13, Theorem 6.1]	[13, Theorem 6.1]
	$k=2$	[11, Section 1]	Proposition 3.12	
	$k=3$	[13, Theorem 2.3]	[13, Theorem 2.3]	[13, Theorem 2.3]
	$k=4$	[13, Section 2.4]	[13, Section 2.4]	[13, Section 2.4]
	$k\geq 5$	[30, Theorem A.2]	[30, Theorem A.2]	[10, Theorem 3.2]
				[30, Theorem A.2]

Table 3: History of dimension formulas for congruence subgroups of Iwahori-type. Earlier references appear above later references.

Theorem 3.5. *Let $p\geq 5$ be a prime and $k\geq 3$. Then*

$$\begin{aligned}
s_k(p,\mathrm{IIa}) = {} & \frac{p^2-1}{2^7 3^3 5}(k-2)(k-1)(2k-3) + \frac{-4(\frac{-3}{p})-3(\frac{-1}{p})+p-3}{2^3 3} + \frac{bh}{2^2} - \delta_{k,3}\\
& + \frac{16(p+3)(\frac{-3}{p})+9(p+4)(\frac{-1}{p})-84p+119}{2^7 3^3}(2k-3)\\
& + \frac{(16(\frac{-3}{p})-p+12)((\frac{-1}{p})-1)+3(p-49)}{2^7 3}(-1)^k + \frac{(-1)^k(2k-3)}{2^3 3}\\
& + \frac{((\frac{-3}{p})+1)(9(\frac{-1}{p})+p-6)-4(p-8)}{2^3 3^3}\hat{c}_3(k) + \frac{(-1)^k \hat{c}_3(k)}{2\cdot 3}\\
& - \frac{c_4(k)}{2^2}\begin{cases} 0, & p\equiv 1,7 \pmod 8,\\ 1, & p\equiv 3,5 \pmod 8,\end{cases} - \frac{c_5(k)}{5}\begin{cases} 1, & p=5,\\ 0, & p\equiv 1,4 \pmod 5,\\ 2, & p\equiv 2,3 \pmod 5.\end{cases}
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\sum_{k\geq 3} s_k(p,\mathrm{IIa})t^k & \\
= \Bigg[& \frac{(p^2-1)(1+t)}{2^6 3^2 5(1-t)^4} - \frac{(p-1)(5+13t+17t^2+12t^3+12t^4)}{2^5 3^2(1-t^2)(1-t^3)} + \frac{t^7}{(1-t^2)(1-t^6)} + \frac{bh}{2^2(1-t)} - 1\\
& + \left(\frac{p(1+t)}{2^3 3^2(1-t)(1-t^3)} + \frac{-3+t+4t^3}{2^3 3(1-t)(1-t^3)}\right)\left(\left(\frac{-3}{p}\right)-1\right)\\
& + \left(\frac{p}{2^5 3(1-t)(1-t^2)} + \frac{-4+2t-t^2+3t^3+3t^4}{2^3 3(1-t^2)(1-t^3)}\right)\left(\left(\frac{-1}{p}\right)-1\right) - \frac{((\frac{-3}{p})-1)((\frac{-1}{p})-1)}{2^3 3(1+t)(1+t+t^2)}\Bigg]t^3
\end{aligned}$$

$$+\frac{t^3}{2^2(1+t)(1+t^2)}\begin{cases}0, & p\equiv 1,7 \pmod 8,\\ 1, & p\equiv 3,5 \pmod 8,\end{cases}+\frac{(1+t)t^3}{5(1+t+t^2+t^3+t^4)}\begin{cases}1, & p=5,\\ 0, & p\equiv 1,4 \pmod 5,\\ 2, & p\equiv 2,3 \pmod 5).\end{cases}$$

Theorem 3.6. *Let $p\geq 5$ be a prime and $k\geq 3$. Then*

$$\begin{aligned}
&s_k(p,\mathrm{IIIa}+\mathrm{VIa/b})\\
&=\frac{(p-1)(p^2+p+2)}{2^8 3^3 5}(k-2)(k-1)(2k-3)+\frac{3(\frac{-1}{p})-p-2}{2^4 3}-\frac{bh}{2^3}-\frac{b^2h^2-2bh}{2^4}(-1)^k-\delta_{k,3}\\
&\quad+\frac{7(p-1)(p+3)(-1)^k}{2^8 3^2}(k-2)(k-1)-\frac{(p-1)(-32(\frac{-3}{p})-27(\frac{-1}{p})+12p-97)}{2^8 3^3}(2k-3)\\
&\quad-\frac{(32(\frac{-3}{p})-5p-3)(9(\frac{-1}{p})-17)-40(p+7)}{2^8 3^3}(-1)^k-\frac{p-1}{2^3 3}(-1)^k(2k-3)\\
&\quad+\frac{1-(\frac{-3}{p})}{2\cdot 3}c_3(k)+\frac{(p+5)(1-(\frac{-3}{p}))}{2^2 3^3}\hat{c}_3(k)+\frac{1-(\frac{-3}{p})}{2^2 3}\hat{c}_3(k)(-1)^k\\
&\quad+\frac{(p+1)(\frac{-1}{p})+p-3}{2^6 3}f_4(k)+\frac{1-(\frac{-1}{p})}{2^3}c_4'(k)+\frac{(p+1)(\frac{-3}{p})+p-3}{2^3 3^3}f_6(k)\\
&\quad+\frac{2((\frac{-3}{p})-2)}{3^3}c_6(k)+\frac{5(\frac{-3}{p})-13}{2^2 3^3}\hat{c}_6(k)+\frac{2((\frac{-3}{p})+1)}{3^3}c_6'(k)\\
&\quad+\frac{((\frac{-3}{p})+1)((\frac{-1}{p})+1)-4}{2^3 3}c_{12}(k)\\
&\quad-\frac{c_4(k)}{2^2}\begin{cases}1, & p\equiv 7 \pmod 8,\\ 0 & \text{otherwise},\end{cases}-\frac{c_5(k)}{5}\begin{cases}1, & p=5,\\ 0, & p\equiv 1 \pmod 5,\\ 1, & p\equiv 2,3 \pmod 5,\\ 2, & p\equiv 4 \pmod 5).\end{cases}
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\sum_{k\geq 3}s_k(p,\mathrm{IIIa}+\mathrm{VIa/b})t^k\\
&=\Bigg[\frac{(p-1)(p^2+p+2)(1+t)}{2^7 3^2 5(1-t)^4}-\frac{(p-1)(p+3)(13+2t+19t^2-2t^4)}{2^7 3^2(1+t)(1-t^2)^2}+\frac{(p-1)N(t)}{2^4 3^2(1+t^2)^2(1-t^6)^2}-1\\
&\qquad-\bigg(\frac{(p+1)C_{-3,1}(t)}{2^3 3^2(1-t)^2(1+t^2+t^4)^2}+\frac{C_{-3,2}(t)}{2^2 3^2(1-t)(1-t^2+t^4)(1-t^6)}\bigg)\bigg(\Big(\frac{-3}{p}\Big)-1\bigg)\\
&\qquad-\bigg(\frac{(p+1)C_{-1,1}(t)}{2^6 3(1-t)(1+t^2)(1-t^4)}+\frac{C_{-1,2}(t)}{2^5 3(1-t)(1-t^4)(1-t^2+t^4)}\bigg)\bigg(\Big(\frac{-1}{p}\Big)-1\bigg)\\
&\qquad+\frac{t(-2-2t+t^3)}{2^3 3(1+t)(1-t^2+t^4)}\bigg(\Big(\frac{-3}{p}\Big)-1\bigg)\bigg(\Big(\frac{-1}{p}\Big)-1\bigg)+\frac{b^2h^2}{2^4(1+t)}-\frac{bh}{2^2(1-t^2)}\Bigg]t^3\\
&\quad+\frac{t^3}{2^2(1+t)(1+t^2)}\begin{cases}1, & p\equiv 7 \pmod 8,\\ 0 & \text{otherwise},\end{cases}+\frac{t^3(1+t)}{5(1+t+t^2+t^3+t^4)}\begin{cases}1, & p=5,\\ 0, & p\equiv 1 \pmod 5),\\ 1, & p\equiv 2,3 \pmod 5),\\ 2, & p\equiv 4 \pmod 5),\end{cases}
\end{aligned}$$

where

$$\begin{aligned}
N(t)&=34-6t+133t^2-35t^3+264t^4-88t^5+344t^6-120t^7+342t^8\\
&\quad-58t^9+224t^{10}+86t^{12}+14t^{13}+13t^{14}+5t^{15},\\
C_{-3,1}(t)&=-2+2t-6t^2+6t^3-5t^4+3t^5-5t^6+3t^7-3t^8+t^9,\\
C_{-3,2}(t)&=14-8t-10t^2+6t^3-5t^4-2t^5+20t^6-6t^7-16t^8+4t^9+7t^{10},\\
C_{-1,1}(t)&=-3-4t+4t^2-8t^3+t^4-4t^5+2t^6,\\
C_{-1,2}(t)&=12+10t-43t^2+40t^4-36t^6+10t^7+13t^8.
\end{aligned}$$

Theorem 3.7. *Let $p \geq 5$ be a prime and $k \geq 3$. Then*

$$\begin{aligned}
s_k(p,\mathrm{IVa}) = {} & \frac{(p-1)(p^3-1)}{2^7 3^3 5}(k-2)(k-1)(2k-3) + \frac{7(p-1)^2(-1)^k}{2^7 3^2}(k-2)(k-1) + \delta_{k,3} \\
& + \frac{(p-1)(16(\frac{-3}{p}) + 9(\frac{-1}{p}) - 25)}{2^7 3^3}(2k-3) + \frac{((\frac{-3}{p})-1)(9(\frac{-1}{p}) + p - 10)}{2^3 3^3}\hat{c}_3(k) \\
& + \frac{(16(\frac{-3}{p}) - p - 15)(9(\frac{-1}{p}) - 25) - 16(p+31)}{2^7 3^3}(-1)^k + \frac{(\frac{-1}{p}) - 1}{2^5 3}(p-1)f_4(k) \\
& + \frac{(\frac{-3}{p}) - 1}{2^2 3^3}(p-1)f_6(k) - \frac{4((\frac{-3}{p}) - 2)}{3^3}c_6(k) + \frac{2((\frac{-3}{p}) + 1)}{3^3}\hat{c}_6(k) - \frac{4((\frac{-3}{p}) + 1)}{3^3}c_6'(k) \\
& + \frac{((\frac{-3}{p}) - 1)((\frac{-1}{p}) - 1)}{2^2 3}c_{12}(k) \\
& + \frac{c_4(k)}{2}\begin{cases} 1, & p \equiv 7 \ (\mathrm{mod}\ 8), \\ 0 & \text{otherwise}, \end{cases} + \frac{c_5(k)}{5}\begin{cases} 1, & p = 5, \\ 0, & p \equiv 1 \ (\mathrm{mod}\ 5), \\ 2, & p \equiv 2,3 \ (\mathrm{mod}\ 5), \\ 4, & p \equiv 4 \ (\mathrm{mod}\ 5). \end{cases}
\end{aligned}$$

Equivalently,

$$\begin{aligned}
& \sum_{k \geq 3} s_k(p,\mathrm{IVa}) t^k \\
& \quad = \Bigg[\frac{(p-1)(p^3-1)(t+1)}{2^6 3^2 5 (1-t)^4} - \frac{7(p-1)^2}{2^6 3^2 (1+t)^3} + 1 \\
& \qquad + \bigg(\frac{(p-1)(1+t)(3 - 5t + 10t^2 - 13t^3 + 10t^4 - 5t^5 + 3t^6)}{2^3 3^2 (1-t)^2 (1+t^2+t^4)^2} + \frac{2}{3^2(1+t^3)} \bigg)\bigg(\Big(\frac{-3}{p}\Big) - 1 \bigg) \\
& \qquad + \frac{(p-1)(3 - 2t^2 + 3t^4)((\frac{-1}{p}) - 1)}{2^5 3 (1-t)(1+t^2)(1-t^4)} - \frac{(3 + 6t + 7t^2 + 6t^3 + 3t^4)((\frac{-3}{p}) - 1)((\frac{-1}{p}) - 1)}{2^3 3 (1+t)(1+t+t^2)(1-t^2+t^4)} \Bigg] t^3 \\
& \qquad - \frac{t^3}{2(1+t)(1+t^2)}\begin{cases} 1, & p \equiv 7 \ (\mathrm{mod}\ 8), \\ 0 & \text{otherwise}, \end{cases} - \frac{t^3(1+t)}{5(1+t+t^2+t^3+t^4)}\begin{cases} 1, & p = 5, \\ 0, & p \equiv 1 \ (\mathrm{mod}\ 5), \\ 2, & p \equiv 2,3 \ (\mathrm{mod}\ 5), \\ 4, & p \equiv 4 \ (\mathrm{mod}\ 5). \end{cases}
\end{aligned}$$

Theorem 3.8. *Let $p \geq 5$ be a prime and $k \geq 3$. Then*

$$\begin{aligned}
s_k(p,\mathrm{Va}) = {} & \frac{p(p-1)^2}{2^8 3^3 5}(k-2)(k-1)(2k-3) + \frac{1 - (\frac{-1}{p})}{2^4} - \frac{bh}{2^3} + \frac{b^2h^2 - 2bh}{2^4}(-1)^k \\
& - \frac{7(p-1)^2(-1)^k}{2^8 3^2}(k-2)(k-1) + \frac{p-1}{2^4 3}(2k-3)(-1)^k - \frac{1 - (\frac{-3}{p})}{2^2 3}\hat{c}_3(k)(-1)^k \\
& - \frac{(p-1)(-32(\frac{-3}{p}) - 27(\frac{-1}{p}) + 12p - 97)}{2^8 3^3}(2k-3) - \frac{(p-4)((\frac{-3}{p}) - 1)}{2^2 3^3}\hat{c}_3(k) \\
& + \frac{(32(\frac{-3}{p}) - 5p - 3)(-9(\frac{-1}{p}) + 1) - 40(p+7)}{2^8 3^3}(-1)^k - \frac{(p-1)((\frac{-1}{p}) - 1)}{2^6 3}f_4(k) \\
& - \frac{(p-1)((\frac{-3}{p}) - 1)}{2^3 3^3}f_6(k) + \frac{2(2(\frac{-3}{p}) - 1)}{3^3}c_6(k) + \frac{(\frac{-3}{p}) + 1}{3^3}\hat{c}_6(k) - \frac{2((\frac{-3}{p}) + 1)}{3^3}c_6'(k) \\
& - \frac{((\frac{-3}{p}) - 1)((\frac{-1}{p}) - 1)}{2^3 3}c_{12}(k) \\
& - \frac{c_4(k)}{2^2}\begin{cases} 1, & p \equiv 7 \ (\mathrm{mod}\ 8), \\ 0 & \text{otherwise}, \end{cases} + \frac{c_5(k)}{5}\begin{cases} 1, & p \equiv 2,3 \ (\mathrm{mod}\ 5), \\ -2, & p \equiv 4 \ (\mathrm{mod}\ 5), \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}$$

Equivalently,

$$\sum_{k\geq 3} s_k(p,\mathrm{Va})t^k = \Big[\frac{p(p-1)^2(1+t)}{2^7 3^2 5(1-t)^4} + \frac{(p-1)^2(1-30t-5t^2+2t^4)}{2^7 3^2(1-t)^2(1+t)^3} + \frac{(p-1)(5-t^2)t}{2^3 3(1-t^2)^2} - \frac{b^2h^2}{2^4(1+t)} - \frac{bht}{2^2(1-t^2)}$$
$$+ \Big(\frac{(p-1)(2-2t+9t^2-7t^3+7t^4-5t^5+3t^6-t^7)t^2}{2^3 3^2(1-t)^2(1+t^2+t^4)^2} + \frac{-2+t^2+3t^3+2t^4}{2\cdot 3^2(1+t)(1+t^2+t^4)}\Big)\Big(\Big(\frac{-3}{p}\Big)-1\Big)$$
$$- \Big(\frac{(p-1)(1-4t-4t^2-8t^3+5t^4-4t^5+2t^6)}{2^6 3(1-t)(1+t^2)(1-t^4)} + \frac{1+3t}{2^5(1-t^2)}\Big)\Big(\Big(\frac{-1}{p}\Big)-1\Big)$$
$$+ \frac{2+2t+t^4}{2^3 3(1+t)(1-t^2+t^4)}\Big(\Big(\frac{-3}{p}\Big)-1\Big)\Big(\Big(\frac{-1}{p}\Big)-1\Big)\Big]t^3$$
$$+ \frac{t^3}{2^2(1+t)(1+t^2)}\begin{cases}1, & p\equiv 7 \ (\mathrm{mod}\ 8),\\ 0 & \text{otherwise},\end{cases} - \frac{t^3(1+t)}{5(1+t+t^2+t^3+t^4)}\begin{cases}1, & p\equiv 2,3 \ (\mathrm{mod}\ 5),\\ -2, & p\equiv 4 \ (\mathrm{mod}\ 5),\\ 0 & \text{otherwise}.\end{cases}$$

Proof of Theorems 3.5–3.8. By (2.1) and [21, Table 3], we obtain

$$\begin{aligned}
\dim_{\mathbb{C}} S_k(\mathrm{K}(p)) &= 2s_k(p,\mathrm{I}) + s_k(p,\mathrm{IIa}) + s_k(p,\mathrm{IIb}) + s_k(p,\mathrm{Vb}) + s_k(p,\mathrm{VIc}),\\
\dim_{\mathbb{C}} S_k(\Gamma_0(p)) &= 4s_k(p,\mathrm{I}) + s_k(p,\mathrm{IIa}) + 3s_k(p,\mathrm{IIb}) + 2s_k(p,\mathrm{IIIa}) + s_k(p,\mathrm{Vb})\\
&\quad + s_k(p,\mathrm{VIa}) + s_k(p,\mathrm{VIb}),\\
\dim_{\mathbb{C}} S_k(\Gamma_0'(p)) &= 4s_k(p,\mathrm{I}) + 2s_k(p,\mathrm{IIa}) + 2s_k(p,\mathrm{IIb}) + s_k(p,\mathrm{IIIa}) + s_k(p,\mathrm{Va})\\
&\quad + s_k(p,\mathrm{Vb}) + s_k(p,\mathrm{VIa}) + s_k(p,\mathrm{VIc}),\\
\dim_{\mathbb{C}} S_k(B(p)) &= 8s_k(p,\mathrm{I}) + 4s_k(p,\mathrm{IIa}) + 4s_k(p,\mathrm{IIb}) + 4s_k(p,\mathrm{IIIa}) + s_k(p,\mathrm{IVa})\\
&\quad + 2s_k(p,\mathrm{Va}) + 2s_k(p,\mathrm{Vb}) + 3s_k(p,\mathrm{VIa}) + s_k(p,\mathrm{VIb}) + s_k(p,\mathrm{VIc}).
\end{aligned} \tag{3.12}$$

Let us replace $s_k(p,\mathrm{VIb})$ by $s_k^{(\mathbf{G})}(p,\mathrm{VIb}) + s_k^{(\mathbf{Y})}(p,\mathrm{VIb}) + s_k^{(\mathbf{P})}(p,\mathrm{VIb})$. Observing (2.5) and (2.6), we get the system of equations

$$\begin{aligned}
s_k(p,\mathrm{IIa}) &= \dim_{\mathbb{C}} S_k(\mathrm{K}(p)) - 2s_k(p,\mathrm{I}) - s_k(p,\mathrm{IIb}) - s_k(p,\mathrm{Vb}) - s_k(p,\mathrm{VIc}),\\
s_k(p,\mathrm{IIIa}+\mathrm{VIa/b}) &= \frac{1}{2}\dim_{\mathbb{C}} S_k(\Gamma_0(p)) - \frac{1}{2}\dim_{\mathbb{C}} S_k(\mathrm{K}(p))\\
&\quad - s_k(p,\mathrm{I}) - s_k(p,\mathrm{IIb}) - \frac{1}{2}s_k^{(\mathbf{P})}(p,\mathrm{VIb}) - \frac{1}{2}s_k^{(\mathbf{Y})}(p,\mathrm{VIb}) + \frac{1}{2}s_k(p,\mathrm{VIc}),\\
s_k(p,\mathrm{Va}) &= \dim_{\mathbb{C}} S_k(\Gamma_0'(p)) - \frac{1}{2}\dim_{\mathbb{C}} S_k(\Gamma_0(p)) - \frac{3}{2}\dim_{\mathbb{C}} S_k(\mathrm{K}(p))\\
&\quad + s_k(p,\mathrm{I}) + s_k(p,\mathrm{IIb}) + s_k(p,\mathrm{Vb}) + \frac{1}{2}s_k^{(\mathbf{P})}(p,\mathrm{VIb}) + \frac{1}{2}s_k^{(\mathbf{Y})}(p,\mathrm{VIb}) + \frac{1}{2}s_k(p,\mathrm{VIc}),\\
s_k(p,\mathrm{IVa}) &= \dim_{\mathbb{C}} S_k(B(p)) + \dim_{\mathbb{C}} S_k(\mathrm{K}(p)) - \dim_{\mathbb{C}} S_k(\Gamma_0(p)) - 2\dim_{\mathbb{C}} S_k(\Gamma_0'(p))\\
&\quad + 2s_k(p,\mathrm{I}) + 2s_k(p,\mathrm{IIb}).
\end{aligned}$$

We derive the explicit formulas of $s_k(p,\Omega)$ for $\Omega\in\{\mathrm{IIa},\mathrm{IIIa}+\mathrm{VIa/b},\mathrm{IVa},\mathrm{Va}\}$ using Theorems 3.1, 3.3, 3.4 and the global dimension formulas of $S_k(\mathrm{K}(p))$, $S_k(\Gamma_0(p))$, $S_k(\Gamma_0'(p))$ and $S_k(B(p))$. The generating series $\sum_{k\geq 3} s_k(p,\Omega)t^k$ follow in a straightforward way. □

Note that the quantities $s_k(p,\mathrm{IVa})$ and $s_k(p,\mathrm{Va})$ are the same as the quantities

$$n(D^{\mathrm{Hol}}_{k-1,k-2},\mathrm{St},p) \quad \text{and} \quad n(D^{\mathrm{Hol}}_{k-1,k-2},\mathrm{Va},p)$$

in [30], respectively.

For $p = 2, 3$

Here we give rational expression for the generating function of $s_k(p, \Omega)$ for $p = 2, 3$. We compute them separately as follows because the formulas in Lemma 3.2 are different for these two primes:

$$\sum_{k\geq 2} s_k(2,\mathrm{Vb})t^k = \frac{t^8}{(1-t^4)(1-t^6)}, \quad \sum_{k\geq 2} s_k^{(\mathbf{P})}(2,\mathrm{VIb})t^k = \frac{t^6+t^8-t^{12}}{(1-t^4)(1-t^6)},$$

$$\sum_{k\geq 2} s_k(3,\mathrm{Vb})t^k = \frac{t^6}{(1-t^2)(1-t^6)}, \quad \sum_{k\geq 2} s_k^{(\mathbf{P})}(3,\mathrm{VIb})t^k = \frac{t^4+t^8-t^{10}}{(1-t^2)(1-t^6)},$$

$$\sum_{k\geq 2} s_k(2,\mathrm{VIc})t^k = \frac{t^{11}}{(1-t^4)(1-t^6)}, \quad \sum_{k\geq 2} s_k^{(\mathbf{Y})}(2,\mathrm{VIb})t^k = 0,$$

$$\sum_{k\geq 2} s_k(3,\mathrm{VIc})t^k = \frac{t^9}{(1-t^2)(1-t^6)}, \quad \sum_{k\geq 2} s_k^{(\mathbf{Y})}(3,\mathrm{VIb})t^k = 0,$$

$$\sum_{k\geq 3} s_k(2,\mathrm{IIa})t^k = \frac{t^{19}(-t^8-t^6+t^4+t^2+1)}{(1-t^4)^2(1-t^6)(1-t^{10})} + \frac{t^{16}(t^8-t^6-t^4+t^2+1)}{(1-t^4)^2(1-t^6)(1-t^{10})},$$

$$\sum_{k\geq 3} s_k(3,\mathrm{IIa})t^k = \frac{t^{15}(-t^{14}-t^{12}-t^{10}+2t^8+2t^6+t^4+t^2+1)}{(1-t^4)(1-t^6)^2(1-t^{10})} + \frac{t^{12}(t^{14}-t^{12}-t^{10}+2t^6+t^4+t^2+1)}{(1-t^4)(1-t^6)^2(1-t^{10})},$$

$$\sum_{k\geq 3} s_k(2,\mathrm{IIIa}+\mathrm{VIa/b})t^k = \frac{t^{25}(-t^{10}+t^8+t^6+t^4+t^2+1)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} + \frac{t^{12}(t^{22}-t^{18}-t^{16}-t^{14}-2t^{12}+t^8+2t^6+2t^4+2t^2+1)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

$$\sum_{k\geq 3} s_k(3,\mathrm{IIIa}+\mathrm{VIa/b})t^k = \frac{t^{17}(-t^{18}+t^{16}+2t^{14}+2t^{12}+2t^{10}+3t^8+2t^6+t^4+t^2+1)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} + \frac{t^8(t^{26}-t^{22}-2t^{20}-2t^{18}-2t^{16}+t^{12}+4t^{10}+5t^8+4t^6+3t^4+2t^2+1)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

$$\sum_{k\geq 3} s_k(2,\mathrm{IVa})t^k = \frac{t^{13}(t^{22}-t^{18}-t^{16}+t^{12}+2t^8+2t^6+t^4+t^2+1)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} + \frac{t^{10}(-t^{22}+t^{18}+t^{16}+t^{12}+2t^{10}+t^4+t^2+1)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

$$\sum_{k\geq 3} s_k(3,\mathrm{IVa})t^k = \frac{t^9(t^{26}-t^{22}+2t^{18}+5t^{16}+5t^{14}+9t^{12}+9t^{10}+8t^8+6t^6+5t^4+2t^2+1)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} + \frac{t^6(-t^{26}+t^{22}+2t^{20}+2t^{18}+5t^{16}+7t^{14}+7t^{12}+7t^{10}+8t^8+6t^6+5t^4+2t^2+1)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

$$\sum_{k\geq 3} s_k(2,\mathrm{Va})t^k = \frac{t^{15}(-t^{12}+t^2+1)+t^{30}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

$$\sum_{k\geq 3} s_k(3,\mathrm{Va})t^k = \frac{t^{11}(-t^{18}-t^{16}+2t^8+2t^6+2t^4+t^2+1)+t^{16}(1+t^4+t^6)(1+t^8)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

Note that the series $\sum_{k\geq 3} k^n t^k$ has a pole of order $n+1$ at $t = 1$. The pole of order 4 at $t = 1$ in the rational expression of $\sum_{k\geq 3} s_k(p, \Omega)t^k$ for $\Omega \in \{\mathrm{I}, \mathrm{IIa}, \mathrm{IIIa} + \mathrm{VIa/b}, \mathrm{IVa}, \mathrm{Va}\}$ is coming from the term $(k-2)(k-1)(2k-3)$ in $s_k(p, \Omega)$. In fact, we have the following results.

Corollary 3.9. *Let $p \geq 2$ be a prime and $k \geq 3$. Then, for $\Omega \in \{\mathrm{I}, \mathrm{IIa}, \mathrm{IIIa} + \mathrm{VIa/b}, \mathrm{IVa}, \mathrm{Va}\}$,*

$$s_k(p,\Omega) = a_\Omega \cdot \frac{(k-2)(k-1)(\frac{k}{3}-\frac{1}{2})}{2^6 3^2 5} + b_\Omega \cdot \frac{7(-1)^k}{2^7 3^2}(k-2)(k-1) + O(k), \tag{3.13}$$

where a_Ω and b_Ω are given in Table 4.

Ω	I	IIa	IIIa + VIa/b	IVa	Va
a_Ω	1	p^2-1	$\frac{(p-1)(p^2+p+2)}{2}$	$(p-1)(p^3-1)$	$\frac{p(p-1)^2}{2}$
b_Ω	1	0	$\frac{(p-1)(p+3)}{2}$	$(p-1)^2$	$-\frac{(p-1)^2}{2}$

Table 4

Proof. For $p \geq 5$, the result follows from Theorems 3.1, 3.5–3.8. For $p = 2, 3$, we note that the second term of (3.13) does not contribute a pole at $t = 1$ in $\sum_{k\geq 3} s_k(p, \Omega)t^k$, and the rational expression for

$$\sum_{k\geq 3} s_k(p, \Omega)t^k - \sum_{k\geq 3} a_\Omega \cdot \frac{(k-2)(k-1)(\frac{k}{3}-\frac{1}{2})}{2^6 3^2 5} t^k$$

has a pole of order 2 at $t = 1$. Hence we obtain (3.13) for $p = 2, 3$ as well. □

In Section 4.1, we will show that a_Ω equals the total Plancherel measure of the tempered Iwahori-spherical representations of $\mathrm{PGSp}(4, \mathbb{Q}_p)$ of type Ω. We do not know a similar interpretation for the quantity b_Ω.

3.3 The cases $k = 1$ and $k = 2$

The formulas in the theorems in the previous section hold for $k \geq 3$. We now consider $k = 1$ and $k = 2$.

Proposition 3.10. *We have $s_1(p, \Omega) = 0$ for all Ω in Table 1.*

Proof. By [13, Theorem 6.1], the left-hand sides of the equations in (3.12) are all zero. Since all the quantities on the right-hand sides are non-negative numbers, this implies our assertion. □

We cannot determine the numbers $s_2(p, \Omega)$ in general because of a lack of global dimension formulas, but we can at least treat the cases $p = 2$ and $p = 3$ (see Table 3). The following lemma is useful for proving that $\dim_{\mathbb{C}} S_2(B(3)) = 0$.

Lemma 3.11. *Let k be a positive integer, and let Γ be either $\Gamma_0(p)$ or $\Gamma_0'(p)$. Suppose $f \in S_k(B(p))$. If $f^2 \in S_{2k}(\Gamma)$, then $f \in S_k(\Gamma)$.*

Proof. Suppose $f \in S_k(B(p))$ and $f^2 \in S_{2k}(\Gamma)$. We have the slash operator $(f|_k\gamma)(Z) := (CZ + D)^{-k} f(\gamma Z)$ for $\gamma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}(4, \mathbb{Z})$. Since $(f|_k\gamma)^2 = f^2|_{2k}\gamma = f^2$ for $\gamma \in \Gamma$, there exists a sign $\varepsilon_\gamma \in \{\pm 1\}$ such that $f|_k\gamma = \varepsilon_\gamma f$. Since $f|_k\gamma_1\gamma_2 = (f|_k\gamma_1)|_k\gamma_2$, we see that $\varepsilon\colon \Gamma \to \{\pm 1\}$ is a homomorphism and $f \in S_k(\Gamma, \varepsilon)$.

We will show that ε is trivial. Since $f \in S_k(B(p))$, the kernel H of ε contains $B(p)$. Hence H is a normal subgroup of Γ with $B(p) \subset H \subset \Gamma$. This implies $H = \Gamma$, which is maybe most easily seen by applying the projection mod p. □

Proposition 3.12. *We have $s_2(p, \Omega) = 0$ for $p \in \{2, 3\}$ and all Ω in Table 1.*

Proof. We already know this is true for $\Omega = \mathrm{I}$ and for all the Saito–Kurokawa and Yoshida types. Hence the formulas in (3.12) become

$$\begin{aligned}
\dim_{\mathbb{C}} S_2(\mathrm{K}(p)) &= s_2(p, \mathrm{IIa}),\\
\dim_{\mathbb{C}} S_2(\Gamma_0(p)) &= s_2(p, \mathrm{IIa}) + 2s_2(p, \mathrm{IIIa+VIa/b}),\\
\dim_{\mathbb{C}} S_2(\Gamma_0'(p)) &= 2s_2(p, \mathrm{IIa}) + s_2(p, \mathrm{IIIa+VIa/b}) + s_2(p, \mathrm{Va}),\\
\dim_{\mathbb{C}} S_2(B(p)) &= 4s_2(p, \mathrm{IIa}) + 4s_2(p, \mathrm{IIIa+VIa/b}) + s_2(p, \mathrm{IVa}) + 2s_2(p, \mathrm{Va}).
\end{aligned}$$

Most of the dimensions on the left, except for $S_2(B(3))$, are known to be zero by [14, Section 5.3] and [11, Section 1]. It follows that the only potentially non-zero number is

$$\dim_{\mathbb{C}} S_2(B(3)) = s_2(3, \mathrm{IVa}).$$

Suppose that there exists a non-zero $f \in S_2(B(3))$. We have

$$\dim_{\mathbb{C}} S_4(B(3)) = \dim_{\mathbb{C}} S_4(\Gamma_0(3)) = 1$$

by [13, Section 2.4]. In particular, $S_4(B(3)) = S_4(\Gamma_0(3))$. The function f^2 spans this 1-dimensional space. Lemma 3.11 then implies that $f \in S_2(\Gamma_0(3))$. But $S_2(\Gamma_0(3)) = 0$ by [14, Section 5.3], a contradiction. So $s_2(3, \mathrm{IVa}) = \dim_{\mathbb{C}} S_2(B(3)) = 0$. □

For $\Omega = \{\mathrm{IIa}, \mathrm{IIIa} + \mathrm{VIa/b}, \mathrm{IVa}, \mathrm{Va}\}$, we give some numerical examples of $s_k(p, \Omega)$, $2 \le p < 20$, in Appendix A.

3.4 Dimensions of the spaces of newforms

In this section, we discuss some of the available notions of newforms for the spaces of Siegel modular forms and write their dimensions in terms of the quantities $s_k(p, \Omega)$.

There is no uniform definition of the spaces of newforms for Siegel modular forms for all the congruence subgroups in (1.3), but there have been several attempts in the literature to define a good notion of Siegel modular newforms. Ibukiyama defines old- and newforms for the minimal congruence subgroup $B(p)$ in [11], and for the paramodular group $\mathrm{K}(p)$ in [12]. He provides further evidence to support these definitions. There is a definition of newforms for $\Gamma_0(N)$ for any N by Andrianov [1]. These definitions coincide with the notion of newforms in [21] for $B(p)$, $\Gamma_0(p)$ and $\mathrm{K}(p)$.

Now there is a well established newform theory for Siegel modular forms with respect to the paramodular group $\mathrm{K}(p)$; see [18, 21]. A Siegel modular form with respect to the paramodular group is called a paramodular form. Let $S_k^{\mathrm{new}}(\mathrm{K}(p))$ be the space of paramodular newforms, and let $S_k^{\mathrm{new},(\mathbf{G})}(\mathrm{K}(p))$ be the space of paramodular newforms of $(\mathbf{G})$ type. Using the local and global newform theory, we can write $\dim_{\mathbb{C}} S_k^{\mathrm{new}}(\mathrm{K}(p))$ and $\dim_{\mathbb{C}} S_k^{\mathrm{new},(\mathbf{G})}(\mathrm{K}(p))$ in terms of the quantities $s_k(p, \Omega)$ as follows.

Proposition 3.13. *Suppose $k \ge 1$ and p is a prime. Then*

$$\dim_{\mathbb{C}} S_k^{\mathrm{new}}(\mathrm{K}(p)) = s_k(p, \mathrm{IIa}) + s_k(p, \mathrm{Vb}) + s_k(p, \mathrm{VIc}),$$
$$\dim_{\mathbb{C}} S_k^{\mathrm{new},(\mathbf{G})}(\mathrm{K}(p)) = s_k(p, \mathrm{IIa}).$$

Proof. By [21, Theorem 3.3.12], there is a paramodular newform $f \in S_k^{\mathrm{new}}(\mathrm{K}(p))$ that corresponds to a cuspidal automorphic representation $\pi \in S_k(p, \Omega)$ such that Ω is one of the types IIa, Vb and VIc. Out of these three types, only a representation of type IIa can appear as a local component of a cuspidal automorphic representation of $(\mathbf{G})$ type. The assertions follow. □

There is a definition of the space $S_k^{\mathrm{new}}(B(p))$ of newforms with respect to the Iwahori subgroup $B(p)$ in [21, Section 3.3]. Using this definition and [21, Theorem 3.3.2], it is easy to see that if $\pi \in S_k(p, \Omega)$ is the cuspidal automorphic representation associated to $f \in S_k^{\mathrm{new}}(B(p))$, then Ω is of type IVa. Hence we get the following result.

Proposition 3.14. *Suppose $k \ge 1$ and p is a prime. Then* $\dim_{\mathbb{C}} S_k^{\mathrm{new}}(B(p)) = s_k(p, \mathrm{IVa})$.

Similarly, using the definition of the space $S_k^{\mathrm{new}}(\Gamma_0(p))$ of newforms with respect to the Siegel congruence subgroup $\Gamma_0(p)$ and [21, Theorem 3.3.9], we get the following result.

Proposition 3.15. *Suppose $k \ge 1$ and p is a prime. Then*

$$\dim_{\mathbb{C}} S_k^{\mathrm{new}}(\Gamma_0(p)) = s_k(p, \mathrm{IIa}) + 2s_k(p, \mathrm{IIIa} + \mathrm{VIa/b}) + s_k(p, \mathrm{Vb}) + s_k^{(\mathbf{P})}(p, \mathrm{VIb}) + s_k^{(\mathbf{Y})}(p, \mathrm{VIb}).$$

There is no definition of the newforms with respect to the Klingen congruence subgroup given in [21]. But one can define the space $S_k^{\mathrm{new}}(\Gamma_0'(p))$ in a similar manner as the space $S_k^{\mathrm{new}}(\Gamma_0(p))$ is defined in [21]. In that case, one would have the following.

Remark 3.16. For $k \ge 1$ and any prime p, $\dim_{\mathbb{C}} S_k^{\mathrm{new}}(\Gamma_0'(p)) = s_k(p, \mathrm{IIIa} + \mathrm{VIa/b}) + s_k(p, \mathrm{Va})$.

4 Plancherel measures

In this section, we relate the global quantity $s_k(p,\Omega)$ to the total Plancherel measure of the tempered Iwahori-spherical representations of $\mathrm{PGSp}(4,\mathbb{Q}_p)$ of type Ω, which is a local quantity. In Section 4.1, we compute the local Plancherel measures of the tempered Iwahori-spherical representations of $\mathrm{PGSp}(4,\mathbb{Q}_p)$. In Section 4.2, we derive that the local representation types at $v=p$ as $\pi\cong\bigotimes_v\pi_v$ varies in $S_k(p,\Omega)$ are equidistributed with respect to their Plancherel measure. We establish a similar result for the vector-valued case using the automorphic Plancherel density theorem from [27].

4.1 Calculation of local Plancherel measures

Let F be a non-archimedean local field of characteristic zero with residual characteristic p. Let $\mathfrak{o}$ be the ring of integers of F with the maximal ideal $\mathfrak{p}$, and let q be the order of the residue field of F. In this section, let $G=\mathrm{PGSp}(4,F)$. Let K be the image of $\mathrm{GSp}(4,\mathfrak{o})$ in G. We fix a Haar measure μ on G for which K has volume 1. Let $\hat{G}$ be the tempered unitary dual of G. There is a unique Borel measure $\hat{\mu}$ on $\hat{G}$, called the *Plancherel measure* with respect to μ, characterized by

$$f(1)=\int\limits_{\hat{G}}\mathrm{Tr}(\pi(f))\,d\hat{\mu}(\pi) \tag{4.1}$$

for all locally constant, compactly supported functions $f\colon G\to\mathbb{C}$ and $\pi\in\hat{G}$. It is well known that the Plancherel measure is supported on the tempered dual $\hat{G}^{\mathrm{temp}}$, and that a representation π is square-integrable if and only if the point π in $\hat{G}$ has positive Plancherel measure.

For $\Omega\in\{\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV},\mathrm{V},\mathrm{VI}\}$, let $\hat{G}_\Omega$ be the part of the unitary dual that consists of all Iwahori-spherical representations of type Ω. Let m_Ω be the total Plancherel measure of the Iwahori spherical representations of type Ω, defined by

$$m_\Omega=\int\limits_{\hat{G}_\Omega}d\hat{\mu}(\pi). \tag{4.2}$$

For example, m_{I} denotes the total Plancherel measure of the unramified dual. The quantity m_{II} is the total Plancherel measure of all representations of the form $\chi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$, where χ,σ are unramified, unitary characters of $F^\times$ satisfying $\chi^2\sigma^2=1$. Note that these are of type IIa, and that the non-tempered representations of type IIb do not contribute to m_{II}. Similarly, only representation type IIIa contributes to m_{III}, only representation type IVa contributes to m_{IV}, and only representation type Va contribute to m_{V}. Also,

$$m_{\mathrm{VI}}=0, \tag{4.3}$$

since the four tempered representations of type VIa/b are not square-integrable.

Now we consider the following open-compact subgroups of $\mathrm{GSp}(4,F)$ defined in (1.2): $K=\mathrm{GSp}(4,\mathfrak{o})$, $\mathrm{K}(\mathfrak{p})$, $\mathrm{Kl}(\mathfrak{p})$, $\mathrm{Si}(\mathfrak{p})$ and I. We use the same symbol for the images of these groups in $G=\mathrm{PGSp}(4,F)$.

Lemma 4.1. *We have*

$$\mathrm{Vol}_\mu(\mathrm{Kl}(\mathfrak{p}))=\frac{1}{(1+q)(1+q^2)},\quad \mathrm{Vol}_\mu(\mathrm{K}(\mathfrak{p}))=\frac{1}{1+q^2},$$
$$\mathrm{Vol}_\mu(\mathrm{Si}(\mathfrak{p}))=\frac{1}{(1+q)(1+q^2)},\quad \mathrm{Vol}_\mu(I)=\frac{1}{(1+q)^2(1+q^2)}.$$

Proof. For $\mathrm{Kl}(\mathfrak{p})$ and $\mathrm{K}(\mathfrak{p})$, see [19, Lemma 3.3.3]. One can prove the cases for $\mathrm{Si}(\mathfrak{p})$ and I in a similar manner. □

Let H be one of these five subgroups of G. Let (π,V) be an Iwahori-spherical representation of G, and let V^H be the space of H-fixed vectors. Let f be the characteristic function of H. Then

$$\mathrm{Tr}(\pi(f))=\mathrm{Vol}_\mu(H)\dim(V^H).$$

Here $\mathrm{Vol}_\mu(H)$ is the volume of H with respect to μ. The quantity $\dim(V^H)$ is the same across all tempered representations in $\hat{G}_\Omega$ if, for type VI, we view the L-packet VIa/b as one representation. We denote this common dimension by $d_{H,\Omega}$. Then, by (4.1) and (4.2), we get

$$1 = \sum_{\Omega\in\{\mathrm{I,II,III,IV,V,VI}\}} \int_{\hat{G}_\Omega} \mathrm{Tr}(\pi(f))\, d\hat{\mu}(\pi) = \mathrm{Vol}_\mu(H) \sum_{\Omega\in\{\mathrm{I,II,III,IV,V,VI}\}} d_{H,\Omega}\cdot m_\Omega. \tag{4.4}$$

We can now compute all the m_Ω.

Theorem 4.2. *The total Plancherel measure m_Ω of the tempered Iwahori-spherical representations of the group* PGSp(4, F) *of type Ω is given in Table 5.*

Ω	I	IIa	IIIa	IVa	Va	VIa/b
m_Ω	1	q^2-1	$\frac{(q-1)(q^2+q+2)}{2}$	$(q-1)(q^3-1)$	$\frac{q(q-1)^2}{2}$	0

Table 5

Proof. Running through all $H \in \{K, \mathrm{K}(\mathfrak{p}), \mathrm{Kl}(\mathfrak{p}), \mathrm{Si}(\mathfrak{p}), I\}$, we get the following system of equations using [21, Table 3], Lemma 4.1 and (4.4):

$$\begin{aligned}
&K: & 1 &= m_{\mathrm{I}},\\
&\mathrm{K}(\mathfrak{p}): & (1+q^2) &= 2m_{\mathrm{I}} + m_{\mathrm{II}},\\
&\mathrm{Kl}(\mathfrak{p}): & (1+q)(1+q^2) &= 4m_{\mathrm{I}} + 2m_{\mathrm{II}} + m_{\mathrm{III}} + m_{\mathrm{V}} + m_{\mathrm{VI}},\\
&\mathrm{Si}(\mathfrak{p}): & (1+q)(1+q^2) &= 4m_{\mathrm{I}} + m_{\mathrm{II}} + 2m_{\mathrm{III}} + 2m_{\mathrm{VI}},\\
&I: & (1+q)^2(1+q^2) &= 8m_{\mathrm{I}} + 4m_{\mathrm{II}} + 4m_{\mathrm{III}} + m_{\mathrm{IV}} + 2m_{\mathrm{V}} + 4m_{\mathrm{VI}}.
\end{aligned}$$

Then, using (4.3), the discussion after (4.2), and the above system of equations, we get Table 5. □

Let us combine, as before, the types III and VI. Hence we define $m_\Omega = m_{\mathrm{III}} + m_{\mathrm{VI}}$ $(= m_{\mathrm{III}})$ for $\Omega = \mathrm{III} + \mathrm{VI}$. Assume that $F = \mathbb{Q}_p$. Then, comparing Tables 4 and 5, we see that the coefficient a_Ω appearing in (3.13) equals the Plancherel mass m_Ω:

$$a_\Omega = m_\Omega \quad \text{for } \Omega \in \{\mathrm{I, II, III + IV, IV, V}\}. \tag{4.5}$$

Here we allowed ourselves to write a_{II} for a_{IIa}, etc.

4.2 More general limit multiplicity formulas

Let $k \geq 3$ and $j \geq 0$ be integers. In the following, let $\xi_{k,j}$ be the irreducible, finite-dimensional representation of $\mathrm{Sp}(4,\mathbb{C})$ with highest weight $(k+j-3, k-3)$, where the weight is defined as in [23]. By [5], it is known that

$$\dim \xi_{k,j} = \frac{(j+1)(k-2)(k+j-1)(2k+j-3)}{6}. \tag{4.6}$$

The infinitesimal character of $\xi_{k,j}$ is $(k+j-1, k-2)$. Let $\Pi_{\mathrm{disc}}(k,j)$ be the L-packet consisting of the discrete series representations of $\mathrm{PGSp}(4,\mathbb{R})$ with the same infinitesimal character $(k+j-1, k-2)$. It consists of a holomorphic (non-generic) discrete series representation $\mathcal{D}^{\mathrm{hol}}(k,j)$ with minimal K-type $(k+j, k)$ and a large (generic) discrete series representation $\mathcal{D}^{\mathrm{gen}}(k,j)$ with minimal K-type $(k+j, 2-k)$. The representation $\mathcal{D}^{\mathrm{hol}}(k,j)$ is the archimedean component of the automorphic representations underlying vector-valued Siegel modular forms of weight (k,j).

Note that, by Corollary 3.9 and equations (4.5) and (4.6),

$$\lim_{k\to\infty} \frac{s_k(p,\Omega)}{2^{-6}3^{-2}5^{-1}\dim \xi_{k,0}} = m_\Omega. \tag{4.7}$$

In this form, the result is reminiscent of the automorphic Plancherel density theorem in [27] for $G = \mathrm{PGSp}(4)$, more precisely, [27, Corollary 4.12]. Roughly speaking, the automorphic Plancherel density theorem says that, in a family of global representations for which the archimedean parameter tends to infinity and which is unramified outside a finite set of finite places T, the local representations at the places in T are equidistributed according to the Plancherel measure. However, there is a difference in that [27] considers archimedean L-packets, whereas our representations are all holomorphic at infinity.

To clarify the relationship, we assume that $k \geq 3$ and define the set $\tilde{S}_k(p, \Omega)$ as in Definition 1.1, except we replace condition (i) by "$\pi_\infty \in \Pi_{\mathrm{disc}}(k, 0)$". Let $\tilde{s}_k(p, \Omega) = \#\tilde{S}_k(p, \Omega)$. We also define Arthur type versions of these quantities, as in (2.3) and (2.4). Then, by the stability of the packets of type (**G**), we have

$$\tilde{s}_k^{(\mathbf{G})}(p, \Omega) = 2s_k^{(\mathbf{G})}(p, \Omega).$$

We note that $s_k(p, \Omega)$ can be replaced by $s_k^{(\mathbf{G})}(p, \Omega)$ in (4.7). The reason is that, similarly to (3.10) and (3.11), the $s_k^{(*)}(p, \Omega)$ for the non-(**G**) types grow at most linearly or quadratically in k. Substituting into (4.7) and reverting back to all packet types with the same argument, we see that (4.7) is equivalent to

$$\lim_{k \to \infty} \frac{\tilde{s}_k(p, \Omega)}{2^{-5}3^{-2}5^{-1} \dim \xi_{k,0}} = m_\Omega. \tag{4.8}$$

In this form, the statement almost follows from [27, Corollary 4.12] (see the proof of Theorem 4.5 below), except that the highest weights of $\xi_{k,0}$ lie on a wall, and hence this particular sequence of finite-dimensional representations does not satisfy the conditions in [27, Definition 3.5]. However, [27, Lemma 4.9], where this hypothesis is used, can easily be proven directly for the $\xi_{k,0}$, using the known weight multiplicities from [5].

Remark 4.3. For $G = \mathrm{PGSp}(4)$, the signed measure $\bar{\mu}(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q}))$ appearing in [27, Definition 3.5] is given by

$$\bar{\mu}(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})) = \frac{-1}{2^5 3^2 5}. \tag{4.9}$$

This can be seen in two different ways. First, one can reduce it to a question for Sp(4) and then use [26, Theorems 4 and 5]. Second, by considering the vector-valued case, one can reverse engineer the constant by letting $\hat{U}$ be the set of spherical, tempered representations at a single place in [27, Theorem 4.11] and comparing with the known formula [29, Theorem 7.1].

Hence we see that (4.7) does indeed follow from [27, Corollary 4.12]. In fact, we can use this corollary and the same arguments to expand the result (4.7) so as to include more general weights and more than one ramified place. To this end, we first make Definition 1.1 more general.

Definition 4.4. Let $k \geq 3$ and $j \geq 0$ be integers, and let T be a finite set of finite primes. For $p \in T$, let $\Omega_p \in \{\mathrm{I}, \mathrm{II}, \mathrm{III} + \mathrm{VI}, \mathrm{IV}, \mathrm{V}\}$, and let $\Omega_T = (\Omega_p)_{p \in T}$. We denote by $S_{k,j}(T, \Omega_T)$ the set of cuspidal automorphic representations $\pi \cong \bigotimes_{v \leq \infty} \pi_v$ of $G(\mathbb{A}_\mathbb{Q})$ satisfying the following properties:
(i) $\pi_\infty = \mathcal{D}^{\mathrm{hol}}(k, j)$.
(ii) For each $v \notin T \cup \{\infty\}$, π_v is unramified.
(iii) For each $p \in T$, π_p is an Iwahori-spherical representation of $G(\mathbb{Q}_p)$ of type Ω_p.
Let $s_{k,j}(T, \Omega_T) = \#S_{k,j}(T, \Omega_T)$.

Theorem 4.5. *Let T and Ω_T be as in Definition 4.4. Let m_{Ω_p} be the total Plancherel measure of the tempered Iwahori-spherical representations of* $\mathrm{PGSp}(4, \mathbb{Q}_p)$ *of type Ω_p, given in Table 5. Then*

$$\lim_{k+j \to \infty} \frac{s_{k,j}(T, \Omega_T)}{2^{-6}3^{-2}5^{-1} \dim \xi_{k,j}} = \prod_{p \in T} m_{\Omega_p}, \tag{4.10}$$

where $\dim \xi_{k,j}$ *is given in* (4.6).

Proof. We will use the notations of [27, Theorem 4.11]. Let $\phi^{S,\infty}$ be the characteristic function of $\prod_{p \notin T} G(\mathbb{Z}_p)$. Let $\hat{f}_T$ be the characteristic function, on the unitary dual of $\prod_{p \in T} G(\mathbb{Q}_p)$, of those representations that are

Iwahori-spherical of type Ω_p at $p \in T$. Then, unraveling the definition of the measure $\widehat{\mu}^{\mathrm{cusp}}_{\phi^{S,\infty},\xi_{k,j}}$ in [27, equation (3.5)], we see

$$\widehat{\mu}^{\mathrm{cusp}}_{\phi^{S,\infty},\xi_{k,j}}(\hat{f}_T) = \frac{(-1)^{\dim(G(\mathbb{R})/K_\infty)/2}}{\bar{\mu}(G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}}))\dim \xi_{k,j}}[\tilde{s}^{(\mathbf{G})}_{k,j}(T,\Omega_T) + \tilde{s}^{(\mathbf{P})}_{k,j}(T,\Omega_T) + \tilde{s}^{(\mathbf{Y})}_{k,j}(T,\Omega_T)].$$

Here K_∞ is the maximal compact subgroup of $G(\mathbb{R})$, and we have $\dim(G(\mathbb{R})/K_\infty) = 6$. The $\tilde{s}^{(*)}_{k,j}(T,\Omega_T)$ are defined similarly to the scalar-valued case above. Then, using (4.9) and an argument similar to the one above that connected (4.7) and (4.8), we obtain (4.10). □

Remark 4.6. Instead of Iwahori-spherical representations, one can work with any relatively quasi-compact subset of the unitary dual of $\prod_{p\in T} G(\mathbb{Q}_p)$. We restricted ourselves to Iwahori-spherical representations because for these we know the right-hand side of (4.10) explicitly.

A Some numerical examples

Here we give some numerical values of $s_k(p,\Omega)$ for $\Omega \in \{\mathrm{IIa}, \mathrm{IIIa} + \mathrm{VIa/b}, \mathrm{Va}, \mathrm{IVa}\}$ using Theorems 3.5–3.8. We have used Mathematica [31] to compute these numerical examples.

p \ k	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**	**21**	**23**	**25**	**27**	**29**	**31**	**33**	**35**
2	0	0	0	0	0	0	0	0	1	1	3	2	5	5	8	9	12
3	0	0	0	0	0	0	1	1	2	5	6	7	13	14	18	26	28
5	0	0	0	0	1	2	4	7	11	16	23	30	40	51	64	79	96
7	0	0	0	1	3	6	10	17	25	36	50	65	85	108	134	165	199
11	0	0	0	2	7	14	26	42	63	90	124	164	213	270	336	412	499
13	0	0	2	6	14	26	43	67	98	137	186	243	313	394	488	596	718
17	0	0	2	9	22	42	71	112	164	231	314	412	531	670	830	1015	1224
19	0	1	4	13	30	55	93	144	210	294	398	522	671	845	1046	1277	1540

p \ k	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**	**32**	**34**
2	0	0	0	0	0	0	1	1	1	2	3	3	6	5	8	9
3	0	0	0	0	1	1	2	5	4	7	11	12	16	22	24	30
5	0	0	0	1	2	4	7	11	15	22	29	38	49	61	75	92
7	0	0	1	2	5	9	15	23	32	45	60	78	100	124	153	186
11	0	0	1	5	11	21	35	54	78	109	146	191	244	306	377	459
13	0	1	4	10	20	35	56	84	118	163	216	280	356	443	544	660
17	0	1	6	15	32	57	92	139	198	273	364	473	602	751	924	1121
19	0	2	8	21	42	74	118	177	252	346	460	597	758	946	1162	1409

Table 6: $s_k(p, \mathrm{IIa})$ for $3 \le k \le 35$ and $2 \le p < 20$.

p \ k	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**	**21**	**23**	**25**	**27**	**29**
2	0	0	0	0	0	0	0	0	0	0	0	1	1	2
3	0	0	0	0	0	0	0	1	1	2	4	6	8	13
5	0	0	0	0	1	2	6	9	17	24	37	50	70	89
7	0	0	0	2	6	11	24	40	59	91	128	170	230	297
11	0	0	3	13	33	68	121	195	295	424	585	785	1023	1306
13	0	1	8	28	68	124	224	350	521	744	1026	1350	1770	2242
17	0	2	23	70	165	312	537	837	1247	1756	2404	3178	4120	5211
19	0	5	32	107	241	451	772	1207	1771	2511	3418	4509	5839	7391

Table 7: $s_k(p, \mathrm{IIIa} + \mathrm{VIa/b})$ for $3 \le k \le 30$ and $2 \le p < 20$.

p \ *k*	4	6	8	10	12	14	16	18	20	22	24	26	28	30
2	0	0	0	0	1	2	3	5	6	8	10	14	16	20
3	0	0	1	2	4	7	11	15	20	27	33	43	53	63
5	0	1	4	8	16	25	39	54	75	97	127	159	199	240
7	0	3	9	21	38	60	93	133	178	241	311	391	491	602
11	0	9	30	67	123	202	308	444	614	822	1071	1367	1710	2107
13	3	18	52	111	203	323	500	715	987	1324	1732	2195	2766	3401
17	5	35	106	226	417	681	1047	1510	2104	2821	3699	4725	5942	7330
19	7	49	142	311	569	931	1434	2079	2881	3889	5092	6509	8193	10127

Table 7 (continued)

p \ *k*	3	5	7	9	11	13	15	17	19	21	23	25	27	29
2	0	0	0	0	0	1	1	2	4	5	6	11	11	15
3	0	0	0	1	2	6	9	16	24	35	46	66	81	106
5	0	1	7	17	39	75	121	188	279	385	522	693	884	1116
7	0	8	31	88	181	332	541	832	1201	1678	2253	2962	3789	4774
11	2	56	235	610	1255	2260	3661	5576	8055	11170	14995	19640	25101	31536
13	8	118	477	1232	2529	4514	7335	11136	16065	22268	29891	39082	49985	62748
17	23	362	1456	3728	7632	13606	22058	33456	48240	66800	89622	117146	149744	187920
19	38	578	2295	5892	12033	21430	34739	52680	75897	105126	140995	184256	235521	295554

p \ *k*	4	6	8	10	12	14	16	18	20	22	24	26	28
2	0	0	0	1	1	2	2	3	6	7	9	13	14
3	0	1	2	6	9	16	22	33	46	62	79	104	126
5	1	7	20	41	76	128	193	282	398	532	700	904	1132
7	5	26	73	162	297	498	767	1126	1575	2138	2811	3624	4567
11	25	150	445	984	1839	3100	4805	7070	9945	13504	17819	23000	29045
13	51	292	869	1930	3619	6084	9471	13926	19597	26628	35167	45360	57353
17	144	848	2550	5674	10672	17984	28016	41238	58090	78960	104336	134656	170294
19	225	1326	3979	8888	16713	28170	43911	64660	91061	123846	163647	211212	267157

Table 8 $s_k(p, \text{IVa})$ for $3 \le k \le 29$ and $2 \le p < 20$.

p \ *k*	3	5	7	9	11	13	15	17	19	21	23	25	27	29
2	0	0	0	0	0	0	1	1	1	2	2	3	4	5
3	0	0	0	0	1	1	3	4	6	7	11	13	17	21
5	0	0	1	4	7	11	19	27	36	51	66	82	106	130
7	0	1	5	10	21	33	54	76	109	144	192	243	309	378
11	0	3	15	38	76	125	205	298	420	573	761	970	1240	1533
13	0	8	28	66	127	216	339	500	705	959	1267	1635	2067	2569
17	0	15	56	141	274	467	746	1106	1560	2141	2837	3661	4653	5794
19	1	20	81	192	381	652	1037	1536	2187	2982	3967	5126	6513	8106

p \ *k*	4	6	8	10	12	14	16	18	20	22	24	26	28	30
2	0	0	0	0	0	0	0	0	0	0	0	0	0	1
3	0	0	0	0	0	0	1	0	2	2	3	4	7	7
5	0	0	0	1	2	3	7	11	15	24	33	42	58	74
7	0	0	1	3	8	14	26	39	61	84	118	154	203	255
11	0	0	5	18	43	75	135	205	300	423	578	750	980	1230
13	0	3	14	39	82	149	245	375	545	759	1023	1342	1721	2166
17	0	8	35	101	208	368	608	923	1325	1848	2480	3232	4147	5205
19	0	12	58	146	306	540	882	1330	1922	2652	3564	4644	5944	7442

Table 9 $s_k(p, \text{Va})$ for $3 \le k \le 30$ and $2 \le p < 20$.

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