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ZETA INTEGRALS FOR GSP(4) VIA BESSEL MODELS

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We give a revised treatment of Piatetski-Shapiro's theory of zeta integrals and L-factors for irreducible, admissible representations of GSp(4, F) via Bessel models. We explicitly calculate the local L-factors in the nonsplit case for all representations. In particular, we introduce the new concept of Jacquet–Waldspurger modules which play a crucial role in our calculations.

1. Introduction

An irreducible, admissible representation of an algebraic reductive group over a local field is called *generic* if it has a Whittaker model. Whittaker models are one of the main tools to define local and global L-functions and ε-factors of representations. The theory was developed by Jacquet and Langlands for GL(2) following ideas of Tate's thesis for GL(1). The general case of GL(n) was developed in a series of works by Jacquet, Piatetski-Shapiro and Shalika. It is well known that any infinite dimensional irreducible, admissible representation of GL(2) is always generic.

Let F be a nonarchimedean local field of characteristic zero. Takloo-Bighash [2000] computed L-functions for all generic representations of the group GSp(4, F). It is similar to the theory of GL(n) in that the approach is based on the existence of Whittaker models and zeta integrals. The method was first introduced by Novodvorsky [1979] in the Corvallis conference. However, it turns out that there are many irreducible, admissible representations of GSp(4, F) which are not generic.

In the 1970s, Novodvorsky and Piatetski-Shapiro introduced the concept of Bessel models. In contrast to Whittaker models, every irreducible, admissible, infinite-dimensional representation of GSp(4, F) admits a Bessel model of some

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kind; see Theorem 6.1.4 of [Roberts and Schmidt 2016]. Piatetski-Shapiro [1997] defined a new type of zeta integral with respect to Bessel models which led to a parallel method to the GL(2) case of defining local factors. However, some of his results were only sketched, and not many factors were calculated explicitly.

Danişman calculated many Piatetski-Shapiro L-factors explicitly in the case of nonsplit Bessel models. In [Danişman 2014], representations were treated whose Jacquet module with respect to the Siegel parabolic has at most length 2. In [Danişman 2015a], this was extended to length at most 3. Nongeneric supercuspidals were the topic of [Danişman 2015b].

In this work we revisit both Piatetski-Shapiro's original theory and Danişman's explicit calculations. We generalize the theory of [Piatetski-Shapiro 1997] in that we do not restrict ourselves to unitary representations. We also fill in some of the missing proofs, for example in the argument that generic representations do not admit "exceptional poles".

Generalizing Danişman's approach, we give a unified treatment of the asymptotics of Bessel functions in the nonsplit case which works for all representations. The key here is to consider a new type of finite-dimensional module $V_{N,T,\Lambda}$ associated to an irreducible, admissible representation (π, V) of $\mathrm{GSp}(4, F)$. These *Jacquet–Waldspurger modules* control the asymptotics of Bessel functions. Table 2 contains the semisimplifications of all Jacquet–Waldspurger modules, and Table 3 contains their precise algebraic structure as $F^\times$-modules. A key lemma in the nonsplit case is due to Danişman; see Proposition 4.3.3.

Once the asymptotic behavior is known, it is easy to calculate the *regular part* $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$ of the Piatetski-Shapiro L-factor; see Table 5. Our results show that in all generic cases, $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$ coincides with the usual spin Euler factor defined via the local Langlands correspondence, but for nongeneric representations these factors generally disagree. The results of Table 5 also imply that $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$ is independent of the choice of Bessel model.

2. Definitions and notations

Let F be a nonarchimedean local field of characteristic zero. Let $\mathfrak{o}$ be its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and ϖ a generator of $\mathfrak{p}$. Let q be the cardinality of $\mathfrak{o}/\mathfrak{p}$. We fix a nontrivial character ψ of F. Let v be the normalized valuation on F, and let ν or $|\cdot|$ be the normalized absolute value on F. Hence $\nu(x) = q^{-v(x)}$ for $x \in F^\times$.

Let $\mathrm{GSp}(4, F) := \{g \in \mathrm{GL}(4, F) : {}^t gJg = \lambda J, \text{ for some } \lambda = \lambda(g) \in F^\times\}$ be defined with respect to the symplectic form

$$J = \begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix}. \tag{1}$$

Let $P = MN$ be the Levi decomposition of the Siegel parabolic subgroup P, where

$$(2)\qquad P = \mathrm{GSp}(4, F) \cap \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}, \quad N = \left\{ \begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix} : x, y, z \in F \right\}$$

and $M = \left\{ \left[\begin{smallmatrix} xA & \\ & {}^tA^{-1} \end{smallmatrix}\right] : A \in \mathrm{GL}(2, F),\ x \in F^\times \right\}$. We let

$$(3)\qquad H := \left\{ \begin{bmatrix} xI_2 & \\ & I_2 \end{bmatrix} : x \in F^\times \right\} \cong F^\times.$$

Let

$$(4)\qquad \beta = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \quad a, b, c \in F$$

be a symmetric matrix. Then β determines a character ψ_β of N by

$$(5)\qquad \psi_\beta\left(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \right) = \psi(\mathrm{tr}(\beta X)), \quad X = \begin{bmatrix} x & y \\ y & z \end{bmatrix}.$$

Every character of N is of this form for a uniquely determined β. We say that ψ_β is *nondegenerate* if $\beta \in \mathrm{GL}(2, F)$.

Attached to a nondegenerate ψ_β is a quadratic extension L/F. If $-\det(\beta) \notin F^{\times 2}$, we set $L = F(\sqrt{-\det(\beta)})$; this is the *nonsplit case*. If $-\det(\beta) \in F^{\times 2}$, we set $L = F \oplus F$; this is the *split case*. Let

$$(6)\qquad \begin{aligned} A_\beta &= \{ g \in M_2(F) : {}^t g \beta g = \det(g) \beta \} \\ &= \left\{ \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix} : x, y \in F \right\}. \end{aligned}$$

Then A_β is an F-algebra isomorphic to L via the map

$$(7)\qquad \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix} \longmapsto x + y\Delta,$$

where $\Delta = \sqrt{-\det(\beta)}$ in the nonsplit case, and $\Delta = (-\delta, \delta)$ if $-\det(\beta) = \delta^2$.

Let T be the connected component of the stabilizer of ψ_β in M. It is easy to check that $T \cong A_\beta^\times \cong L^\times$. We always consider T a subgroup of $\mathrm{GSp}(4, F)$ via

$$(8)\qquad T \ni g \longmapsto \begin{bmatrix} g & \\ & \det(g)\, {}^t g^{-1} \end{bmatrix}.$$

Explicitly, T consists of all elements

$$\begin{bmatrix} x+yb/2 & yc & & \\ -ya & x-yb/2 & & \\ & & x-yb/2 & ya \\ & & -yc & x+yb/2 \end{bmatrix}, \qquad x, y \in F,\ x^2 - y^2\Delta^2 \neq 0. \tag{9}$$

Let $R := TN$ be the *Bessel subgroup* of $\mathrm{GSp}(4, F)$. If Λ is a character of T, then we can define a character $\Lambda \otimes \psi_\beta$ of R by $tn \mapsto \Lambda(t)\psi_\beta(n)$ for $t \in T$ and $n \in N$.

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$. Nonzero elements of $\mathrm{Hom}_R(V, \mathbb{C}_{\Lambda\otimes\psi_\beta})$ are called (Λ, β)-*Bessel functionals*. It is known that if such a Bessel functional ℓ exists, then $\mathrm{Hom}_R(V, \mathbb{C}_{\Lambda\otimes\psi_\beta})$ is one-dimensional. In this case the space of functions

$$\mathcal{B}(\pi, \Lambda, \beta) := \{B_v : g \mapsto \ell(\pi(v)g) : v \in V\}, \tag{10}$$

endowed with the action of $\mathrm{GSp}(4, F)$ given by right translations, is called the (Λ, β)-*Bessel model* of π.

3. Jacquet–Waldspurger modules

In this section we introduce a certain finite-dimensional $F^\times$-module attached to an irreducible, admissible representation of $\mathrm{GSp}(4, F)$. Since it is derived from the usual Jacquet module by applying a Waldspurger functor, we call it a *Jacquet–Waldspurger module*. Its relevance is that it controls the asymptotics of Bessel functions along the subgroup H defined in (3). The main result of this section is Table 2, which lists the semisimplifications of the Jacquet–Waldspurger modules in the nonsplit case for all representations.

3.1. ***Jacquet modules.*** Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$,

$$V(N) = \langle \pi(n)v - v \mid v \in V,\ n \in N\rangle \quad \text{and} \quad V_N = V/V(N)$$

be the usual Jacquet module with respect to the Siegel parabolic subgroup. We identify M with $\mathrm{GL}(2, F) \times \mathrm{GL}(1, F)$ via the map

$$(A, x) \longmapsto \begin{bmatrix} xA & \\ & \det(A)\,{}^tA^{-1} \end{bmatrix}, \qquad A \in \mathrm{GL}(2, F),\ x \in F^\times, \tag{11}$$

so V_N carries an action of M, and thus an action of $\mathrm{GL}(2, F) \times \mathrm{GL}(1, F)$ via this isomorphism. We have tabulated the semisimplifications of these Jacquet modules in Table 1. Note that this table differs from Table A.3 of [Roberts and Schmidt 2007] in three ways:

- Roberts and Schmidt used a different version of $\mathrm{GSp}(4, F)$. Switching the last two rows and columns provides an isomorphism.
- The Jacquet modules listed in [Roberts and Schmidt 2007, Table A.3] are normalized, while the Jacquet modules listed in Table 1 are not. The normalized Jacquet module is obtained from the unnormalized one by twisting by $\delta_P^{-1/2}$, where
$$\delta_P\left(\begin{bmatrix} A & \\ & x\,{}^tA^{-1}\end{bmatrix}\right) = |x^{-1}\det(A)|^3.$$
Hence, we replace each component $\tau\otimes\sigma$ in [Roberts and Schmidt 2007, Table A.3] by $(\nu^{3/2}\tau)\otimes(\nu^{-3/2}\sigma)$ in order to obtain the unnormalized Jacquet modules.
- Roberts and Schmidt used the isomorphism

$$(A, x)\longmapsto \begin{bmatrix} A & \\ & x\,{}^tA^{-1}\end{bmatrix},\qquad A\in \mathrm{GL}(2, F),\ x\in F^\times. \tag{12}$$

Calculations show that we have to replace each component $(\nu^{3/2}\tau)\otimes(\nu^{-3/2}\sigma)$ of the unnormalized Jacquet module by $(\sigma\tau)\otimes(\nu^{3/2}\omega_\tau\sigma)$.

3.2. *Waldspurger functionals for* GL(2). Recall the algebra $A_\beta\subset M_2(F)$ defined in (6), and its unit group $T\subset \mathrm{GL}(2, F)$. Let Λ be a character of T. Let (τ, V) be a smooth representation of $\mathrm{GL}(2, F)$ admitting a central character ω_τ. A Λ-*Waldspurger functional* on τ is a nonzero linear map $\delta : V\to\mathbb{C}$ such that
$$\delta(\tau(t)v) = \Lambda(t)\delta(v)\quad \text{for all } v\in V \text{ and } t\in T.$$
Since T contains the center Z of $\mathrm{GL}(2, F)$, a necessary condition for such a δ to exist is that $\Lambda|_{F^\times} = \omega_\tau$. As in the case of Bessel functionals, we call a Waldspurger functional *split* if $-\det(\beta)\in F^{\times 2}$, otherwise *nonsplit*.

The (Λ, β)-Waldspurger functionals are the nonzero elements of the space $\mathrm{Hom}_T(\tau, \mathbb{C}_\Lambda)$. If we put

$$V(T, \Lambda) = \langle \tau(t)v - \Lambda(t)v : v\in V,\ t\in T\rangle \quad\text{and}\quad V_{T,\Lambda} = V/V(T, \Lambda), \tag{13}$$

then $\mathrm{Hom}_T(\tau, \mathbb{C}_\Lambda)\cong \mathrm{Hom}(V_{T,\Lambda}, \mathbb{C})$. Note that if L is a field, so that T/Z is compact, then the space $V(T, \Lambda)$ can also be characterized as

$$V(T, \Lambda) = \left\{ v\in V : \int_{T/Z}\Lambda(t)^{-1}\tau(t)v\,dt = 0\right\}. \tag{14}$$

The map $V\mapsto V_{T,\Lambda}$ defines a functor, called the *Waldspurger functor*, from the category of smooth representations of $\mathrm{GL}(2, F)$ to the category of $F^\times$-modules. This can be seen just as the analogous statement in the case of Jacquet modules. In particular, if L is a field, then the Waldspurger functor is exact; this follows from (14) with similar arguments as in [Bernstein and Zelevinskii 1976, Proposition 2.35].

		representation	semisimplification
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$\sigma(\chi_1\times\chi_2)\otimes\nu^{3/2}\chi_1\chi_2\sigma+\sigma(\chi_2\times\chi_1)\otimes\nu^{3/2}\sigma$ $+\sigma(\chi_1\chi_2\times 1_{F^\times})\otimes\nu^{3/2}\chi_1\sigma+\sigma(\chi_1\chi_2\times 1_{F^\times})\otimes\nu^{3/2}\chi_2\sigma$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$\sigma\chi\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{3/2}\chi^2\sigma+\sigma\chi\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{3/2}\sigma+(\chi^2\sigma\times\sigma)\otimes\nu^2\chi\sigma$
	b	$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	$\sigma\chi 1_{\mathrm{GL}(2)}\otimes\nu^{3/2}\chi^2\sigma+\sigma\chi 1_{\mathrm{GL}(2)}\otimes\nu^{3/2}\sigma+(\chi^2\sigma\times\sigma)\otimes\nu\chi\sigma$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\sigma(\chi\nu^{-1/2}\times\nu^{1/2})\otimes\chi\nu^2\sigma+\sigma(\chi\nu^{1/2}\times\nu^{-1/2})\otimes\nu^2\sigma$
	b	$\chi\rtimes\sigma 1_{\mathrm{GSp}(2)}$	$\sigma(\chi\nu^{1/2}\times\nu^{-1/2})\otimes\chi\nu\sigma+\sigma(\chi\nu^{-1/2}\times\nu^{1/2})\otimes\nu\sigma$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^3\sigma$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\sigma 1_{\mathrm{GL}(2)}\otimes\nu^3\sigma+\sigma(\nu^{3/2}\times\nu^{-3/2})\otimes\nu\sigma$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\sigma+\sigma(\nu^{3/2}\times\nu^{-3/2})\otimes\nu^2\sigma$
	d	$\sigma 1_{\mathrm{GSp}(4)}$	$\sigma 1_{\mathrm{GL}(2)}\otimes\sigma$
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$\sigma\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^2\sigma+\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\xi\nu^2\sigma$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\sigma\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu\sigma+\sigma 1_{\mathrm{GL}(2)}\otimes\xi\nu^2\sigma$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\xi\nu\sigma+\sigma\xi 1_{\mathrm{GL}(2)}\otimes\nu^2\sigma$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$\sigma 1_{\mathrm{GL}(2)}\otimes\xi\nu\sigma+\sigma\xi 1_{\mathrm{GL}(2)}\otimes\nu\sigma$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$2\cdot(\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^2\sigma)+\sigma 1_{\mathrm{GL}(2)}\otimes\nu^2\sigma$
	b	$\tau(T,\nu^{-1/2}\sigma)$	$\sigma 1_{\mathrm{GL}(2)}\otimes\nu^2\sigma$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu\sigma$
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$2\cdot(\sigma 1_{\mathrm{GL}(2)}\otimes\nu\sigma)+\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu\sigma$
VII		$\chi\rtimes\pi$	0
VIII	a	$\tau(S,\pi)$	0
	b	$\tau(T,\pi)$	0
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi(\mu))$	0
	b	$L(\nu\xi,\nu^{-1/2}\pi(\mu))$	0
X		$\pi\rtimes\sigma$	$\sigma\pi\otimes\nu^{3/2}\omega_\pi\sigma+\sigma\pi\otimes\nu^{3/2}\sigma$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\sigma\pi\otimes\nu^2\sigma$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\sigma\pi\otimes\nu\sigma$
		supercuspidal	0

Table 1. Jacquet modules with respect to P, using the isomorphism (11).

Now assume that (τ, V) is irreducible and admissible. Then it is known by [Tunnell 1983; Saito 1993; Waldspurger 1985, Lemme 8] that the space $\mathrm{Hom}_T(\tau, \mathbb{C}_\Lambda)$ is at most one-dimensional. It follows that

$$\dim V_{T,\Lambda} \le 1. \tag{15}$$

The following facts are known for any character Λ of T such that $\Lambda|_{F^\times} = \omega_\tau$:

- For principal series representations, we have

$$\dim(\mathrm{Hom}_T(\chi_1 \times \chi_2, \mathbb{C}_\Lambda)) = 1 \quad \text{for all } \Lambda; \tag{16}$$

  see [Tunnell 1983, Proposition 1.6 and Theorem 2.3].

- For twists of the Steinberg representation, we have

$$\dim(\mathrm{Hom}_T(\sigma \mathrm{St}_{\mathrm{GL}(2)}, \mathbb{C}_\Lambda)) = \begin{cases} 0 & \text{if } L \text{ is a field and } \Lambda = \sigma \circ \mathrm{N}_{L/F}, \\ 1 & \text{otherwise;} \end{cases} \tag{17}$$

  see [Tunnell 1983, Proposition 1.7 and Theorem 2.4].

- If τ is infinite-dimensional and $L = F \times F$, then

$$\dim(\mathrm{Hom}_T(\pi, \mathbb{C}_\Lambda)) = 1 \quad \text{for all } \Lambda; \tag{18}$$

  see Lemme 8 of [Waldspurger 1985].

- For one-dimensional representations, we have

$$\dim(\mathrm{Hom}_T(\sigma 1_{\mathrm{GL}(2)}, \mathbb{C}_\Lambda)) = \begin{cases} 1 & \text{if } \Lambda = \sigma \circ \mathrm{N}_{L/F}, \\ 0 & \text{otherwise;} \end{cases} \tag{19}$$

  this is obvious.

3.3. ***Jacquet–Waldspurger modules.*** Recall the groups N and T defined in (2) and (9), respectively. Let (π, V) be an admissible representation of $\mathrm{GSp}(4, F)$. We now consider

$$\begin{aligned} V(N, T, \Lambda) &= \langle \pi(tn)v - \Lambda(t)v \,:\, v \in V,\ t \in T,\ n \in N \rangle \\ V_{N,T,\Lambda} &= V / V(N, T, \Lambda). \end{aligned} \tag{20}$$

Evidently, there is a surjective map $V_N \to V_{N,T,\Lambda}$ which induces an isomorphism

$$(V_N)_{T,\Lambda} \cong V_{N,T,\Lambda}. \tag{21}$$

Here, on the left we use the notation (13) for the $\mathrm{GL}(2, F)$-module V_N. Note that, in view of (8), we have to embed $\mathrm{GL}(2, F)$ into $\mathrm{GSp}(4, F)$ via the map

$$\mathrm{GL}(2, F) \ni g \longmapsto \begin{bmatrix} g & \\ & \det(g)\, {}^t g^{-1} \end{bmatrix}, \tag{22}$$

and consider V_N a $\mathrm{GL}(2, F)$-module via this embedding. We call $V_{N,T,\Lambda}$ the *Jacquet–Waldspurger module* of π. This module retains an action of $F^\times$, coming from the action of the group $\{\mathrm{diag}(x, x, 1, 1) : x \in F^\times\}$ on V. The map $V \mapsto V_{N,T,\Lambda}$ defines a functor, called the *Jacquet–Waldspurger functor*, from the category of admissible $\mathrm{GSp}(4, F)$-representations to the category of $F^\times$-modules.

Lemma 3.3.1. *Let V, V', V'' be admissible representations of* $\mathrm{GSp}(4, F)$.

(i) *If $V = V' \oplus V''$ is a direct sum, then*

$$V_{N,T,\Lambda} = V'_{N,T,\Lambda} \oplus V''_{N,T,\Lambda}. \tag{23}$$

(ii) *The Jacquet–Waldspurger functor is right exact, i.e, if $0 \to V' \to V \to V'' \to 0$ is exact, then*

$$V'_{N,T,\Lambda} \to V_{N,T,\Lambda} \to V''_{N,T,\Lambda} \to 0 \tag{24}$$

is exact. Moreover, if we are in the nonsplit case, then the Jacquet–Waldspurger functor is exact.

Proof. These are general properties of Jacquet-type functors. See Proposition 2.35 of [Bernstein and Zelevinskii 1976]. □

Lemma 3.3.2. *Let (π, V) be an admissible representation of* $\mathrm{GSp}(4, F)$ *of finite length. Then the $F^\times$-module $V_{N,T,\Lambda}$ is finite-dimensional. More precisely, if n is the length of the* $\mathrm{GL}(2, F)$*-module V_N, then* $\dim V_{N,T,\Lambda} \le n$.

Proof. The proof is by induction on n. If $n = 1$, then V_N is an irreducible, admissible representation of $\mathrm{GL}(2, F)$. In this case the assertion follows from (15).

Assume that $n > 1$. Let V' be a submodule of V_N of length $n - 1$. Then $V'' := V_N / V'$ is irreducible. By (24), we have an exact sequence

$$V'_{T,\Lambda} \xrightarrow{\alpha} V_{N,T,\Lambda} \to V''_{T,\Lambda} \to 0. \tag{25}$$

By induction and (15), it follows that

$$\dim V_{N,T,\Lambda} = \dim \mathrm{im}(\alpha) + \dim V''_{T,\Lambda} \le n - 1 + 1 = n. \tag{26}$$

This concludes the proof. □

Assume that we are in the nonsplit case, i.e., the quadratic extension L is a field. Then the semisimplifications of the $V_{N,T,\Lambda}$ can easily be calculated from V_N using (21). We already noted that in the nonsplit case the Waldspurger functor is exact. Therefore, to calculate the $V_{N,T,\Lambda}$, we can simply take $(\tau \otimes \sigma)_{T,\Lambda}$ for each constituent $\tau \otimes \sigma$ occurring in Table 1. If $\tau_{T,\Lambda}$ is one-dimensional, then $(\tau \otimes \sigma)_{T,\Lambda} = \sigma 1_{F^\times}$ as an $F^\times$-module, and if $\tau_{T,\Lambda} = 0$, then $(\tau \otimes \sigma)_{T,\Lambda} = 0$. We have listed the semisimplifications of the $V_{N,T,\Lambda}$ for all irreducible, admissible representations in Table 2.

		representation	semisimplification of $V_{N,T,\Lambda}$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$\nu^{3/2}\chi_1\chi_2\sigma 1_{F^\times} + \nu^{3/2}\sigma 1_{F^\times} + \nu^{3/2}\chi_1\sigma 1_{F^\times} + \nu^{3/2}\chi_2\sigma 1_{F^\times}$
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\nu^{3/2}\chi^2\sigma 1_{F^\times} + \nu^{3/2}\sigma 1_{F^\times} + \nu^2\chi\sigma 1_{F^\times}$
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$\nu^{3/2}\chi^2\sigma 1_{F^\times} + \nu^{3/2}\sigma 1_{F^\times} + \nu\chi\sigma 1_{F^\times}$
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	$\chi\nu^2\sigma 1_{F^\times} + \nu^2\sigma 1_{F^\times}$
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	—
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^3\sigma 1_{F^\times}$
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	$\nu^3\sigma 1_{F^\times} + \nu\sigma 1_{F^\times}$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	—
	d	$\sigma 1_{\mathrm{GSp}(4)}$	—
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\nu^2\sigma 1_{F^\times} + \xi\nu^2\sigma 1_{F^\times}$
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\nu\sigma 1_{F^\times} + \xi\nu^2\sigma 1_{F^\times}$
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\xi\sigma)$	$\xi\nu\sigma 1_{F^\times} + \nu^2\sigma 1_{F^\times}$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	$\xi\nu\sigma 1_{F^\times} + \nu\sigma 1_{F^\times}$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	$2 \cdot (\nu^2\sigma 1_{F^\times})$
	b	$\tau(T, \nu^{-1/2}\sigma)$	$\nu^2\sigma 1_{F^\times}$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	—
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	—
VII		$\chi \rtimes \pi$	0
VIII	a	$\tau(S, \pi)$	0
	b	$\tau(T, \pi)$	0
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi(\mu))$	0
	b	$L(\nu\xi, \nu^{-1/2}\pi(\mu))$	0
X		$\pi \rtimes \sigma$	$\nu^{3/2}\omega_\pi\sigma 1_{F^\times} + \nu^{3/2}\sigma 1_{F^\times}$
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\nu^2\sigma 1_{F^\times}$
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\nu\sigma 1_{F^\times}$
		supercuspidal	0

Table 2. The semisimplifications of Jacquet–Waldspurger modules. It is assumed that L is a field, and that the representation of GSp(4, F) admits a (Λ, β)-Bessel functional. A "—" indicates that no such Bessel functional exists.

4. Asymptotic behavior

We begin this section by developing a simple theory of finite-dimensional $F^\times$-modules, which applies to the Jacquet–Waldspurger modules of the previous section. In Section 4.2 we clarify the notion of "asymptotic function". Using our previous results on Jacquet–Waldspurger modules, as well as a result of Danişman in the nonsplit case (Proposition 4.3.3), we can calculate the asymptotic behavior of all Bessel functions of all representations; see Table 4. Simultaneously, we obtain the precise structure as an $F^\times$-module of the Jacquet–Waldspurger modules; see Table 3.

4.1. *Finite-dimensional $F^\times$-modules.* Recall that $F^\times = \langle\varpi\rangle \times \mathfrak{o}^\times$. We consider representations of $F^\times$ on finite-dimensional complex vector spaces. All such representations are assumed to be continuous.

Let n be a positive integer and U be an n-dimensional complex vector space with basis $e_1, \dots, e_n$. We define an action of $F^\times$ on U as follows:

- $\mathfrak{o}^\times$ acts trivially on all of U.
- ϖ acts by sending e_j to $e_j + e_{j-1}$ for all $j \in \{1, \dots, n\}$, where we understand $e_0 = 0$. In other words, the matrix of ϖ with respect to the basis $e_1, \dots, e_n$ is a Jordan block

$$\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}. \tag{27}$$

We denote the equivalence class of the $F^\times$-module thus defined by $[n]$. Note that $[n]$ is canonically defined, even though ϖ is not. Clearly, $[n]$ is an indecomposable $F^\times$-module. If σ is a character of $F^\times$, then $\sigma[n] := \sigma \otimes [n]$ is also indecomposable.

Lemma 4.1.1. *Every finite-dimensional indecomposable $F^\times$-module is of the form $\sigma[n]$ for some character σ of $F^\times$ and positive integer n.*

Proof. Let (φ, U) be an indecomposable $F^\times$-module. We may decompose U over $\mathfrak{o}^\times$, i.e.,

$$U = \bigoplus_{i=1}^{r} U(\sigma_i), \tag{28}$$

where the σ_i are pairwise distinct characters of $F^\times$, and

$$U(\sigma_i) = \{u \in U \,:\, \varphi(x)u = \sigma_i(x)u \ \text{ for all } x \in \mathfrak{o}^\times\}. \tag{29}$$

Let $f = \varphi(\varpi)$. Since each $U(\sigma_i)$ is f-invariant and U is indecomposable, it follows that $r = 1$, i.e., $U = U(\sigma)$ for some character σ of $\mathfrak{o}^\times$. Indecomposability implies that the Jordan normal form of f consists of only one Jordan block

$$\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \qquad \lambda \in \mathbb{C}^\times, \tag{30}$$

of size n. Extend σ to a character of $F^\times$ by setting $\sigma(\varpi) = \lambda$. Then it is easy to see that $\varphi \cong \sigma[n]$. □

Lemma 4.1.2. *Let U be a finite-dimensional $F^\times$-module. Then*

$$U \cong \bigoplus_{i=1}^{r} \sigma_i[n_i] \tag{31}$$

with characters σ_i of $F^\times$ and positive integers n_i. A decomposition as in (31) is unique up to permutation of the summands.

Proof. A decomposition as in (31) exists by Lemma 4.1.1. To prove uniqueness, assume that

$$\bigoplus_{i=1}^{r} \sigma_i[n_i] \cong \bigoplus_{j=1}^{s} \tau_j[m_j]. \tag{32}$$

By considering isotypical components with respect to characters of $\mathfrak{o}^\times$, we may assume that all σ_i and τ_j agree when restricted to $\mathfrak{o}^\times$. After appropriate tensoring we may assume this restriction is trivial. The uniqueness statement then follows from the uniqueness of Jordan normal forms. □

Lemma 4.1.3. *Let σ be a character of $F^\times$, and n a positive integer. Let $m \in \{0, \ldots, n\}$.*

(i) *There exists exactly one $F^\times$-invariant submodule U_m of $\sigma[n]$ of dimension m. We have $U_k \subset U_m$ for $k \leq m$.*

(ii) *The representation of $F^\times$ on U_m is isomorphic to $\sigma[m]$.*

(iii) *The representation of $F^\times$ on $\sigma[n]/U_m$ is isomorphic to $\sigma[n-m]$.*

Proof. (i) Since the invariant subspaces of $[n]$ and $\sigma[n]$ coincide, we may assume that $\sigma = 1$, so that $\sigma[n] = [n]$. Let $e_1, \ldots, e_n$ be a basis of $[n]$ with respect to which ϖ acts via the matrix (27). Let $U_m = \langle e_1, \ldots, e_m \rangle$. Then U_m is invariant and isomorphic to $[m]$ as an $F^\times$-module.

Conversely, let $U \subset [n]$ be any nonzero invariant subspace. Then U is also invariant under the endomorphism f with matrix

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}. \tag{33}$$

The effect of f on a column vector u is to shift its entries "up" and fill in a 0 at the bottom. Let m be maximal with the property that there exists a $u \in U$ of the form

$$u = {}^t[u_1, \dots, u_m, 0, \dots, 0] \quad \text{with } u_m \neq 0.$$

The vector $f^{m-1}u$ is a nonzero multiple of e_1, showing that $e_1 \in U$. Considering $f^{m-2}u$, we see that $e_2 \in U$ as well. Continuing, we see that $e_1, \dots, e_m \in U$. The maximality of m implies that $U = U_m$.

(ii) We already saw that the subspace U_m of $[n]$ is isomorphic to $[m]$. Hence the subspace $\sigma \otimes U_m$ of $\sigma[n]$ is isomorphic to $\sigma[m]$.

(iii) Clearly $[n]/U_m$ is isomorphic to $[n-m]$. Hence $\sigma[n]/(\sigma \otimes U_m)$ is isomorphic to $\sigma[n-m]$. $\square$

Let U be a finite-dimensional $F^\times$-module. For a character σ of $F^\times$, let U_σ be the sum of all submodules of U isomorphic to $\sigma[n]$ for some n. We call U_σ the *σ-component* of U. By (31), U is the direct sum of its σ-components. A homomorphism $U \to V$ of finite-dimensional $F^\times$-modules induces a map $U_\sigma \to V_\sigma$ for all σ; this follows from Lemma 4.1.3.

4.2. *Asymptotic functions.* Let $\mathcal{L}$ be the vector space of functions $f : F^\times \to \mathbb{C}$ with the following properties:

(i) There exists an open-compact subgroup Γ of $F^\times$ such that $f(u\gamma) = f(u)$ for all $u \in F^\times$ and all $\gamma \in \Gamma$.

(ii) $f(u) = 0$ for $v(u) \ll 0$.

Such f arise if we restrict Bessel functions on $\mathrm{GSp}(4, F)$ to the subgroup

$$\{\mathrm{diag}(u, u, 1, 1) : u \in F^\times\} \cong F^\times.$$

Clearly $\mathcal{L}$ contains the Schwartz space $\mathcal{S}(F^\times)$, i.e., the space of locally constant, compactly supported functions $F^\times \to \mathbb{C}$. We may think of the quotient $\mathcal{L}/\mathcal{S}(F^\times)$ as a space of "asymptotic functions", in the sense that the image of some $f \in \mathcal{L}$ in this quotient is determined by the values $f(u)$ for $v(u) \gg 0$.

There is an action $\bar{\pi}$ of $F^\times$ on $\mathcal{L}$ given by translation: $(\bar{\pi}(x)f)(u) = f(ux)$ for $x, u \in F^\times$. This is a smooth action by the properties of the elements of $\mathcal{L}$. The action preserves the subspace $\mathcal{S}(F^\times)$, so that we get an action on the quotient $\mathcal{L}/\mathcal{S}(F^\times)$.

For the proof of the following lemma, we will use the formula

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k P(k)=0 \quad \text{for all } P\in\mathbb{C}[X] \text{ with } \deg(P)<n. \tag{34}$$

This formula follows by differentiating the identity $(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$ repeatedly and setting $x=-1$.

Lemma 4.2.1. *Let $\beta\in\mathbb{C}^\times$. For a positive integer n, let $\mathcal{F}_n(\beta)$ be the space of functions $f:\mathbb{Z}_{\geq 0}\to\mathbb{C}$ satisfying*

$$\sum_{k=0}^{n}\binom{n}{k}(-\beta)^{n-k}f(m+k)=0 \quad \textit{for all } m\geq 0. \tag{35}$$

Then $\dim\mathcal{F}_n(\beta)=n$, *and a basis of $\mathcal{F}_n(\beta)$ is given by the functions*

$$f_j(m)=m^j\beta^m,\qquad m\geq 0, \tag{36}$$

for $j=0,\dots,n-1$.

Proof. It is clear from (35) that any $f\in\mathcal{F}_n(\beta)$ is determined by the values $f(0),\dots,f(n-1)$. Hence $\dim\mathcal{F}_n(\beta)\leq n$, and we only need to show that the functions f_j lie in $\mathcal{F}_n(\beta)$ and are linearly independent. The fact that the functions f_j lie in $\mathcal{F}_n(\beta)$ follows from (34). It is easy to prove that they are linearly independent. □

Proposition 4.2.2. *Let $\mathcal{K}$ be an $F^\times$-invariant subspace of $\mathcal{L}$ which contains $\mathcal{S}(F^\times)$ with finite codimension n. Assume that, as an $F^\times$-module, the quotient $\mathcal{K}/\mathcal{S}(F^\times)$ is isomorphic to $\sigma[n]$ for some character σ of $F^\times$. Then there exist $f_0,\dots,f_{n-1}\in\mathcal{K}$ with the following properties*:

(i) *The images of $f_0,\dots,f_{n-1}$ in $\mathcal{K}/\mathcal{S}(F^\times)$ are a basis of the quotient space.*

(ii) *f_j has asymptotic behavior*

$$f_j(x)=v(x)^j\sigma(x)\quad \textit{for all } x\in F^\times \textit{ with } v(x)\gg 0, \tag{37}$$

for all $j\in\{0,\dots,n-1\}$.

Proof. It suffices to show that every $f\in\mathcal{K}$ has the asymptotic form

$$f(x)=\sum_{k=0}^{n-1}c_k\,v(x)^k\sigma(x)\quad \text{for all } x\in F^\times \text{ with } v(x)\gg 0 \tag{38}$$

for some constants c_k. We have $\sigma[n](u)=\sigma(u)\mathrm{id}$ for $u\in\mathfrak{o}^\times$ on all of $\sigma[n]$. Hence, for a fixed unit $u\in\mathfrak{o}^\times$,

$$\bar{\pi}(u)f-\sigma(u)f\in\mathcal{S}(F^\times). \tag{39}$$

It follows that there exists a $j_0 \geq 0$ such that

$$f(u\varpi^{m+j_0}) = \sigma(u) f(\varpi^{m+j_0}) \quad \text{for all } m \geq 0. \tag{40}$$

Since $\mathfrak{o}^\times$ is compact and both sides of (40) are locally constant, we may choose j_0 large enough so that (40) holds for all $u \in \mathfrak{o}^\times$.

Every vector in $\sigma[n]$ is annihilated by $(\sigma[n](\varpi) - \lambda\,\mathrm{id})^n$, where we abbreviate $\lambda = \sigma(\varpi)$. Hence

$$(\bar{\pi}(\varpi) - \lambda\,\mathrm{id})^n f \in \mathcal{S}(F^\times) \tag{41}$$

for all $f \in \mathcal{K}$, or

$$\sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \bar{\pi}(\varpi^k) f \in \mathcal{S}(F^\times). \tag{42}$$

It follows that there exists a $j_0 \geq 0$ such that

$$\sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} f(\varpi^{m+k+j_0}) = 0 \quad \text{for all } m \geq 0. \tag{43}$$

We may assume that the same j_0 works for both (40) and (43). Setting $h(m) := f(\varpi^{m+j_0})$, (43) reads

$$\sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} h(m+k) = 0 \quad \text{for all } m \geq 0. \tag{44}$$

By Lemma 4.2.1, there exist constants $d_0, \dots, d_{n-1}$ such that

$$h(m) = \sum_{k=0}^{n-1} d_k m^k \lambda^m \quad \text{for all } m \geq 0. \tag{45}$$

We can then also find constants $c_0, \dots, c_{n-1}$ such that

$$h(m) = \sum_{k=0}^{n-1} c_k (m+j_0)^k \lambda^{m+j_0} \quad \text{for all } m \geq 0. \tag{46}$$

(To get the c_k's from the d_k's, expand $m^k = ((m+j_0) - j_0)^k$ in (45).) For $x \in F^\times$ with $v(x) \geq j_0$, write $x = u\varpi^j$ with $u \in \mathfrak{o}^\times$ and $j \geq j_0$. Then

$$\begin{aligned} f(x) &= \sigma(u) f(\varpi^j) && \text{by (40)} \\ &= \sigma(u) \sum_{k=0}^{n-1} c_k j^k \lambda^j && \text{by (46)} \\ &= \sum_{k=0}^{n-1} c_k\, v(x)^k \sigma(x). \end{aligned}$$

□

Corollary 4.2.3. *Let U be a finite-dimensional submodule of $\mathcal{L}/\mathcal{S}(F^\times)$. Then each σ-component of U is indecomposable.*

Proof. Let $\mathcal{K}$ be the preimage of U under the projection $\mathcal{L} \to \mathcal{L}/\mathcal{S}(F^\times)$. Assume that there exists a σ for which U_σ is decomposable. Then U_σ contains a direct sum $\sigma[n] \oplus \sigma[n']$ with $n, n' > 0$. By Proposition 4.2.2, there exist two functions $f, f' \in \mathcal{K}$ such that the image of f in

$$U = \mathcal{K}/\mathcal{S}(F^\times)$$

lies in $\sigma[n]$, the image of f' lies in $\sigma[n']$, and such that

$$f(x) = \sigma(x) \text{ and } f'(x) = \sigma(x) \quad \text{for all } x \in F^\times \text{ with } v(x) \gg 0. \tag{47}$$

It follows from (47) that f and f' have the same image in $\mathcal{K}/\mathcal{S}(F^\times)$, which is a contradiction. □

4.3. *Asymptotic behavior of Bessel functions.* Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$. Assume that V is the (Λ, β)-Bessel model of π with respect to a character Λ of T. We associate with each Bessel function $B \in V$ the function $\varphi_B : F^\times \to \mathbb{C}$ defined by

$$\varphi_B(u) = B(\mathrm{diag}(u, u, 1, 1)).$$

Let $\mathcal{K}$ be the space spanned by all functions φ_B.

Lemma 4.3.1. *$\mathcal{K}$ contains $\mathcal{S}(F^\times)$.*

Proof. This follows by the same arguments as in Lemma 4.1 of [Danişman 2014]. □

An easy argument as in Proposition 4.7.2 of [Bump 1997], or as in Proposition 3.1 of [Danişman 2014], shows that if $B \in V(N)$, then φ_B has compact support. It is also true, and equally easy to see, that

$$B \in V(N, T, \Lambda) \quad \Longrightarrow \quad \varphi_B \text{ has compact support in } F^\times.$$

It follows that the linear map $B \mapsto \varphi_B$ induces a surjection

$$V_{N,T,\Lambda} \to \mathcal{K}/\mathcal{S}(F^\times). \tag{48}$$

Lemma 4.3.2. *Assume that the map (48) is an isomorphism. Then every σ-component of $V_{N,T,\Lambda}$ is indecomposable as an $F^\times$-module.*

Proof. The map (48) induces an isomorphism of the respective σ-components. Hence the assertion follows from Corollary 4.2.3. □

Proposition 4.3.3. *Suppose we are in the nonsplit case. Then the map (48) is an isomorphism.*

Proof. See Theorem 4.9 of [Danişman 2014]. □

Recall that in Table 2 we determined the semisimplifications of the Jacquet–Waldspurger modules for all irreducible, admissible representations. In the nonsplit case, we can now determine the precise algebraic structure of these modules.

Corollary 4.3.4. *The algebraic structure of the Jacquet–Waldspurger modules $V_{N,T,\Lambda}$ for all irreducible, admissible representations of* $\mathrm{GSp}(4, F)$ *is given in Table 3, under the assumption that the representation* (π, V) *admits a nonsplit* (Λ, β)*-Bessel functional.* (A "—" indicates that no such Bessel functional exists.)

Proof. By Proposition 4.3.3 and Lemma 4.3.2, every σ-component of $V_{N,T,\Lambda}$ is indecomposable. This information, together with the semisimplifications from Table 2, gives the precise structure. □

For type I, we have to distinguish various cases, depending on the regularity of the inducing character:

$$V_{N,T,\Lambda} = \begin{cases} \nu^{3/2}\chi_1\chi_2\sigma \oplus \nu^{3/2}\chi_1\sigma \oplus \nu^{3/2}\chi_2\sigma \oplus \nu^{3/2}\sigma & \text{if } \chi_1\chi_2, \chi_1, \chi_2, 1 \\ & \qquad \text{are pairwise different,} \\ \nu^{3/2}\chi^2\sigma \oplus (\nu^{3/2}\chi\sigma)[2] \oplus \nu^{3/2}\sigma & \text{if } \chi := \chi_1 = \chi_2 \neq 1,\ \chi^2 \neq 1, \\ (\nu^{3/2}\chi\sigma)[2] \oplus (\nu^{3/2}\sigma)[2] & \text{if } \chi := \chi_1 = \chi_2 \neq 1,\ \chi^2 = 1, \\ (\nu^{3/2}\chi\sigma)[2] \oplus (\nu^{3/2}\sigma)[2] & \text{if } \{\chi_1, \chi_2\} = \{\chi \neq 1, 1\} \\ (\nu^{3/2}\sigma)[4] & \text{if } \chi_1 = \chi_2 = 1. \end{cases} \tag{49}$$

Corollary 4.3.5. *Table 4 shows the asymptotic behavior of the functions*

$$B(\mathrm{diag}(u, u, 1, 1))$$

for all irreducible, admissible representations (π, V) *of* $\mathrm{GSp}(4, F)$, *where B runs through a nonsplit* (Λ, β)*-Bessel model of* π. (A "—" indicates that no such Bessel model exists.)

Proof. By Proposition 4.3.3, the map (48) is an isomorphism. We can thus use Proposition 4.2.2, which translates the algebraic structure of $V_{N,T,\Lambda}$ given in Table 3 into the asymptotic behavior of Bessel functions. □

Remark. This result is to be understood in the sense that all the constants given in Table 4 are necessary, i.e., for any choice of $C_1, C_2, \ldots$ there exists a Bessel function B such that $B(\mathrm{diag}(u, u, 1, 1))$ has the asymptotic behavior given by this choice of constants.

		representation		$V_{N,T,\Lambda}$
I		$\chi_1 \times \chi_2 \rtimes \sigma$		see (49)
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2 \neq 1$	$\nu^2\chi\sigma \oplus \nu^{3/2}\chi^2\sigma \oplus \nu^{3/2}\sigma$
			$\chi^2 = 1$	$\nu^2\chi\sigma \oplus (\nu^{3/2}\sigma)[2]$
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2 \neq 1$	$\nu\chi\sigma \oplus \nu^{3/2}\chi^2\sigma \oplus \nu^{3/2}\sigma$
			$\chi^2 = 1$	$\nu\chi\sigma \oplus (\nu^{3/2}\sigma)[2]$
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$		$\chi\nu^2\sigma \oplus \nu^2\sigma$
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$		—
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$		$\nu^3\sigma$
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$		$\nu^3\sigma \oplus \nu\sigma$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$		—
	d	$\sigma 1_{\mathrm{GSp}(4)}$		—
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$		$\nu^2\sigma \oplus \xi\nu^2\sigma$
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		$\nu\sigma \oplus \xi\nu^2\sigma$
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\xi\sigma)$		$\xi\nu\sigma \oplus \nu^2\sigma$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		$\xi\nu\sigma \oplus \nu\sigma$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$		$(\nu^2\sigma)[2]$
	b	$\tau(T, \nu^{-1/2}\sigma)$		$\nu^2\sigma$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		—
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		—
VII		$\chi \rtimes \pi$		0
VIII	a	$\tau(S, \pi)$		0
	b	$\tau(T, \pi)$		0
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi(\mu))$		0
	b	$L(\nu\xi, \nu^{-1/2}\pi(\mu))$		0
X		$\pi \rtimes \sigma$	$\omega_\pi \neq 1$	$\nu^{3/2}\omega_\pi\sigma \oplus \nu^{3/2}\sigma$
			$\omega_\pi = 1$	$(\nu^{3/2}\sigma)[2]$
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$\nu^2\sigma$
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$\nu\sigma$
		supercuspidal		0

Table 3. Jacquet–Waldspurger modules $V_{N,T,\Lambda}$. It is assumed that L is a field, and that the representation of $\mathrm{GSp}(4, F)$ admits a (Λ, ψ_β)-Bessel functional. A "—" indicates that no nonsplit Bessel functional exists.

		representation		$\lvert u\rvert^{-3/2}\,B(\mathrm{diag}(u,u,1,1))$
I		$\chi_1\times\chi_2\rtimes\sigma$		see (50)
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$\chi^2\neq 1$	$C_1(\nu^{1/2}\chi\sigma)(u)+C_2(\chi^2\sigma)(u)+C_3\sigma(u)$
			$\chi^2=1$	$C_1(\nu^{1/2}\chi\sigma)(u)+(C_2+C_3v(u))\sigma(u)$
	b	$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	$\chi^2\neq 1$	$C_1(\nu^{-1/2}\chi\sigma)(u)+C_2(\chi^2\sigma)(u)+C_3\sigma(u)$
			$\chi^2=1$	$C_1(\nu^{-1/2}\chi\sigma)(u)+(C_2+C_3v(u))\sigma(u)$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$		$C_1(\nu^{1/2}\chi\sigma)(u)+C_2(\nu^{1/2}\sigma)(u)$
	b	$\chi\rtimes\sigma 1_{\mathrm{GSp}(2)}$		—
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$		$C(\nu^{3/2}\sigma)(u)$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$		$C_1(\nu^{3/2}\sigma)(u)+C_2(\nu^{-1/2}\sigma)(u)$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$		—
	d	$\sigma 1_{\mathrm{GSp}(4)}$		—
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$		$C_1(\nu^{1/2}\xi\sigma)(u)+C_2(\nu^{1/2}\sigma)(u)$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$		$C_1(\nu^{1/2}\xi\sigma)(u)+C_2(\nu^{-1/2}\sigma)(u)$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\xi\sigma)$		$C_1(\nu^{-1/2}\xi\sigma)(u)+C_2(\nu^{1/2}\sigma)(u)$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$		$C_1(\nu^{-1/2}\xi\sigma)(u)+C_2(\nu^{-1/2}\sigma)(u)$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$		$(C_1+C_2v(u))(\nu^{1/2}\sigma)(u)$
	b	$\tau(T,\nu^{-1/2}\sigma)$		$C(\nu^{1/2}\sigma)(u)$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$		—
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$		—
VII		$\chi\rtimes\pi$		0
VIII	a	$\tau(S,\pi)$		0
	b	$\tau(T,\pi)$		0
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi(\mu))$		0
	b	$L(\nu\xi,\nu^{-1/2}\pi(\mu))$		0
X		$\pi\rtimes\sigma$	$\omega_\pi\neq 1$	$C_1(\omega_\pi\sigma)(u)+C_2\sigma(u)$
			$\omega_\pi=1$	$(C_1+C_2v(u))\sigma(u)$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$		$C(\nu^{1/2}\sigma)(u)$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$		$C(\nu^{-1/2}\sigma)(u)$
		supercuspidal		0

Table 4. Asymptotic behavior of $B(\mathrm{diag}(u,u,1,1))$ in the nonsplit case. A "—" indicates that no nonsplit Bessel functional exists.

Again, for type I we have to distinguish various cases:

$$(50)\quad |u|^{-3/2}\,B(\mathrm{diag}(u,u,1,1))$$
$$=\begin{cases} C_1(\chi_1\chi_2\sigma)(u) & \text{if } \chi_1\chi_2,\chi_1,\chi_2,1 \\ \qquad +C_2(\chi_1\sigma)(u)+C_3(\chi_2\sigma)(u)+C_4\sigma(u) & \qquad \text{are pairwise different,} \\ C_1(\chi^2\sigma)(u) & \text{if } \chi:=\chi_1=\chi_2\neq 1,\ \chi^2\neq 1, \\ \qquad +(C_2+C_3v(u))(\chi\sigma)(u)+C_4\sigma(u) & \\ (C_1+C_2v(u))(\chi\sigma)(u)+(C_3+C_4v(u))\sigma(u) & \text{if } \chi:=\chi_1=\chi_2\neq 1,\ \chi^2=1, \\ (C_1+C_2v(u))(\chi\sigma)(u)+(C_3+C_4v(u))\sigma(u) & \text{if } \{\chi_1,\chi_2\}=\{\chi\neq 1,1\}, \\ (C_1+C_2v(u)+C_3v^2(u)+C_4v^3(u))\sigma(u) & \text{if } \chi_1=\chi_2=1. \end{cases}$$

Remark 4.3.6. The proof of Proposition 4.3.3 given in [Danişman 2014] is based on the exactness of the Waldspurger functor, which is only true in the nonsplit case. Assume that (π, V) is an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ which admits a *split* Bessel model $\mathcal{B}(\pi,\Lambda,\beta)$. Then we still have the surjection (48), which implies that the space of asymptotic functions $\mathcal{K}/\mathcal{S}(F^\times)$, as an $F^\times$-module, is a quotient of the Jacquet–Waldspurger module $V_{N,T,\Lambda}$. Starting from the $V_{N,T}$ given in Table 1, the $V_{N,T,\Lambda}$ can be calculated in many cases, but some of them pose difficulties, again due to the fact that the Waldspurger functor in the split case is not exact. Thus, complete results in the split case would follow from the solution of the following two problems:

- Calculate the Jacquet–Waldspurger modules $V_{N,T,\Lambda}$ in all cases.
- Control the kernel of the map (48).

The current methods still allow for some preliminary results on the asymptotic behavior of the functions $B(\mathrm{diag}(u,u,1,1))$ in the split case. More precisely, it is not difficult to create a table similar to Table 4, but it is unclear if all the constants C_i in such a table are really necessary. What is clear is that every $B(\mathrm{diag}(u,u,1,1))$ is of the general form

$$(51)\qquad B(\mathrm{diag}(u,u,1,1))=\sum_{i=1}^{n} C_i\,v(u)^{k_i}\sigma_i(u)\quad \text{for } v(u)\gg 0$$

with k_i nonnegative integers, σ_i characters of $F^\times$, and $C_i\in\mathbb{C}$.

5. Local zeta integrals and L-factors

Given an irreducible, admissible, unitary representation π of $\mathrm{GSp}(4, F)$ and a character μ of $F^\times$, a certain type of zeta integral was introduced in Section 3 of [Piatetski-Shapiro 1997] and used to define an L-factor $L^{\mathrm{PS}}(s,\pi,\mu)$. These zeta integrals depend on a choice of Bessel model for π, and hence the L-factor may also depend on this choice. In many cases, though, one can prove that $L^{\mathrm{PS}}(s,\pi,\mu)$ is independent of the choice of Bessel data.

In Section 5.1 we introduce a simplified type of zeta integral and use it to define the *regular part* $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ of the Piatetski-Shapiro L-factor. The simplified zeta integrals also depend on the choice of a Bessel model for π. Using the asymptotic behavior given in Table 4, we explicitly calculate $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ in the nonsplit case for all representations. It turns out that $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ is independent of the choice of Bessel model, and coincides with the usual degree-4 (spin) Euler factor if π is generic. For nongeneric representations, however, the two factors do not agree in general.

We then investigate the Piatetski-Shapiro zeta integrals (78). Their definition involves a certain subgroup G of $\mathrm{GSp}(4,F)$, to which we dedicate Section 5.2. The resulting L-factor $L^{\mathrm{PS}}(s,\pi,\mu)$ is either equal to $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$, or has an additional factor $L(s+1/2,\Lambda_\mu)$, where $\Lambda_\mu=\Lambda\cdot(\mu\circ N_{L/F})$ depends on the Bessel data. In Section 5.5 we will identify several cases where $L^{\mathrm{PS}}(s,\pi,\mu)=L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$.

Overall in this section we closely follow [Piatetski-Shapiro 1997]. However, we treat all representations, not only unitary ones. Our notion of exceptional pole is slightly more general than the one given in [Piatetski-Shapiro 1997]. Also, we fill in some of the missing proofs of that paper.

5.1. ***The simplified zeta integrals.*** Let π be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$. Let $\mathcal{B}(\pi,\Lambda,\beta)$ be a (Λ,β)-Bessel model for π. Let μ be a character of $F^\times$. For $B\in\mathcal{B}(\pi,\Lambda,\beta)$ and $s\in\mathbb{C}$, we define the *simplified zeta integrals*

$$\zeta(s,B,\mu)=\int_{F^\times} B\left(\begin{bmatrix} x & \\ & 1\end{bmatrix}\right)\mu(x)|x|^{s-3/2}\,d^\times x. \tag{52}$$

The same integrals appear in Proposition 18 of [Danişman 2015b]. Using the general form (51) of the functions $B\left(\left[\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right]\right)$, which holds both in the split and the nonsplit case, it is easy to see that $\zeta(s,B,\mu)$ converges to an element of $\mathbb{C}(q^{-s})$ for real part of s large enough. Let $I(\pi,\mu)$ be the $\mathbb{C}$-vector subspace of $\mathbb{C}(q^{-s})$ spanned by all $\zeta(s,B,\mu)$ as B runs through $\mathcal{B}(\pi,\Lambda,\beta)$.

Proposition 5.1.1. *Let π be an irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *admitting a (Λ,β)-Bessel model with β as in* (4). *Then $I(\pi,\mu)$ is a nonzero $\mathbb{C}[q^{-s},q^s]$ module containing $\mathbb{C}$, and there exists $R(X)\in\mathbb{C}[X]$ such that*

$$R(q^{-s})I(\pi,\mu)\subset\mathbb{C}[q^{-s},q^s],$$

so that $I(\pi,\mu)$ is a fractional ideal of the principal ideal domain $\mathbb{C}[q^{-s},q^s]$ whose quotient field is $\mathbb{C}(q^{-s})$. The fractional ideal $I(\pi,\mu)$ admits a generator of the form $1/Q(q^{-s})$ with $Q(0)=1$, where $Q(X)\in\mathbb{C}[X]$.

Proof. One can argue as in the proof of Proposition 2.6.4 of [Roberts and Schmidt 2007]. One step in the proof is to show that $I(\pi,\mu)$ contains $\mathbb{C}$. This follows from Lemma 4.3.1. □

Using the notation of this proposition, we set

$$L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu):=1/Q(q^{-s}) \tag{53}$$

and call this the *regular part of the Piatetski-Shapiro L-factor*; see [Piatetski-Shapiro 1997]. As the notation indicates, $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ does not depend on the Bessel data β and Λ. This is implied by the following result.

Theorem 5.1.2. *Table 5 shows the factors $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ for all irreducible, admissible representations (π,V) of* $\mathrm{GSp}(4,F)$ *in the nonsplit case.* (A "—" indicates that no nonsplit Bessel functional exists.)

Proof. Up to an element of $\mathcal{S}(F^\times)$, the functions $x\mapsto B\bigl(\bigl[\begin{smallmatrix}x&\\&1\end{smallmatrix}\bigr]\bigr)$, where $B\in\mathcal{B}(\pi,\Lambda,\beta)$, are listed in Table 4. Using the fact that

$$\sum_{m=m_0}^{\infty} m^j z^m = g(z)\frac{1}{(1-z)^{j+1}} \tag{54}$$

with a function $g(z)$ which is holomorphic and nonvanishing at $z=1$, the integrals in (52) are thus easily calculated up to elements of $\mathbb{C}[q^s,q^{-s}]$. □

Also indicated in Table 5 are the generic representations (i.e., those that admit a Whittaker model); supercuspidals may or may not be generic. We see that for all generic representations $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)=L(s,\varphi)$ if $\mu=1_{F^\times}$. Here $L(s,\varphi)$ is the L-factor of the Langlands parameter φ of π, as listed in Table A.8 of [Roberts and Schmidt 2007].

5.2. ***The group G.*** We now recall the setup of [Piatetski-Shapiro 1997]. Let L be the quadratic extension of F as in Section 2. Let $V=L^2$, which we consider as a space of row vectors. We endow V with the skew-symmetric F-linear form

$$\rho(x,y)=\mathrm{Tr}_{L/F}(x_1y_2-x_2y_1),\qquad x=(x_1,x_2),\ y=(y_1,y_2). \tag{55}$$

Let

$$\mathrm{GSp}_\rho=\bigl\{g\in\mathrm{GL}(4,F):\ \rho(xg,yg)=\lambda\rho(x,y),\ \text{for some } \lambda=\lambda(g)\in F^\times,\ \text{for all } x,y\in V\bigr\}$$

be the symplectic similitude group of the form ρ. Let

$$G=\{g\in\mathrm{GL}(2,L):\ \det(g)\in F^\times\}. \tag{56}$$

The group G acts on V by matrix multiplication from the right. A calculation shows

$$\rho(xg,yg)=\det(g)\rho(x,y) \tag{57}$$

		representation	$L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$	generic
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$L(s,\chi_1\chi_2\sigma\mu)L(s,\sigma\mu)L(s,\chi_1\sigma\mu)L(s,\chi_2\sigma\mu)$	•
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\nu^{1/2}\chi\sigma\mu)L(s,\chi^2\sigma\mu)L(s,\sigma\mu)$	•
	b	$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\nu^{-1/2}\chi\sigma\mu)L(s,\chi^2\sigma\mu)L(s,\sigma\mu)$	
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$L(s,\nu^{1/2}\chi\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	•
	b	$\chi\rtimes\sigma 1_{\mathrm{GSp}(2)}$	—	
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(s,\nu^{3/2}\sigma\mu)$	•
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$L(s,\nu^{3/2}\sigma\mu)L(s,\nu^{-1/2}\sigma\mu)$	
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	—	
	d	$\sigma 1_{\mathrm{GSp}(4)}$	—	
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\xi\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	•
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\xi\sigma\mu)L(s,\nu^{-1/2}\sigma\mu)$	
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\xi\sigma)$	$L(s,\nu^{-1/2}\xi\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\xi\sigma\mu)L(s,\nu^{-1/2}\sigma\mu)$	
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma\mu)^2$	•
	b	$\tau(T,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma\mu)$	
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	—	
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	—	
VII		$\chi\rtimes\pi$	1	•
VIII	a	$\tau(S,\pi)$	1	•
	b	$\tau(T,\pi)$	1	
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi(\mu))$	1	•
	b	$L(\nu\xi,\nu^{-1/2}\pi(\mu))$	1	
X		$\pi\rtimes\sigma$	$L(s,\omega_\pi\sigma\mu)L(s,\sigma\mu)$	•
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma\mu)$	•
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma\mu)$	
		supercuspidal	1	∘

Table 5. Regular parts of Piatetski-Shapiro L-factors (nonsplit case).

for $x, y \in V$ and $g \in G$. Hence, $G \subset \mathrm{GSp}_\rho$. Since all four-dimensional symplectic F-spaces are isomorphic to the standard space F^4 with the form (1), the groups GSp_ρ and $\mathrm{GSp}(4, F)$ are isomorphic; here, we think of $\mathrm{GSp}(4, F)$ as acting on the right on the space of row vectors F^4. We wish to find one such isomorphism under which the group G takes on a particularly simple shape inside $\mathrm{GSp}(4, F)$.

For this we assume that the matrix β in (4) is diagonal and nondegenerate, i.e., $b = 0$ and $a, c \neq 0$; after a suitable conjugation, every nondegenerate β can be brought into this form. Consider the following F-basis of V,

$$f_1 = (1, 0), \quad f_2 = (\Delta/c, 0), \quad f_3 = (0, 1/2), \quad f_4 = (0, c/(2\Delta)). \tag{58}$$

Let $e_1, \ldots, e_4$ be the standard basis of F^4. Then the map $f_i \mapsto e_i$ establishes an isomorphism $V \cong F^4$ preserving the symplectic form on both spaces (the form ρ on V, and the form J defined in (1) on F^4). The resulting isomorphism $\mathrm{GSp}_\rho \cong \mathrm{GSp}(4, F)$ has the following properties:

$$G \ni \begin{bmatrix} x & \\ & 1 \end{bmatrix} \longmapsto \begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix}, \tag{59}$$

$$G \ni \begin{bmatrix} 1 & \\ & x \end{bmatrix} \longmapsto \begin{bmatrix} 1 &&& \\ & 1 && \\ && x & \\ &&& x \end{bmatrix}, \tag{60}$$

$$G \ni \begin{bmatrix} t & \\ & \bar{t} \end{bmatrix} \longmapsto \begin{bmatrix} x & yc && \\ -ya & x && \\ && x & ya \\ && -yc & x \end{bmatrix} \quad \text{for } t = x + y\Delta \in L^\times, \tag{61}$$

$$G \ni \begin{bmatrix} 1 & x + y\Delta \\ & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 && 2x & -2ay \\ & 1 & -2ay & -2ac^{-1}x \\ && 1 & \\ &&& 1 \end{bmatrix}. \tag{62}$$

Here, $\bar{t} = x - y\Delta$ is the Galois conjugate of t. Recall from (9) that the matrices on the right hand side of (61) are precisely the elements of T. It is easy to verify that the matrices on the right-hand side of (62) are precisely those elements of N that lie in

$$N_0 = \left\{ \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} : \mathrm{tr}(\beta X) = 0 \right\} = \left\{ \begin{bmatrix} 1 && x & y \\ & 1 & y & z \\ && 1 & \\ &&& 1 \end{bmatrix} : ax + by + cz = 0 \right\}. \tag{63}$$

In particular, if we consider G a subgroup of $\mathrm{GSp}(4, F)$, then we see that

$$G \cap R = T N_0;$$

see Proposition 2.1 of [Piatetski-Shapiro 1997]. We define the following subgroups of G:

$$A^G = G \cap \begin{bmatrix} * & \\ & * \end{bmatrix} = \left\{ \begin{bmatrix} xt & \\ & \bar{t} \end{bmatrix} \in \mathrm{GL}(2, L) \,:\, x \in F^\times,\, t \in L^\times \right\}, \tag{64}$$

$$N_0 = G \cap \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \in \mathrm{GL}(2, L) \,:\, b \in L \right\}, \tag{65}$$

$$B^G = G \cap \begin{bmatrix} * & * \\ & * \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ & d \end{bmatrix} \in \mathrm{GL}(2, L) \,:\, ad \in F^\times \right\}, \tag{66}$$

$$K^G = G \cap \mathrm{GL}(2, \mathfrak{o}_L) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathfrak{o}_L) \,:\, ad - bc \in F^\times \right\}. \tag{67}$$

By our remarks above, when embedded into $\mathrm{GSp}(4, F)$, the group N_0 coincides with the group introduced in (63), so that the notation is consistent. The Iwasawa decomposition for $\mathrm{GL}(2, L)$ implies that $G = B^G K^G$. The modular factor for B^G is $\delta\big(\big[\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\big]\big) = |a/d|_L$, where $|\cdot|_L$ is the normalized absolute value on L. Note that $|t|_L = |N_{L/F}(t)|_F$ for $t \in L^\times$. Let dn be the Haar measure on N_0 that gives $N_0 \cap K^G$ volume 1. Let da be the Haar measure on A^G that gives $A^G \cap K^G$ volume 1. Let dk be the Haar measure on K^G with total volume 1. There is a Haar measure on G given by

$$\int\limits_{N_0} \int\limits_{A^G} \int\limits_{K^G} f(nak)\delta(a)^{-1}\, dk\, da\, dn. \tag{68}$$

The measure (68) gives K^G volume 1. We will also use the integration formula

$$\int\limits_{N_0 \backslash G} f(g)\, dg = \int\limits_{B^G} f(wb)\, db = \int\limits_{N_0} \int\limits_{A^G} f(wna)\, da\, dn \tag{69}$$

for a function f on G that is left N_0-invariant (the db in the middle integral is a *right* Haar measure on B^G). Here, $w = \big[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\big] \in G$, which is embedded into $\mathrm{GSp}(4, F)$ as

$$w \longmapsto \begin{bmatrix} 2 & & & \\ & -2ac^{-1} & & \\ & & \frac{1}{2} & \\ & & & -\frac{1}{2}ca^{-1} \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}. \tag{70}$$

Principal series representations of G. Let Λ be a character of $L^\times$, let μ be a character of $F^\times$, and $s \in \mathbb{C}$. We denote by $\mathcal{J}(\Lambda, \mu, s)$ the induced representation $\mathrm{ind}_{B^G}^G(\chi)$ (unnormalized induction), where

$$\chi\left(\begin{bmatrix} xt & * \\ & \bar{t} \end{bmatrix}\right) = \mu(x)|x|^{s+1/2}\Lambda(t)^{-1}. \tag{71}$$

It is easy to see that the contragredient of $\mathcal{J}(\Lambda, \mu, s)$ is $\mathcal{J}(\Lambda^{-1}, \mu^{-1}, 1-s)$.

Let $V = L^2$, considered as a space of row vectors. Let $\mathcal{S}(V)$ be the space of Schwartz–Bruhat functions on V, i.e., the space of locally constant functions with compact support. For $g \in G$, $\Phi \in \mathcal{S}(V)$ and a complex number s, we define

$$f^{\Phi}(g, \mu, \Lambda, s) := \mu(\det(g))|\det(g)|^{s+1/2} \int_{L^\times} \Phi((0, \bar{t})g)|t\bar{t}|^{s+1/2}\mu(t\bar{t})\Lambda(t)\, d^\times t. \tag{72}$$

This is the same definition as on page 265 of [Piatetski-Shapiro 1997], except we have $(0, \bar{t})$ instead of $(0, t)$, in order to be compatible with our conventions about Bessel models. Assuming convergence, a calculation shows that $f^{\Phi} \in \mathcal{J}(\Lambda, \mu, s)$.

Let $\mathcal{S}_0(V)$ be the subspace of $\Phi \in \mathcal{S}(V)$ for which $\Phi(0, 0) = 0$. If $\Phi \in \mathcal{S}_0(V)$ and $g \in G$, then $\Phi((0, \bar{t})g) = 0$ for t outside a compact set of $L^\times$. It follows that the integral (72) converges absolutely for $\Phi \in \mathcal{S}_0(V)$, for any $s \in \mathbb{C}$.

Lemma 5.2.1. $\mathcal{J}(\Lambda, \mu, s) = \{f^{\Phi}(\,\cdot\,, \mu, \Lambda, s) : \Phi \in \mathcal{S}_0(V)\}$.

Proof. Given $f \in \mathcal{J}(\Lambda, \mu, s)$, we need to find $\Phi \in \mathcal{S}_0(V)$ such that $f^{\Phi} = f$. We define Φ by

$$\Phi(x, y) = \begin{cases} \mu^{-1}(\det(k))f(k) & \text{if } (x, y) = (0, 1)k \text{ for some } k \in K^G, \\ 0 & \text{if } (x, y) \notin (0, 1)K^G. \end{cases} \tag{73}$$

It is straightforward to verify that Φ is well defined, that $\Phi \in \mathcal{S}_0(V)$, and that f^{Φ} is a multiple of f. $\square$

Lemma 5.2.2. *Let $\Lambda_\mu = \Lambda \cdot (\mu \circ N_{L/F})$.*

(i) *The representation $\mathcal{J}(\Lambda, \mu, s)$ contains a one-dimensional G-invariant subspace if and only if*

$$\Lambda_\mu(t) = |t|_L^{-s-1/2} \quad \textit{for all } t \in L^\times. \tag{74}$$

In this case the function

$$f(g) = \mu(\det(g))|\det(g)|^{s+1/2}, \qquad g \in G, \tag{75}$$

spans a one-dimensional G-invariant subspace of $\mathrm{ind}_{B^G}^G(\chi)$.

(ii) *The representation $\mathcal{J}(\Lambda, \mu, s)$ contains a one-dimensional G-invariant quotient if and only if*

$$\Lambda_\mu(t) = |t|_L^{-s+3/2} \quad \textit{for all } t \in L^\times. \tag{76}$$

Proof. Part (i) is an easy exercise. Part (ii) follows from (i), observing that the contragredient of $\mathcal{J}(\Lambda, \mu, s)$ is $\mathcal{J}(\Lambda^{-1}, \mu^{-1}, 1-s)$. $\square$

Note that condition (74) is equivalent to saying that s is a pole of $L(s+1/2, \Lambda_\mu)$. Later we will define the notion of *exceptional pole*; see (92). The exceptional poles will be among the poles of $L(s+1/2, \Lambda_\mu)$. Note that, by (73), the function f in (75) is a multiple of f^Φ, where

$$\Phi(x, y) = \begin{cases} 1 & \text{if } (x, y) = (0, 1)k \text{ for some } k \in K^G, \\ 0 & \text{if } (x, y) \notin (0, 1)K^G. \end{cases} \tag{77}$$

Hence, in the nonsplit case, Φ is the characteristic function of $(\mathfrak{o}_L \oplus \mathfrak{o}_L) \setminus (\mathfrak{p}_L \oplus \mathfrak{p}_L)$.

5.3. ***The zeta integrals.*** Let Λ be a character of $T \cong L^\times$, and let μ be a character of $F^\times$. Recall the definition of the functions $f^\Phi(g, \mu, \Lambda, s)$ in (72). Let π be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$. Let $\mathcal{B}(\pi, \Lambda, \beta)$ be a (Λ, β)-Bessel model for π. For $B \in \mathcal{B}(\pi, \Lambda, \beta)$ and $s \in \mathbb{C}$, let

$$Z(s, B, \Phi, \mu) = \int\limits_{TN_0\backslash G} B(g) f^\Phi(g, \mu, \Lambda, s)\, dg, \tag{78}$$

provided this integral converges. (In [Piatetski-Shapiro 1997] this integral was denoted by $L(W, \Phi, \mu, s)$.) Substituting the definition of $f^\Phi(g, \mu, \Lambda, s)$ and unfolding the integral shows that

$$Z(s, B, \Phi, \mu) = \int\limits_{N_0\backslash G} B(g)\Phi((0, 1)g)\mu(\det(g))|\det(g)|^{s+1/2}\, dg. \tag{79}$$

By (68), we have

$$\begin{aligned} &Z(s, B, \Phi, \mu) \\ &\qquad = \int\limits_{A^G}\int\limits_{K^G} \delta(a)^{-1} B(ak)\Phi((0, 1)ak)\mu(\det(ak))|\det(ak)|^{s+1/2}\, dk\, da. \end{aligned} \tag{80}$$

Recall that $\mathcal{S}_0(V)$ is the space of $\Phi \in \mathcal{S}(V)$ satisfying $\Phi(0, 0) = 0$. Let $\Phi_1 \in \mathcal{S}(V)$ be the characteristic function of $\mathfrak{o}_L \oplus \mathfrak{o}_L$. Then every $\Phi \in \mathcal{S}(V)$ can be written in a unique way as $\Phi = \Phi_0 + c\Phi_1$ with $\Phi_0 \in \mathcal{S}_0(V)$ and $c \in \mathbb{C}$. We will first investigate $Z(s, B, \Phi, \mu)$ for $\Phi \in \mathcal{S}_0(V)$.

Lemma 5.3.1. *Let the notations and hypotheses be as above.*

(i) *For any $B \in \mathcal{B}(\pi, \Lambda, \beta)$ and $\Phi \in \mathcal{S}_0(V)$, the function $Z(s, B, \Phi, \mu)$ converges for real part of s large enough to an element of $\mathbb{C}(q^{-s})$. This element lies in the ideal $I(\pi, \mu)$ generated by all simplified zeta integrals*; *see Proposition 5.1.1.*

(ii) *For any $B \in \mathcal{B}(\pi, \Lambda, \beta)$, there exists $\Phi \in \mathcal{S}_0(V)$ such that $Z(s, B, \Phi, \mu) = \zeta(s, B, \mu)$.*

Hence, the integrals $Z(s, B, \Phi, \mu)$, as B runs through $\mathcal{B}(\pi, \Lambda, \beta)$ and Φ runs through $\mathcal{S}_0(V)$, generate the ideal $I(\pi, \mu)$ already exhibited in Proposition 5.1.1.

Proof. (i) Let $\Phi \in \mathcal{S}_0(V)$. We have

$$\Phi((0,1)ak) = \Phi(\bar{t}k_3, \bar{t}k_4) \quad \text{if } a = \begin{bmatrix} xt & \\ & \bar{t} \end{bmatrix} \in A^G,\ k = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \in K^G. \tag{81}$$

Since one of k_3 or k_4 is a unit and $\Phi(0,0) = 0$, it follows that $\Phi((0,1)ak) = 0$ if t is outside a compact set of $L^\times$. As a consequence, there exists a small subgroup Γ of K^G such that

$$\Phi((0,1)ak\gamma) = \Phi((0,1)ak)$$

for all $a \in A^G$, $k \in K^G$ and $\gamma \in \Gamma$. By making Γ even smaller, we may assume that B and $\mu \circ \det$ are right Γ-invariant. It follows that $Z(s, B, \Phi, \mu)$ as in (80) is a finite sum of integrals of the form

$$I(s, B, \Phi, \mu) = \int\limits_{A^G} \delta(a)^{-1} B(a) \Phi((0,1)a) \mu(\det(a)) |\det(a)|^{s+1/2}\, da, \tag{82}$$

with different B and $\Phi \in \mathcal{S}_0(V)$. Using coordinates on A^G, we have

$$\begin{aligned} &I(s, B, \Phi, \mu) \\ &= \int\limits_{F^\times} \int\limits_{L^\times} |xt\bar{t}^{-1}|_L^{-1} B\left(\begin{bmatrix} xt & \\ & \bar{t} \end{bmatrix}\right) \Phi(0, \bar{t}) \mu(xt\bar{t}) |xt\bar{t}|^{s+1/2}\, d^\times t\, d^\times x \\ &= \int\limits_{F^\times} \int\limits_{L^\times} |x|^{-2} \Lambda(t) B\left(\begin{bmatrix} x & \\ & 1 \end{bmatrix}\right) \Phi(0, \bar{t}) \mu(xt\bar{t}) |xt\bar{t}|^{s+1/2}\, d^\times t\, d^\times x \\ &= \Bigg(\int\limits_{F^\times} B\left(\begin{bmatrix} x & \\ & 1 \end{bmatrix}\right) \mu(x) |x|^{s-3/2}\, d^\times x\Bigg) \Bigg(\int\limits_{L^\times} \Lambda(t) \Phi(0, \bar{t}) \mu(t\bar{t}) |t\bar{t}|^{s+1/2}\, d^\times t\Bigg). \end{aligned} \tag{83}$$

The first integral is precisely $\zeta(s, B, \mu)$; see (52). Since the integration in the second integral is over a compact subset of $L^\times$, this integral is in $\mathbb{C}[q^s, q^{-s}]$. It follows that $I(s, B, \Phi, \mu)$ lies in the ideal $I(\pi, \mu)$.

(ii) By (79) and (69), we have

$$\begin{aligned} Z(s, B, \Phi, \mu) &= \int\limits_{N_0} \int\limits_{A^G} B(wna) \Phi((0,1)wna) \mu(\det(a)) |\det(a)|^{s+1/2}\, da\, dn \\ &= \int\limits_{N_0} \int\limits_{A^G} B(wna) \Phi((-1,0)na) \mu(\det(a)) |\det(a)|^{s+1/2}\, da\, dn \end{aligned}$$

$$= \int\limits_L \int\limits_{F^\times} \int\limits_{L^\times} B\left(w\begin{bmatrix}1 & n\\ & 1\end{bmatrix}\begin{bmatrix}xt & \\ & \bar t\end{bmatrix}\right)\Phi\left((-1,0)\begin{bmatrix}1 & n\\ & 1\end{bmatrix}\begin{bmatrix}xt & \\ & \bar t\end{bmatrix}\right) \times \mu(xt\bar t)|xt\bar t|^{s+1/2}\,d^\times t\,d^\times x\,dn$$

$$= \int\limits_L \int\limits_{F^\times} \int\limits_{L^\times} B\left(w\begin{bmatrix}xt & \bar t n\\ & \bar t\end{bmatrix}\right)\Phi(-xt,-\bar t n)\mu(xt\bar t)|x|^{s+1/2}|t|_L^{s+1/2}\,d^\times t\,d^\times x\,dn$$

$$= \int\limits_L \int\limits_{F^\times} \int\limits_{L^\times} B\left(w\begin{bmatrix}xt & n\\ & \bar t\end{bmatrix}\right)\Phi(-xt,-n)\mu(xt\bar t)|x|^{s+1/2}|t|_L^{s-1/2}\,d^\times t\,d^\times x\,dn$$

$$= \int\limits_L \int\limits_{F^\times} \int\limits_{L^\times} B\left(w\begin{bmatrix}1 & \\ & x^{-1}\end{bmatrix}\begin{bmatrix}t & n\\ & \bar t\end{bmatrix}\right)\Phi(-t,-n) \times \mu(x)^{-1}\mu(t\bar t)|x|^{3/2-s}|t|_L^{s-1/2}\,d^\times t\,d^\times x\,dn.$$

Now choose Φ such that $\Phi(-t,-n)$ is zero unless t is close to 1 and n is close to 0. If the support of Φ is chosen small enough, then, after appropriate normalization,

$$Z(s,B,\Phi,\mu) = \int\limits_{F^\times} B\left(\begin{bmatrix}x^{-1} & \\ & 1\end{bmatrix}w\right)\mu(x)^{-1}|x|^{3/2-s}\,d^\times x.$$

This is just $\zeta(s,wB,\mu)$. The assertion follows. □

We see from Lemma 5.3.1 that, instead of (53), we could have chosen to define $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ as the gcd of all $Z(s,B,\Phi,\mu)$, as B runs through $\mathcal{B}(\pi,\Lambda,\beta)$ and Φ runs through $\mathcal{S}_0(V)$. The same observation was made in [Danişman 2015b, Proposition 18(i)].

Next we investigate $Z(s,B,\Phi_1,\mu)$, where we recall Φ_1 is the characteristic function of $\mathfrak{o}_L\oplus\mathfrak{o}_L$. In the split case, a character Λ of $L^\times = F^\times\times F^\times$ is a pair (λ_1,λ_2) of characters of $F^\times$, and by $L(s,\Lambda)$ we mean $L(s,\lambda_1)L(s,\lambda_2)$.

Lemma 5.3.2. *Let $\Lambda_\mu = \Lambda\cdot(\mu\circ N_{L/F})$.*

(i) *Assume that Λ_μ is ramified. Then $Z(s,B,\Phi_1,\mu)=0$.*

(ii) *Assume that Λ_μ is unramified. Then*

$$Z(s,B,\Phi_1,\mu) = \zeta(s,B_\mu,\mu)L(s+1/2,\Lambda_\mu), \tag{84}$$

where

$$B_\mu(g) := \int\limits_{K^G} B(gk)\mu(\det(k))\,dk,\quad g\in \mathrm{GSp}(4,F). \tag{85}$$

Proof. Evidently, $\Phi_1((x,y)k)=\Phi_1(x,y)$ for all $(x,y)\in V$ and $k\in K^G$. Therefore, from (80), we get

$$
\begin{aligned}
(86)\quad & Z(s,B,\Phi_1,\mu)\\
&=\int\limits_{A^G}\int\limits_{K^G}\delta(a)^{-1}B(ak)\Phi_1((0,1)a)\mu(\det(ak))|\det(a)|^{s+1/2}\,dk\,da\\
&=\int\limits_{A^G}\delta(a)^{-1}B_\mu(a)\Phi_1((0,1)a)\mu(\det(a))|\det(a)|^{s+1/2}\,da.
\end{aligned}
$$

Clearly, B_μ is an element of $\mathcal{B}(\pi,\Lambda,\beta)$ satisfying

$$B_\mu(gk)=\mu^{-1}(\det(k))B_\mu(g)$$

for $k\in K^G$. Using coordinates on A^G, we have

$$
\begin{aligned}
(87)\quad & Z(s,B,\Phi_1,\mu)\\
&=\int\limits_{F^\times}\int\limits_{L^\times}|xt\bar t^{-1}|_L^{-1}B_\mu(a)\Phi_1((0,\bar t))\mu(xt\bar t)|xt\bar t|^{s+1/2}\,d^\times t\,d^\times x\\
&=\int\limits_{F^\times}\int\limits_{L^\times}B_\mu\left(\begin{bmatrix}xt&\\&\bar t\end{bmatrix}\right)\Phi_1((0,\bar t))\mu(xt\bar t)|t\bar t|^{s+1/2}|x|^{s-3/2}\,d^\times t\,d^\times x\\
&=\int\limits_{F^\times}\int\limits_{L^\times\cap\mathfrak{o}_L}\Lambda(t)B_\mu\left(\begin{bmatrix}x&\\&1\end{bmatrix}\right)\mu(xt\bar t)|t\bar t|^{s+1/2}|x|^{s-3/2}\,d^\times t\,d^\times x\\
&=\zeta(s,B_\mu,\mu)\int\limits_{L^\times\cap\mathfrak{o}_L}\Lambda(t)\mu(t\bar t)|t\bar t|^{s+1/2}\,d^\times t.
\end{aligned}
$$

It is straightforward to calculate that

$$
(88)\quad \int\limits_{L^\times\cap\mathfrak{o}_L}\Lambda(t)\mu(t\bar t)|t\bar t|^{s+1/2}\,d^\times t=\begin{cases}L(s+1/2,\Lambda_\mu) & \text{if } \Lambda_\mu \text{ is unramified,}\\ 0 & \text{if } \Lambda_\mu \text{ is ramified.}\end{cases}\qquad\square
$$

We see from Lemma 5.3.1 and Lemma 5.3.2 that $Z(s,B,\Phi,\mu)$ converges for real part of s large enough to an element of $\mathbb{C}(q^{-s})$, for any $B\in\mathcal{B}(\pi,\Lambda,\beta)$ and $\Phi\in\mathcal{S}(V)$. Let $I_{\Lambda,\beta}(\pi,\mu)$ be the $\mathbb{C}$-vector subspace of $\mathbb{C}(q^{-s})$ spanned by all $\zeta(s,B,\mu)$ as B runs through $\mathcal{B}(\pi,\Lambda,\beta)$.

Proposition 5.3.3. *Let π be an irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *admitting a (Λ,β)-Bessel model with β as in (4). Then $I_{\Lambda,\beta}(\pi,\mu)$ is a nonzero $\mathbb{C}[q^{-s},q^s]$ module containing $\mathbb{C}$, and there exists $R(X)\in\mathbb{C}[X]$ such that*

$$R(q^{-s})I_{\Lambda,\beta}(\pi,\mu)\subset\mathbb{C}[q^{-s},q^s],$$

so that $I_{\Lambda,\beta}(\pi,\mu)$ *is a fractional ideal of the principal ideal domain* $\mathbb{C}[q^{-s},q^s]$ *whose quotient field is* $\mathbb{C}(q^{-s})$. *The fractional ideal* $I_{\Lambda,\beta}(\pi,\mu)$ *admits a generator of the form* $1/Q(q^{-s})$ *with* $Q(0)=1$, *where* $Q(X)\in\mathbb{C}[X]$.

Proof. The proof is similar to that of Proposition 5.1.1. It follows easily from (79) that $I_{\Lambda,\beta}(\pi,\mu)$ is a $\mathbb{C}[q^s,q^{-s}]$-module. It follows from Proposition 5.1.1 and Lemma 5.3.1 that $I_{\Lambda,\beta}(\pi,\mu)$ contains $\mathbb{C}$. □

Using the notation of this proposition, we set

$$L_\Lambda^{\mathrm{PS}}(s,\pi,\mu):=1/Q(q^{-s}). \tag{89}$$

This is the Piatetski-Shapiro L-factor, as defined in [Piatetski-Shapiro 1997]. Our notation indicates that these factors may depend on Λ (and β, which we suppress from the notation).

We now distinguish two cases. In the first, assume

$$\frac{Z(s,B,\Phi,\mu)}{L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu)} \text{ is entire for all } B\in\mathcal{B}(\pi,\Lambda,\beta) \text{ and } \Phi\in\mathcal{S}(V). \tag{90}$$

Being entire is equivalent to lying in $\mathbb{C}[q^s,q^{-s}]$. Hence, in this case the fractional ideal generated by all $Z(s,B,\Phi,\mu)$ is generated by $L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu)$, and we have

$$L_\Lambda^{\mathrm{PS}}(s,\pi,\mu)=L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu). \tag{91}$$

In particular, the Piatetski-Shapiro L-factor does not depend on Λ in this case.

For the second case, assume

$$\frac{Z(s,B,\Phi,\mu)}{L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu)} \text{ has a pole for some } B\in\mathcal{B}(\pi,\Lambda,\beta) \text{ and } \Phi\in\mathcal{S}(V). \tag{92}$$

Such poles are called *exceptional poles*. By (84), exceptional poles are precisely the poles of

$$\frac{\zeta(s,B_\mu,\mu)}{L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu)}L(s+1/2,\Lambda_\mu), \tag{93}$$

as B runs through $\mathcal{B}(\pi,\Lambda,\beta)$. Since the fraction in (93) is entire, exceptional poles are found among the poles of $L(s+1/2,\Lambda_\mu)$. If we write

$$L(s,\Lambda_\mu)=\frac{1}{(1-\gamma_1q^{-s})(1-\gamma_2q^{-s})}, \tag{94}$$

where one of the complex numbers γ_1,γ_2 may be zero, then

$$L^{\mathrm{PS}}(s,\pi,\mu)=L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu)\frac{1}{P(q^{-s-1/2})}, \tag{95}$$

where $P\in\mathbb{C}[X]$ is either $1-\gamma_iX$ or $(1-\gamma_1X)(1-\gamma_2X)$.

Remark. Our definition of exceptional pole is slightly more general than the one given in [Piatetski-Shapiro 1997]. Therein, a complex number s_0 is called an exceptional pole if s_0 is a pole of $L^{\mathrm{PS}}(s, \pi, \mu)$ but not of $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$. It follows easily that an exceptional pole according to Piatetski-Shapiro is also an exceptional pole according to our definition. However, the two notions may not coincide if there is overlap between the poles of $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$ and the poles of $L(s+1/2, \Lambda_\mu)$.

The *regular poles* are the poles of $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$. According to our definition, an exceptional pole can also be regular, while in [Piatetski-Shapiro 1997] the two notions are exclusive. Our definition is designed in such a way that $L^{\mathrm{PS}}(s, \pi, \mu) \neq L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$ precisely if there exist exceptional poles.

5.4. ***Double coset decompositions.*** We first prove the following double coset decomposition for $\mathrm{GL}(2, F)$. Let β be as in (4), and let T be the group of all

$$\begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} \in \mathrm{GL}(2, F), \qquad x^2 - y^2\Big(\frac{b^2}{4} - ac\Big) \neq 0. \tag{96}$$

Recall that we are in the *split case* if and only if $b^2 - 4ac \in F^{\times 2}$. We can and will make the assumption that

$$a, c \neq 0. \tag{97}$$

In the split case, let $r_1, r_2 \in F^\times$ be the two roots of the equation

$$ar^2 + br + c = 0. \tag{98}$$

Let B_1 be the subgroup of $\mathrm{GL}(2, F)$ consisting of all elements of the form $\left[\begin{smallmatrix} 1 & * \\ & * \end{smallmatrix}\right]$, and let B_2 be the subgroup consisting of all elements of the form $\left[\begin{smallmatrix} 1 & \\ * & * \end{smallmatrix}\right]$.

Lemma 5.4.1. (i) *In the nonsplit case*, $\mathrm{GL}(2, F) = TB_1 = TB_2$.

(ii) *In the split case*,

$$\begin{aligned} \mathrm{GL}(2, F) &= TB_1 \sqcup Tg_1sB_1 \sqcup Tg_2sB_1 \\ &= TB_2 \sqcup Tg_1B_2 \sqcup Tg_2B_2, \quad \textit{where } g_i = \begin{bmatrix} 1 & r_i \\ & 1 \end{bmatrix}, \ s = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}. \end{aligned} \tag{99}$$

The set TB_1 (resp. TB_2) is open and dense in $\mathrm{GL}(2, F)$, *and consists of all* $\left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right] \in \mathrm{GL}(2, F)$ *with* $aa_1^2 + ba_1a_3 + ca_3^2 \neq 0$ (*resp.* $aa_2^2 + ba_2a_4 + ca_4^2 \neq 0$). *For $i = 1$ or 2, the set Tg_isB_1 (resp. Tg_iB_2) consists of all* $\left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right] \in \mathrm{GL}(2, F)$ *with $a_1 = a_3r_i$ (resp. $a_2 = a_4r_i$).*

Proof. Calculations show that if $aa_1^2 + ba_1a_3 + ca_3^2 \neq 0$, then the equation

$$\begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} \begin{bmatrix} 1 & z \\ & d \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

can be solved for x, y, z, d. Assume that $aa_1^2+ba_1a_3+ca_3^2=0$. Then $a_1=a_3r_i$ for $i=1$ or $i=2$. Calculations show that the equation

$$\begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} g_i s \begin{bmatrix} 1 & z \\ & d \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

can be solved for x, y, z, d. This proves the statements for B_1, and the proof for B_2 is similar. □

Let P be the (F-points of the) Siegel parabolic subgroup of $\mathrm{GSp}(4, F)$; see (2). Let G be the group defined in (56). We assume that $\beta = \left[\begin{smallmatrix} a & \\ & c \end{smallmatrix}\right]$ with $ac \neq 0$, and embed G into $\mathrm{GSp}(4, F)$ such that (59) to (62) hold. More generally, if

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in G,$$

then a calculation shows that, as an element of $\mathrm{GSp}(4, F)$,

$$g = \begin{bmatrix} \alpha_1 & c\alpha_2 & 2\beta_1 & -2a\beta_2 \\ -a\alpha_2 & \alpha_1 & -2a\beta_2 & -\frac{2a}{c}\beta_1 \\ \frac{1}{2}\gamma_1 & \frac{c}{2}\gamma_2 & \delta_1 & -a\delta_2 \\ \frac{c}{2}\gamma_2 & -\frac{c}{2a}\gamma_1 & c\delta_2 & \delta_1 \end{bmatrix}. \tag{100}$$

Here, $\alpha = \alpha_1 + \Delta\alpha_2$ etc., with Δ as defined after (7). The following result is a more precise version of a remark made in the proof of Theorem 4.3 of [Piatetski-Shapiro 1997].

Lemma 5.4.2. *Assume the above notations and hypotheses. Let*

$$s_2 = \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}. \tag{101}$$

Then

$$\mathrm{GSp}(4, F) = GP \sqcup Gs_2P. \tag{102}$$

The double coset Gs_2P is open and dense in $\mathrm{GSp}(4, F)$, *and*

$$s_2^{-1}Gs_2 \cap P = \left\{ \begin{bmatrix} A & \\ & \det(A)\,{}^tA^{-1} \end{bmatrix} : A \in \mathrm{GL}(2, F) \right\}. \tag{103}$$

We have $Gs_2P = Gs_2HN$, where H and N are defined in (3) *and* (2), *respectively. Furthermore,*

$$GP = \begin{cases} GB_2N & \text{in the nonsplit case,} \\ GB_2N \sqcup Gg_1B_2N \sqcup Gg_2B_2N & \text{in the split case,} \end{cases} \tag{104}$$

where

$$(105)\qquad B_2=\left\{\begin{bmatrix}1&&&\\x&y&&\\&&y&-x\\&&&1\end{bmatrix}:x\in F,\ y\in F^\times\right\},\qquad g_i=\begin{bmatrix}1&r_i&&\\&1&&\\&&1&\\&&-r_i&1\end{bmatrix},$$

with $r_1,r_2\in F^\times$ *being the two roots of the equation* $ar^2+c=0$.

Proof. Using the description (100) of the elements of G, it is easy to verify (103). As a consequence, $Gs_2P=Gs_2HN$. Equation (104) follows from (99); the disjointness in the split case is easy to verify.

By the Bruhat decomposition,

$$(106)\quad \mathrm{GSp}(4,F)=P\sqcup\begin{bmatrix}1&&*&\\&1&&\\&&1&\\&&&1\end{bmatrix}s_2P\sqcup\begin{bmatrix}1&&&\\ *&1&&*\\&&1&*\\&&&1\end{bmatrix}s_1s_2P\sqcup\begin{bmatrix}1&&*&*\\&1&*&*\\&&1&\\&&&1\end{bmatrix}s_2s_1s_2P.$$

Calculations show that

$$(107)\quad Gs_2P\cap\begin{bmatrix}1&&*&*\\&1&*&*\\&&1&\\&&&1\end{bmatrix}s_2s_1s_2P=\left\{\begin{bmatrix}1&X\\&1\end{bmatrix}s_2s_1s_2p\ :\ p\in P,\ \mathrm{tr}(\beta X)\neq 0\right\},$$

$$(108)\quad Gs_2P\cap\begin{bmatrix}1&&&\\ *&1&&*\\&&1&*\\&&&1\end{bmatrix}s_1s_2P=\left\{\begin{bmatrix}1&&&\\x&1&&z\\&&1&-x\\&&&1\end{bmatrix}s_1s_2p\ :\ p\in P,\ x^2\neq -a/c\right\},$$

$$(109)\quad Gs_2P\cap\begin{bmatrix}1&&*&\\&1&&\\&&1&\\&&&1\end{bmatrix}s_2P=\begin{bmatrix}1&&*&\\&1&&\\&&1&\\&&&1\end{bmatrix}s_2P,$$

$$(110)\quad Gs_2P\cap P=\varnothing,$$

and

$$(111)\quad GP\cap\begin{bmatrix}1&&*&*\\&1&*&*\\&&1&\\&&&1\end{bmatrix}s_2s_1s_2P=\left\{\begin{bmatrix}1&X\\&1\end{bmatrix}s_2s_1s_2p\ :\ p\in P,\ \mathrm{tr}(\beta X)=0\right\},$$

$$GP \cap \begin{bmatrix} 1 &&& \\ * & 1 && * \\ && 1 & * \\ &&& 1 \end{bmatrix} s_1 s_2 P = \left\{ \begin{bmatrix} 1 &&& \\ x & 1 && z \\ && 1 & -x \\ &&& 1 \end{bmatrix} s_1 s_2 p \, : \, p \in P, \, x^2 = -a/c \right\}, \tag{112}$$

$$GP \cap \begin{bmatrix} 1 && * & \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2 P = \varnothing, \tag{113}$$

$$GP \cap P = P. \tag{114}$$

It follows that $\mathrm{GSp}(4, F) = GP \sqcup Gs_2 P$. Since the big Bruhat cell is dense in $\mathrm{GSp}(4, F)$, (107) implies that $Gs_2 P$ is also dense in $\mathrm{GSp}(4, F)$. Since $GP = K^G B^G P = K^G P$ is the product of a compact and a closed set, it is closed in $\mathrm{GSp}(4, F)$. □

In the proof of the following lemma we will make use of the fact that a continuous bijection $X \to Y$ between p-adic spaces is a homeomorphism. This is because we can cover X with open-compact subsets, and a continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism.

For a locally compact, totally disconnected space X, we denote by $\mathcal{S}(X)$ the space of locally constant functions $X \to \mathbb{C}$ with compact support. If X is a group, $h \in X$ and $\phi \in \mathcal{S}(X)$, we denote by $R_h \phi$ the element of $\mathcal{S}(X)$ given by $x \mapsto \phi(xh)$, and by $L_h \phi$ the element of $\mathcal{S}(X)$ given by $x \mapsto \phi(h^{-1}x)$.

Let U be the unipotent radical of the Borel subgroup of $\mathrm{GSp}(4, F)$. Then U consists of all matrices of the form

$$\begin{bmatrix} 1 && * & * \\ * & 1 & * & * \\ && 1 & * \\ &&& 1 \end{bmatrix}$$

in $\mathrm{GSp}(4, F)$. For $c_1, c_2 \in F$, we define a character ψ_{c_1,c_2} of U by

$$\psi_{c_1,c_2}\left(\begin{bmatrix} 1 && y & * \\ x & 1 & * & * \\ && 1 & -x \\ &&& 1 \end{bmatrix} \right) = \psi(c_1 x + c_2 y). \tag{115}$$

The statement of the following result was mentioned in the proof of Theorem 4.3 of [Piatetski-Shapiro 1997].

Lemma 5.4.3. *Let* $D : \mathcal{S}(\mathrm{GSp}(4, F)) \to \mathbb{C}$ *be a distribution on* $\mathrm{GSp}(4, F)$ *with the following properties*:

- *There exist $c_1, c_2 \in F^\times$ such that*

$$D(R_u\phi) = \psi_{c_1,c_2}(u)D(\phi) \quad \textit{for all } u \in U \tag{116}$$

and all $\phi \in \mathcal{S}(\mathrm{GSp}(4, F))$.

- *There exists a character β of G such that*

$$D(L_h\phi) = \beta(h)D(\phi) \quad \textit{for all } h \in G \tag{117}$$

and all $\phi \in \mathcal{S}(\mathrm{GSp}(4, F))$.

Then $D = 0$.

Proof. Since $\mathrm{GSp}(4, F) = GP \sqcup Gs_2P$, it suffices to show that a distribution on $\mathcal{S}(Gs_2P)$ with the properties (116) and (117) is zero, and a distribution on $\mathcal{S}(GP)$ with those properties is also zero.

(1) First we prove that a distribution D on Gs_2P with the properties (116) and (117) must be zero. For $x \in F^\times$, let $h_x = \mathrm{diag}(x, x, 1, 1)$. By Lemma 5.4.2, $Gs_2P = Gs_2HN$. In fact, every element of Gs_2P can be written in the form gs_2h_xn with $g \in G$ and uniquely determined $x \in F^\times$ and $n \in N$. Hence Gs_2P is homeomorphic to $G \times H \times N$. We consider the continuous map

$$p : Gs_2P \to F^\times \qquad \text{defined by } \; gs_2h_xn \longmapsto x.$$

The set Gs_2P is invariant under the left action of G and the right action of U. It is easy to see that every fiber $p^{-1}(x)$ is $G \times U$-invariant. By Corollary 2.1 of [Aizenbud et al. 2010], Bernstein's localization principle, it is sufficient to prove that any distribution D on $\mathcal{S}(p^{-1}(x))$ with the properties (116) and (117) vanishes, for all $x \in F^\times$.

We apply Proposition 4.3.2 of [Bump 1997] with

$$G \times N \cong Gs_2h_xN = p^{-1}(x).$$

It shows that there exists a constant $c_1 \in \mathbb{C}$ such that

$$D(\phi) = c_1 \int_G \int_N \beta(g)\, \psi_{c_1,c_2}^{-1}(n)\, \phi(gs_2h_xn)\, dn\, dg$$

for all $\phi \in \mathcal{S}(p^{-1}(x))$. We may choose some $z \in F$ such that

$$\psi_{c_1,c_2}(u_z) \neq 1 \quad \text{for } u_z = \begin{bmatrix} 1 & & & \\ z & 1 & & \\ & & 1 & -z \\ & & & 1 \end{bmatrix}.$$

By (62),

$$n_z := s_2 u_z s_2^{-1} = \begin{bmatrix} 1 & & & -z \\ & 1 & -z & \\ & & 1 & \\ & & & 1 \end{bmatrix} \in N_0 \subset G,$$

so that $D(L_{n_z^{-1}}\phi) = \beta(n_z^{-1})D(\phi) = D(\phi)$ by (117). On the other hand, the substitution $g \mapsto n_z^{-1} g n_z$ shows that

$$\begin{aligned} D(L_{n_z^{-1}}\phi) &= c_1 \int\limits_G \int\limits_N \phi(n_z g s_2 h_x n)\beta(g)\psi_{c_1,c_2}^{-1}(n)\,dn\,dg \\ &= c_1 \int\limits_G \int\limits_N \phi(g n_z s_2 h_x n)\beta(g)\psi_{c_1,c_2}^{-1}(n)\,dn\,dg \\ &= c_1 \int\limits_G \int\limits_N \Phi(g s_2 u_z h_x n)\beta(g)\psi_{c_1,c_2}^{-1}(n)\,dn\,dg \\ &= c_1 \int\limits_G \int\limits_N \Phi(g s_2 h_x n u_z)\beta(g)\psi_{c_1,c_2}^{-1}(n)\,dn\,dg \\ &= \psi_{c_1,c_2}(u_z) c_1 \int\limits_G \int\limits_N \Phi(g s_2 h_x n)\beta(g)\psi_{c_1,c_2}^{-1}(n)\,dn\,dg. \end{aligned}$$

In the last step we used (116). Hence $D(\phi) = \psi_{c_1,c_2}(u_z)D(\phi)$, which implies $D = 0$ on $\mathcal{S}(p^{-1}(x))$.

(2) Next, using the decomposition (104), we prove that a distribution D on GP with the properties (116) and (117) must be zero.

(2.1) We will first show that a distribution D on GB_2N with the properties (116) and (117) must be zero. We define the groups

$$(118)\quad H_1 := \left\{ k_x = \begin{bmatrix} 1 & & & \\ & x & & \\ & & x & \\ & & & 1 \end{bmatrix} : x \in F^\times \right\}, \qquad U_1 := \begin{bmatrix} 1 & & & \\ * & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix} \cap \mathrm{GSp}(4, F).$$

Then, with N_0 as in (63),

$$(119)\qquad GB_2N = GUH_1 = GN_0U_1H_1 = GU_1H_1 = GH_1U_1.$$

In fact, it is not difficult to see that any element of GP can be written in the form gk_xu with uniquely determined $g \in G$, $x \in F^\times$ and $u \in U_1$. Hence GB_2N is homeomorphic to $G \times H_1 \times U_1$. We consider the continuous map

$$p : GB_2N \to F^\times \quad \text{defined by } gk_xu \longmapsto x.$$

The set GB_2N is invariant under the left action of G and the right action of U. It is easy to see that every fiber $p^{-1}(x)$ is $G\times U$-invariant. By Bernstein's localization principle, it is enough to show that a distribution D on $p^{-1}(x)$ with the properties (116) and (117) vanishes.

We apply Proposition 4.3.2 of [Bump 1997] to

$$G\times U_1 \cong Gk_xU_1 = p^{-1}(x).$$

It shows that there exists a constant $c_2\in\mathbb{C}$ such that

$$D(\phi)=c_2\int\limits_G\int\limits_{U_1}\beta(g)\,\psi^{-1}_{c_1,c_2}(u_1)\,\phi\left(g\begin{bmatrix}1&&&\\&x&&\\&&x&\\&&&1\end{bmatrix}u_1\right)du_1\,dg \tag{120}$$

for any $\phi\in\mathcal{S}(p^{-1}(x))$. Let $t\in F^\times$ be such that $\psi(c_22tx)\neq 1$,

$$n:=\begin{bmatrix}1&2t&&\\&1&&-2ac^{-1}t\\&&1&\\&&&1\end{bmatrix}\in N_0\subset G\quad\text{and}\quad u:=\begin{bmatrix}1&2tx&&\\&1&&\\&&1&\\&&&1\end{bmatrix}. \tag{121}$$

Hence,

$$\psi_{c_1,c_2}(u)=\psi(c_22tx)\neq 1.$$

Much as above, we calculate

$$\begin{aligned}
&D(L_{n^{-1}}\phi)\\
&=c_2\int\limits_G\int\limits_{U_1}\beta(g)\,\psi^{-1}_{c_1,c_2}(u_1)\,\phi(gnk_xu_1)\,du_1\,dg\\
&=c_2\int\limits_G\int\limits_{U_1}\beta(g)\,\psi^{-1}_{c_1,c_2}(u_1)\,\phi\left(gk_x\begin{bmatrix}1&2tx&&\\&1&&-2ac^{-1}tx^{-1}\\&&1&\\&&&1\end{bmatrix}u_1\right)du_1\,dg\\
&=c_2\int\limits_G\int\limits_F\int\limits_F\beta(g)\,\psi^{-1}(c_1y)\,\phi\left(gk_x\begin{bmatrix}1&2tx&&\\&1&&-2ac^{-1}tx^{-1}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y&1&&z\\&&1&-y\\&&&1\end{bmatrix}\right)dy\,dz\,dg\\
&=c_2\int\limits_G\int\limits_F\int\limits_F\beta(g)\,\psi^{-1}(c_1y)\,\phi\left(g\begin{bmatrix}1&&&-2txy\\&1&-2txy&\\&&1&\\&&&1\end{bmatrix}k_x\begin{bmatrix}1&&&\\y&1&&z\\&&1&-y\\&&&1\end{bmatrix}\begin{bmatrix}1&2tx&&\\&1&&\\&&1&\\&&&1\end{bmatrix}\right)dy\,dz\,dg
\end{aligned}$$

$$= c_2 \int_G \int_F \int_F \beta\left(g \begin{bmatrix} 1 & & & 2txy \\ & 1 & 2txy & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \psi^{-1}(c_1 y)\, \phi\left(g k_x \begin{bmatrix} 1 & & & \\ y & 1 & & z \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 2tx & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) dy\, dz\, dg$$

$$= c_2 \int_G \int_{U_1} \beta(g)\, \psi^{-1}(c_1 y)\, \phi\left(g k_x u_1 \begin{bmatrix} 1 & 2tx & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) du_1\, dg$$

$$= D(R_u \phi).$$

Hence, by (116) and (117),

$$\begin{aligned} D(\phi) = D(L_{n^{-1}}\phi) &= D(R_u \phi) \\ &= \psi(c_2 2tx) D(\phi). \end{aligned}$$

It follows that $D(\phi) = 0$.

(2.2) Now assume we are in the split case. Let $i \in \{1, 2\}$. We will show that a distribution D on $Gg_i B_2 N$ with the properties (116) and (117) must be zero. Calculations in coordinates verify that

$$g_i^{-1} G g_i \cap B_2 = \left\{ \begin{bmatrix} 1 & & & \\ \frac{y-1}{2r_i} & y & & \\ & & y & \frac{1-y}{2r_i} \\ & & & 1 \end{bmatrix} : y \in F^\times \right\}. \tag{122}$$

It follows that

$$Gg_i B_2 N = Gg_i H_1 N \sqcup Gg_i \tilde{g}_i N, \quad \text{where } \tilde{g}_i = \begin{bmatrix} 1 & & & \\ -\frac{1}{2r_i} & 1 & & \\ & & 1 & \frac{1}{2r_i} \\ & & & 1 \end{bmatrix}, \tag{123}$$

and H_1 is as in (118). We will proceed to show that a distribution D on $Gg_i B_2 N$ with the properties (117) and

$$D(R_u \phi) = \psi(c_2 x) D(\phi) \quad \text{for all } u = \begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix} \in N \tag{124}$$

must be zero.

(2.2.1) We will first show that a distribution D on Gg_iH_1N with the properties (117) and (124) vanishes. We have

$$g_i^{-1}Gg_i \cap H_1N = \left\{ \begin{bmatrix} 1 & & -2r_iu & u \\ & 1 & u & v \\ & & 1 & \\ & & & 1 \end{bmatrix} : u, v \in F \right\}. \tag{125}$$

Hence

$$Gg_iH_1N = Gg_iH_1U_2, \quad \text{where } U_2 = \begin{bmatrix} 1 & & * & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{126}$$

In fact, every element of Gg_iH_1N can be written in the form gg_ik_xu with uniquely determined $x \in F^\times$ and $u \in U_2$. We consider the continuous map

$$p : Gg_iH_1N \to F^\times \quad \text{defined by } gg_ik_xu \longmapsto x.$$

It is easy to see that every fiber $p^{-1}(x)$ is $G \times N$-invariant. By Bernstein's localization principle, it is enough to show that a distribution D on $p^{-1}(x)$ with the properties (117) and (124) vanishes. We apply Proposition 4.3.2 of [Bump 1997] to

$$G \times U_2 \cong Gg_ik_xU_2 = p^{-1}(x).$$

It shows that there exists a constant $c_3 \in \mathbb{C}$ such that

$$D(\phi) = c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2z)\, \phi\left(gg_ik_x \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dz\, dg \tag{127}$$

for all $\phi \in \mathcal{S}(p^{-1}(x))$. Now, for any $y \in F$,

$$\begin{aligned} D(\phi) &= c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2z)\, \phi\left(gg_ik_x \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dz\, dg \\ &= c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2z)\, \phi\left(gg_i \begin{bmatrix} 1 & & & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} k_x \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dz\, dg \end{aligned}$$

$$= c_3 \int_G \int_F \beta(g)\,\psi^{-1}(c_2 z)\,\phi\left(g g_i \begin{bmatrix} 1 & -2r_i y & y & \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} g_i^{-1} g_i \begin{bmatrix} 1 & 2r_i y & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} k_x \begin{bmatrix} 1 & z & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dz\,dg$$

$$= c_3 \int_G \int_F \beta(g)\,\psi^{-1}(c_2 z)\,\phi\left(g g_i \begin{bmatrix} 1 & 2r_i y & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} k_x \begin{bmatrix} 1 & z & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dz\,dg$$

$$= c_3 \int_G \int_F \beta(g)\,\psi^{-1}(c_2 z)\,\phi\left(g g_i k_x \begin{bmatrix} 1 & z+2r_i x y & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dz\,dg$$

$$= \psi(c_2 2 r_i x y) c_3 \int_G \int_F \beta(g)\,\psi^{-1}(c_2 z)\,\phi\left(g g_i k_x \begin{bmatrix} 1 & z & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dz\,dg$$

$$= \psi(c_2 2 r_i x y) D(\phi).$$

It follows that $D(\phi) = 0$.

(2.2.2) Finally, we will show that a distribution D on $G g_i \tilde{g}_i N$ with the properties (117) and (124) vanishes. We have

$$(g_i \tilde{g}_i)^{-1} G g_i \tilde{g}_i \cap N = \left\{ \begin{bmatrix} 1 & & u & \\ & 1 & & v \\ & & 1 & \\ & & & 1 \end{bmatrix} : u, v \in F \right\}. \tag{128}$$

Hence

$$G g_i \tilde{g}_i N = G g_i \tilde{g}_i U_3, \quad \text{where } U_3 = \begin{bmatrix} 1 & & & * \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{129}$$

We apply Proposition 4.3.2 of [Bump 1997] to

$$G \times U_3 \cong G g_i \tilde{g}_i U_3.$$

It shows that there exists a constant $c_4 \in \mathbb{C}$ such that

$$D(\phi) = c_4 \int_G \int_F \beta(g)\,\phi\left(g g_i \tilde{g}_i \begin{bmatrix} 1 & & & z \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dz\,dg \tag{130}$$

for any $\phi \in \mathcal{S}(Gg_i\tilde{g}_iN)$. Then, for any $x \in F$,

$$\begin{aligned}
\psi(c_2x)D(\phi) &= c_4 \int_G\int_F \beta(g)\phi\left(gg_i\tilde{g}_i\begin{bmatrix}1&&&z\\&1&z&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&x&\\&1&&\\&&1&\\&&&1\end{bmatrix}\right)dz\,dg\\
&= c_4 \int_G\int_F \beta(g)\phi\left(gg_i\tilde{g}_i\begin{bmatrix}1&&x&\\&1&&\\&&1&\\&&&1\end{bmatrix}(g_i\tilde{g}_i)^{-1}g_i\tilde{g}_i\begin{bmatrix}1&&&z\\&1&z&\\&&1&\\&&&1\end{bmatrix}\right)dz\,dg\\
&= c_4 \int_G\int_F \beta(g)\phi\left(gg_i\tilde{g}_i\begin{bmatrix}1&&&z\\&1&z&\\&&1&\\&&&1\end{bmatrix}\right)dz\,dg\\
&= D(\phi).
\end{aligned}$$

It follows that $D(\phi) = 0$. This concludes the proof. $\square$

5.5. *Some cases with no exceptional poles.* The following is Theorem 4.2 of [Piatetski-Shapiro 1997], with a slightly modified proof to accommodate our more general notion of exceptional pole.

Theorem 5.5.1. *Let μ be a character of $F^\times$. Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *admitting a (Λ, β)-Bessel model. Assume that s_0 is an exceptional pole for the datum π, Λ, β, μ, as defined in the previous section. Then there exists a nonzero functional $\ell : V \to \mathbb{C}$ with the property*

(131) $\quad \ell(\pi(g)v) = \mu^{-1}(\det(g))|\det(g)|^{-s_0-1/2}\,\ell(v)$ *for all $v \in V$ and $g \in G$.*

Proof. By definition, the function

$$\frac{Z(s, B, \Phi, \mu)}{L^{\mathrm{PS}}_\Lambda(s, \pi, \mu)} = \frac{Z(s, B, \Phi, \mu)}{L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)L(s+1/2, \Lambda_\mu)} \tag{132}$$

lies in $\mathbb{C}[q^s, q^{-s}]$, for any choice of $B \in \mathcal{B}(\pi, \Lambda, \beta)$ and $\Phi \in \mathcal{S}(V)$. In particular, we may evaluate at s_0. We note that

$$\left.\frac{Z(s, B, \Phi, \mu)}{L^{\mathrm{PS}}_\Lambda(s, \pi, \mu)}\right|_{s=s_0} = 0 \quad \text{if } \Phi \in \mathcal{S}_0(V). \tag{133}$$

This follows from Lemma 5.3.1(i), and the fact that s_0 is a pole of $L(s+1/2, \Lambda_\mu)$. We now define

$$\ell(B) = \left.\frac{Z(s, B, \Phi_1, \mu)}{L^{\mathrm{PS}}_\Lambda(s, \pi, \mu)}\right|_{s=s_0}, \tag{134}$$

where, as before, Φ_1 is the characteristic function of $\mathfrak{o}_L \oplus \mathfrak{o}_L$. Since $Z(s, B, \Phi, \mu) = L_\Lambda^{\mathrm{PS}}(s, \pi, \mu)$ for some choice of B and Φ, (133) implies that ℓ is a nonzero functional. It follows from (79) that

$$Z(s, \pi(g)B, g.\Phi, \mu) = Z(s, B, \Phi, \mu)\mu^{-1}(\det(g))|\det(g)|^{-s-1/2} \quad \text{for all } g \in G, \tag{135}$$

where $(g.\Phi)(x, y) = \Phi((x, y)g)$. Consequently,

$$\frac{Z(s, \pi(g)B, g.\Phi_1, \mu)}{L_\Lambda^{\mathrm{PS}}(s, \pi, \mu)}\bigg|_{s=s_0} = \frac{Z(s, B, \Phi_1, \mu)}{L_\Lambda^{\mathrm{PS}}(s, \pi, \mu)}\bigg|_{s=s_0} \mu^{-1}(\det(g))|\det(g)|^{-s_0-1/2}. \tag{136}$$

Since $g.\Phi - \Phi \in \mathcal{S}_0(V)$, property (133) allows us to replace $g.\Phi$ on the left-hand side by Φ. It follows that ℓ has the asserted property (131). $\square$

Let $c_1, c_2 \in F^\times$. Recall from (115) the definition of the character ψ_{c_1,c_2} of U. An irreducible, admissible representation (π, V) of $\mathrm{GSp}(4, F)$ is called *generic* if it admits a nonzero functional $L : V \to \mathbb{C}$ satisfying

$$L(\pi(u)v) = \psi_{c_1,c_2}(u)L(v) \quad \text{for all } v \in V,\ u \in U. \tag{137}$$

Such an L is called a ψ_{c_1,c_2}-Whittaker functional.

The proof of (ii) of the following result has been sketched in Theorem 4.3 of [Piatetski-Shapiro 1997]; here, we provide the details.

Corollary 5.5.2. *There are no exceptional poles for π, Λ, β, μ if one of the following conditions is satisfied.*

(i) *The character $\Lambda_\mu = \Lambda \cdot (\mu \circ N_{L/F})$ is ramified.*

(ii) *π is generic.*

Hence, in these cases we have $L_\Lambda^{\mathrm{PS}}(s, \pi, \mu) = L_{\mathrm{reg}}^{\mathrm{PS}}(s, \pi, \mu)$, and in particular the Piatetski-Shapiro L-factor is independent of the choice of Bessel model for π.

Proof. (i) This is immediate from Lemma 5.3.2(i).

(ii) Let (π, V) be an irreducible, admissible, generic representation of $\mathrm{GSp}(4, F)$. Let $(\pi^\vee, V^\vee)$ be the contragredient representation. Then $\pi^\vee$ is also generic. Let L be a ψ_{c_1,c_2}-Whittaker functional on $V^\vee$.

Assume that π admits an exceptional pole; we will obtain a contradiction. By Theorem 5.5.1, there exists a character β of G and a functional $\ell : V \to \mathbb{C}$ such that

$$\ell(\pi(g)v) = \beta(g)v \tag{138}$$

for all $v \in V$ and $g \in G$. We define a linear map

$$\Delta : \mathcal{S}(\mathrm{GSp}(4, F)) \to V^\vee \tag{139}$$

by

$$\Delta(\phi)(v) = \int\limits_{\mathrm{GSp}(4,F)} \phi(g)\ell(\pi(g)v)\,dg, \tag{140}$$

where $\phi \in \mathcal{S}(\mathrm{GSp}(4, F))$, $v \in V$, and ℓ is a functional as in (131). Since ℓ is nonzero, it is easy to see that Δ is nonzero. One readily verifies that

$$\Delta(R_h\phi) = \pi^\vee(h)\Delta(\phi) \quad \text{for all } h \in \mathrm{GSp}(4, F). \tag{141}$$

In particular, the image of Δ is an invariant subspace of $V^\vee$. Consequently, Δ is surjective. This allows us to define a nonzero distribution $D : \mathcal{S}(\mathrm{GSp}(4, F)) \to \mathbb{C}$ by

$$D(\phi) = L(\Delta(\phi)), \quad \phi \in \mathcal{S}(\mathrm{GSp}(4, F)). \tag{142}$$

Since L is a ψ_{c_1,c_2}-Whittaker functional on $V^\vee$, it follows from (141) that

$$D(R_u\phi) = \psi_{c_1,c_2}(u)D(\phi) \quad \text{for all } u \in U. \tag{143}$$

For $h \in G$, we have

$$\begin{aligned}\Delta(L_h\phi)(v) &= \int\limits_{\mathrm{GSp}(4,F)} \phi(h^{-1}g)\ell(\pi(g)v)\,dg \\ &= \int\limits_{\mathrm{GSp}(4,F)} \phi(g)\ell(\pi(hg)v)\,dg \\ &= \beta(h) \int\limits_{\mathrm{GSp}(4,F)} \phi(g)\ell(\pi(g)v)\,dg\end{aligned}$$

by (138). Hence $\Delta(L_h\phi) = \beta(h)\Delta(\phi)$, and thus

$$D(L_h\phi) = \beta(h)D(\phi) \quad \text{for all } h \in G. \tag{144}$$

By Lemma 5.4.3, properties (143) and (144) imply that $D = 0$, a contradiction. $\square$

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