On Modular Forms for the Paramodular Group

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Contents

1 Introduction

Let F be a $\mathfrak{p}$–adic field, and let G be the algebraic F–group GSp(4). In our paper [RS] we presented a conjectural theory of local newforms for irreducible, admissible, generic representations of $G(F)$ with trivial central character. The main feature of this theory is that it considers fixed vectors under the *paramodular groups* $\mathrm{K}(\mathfrak{p}^n)$, a certain family of compact-open subgroups. The group $\mathrm{K}(\mathfrak{p}^0)$ is equal to the standard maximal compact subgroup $G(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers of F. In fact, $\mathrm{K}(\mathfrak{p}^0)$ and $\mathrm{K}(\mathfrak{p}^1)$ represent the two conjugacy classes of maximal compact subgroups of $G(F)$. In general $\mathrm{K}(\mathfrak{p}^n)$ can be conjugated into $\mathrm{K}(\mathfrak{p}^0)$ if n is even, and into $\mathrm{K}(\mathfrak{p}^1)$ if n is odd. Our theory is analogous to Casselman's well-known theory for representations of $\mathrm{GL}(2,F)$; see [Cas]. The main conjecture made in [RS] states that for each irreducible, admissible, generic representation (π, V) of $\mathrm{PGSp}(4,F)$ there exists an n such that the space $V(n)$ of $\mathrm{K}(\mathfrak{p}^n)$ invariant vectors is non-zero; if n_0 is the minimal such n then $\dim_{\mathbb{C}}(V(n_0)) = 1$; and the Novodvorski zeta integral of a suitably normalized vector in $V(n_0)$ computes the L–factor $L(s,\pi)$ (for this last statement we assume that V is the Whittaker model of π).

We recently proved all parts of this conjecture; it is now a theorem[1]. Parts of the main theorem have been generalized to include non-generic representations. In addition, there is a description of *oldforms*, that is, the spaces $V(n)$ for $n > n_0$. This description is based on certain linear operators $\theta, \theta' : V(n) \to V(n+1)$ and $\eta : V(n) \to V(n+2)$, which we call *level raising operators* and which play a prominent role in our theory.

Now $G = \mathrm{GSp}(4)$ is the group behind classical Siegel modular forms of degree 2, in the sense that such a modular form can be considered as a function on the adelic group $G(\mathbb{A}_\mathbb{Q})$, where it generates an automorphic representation of this group. Exploiting this link between modular forms and representations, we shall explore in this paper the consequences of our local newform theory for Siegel modular forms of degree 2 with respect to paramodular groups. We shall explain how our local theory will imply a global Atkin–Lehner style theory of old– and newforms for paramodular cusp forms, provided we accept some global results on the discrete spectrum of $G(\mathbb{A})$, which have been announced but not yet published.

We shall start in a classical setting, defining the paramodular groups $\Gamma^{\mathrm{para}}(N)$ for positive integers N, and the corresponding spaces $S_k(N)$ of cusp forms of degree 2. We shall then define, for a prime number p, level raising operators θ_p and θ'_p, which multiply the level by p, and η_p, which multiplies the level by p^2. These operators are compatible with the local operators mentioned above, and the connection will be explained. Perhaps surprisingly, the η_p and θ_p operator are compatible, via the Fourier–Jacobi expansion, with the well-known U_p and V_p operators from the theory of Jacobi forms. Paramodular oldforms will be defined, roughly speaking, as those modular forms that can be obtained by repeatedly applying the three level raising operators. The space of newforms is defined as the orthogonal complement of the oldforms with respect to the Petersson inner product. We shall formulate conjectural Atkin–Lehner type results for the newforms thus defined, and explain how these results would follow from our local theory together with some plausible global results that are not yet fully available.

Examples of paramodular cusp forms are provided by the Saito–Kurokawa lifting. There is a classical construction available, combining results of Skoruppa, Zagier and Gritsenko, which produces elements of $S_k(N)$ from elliptic modular forms of level N and weight $2k-2$. However, we propose an alternative group theoretic construction, which gives the additional information that the Saito–Kurokawa liftings we obtain from elliptic newforms are paramodular newforms as defined above. In other words, there is a level-preserving Hecke-equivariant Saito–Kurokawa lifting from cuspidal elliptic newforms (with a "−" sign in the functional equation of the L–function) to cuspidal paramodular newforms of degree 2. We shall explain how this map can be extended to the "certain space" of modular forms defined by Skoruppa and Zagier in [SZ].

In the final section of this paper we will consider two seemingly unrelated theorems on paramodular

[1]It was still a conjecture at the time of the Arakawa conference.

cusp forms. One says that the θ operator defined before is injective. The other one says that paramodular cusp forms of weight 1 do not exist. We shall translate these theorems into group theoretic statements, where it turns out that the second one is the exact archimedean analogue of the first one.

2 Definitions

Paramodular groups

In the following we let G be the algebraic $\mathbb{Q}$–group $\mathrm{GSp}(4)$, realized as the set of all $g \in \mathrm{GL}(4)$ such that ${}^tgJg = xJ$ for some $x \in \mathrm{GL}(1)$, where $J = \begin{bmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{bmatrix}$. The element x is called the multiplier of g and denoted by $\lambda(g)$. The kernel of the homomorphism $\lambda:\ \mathrm{GSp}(4) \to \mathrm{GL}(1)$ is the symplectic group $\mathrm{Sp}(4)$.

Let N be a positive integer. The *Klingen congruence subgroup* of level N is the set of all $\gamma \in \mathrm{Sp}(4, \mathbb{Z})$ such that

$$\gamma \in \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

(That this is a subgroup becomes obvious by switching the first two rows and first two columns, which amounts to an isomorphism with a more symmetric version of the symplectic group.) This group can be enlarged to the *paramodular group* of level N by allowing certain denominators. Namely, we define

$$\Gamma^{\mathrm{para}}(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4, \mathbb{Q}).$$

Note that $\Gamma^{\mathrm{para}}(N)$ is not contained in $\Gamma^{\mathrm{para}}(M)$ if $M|N$. In fact, no paramodular group contains any other paramodular group, since the element

$$\begin{bmatrix} 1 & & & \\ & & & N^{-1} \\ & & -1 & \\ & N & & \end{bmatrix}$$

is contained in $\Gamma^{\mathrm{para}}(N)$ only. We also define local paramodular groups. Let F be a non-archimedean local field, $\mathfrak{o}$ its ring of integers and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. We define $\mathrm{K}(\mathfrak{p}^n)$

as the group of all $g \in \mathrm{GSp}(4,F)$ such that

$$g \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{o} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix} \qquad \text{and} \qquad \det(g) \in \mathfrak{o}^*. \tag{1}$$

These are the local analogues of the groups $\Gamma^{\mathrm{para}}(N)$. In fact, if $F = \mathbb{Q}$, then

$$\Gamma^{\mathrm{para}}(N) = G(\mathbb{Q}) \cap G(\mathbb{R})^+ \prod_p \mathrm{K}(p^{v_p(N)}), \tag{2}$$

where $p^{v_p(N)}$ is the exact power of p dividing N (if $p \nmid N$, then we understand $\mathrm{K}(p^{v_p(N)}) = G(\mathbb{Z}_p)$).

Modular forms

Let $\mathbb{H}_2$ be the Siegel upper half plane of degree 2. The group $G(\mathbb{R})^+ = \{g \in \mathrm{GSp}(4,\mathbb{R}) : \lambda(g) > 0\}$, which is the identity component of $G(\mathbb{R})$, acts on $\mathbb{H}_2$ by linear fractional transformations $Z \mapsto g\langle Z\rangle$. We define the usual modular factor

$$j(g,Z) = \det(CZ+D) \qquad \text{for } Z \in \mathbb{H}_2 \text{ and } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G(\mathbb{R})^+.$$

We fix a weight k, which is a positive integer. The slash operator $\big|_k$ or simply $\big|$ on functions $F : \mathbb{H}_2 \to \mathbb{C}$ is defined as

$$(F\big|g)(Z) = \lambda(g)^k j(g,Z)^{-k} F(g\langle Z\rangle) \qquad \text{for } g \in G(\mathbb{R})^+.$$

The factor $\lambda(g)^k = \det(g)^{k/2}$ ensures that the center of $G(\mathbb{R})^+$ acts trivially. A *modular form* F (always of degree 2) of weight k with respect to $\Gamma^{\mathrm{para}}(N)$ is a holomorphic function on $\mathbb{H}_2$ such that $F\big|\gamma = F$ for all $\gamma \in \Gamma^{\mathrm{para}}(N)$. We denote the space of such modular forms by $M_k(N)$, and the subspace of cusp forms by $S_k(N)$. Modular forms for the paramodular group have been considered by various authors; see, for example, [IO] and the references therein. In this paper we shall fix the weight k and vary the level N.

We shall often write modular forms as $F(\tau,z,\tau')$, where $\begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix} \in \mathbb{H}_2$. Note that elements $F \in M_k(N)$ have the invariance property $F(\tau,z,\tau'+t) = F(\tau,z,\tau')$ for $t \in N^{-1}\mathbb{Z}$. In particular, F has a Fourier–Jacobi expansion

$$F(\tau,z,\tau') = \sum_{m=0}^{\infty} f_m(\tau,z) e^{2\pi i m\tau'}. \tag{3}$$

Here $f_m \in J_{k,m}$ is a Jacobi form of weight k and index m, as in [EZ]. Since F depends only on τ' modulo $N^{-1}\mathbb{Z}$, we have $f_m = 0$ for $N \nmid m$.

We shall attach to a given $F \in M_k(N)$ an adelic function $\Phi : G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ in the following way. Let K_N be the compact group $\prod_{p<\infty} \mathrm{K}(p^{v_p(N)})$. Since the local multiplier maps $\mathrm{K}(p^{v_p(N)}) \to \mathbb{Z}_p^*$ are all surjective, it follows from strong approximation for Sp(4) that $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+K_N$. Decomposing a given $g \in G(\mathbb{A})$ accordingly as $g = \rho h \kappa$, we define

$$\Phi(g) = (F\big|_k h)(I), \qquad g = \rho h \kappa \text{ with } \rho \in G(\mathbb{Q}),\ h \in G(\mathbb{R})^+,\ \kappa \in K_N. \tag{4}$$

Here I is the element $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ of $\mathbb{H}_2$. In view of (2), the function Φ is well-defined. It obviously has the invariance properties

$$\Phi(\rho g \kappa z) = \Phi(g) \qquad \text{for all } g \in G(\mathbb{A}),\ \rho \in G(\mathbb{Q}),\ \kappa \in K_N,\ z \in Z(\mathbb{A}),$$

where Z is the center of GSp(4). In fact, Φ is an automorphic form on $\mathrm{PGSp}(4, \mathbb{A})$. One can show that Φ is a cuspidal automorphic form if and only if $F \in S_k(N)$. Assuming this is the case, we consider the cuspidal automorphic representation $\pi = \pi_F$ generated by Φ. This representation may not be irreducible, but it always decomposes as a finite direct sum $\pi = \oplus_i \pi_i$ with irreducible automorphic representations π_i.

Atkin–Lehner involutions

We first consider local Atkin–Lehner involutions. Let again F be a non-archimedean local field, and let $\mathfrak{o}$ and $\mathfrak{p}$ be as above. Let ϖ be a generator of $\mathfrak{p}$. The element

$$u_n = \begin{bmatrix} &&&1 \\ &&1& \\ &\varpi^n&& \\ \varpi^n&&& \end{bmatrix} \tag{5}$$

is called the *Atkin–Lehner element* of level n. It is easily checked that u_n normalizes the local paramodular group $\mathrm{K}(\mathfrak{p}^n)$. Therefore, if (π, V) is an admissible representation of $G(F)$, the operator $\pi(u_n)$ induces an endomorphism of the (finite-dimensional) space $V(n)$ of $\mathrm{K}(\mathfrak{p}^n)$–invariant vectors. Assume in addition that π has trivial central character. Then, since u_n^2 is central, this endomorphism on $V(n)$ is an involution, the *Atkin–Lehner involution* of level n (or $\mathfrak{p}^n$). It splits the space $V(n)$ into ±1 eigenspaces.

To define the global involutions, let N be a positive integer and let p be a prime dividing N. Let

p^j be the exact power of p dividing N. Choose a matrix $\gamma_p \in \mathrm{Sp}(4,\mathbb{Z})$ such that

$$\gamma_p \equiv \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix} \mod p^j \qquad \text{and} \qquad \gamma_p \equiv \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix} \mod Np^{-j},$$

and let

$$u_p := \gamma_p \begin{bmatrix} p^j&&&\\ &p^j&&\\ &&1&\\ &&&1 \end{bmatrix}.$$

We call u_p an *Atkin–Lehner element.* A different choice of γ_p results in multiplying u_p from the left with an element of the principal congruence subgroup $\Gamma(N)$. Therefore the action of u_p on modular forms for $\Gamma(N)$ is unambiguously defined. It is easily checked using (2) that u_p normalizes $\Gamma^{\rm para}(N)$. Consequently the map $F \mapsto F\big|u_p$ defines an endomorphism of $M_k(N)$. Its restriction to cusp forms defines an endomorphism of $S_k(N)$. These endomorphisms are involutions since $u_p^2 \in p^j\Gamma^{\rm para}(N)$, as is easily checked. To summarize, for a given level N, we can define *Atkin–Lehner involutions* $u_p(F) := F\big|_k u_p$ on $M_k(N)$ and $S_k(N)$ for each $p|N$.

The relation between the local and global Atkin–Lehner involutions is as follows. Let $F \in M_k(N)$ and Φ the corresponding adelic function defined above. Then $u_p(F)$ corresponds to the right translate of Φ by the local Atkin–Lehner element $\underline{u}_{p^j} \in G(\mathbb{Q}_p)$, where p^j is the exact power of p dividing N:

$$(u_p(F)\big|g)(I) = \Phi(g\underline{u}_{p^j}), \qquad g \in G(\mathbb{R})^+,\ \underline{u}_{p^j} = \begin{bmatrix} &&&1\\ &&1&\\ &p^j&&\\ p^j&&& \end{bmatrix}. \tag{6}$$

3 Linear independence at different levels

We shall prove an easy but useful result on modular forms for the paramodular group, starting with an analogous local statement. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p}$. Let ϖ be a generator of $\mathfrak{p}$. We define

$$t_n := \begin{bmatrix} 1&&&\\ &&&-\varpi^{-n}\\ &&1&\\ &\varpi^n&& \end{bmatrix}.$$

3.1 Lemma. *Let $0 \le n_1 < \cdots < n_r$ be integers. Let $m \ge 0$ be an integer such that $m < n_1$. Then the subgroup H generated by $\mathrm{K}(\mathfrak{p}^{n_1}) \cap \cdots \cap \mathrm{K}(\mathfrak{p}^{n_r})$ and t_m contains $\mathrm{Sp}(4, F)$.*

Proof: The proof will be easy once we can show that H contains all elements

$$\begin{bmatrix}1&&&\\&a&&b\\&&1&\\&c&&d\end{bmatrix}, \qquad \text{where } \begin{bmatrix}a&b\\c&d\end{bmatrix} \in \mathrm{SL}(2, F).$$

By hypothesis the group H contains the elements

$$\begin{bmatrix}1&&&\\&a&&b\varpi^{-n_1}\\&&1&\\&c\varpi^{n_r}&&d\end{bmatrix} \qquad \text{such that } a,b,c,d \in \mathfrak{o} \text{ and } \begin{bmatrix}a&b\varpi^{-n_1}\\c\varpi^{n_r}&d\end{bmatrix} \in \mathrm{SL}(2, F).$$

Since H also contains t_m, it will suffice to show that the subgroup H' of $\mathrm{SL}(2, F)$ generated by

$$\begin{bmatrix}&-\varpi^{-m}\\\varpi^m&\end{bmatrix}, \quad \begin{bmatrix}a&b\varpi^{-n_1}\\c\varpi^{n_r}&d\end{bmatrix}, \quad a,b,c,d \in \mathfrak{o},\ ad - bc\varpi^{n_r-n_1} = 1$$

is $\mathrm{SL}(2, F)$. We shall show that the conjugate subgroup $H'' := \begin{bmatrix}\varpi^m&\\&1\end{bmatrix} H' \begin{bmatrix}\varpi^{-m}&\\&1\end{bmatrix}$ is equal to $\mathrm{SL}(2, F)$, which is equivalent. This subgroup H'' is generated by

$$\begin{bmatrix}&-1\\1&\end{bmatrix}, \quad \begin{bmatrix}a&b\varpi^{m-n_1}\\c\varpi^{n_r-m}&d\end{bmatrix}, \quad a,b,c,d \in \mathfrak{o},\ ad - bc\varpi^{n_r-n_1} = 1.$$

In particular, H'' contains $\begin{bmatrix}&-1\\1&\end{bmatrix}$ and $\begin{bmatrix}1&\mathfrak{o}\\&1\end{bmatrix}$, and therefore $\mathrm{SL}(2, \mathfrak{o})$. It is not hard to show that the group generated by $\mathrm{SL}(2, \mathfrak{o})$ and $\begin{bmatrix}1&\mathfrak{p}^{-1}\\&1\end{bmatrix}$ is all of $\mathrm{SL}(2, F)$. ■

3.1 Proposition. *Let F be a non-archimedean local field, and let (π, V) be an admissible representation of $G(F)$ with trivial central character that has no non-zero $\mathrm{Sp}(4, F)$ invariant vectors.*[2] *Then paramodular vectors in V of different levels are linearly independent. More precisely, for $i = 1, \ldots, r$ let $v_i \in V$ be fixed by the paramodular group $\mathrm{K}(\mathfrak{p}^{n_i})$, where $n_i \ne n_j$ for $i \ne j$. Then $v_1 + \ldots + v_r = 0$ implies $v_1 = \ldots = v_r = 0$.*

[2] For example, π could be an irreducible, infinite-dimensional representation.

Proof: We may assume that $n_1 < \ldots < n_r$. From $v_1 + \cdots + v_r = 0$ we obtain $-v_1 = v_2 + \cdots + v_r$. This element is invariant under t_{n_1} and $\mathrm{K}(\mathfrak{p}^{n_2}) \cap \cdots \cap \mathrm{K}(\mathfrak{p}^{n_r})$. Since $n_1 < n_2$, by Lemma 3.1, it is invariant under $\mathrm{Sp}(4, F)$; hence, $v_1 = v_2 + \cdots + v_r = 0$. Applying the same argument successively gives $v_2 = \cdots = v_r = 0$. ■

This local result has the following global analogue. Note that the corresponding statement for $\Gamma_0(N)$ congruence subgroups is obviously wrong.

3.2 Proposition. *Modular forms for the paramodular group of different levels are linearly independent. More precisely, for $i = 1, \ldots, r$ let $F_i \in M_k(N_i)$, where $N_i \neq N_j$ for $i \neq j$. Then $F_1 + \ldots + F_r = 0$ implies $F_1 = \ldots = F_r = 0$.*

Proof: One can either exploit the relationship between modular forms and representations and use Proposition 3.1, or one can give a direct proof along the lines of the local proofs. ■

An important consequence of Proposition 3.2 is the following. Soon we will have reason to consider the spaces $M_k(\Gamma^{\mathrm{para}}) := \bigoplus_{N=1}^{\infty} M_k(N)$, see (18). Proposition 3.2 implies that this abstract direct sum is the same as the sum of the spaces $M_k(N)$ taken inside the vector space of all complex-valued functions on $\mathbb{H}_2$.

4 The level raising operators

As before let $M_k(N)$ be the space of modular forms of weight k with respect to the paramodular group of level N. Since no $\Gamma^{\mathrm{para}}(N)$ is contained in any other $\Gamma^{\mathrm{para}}(M)$ ($M \neq N$), there are no inclusions between (the non-zero ones of) the spaces $M_k(N)$. In particular, for $N|M$, the space $M_k(N)$, if not zero, is not a subspace of $M_k(M)$. However, we shall see that there are natural operators raising the level. For a prime number p, which may or may not divide N, we shall define linear operators θ_p and θ'_p from $M_k(N)$ to $M_k(Np)$. We shall also define an operator η_p from $M_k(N)$ to $M_k(Np^2)$.

The η operator

We begin by defining η_p, since this is easiest. For $F \in M_k(N)$ let

$$\eta_p F := F\big|\underline{\eta}_p^{-1}, \qquad \text{where} \quad \underline{\eta}_p = \begin{bmatrix} 1 & & & \\ & p^{-1} & & \\ & & 1 & \\ & & & p \end{bmatrix}. \tag{7}$$

One easily checks that $\underline{\eta}_p \Gamma^{\mathrm{para}}(N) \underline{\eta}_p^{-1} \supset \Gamma^{\mathrm{para}}(Np^2)$. Hence $\eta_p(F) \in M_k(Np^2)$, and we get linear operators

$$\eta_p:\ M_k(N) \longrightarrow M_k(Np^2) \qquad \text{and} \qquad \eta_p:\ S_k(N) \longrightarrow S_k(Np^2).$$

Explicitly, we have $(\eta_p F)(\tau,z,\tau') = p^k F(\tau,pz,p^2\tau')$. If the Fourier–Jacobi expansion of F is written as in (3), then the Fourier–Jacobi expansion of $\eta_p F$ is given by

$$(\eta_p F)(\tau,z,\tau') = p^k \sum_{m=0}^{\infty} f_m(\tau,pz) e^{2\pi i m p^2 \tau'} = p^k \sum_{m=0}^{\infty} (U_p f_m)(\tau,z) e^{2\pi i m p^2 \tau'}. \tag{8}$$

Here $(U_p f_m)(\tau,z) = f_m(\tau,pz)$ is the operator from $J_{k,m}$ to J_{k,mp^2} defined in section I.4 of [EZ]. If Φ is the adelic function corresponding to F defined in (4), then a straightforward calculation shows that

$$((\eta_p F)\big|g)(I) = \Phi(g\underline{\eta}_p), \qquad g \in G(\mathbb{R})^+,\ \underline{\eta}_p = \begin{bmatrix} 1 &&& \\ & p^{-1} && \\ && 1 & \\ &&& p \end{bmatrix} \in G(\mathbb{Q}_p). \tag{9}$$

In other words, the adelic function corresponding to $\eta_p F$ is the right translate of Φ by the p–adic matrix η_p. From the local descriptions (6) and (9) and the matrix identity

$$p\begin{bmatrix} 1 &&& \\ & p^{-1} && \\ && 1 & \\ &&& p \end{bmatrix}\begin{bmatrix} &&& 1 \\ && 1 & \\ & p^n && \\ p^n &&& \end{bmatrix} = \begin{bmatrix} &&& 1 \\ && 1 & \\ & p^{n+2} && \\ p^{n+2} &&& \end{bmatrix}\begin{bmatrix} 1 &&& \\ & p^{-1} && \\ && 1 & \\ &&& p \end{bmatrix}$$

it is immediate that the η operator commutes with Atkin–Lehner involutions: $u_p \circ \eta_p = \eta_p \circ u_p$. Note that the u_p on the right acts on $M_k(N)$, and the u_p on the left acts on $M_k(Np^2)$.

The θ operator

It is not possible to conjugate the group $\Gamma^{\rm para}(Np)$ into $\Gamma^{\rm para}(N)$. Consequently there is no simple operator from $M_k(N)$ to $M_k(Np)$ given by applying a single matrix as in the case of η_p. We can however define an operator by applying ${\rm diag}(1,1,p^{-1},p^{-1})$ and then average to restore the paramodular invariance. More precisely, for $F \in M_k(N)$ we define

$$\theta_p F = \sum_{\gamma \in \Gamma_0(p)\backslash {\rm SL}(2,\mathbb{Z})} F\big|_k \Big(\begin{bmatrix} 1 &&& \\ & 1 && \\ && p^{-1} & \\ &&& p^{-1} \end{bmatrix}\begin{bmatrix} a && b & \\ & 1 && \\ c && d & \\ &&& 1 \end{bmatrix}\Big) \qquad (\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}). \tag{10}$$

It is easy to check that $\theta_p F$ is well-defined and indeed is an element of $M_k(Np)$. Hence we get linear operators

$$\theta_p:\ M_k(N) \longrightarrow M_k(Np) \qquad \text{and} \qquad \theta_p:\ S_k(N) \longrightarrow S_k(Np).$$

Assume that the Fourier–Jacobi expansion of $F \in M_k(N)$ is given by (3). Then a straightforward calculation shows that

$$(\theta_p F)(\tau, z, \tau') = p \sum_{m=0}^{\infty} (V_p f_m)(\tau, z) e^{2\pi i m p \tau'} \tag{11}$$

with

$$(V_p f_m)(\tau, z) = p^{k-1} \sum_{\gamma \in \Gamma_0(p) \backslash \mathrm{SL}(2,\mathbb{Z})} (c\tau + d)^{-k} \, e^{-2\pi i m p \frac{cz^2}{c\tau+d}} \, f_m\Big(p \frac{a\tau + b}{c\tau + d}, \frac{pz}{c\tau + d}\Big).$$

Note that there is a bijection $\Gamma_0(p)\backslash\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z})\backslash\{A \in M_2(\mathbb{Z}) : \det(A) = p\}$ given by $\gamma \mapsto \mathrm{diag}(p, 1)\gamma$. Hence $V_p f_m$ is exactly the function defined in (2) of section I.4 of [EZ]. The V_p operator is a linear map from $J_{k,m}$ to $J_{k,mp}$. To summarize equations (8) and (11), the operator η_p on $M_k(N)$ is compatible with the operator U_p on Jacobi forms, and θ_p is compatible with the operator V_p.

We now define an adelic version of the θ_p operator. If Φ is a function on $G(\mathbb{A}_\mathbb{Q})$ that is right invariant under the paramodular group $\mathrm{K}(p^j) \subset G(\mathbb{Q}_p)$ of some level p^j, we define a new function $\theta_p\Phi$ by

$$(\theta_p\Phi)(g) = \sum_{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z}_p)/\Gamma_0(p)} \Phi\Big(g \underbrace{\begin{bmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix}}_{\in G(\mathbb{Q}_p)}\Big) \qquad (g \in G(\mathbb{A})). \tag{12}$$

Then $\theta_p\Phi$ is right invariant under $\mathrm{K}(p^{j+1})$. A standard calculation shows that if Φ corresponds to F as in (4), then $\theta_p\Phi$ corresponds to $\theta_p F$. In other words,

$$((\theta_p F)\big|g)(I) = (\theta_p\Phi)(g), \qquad g \in G(\mathbb{R})^+. \tag{13}$$

The θ' operator

While the η operator commutes with Atkin–Lehner involutions, this is no longer true for the θ operator. We use this fact to define a new operator θ'_p from $M_k(N)$ to $M_k(Np)$ by

$$\theta'_p := u_p \circ \theta_p \circ u_p \qquad \text{(the } u_p \text{ are Atkin–Lehner involutions).} \tag{14}$$

Note that the u_p on the right acts on $M_k(N)$, and the u_p on the left acts on $M_k(Np)$. We obtain linear operators

$$\theta'_p : \ M_k(N) \longrightarrow M_k(Np) \qquad \text{and} \qquad \theta'_p : \ S_k(N) \longrightarrow S_k(Np).$$

It is clear from (13) and (6) that if $F \in M_k(N)$ corresponds to the adelic function Φ, then θ'_pF corresponds to the function

$$(\theta'_p\Phi)(g) := \sum_{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z}_p)/\Gamma_0(p)} \Phi\big(g \underbrace{\underline{u}_{p^{j+1}} \begin{bmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \underline{u}_{p^j}}_{\in G(\mathbb{Q}_p)}\big) \quad (g \in G(\mathbb{A})). \tag{15}$$

Here p^j is the exact power of p dividing N, and $\underline{u}_{p^j}$ is as in (6). The operator θ'_p has the following simple description on an element $F \in M_k(N)$:

$$\theta'_pF = \eta_pF + \sum_{c \in \mathbb{Z}/p\mathbb{Z}} F\big|_k \begin{bmatrix} 1 & & & \\ & 1 & & cp^{-1}N^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{16}$$

To prove this formula, consider the local description (15) and the matrix identity

$$\underline{u}_{p^{j+1}} \begin{bmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \underline{u}_{p^j} = p^{j+1} \begin{bmatrix} 1 & & & \\ & d & & cp^{-j-1} \\ & & 1 & \\ & bp^{j+1} & & a \end{bmatrix}.$$

As a system of representatives for $\mathrm{SL}(2,\mathbb{Z}_p)/\Gamma_0(p)$ we can choose $\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}$, $c \in \mathbb{Z}/p\mathbb{Z}$, together with the matrix $\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$. The first type of representatives leads to the summation in (16). As for $\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$, note that

$$\begin{bmatrix} 1 & & & \\ & & & -p^{-j-1} \\ & & 1 & \\ & p^{j+1} & & \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & p^{-1} & & \\ & & 1 & \\ & & & p \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & & -p^{-j} \\ & & 1 & \\ & p^j & & \end{bmatrix}$$

and that Φ is invariant under the rightmost matrix. In view of (9), this proves (16). Actually, the matrices in (16) are a system of representatives for $\Gamma^{\mathrm{para}}(Np) \cap \Gamma^{\mathrm{para}}(N) \backslash \Gamma^{\mathrm{para}}(Np)$. Hence the θ'_p operator is nothing but the natural trace operator from $M_k(N)$ to $M_k(Np)$. Using formula (16), it is now easy to compute the Fourier–Jacobi expansion of θ'_pF. If that of F is given by (3), then

$$(\theta'_pF)(\tau,z,\tau') = \sum_{m=0}^{\infty} \big(p^k\,(U_pf_{m/p})(\tau,z) + p\,\tilde{f}_{mp}(\tau,z)\big)e^{2\pi imp\tau'}. \tag{17}$$

Here we understand that $f_{m/p} = 0$ if $p \nmid m$, and

$$\tilde{f}_{mp}(\tau, z) := \begin{cases} f_{mp}(\tau, z) & \text{if } N \mid m, \\ 0 & \text{if } N \nmid m. \end{cases}$$

The algebra of operators

For each prime number p we have now defined operators θ_p, θ'_p and η_p on $M_k(N)$ multiplying the level by p and p^2, respectively. Let us put

$$M_k(\Gamma^{\text{para}}) := \bigoplus_{N=1}^{\infty} M_k(N), \qquad S_k(\Gamma^{\text{para}}) := \bigoplus_{N=1}^{\infty} S_k(N). \tag{18}$$

By definition these are abstract direct sums, but see Proposition 3.2 and the remark thereafter. The collection of operators θ_p for different levels N define endomorphisms of $M_k(\Gamma^{\text{para}})$ and $S_k(\Gamma^{\text{para}})$, and similarly for θ'_p and η_p.

4.1 Lemma. *The operators θ_p, θ'_p and η_p commute pairwise.*

Proof: The matrix $\underline{\eta}_p$ in (7) used to define η_p commutes with the matrices in (10). Hence η_p commutes with θ_p. We already noted before that η_p commutes with Atkin–Lehner involutions. By the definition in (14) it follows that η_p commutes with θ'_p (this can also be seen from (16)). That θ_p commutes with θ'_p is easily proved using (16). ■

The lemma states that the algebra $\mathcal{A}_p$ generated by the endomorphisms θ_p, θ'_p and η_p of $M_k(N)$ is commutative. Moreover, it is clear by the local descriptions we have given that for different prime numbers p and q the p operators commute with the q operators. Hence the algebra $\mathcal{A}$ generated by all these operators acting on $M_k(\Gamma^{\text{para}})$ and $S_k(\Gamma^{\text{para}})$ is commutative.

Local representations

Let F be a local non-archimedean field and $\mathfrak{o}$, $\mathfrak{p}$ and ϖ as before. Let (π, V) be an irreducible, admissible representation of $G(F)$ ($G = \text{GSp}(4)$) with trivial central character. Let $V(n) \subset V$ be the subspace of vectors fixed by the paramodular group $\text{K}(\mathfrak{p}^n)$ as defined in (1). We already defined the local Atkin–Lehner involutions on $V(n)$, see (5). We further define:

- An operator $\eta : V(n) \to V(n+2)$. It is defined by applying $\pi(\text{diag}(1, \varpi^{-1}, 1, \varpi))$.
- An operator $\theta : V(n) \to V(n+1)$. It is defined by a similar summation as in (12).
- An operator $\theta' : V(n) \to V(n+1)$. It is defined as in (14) or, alternatively, by a formula as in (16).

Just as in the global case these operators generate a commutative algebra of linear operators on the space of paramodular vectors. Now assume that the modular form $F \in M_k(N)$ corresponds to the adelic function Φ, and that Φ generates an irreducible, automorphic representation $\pi = \otimes_{p\leq\infty}\pi_p$ of $G(\mathbb{A})$. Then it is clear that each π_p ($p < \infty$) contains paramodular invariant vectors. It is further clear that the local operators η, θ and θ' are compatible with the global operators. It is our intention to use local representation theoretic results on paramodular vectors to obtain results on classical modular forms.

5 Oldforms and newforms

The main purpose of the operators introduced in the previous section is to define oldforms and newforms. Roughly speaking, all the modular forms in the images of our operators should be considered "old". A modular form that is orthogonal to all the oldforms is "new". Recall the definition (18) of the space $M_k(\Gamma^{\rm para})$ and the algebra $\mathcal{A}$ acting on it. Let $\mathcal{I} \subset \mathcal{A}$ be the ideal generated by η_p, θ_p and θ'_p, where p runs through all prime numbers. Then we define

$$M_k^{\rm old}(\Gamma^{\rm para}) := \mathcal{I}\, M_k(\Gamma^{\rm para}), \qquad M_k^{\rm old}(N) := M_k^{\rm old}(\Gamma^{\rm para}) \cap M_k(N).$$

Similar definitions are made for cusp forms. Elements of these spaces are called *oldforms*. On the spaces $S_k(N)$ we have the Petersson scalar product, which allows us to define the subspace of *newforms* as the orthogonal complement of the oldforms:

$$S_k^{\rm new}(N) := S_k^{\rm old}(N)^\perp.$$

We conjecture that paramodular cusp forms have a newform theory that is as nice as the well-known newform theory for elliptic modular forms:

5.1 Conjecture. (Newforms Conjecture) *Let N be a nonnegative integer.*

i) Assume that $F \in S_k^{\rm new}(N)$ is an eigenform for the unramified local Hecke algebra $\mathcal{H}_p$ for almost all p not dividing N. Then F is an eigenform for $\mathcal{H}_p$ for all $p \nmid N$.

ii) Let $F_i \in S_k^{\rm new}(N_i)$, $i = 1, 2$, be two non-zero cusp forms. Assume that F_1 and F_2 are both eigenforms for the unramified local Hecke algebra $\mathcal{H}_p$ for almost all p. Assume further that for almost all p the Hecke eigenvalues of F_1 and F_2 coincide. Then $N_1 = N_2$, and F_1 is a multiple of F_2.

Our belief in the Newforms Conjecture is based on an analogous local statement and the conjectural structure of the discrete spectrum of PGSp(4). The local statement is as follows.

5.1 Theorem. (Local New- and Oldforms Theorem) *Let F be a non-archimedean local field of characteristic zero, $\mathfrak{o}$ its ring of integers, and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. Let (π, V) be an irreducible, admissible representation of $G(F)$ with trivial central character. For n a nonnegative integer, let $V(n)$ be the space of vectors fixed by the local paramodular group $\mathrm{K}(\mathfrak{p}^n)$. Assume that for some n we have $V(n) \neq 0$.*

i) (Local multiplicity one) If n_0 is the minimal n such that $V(n) \neq 0$, then $\dim(V(n_0)) = 1$.

ii) (Local oldforms theorem) For any $n > n_0$, the space $V(n)$ is spanned by vectors obtained by repeatedly applying the operators θ, θ' and η to the elements of $V(n_0)$.

Part i) of this theorem states that there is always a *local newform* that is unique up to scalars, provided there are paramodular vectors at all. We completed the proof of Theorem 5.1 just recently. We also proved that every *generic* irreducible representation has non-zero paramodular invariant vectors, and that for *tempered* representations this condition is also necessary.

We shall indicate further below how the (global) Newforms Conjecture follows from the local Theorem 5.1 and the following two global statements.

5.2 Conjecture. (Weak Multiplicity One) *If π is an irreducible admissible representation of $\mathrm{PGSp}(4, \mathbb{A}_F)$, where F is any number field, then π occurs with multiplicity at most one in the discrete spectrum of $\mathrm{PGSp}(4, \mathbb{A}_F)$.*

Proofs of this conjecture have been announced by several authors, but currently there is no published proof. Note that while this conjecture is assumed to be true over any number field, the following conjecture depends on the arithmetic of $\mathbb{Q}$ and is in general wrong over other number fields.

5.3 Conjecture. (Paramodular Strong Multiplicity One) *If $\pi \cong \otimes_{p \leq \infty} \pi_p$ is an irreducible discrete automorphic representation of $\mathrm{PGSp}(4, \mathbb{A}_\mathbb{Q})$ and π is paramodular, i.e., π_p admits a nonzero vector invariant under some paramodular group for all finite p, then π is determined, up to equivalence, by π_∞ and all but finitely many of the π_p for finite p.*

Generally speaking, strong multiplicity one should not hold for irreducible discrete automorphic representations of a connected reductive algebraic group over a number field, and it does not hold for $\mathrm{PGSp}(4)$. To explain our reasoning as to why it should hold for the smaller class of paramodular irreducible discrete automorphic representations of $\mathrm{PGSp}(4, \mathbb{A}_\mathbb{Q})$, let $\pi \cong \otimes_{p \leq \infty} \pi_p$ be such a representation. Let $[\pi]_{\mathrm{near}}$ be the discrete near equivalence class of π, i.e., the set of all irreducible admissible representations $\pi' \cong \otimes_{p \leq \infty} \pi'_p$ of $\mathrm{PGSp}(4, \mathbb{A}_\mathbb{Q})$ that occur in the discrete spectrum of $\mathrm{PGSp}(4, \mathbb{A}_\mathbb{Q})$ and for which $\pi'_p \cong \pi_p$ for almost all p. To verify the conjecture it would suffice to prove that $[\pi]_{\mathrm{near}}$ contains exactly one paramodular element, namely π. Conjecturally, $[\pi]_{\mathrm{near}}$ is the set of automorphic elements of a conjectural Arthur packet $\Pi(\phi)$ corresponding to an

Arthur parameter $\phi : \mathrm{L}_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C}) \to {}^L\mathrm{PGSp}(4)$, where $\mathrm{L}_{\mathbb{Q}}$ is the conjectural Langlands group of $\mathbb{Q}$. First, assume ϕ is tempered. Then, conjecturally, all the elements of $[\pi]_{\mathrm{near}}$ are tempered. We can prove that if $p < \infty$, then an irreducible tempered admissible representation of $\mathrm{PGSp}(4,\mathbb{Q}_p)$ is paramodular if and only if it is generic. It follows that the only paramodular element of $[\pi]_{\mathrm{near}}$ with infinity type π_∞ is π (π_p is the generic base point of the local tempered Arthur packet $\Pi(\phi_p)$ for all $p < \infty$). Next, assume ϕ is not tempered. Then, by the Ramanujan conjecture (see 6.1 further below), π is CAP (cuspidal associated to parabolic) with respect to the Borel subgroup B, the Klingen parabolic subgroup Q or the Siegel parabolic subgroup P of $\mathrm{PGSp}(4)$. We can prove that no paramodular irreducible discrete automorphic representation of $\mathrm{PGSp}(4,\mathbb{A}_{\mathbb{Q}})$ is CAP with respect to B or Q (it is here that we need the assumption that we are working over $\mathbb{Q}$). Hence, π is CAP with respect to P. Conjecturally the elements of $[\pi]_{\mathrm{near}}$ form what is called a Saito–Kurokawa packet, and we can prove that the only paramodular element of $[\pi]_{\mathrm{near}}$ with infinity type π_∞ is π (as in the tempered case, π_p is the base point of the local nontempered Arthur packet $\Pi(\phi_p)$ for all $p < \infty$). See the next section for more on Saito–Kurokawa packets.

“Proof” of the Newforms Conjecture

We shall now indicate how to obtain a proof of Conjecture 5.1 from Theorem 5.1 and the Conjectures 5.2 and 5.3. Let $F \in S_k^{\mathrm{new}}(N)$ be an eigenform for almost all of the unramified local Hecke algebras. Let Φ be the corresponding adelic function $G(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$, and let π be the representation generated by Φ. Then π is a finite direct sum of irreducible cuspidal automorphic representations π_i. Let $\pi_i = \otimes\pi_{i,p}$ with $\pi_{i,p}$ an irreducible, admissible representation of $G(\mathbb{Q}_p)$. The archimedean representations $\pi_{i,\infty}$ all have scalar minimal K–type of weight k and are therefore isomorphic. Since F is $\Gamma^{\mathrm{para}}(N)$ invariant, each $\pi_{i,p}$ for $p < \infty$ has non-zero paramodular vectors. The eigenform condition implies that the local representations $\pi_{i,p}$ and $\pi_{j,p}$ are isomorphic for almost all p and all i, j. By Conjecture 5.3 the representations π_i are all isomorphic. But then, by Conjecture 5.2, there can be only one i; in other words, π is irreducible. This implies part i) of Conjecture 5.1.

Now let F_1 and F_2 be as in ii) of Conjecture 5.1. We just proved that F_1 and F_2 generate irreducible cuspidal automorphic representations $\pi_1 = \otimes\pi_{1,p}$ and $\pi_2 = \otimes\pi_{2,p}$. The condition of F_1 and F_2 having the same Hecke eigenvalues almost everywhere translates into $\pi_{1,p} \cong \pi_{2,p}$ for almost all p. By Conjecture 5.3 it follows that $\pi_1 \cong \pi_2$, and then $\pi_1 = \pi_2$ as spaces of automorphic forms by Conjecture 5.2. We shall write π for $\pi_1 = \pi_2$ and π_p for $\pi_{1,p} = \pi_{2,p}$. Let V_p be a model of π_p. Let $v_\infty \in V_\infty$ be a lowest weight vector (generating the scalar minimal K–type). For $p < \infty$ let $v_p \in V_p$ be the essentially unique local newform according to Theorem 5.1 i). Let F be the function on the upper half plane corresponding to the vector $\otimes v_p \in \otimes\pi_p$. Then F is a paramodular cusp form of weight k. Since v_p is the local newform at every place, the level of F is at least as “good” as the level of F_1, in the sense that $F \in S_k(N)$ with $N|N_1$.

Part ii) of Theorem 5.1 says that every paramodular vector in V_p can be obtained from the local newform v_p by repeatedly applying the local level raising operators and taking linear combinations.

Since local and global level raising operators are compatible, this implies that $F_1 = \Theta F$, where Θ is an element of the algebra $\mathcal{A}$ introduced in the previous section. This element Θ cannot be in the ideal $\mathcal{I}$ generated by $\theta_p, \theta'_p, \eta_p$ for all primes p, since otherwise F_1 would be an oldform. Hence Θ is a scalar and F_1 is a multiple of F. The same argument applies to F_2, proving that F_1 and F_2 are multiplies of each other and that $N_1 = N_2 = N$.

6 Saito–Kurokawa liftings

Examples of modular forms for the paramodular group are obtained by the Saito–Kurokawa lifting. Let k be a positive integer. Let $f \in S_{2k-2}(\Gamma_0(N))$ be an elliptic cusp form, which we also assume to be a newform. We also assume that the sign in the functional equation of $L(s,f)$ is -1. Then f corresponds to a cuspidal Jacobi form $\tilde{f} \in J_{k,N}$ via the Skoruppa–Zagier lifting; see [SZ]. From $\tilde{f}$ we can construct a Siegel modular form $F \in S_k(N)$ via Gritsenko's "arithmetical lifting", which is a generalization of the Maaß construction; see [Gr]. The map $f \mapsto F$ extends to an injective linear map

$$\mathrm{SK}:\ S_{2k-2}^{\mathrm{new},-}(\Gamma_0(N)) \longrightarrow S_k(N).$$

Here, the "$-$" indicates the subspace of cusp forms for which the sign in the functional equation is -1. We propose an alternative group theoretic construction of this lifting which gives a little bit more information.

6.1 Theorem. *Let k and N be positive integers. Let $f \in S_{2k-2}^{\mathrm{new}}(\Gamma_0(N))$ be an elliptic cuspidal newform, assumed to be an eigenform for almost all Hecke operators. We assume that the sign in the functional equation of $L(s,f)$ is -1. Then there exists a paramodular Siegel cusp form $F \in S_k^{\mathrm{new}}(N)$ such that the incomplete spin L–function of F is given by*

$$L^S(s,F) = L^S(s,f)\, Z^S(s-1/2)\, Z^S(s+1/2). \tag{19}$$

Such an F is unique up to scalars.

We give an outline of the proof, whose details will appear elsewhere. Let π be the cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ associated to the modular form f (it is generated by an adelic function on $\mathrm{GL}(2,\mathbb{A})$ constructed from f by a similar formula as in (4)). Our hypothesis on $L(s,f)$ assures that there exists a Saito–Kurokawa lifting to $\mathrm{GSp}(4)$, meaning a cuspidal automorphic representation Π on $G(\mathbb{A})$ with trivial central character such that $L^S(s,\Pi) = L^S(s,\pi)Z^S(s-1/2)Z^S(s+1/2)$. The construction of Π is carried out in [Sch3] and further investigated in [Sch4]. In fact, there may exist a whole (finite) packet of such Π, but exactly one element in the packet is distinguished in the sense that each of its local components Π_p ($p < \infty$) contains non-zero paramodular vectors. Hence we can extract a Siegel modular form $F \in S_k(N')$ from Π for some level N' (the archimedean component Π_∞ is such that F is holomorphic of weight k).

Further analysis of the Π_p shows that they have a unique paramodular vector at the "right" level and at no better level; see Theorem 6.2 below for more details. In other words, we can actually extract an $F \in S_k(N)$ from Π, which is unique up to scalars, and since the local representations contain no paramodular vectors at lower levels, this F must be a newform.

Theorem 6.1 can be reformulated by saying that there is a Hecke-equivariant injection

$$\mathrm{SK}:\ S_{2k-2}^{\mathrm{new},-}(\Gamma_0(N)) \longrightarrow S_k^{\mathrm{new}}(N). \tag{20}$$

Here "Hecke-equivariant" has the following meaning. Let $T(p)$ be the usual Hecke operator on $S_{2k-2}(\Gamma_0(N))$. Let $T_S(p)$ and $T_S'(p)$ be the generators for the local Hecke algebra $\mathcal{H}_p$ for Siegel modular forms as in [EZ], §6. We define a homomorphism ι of local Hecke algebras by

$$\iota(T_S(p)) = T_J(p) + p^{k-1} + p^{k-2},$$
$$\iota(T_S'(p)) = (p^{k-1} + p^{k-2})T_J(p) + 2p^{2k-3} + p^{2k-4}.$$

Then the map (20) is Hecke-equivariant in the sense that $T(\mathrm{SK}(f)) = \mathrm{SK}(\iota(T)f)$ for all elements $T \in \mathcal{H}_p$, for any $p \nmid N$.

Local liftings

We now present the local ingredient to Theorem 6.1 in more detail. Let F be a p–adic field, and let π be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character. In [Sch3], a local Saito–Kurokawa lifting $\mathrm{SK}(\pi)$ has been attached to π. It can be constructed as the unique irreducible quotient of the induced representation $\nu^{1/2}\pi \rtimes \nu^{-1/2}$; we refer to [ST] for the notation and the fact that this induced representation has exactly two irreducible constituents. The representation $\mathrm{SK}(\pi)$ thus constructed is a non-generic, non-tempered, irreducible, admissible representation of $\mathrm{PGSp}(4,F)$. One can prove the following result, which is a local version of the Saito–Kurokawa lifting. The symbol $\mathrm{K}(\mathfrak{p}^n)$ stands for the local paramodular group; see (1).

6.2 Theorem. *Let (π, V) be an irreducible, admissible, infinite-dimensional representation of $\mathrm{PGL}(2,F)$, and let $(\mathrm{SK}(\pi), W)$ be the local Saito–Kurokawa lifting of π as explained above. Let $V(n) \subset V$ be the subspace of $\Gamma_0(\mathfrak{p}^n)$–invariant vectors, and let $W(n) \subset W$ be the subspace of $\mathrm{K}(\mathfrak{p}^n)$ invariant vectors. Let n_0 be the minimal n such that $V(n) \neq 0$.*

i) The integer n_0 is also the minimal n such that $W(n) \neq 0$.

ii) $\dim(W(n_0)) = 1$.

iii) For any $n \geq n_0$, we have

$$W(n) = \bigoplus_{\substack{d,e \geq 0 \\ d+2e = n-n_0}} \theta^d \eta^e W(n_0),$$

where θ and η are the local level raising operators defined earlier.

iv) *All paramodular vectors $w \in W(n)$ are Atkin–Lehner eigenvectors with the same eigenvalue (as some $w_0 \in W(n_0)$).*

Thus, the local lifting Π has a unique newform at the same level as π. This explains the existence and part of the uniqueness assertion in Theorem 6.1 (one also needs to know global multiplicity one in the Saito–Kurokawa space), and the assertion that the lifting is a newform.

Extension to oldforms

We would like to extend the map (20) to include oldforms. However, part iii) of Theorem 6.2 shows that the structure of oldforms in a local representation (π, V) and in its lifting $(\mathrm{SK}(\pi), W)$ is different. While in V the dimensions of the spaces $V(n)$, $n \geq n_0$, grow (by [Cas]) like $1, 2, 3, \ldots$, the dimensions of the spaces $W(n)$ grow like $1, 1, 2, 2, 3, 3, \ldots$ (see Table 1 in the appendix, where we can observe these dimensions in the representations IIb, Vb and VIc, which are local Saito–Kurokawa liftings). This suggests that only a *subspace* of the space of oldforms in V can be matched to the oldforms in W. Part iv) of Theorem 6.2 provides the clue that this subspace should consist of the newforms and all oldforms with the same Atkin–Lehner eigenvalue as the newform. This local situation is compatible with the work of Skoruppa and Zagier, which shows that the map (20) can be extended to the "certain space" in the title of [SZ]; see further below for a precise definition.

We shall first describe the local analogue of the "certain space" more precisely. As above let (π, V) be an irreducible, admissible, infinite-dimensional representation of $\mathrm{PGL}(2, F)$, where F is a non-archimedean local field, and let $V(n) \subset V$ be the subspace of vectors fixed under the local congruence subgroup $\Gamma_0(\mathfrak{p}^n)$. Let n_0 be the minimal n such that $V(n) \neq 0$. Then, by Casselman's theory, $\dim(V(n)) = n - n_0 + 1$ for $n \geq n_0$. The local Atkin–Lehner involution

$$u_n = \begin{bmatrix} & 1 \\ \varpi^n & \end{bmatrix} \qquad (\varpi \text{ a uniformizer})$$

splits $V(n)$ into ± 1 eigenspaces $V(n)^{\pm}$. The eigenvalue ε at the minimal level n_0 coincides with the value $\varepsilon(1/2, \pi)$ of the ε–factor at $1/2$. Locally, the "certain space" is $\oplus_{n=n_0}^{\infty} V(n)^{\varepsilon}$, i.e., it consists of the newform and all those oldforms with the same Atkin–Lehner eigenvalue as the newform. These oldforms are obtained by repeated application of the operators

$$\alpha : V(n) \longrightarrow V(n+1), \qquad v \longmapsto v + \pi(\begin{bmatrix} \varpi^{-1} & \\ & 1 \end{bmatrix})v \tag{21}$$

and

$$\beta : V(n) \longrightarrow V(n+2), \qquad v \longmapsto \pi(\begin{bmatrix} \varpi^{-1} & \\ & 1 \end{bmatrix})v \tag{22}$$

to $V(n_0)$ (it is immediately verified that α and β commute with Atkin–Lehner involutions). One can check that $\alpha^2 v$ is not a multiple of βv, and more generally that

$$V(n)^\varepsilon = \bigoplus_{\substack{d,e\geq 0\\ d+2e=n-n_0}} \alpha^d\beta^e V(n_0). \tag{23}$$

We see that the "certain space" can be matched exactly with the complete space of paramodular vectors in $(\mathrm{SK}(\pi), W)$, whose structure is given in Theorem 6.2 iii). More precisely, we can define a local Saito–Kurokawa map

$$\mathrm{SK}:\ \bigoplus_{n=n_0}^{\infty} V(n)^\varepsilon \longrightarrow \bigoplus_{n=n_0}^{\infty} W(n) \tag{24}$$

by mapping a non-zero vector $v \in V(n_0)$ to a non-zero vector $w \in W(n_0)$ and requiring that $\mathrm{SK}\circ\alpha = \theta\circ\mathrm{SK}$ and $\mathrm{SK}\circ\beta = \eta\circ\mathrm{SK}$. Then SK is a linear isomorphism. We note that SK is *not* canonically defined: Not only do we have a freedom in the normalization of the newforms and the operators, but we could also replace the operator β by a linear combination of β and α^2.

The global version of the "certain space" is defined as follows. On the spaces $S_k(\Gamma_0(N))$ (elliptic modular forms) we have, for any prime number p, the Atkin–Lehner involutions u_p, defined analogously as above in the degree 2 case. If $p \nmid N$, we let u_p be the identity. We are looking for level raising operators $S_k(\Gamma_0(N)) \to S_k(\Gamma_0(Np))$ commuting with Atkin–Lehner involutions; here the prime number p may or may not divide N. The two natural operators $f(z) \mapsto f(z)$ and $f(z) \mapsto f(pz)$ do not have this property, but a computation shows that a certain linear combination has. More precisely, put

$$\alpha_p:\ S_k(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(Np)), \qquad f(z) \longmapsto f(z) + p^{k/2} f(pz). \tag{25}$$

Then $\alpha_p \circ u_p = u_p \circ \alpha_p$. Furthermore, another computation shows that $f \mapsto f(pz)$ *does* commute with Atkin–Lehner involutions if we consider $f(pz)$ an element of $S_k(\Gamma_0(Np^2))$. We therefore define

$$\beta_p:\ S_k(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(Np^2)), \qquad f(z) \longmapsto p^{k/2} f(pz). \tag{26}$$

Then $\beta_p \circ u_p = u_p \circ \beta_p$. If Φ is the adelic function on $\mathrm{GL}(2,\mathbb{A})$ corresponding to f, then the adelic functions corresponding to $\alpha_p f$ and $\beta_p f$ are

$$g \longmapsto \Phi(g) + \Phi\Big(g\begin{bmatrix} p^{-1} & \\ & 1\end{bmatrix}\Big) \qquad \text{and} \qquad g \longmapsto \Phi\Big(g\begin{bmatrix} p^{-1} & \\ & 1\end{bmatrix}\Big),$$

respectively. In other words, α_p and β_p are compatible with the local operators (21) and (22). We shall now define the "certain space", which will be denoted by $\mathcal{S}_k(\Gamma_0(N))$. Let

$$S_k^{\mathrm{new}}(\Gamma_0) := \bigoplus_{N=1}^{\infty} S_k^{\mathrm{new}}(\Gamma_0(N)), \qquad S_k(\Gamma_0) := \bigoplus_{N=1}^{\infty} S_k(\Gamma_0(N)).$$

We consider the operators α_p and β_p as endomorphisms of $S_k(\Gamma_0)$. They obviously commute, and operators for different p also commute. Hence we get a commutative algebra $\mathcal{B}$ of endomorphisms of $S_k(\Gamma_0)$ generated by all these operators for all prime numbers p. We define the "certain space" as the image of $S_k^{\mathrm{new}}(\Gamma_0)$ under $\mathcal{B}$,

$$\mathcal{S}_k(\Gamma_0) := \mathcal{B}\, S_k^{\mathrm{new}}(\Gamma_0), \qquad \mathcal{S}_k(\Gamma_0(N)) := \mathcal{S}_k(\Gamma_0) \cap S_k(\Gamma_0(N)).$$

Hence $\mathcal{S}_k(\Gamma_0(N))$ consists of all the newforms of level N plus those oldforms that can be obtained by repeated application of α_p and β_p operators to newforms of lower levels. Those oldforms have the same Atkin–Lehner eigenvalues as the newforms from which they come. If in the above definitions we allow only newforms with a certain sign in the functional equation of their L–function, we obtain the spaces $\mathcal{S}_k^{\pm}(\Gamma_0(N))$

From the local linear independence (23) we derive the global result that

$$\mathcal{S}_k(\Gamma_0(N)) = \bigoplus_{M|N} \bigoplus_{\substack{d,e\geq 1\\ de^2=N/M}} \alpha_d \beta_e S_k^{\mathrm{new}}(\Gamma_0(M)). \tag{27}$$

Here, $\alpha_d = \prod_i \alpha_{p_i}^{\nu_i}$ if $d = \prod_i p_i^{\nu_i}$, and similarly for β_e. Restricting to newforms with a fixed sign in the functional equation, we get

$$\mathcal{S}_k^{\pm}(\Gamma_0(N)) = \bigoplus_{M|N} \bigoplus_{\substack{d,e\geq 1\\ de^2=N/M}} \alpha_d \beta_e S_k^{\mathrm{new},\pm}(\Gamma_0(M)). \tag{28}$$

Now for each M we have the maps (20) from $S_{2k-2}^{\mathrm{new},-}(\Gamma_0(M))$ to $S_k^{\mathrm{new}}(M)$. We put them all together to define a linear map

$$S_{2k-2}^{\mathrm{new},-}(\Gamma_0) \longrightarrow S_k^{\mathrm{new}}(\Gamma^{\mathrm{para}})$$

The direct sum decomposition (28) shows that this linear map can be extended to a linear map

$$\mathrm{SK}:\ \mathcal{S}_{2k-2}^{-}(\Gamma_0) \longrightarrow S_k(\Gamma^{\mathrm{para}})$$

in such a way that

$$\mathrm{SK} \circ \alpha_p = \theta_p \circ \mathrm{SK} \qquad \text{and} \qquad \mathrm{SK} \circ \beta_p = \eta_p \circ \mathrm{SK}.$$

The image of SK is called the *Maaß space* and denoted by $\mathcal{S}_k(\Gamma^{\mathrm{para}})$. Restricting to a fixed level we get a Saito–Kurokawa lifting SK from $\mathcal{S}_{2k-2}^{-}(\Gamma_0(N))$ to $\mathcal{S}_k(N) = \mathcal{S}_k(\Gamma^{\mathrm{para}}) \cap S_k(N)$. Since SK is compatible with the local isomorphisms (24), we obtain the following result.

6.3 Theorem. *The Saito–Kurokawa lifting (20) can be extended to a Hecke–equivariant isomorphism*

$$\mathrm{SK}:\ \mathcal{S}_{2k-2}^{-}(\Gamma_0(N)) \longrightarrow \mathcal{S}_k(N).$$

Characterizations of the Maaß space

The following version of the Ramanujan conjecture is believed to be true, but currently there is no published proof.

6.1 Conjecture. (Ramanujan Conjecture for GSp(4)**)** *Let $\pi = \otimes\pi_v$ be a cuspidal automorphic representation of* $\mathrm{GSp}(4, \mathbb{A}_F)$*, where F is any number field. If π is not a CAP representation, then each π_v is tempered.*

The Ramanujan conjecture has the following relevance for the characterization of the Maaß space. We note that in the classical case the characterization of eigenforms in the Maaß space by their spin L–functions having poles was obtained by ODA [Oda] and EVDOKIMOV [Ev].

6.4 Theorem. *Write $F \in S_k(N)$ as a sum $F = \sum_i F_i$, where each $F_i \in S_k(N)$ is a Hecke eigenform for almost all Hecke operators. Then the following statements are equivalent.*

i) F is an element of the Maaß space $\mathcal{S}_k(N)$.

ii) Each of the incomplete spin L–functions $L(s, F_i)$ has a pole at $s = 3/2$.

iii) Each F_i corresponds to a vector in an irreducible cuspidal automorphic representation of $\mathrm{PGSp}(4, \mathbb{A}_{\mathbb{Q}})$ *that is CAP with respect to the Siegel parabolic subgroup.*

Each of these conditions implies the following.

iv) $\theta_p F = \theta'_p F$ for each prime number p.

If the Ramanujan conjecture holds, then the following condition implies the others.

v) There exists a prime number p such that $\theta_p F$ is a multiple of $\theta'_p F$.

Sketch of proof: i) $\Rightarrow$ ii) follows from the shape of the L–function in (19). ii) $\Rightarrow$ iii) follows from the characterization of CAP automorphic representations in [PS] and local results showing that a) global Saito–Kurokawa packets contain at most one element that is paramodular at every finite place, and b) Borel–CAP representations do not have paramodular vectors at every finite place. iii) $\Rightarrow$ i) follows from the fact that the local lifting (24) is onto, meaning SK exhausts the space of paramodular vectors.

i) $\Rightarrow$ iv): Let $\Pi = \otimes\Pi_p$ be an irreducible constituent of the space of cusp forms on $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ generated by the adelic function Φ attached to F. Then, by the group theoretic construction of Saito–Kurokawa liftings indicated after Theorem 6.1, each Π_p for $p < \infty$ is of the form $\mathrm{SK}(\pi)$ for an irreducible, admissible, infinite-dimensional representations π of $\mathrm{PGL}(2, \mathbb{Q}_p)$. One can prove by local computations that $\theta - \theta'$ annihilates the space of paramodular vectors in $\mathrm{SK}(\pi)$. This implies iv) since the local and global operators are compatible.

v) $\Rightarrow$ i): Let $\Pi = \otimes\Pi_p$ be as in the previous paragraph. The hypothesis implies that for some p the local representation Π_p contains a paramodular vector that is annihilated by $\theta - \theta'$. One can prove by local methods that Π_p must be one of the representations in the following list (see the table in the appendix):

- An unramified twist of the trivial representation.
- A representation of type IVb.
- A representation of type Vd or VId.
- A representation of the form $\mathrm{SK}(\pi)$ as in Theorem 6.2.

It is known that one-dimensional representations do not occur in global cusp forms. Representations of type IVb do also not occur in global cusp forms because they are not unitarizable. Representations of type Vd and VId are not tempered. Hence, if Π_p is one of these representations, and if the Ramanujan conjecture is true, then Π must be a CAP representation. One can give a complete description of the local components of CAP representations, and this description shows that Π must be CAP with respect to B, the minimal parabolic subgroup, or Q, the Klingen parabolic subgroup. But one can further show that in such CAP representations there is always at least one place for which the local component has no paramodular vectors (it is essential here that the ground field is $\mathbb{Q}$; the statement is wrong for other number fields). This proves that Π_p cannot be of type Vd or VId, and must consequently be a local Saito–Kurokawa representation of the form $\mathrm{SK}(\pi)$. These representations are also non-tempered, so that by the Ramanujan conjecture Π must be a CAP representation. By the above mentioned explicit description of CAP representation, Π must indeed be CAP with respect to P, the Siegel parabolic subgroup. In other words, Π lies in the Saito–Kurokawa space. ∎

In view of the definition (14) of the θ'_p operator we see that, at least if the Ramanujan conjecture is true, the Maaß space can be characterized as the subspace of $S_k(N)$ where θ_p commutes with Atkin–Lehner involutions. For a classical Saito–Kurokawa lifting $F \in S_k(1)$ (full modular group) the condition v) in Theorem 6.4 means

$$V_p f_m = p^{k-1} U_p f_{m/p} + f_{mp} \qquad \text{for } m \geq 1,$$

where we understand $f_{m/p} = 0$ for $p \nmid m$; see (11) and (17). In terms of the Fourier expansion $F(\tau, z, \tau') = \sum_{n,r,m} a(n,r,m) e^{2\pi i(n\tau + rz + m\tau')}$ this translates into the conditions

$$a(np, r, m) + p^{k-1} a\Big(\frac{n}{p}, \frac{r}{p}, m\Big) = p^{k-1} a\Big(n, \frac{r}{p}, \frac{m}{p}\Big) + a(n, r, mp) \qquad \text{for } n, r, m \in \mathbb{Z}, \tag{29}$$

with the convention that $a(\alpha, \beta, \gamma) = 0$ if $(\alpha, \beta, \gamma) \notin \mathbb{Z}^3$. The Maaß space for the full modular group is defined by the more general relations

$$a(n, r, m) = \sum_{d|(n,r,m)} d^{k-1} a\Big(\frac{nm}{d^2}, \frac{r}{d}, 1\Big) \qquad \text{for } n, r, m \in \mathbb{Z}; \tag{30}$$

see [Ma] or [EZ] §6. To see that the Maaß relations (30) are indeed more general, substitute (30) into (29). Conversely, for m a power of p, the condition (30) is implied by (29). Theorem 6.4 says that if the Ramanujan conjecture holds, then (29) and (30) are actually equivalent:

6.1 Corollary. *Suppose that the Ramanujan conjecture 6.1 holds. Let $F \in S_k(1)$ be a cusp form for $\mathrm{Sp}(4,\mathbb{Z})$ with Fourier expansion $F(\tau, z, \tau') = \sum_{n,r,m} a(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$. Then F is in the Maaß space if and only if there is a prime number p such that (29) holds.*

We stress that the prime number p in this corollary is completely arbitrary.

7 Two theorems

As before, let $S_k(N)$ be the space of cusp forms of weight k with respect to the paramodular group $\Gamma^{\mathrm{para}}(N)$. In this section we shall elaborate on the representation theoretic meaning of the following two theorems.

7.1 Theorem. *There are no paramodular cusp forms of weight 1: The spaces $S_1(N)$ are zero for any N.*

7.2 Theorem. *The operators θ_p and θ'_p from $S_k(N)$ to $S_k(Np)$ are injective for any N and any prime p.*

Both theorems are quickly proved using results on Jacobi forms. Theorem 7.1 follows immediately from the Fourier–Jacobi expansion and a result of Skoruppa stating that $J_{1,m} = 0$ for any m; see Theorem 5.7 in [EZ]. For Theorem 7.2, note that in view of the definition (14) it is enough to prove the result for θ_p. By (11), the θ_p operator is compatible with the operator V_p on Jacobi forms. But it is a consequence of the results of Skoruppa and Zagier [SZ] that the operator $V_p : J_{k,m}^{\mathrm{cusp}} \to J_{k,mp}^{\mathrm{cusp}}$ on cuspidal Jacobi forms is injective (see Lemma 1.10 of [Sch2] for a corresponding local statement). Theorem 7.2 follows.

We shall now reformulate Theorem 7.1 in terms of representations. For representations of $G(F)$, where F is a local field, we shall employ the notation of [ST] (this paper treats non-archimedean representations, but the notation can also be used for $F = \mathbb{R}$). The symbol ν stands for the normalized absolute value, which in the case $F = \mathbb{R}$ is the usual absolute value $|\,|$. Let ξ_0 be a character of F^* of order 2. Then, by [ST] Lemma ..., the induced representation $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}$ decomposes into four irreducible components. We are interested in the Langlands quotient $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$. In the archimedean case $F = \mathbb{R}$, where ξ_0 is the sign character, it can be shown that $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$ has a minimal K–type of weight $(1, 1)$. In other words, this is the archimedean representation underlying Siegel modular forms of weight one. Theorem 7.1 is therefore equivalent to the archimedean part of the following statement.

7.1 Corollary. *Let $F \in S_k(N)$, and let Φ be the adelic function corresponding to F. Let $\pi = \oplus \pi_i$ be the cuspidal automorphic representation of* $\mathrm{PGSp}(4, \mathbb{A})$ *generated by Φ, and let $\pi_i = \otimes \pi_{i,p}$ be the tensor product decomposition of the irreducible component π_i of π. Then no $\pi_{i,p}$ ($p \leq \infty$) is equal to $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$, where ξ_0 is a local character of order* 2.

Let us now focus on the non-archimedean content of Corollary 7.1. The appendix contains a table with the complete list of Iwahori–spherical representations of $\mathrm{GSp}(4, F)$ and the dimensions of their spaces of fixed vectors under the paramodular groups $\mathrm{K}(\mathfrak{p}^n)$ for any level $\mathfrak{p}^n$. We see that the dimensions of these spaces are always growing with growing n, except for the representation $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$ of type Vd; here ξ_0 is the unique non-trivial unramified quadratic character of F^*. For this representation the dimensions are $1, 0, 1, 0, \ldots$ (one can show that if ξ_0 is ramified, then Vd contains no paramodular vectors at all). Hence the corresponding local statement to Theorem 7.2 is not always true: In the Vd type representations, the local θ and θ' operators from $V(n)$ to $V(n+1)$ are zero (here $V(n)$ is the space of vectors fixed under $\mathrm{K}(\mathfrak{p}^n)$). It follows that these representations cannot occur as local components in automorphic representations generated by elements of $S_k(N)$, which is exactly the statement of Corollary 7.1 for $p < \infty$.

We mention the following local result, which says that the representations of type Vd are the *only* counterexamples to the injectivity of θ and θ'.

7.3 Theorem. *Let F be a p–adic field. Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character. Let $V(n)$ be the space of vectors fixed under the paramodular group* $\mathrm{K}(\mathfrak{p}^n)$ *as in (1). Assume that π is not isomorphic to a representation $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma)$ of type Vd, where ξ_0 is the unramified character of order* 2. *Then the operators θ and θ' from $V(n)$ to $V(n+1)$ are injective, for any n.*

This theorem is analogous to Lemma 1.10 of [Sch2], which says that certain Weil representations are the only counterexamples to the injectivity of a local V operator on representations of the Jacobi group. The two results are actually related since these Weil representations are Fourier–Jacobi models of the Vd type representations. Theorem 7.3 says that Vd type representations are the *only* local representations excluded by Theorem 7.2 (in automorphic representations generated by elements of $S_k(N)$). Therefore Theorem 7.2 is the *exact* non-archimedean analogue of Theorem 7.1.

Note that Corollary 7.1 is a Ramanujan type result: The representations of type Vd are non-tempered, and the corollary says that they do not occur in certain cuspidal automorphic representations of $\mathrm{PGSp}(4, \mathbb{A})$. There *are* actually cuspidal automorphic representations of this group, namely certain CAP representations, that contain

- $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$ as archimedean component, and moreover
- $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$ at almost every place.

But one can show that there is always at least one non-archimedean place where the local representation has no paramodular vectors. In other words, cusp forms of weight 1 *do* exist, but not for the paramodular group.

Appendix: Paramodular vectors in Iwahori–spherical representations

The following table lists the dimensions of the spaces of paramodular vectors of any level for each irreducible, admissible representation of $\mathrm{PGSp}(4, F)$ which admits a nonzero vector fixed by the Iwahori subgroup I.

The first column. By [Bo], these representations are exactly the irreducible subquotients of the representations of $\mathrm{PGSp}(4, F)$ induced from unramified quasi-characters of the Borel subgroup. The basic reference on representations of $\mathrm{GSp}(4, F)$ induced from a quasi-character of the Borel subgroup is section 3 of [ST], and we will use the notation of that paper. Thus, St is the Steinberg representation, **1** is the trivial representation, and $\nu = |\cdot|$. It is also useful to consult section 4.1 of [T-B]. Let χ_1, χ_2 and σ be unramified quasi-characters of $F^\times$ with $\chi_1\chi_2\sigma^2 = 1$, so that the representation $\chi_1 \times \chi_2 \rtimes \sigma$ of $\mathrm{GSp}(4, F)$ induced from the quasi-character $\chi_1 \otimes \chi_2 \otimes \sigma$ has trivial central character. Of course, $\chi_1 \times \chi_2 \rtimes \sigma$ may be reducible. It turns out that by section 3 of [ST], there are six types of $\chi_1 \times \chi_2 \rtimes \sigma$ such that every irreducible admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character which contains a nonzero vector fixed by I is an irreducible subquotient of a representative of one of these six types, and that no two representatives of two different types share a common irreducible subquotient. The first column gives the name of the type. In the table we choose a representative for a type with the notation as below, and in subsequent columns we give information about the irreducible subquotients of that representative. The types are described as follows. *Type I:* These are the $\chi_1 \times \chi_2 \rtimes \sigma$ where χ_1, χ_2 and σ are unramified quasi-characters of $F^\times$ such that $\chi_1\chi_2\sigma^2 = 1$ and $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible. See Lemma 3.2 of [ST]. *Type II:* These are the $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ where χ and σ are unramified quasi-characters of $F^\times$ such that $\chi^2\sigma^2 = 1$. See Lemmas 3.3 and 3.7 of [ST]. *Type III:* These are the $\chi \times \nu \rtimes \nu^{-1/2}\sigma$ where χ and σ are unramified quasi-characters of $F^\times$ such that $\chi\sigma^2 = 1$. See Lemmas 3.4 and 3.9 of [ST]. *Type IV:* These are the $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$ where σ is an unramified quasi-character of $F^\times$ such that $\sigma^2 = 1$. See Lemma 3.5 of [ST]. *Type V:* These are the $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$ where ξ_0 and σ are unramified quasi-characters of $F^\times$ such that ξ_0 has order two and $\sigma^2 = 1$. See Lemma 3.6 of [ST]. *Type VI:* These are the $\nu \times 1 \rtimes \nu^{-1/2}\sigma$ where σ is an unramified quasi-character of $F^\times$ such that $\sigma^2 = 1$. See Lemma 3.8 of [ST].

The second column Choose a type as in the first column, and choose a representative $\chi_1 \times \chi_2 \rtimes \sigma$ of that type. Then $\chi_1 \times \chi_2 \rtimes \sigma$ admits a finite number of irreducible subquotients, and this number depends only on the type of $\chi_1 \times \chi_2 \rtimes \sigma$. We index the irreducible subquotients by lower case Roman letters. The letter "a" is reserved for the generic irreducible subquotient.

The representation column. This column lists the irreducible subquotients of the representative of the type of the first column. We use the specific notation as in the discussion of the first column.

The N and $\epsilon(1/2, \pi)$ columns. Suppose π is an entry of the third column, and let φ be the L-parameter associated to π by [KL]. We define N and $\epsilon(1/2, \pi)$ by the equation $\epsilon(s, \varphi, \psi, \mathrm{d}x_\psi) = \epsilon(1/2, \pi)q^{-N(s-1/2)}$.

The K(0)**,** K(1)**,** K(2)**,** K(3) **and** K(n) **columns.** The numbers in the columns give the dimensions of the spaces of $\mathrm{K}(\mathfrak{p}^n)$ fixed vectors for $n = 0, 1, 2, 3$ and arbitrary $n \geq 0$. Note that to save space we have abbreviated $\mathrm{K}(\mathfrak{p}^n)$ by $\mathrm{K}(n)$. The signs under the numbers in the K(0), K(1), K(2) and K(3) columns indicate how these spaces of $\mathrm{K}(\mathfrak{p}^n)$ fixed vectors split under the action of the Atkin-Lehner operator $\pi(u_n)$. The signs are correct if in the type II case, where the central character of π is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial, and in the type IV, V, and IV cases, where the central character of π is σ^2, the character σ is trivial. If these assumptions are not met, then the plus and minus signs must be interchanged to obtain the correct signs.

		representation	N	$\varepsilon(1/2,\pi)$	$K(0)$	$K(1)$	$K(2)$	$K(3)$	$K(n)$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	0	1	$\mathbf{1}$ $+$	2 $+-$	4 $+++-$	6 $+++$ $---$	$\left[\frac{(n+2)^2}{4}\right]$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	1	$-(\sigma\chi)(\varpi)$	0	$\mathbf{1}$ $-$	2 $+-$	4 $++--$	$\left[\frac{(n+1)^2}{4}\right]$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	0	1	$\mathbf{1}$ $+$	1 $+$	2 $++$	2 $++$	$\left[\frac{n+2}{2}\right]$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	2	1	0	0	$\mathbf{1}$ $+$	2 $+-$	$\left[\frac{n^2}{4}\right]$
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	0	1	$\mathbf{1}$ $+$	2 $+-$	3 $++-$	4 $++--$	$n+1$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	3	$-\sigma(\varpi)$	0	0	0	$\mathbf{1}$ $-$	$\left[\frac{(n-1)^2}{4}\right]$
	b	$L((\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}))$	2	1	0	0	$\mathbf{1}$ $+$	1 $+$	$\left[\frac{n}{2}\right]$
	c	$L((\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$	1	$-\sigma(\varpi)$	0	$\mathbf{1}$ $-$	2 $+-$	3 $+--$	n
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	0	1	$\mathbf{1}$ $+$	1 $+$	1 $+$	1 $+$	1
V	a	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$	2	-1	0	0	$\mathbf{1}$ $-$	2 $+-$	$\left[\frac{n^2}{4}\right]$
	b	$L((\nu^{1/2}\xi_0\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	1	$\sigma(\varpi)$	0	$\mathbf{1}$ $+$	1 $+$	2 $++$	$\left[\frac{n+1}{2}\right]$
	c	$L((\nu^{1/2}\xi_0\mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$	1	$-\sigma(\varpi)$	0	$\mathbf{1}$ $-$	1 $+$	2 $--$	$\left[\frac{n+1}{2}\right]$
	d	$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$	0	1	$\mathbf{1}$ $+$	0	1 $+$	0	$\frac{1+(-1)^n}{2}$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	2	1	0	0	$\mathbf{1}$ $+$	2 $+-$	$\left[\frac{n^2}{4}\right]$
	b	$\tau(T, \nu^{-1/2}\sigma)$	2	1	0	0	0	0	0
	c	$L((\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	1	$-\sigma(\varpi)$	0	$\mathbf{1}$ $-$	1 $-$	2 $--$	$\left[\frac{n+1}{2}\right]$
	d	$L((\nu, \mathbf{1}_{F^*} \rtimes \nu^{-1/2}\sigma))$	0	1	$\mathbf{1}$ $+$	1 $+$	2 $++$	2 $++$	$\left[\frac{n+2}{2}\right]$

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