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Dimension formulas for Siegel modular forms of level 4

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With an appendix by Cris Poor and David S. Yuen

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Abstract

We prove several dimension formulas for spaces of scalar-valued Siegel modular forms of degree 2 with respect to certain congruence subgroups of level 4. In case of cusp forms, all modular forms considered originate from cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A})$ whose local component at $p = 2$ admits nonzero fixed vectors under the principal congruence subgroup of level 2. Using known dimension formulas combined with dimensions of spaces of fixed vectors in local representations at $p = 2$, we obtain formulas for the number of relevant automorphic representations. These, in turn, lead to new dimension formulas, in particular for Siegel modular forms with respect to the Klingen congruence subgroup of level 4.

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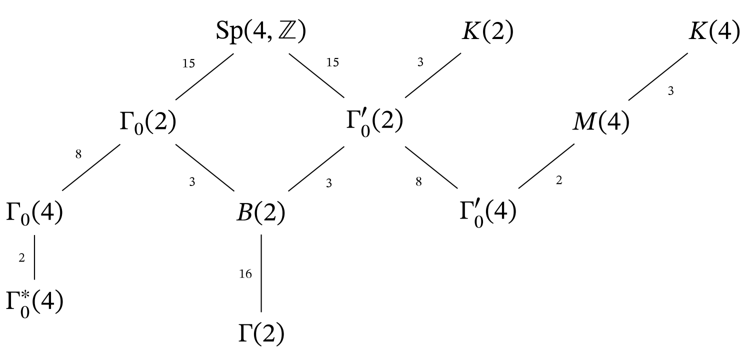
Contents

1 | INTRODUCTION

In this work, we consider dimension formulas for spaces of scalar-valued Siegel modular forms of degree 2, weight k, and level dividing 4. The notion of level is ambiguous; for example, level 4 could refer to modular forms with respect to the paramodular group $K(4)$, the Siegel congruence subgroup $\Gamma_0(4)$, the Klingen congruence subgroup $\Gamma_0'(4)$, or others. We consider the following 11 congruence subgroups of $\mathrm{Sp}(4,\mathbb{Q})$, all of which are in some sense level 1, 2, or 4:



(1)

The connecting lines indicate inclusions (with the bigger group on top), and their labels show indices. The group $\Gamma(2)$ is the principal congruence subgroup of level 2, $B(2)$ is the Borel congruence subgroup of level 2, $\Gamma_0^*(4)$ is a certain subgroup of index 2 in $\Gamma_0(4)$, and $M(4)$ is the "middle" group, which lies between $\Gamma_0'(4)$ and $K(4)$. For precise definitions, see Table 1 in the notations section.

For many of the subgroups Γ in (1), the dimension of the space of Siegel modular forms $M_k(\Gamma)$ and the subspace of cusp forms $S_k(\Gamma)$ is known; Table B.1 gives some references. In case a conjugate of Γ lies between $\Gamma(2)$ and $\mathrm{Sp}(4,\mathbb{Z})$, there is a well-known method based on Igusa's classic

TABLE 1 Global and local congruence subgroups.

Global groups		Local groups		
Symbol	**Definition**	**Symbol**	**Definition**	**Name**
$\Gamma(N)$	$\mathrm{Sp}(4,\mathbb{Z})\cap\begin{bmatrix}1+N\mathbb{Z}&N\mathbb{Z}&N\mathbb{Z}&N\mathbb{Z}\\N\mathbb{Z}&1+N\mathbb{Z}&N\mathbb{Z}&N\mathbb{Z}\\N\mathbb{Z}&N\mathbb{Z}&1+N\mathbb{Z}&N\mathbb{Z}\\N\mathbb{Z}&N\mathbb{Z}&N\mathbb{Z}&1+N\mathbb{Z}\end{bmatrix}$	$\Gamma(\mathfrak{p}^n)$	$K\cap\begin{bmatrix}1+\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n\\\mathfrak{p}^n&1+\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n\\\mathfrak{p}^n&\mathfrak{p}^n&1+\mathfrak{p}^n&\mathfrak{p}^n\\\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n&1+\mathfrak{p}^n\end{bmatrix}$	principal congruence subgroup
$K(N)$	$\mathrm{Sp}(4,\mathbb{Q})\cap\begin{bmatrix}\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}&\mathbb{Z}&N^{-1}\mathbb{Z}\\\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\N\mathbb{Z}&N\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$K(\mathfrak{p}^n)$	$G^1\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-n}\\\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}\end{bmatrix}$	paramodular group
$\Gamma_0(N)$	$\mathrm{Sp}(4,\mathbb{Z})\cap\begin{bmatrix}\mathbb{Z}&\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\N\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\N\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\Gamma_0(\mathfrak{p}^n)$	$K\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\end{bmatrix}$	Siegel congruence subgroup
$\Gamma_0'(N)$	$\mathrm{Sp}(4,\mathbb{Z})\cap\begin{bmatrix}\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\N\mathbb{Z}&N\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\Gamma_0'(\mathfrak{p}^n)$	$K\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}\end{bmatrix}$	Klingen congruence subgroup
$B(N)$	$\mathrm{Sp}(4,\mathbb{Z})\cap\begin{bmatrix}\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\N\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\N\mathbb{Z}&N\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$B(\mathfrak{p}^n)$	$K\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}\end{bmatrix}$	Borel congruence subgroup
$M(4)$	$\mathrm{Sp}(4,\mathbb{Q})\cap\begin{bmatrix}\mathbb{Z}&4\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}&\mathbb{Z}&2^{-1}\mathbb{Z}\\\mathbb{Z}&4\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&4\mathbb{Z}&4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$M(\mathfrak{p}^2)$	$G^1\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}^2&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^2&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{o}\end{bmatrix}$	The "middle" group
$\Gamma_0^*(4)$	$\left\{\begin{bmatrix}A&B\\C&D\end{bmatrix}\in\Gamma_0(4):D \bmod 2 \in\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix}\right\}\right\}$	$\Gamma_0^*(\mathfrak{p}^2)$	$\left\{\begin{bmatrix}A&B\\C&D\end{bmatrix}\in\Gamma_0(\mathfrak{p}^2):D \bmod \mathfrak{p} \in\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix}\right\}\right\}$	

paper [20]. Theorem 2 of [20] gives the character of the representation of $\mathrm{Sp}(4,\mathbb{Z}/2\mathbb{Z}) \cong S_6$ on $M_k(\Gamma(2))$. Using standard character theory, one can thus easily calculate $\dim M_k(\Gamma)$. This method works for all groups in (1) except $K(2)$ and $\Gamma_0'(4)$. The results are summarized in Table B.2. All the dimension formulas in this paper are packaged into generating series like $\sum_{k=0}^{\infty} \dim M_k(\Gamma)t^k$.

The result for $K(2)$ in Table B.2 is taken from the literature. The result for $\Gamma_0'(4)$, which follows from our considerations using automorphic representation theory, is new. In fact, calculating $\dim M_k(\Gamma_0'(4))$ and $\dim S_k(\Gamma_0'(4))$ provided the original motivation for the present work.

At least for $k \geqslant 6$, the codimension $\dim M_k(\Gamma) - \dim S_k(\Gamma)$ can be determined from the cusp structure of the Satake compactification and Satake's characterization of the image of the global Φ-map. In degree 2, the method distills down to a simple formula, which we record in Theorem 4.3. We thus obtain codimension formulas for all the groups in (1); see Table 8. Together with the information about $\dim M_k(\Gamma)$ in Table B.2, we get the dimension formulas for $\dim S_k(\Gamma)$ in Table B.3 for all Γ except $\Gamma_0'(4)$.

To obtain further results, we consider the automorphic representations π generated by the eigenforms in $S_k(\Gamma)$ for Γ in (1). If we factor an irreducible such π into local representations $\pi \cong \otimes \pi_v$ with irreducible, admissible representations π_v of $\mathrm{GSp}(4,\mathbb{Q}_v)$, then π_∞ is a "holomorphic" representation of lowest weight k, and π_p is unramified for all primes $p \geqslant 3$. If Γ is not equal to $\Gamma_0'(4)$, then Γ contains a conjugate of $\Gamma(2)$, and consequently, π_2 will have nonzero fixed vectors under the local principal congruence subgroup $\Gamma(\mathfrak{p})$, where $\mathfrak{p} = 2\mathbb{Z}_2$. A complete determination of such π_2, which are also known as representations with nonzero hyperspecial parahoric restriction, has been achieved in [28]. We reproduce the list of irreducible, admissible representations of $\mathrm{GSp}(4,\mathbb{Q}_2)$ with nonzero hyperspecial parahoric restriction in Table 4. They are organized into types I, IIa, IIb, …. We let $S_k(\Omega)$ be the set of cuspidal, automorphic representations that are holomorphic of weight k at the archimedean place, unramified outside 2, and are of type Ω with nonzero hyperspecial parahoric restriction at $p = 2$; see Definition 5.1 for more details. We note that $S_k(\Omega)$ is a finite set; see [5].

A key result which allows us to get information about $\Gamma_0'(4)$ is [47, Lemma 4]. It implies that if an irreducible, admissible representation of $\mathrm{GSp}(4,\mathbb{Q}_2)$ has nonzero $\Gamma_0'(\mathfrak{p}^2)$-invariant vectors, then it also has nonzero $\Gamma(\mathfrak{p})$-invariant vectors, and hence, appears in Table 4. Therefore, eigenforms in $S_k(\Gamma_0'(4))$ also generate elements of $S_k(\Omega)$ for some Ω. Conversely, given an automorphic representation $\pi \cong \otimes\pi_v$ in $S_k(\Omega)$ and a nonzero vector in π_2 invariant under the local congruence subgroup C analogous to Γ for some Γ in (1), we can construct an element of $S_k(\Gamma)$ by "descending" to the Siegel upper half space $\mathcal{H}_2$. We thus get the relation

$$\dim S_k(\Gamma) = \sum_{\Omega} s_k(\Omega) d_{C,\Omega}, \tag{2}$$

where $s_k(\Omega) = |S_k(\Omega)|$ and $d_{C,\Omega}$ is the common dimension of the space of C-invariant vectors in representations of type Ω occurring in Table 4. The numbers $d_{C,\Omega}$ can all be calculated and are listed in Table 5.

Observe that (2) is a system of linear equations relating the $\dim S_k(\Gamma)$ for all Γ and the $s_k(\Omega)$ for all Ω. Recall that the $\dim S_k(\Gamma)$ are already known for all Γ except $\Gamma_0'(4)$. Essentially now what happens is that as Γ runs through the subgroups in (1) except $\Gamma_0'(4)$ the system (2) provides enough equations in order to determine the $s_k(\Omega)$. Once these are known we use (2) again, this time for $\Gamma = \Gamma_0'(4)$, to determine $\dim S_k(\Gamma_0'(4))$.

In full detail, the situation is slightly more complicated because the system (2) has more unknowns than equations. This hurdle is overcome by exploiting that automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$ are categorized into six different kinds of Arthur packets. In Proposition 5.3, we will prove that only packets of “general type,” denoted by **(G)**, and packets of Saito–Kurokawa type, denoted by **(P)**, are relevant. Considering the **(G)** version and the **(P)** version of (2) separately reduces the number of equations and makes the method work. As a by-product, we obtain refined dimension formulas for the spaces of nonlifts $S_k^{(\mathbf{G})}(\Gamma)$ and the spaces of lifts $S_k^{(\mathbf{P})}(\Gamma)$ (see Section 5.1 for a more precise definition of these spaces). We remark that we included the groups $\Gamma_0^*(4)$ and $M(4)$ in our list (1) in order to obtain two more linear equations; without these, the system (2) would still be underdetermined.

There are only two spaces which are not accessible with the above methods, namely, $M_2(\Gamma_0'(4))$ and $M_4(\Gamma_0'(4))$. Their dimensions have been determined by Cris Poor and David S. Yuen in Appendix A.

As mentioned above, many dimension formulas for Siegel modular forms are already contained in the literature. The new contributions of the present work are as follows.

- Siegel modular forms for the groups $\Gamma_0^*(4)$ and $M(4)$ have not previously received much attention in the literature. (See, however, the “paramodular groups with level” defined in [6].)
- The dimension formulas for $\Gamma_0'(4)$ are new. Until now, the literature only contains dimension formulas for $\Gamma_0'(p)$ where p is prime; see [12, 15, 18, 46].
- We obtain the refined dimension formulas for the spaces of lifts $S_k^{(\mathbf{P})}(\Gamma)$ and nonlifts $S_k^{(\mathbf{G})}(\Gamma)$.
- We obtain formulas for $s_k(\Omega)$, the number of cuspidal automorphic representations of $\mathrm{PGSp}(4,\mathbb{A})$ of weight k, unramified outside 2, and with a representation of type Ω at $p=2$ admitting nonzero $\Gamma(\mathfrak{p})$-invariant vectors.

The paper is organized as follows. In Section 3, we collect the necessary facts from local representation theory. The main outcomes are Table 4, the complete list of all relevant representations, and Table 5, which contains the dimensions of the spaces of fixed vectors in these representations under all relevant local congruence subgroups. In Section 4, which is largely independent from Section 3, we first utilize Satake's method to obtain codimension formulas for all Γ in (1). Combined with Igusa's result, we thus obtain dimension formulas for $M_k(\Gamma)$ and $S_k(\Gamma)$ for all Γ in (1) except $K(2)$ and $\Gamma_0'(4)$. The formulas for $K(2)$ are already known, and the ones for $\Gamma_0'(4)$ will follow as a consequence of our other results. Section 5 begins with a review of Arthur packets for GSp(4). We make the connection between Siegel modular forms and representations, resulting in the system of linear equations (2). We then derive the numbers $s_k(\Omega)$, first for Saito–Kurokawa lifts, then for representations of general type. Finally, as an application, we obtain the desired dimension formulas for $\Gamma_0'(4)$.

Most of our results are summarized in table form in Appendix B. More precisely, Tables B.2 and B.3 contain dimension formulas for $M_k(\Gamma)$ and $S_k(\Gamma)$, respectively. Tables B.4 and B.5 are for dimension formulas of $S_k^{(\mathbf{P})}(\Gamma)$ and $S_k^{(\mathbf{G})}(\Gamma)$, respectively. Tables B.10 and B.11 contain formulas for $s_k^{(\mathbf{P})}(\Omega)$ and $s_k^{(\mathbf{G})}(\Omega)$, respectively. Here, $s_k^{(*)}(\Omega) = |S_k^{(*)}(\Omega)|$ (see Section 5.1). Tables B.6–B.9 and B.12 provide numerical examples for weight $k \leqslant 20$. Appendix A, provided by Cris Poor and David S. Yuen, fills the final gap by calculating $\dim M_k(\Gamma_0'(4))$ for $k=2$ and $k=4$.

2 | NOTATION AND PRELIMINARIES

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ have the usual meaning. The symbol $\mathbb{Q}_p$ stands for the field of p-adic numbers. We will write $\mathbb{F}_p$ for the field with p elements; only $\mathbb{F}_2$ is needed in this work.

Let J be a 4×4 antisymmetric matrix over a field F. We consider the symplectic similitude group

$$G = \mathrm{GSp}(4) := \{g \in \mathrm{GL}(4) : \ {}^t gJg = \lambda(g)J,\ \lambda(g) \in \mathrm{GL}(1)\}, \tag{3}$$

which is an algebraic F-group. The function λ is called the multiplier homomorphism. The kernel of this function is the symplectic group Sp(4). Let Z be the center of GSp(4) and $\mathrm{PGSp}(4) := \mathrm{GSp}(4)/Z$.

While all choices of J lead to isomorphic groups, one or the other choice might be more convenient depending on the context. When working with classical Siegel modular forms, the usual choice for J is[†]

$$J_1 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}, \tag{4}$$

leading to the "classical" version of the symplectic group. When working with local representations, it is often more convenient to use

$$J_2 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}, \tag{5}$$

resulting in the "symmetric" version of the symplectic group. For example, the standard Borel subgroup in the second version consists of upper triangular matrices. We will allow ourselves to use both versions of GSp(4). An isomorphism between them is obtained by switching the first two rows and columns.

We will utilize the following representatives for elements of the Weyl group,

$$s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \quad s_2 = \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}, \tag{6}$$

given in the J_1 version of Sp(4).

Local and global congruence subgroups. For a positive integer N, we define $\Gamma_0^{(1)}(N) := \mathrm{SL}(2,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{bmatrix}$. In degree 2, we will work with a number of congruence subgroups of "level 2" and "level 4." The global subgroups are contained in $\mathrm{Sp}(4,\mathbb{Q})$, and except for the paramodular group $K(2)$, all of them can be conjugated into the full modular group $\mathrm{Sp}(4,\mathbb{Z})$. Locally, we work over the field $\mathbb{Q}_2$ and denote by $\mathfrak{o}$ its ring of integers $\mathbb{Z}_2$ and by $\mathfrak{p}$ the maximal ideal $2\mathbb{Z}_2$ of $\mathfrak{o}$. All our subgroups will be contained in $G^1 := \{g \in \mathrm{GSp}(4,\mathbb{Q}_2) : \lambda(g) \in \mathfrak{o}^\times\}$, and except for the paramodular group $K(\mathfrak{p})$, all of them can be conjugated into the hyperspecial maximal compact subgroup $K := \mathrm{GSp}(4,\mathbb{Z}_2)$. Table 1 shows the notations we use for various congruence subgroups. Note that for the global groups, we use the symplectic form J_1, and for the local groups, we use the symplectic form J_2.

Siegel modular forms

Let $\mathcal{H}_n$ be the Siegel upper half space of degree n, that is, $\mathcal{H}_n$ consists of all symmetric complex $n \times n$ matrices whose imaginary part is positive definite. The principal congruence subgroup $\Gamma(N)$ of $\mathrm{Sp}(2n,\mathbb{Z})$ is the kernel of the reduction map $\mathrm{Sp}(2n,\mathbb{Z}) \to \mathrm{Sp}(2n,\mathbb{Z}/N\mathbb{Z})$. By a congruence

[†] Empty entries in matrices mean zeros.

subgroup of $\mathrm{Sp}(2n, \mathbb{Q})$, we mean a subgroup of $\mathrm{Sp}(2n, \mathbb{Q})$ which, for some N, contains $\Gamma(N)$ with finite index.

Definition 2.1. A Siegel modular form of degree n and weight k with respect to a congruence subgroup Γ of $\mathrm{Sp}(2n, \mathbb{Q})$ is a holomorphic function $f : \mathcal{H}_n \to \mathbb{C}$ with the transformation property

$$(f|_k g)(Z) = j(g, Z)^{-k} f((AZ + B)(CZ + D)^{-1}) = f(Z) \text{ for } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma,$$

where $j(g, Z) = \det(CZ + D)$, and which satisfies the usual moderate growth condition if $n = 1$.

We call a Siegel modular form f a *cusp form* if

$$\lim_{\lambda \to \infty} (f|_k g)\left(\begin{bmatrix} \tau & \\ & i\lambda \end{bmatrix}\right) = 0 \quad \text{for all } g \in \mathrm{Sp}(2n, \mathbb{Q}) \text{ and } \tau \in \mathcal{H}_{n-1}.$$

In this work, we will primarily consider Siegel modular forms of degree 2, and occasionally modular forms of degree 1. We denote by $M_k(\Gamma)$ the space of Siegel modular forms of degree 2 and weight k with respect to the congruence subgroup Γ of $\mathrm{Sp}(4, \mathbb{Q})$, and by $S_k(\Gamma)$ its subspace of cusp forms. We denote by $M_k^{(1)}(\Gamma)$ the space of modular forms of degree 1 and weight k with respect to the congruence subgroup Γ of $\mathrm{SL}(2, \mathbb{Q})$, and by $S_k^{(1)}(\Gamma)$ its subspace of cusp forms.

A lemma on rational points and integral points. For lack of a good reference, we include a proof of the following result. It will be used in Section 4.1.

Lemma 2.2. *Let n be a positive integer. Let R be any standard parabolic subgroup of* $\mathrm{Sp}(2n)$. *Then*

$$\mathrm{Sp}(2n, \mathbb{Q}) = R(\mathbb{Q})\mathrm{Sp}(2n, \mathbb{Z}).$$

Proof. Let $g \in \mathrm{Sp}(2n, \mathbb{Q})$. For any place p, let K_p be the standard maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{Q}_p)$. Let $K = \prod K_p$. Use the Iwasawa decomposition to write $g = r_p \kappa_p$, with $r_p \in R(\mathbb{Q}_p)$ and $\kappa_p \in K_p$. Then $g = r\kappa$, where $r = (r_p)$ and $\kappa = (\kappa_p)$. Let $R = MN$ be the Levi decomposition of R. Write $r = mn$ with $m \in M(\mathbb{A})$ and $n \in N(\mathbb{A})$. By strong approximation, we may write

$$m = m_{\mathbb{Q}} m_{\mathbb{R}} m_K \quad \text{with } m_{\mathbb{Q}} \in M(\mathbb{Q}),\ m_{\mathbb{R}} \in M(\mathbb{R}),\ m_K \in M(\mathbb{A}) \cap K.$$

The element m_K may be absorbed into K (possibly modifying n), and may therefore assumed to be 1. Using $\mathbb{A} = \mathbb{Q} + \mathbb{R} + \prod_{p<\infty} \mathbb{Z}_p$, we can write

$$n = n_{\mathbb{Q}} n_{\mathbb{R}} n_K, \quad n_{\mathbb{Q}} \in N(\mathbb{Q}),\ n_{\mathbb{R}} \in N(\mathbb{R}),\ n_K \in N(\mathbb{A}) \cap K.$$

The element n_K may be absorbed into K, and therefore, assumed to be 1. Summarizing, we see that we can write

$$g = r_{\mathbb{Q}} r_{\mathbb{R}} \kappa, \quad r_{\mathbb{Q}} \in R(\mathbb{Q}),\ r_{\mathbb{R}} \in R(\mathbb{R}),\ \kappa \in K.$$

The matrix $r_{\mathbb{Q}}^{-1} g$ lies in $\mathrm{Sp}(2n, \mathbb{Z}_p)$ for all finite p, and hence in $\mathrm{Sp}(2n, \mathbb{Z})$. This concludes the proof. ◻

3 | LOCAL DIMENSIONS

We will start this section by collecting some facts about the symmetric group S_6. In Subsections 3.2 and 3.3, we work over the p-adic field $\mathbb{Q}_2$ and write $\mathfrak{o}$ for its ring of integers and $\mathfrak{p}$ for the maximal ideal of $\mathfrak{o}$. In this local context, it is convenient to work with the "symmetric" version of the symplectic group, defined by the symplectic form J_2 given in (5).

3.1 | Preliminaries on S_6

Let (ρ, U) be a representation of a finite group G, and let H be a subgroup of G. Then $p = \frac{1}{|H|}\sum_{h\in H}\rho(h)$ is a projector onto the subspace U^H of H-fixed vectors. Hence,

$$\dim U^H = \mathrm{Tr}(p) = \frac{1}{|H|}\sum_{h\in H}\chi_\rho(h), \tag{7}$$

where χ_ρ is the character of ρ. More generally, if τ is a representation of H, then the multiplicity of τ in $\rho|_H$ is

$$\mathrm{mult}_\rho(\tau) = \frac{1}{|H|}\sum_{h\in H}\overline{\chi_\tau(h)}\chi_\rho(h). \tag{8}$$

We will apply this principle to the finite group $\mathrm{Sp}(4,\mathbb{Z})/\Gamma(2) \cong \mathrm{Sp}(4,\mathbb{F}_2)$. In order to do so, we will exhibit an explicit isomorphism with the symmetric group S_6.

Consider the natural permutation action of S_6 on the space of column vectors $(\mathbb{F}_2)^6$. Let W be the five-dimensional subspace of vectors whose coordinates add up to zero. Let U be the subspace of W spanned by $u = {}^t(1,1,1,1,1,1)$. Then W and U are both invariant under the action of S_6, so that we get an action on the four-dimensional space W/U. There is a symplectic (and symmetric) form on W given by

$$\langle x, y\rangle = \sum_{i=1}^{6} x_i y_i, \quad x = {}^t(x_1,\dots,x_6),\ y = {}^t(y_1,\dots,y_6).$$

This form is degenerate with radical U, thus inducing a nondegenerate symplectic form on the quotient W/U. Evidently, this form is invariant under the action of S_6. We thus obtain a nontrivial homomorphism $S_6 \to \mathrm{Sp}(4,\mathbb{F}_2)$. Since the image of this map has more than two elements, and since A_6 is the only nontrivial, proper, normal subgroup of S_6, the map is injective. Since both groups have the same number of elements, it is an isomorphism. To make the isomorphism more explicit, let

$$e_1 = \begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix},\quad e_2 = \begin{bmatrix}0\\0\\0\\0\\1\\1\end{bmatrix},\quad f_2 = \begin{bmatrix}0\\0\\0\\1\\1\\0\end{bmatrix},\quad f_1 = \begin{bmatrix}0\\1\\1\\0\\0\\0\end{bmatrix}.$$

Then, $W = \langle e_1, e_2, f_2, f_1, u\rangle$. The images of e_1, e_2, f_2, f_1 form a basis of W/U, with respect to which the form $\langle\ ,\ \rangle$ has matrix J_2 defined in (5). Easy calculations then show that on certain elements, the isomorphism $S_6 \to \mathrm{Sp}(4, \mathbb{F}_2) = \{g \in \mathrm{GL}(4, \mathbb{F}_2)\ :\ {}^t g J_2 g = J_2\}$ has the following explicit description.

$$(16)(25)(34) \longmapsto \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \qquad (46) \longmapsto \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}, \tag{9}$$

$$(13)(46) \longmapsto \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}, \qquad (12)(36)(45) \longmapsto \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}, \tag{10}$$

$$(12)(34)(56) \longmapsto \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}, \qquad (12) \longmapsto \begin{bmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \tag{11}$$

$$(135)(246) \longmapsto \begin{bmatrix} 1 & 1 & & \\ 1 & & & \\ & & 1 & 1 \\ & & 1 & \end{bmatrix}, \qquad (153)(264) \longmapsto \begin{bmatrix} & 1 & & \\ 1 & 1 & & \\ & & & 1 \\ & & 1 & 1 \end{bmatrix}. \tag{12}$$

Using such a description, it is easy to determine the number of elements of a given cycle type in certain subgroups of $\mathrm{Sp}(4, \mathbb{F}_2) \cong S_6$. Table 2 shows such data for a number of subgroups of $\mathrm{Sp}(4, \mathbb{F}_2)$ (the first one of which is the trivial and the second one of which is the full subgroup). All these subgroups are obtained as the image of a conjugate of a congruence subgroup Γ of $\mathrm{Sp}(4, \mathbb{Q})$, this conjugate lying between $\Gamma(2)$ and $\mathrm{Sp}(4, \mathbb{Z})$, under the projection map $\mathrm{Sp}(4, \mathbb{Z}) \to \mathrm{Sp}(4, \mathbb{F}_2)$; the first column of Table 2 shows the congruence subgroup Γ.

Both the conjugacy classes and the irreducible characters of S_6 (also referred to as S_6-types) are parametrized by partitions of 6. We write $[n_1, \dots, n_r]$ for the irreducible character of S_6 corresponding to the partition $6 = n_1 + \dots + n_r$. For example, [6] is the trivial character and [1,1,1,1,1,1] is the sign character of S_6. The character table of S_6 is given in [20, p. 400]. Using formula (7), the data in Table 2, and the character table, we obtain the dimension of the space of fixed vectors in each S_6-type under the subgroups listed in Table 2. The results are summarized in Table 3. The last two rows of Table 3 indicate the generic representations (i.e., those which admit a nonzero Whittaker functional) and the cuspidal representations (i.e., those with no nonzero fixed vectors under the unipotent radicals of the parabolic subgroups).

3.2 | Parahoric restriction

In this section, we consider irreducible, admissible representations of $\mathrm{GSp}(4, \mathbb{Q}_2)$ with trivial central character that have nonzero fixed vectors under the principal congruence subgroup $\Gamma(2)$.

TABLE 2 The number of elements of a given cycle type in some subgroups of $\mathrm{Sp}(4, \mathbb{F}_2) \cong S_6$. Here, we use the "symmetric" form of Sp(4), that is, the one defined with the symplectic form J_2 as in (5). The third column shows the cardinality of the subgroup.

Γ	**⊂ Sp(4, $\mathbb{F}_2$)**	**#**	**1**	**(12)**	**(12)(34)**	**(12)(34)(56)**	**(123)**	**(123)(45)**	**(123)(456)**	**(1234)**	**(1234)(56)**	**(12345)**	**(123456)**
$\Gamma(2)$	$\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix}$	1	1	0	0	0	0	0	0	0	0	0	0
$\mathrm{Sp}(4, \mathbb{Z})$	$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$	720	1	15	45	15	40	120	40	90	90	144	120
$K(4)$	$\begin{bmatrix} * & & & * \\ & * & * & \\ & * & * & \\ * & & & * \end{bmatrix}$	36	1	6	9	0	4	12	4	0	0	0	0
$\Gamma_0(2)$	$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}$	48	1	3	9	7	0	0	8	6	6	0	8
$\Gamma_0(4)$	$\begin{bmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{bmatrix}$	6	1	0	0	3	0	0	2	0	0	0	0
$\Gamma_0^*(4)$	index 2 in $\begin{bmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{bmatrix}$	3	1	0	0	0	0	0	2	0	0	0	0
$\Gamma_0'(2)$	$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}$	48	1	7	9	3	8	8	0	6	6	0	0
$M(4)$	$\begin{bmatrix} * & & & \\ & * & * & \\ & * & * & \\ * & & & * \end{bmatrix}$	12	1	4	3	0	2	2	0	0	0	0	0
$B(2)$	$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$	16	1	3	5	3	0	0	0	2	2	0	0

TABLE 3 The dimensions of the spaces of fixed vectors in each S_6-type under some subgroups of $\mathrm{Sp}(4,\mathbb{F}_2) \cong S_6$. Here, we use the “symmetric” form of Sp(4), that is, the one defined with the symplectic form J_2 as in (5). The “$\Gamma(2)$” row shows the dimensions of each S_6-type.

Γ	$\subset \mathbf{Sp}(4,\mathbb{F}_2)$	[6]	[5,1]	[4,2]	[4,1,1]	[3,3]	[3,2,1]	[3,1,1,1]	[2,2,2]	[2,2,1,1]	[2,1,1,1,1]	[1,1,1,1,1,1]
$\Gamma(2)$	$\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix}$	1	5	9	10	5	16	10	5	9	5	1
$\mathrm{Sp}(4,\mathbb{Z})$	$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$	1	0	0	0	0	0	0	0	0	0	0
$K(4)$	$\begin{bmatrix} * & & & * \\ & * & * & \\ & * & * & \\ * & & & * \end{bmatrix}$	1	1	1	0	1	0	0	0	0	0	0
$\Gamma_0(2)$	$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}$	1	0	1	0	0	0	0	1	0	0	0
$\Gamma_0(4)$	$\begin{bmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{bmatrix}$	1	0	3	1	0	2	3	3	0	1	0
$\Gamma_0^*(4)$	index 2 in $\begin{bmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{bmatrix}$	1	1	3	4	3	4	4	3	3	1	1
$\Gamma_0'(2)$	$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}$	1	1	1	0	0	0	0	0	0	0	0
$M(4)$	$\begin{bmatrix} * & & & \\ & * & * & \\ & * & * & \\ * & & & * \end{bmatrix}$	1	2	2	1	1	1	0	0	0	0	0
$B(2)$	$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$	1	1	2	0	0	1	0	1	0	0	0
generic					•		•	•		•		
cuspidal										•		•

TABLE 4 Hyperspecial parahoric restriction for $\mathrm{GSp}(4,\mathbb{Q}_2)$. All characters $\chi,\chi_1,\chi_2,\sigma,\xi$ are assumed to be unramified, and the supercuspidal representation π of $\mathrm{GL}(2,\mathbb{Q}_2)$ has depth 0.

		Representation	a	ε	Temp	Para	Parahoric restriction	(G)	(P)
I		$\chi_1\times\chi_2\rtimes\sigma$ (irred.)	0	+	•	•	[6]+[5,1]+2[4,2]+[3,2,1]+[2,2,2]	•	
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	1	±	•	•	[5,1]+[4,2]+[3,2,1]	•	
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	0	+		•	[6]+[4,2]+[2,2,2]		•
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	2	+	•	•	[4,2]+[3,2,1]+[2,2,2]	•	
	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	0	+		•	[6]+[5,1]+[4,2]		
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	3	±	•	•	[3,2,1]	•	
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	2	+		•	[4,2]+[2,2,2]		
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	1	±		•	[4,2]+[5,1]		
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	0	+		•	[6]		
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	2	–	•	•	[5,1]+[3,2,1]	•	
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	1	±		•	[4,2]		•
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	1	±		•	[4,2]		•
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	0	+		•	[6]+[2,2,2]		
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	2	+	•	•	[4,2]+[3,2,1]	•	
	b	$\tau(T,\nu^{-1/2}\sigma)$	2	+	•		[2,2,2]	•	•
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	1	±		•	[5,1]		•
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	0	+		•	[6]+[4,2]		
VII		$\chi\rtimes\pi$	4	+	•	•	[3,1,1,1]+[2,1,1,1,1]	•	
VIII	a	$\tau(S,\pi)$	4	+	•	•	[3,1,1,1]	•	
	b	$\tau(T,\pi)$	4	+	•		[2,1,1,1,1]	•	
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	4	+	•	•	[3,1,1,1]	•	
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	4	+			[2,1,1,1,1]		
X		$\pi\rtimes\sigma$	2	–	•	•	[4,1,1]+[3,3]	•	
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	3	±	•	•	[4,1,1]	•	
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	2	–		•	[3,3]		•
Va*		$\delta^*([\xi,\nu\xi],\nu^{-1/2}\sigma)$	2	–	•		[1,1,1,1,1,1]	•	•
sc(16)			4	–	•	•	[2,2,1,1]	•	

Table 4 contains a complete list of such representations. Their central characters are necessarily unramified, so after a twist, we may assume that the central character is trivial. All characters in the representations appearing in Table 4 are assumed to be unramified.

Let (π,V) be an irreducible, admissible representation of $\mathrm{GSp}(4,\mathbb{Q}_2)$. The hyperspecial maximal compact subgroup $K=\mathrm{GSp}(4,\mathbb{Z}_2)$ of $\mathrm{GSp}(4,\mathbb{Q}_2)$ normalizes $\Gamma(\mathfrak{p})$. Hence, K acts on the space $V^{\Gamma(\mathfrak{p})}$ of $\Gamma(\mathfrak{p})$-fixed vectors. The resulting representation of $K/\Gamma(\mathfrak{p})\cong\mathrm{Sp}(4,\mathbb{F}_2)$ is called the hyperspecial parahoric restriction of π and denoted by $r_K(\pi)$. It has been calculated for all π in [28, 29].

Table 4 contains a list of all irreducible, admissible representations of $\mathrm{PGSp}(4,\mathbb{Q}_2)$ for which $r_K(\pi)\neq 0$, using notations as in [27, 32]. Since hyperspecial parahoric restriction commutes with induction by [28, Theorem 2.19], all the parameters in Table 4 must have nonzero parahoric restriction on GL(1) or GL(2). This means the characters $\chi,\chi_1,\chi_2,\sigma,\xi$ of $\mathbb{Q}_2^\times$ are assumed to be unramified, and the supercuspidal representation π in types VII–XIb is an unramified twist of the unique supercuspidal representation π of $\mathrm{PGL}(2,\mathbb{Q}_2)$ of conductor exponent 2. The "a" column shows the (exponent of the) conductor of the representation. The "ε" column shows the possibilities for the value of the ε-factor at $1/2$. The "temp" column indicates the tempered representations, under the assumption that the inducing data is unitary. The "para" column indicates the representations that have nonzero paramodular vectors (in which case the minimal paramodular level coincides with the conductor). The hyperspecial parahoric restriction for nonsupercuspidal representations is given in [29, Table 3] and [28, Table 3.1]; see [28, p. 103] for the translation of Enomoto's notation to standard S_6 notation.

There are two supercuspidal representations in Table 4, the nongeneric $\delta^*([\xi,\nu\xi],\nu^{-1/2}\sigma)$ of type Va* and the generic sc(16). The representation $\delta^*([\xi,\nu\xi],\nu^{-1/2}\sigma)$ is invariant under twisting by ξ, so there is only one representation of type Va*. It shares an L-packet with the unique representation of type Va; both have L-parameter $\mathrm{st}_2 \oplus \xi\mathrm{st}_2$, where st_2 is the L-parameter of the Steinberg representation $\mathrm{St}_{\mathrm{GL}(2)}$ of $\mathrm{GL}(2,\mathbb{Q}_2)$. So, the parahoric restriction information for Va* comes from [28, Tables 4.2 and 5.2]. By Frobenius reciprocity and the proposition in [22, Section 1.4], it follows that

$$\delta^*([\xi,\nu\xi],\nu^{-1/2}\sigma) = \text{c-Ind}_{ZK}^{G}([1,1,1,1,1,1]), \tag{13}$$

where we inflate [1,1,1,1,1,1], the sign character of $K/\Gamma(2) \cong \mathrm{Sp}(4,\mathbb{F}_2)$ to K and then extend it to ZK by having Z act trivially. By [28, Table 4.2], there are no nongeneric supercuspidals of $\mathrm{PGSp}(4,\mathbb{Q}_2)$ with nonzero hyperspecial parahoric restriction besides Va*.

By [23] and [28, Proposition 2.16], there are no generic supercuspidals of $\mathrm{PGSp}(4,\mathbb{Q}_2)$ with nonzero hyperspecial parahoric restriction besides sc(16). In this case, we have

$$\mathrm{sc}(16) = \text{c-Ind}_{ZK}^{G}([2,2,1,1]). \tag{14}$$

The parahoric restriction for sc(16) follows from [28, Lemma 2.18].

3.3 | Local fixed vectors

Table 5 lists the dimensions of the space of fixed vectors under various congruence subgroups for the same class of representations as in Table 4. These are the irreducible, admissible representations π of $\mathrm{GSp}(4,\mathbb{Q}_2)$ for which the hyperspecial parahoric restriction $r_K(\pi)$ is nonzero, that is, which have nonzero vectors fixed under the principal congruence subgroup $\Gamma(\mathfrak{p})$.

Theorem 3.1. *Let (π,V) be an irreducible, admissible representation of* $\mathrm{GSp}(4,\mathbb{Q}_2)$. *Let H be one of the congruence subgroups listed in the first row of Table 5.*

- (i) *If $r_K(\pi)\neq 0$, so that π occurs among the representations in Tables 4, then* $\dim V^H$ *is given as in Table 5.*
- (ii) *If* $\dim V^H \neq 0$, *then $r_K(\pi)\neq 0$, so that π occurs among the representations in Tables 4.*

Proof.

- (i) For $H=\mathrm{K}(\mathfrak{p})$, see [27, Section A.8]. For $H=\Gamma_0'(\mathfrak{p}^2)$, see [47, Table 1]. For every other H, there exists a conjugate subgroup $\tilde{H}$ such that $\Gamma(\mathfrak{p})\subset\tilde{H}\subset K$. If $r_K(\pi)=\rho_1\oplus\ldots\oplus\rho_m$ with S_6-types

TABLE 5 The dimensions of the spaces of fixed vectors under various congruence subgroups of the irreducible, admissible representations of $\mathrm{GSp}(4,\mathbb{Q}_2)$ with nonzero hyperspecial parahoric restriction. (See Table 4 for the precise notation for these representations.).

Ω		$\Gamma(\mathfrak{p})$	K	$\mathrm{K}(\mathfrak{p})$	$\mathrm{K}(\mathfrak{p}^2)$	$\Gamma_0(\mathfrak{p})$	$\Gamma_0(\mathfrak{p}^2)$	$\Gamma_0^*(\mathfrak{p}^2)$	$\Gamma_0'(\mathfrak{p})$	$\Gamma_0'(\mathfrak{p}^2)$	$M(\mathfrak{p}^2)$	$B(\mathfrak{p})$
I		45	1	2	4	4	12	15	4	11	8	8
II	a	30	0	1	2	1	5	8	2	7	5	4
	b	15	1	1	2	3	7	7	2	4	3	4
III	a	30	0	0	1	2	8	10	1	5	3	4
	b	15	1	2	3	2	4	5	3	6	5	4
IV	a	16	0	0	0	0	2	4	0	2	1	1
	b	14	0	0	1	2	6	6	1	3	2	3
	c	14	0	1	2	1	3	4	2	5	4	3
	d	1	1	1	1	1	1	1	1	1	1	1
V	a	21	0	0	1	0	2	5	1	5	3	2
	b	9	0	1	1	1	3	3	1	2	2	2
	c	9	0	1	1	1	3	3	1	2	2	2
	d	6	1	0	1	2	4	4	1	2	1	2
VI	a	25	0	0	1	1	5	7	1	5	3	3
	b	5	0	0	0	1	3	3	0	0	0	1
	c	5	0	1	1	0	0	1	1	2	2	1
	d	10	1	1	2	2	4	4	2	4	3	3
VII		15	0	0	0	0	4	5	0	2	0	0
VIII	a	10	0	0	0	0	3	4	0	2	0	0
	b	5	0	0	0	0	1	1	0	0	0	0
IX	a	10	0	0	0	0	3	4	0	1	0	0
	b	5	0	0	0	0	1	1	0	1	0	0
X		15	0	0	1	0	1	7	0	3	2	0
XI	a	10	0	0	0	0	1	4	0	2	1	0
	b	5	0	0	1	0	0	3	0	1	1	0
Va*		1	0	0	0	0	0	1	0	0	0	0
sc(16)		9	0	0	0	0	0	3	0	1	0	0

(ρ_i, U_i), then

$$\dim V^H = \dim r_K(\pi)^{\bar{H}} = \sum_{i=1}^{m} \dim U_i^{\bar{H}}, \tag{15}$$

where $\bar{H}$ is the image of $\tilde{H}$ in $K/\Gamma(\mathfrak{p}) \cong \mathrm{Sp}(4,\mathbb{F}_2)$. The dimensions $U^{\bar{H}}$ are listed in Table 3, for each S_6-type (ρ, U). We thus get the desired dimensions from the $r_K(\pi)$ listed in Table 4.

(ii) If $H \neq \Gamma_0'(\mathfrak{p}^2)$, then a conjugate of H contains $\Gamma(\mathfrak{p})$, so that $r_K(\pi) = V^{\Gamma(\mathfrak{p})} \supset V^H \neq 0$. If $H = \Gamma_0'(\mathfrak{p}^2)$, then [47, Lemma 4] shows that $V^{\Gamma(\mathfrak{p})} \neq 0$. □

We remark that for most of the congruence subgroups, the dimensions in Table 5 appear elsewhere in the literature. For all the subgroups containing $B(\mathfrak{p})$, see [35]. For the paramodular groups, see [27]. For $M(\mathfrak{p}^2)$ and $\Gamma_0'(\mathfrak{p}^2)$, see [47].

4 | GLOBAL DIMENSIONS AND CODIMENSIONS

The goal of this section is to derive the dimension formulas in Tables B.2 and B.3 for all congruence subgroups except $\Gamma_0'(4)$. Most of these formulas are already contained in the literature, but we give a unified approach. First, we derive a general formula in degree 2 for the codimension $\dim M_k(\Gamma) - \dim S_k(\Gamma)$ based on the Satake compactification and the global Φ map; see Theorem 4.3. We thus obtain the codimensions summarized in Table 8 for all congruence subgroups of interest to us.

To obtain the actual dimensions, we note that most of our congruence subgroups Γ, after an appropriate conjugation, lie between $\Gamma(2)$ and $\mathrm{Sp}(4,\mathbb{Z})$. One can thus use Igusa's result [20, Theorem 2] to calculate $\dim M_k(\Gamma)$; see Section 4.3. The only subgroup other than $\Gamma_0'(4)$ for which this does not work is $K(2)$, for which the result is already contained in the literature.

4.1 | A general codimension formula

In this section, we will find a general formula for calculating the codimension of $S_k(\Gamma)$ in $M_k(\Gamma)$ for a congruence subgroup Γ of $\mathrm{Sp}(4,\mathbb{Q})$. A summary of the method for any degree is given in [25, Section 3]. It is based on [33] and the surjectivity of the global Φ operator proven in [34]. We specialize to the degree 2 case, resulting in the formula in Theorem 4.3 below.

We define the symplectic group $\mathrm{Sp}(4)$ with respect to the form J_1 given in (4), and use the following parabolic subgroups;

$$B=\begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\cap \mathrm{Sp}(4),\quad P=\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}\cap \mathrm{Sp}(4),\quad Q=\begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\cap \mathrm{Sp}(4). \tag{16}$$

Consider the homomorphisms $\omega: Q(\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ and $\iota: \mathrm{SL}(2,\mathbb{R}) \to Q(\mathbb{R})$ given by

$$\omega: Q(\mathbb{R}) \longrightarrow \mathrm{SL}(2,\mathbb{R}), \qquad \iota: \mathrm{SL}(2,\mathbb{R}) \longrightarrow Q(\mathbb{R}), \tag{17}$$

$$\begin{bmatrix} a & 0 & b & * \\ * & * & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & * \end{bmatrix} \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let Γ be a congruence subgroup of $\mathrm{Sp}(4,\mathbb{Q})$. We will describe how the geometry of the Satake compactification $S(\Gamma\backslash\mathcal{H}_2)$ is reflected algebraically via double cosets.

Let X be a fixed set of representatives for the double cosets $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/P(\mathbb{Q})$, and let Y be a fixed set of representatives for the double cosets $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/Q(\mathbb{Q})$. (Note that the quotient $\mathrm{Sp}(4,\mathbb{R})^{\mathrm{pr}}/H(\mathbb{R})^{\mathrm{pr}}$ appearing in [25, p. 451] simplifies to $\mathrm{Sp}(4,\mathbb{Q})/H(\mathbb{Q})$ for any of the subgroups H in (16)). Since $\mathrm{Sp}(4,\mathbb{Q}) = \mathrm{Sp}(4,\mathbb{Z})B(\mathbb{Q})$ (by taking inverses in Lemma 2.2), we may assume that $X, Y \subset \mathrm{Sp}(4,\mathbb{Z})$. There is a bijection between X and the zero-dimensional cusps of Γ. Similarly, there is a bijection between Y and the one-dimensional cusps of Γ. For $y \in Y$, let C_y be the 1-cusp corresponding to y, and let

$$\Gamma_y = \omega\big(y^{-1}\Gamma y \cap Q(\mathbb{Q})\big), \tag{18}$$

which is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Q})$. Let R_y be a fixed set of representatives for the double cosets $\Gamma_y\backslash\mathrm{SL}(2,\mathbb{Q})/B_1(\mathbb{Q})$, where B_1 is the upper triangular subgroup of SL(2). As is well known, there is a bijection between R_y and the set of cusps of Γ_y, which are points in the Satake compactification $\mathcal{S}(\Gamma_y\backslash\mathcal{H}_1)$. There is an embedding $\Gamma_y\backslash\mathcal{H}_1 \to \Gamma\backslash\mathcal{H}_2$, which extends to a continuous map $\mathcal{S}(\Gamma_y\backslash\mathcal{H}_1) \to \mathcal{S}(\Gamma\backslash\mathcal{H}_2)$. Let $C_{y,\rho}$ be the image of the cusp corresponding to $\rho \in R_y$ under this map. It is a 0-cusp of $\mathcal{S}(\Gamma\backslash\mathcal{H}_2)$ lying on the 1-cusp C_y. The double coset corresponding to $C_{y,\rho}$ is $\Gamma y\iota(\rho)P(\mathbb{Q})$. We see:

- If $\Gamma y_1\iota(\rho_1)P(\mathbb{Q}) = \Gamma y_2\iota(\rho_2)P(\mathbb{Q})$ for two distinct $y_1, y_2 \in Y$ and some $\rho_1 \in R_{y_1}, \rho_2 \in R_{y_2}$, then it means that C_{y_1} and C_{y_2} intersect at $C_{y_1,\rho_1} = C_{y_2,\rho_2}$.
- If $\Gamma y\iota(\rho_1)P(\mathbb{Q}) = \Gamma y\iota(\rho_2)P(\mathbb{Q})$ for $y \in Y$ and distinct $\rho_1, \rho_2 \in R_y$, then it means that C_y has a self-intersection at $C_{y,\rho_1} = C_{y,\rho_2}$.

In this way, we find the cusp structure diagram for Γ. It consists of $|Y|$ curves representing the 1-cusps C_y, and $|X|$ points representing the 0-cusps $C_{y,\rho}$ for $\rho \in R_y$, with $C_{y,\rho}$ lying on C_y indicating the intersections and self-intersections.

For $f \in M_k(\Gamma)$, the Siegel Φ-operator produces a function Φf on $\mathcal{H}_1$ defined by

$$(\Phi f)(\tau) = \lim_{\lambda\to\infty} f\left(\begin{bmatrix}\tau & \\ & i\lambda\end{bmatrix}\right), \quad \tau \in \mathcal{H}_1. \tag{19}$$

It follows from the Fourier expansion of f that, in fact,

$$(\Phi f)(\tau) = \lim_{\lambda\to\infty} f\left(\begin{bmatrix}\tau & z\\ z & i\lambda\end{bmatrix}\right) \quad \text{for any } z \in \mathbb{C}. \tag{20}$$

We also define a Φ-operator on modular forms on $\mathcal{H}_1$. If f is such a modular form, then Φf is simply the number $\lim_{\lambda\to\infty} f(i\lambda)$.

Lemma 4.1. *Let*

$$u = \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & r \end{bmatrix} \in Q(\mathbb{Q}) \tag{21}$$

and f be a modular form of weight k on $\mathcal{H}_2$ with respect to some congruence subgroup. Then

$$\Phi(f|u) = r^{-k}(\Phi f)|\omega(u). \tag{22}$$

Proof. See the calculation in [24, p. 2464] to obtain (22). ☐

Lemma 4.2. *Let Γ be a congruence subgroup of* $\mathrm{Sp}(4,\mathbb{Q})$ *and $f \in M_k(\Gamma)$. Let $y \in Y$.*

(i) *If* $k \geqslant 1$ *is even, then* $\Phi(f|y) \in M_k^{(1)}(\Gamma_y)$*, where* Γ_y *is the group defined in (18).*

(ii) *If* $k \geqslant 1$ *is odd and* $\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \in y^{-1}\Gamma y$*, then* $\Phi(f|y) = 0$*.*

Proof. Let

$$u = \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & r \end{bmatrix} \in y^{-1}\Gamma y \cap Q(\mathbb{Q}). \tag{23}$$

We claim that $r \in \{\pm 1\}$. Indeed, the map $y^{-1}\Gamma y \cap Q(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ that sends any matrix to its (4,4)-coefficient is a continuous group homomorphism. Since $y^{-1}\Gamma y \cap Q(\mathbb{Q}_p)$ lies in a compact subset of $Q(\mathbb{Q}_p)$, the image of this homomorphism lies in $\mathbb{Z}_p^\times$. This is true for all p, and hence $r \in \{\pm 1\}$.

Applying Lemma 4.1 to $g := f|y$ instead of f and u as in (23), we see that

$$\Phi g = r^{-k}(\Phi g)|\omega(u). \tag{24}$$

Now both (i) and (ii) follow easily. □

Theorem 4.3. *Let* Γ *be a congruence subgroup of* $\mathrm{Sp}(4, \mathbb{Q})$*. Let* X*,* Y *and* Γ_y *be as defined above. Then, for even* $k \geqslant 6$*, we have*

$$\dim M_k(\Gamma) - \dim S_k(\Gamma) = |X| + \sum_{y \in Y} \dim S_k^{(1)}(\Gamma_y). \tag{25}$$

Proof. Observing Lemma 4.2 (i), we define

$$\widetilde{\Phi} : M_k(\Gamma) \longrightarrow \bigoplus_{y \in Y} M_k^{(1)}(\Gamma_y), \quad f \longmapsto (f_y)_{y \in Y}, \quad \text{where } f_y = \Phi(f|y). \tag{26}$$

One may think of f_y as the restriction of f to C_y. Evidently, $\ker(\tilde{\Phi}) = S_k(\Gamma)$, so that we have an exact sequence

$$0 \longrightarrow S_k(\Gamma) \longrightarrow M_k(\Gamma) \longrightarrow \mathrm{Im}(\tilde{\Phi}) \longrightarrow 0. \tag{27}$$

Hence, our desired codimension equals $\dim \mathrm{Im}(\tilde{\Phi})$. To understand $\mathrm{Im}(\tilde{\Phi})$, note that the f_y satisfy the following compatibility condition: For all $y_1, y_2 \in Y$, $\rho_1 \in R_{y_1}$ and $\rho_2 \in R_{y_2}$,

$$\Gamma y_1 \iota(\rho_1) P(\mathbb{Q}) = \Gamma y_2 \iota(\rho_2) P(\mathbb{Q}) \implies \Phi(f_{y_1}|\rho_1) = \Phi(f_{y_2}|\rho_2). \tag{28}$$

(This amounts to saying that f_{y_1} and f_{y_2} agree on the intersection points of the 1-cusps C_{y_1} and C_{y_2}; see [25, (1)].) Satake [34] proved that $\mathrm{Im}(\tilde{\Phi})$ is characterized by this compatibility condition.

In particular,

$$\bigoplus_{y\in Y} S_k^{(1)}(\Gamma_y) \subset \mathrm{Im}(\tilde{\Phi}). \tag{29}$$

To further study $\mathrm{Im}(\tilde{\Phi})$, we choose for every $x \in X$ a $y_x \in Y$ and a $\rho_x \in R_{y_x}$ such that the 0-cusp represented by x equals C_{y_x,ρ_x}. In terms of double cosets, this means

$$\Gamma x P(\mathbb{Q}) = \Gamma y_x \iota(\rho_x) P(\mathbb{Q}). \tag{30}$$

Then, we define the map

$$\theta : \mathrm{Im}(\tilde{\Phi}) \longrightarrow \mathbb{C}^{|X|}, \quad (f_y)_{y\in Y} \longmapsto \Big(\Phi(f_{y_x}|\rho_x)\Big)_{x\in X}. \tag{31}$$

The compatibility condition (28) assures that θ is independent of the choices of y_x and ρ_x.

It is clear that $\bigoplus_{y\in Y} S_k^{(1)}(\Gamma_y) \subseteq \ker\theta$ by definition of θ. To prove the reverse inclusion, suppose $(f_y)_{y\in Y} \in \mathrm{Im}(\tilde{\Phi})$ lies in the kernel of θ, that is, $\Phi(f_{y_x}|\rho_x) = 0$ for all $x \in X$. We want to show that $\Phi(f_y|\rho) = 0$ for all $y \in Y$ and $\rho \in R_y$. Let $a \in X$ be such that $\Gamma y\iota(\rho)P(\mathbb{Q}) = \Gamma a P(\mathbb{Q})$. Since also $\Gamma a P(\mathbb{Q}) = \Gamma y_a \iota(\rho_a) P(\mathbb{Q})$, we have

$$\Phi(f_y|\rho) = \Phi(f_{y_a}|\rho_a) = 0$$

by the compatibility condition (28). This proves $\ker\theta = \bigoplus_{y\in Y} S_k^{(1)}(\Gamma_y)$.

Next, we show that θ is surjective. Let $x \in X$. It follows from [7, Theorem 3.5.1] that we can find an $f_y \in M_k^{(1)}(\Gamma_y)$ such that, for all $\rho \in R_y$,

$$\Phi(f_y|\rho) = \begin{cases} 1 & \text{if } \Gamma x P(\mathbb{Q}) = \Gamma y \iota(\rho) P(\mathbb{Q}), \\ 0 & \text{otherwise.} \end{cases}$$

The family of f_y thus defined satisfies the compatibility condition (28), so that $(f_y)_{y\in Y}$ lies in the image of $\tilde{\Phi}$. Hence, we constructed an element of $\mathrm{Im}(\tilde{\Phi})$ that does not vanish at the 0-cusp corresponding to x, but vanishes at all other 0-cusps. It follows that θ is surjective.

We proved that there is an exact sequence

$$0 \longrightarrow \bigoplus_{y\in Y} S_k^{(1)}(\Gamma_y) \longrightarrow \mathrm{Im}(\tilde{\Phi}) \longrightarrow \mathbb{C}^{|X|} \longrightarrow 0. \tag{32}$$

Our assertion now follows from (27) and (32). □

4.2 | Codimension formulas for some congruence subgroups

In this section, we determine the codimension $\dim M_k(\Gamma) - \dim S_k(\Gamma)$ for the congruence subgroups Γ listed below in Theorem 4.4. One of them is a group $\Gamma_0^*(4)$ defined as follows. We consider the character of the group $\Gamma_0(4)$ obtained as the composition

$$\Gamma_0(4) \longrightarrow \mathrm{GL}(2,\mathbb{Z}) \longrightarrow \mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z}) \overset{\sim}{\longrightarrow} S_3 \longrightarrow \{\pm 1\}, \tag{33}$$

where the first map is given by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto D$, the second map is reduction modulo 2, the third map is any isomorphism, and the last map is the sign character of the symmetric group S_3. Let $\Gamma_0^*(4)$ be the kernel of this character. Explicitly,

$$\Gamma_0^*(4) = \left\{ g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_0(4) : D \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \bmod 2 \right\}. \tag{34}$$

Evidently, $\Gamma_0(4) = \Gamma_0^*(4) \sqcup \Gamma_0^*(4)s_1$, where s_1 is defined in (6).

For odd weights, we have the following result.

Theorem 4.4. *Suppose that $k \geqslant 1$ is odd, and that Γ is conjugate to one of the congruence subgroups in (1). Then $M_k(\Gamma) = S_k(\Gamma)$.*

Proof. Let $Y \subset \mathrm{Sp}(4,\mathbb{Z})$ be a fixed set of representatives for the double cosets $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/Q(\mathbb{Q})$. If we can verify the condition in Lemma 4.2 (ii) for all $y \in Y$, then $M_k(\Gamma) = S_k(\Gamma)$ will follow from (26) and (27); note that (26) and (27) hold for both even and odd k.

If Γ' is conjugate to Γ by an element of $\mathrm{Sp}(4,\mathbb{Q})$, then $M_k(\Gamma') = S_k(\Gamma')$ if and only if $M_k(\Gamma) = S_k(\Gamma)$. Hence, we need only consider the groups in (1).

The condition in Lemma 4.2 (ii) is satisfied for the normal subgroup $\Gamma(2)$ of $\mathrm{Sp}(4,\mathbb{Z})$, and then also for any subgroup containing $\Gamma(2)$. Up to conjugation, this covers all groups in (1) except $\Gamma_0'(4)$. For $\Gamma_0'(4)$, one can verify the condition directly using the representatives y given in Table 7. □

We turn to even weights, considering the case $k \geqslant 6$. Recall that for the codimension formula in Theorem 4.3, we need $|X|$, which is the cardinality of the double coset space $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/P(\mathbb{Q})$, and we need to know the groups Γ_y defined in (18), where y runs through a system of representatives for the double coset space $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/Q(\mathbb{Q})$.

For Γ equal to the principal congruence subgroup $\Gamma(2)$, it is well known that both double coset spaces have 15 elements, and that each group Γ_y equals $\Gamma^{(1)}(2)$. A quick derivation uses the fact that $\mathrm{Sp}(4,\mathbb{F}_2) \cong S_6$ has 720 elements (see Section 4.3). Note $\mathrm{Sp}(4,\mathbb{Q})/P(\mathbb{Q}) \cong \mathrm{Sp}(4,\mathbb{Z})/P(\mathbb{Z})$ and $\mathrm{Sp}(4,\mathbb{Q})/Q(\mathbb{Q}) \cong \mathrm{Sp}(4,\mathbb{Z})/Q(\mathbb{Z})$. Hence,

$$\Gamma(2)\backslash\mathrm{Sp}(4,\mathbb{Q})/P(\mathbb{Q}) \cong \mathrm{Sp}(4,\mathbb{F}_2)/P(\mathbb{F}_2) \quad \text{and} \quad \Gamma(2)\backslash\mathrm{Sp}(4,\mathbb{Q})/Q(\mathbb{Q}) \cong \mathrm{Sp}(4,\mathbb{F}_2)/Q(\mathbb{F}_2). \tag{35}$$

Since $P(\mathbb{F}_2)$ and $Q(\mathbb{F}_2)$ both have 48 elements, it follows that both double coset spaces have cardinality 15. Moreover, since $\Gamma(2)$ is normal in $\mathrm{Sp}(4,\mathbb{Z})$, each Γ_y equals $\omega(\Gamma \cap Q(\mathbb{Q})) = \mathrm{SL}(2,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} \end{bmatrix}$, which is conjugate (by an element of $\mathrm{SL}(2,\mathbb{Q})$) to $\Gamma_0^{(1)}(4)$. From Theorem 4.3, we thus get $\dim M_k(\Gamma(2)) - \dim S_k(\Gamma(2)) = 15 + 15 \dim S_k(\Gamma_0^{(1)}(4))$ for even $k \geqslant 6$.

For the congruence subgroups in (1) other than $\Gamma(2)$, the last column of Table 6 shows the cardinality of $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/P(\mathbb{Q})$. The table also indicates representatives for this double coset space,

TABLE 6 Double coset representatives for $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/P(\mathbb{Q})$.

Γ	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	#
$\mathrm{Sp}(4,\mathbb{Z})$	•								1
$K(2)$	•								1
$K(4)$	•						•		2
$\Gamma_0(2)$	•	•		•					3
$\Gamma_0(4)$	•		•	•	•	•	•	•	7
$\Gamma_0^*(4)$	•		•	•	•	•	•	•	7
$\Gamma_0'(2)$	•		•						2
$\Gamma_0'(4)$	•		•		•		•		4
$M(4)$	•		•		•				3
$B(2)$	•	•	•	•					4

using the following notations:

$$x_1 = \mathbf{I}_4 \quad x_2 = s_2 = \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix} \quad x_3 = s_1 s_2 = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ -1 & & & \end{bmatrix}$$

$$x_4 = s_2 s_1 s_2 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \quad x_5 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & 2 & & 1 \end{bmatrix} \quad x_6 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ 2 & & 1 & \\ & 2 & & 1 \end{bmatrix}$$

$$x_7 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 2 & 1 & \\ 2 & & & 1 \end{bmatrix} \quad x_8 = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & 2 & & 1 \\ -1 & & & \end{bmatrix}. \tag{36}$$

For the group $\mathrm{Sp}(4,\mathbb{Z})$, the information in Table 6 is trivial, for $K(2)$ and $K(4)$ see [25, Theorem 1.3]. Representatives for $\Gamma_0(2)$ follow from

$$\Gamma_0(2)\backslash\mathrm{Sp}(4,\mathbb{Q})/P(\mathbb{Q}) \cong P(\mathbb{F}_2)\backslash\mathrm{Sp}(4,\mathbb{F}_2)/P(\mathbb{F}_2) \tag{37}$$

and the Bruhat decomposition; similarly, for $\Gamma_0'(2)$ and $B(2)$. For $\Gamma_0'(4)$ and $M(4)$, see [47, Lemma 1, Lemma 2]. For $\Gamma = \Gamma_0(4)$, see [42, Proposition 2.6]. It is an exercise to derive the result for $\Gamma_0^*(4)$ from that for $\Gamma_0(4)$, using that $\Gamma_0(4) = \Gamma_0^*(4) \sqcup \Gamma_0^*(4)s_1$.

Table 7 gives double coset representatives for $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/Q(\mathbb{Q})$. The notation used is

$$y_1 = \mathbf{I}_4 \qquad y_2 = s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \quad y_3 = s_2 s_1 = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & -1 & & \\ & & 1 & \end{bmatrix}$$

TABLE 7 Double coset representatives for $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/Q(\mathbb{Q})$. The groups Γ_y defined in (18) are obtained by intersecting the given sets of 2×2 matrices with $\mathrm{SL}(2,\mathbb{Q})$.

Γ	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	#
$\mathrm{Sp}(4,\mathbb{Z})$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$									1
$K(2)$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&2^{-1}\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$								2
$K(4)$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&4^{-1}\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$				$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$				3
$\Gamma_0(2)$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$		$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$							2
$\Gamma_0(4)$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$			$\begin{bmatrix}\mathbb{Z}&4\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$		$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$			4
$\Gamma_0^*(4)$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$			$\begin{bmatrix}\mathbb{Z}&4\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$		$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$		5
$\Gamma_0'(2)$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$		$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$						3
$\Gamma_0'(4)$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$		$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$			$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	6
$M(4)$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&2^{-1}\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$		$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{bmatrix}$		$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$			$\begin{bmatrix}\mathbb{Z}&2^{-1}\mathbb{Z}\\4\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	5
$B(2)$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$	$\begin{bmatrix}\mathbb{Z}&\mathbb{Z}\\2\mathbb{Z}&\mathbb{Z}\end{bmatrix}$						4

$$y_4 = s_1s_2s_1 = \begin{bmatrix}1&&&\\&&&1\\&&1&\\&-1&&\end{bmatrix}\quad y_5 = \begin{bmatrix}1&&&\\&1&&\\&&1&\\&2&&1\end{bmatrix}\quad y_6 = \begin{bmatrix}1&-2&&\\&1&&\\&&1&\\&&2&1\end{bmatrix}$$

$$y_7 = \begin{bmatrix}1&&&\\&1&&\\&2&1&\\2&&&1\end{bmatrix}\quad y_8 = \begin{bmatrix}&1&&\\1&&&\\&2&&1\\&&1&\end{bmatrix}\quad y_9 = \begin{bmatrix}&1&&\\1&&&\\2&&&1\\&2&1&\end{bmatrix}. \tag{38}$$

A nonempty entry in the row for Γ and the column for y_i indicates that y_i is to be included in the set Y of representatives for $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/Q(\mathbb{Q})$. The entry itself indicates the group Γ_{y_i}, obtained by intersecting the given set with $\mathrm{SL}(2,\mathbb{Q})$. For most of the groups, the references are the same as given above for Table 6. For $\Gamma=\Gamma_0(4)$, see [42, Proposition 2.5]. Again, the result for $\Gamma_0^*(4)$ can be derived from that for $\Gamma_0(4)$.

The following generating series can easily be derived from well-known dimension formulas; see, for example, [7, Theorem 3.5.1].

$$\sum_{k=0}^{\infty} \dim S_k(\mathrm{SL}(2,\mathbb{Z}))t^k = \sum_{\substack{k=6\\k \text{ even}}}^{\infty} \dim S_k(\mathrm{SL}(2,\mathbb{Z}))t^k = \frac{t^{12}}{(1-t^4)(1-t^6)}, \tag{39}$$

TABLE 8 Codimension formulas valid for even weight $k \geqslant 6$, given in the form $\dim M_k(\Gamma) - \dim S_k(\Gamma) = \alpha \dim S_k(\mathrm{SL}(2,\mathbb{Z})) + \beta \dim S_k(\Gamma_0^{(1)}(2)) + \gamma \dim S_k(\Gamma_0^{(1)}(4)) + \delta$.

Γ	α	β	γ	δ	$\sum_{k=6}^{\infty}(\dim M_k(\Gamma) - \dim S_k(\Gamma))t^k$
$\Gamma(2)$	0	0	15	15	$15\frac{t^6(2-t^2)}{(1-t^2)^2}$
$\mathrm{Sp}(4,\mathbb{Z})$	1	0	0	1	$\frac{t^6(1+t^2-t^8)}{(1-t^4)(1-t^6)}$
$K(2)$	2	0	0	1	$\frac{t^6(1+t^2+t^6-t^8)}{(1-t^4)(1-t^6)}$
$K(4)$	2	1	0	2	$\frac{t^6(2+3t^2+t^4+t^6-2t^8)}{(1-t^4)(1-t^6)}$
$\Gamma_0(2)$	0	2	0	3	$\frac{t^6(3+2t^2-3t^4)}{(1-t^2)(1-t^4)}$
$\Gamma_0(4)$	0	0	4	7	$\frac{t^6(11-7t^2)}{(1-t^2)^2}$
$\Gamma_0^*(4)$	0	0	5	7	$\frac{t^6(12-7t^2)}{(1-t^2)^2}$
$\Gamma_0'(2)$	2	1	0	2	$\frac{t^6(2+3t^2+t^4+t^6-2t^8)}{(1-t^4)(1-t^6)}$
$\Gamma_0'(4)$	3	1	2	4	$\frac{t^6(6+9t^2+5t^4+2t^6-4t^8)}{(1-t^4)(1-t^6)}$
$M(4)$	2	3	0	3	$\frac{t^6(3+6t^2+3t^4+2t^6-3t^8)}{(1-t^4)(1-t^6)}$
$B(2)$	0	4	0	4	$4\frac{t^6(1+t^2-t^4)}{(1-t^2)(1-t^4)}$

$$\sum_{k=0}^{\infty} \dim S_k(\Gamma_0^{(1)}(2))t^k = \sum_{\substack{k=6\\k \text{ even}}}^{\infty} \dim S_k(\Gamma_0^{(1)}(2))t^k = \frac{t^8}{(1-t^2)(1-t^4)}, \tag{40}$$

$$\sum_{k=0}^{\infty} \dim S_k(\Gamma_0^{(1)}(4))t^k = \sum_{\substack{k=6\\k \text{ even}}}^{\infty} \dim S_k(\Gamma_0^{(1)}(4))t^k = \frac{t^6}{(1-t^2)^2}. \tag{41}$$

Using these formulas, Theorem 4.3, and the information in Tables 6 and 7, we now get the following result.

Theorem 4.5. *For even* $k \geqslant 6$ *and a congruence subgroup* Γ *as in (1), the quantity* $\dim M_k(\Gamma) - \dim S_k(\Gamma)$ *is given as in Table 8.*

Remark 4.6. After we calculate $\dim M_4(\Gamma)$ and $\dim S_4(\Gamma)$ in the next section, it will turn out that the codimension formulas in Table 8 also hold for $k = 4$. See [4] for other cases in which Satake's method still works for $k = 4$.

4.3 | Dimension formulas for some congruence subgroups

In this section, we determine $\dim M_k(\Gamma)$ and $\dim S_k(\Gamma)$ for all nonnegative integers k and all congruence subgroups Γ in (1) except $\Gamma_0'(4)$. Many of the dimension formulas for these groups have appeared before in the literature, but to the best of our knowledge, the groups $\Gamma_0^*(4)$ and $M(4)$ have not been previously considered. References are contained in Tables B.2 and B.3. Except for

$K(2)$ and $\Gamma_0'(4)$, the dimension of $M_k(\Gamma)$ for Γ in (1) can be determined from [20, Theorem 2]. The method is well known, but we summarize it for completeness.

Let Γ be a congruence subgroup of $\mathrm{Sp}(4,\mathbb{Q})$ for which there exists an element $g \in \mathrm{Sp}(4,\mathbb{Q})$ such that the group $\tilde{\Gamma} := g\Gamma g^{-1}$ satisfies $\Gamma(2) \subset \tilde{\Gamma} \subset \mathrm{Sp}(4,\mathbb{Z})$. Evidently $\dim M_k(\Gamma) = \dim M_k(\tilde{\Gamma})$. The group $\mathrm{Sp}(4,\mathbb{Z})/\Gamma(2) \cong \mathrm{Sp}(4,\mathbb{F}_2) \cong S_6$ acts naturally on the space $M_k(\Gamma(2))$. The character of this action has been determined in [20, Theorem 2]. The space $M_k(\tilde{\Gamma})$ is the fixed space of this action under the subgroup $\tilde{\Gamma}/\Gamma(2)$. Hence, we can use formula (7) and [20, Theorem 2] to calculate $\dim M_k(\tilde{\Gamma})$. All we need to know is how many elements of each conjugacy class of S_6 are contained in $\tilde{\Gamma}/\Gamma(2)$. For our subgroups of interest, we have already summarized this information in Table 2.

Proposition 4.6. *With the possible exception of* $\Gamma = \Gamma_0'(4)$*, the generating series for* $\dim M_k(\Gamma)$ *given in Table B.2 and for* $\dim S_k(\Gamma)$ *given in Table B.3 hold.*

Proof. The dimensions of $M_k(K(2))$ and $S_k(K(2))$ are given in [13, Proposition 2]. (Note that the generating series for $\dim S_k(K(2))$ given in [13, Proposition 2] is missing the odd weights. The correct formula, given in Table B.3, can be derived from the original source [16, Theorem 4].) We may therefore assume that a conjugate of Γ lies between $\Gamma(2)$ and $\mathrm{Sp}(4,\mathbb{Z})$. For such Γ, the dimension of $M_k(\Gamma)$ can be derived from (7) and [20, Theorem 2], as explained above. Hence, we obtain the information in Table B.2.

The quantity $\sum_{k=6}^{\infty} \dim S_k(\Gamma)$ follows from Tables 8 and B.2. It remains to explain $\dim S_k(\Gamma)$ for $k \in \{1,2,3,4,5\}$. For odd k, we have $\dim M_k(\Gamma) = \dim S_k(\Gamma)$ by Theorem 4.4. We have $S_4(\Gamma(2)) = 0$ by [40, p. 882]. Hence also $S_2(\Gamma(2)) = 0$, and $S_2(\Gamma) = S_4(\Gamma) = 0$ for all Γ that contain a conjugate of $\Gamma(2)$. This concludes the proof. □

For illustration, we have listed $\dim M_k(\Gamma)$ and $\dim S_k(\Gamma)$ for weights $k \leqslant 20$ in Tables B.6 and B.7.

5 | COUNTING AUTOMORPHIC REPRESENTATIONS

This section contains our main results. In essence, we will use the dimension formulas proven or quoted so far in order to count the number of certain automorphic representations. Then, we will use these counts to derive more dimension formulas. It is essential to consider the packet structure of the discrete automorphic spectrum of $\mathrm{GSp}(4,\mathbb{A})$, which we recall first.

5.1 | Arthur packets

We recall from [2] that there are six types of automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$ in the discrete spectrum. We are only interested in representations with trivial central character, for which the description simplifies as follows.

- The general type **(G)**: These representations are characterized by the fact that they lift to cusp forms on $\mathrm{GL}(4,\mathbb{A})$ with trivial central character. They consist of finite, tempered, and stable packets. The latter means that if $\pi \cong \otimes\pi_v$ is such a representation, and if one of the local components π_w is part of a local L-packet $\{\pi_w, \pi'_w\}$, then $\pi' := \pi'_w \otimes (\otimes_{v\neq w}\pi_v)$ is also an automorphic representation in the discrete spectrum. All these representations are cuspidal.

- The Yoshida type **(Y)**: These packets are parametrized by pairs of distinct, cuspidal automorphic representations μ_1, μ_2 of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character. The packets are tempered and finite, but they are not stable. If $\pi \cong \otimes\pi_v$ is parametrized by $\mu_1 \cong \otimes\mu_{1,v}$ and $\mu_2 \cong \otimes\mu_{2,v}$, then the local L-parameter of π_v is the direct sum of the L-parameters of $\mu_{1,v}$ and $\mu_{2,v}$. Given μ_1 and μ_2, if the π_v are chosen from the local L-packets parametrized by $\mu_{1,v}$ and $\mu_{2,v}$, then $\pi = \otimes\pi_v$ belongs to the discrete spectrum if and only if the number of nongeneric π_v is even.
- The Saito–Kurokawa type **(P)**: These packets are parametrized by pairs (μ, σ), where μ is a cuspidal, automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character, and σ is a quadratic Hecke character. We will see in Lemma 5.4 below that only the case $\sigma = 1$ is relevant to us, in which case we say that π is a Saito–Kurokawa lift of μ. The packets are finite, nontempered, and not stable. Given μ, if the π_v are chosen from local Arthur packets (listed in [37, Table 2]) parametrized by μ_v, then $\pi = \otimes\pi_v$ belongs to the discrete spectrum if and only if the parity condition

$$\varepsilon(1/2, \mu) = (-1)^n \tag{42}$$

is satisfied, where n is the number of places where π_v is *not* the base point in the local Arthur packet.
- The Soudry type **(Q)**: These are parametrized by self-dual, cuspidal, automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ with nontrivial central character. The packets are nontempered, infinite, and stable. The local Arthur packets are given in [37, Table 3].
- The Howe–Piatetski–Shapiro type **(B)**: The packets are parametrized by pairs of distinct, quadratic Hecke characters. They are nontempered, infinite and unstable. The local Arthur packets are given in [37, Table 1].
- The finite type **(F)**: These are one-dimensional representations. They are not relevant for this work, because they are not cuspidal.

We next determine how these types intersect with the representations of interest to us. For the following definition, let Ω be one of the representation types I, IIa, IIb, …, XIb, Va*, sc appearing in Table 4.

Definition 5.1. Let k be a positive integer. Let $S_k(\Omega)$ be the set of cuspidal automorphic representations $\pi \cong \otimes_v\pi_v$ of $\mathrm{GSp}(4, \mathbb{A})$ with the following properties:

(i) π has trivial central character.
(ii) π_∞ is the lowest weight module with minimal K-type (k, k); it is a holomorphic discrete series representation if $k \geqslant 3$, a holomorphic limit of discrete series representation if $k = 2$, and a nontempered representation if $k = 1$. (It was denoted by $\mathcal{B}_{k,0}$ in [36, Section 3.5].)
(iii) π_p is unramified for each finite $p \neq 2$.
(iv) π_2 is an irreducible, admissible representation of $\mathrm{GSp}(4, \mathbb{Q}_2)$ of type Ω with nontrivial $\Gamma(\mathfrak{p})$-invariant vectors.

We note a peculiarity about representation types Vb and Vc. While these occupy two different rows in Table 4, the resulting sets of representations are identical, if the parameters in Table 4 are allowed to vary over all possibilities. Therefore, $S_k(\mathrm{Vb}) = S_k(\mathrm{Vc})$. In the following, we will work with Vb and ignore Vc.

Proposition 5.2. *Let Γ be one of the congruence subgroups of* $\mathrm{Sp}(4,\mathbb{Q})$ *in (1). Suppose that π is one of the cuspidal, automorphic representations of* $\mathrm{GSp}(4,\mathbb{A})$ *generated by the adelization of some nonzero $f \in S_k(\Gamma)$. Then $\pi \in S_k(\Omega)$ for some Ω.*

Proof. Recall from [36, Section 4.2] (among other places) that the adelization Φ of $f \in S_k(\Gamma)$ is the unique function $G(\mathbb{A}) \to \mathbb{C}$, which is left invariant under $G(\mathbb{Q})$, invariant under the center of $G(\mathbb{A})$, right invariant under

$$C_2 \times \prod_{\substack{p<\infty \\ p\neq 2}} G(\mathbb{Z}_p), \tag{43}$$

and satisfies

$$\Phi(g) = (f|_k g)(\begin{bmatrix} i & \\ & i \end{bmatrix}) \quad \text{for all } g \in \mathrm{Sp}(4,\mathbb{R}). \tag{44}$$

Here, C_2 is the congruence subgroup of $G(\mathbb{Q}_2)$ analogous to Γ, or more precisely, the closure of Γ in $\mathrm{Sp}(4,\mathbb{Q}_2)$ times the group of "multiplier matrices" $\mathrm{diag}(1,1,x,x)$ with $x \in \mathbb{Z}_2^\times$.

If $\pi \cong \otimes\pi_v$ is one of the irreducible components of the representation generated by Φ under right translation, then it is clear that each π_p for primes $p \neq 2$ is spherical, and that π_2 contains nonzero C_2-invariant vectors. By Theorem 3.1 (ii), the representation π_2 contains nonzero $\Gamma(\mathfrak{p})$-invariant vectors. Finally, it follows from the holomorphy of f that π_∞ is a lowest weight representation minimal K-type (k,k); see [3]. Hence, $\pi \in S_k(\Omega)$ for some Ω. □

The point of Proposition 5.2 is that if Γ is one of the congruence subgroups in (1), then no cuspidal, automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$ besides those in $S_k(\Omega)$, where Ω runs through the types occurring in Table 4, will contribute to $S_k(\Gamma)$. Of course, the $S_k(\Omega)$ contribute to $S_k(\Gamma')$ for many other congruence subgroups Γ' (e.g., subgroups or conjugates of any of the Γ's in (1)).

Let $S_k^{(\mathbf{G})}(\Omega)$ be the subset of $\pi \in S_k(\Omega)$ that are of type **(G)**, and similarly for the other Arthur types. Let $s_k(\Omega)$ be the cardinality of $S_k(\Omega)$, and $s_k^{(*)}(\Omega)$ be the cardinality of $S_k^{(*)}(\Omega)$. Evidently,

$$S_k(\Omega) = S_k^{(\mathbf{G})}(\Omega) \sqcup S_k^{(\mathbf{Y})}(\Omega) \sqcup S_k^{(\mathbf{P})}(\Omega) \sqcup S_k^{(\mathbf{Q})}(\Omega) \sqcup S_k^{(\mathbf{B})}(\Omega), \tag{45}$$

so that

$$s_k(\Omega) = s_k^{(\mathbf{G})}(\Omega) + s_k^{(\mathbf{Y})}(\Omega) + s_k^{(\mathbf{P})}(\Omega) + s_k^{(\mathbf{Q})}(\Omega) + s_k^{(\mathbf{B})}(\Omega). \tag{46}$$

It follows from Proposition 5.2 that

$$S_k(\Gamma) = S_k^{(\mathbf{G})}(\Gamma) \oplus S_k^{(\mathbf{Y})}(\Gamma) \oplus S_k^{(\mathbf{P})}(\Gamma) \oplus S_k^{(\mathbf{Q})}(\Gamma) \oplus S_k^{(\mathbf{B})}(\Gamma) \tag{47}$$

for any of the congruence subgroups Γ in (1), the obvious notation being that elements of $S_k^{(*)}(\Omega)$ (for any possible Ω) give rise to elements of $S_k^{(*)}(\Gamma)$. Hence,

$$\dim S_k(\Gamma) = \dim S_k^{(\mathbf{G})}(\Gamma) + \dim S_k^{(\mathbf{Y})}(\Gamma) + \dim S_k^{(\mathbf{P})}(\Gamma) + \dim S_k^{(\mathbf{Q})}(\Gamma) + \dim S_k^{(\mathbf{B})}(\Gamma). \tag{48}$$

Proposition 5.3. *Let k be a positive integer. Then*

$$S_k^{(\mathbf{Y})}(\Omega) = S_k^{(\mathbf{Q})}(\Omega) = S_k^{(\mathbf{B})}(\Omega) = \emptyset \tag{49}$$

for any Ω, and hence,

$$S_k^{(\mathbf{Y})}(\Gamma) = S_k^{(\mathbf{Q})}(\Gamma) = S_k^{(\mathbf{B})}(\Gamma) = 0 \tag{50}$$

for any of the congruence subgroups Γ in (1).

Proof. For types **(Q)** or **(B)**, the proof is analogous to that of [30, Proposition 2.1]. If $\pi \cong \otimes\pi_v$ lies in an Arthur packet of type **(Q)** or **(B)**, then the characters parametrizing the packet are ramified at least at one prime p. A look at [37, Table 1, Table 3] shows that π_p is not among the representations listed in Table 4. (Recall that all the characters appearing in Table 4 are unramified.) Therefore, $\pi \notin S_k(\Omega)$ for any Ω.

Now consider a cuspidal, automorphic representation $\pi = \otimes\pi_v$ of type **(Y)**. Recall that the packet containing π is parametrized by two distinct, cuspidal automorphic representations $\mu_1 = \otimes\mu_{1,v}$ and $\mu_2 = \otimes\mu_{2,v}$ of $\mathrm{GL}(2,\mathbb{A})$ with trivial central character. Chasing through archimedean Langlands parameters, we see that in order for π_∞ to be a lowest weight representation of weight k, the only possibility, up to order, is that $\mu_{1,\infty}$ is a discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ of lowest weight $2k-2$, and $\mu_{2,\infty}$ is a discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ of lowest weight 2. Hence, μ_1 corresponds to a newform $f_1 \in S_{2k-2}(\Gamma_0^{(1)}(N_1))$ and μ_2 corresponds to a newform $f_2 \in S_2(\Gamma_0^{(1)}(N_2))$ for some levels N_1, N_2. If we want π to be in $S_k(\Omega)$ for some Ω, then N_1 and N_2 both have to be powers of 2. Now $S_2(\Gamma_0^{(1)}(4)) = 0$, so that we would need $N_2 = 2^n$ for some $n \geqslant 3$. But then the local component $\mu_{2,2}$, whose L-parameter is a direct summand of the L-parameter of π_2, is such that π_2 is not among the representations listed in Table 4; see [31, Equation (16)] for the possible local Yoshida packets. It follows that π cannot be in $S_k(\Omega)$ for any Ω.

Note that (50) follows from (49) in view of Proposition 5.2. □

As a consequence of Proposition 5.3,

$$S_k(\Omega) = S_k^{(\mathbf{G})}(\Omega) \sqcup S_k^{(\mathbf{P})}(\Omega), \tag{51}$$

so that $s_k(\Omega) = s_k^{(\mathbf{G})}(\Omega) + s_k^{(\mathbf{P})}(\Omega)$, and

$$S_k(\Gamma) = S_k^{(\mathbf{G})}(\Gamma) \oplus S_k^{(\mathbf{P})}(\Gamma), \tag{52}$$

so that $\dim S_k(\Gamma) = \dim S_k^{(\mathbf{G})}(\Gamma) + \dim S_k^{(\mathbf{P})}(\Gamma)$. In Section 5.3, we will determine the numbers $s_k^{(\mathbf{P})}(\Omega)$.

The sets $S_k^{(\mathbf{G})}(\Omega)$ are empty for certain Ω, because the local Arthur packets (which are L-packets in this case) must contain a tempered element. The **(G)** column in Table 4 indicates which Ω can occur in packets of type **(G)**. Similarly, the sets $S_k^{(\mathbf{P})}(\Omega)$ are empty for certain Ω, because the local Arthur packets can only contain the representations listed in [37, Table 2]. The **(P)** column in Table 4 indicates which Ω can occur in packets of type **(P)**. We see that the only representations that can occur in packets of both Arthur types **(G)** and **(P)** or those of type VIb and Va*.

Since Arthur packets of type **(G)** are stable, one can switch within local L-packets and still retain the automorphic property. Most representations in Table 4 constitute singleton L-packets, except {Va, Va*}, {VIa, VIb}, and {VIIIa, VIIIb}, which constitute two-element L-packets (XIa is also part of an L-packet {XIa, XIa*}, but XIa* does not appear in Table 4).

Hence, $s_k^{(\mathrm{G})}(\mathrm{Va}) = s_k^{(\mathrm{G})}(\mathrm{Va}^*)$, $s_k^{(\mathrm{G})}(\mathrm{VIa}) = s_k^{(\mathrm{G})}(\mathrm{VIb})$, $s_k^{(\mathrm{G})}(\mathrm{VIIIa}) = s_k^{(\mathrm{G})}(\mathrm{VIIIb})$ and we denote these common numbers as follows:

$$s_k^{(\mathrm{G})}(\mathrm{Va/a^*}) := s_k^{(\mathrm{G})}(\mathrm{Va}) = s_k^{(\mathrm{G})}(\mathrm{Va}^*), \tag{53}$$

$$s_k^{(\mathrm{G})}(\mathrm{VIa/b}) := s_k^{(\mathrm{G})}(\mathrm{VIa}) = s_k^{(\mathrm{G})}(\mathrm{VIb}), \tag{54}$$

$$s_k^{(\mathrm{G})}(\mathrm{VIIIa/b}) := s_k^{(\mathrm{G})}(\mathrm{VIIIa}) = s_k^{(\mathrm{G})}(\mathrm{VIIIb}). \tag{55}$$

We observe from Table 5 that for each of the congruence subgroups H in this table, the dimension of the space of H-invariant vectors in a type IIIa (resp. VII) representation equals the sum of the dimensions of the spaces of H-invariant vectors for the L-packet VIa/b (resp. VIIIa/b). (The reason is that IIIa is a parabolically induced representation $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$ for an unramified, nontrivial character χ, and VIa/b are the two constituents of the same induced representation when χ is trivial. Similarly, VII is $\chi \rtimes \pi$ for an unramified, nontrivial χ and VIIIa/b are the two constituents of the same induced representation with trivial χ.) Since our methods cannot determine the numbers $s_k^{(\mathrm{G})}(\mathrm{IIIa})$ and $s_k^{(\mathrm{G})}(\mathrm{VIa/b})$ (resp. $s_k^{(\mathrm{G})}(\mathrm{VII})$ and $s_k^{(\mathrm{G})}(\mathrm{VIIIa/b})$) separately, we consider

$$s_k^{(\mathrm{G})}(\mathrm{IIIa + VIa/b}) := s_k^{(\mathrm{G})}(\mathrm{IIIa}) + s_k^{(\mathrm{G})}(\mathrm{VIa/b}), \tag{56}$$

$$s_k^{(\mathrm{G})}(\mathrm{VII + VIIIa/b}) := s_k^{(\mathrm{G})}(\mathrm{VII}) + s_k^{(\mathrm{G})}(\mathrm{VIIIa/b}). \tag{57}$$

5.2 | Siegel modular forms and representations in $S_k(\Omega)$

Consider $\pi \cong \bigotimes_{p\leqslant\infty} \pi_p \in S_k(\Omega)$. Recall that π_2 is an irreducible, admissible representation of $\mathrm{PGSp}(4, \mathbb{Q}_2)$ of type Ω with nonzero hyperspecial parahoric restriction $r_K(\pi_2)$. Let C be one of the compact open subgroups in Table 4, and let Γ be the corresponding congruence subgroup of $\mathrm{Sp}(4, \mathbb{Q})$. More precisely,

$$\Gamma = \mathrm{Sp}(4, \mathbb{Q}) \cap \left(C \times \prod_{\substack{p<\infty \\ p\neq 2}} \mathrm{GSp}(4, \mathbb{Z}_p) \right). \tag{58}$$

Then every eigenform (for the Hecke operators at all odd primes) $f \in S_k(\Gamma)$ arises from a vector in π_2^C, for some $\pi \in S_k(\Omega)$, by a procedure similar to the one explained in [30, Section 2.1]. Thus, we obtain the formula

$$\dim S_k(\Gamma) = \sum_{\Omega} \sum_{\pi \in S_k(\Omega)} \dim \pi_2^C = \sum_{\Omega} s_k(\Omega) d_{C,\Omega}, \tag{59}$$

where $d_{C,\Omega}$ is the common dimension of the space of C-fixed vectors of the representations π_2 of type Ω with $r_K(\pi_2) \neq 0$. The $d_{C,\Omega}$ are the numbers listed in Table 5. Hence, Equations (59) for all

Γ and all Ω are equivalent to the matrix equation

$$
\begin{bmatrix} \dim S_k(\Gamma(2)) \\ \dim S_k(\mathrm{Sp}(4,\mathbb{Z})) \\ \dim S_k(K(2)) \\ \dim S_k(K(4)) \\ \dim S_k(\Gamma_0(2)) \\ \dim S_k(\Gamma_0(4)) \\ \dim S_k(\Gamma_0^*(4)) \\ \dim S_k(\Gamma_0'(2)) \\ \dim S_k(\Gamma_0'(4)) \\ \dim S_k(M(2)) \\ \dim S_k(B(2)) \end{bmatrix}
=
\begin{bmatrix}
45 & 30 & 15 & 30 & 16 & 21 & 9 & 25 & 5 & 5 & 15 & 10 & 5 & 10 & 15 & 10 & 5 & 1 & 9 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
4 & 1 & 3 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 5 & 7 & 8 & 2 & 2 & 3 & 5 & 3 & 0 & 4 & 3 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\
15 & 8 & 7 & 10 & 4 & 5 & 3 & 7 & 3 & 1 & 5 & 4 & 1 & 4 & 7 & 4 & 3 & 1 & 3 \\
4 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 7 & 4 & 5 & 2 & 5 & 2 & 5 & 0 & 2 & 2 & 2 & 0 & 1 & 3 & 2 & 1 & 0 & 1 \\
8 & 5 & 3 & 3 & 1 & 3 & 2 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\
8 & 4 & 4 & 4 & 1 & 2 & 2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} s_k(\mathrm{I}) \\ s_k(\mathrm{IIa}) \\ s_k(\mathrm{IIb}) \\ s_k(\mathrm{IIIa}) \\ s_k(\mathrm{IVa}) \\ s_k(\mathrm{Va}) \\ s_k(\mathrm{Vb}) \\ s_k(\mathrm{VIa}) \\ s_k(\mathrm{VIb}) \\ s_k(\mathrm{VIc}) \\ s_k(\mathrm{VII}) \\ s_k(\mathrm{VIIIa}) \\ s_k(\mathrm{VIIIb}) \\ s_k(\mathrm{IXa}) \\ s_k(\mathrm{X}) \\ s_k(\mathrm{XIa}) \\ s_k(\mathrm{XIb}) \\ s_k(\mathrm{Va}^*) \\ s_k(\mathrm{sc}(16)) \end{bmatrix}. \tag{60}
$$

Here, we have omitted those Ω that do not occur in packets of type **(G)** or **(P)**, because for these $s_k(\Omega) = 0$ by Proposition 5.3. Note also that the class of representations of type Vb is the same as the class of representations of type Vc, since the parameter σ in Table 4 runs through all possibilities. We therefore include only Vb in (60).

The identity (60) still holds if we put a **(G)** or a **(P)** on all the $S_k(\Gamma)$ and all the $s_k(\Omega)$; this is the definition of the spaces $S_k^{(\mathbf{G})}(\Gamma)$ and $S_k^{(\mathbf{P})}(\Gamma)$. More of the $s_k^{(*)}(\Omega)$ will then be zero; see Table 4. We will utilize the **(P)** version of (60) in the proof of Corollary 5.6, and the **(G)** version in the proof of Theorem 5.8. More precisely, we will proceed as follows.

- Exploiting the fact that packets of type **(P)** are parametrized by cuspidal, automorphic representations of $\mathrm{GL}(2, \mathbb{A})$, the numbers $s_k^{(\mathbf{P})}(\Omega)$ can be determined for all Ω from dimension formulas for elliptic modular forms. (Theorem 5.5)
- We then use the **(P)** version of (60) to calculate $\dim S_k^{(\mathbf{P})}(\Gamma)$ for all Γ (Corollary 5.6).
- Since we already determined $\dim S_k(\Gamma)$ for all Γ except $\Gamma_0'(4)$, we can calculate $\dim S_k^{(\mathbf{G})}(\Gamma)$ for all Γ except $\Gamma_0'(4)$. (Proposition 5.7)
- Then we use the **(G)** version of (60), with the row for $\Gamma_0'(4)$ omitted, to determine the $s_k^{(\mathbf{G})}(\Omega)$. Here, it is necessary to combine some types Ω, which cannot be distinguished by their fixed vector dimensions; see (56) and (57). This step reduces the number of unknowns to 10, the same as the number of equations. (Theorem 5.8)
- Next, we use the $\Gamma_0'(4)$-row of the **(G)** version of (60) to determine $\dim S_k^{(\mathbf{G})}(\Gamma_0'(4))$. Since we already have $\dim S_k^{(\mathbf{P})}(\Gamma_0'(4))$, this gives us $\dim S_k(\Gamma_0'(4))$. (Corollary 5.9)

Finally, we will be able to fill in the row for $\dim M_k(\Gamma_0'(4))$ in Table B.2, using the codimension formula for $k \geqslant 6$ given in Table 8, and the low weight results from Appendix A.

TABLE 9 Some spaces of elliptic cusp forms and their Saito–Kurokawa lifts.

Space	k	ε_∞	ε_2	μ_2	S	π_2
$S_{2k-2}(\mathrm{SL}(2,\mathbb{Z}))$	Even	−1	1	Spherical	$\{\infty\}$	IIb
	Odd	1	1	Spherical	No lifting	
$S^{+,\mathrm{new}}_{2k-2}(\Gamma_0^{(1)}(2))$	Even	−1	−1	$\mathrm{St}_{\mathrm{GL}(2)}$	$\{\infty, 2\}$	VIb
	Odd	1	1	$\xi\mathrm{St}_{\mathrm{GL}(2)}$	$\{\infty, 2\}$	Va*
$S^{-,\mathrm{new}}_{2k-2}(\Gamma_0^{(1)}(2))$	Even	−1	1	$\xi\mathrm{St}_{\mathrm{GL}(2)}$	$\{\infty\}$	Vb
	Odd	1	−1	$\mathrm{St}_{\mathrm{GL}(2)}$	$\{\infty\}$	VIc
$S^{+,\mathrm{new}}_{2k-2}(\Gamma_0^{(1)}(4))$	Even	−1	−1	τ_2	$\{\infty, 2\}$	XIa*
	Odd	1	1	No possible μ_2		
$S^{-,\mathrm{new}}_{2k-2}(\Gamma_0^{(1)}(4))$	Even	−1	1	No possible μ_2		
	Odd	1	−1	τ_2	$\{\infty\}$	XIb

5.3 | Saito–Kurokawa type

Recall from Section 5.1 that Arthur packets of type **(P)** are parametrized by pairs (μ, σ), where μ is a cuspidal, automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with trivial central character, and σ is a quadratic Hecke character.

Lemma 5.4. *Suppose that the cuspidal, automorphic representation π lies in a packet of type **(P)**, parametrized by the pair (μ, σ), where μ is a cuspidal, automorphic representation of* $\mathrm{GL}(2,\mathbb{A})$ *with trivial central character, and σ is a quadratic Hecke character. Suppose that also $\pi \in S_k(\Omega)$ for some Ω. Then σ is trivial.*

Proof. We write $\pi = \otimes\pi_v$ and $\sigma = \otimes\sigma_v$. The local representation π_v occurs in [37, Table 2], for any place v. Since π_p is spherical for $p \geqslant 3$, we see from [37, Table 2] that σ_p is unramified. Since π_2 occurs in Table 4, inspecting [37, Table 2] shows that σ_2 is also unramified. Hence, the character σ, being unramified everywhere, must be trivial. □

If π lies in a packet of type **(P)**, parametrized by the pair (μ, σ) with trivial σ as in the lemma, then we say that "π is a Saito–Kurokawa lift of μ." Note that a given μ may have multiple Saito–Kurokawa lifts, depending on the size of the Arthur packet. As the proof of the next result shows, those μ corresponding to eigenforms in $S_k^{\mathrm{new}}(\Gamma_0^{(1)}(N))$ with $N \in \{2, 4\}$ admit a unique *holomorphic* Saito–Kurokawa lift.

Theorem 5.5. *The generating series for the numbers $s_k^{\mathbf{(P)}}(\Omega)$ given in Table B.10 hold. If a representation type Ω is not listed in Table B.10, then $s_k^{\mathbf{(P)}}(\Omega) = 0$ for all k.*

Proof. Table 9 shows several spaces of elliptic modular newforms, and how an eigenform in one of these spaces Saito–Kurokawa lifts to $\mathrm{GSp}(4,\mathbb{A})$. The notation $S^{\pm,\mathrm{new}}_{2k-2}(\Gamma_0^{(1)}(N))$ indicates the subspace of $S^{\mathrm{new}}_{2k-2}(\Gamma_0^{(1)}(N))$ spanned by eigenforms with sign ± 1 in the functional equation of their L-function. If $\mu \cong \otimes\mu_v$ is the cuspidal, automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ corresponding to an eigenform in one of these spaces, then the sign in the functional equation coincides with the global ε-factor $\varepsilon(1/2, \mu) = \varepsilon_\infty\varepsilon_2$, where $\varepsilon_\infty := \varepsilon(1/2, \mu_\infty) = (-1)^{k-1}$ and $\varepsilon_2 := \varepsilon(1/2, \mu_2)$. In the μ_2 column of Table 9, the symbol ξ stands for the unique nontrivial, unramified, quadratic character of $\mathbb{Q}_2^\times$, and τ_2 denotes the unique irreducible, admissible representation of $\mathrm{GL}(2,\mathbb{Q}_2)$ with trivial central character and conductor exponent 2; it is a depth zero supercuspidal.

Now μ_v, or rather the pair $(\mu_v, 1_v)$, where 1_v is the trivial character of $\mathbb{Q}_v^\times$, determines a local Arthur packet consisting of one or two representations, for each place v. These local packets are explicitly given in [37, Table 2], and each one of them contains a "base point." The packet is a singleton if and only if μ_v is not a discrete series representation, in which case the unique representation in the packet is also the base point. Recall from (42) that in order for the global Saito–Kurokawa packet μ to contain the cuspidal, automorphic representation $\pi \cong \otimes\pi_v$ of $\mathrm{GSp}(4,\mathbb{A})$, the parity condition $\varepsilon(1/2,\mu) = (-1)^n$ has to be satisfied, where n is the number of places for which π_v is not the base point in the local Arthur packet. Since we want π to correspond to holomorphic Siegel modular forms, the set S of places where π_v is not the base point must include the archimedean place, the reason being that the nonbase point in the archimedean local packet is the holomorphic discrete series representation of $\mathrm{PGSp}(4,\mathbb{R})$ of lowest weight (k,k). Hence, the set S must be $\{\infty\}$ if $\varepsilon(1/2,\mu) = -1$ and must be $\{\infty, 2\}$ if $\varepsilon(1/2,\mu) = 1$. The final column of Table 9, which can be read off [37, Table 2], shows the type of π_2, the local component at $p = 2$ of the unique cuspidal, automorphic representation π in the global packet parametrized by μ which has the required discrete series representation at the archimedean place.

The upshot is that each newform in one of the spaces given in Table 9 gives rise to a unique "holomorphic" cuspidal representation of $\mathrm{PGSp}(4,\mathbb{A})$, the only exception being that eigenforms in $S_{2k-2}(\mathrm{SL}(2,\mathbb{Z}))$ for odd k cannot be lifted, because it is impossible to satisfy the parity condition. We can thus produce elements of $S_k^{(\mathbf{P})}(\Omega)$ for those types Ω listed in the last column of Table 9. Note that representations of type XIa* do not appear in Table 4, and hence, those Saito–Kurokawa lifts are not relevant for our purposes.

Conversely, suppose that $\pi \cong \otimes\pi_v$ is an element of $S_k^{(\mathbf{P})}(\Omega)$ for some Ω. Then, by Lemma 5.4, the Arthur packet containing π is parametrized by a cuspidal, automorphic representation $\mu \cong \otimes\mu_v$ and the trivial character σ. Looking at the archimedean parameters in [37, Table 2], we see that μ corresponds to a newform of weight $2k-2$. There can be no ramification outside 2, so that the level of this newform is a power of 2. In fact, the level must be 1, 2, or 4, since otherwise a look at the nonarchimedean packets in [37, Table 2] would show that the local component μ_2 would be such that the elements of the local Arthur packet at $p = 2$ would not appear in Table 4. Hence, π is a lift of a newform of one of the spaces appearing in Table 9.

This discussion shows that

$$s_k^{(\mathbf{P})}(\mathrm{IIb}) = \begin{cases} \dim S_{2k-2}(\mathrm{SL}(2,\mathbb{Z})) & \text{if k is even,} \\ 0 & \text{if k is odd,} \end{cases}$$

$$s_k^{(\mathbf{P})}(\mathrm{VIb}) = \begin{cases} \dim S_{2k-2}^{+,\mathrm{new}}(\Gamma_0^{(1)}(2)) & \text{if k is even,} \\ 0 & \text{if k is odd,} \end{cases}$$

$$s_k^{(\mathbf{P})}(\mathrm{Va}^*) = \begin{cases} 0 & \text{if k is even,} \\ \dim S_{2k-2}^{+,\mathrm{new}}(\Gamma_0^{(1)}(2)) & \text{if k is odd,} \end{cases}$$

$$s_k^{(\mathbf{P})}(\mathrm{Vb}) = \begin{cases} \dim S_{2k-2}^{-,\mathrm{new}}(\Gamma_0^{(1)}(2)) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

$$s_k^{(\mathbf{P})}(\mathrm{VIc}) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dim S_{2k-2}^{-,\mathrm{new}}(\Gamma_0^{(1)}(2)) & \text{if } k \text{ is odd,} \end{cases}$$

$$s_k^{(\mathbf{P})}(\mathrm{XIb}) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dim S_{2k-2}^{-,\mathrm{new}}(\Gamma_0^{(1)}(4)) & \text{if } k \text{ is odd.} \end{cases}$$

Now the asserted formulas follow from (39), (40), (41), the dimension formula for $S_k^{\pm,\mathrm{new}}(\Gamma_0^{(1)}(2))$ in [21, Theorem 2.2], and straightforward calculations.

If $\Omega \notin \{\mathrm{IIb}, \mathrm{Vb}, \mathrm{VIb}, \mathrm{VIc}, \mathrm{XIb}, \mathrm{Va}^*, \mathrm{XIa}^*\}$, then $s_k^{(\mathbf{P})}(\Omega) = 0$, because type Ω does not appear in local Arthur packets of type **(P)**; see [37, Table 2]. Furthermore, $s_k^{(\mathbf{P})}(\mathrm{XIa}^*) = 0$ because the hyperspecial parahoric restriction for representations of type XIa* is zero. □

We note that the cases of $s_k(\Omega)$ for $\Omega \in \{\mathrm{IIb}, \mathrm{Vb}, \mathrm{VIb}, \mathrm{VIc}\}$ can be found in [30, (3.6) and Section 3.2].

Corollary 5.6. *The dimension formulas for Saito–Kurokawa cusp forms given in Table B.4 hold.*

Proof. This is immediate from Theorem 5.5 and the following **(P)** version of (60).

$$\begin{bmatrix} \dim S_k^{(\mathbf{P})}(\Gamma(2)) \\ \dim S_k^{(\mathbf{P})}(\mathrm{Sp}(4,\mathbb{Z})) \\ \dim S_k^{(\mathbf{P})}(K(2)) \\ \dim S_k^{(\mathbf{P})}(K(4)) \\ \dim S_k^{(\mathbf{P})}(\Gamma_0(2)) \\ \dim S_k^{(\mathbf{P})}(\Gamma_0(4)) \\ \dim S_k^{(\mathbf{P})}(\Gamma_0^*(4)) \\ \dim S_k^{(\mathbf{P})}(\Gamma_0'(2)) \\ \dim S_k^{(\mathbf{P})}(\Gamma_0'(4)) \\ \dim S_k^{(\mathbf{P})}(M(4)) \\ \dim S_k^{(\mathbf{P})}(B(2)) \end{bmatrix} = \begin{bmatrix} 15 & 9 & 5 & 5 & 5 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 7 & 3 & 3 & 1 & 3 & 1 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 4 & 2 & 0 & 2 & 1 & 0 \\ 3 & 2 & 0 & 2 & 1 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_k^{(\mathbf{P})}(\mathrm{IIb}) \\ s_k^{(\mathbf{P})}(\mathrm{Vb}) \\ s_k^{(\mathbf{P})}(\mathrm{VIb}) \\ s_k^{(\mathbf{P})}(\mathrm{VIc}) \\ s_k^{(\mathbf{P})}(\mathrm{XIb}) \\ s_k^{(\mathbf{P})}(\mathrm{Va}^*) \end{bmatrix}. \tag{61}$$

□

For illustration, we have listed $\dim S_k^{(\mathbf{P})}(\Gamma)$ and $s_k^{(\mathbf{P})}(\Omega)$ for weights $k \leqslant 20$ in Tables B.8 and B.12.

5.4 | General type

In this section, we will determine the numbers $s_k^{(\mathbf{G})}(\Omega)$. As an application, we obtain dimension formulas for the congruence subgroup $\Gamma_0'(4)$.

Proposition 5.7. *With the possible exception of $\Gamma = \Gamma_0'(4)$, the generating series for $\dim S_k^{(\mathbf{G})}(\Gamma)$ given in Table B.5 hold.*

Proof. Recall from Proposition 4.7 that the formulas given in Table B.3 have been proven except for $\Gamma = \Gamma_0'(4)$. The formulas in Table B.4 have been proven for all Γ; see Corollary 5.6. In view of (52), all we have to do is subtract the formulas in Table B.4 from those in Table B.3. □

Theorem 5.8. *The generating series for the numbers* $s_k^{(\mathrm{G})}(\Omega)$ *given in Table B.11 hold. If a representation type* Ω *is not listed in Table B.11, then* $s_k^{(\mathrm{G})}(\Omega) = 0$ *for all* k.

Proof. The **(G)** version of (60), with appropriate columns combined and the row for $\Gamma_0'(4)$ omitted, is

$$\begin{bmatrix} \dim S_k^{(\mathrm{G})}(\Gamma(2)) \\ \dim S_k^{(\mathrm{G})}(\mathrm{Sp}(4,\mathbb{Z})) \\ \dim S_k^{(\mathrm{G})}(K(2)) \\ \dim S_k^{(\mathrm{G})}(K(4)) \\ \dim S_k^{(\mathrm{G})}(\Gamma_0(2)) \\ \dim S_k^{(\mathrm{G})}(\Gamma_0(4)) \\ \dim S_k^{(\mathrm{G})}(\Gamma_0^*(4)) \\ \dim S_k^{(\mathrm{G})}(\Gamma_0'(2)) \\ \dim S_k^{(\mathrm{G})}(M(4)) \\ \dim S_k^{(\mathrm{G})}(B(2)) \end{bmatrix} = \begin{bmatrix} 45 & 30 & 30 & 16 & 22 & 15 & 10 & 15 & 10 & 9 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 5 & 8 & 2 & 2 & 4 & 3 & 1 & 1 & 0 \\ 15 & 8 & 10 & 4 & 6 & 5 & 4 & 7 & 4 & 3 \\ 4 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 5 & 3 & 1 & 3 & 0 & 0 & 2 & 1 & 0 \\ 8 & 4 & 4 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_k^{(\mathrm{G})}(\mathrm{I}) \\ s_k^{(\mathrm{G})}(\mathrm{IIa}) \\ s_k^{(\mathrm{G})}(\mathrm{IIIa{+}VIa/b}) \\ s_k^{(\mathrm{G})}(\mathrm{IVa}) \\ s_k^{(\mathrm{G})}(\mathrm{Va/a^*}) \\ s_k^{(\mathrm{G})}(\mathrm{VII{+}VIIIa/b}) \\ s_k^{(\mathrm{G})}(\mathrm{IXa}) \\ s_k^{(\mathrm{G})}(\mathrm{X}) \\ s_k^{(\mathrm{G})}(\mathrm{XIa}) \\ s_k^{(\mathrm{G})}(\mathrm{sc}(16)) \end{bmatrix}. \tag{62}$$

The 10×10 matrix is invertible, so that we can solve for the $s_k^{(\mathrm{G})}(\Omega)$. □

Corollary 5.9. *For* $\Gamma = \Gamma_0'(4)$*, the results given in Tables B.2, B.3, and B.5 hold.*

Proof. The row for $\Gamma_0'(4)$ in the **(G)** version of (60) is

$$\dim S_k^{(\mathrm{G})}(\Gamma_0'(4)) = \begin{bmatrix} 11 & 7 & 5 & 2 & 5 & 2 & 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} s_k^{(\mathrm{G})}(\mathrm{I}) \\ s_k^{(\mathrm{G})}(\mathrm{IIa}) \\ s_k^{(\mathrm{G})}(\mathrm{IIIa{+}VIa/b}) \\ s_k^{(\mathrm{G})}(\mathrm{IVa}) \\ s_k^{(\mathrm{G})}(\mathrm{Va/a^*}) \\ s_k^{(\mathrm{G})}(\mathrm{VII{+}VIIIa/b}) \\ s_k^{(\mathrm{G})}(\mathrm{IXa}) \\ s_k^{(\mathrm{G})}(\mathrm{X}) \\ s_k^{(\mathrm{G})}(\mathrm{XIa}) \\ s_k^{(\mathrm{G})}(\mathrm{sc}(16)) \end{bmatrix}. \tag{63}$$

The numbers on the right-hand side are all known and given in Table B.11, allowing us to calculate $\dim S_k^{(\mathrm{G})}(\Gamma_0'(4))$. Since $\dim S_k^{(\mathrm{P})}(\Gamma_0'(4))$ is already known by Corollary 5.6, we obtain $\dim S_k(\Gamma_0'(4))$ by (52). We then obtain $\sum_{k=6}^{\infty} \dim M_k(\Gamma_0'(4))t^k$ using the codimensions from Table 8. Evidently, $\dim M_0(\Gamma_0'(4)) = 1$, and $M_k(\Gamma_0'(4)) = 0$ for $k \in \{1,3,5\}$ by Theorem 4.4. Finally, $\dim M_k(\Gamma_0'(4))$ for $k \in \{2,4\}$ are determined in Appendix A. □

For illustration, we have listed $\dim S_k^{(\mathrm{G})}(\Gamma)$ and $s_k^{(\mathrm{G})}(\Omega)$ for weights $k \leqslant 20$ in Tables B.9 and B.12.

APPENDIX A: MODULAR FORMS OF KLINGEN LEVEL 4 AND SMALL WEIGHT by Cris Poor and David S. Yuen

A.1 | Introduction and notation

This appendix proves $\dim M_4(\Gamma_0'(4)) = 4$ and $\dim M_2(\Gamma_0'(4)) = 0$. The proof proceeds by getting upper and lower bounds that agree. The proofs of the upper bounds rely on the known dimensions $\dim M_8(\Gamma_0'(4)) = 12$ and $\dim M_8(M(4)) = 8$. Proving the nontrivial lower bound relies on constructing Gritsenko lifts of linearly independent Jacobi–Eisenstein series.

Let $\mathrm{J}_{k,m}$ denote the space of Jacobi forms of weight k and index m on $\mathrm{SL}(2,\mathbb{Z})$, see [9] for definitions. Jacobi forms of index zero are identified with elliptic modular forms, $\mathrm{J}_{k,0} = M_k(\mathrm{SL}(2,\mathbb{Z}))$, and we will need the Eisenstein series $G_k \in M_k(\mathrm{SL}(2,\mathbb{Z}))$ for even $k \geqslant 4$,

$$G_k(\tau) = \frac{1}{2}\zeta(1-k) + \sum_{n\geqslant 1} \sigma_{k-1}(n) q^n,$$

for $\tau \in \mathcal{H}_1$ and $q = e(\tau) = e^{2\pi i \tau}$. For $z \in \mathbb{C}$ and $y = e(z)$, let

$$E_{k,1}(\tau, z) = \sum_{n,r\in\mathbb{Z}:\, n,4n-r^2\geqslant 0} \frac{H(k-1, 4n-r^2)}{H(k-1,0)} q^n y^r \in \mathrm{J}_{k,1}$$

be the Jacobi–Eisenstein series of even weight $k \geqslant 4$ and index one from Eichler–Zagier [9, p. 22]. Here, the Cohen numbers $H(r,n)$ for $n \geqslant 0, r \geqslant 2$ are directly computed as $H(r,0) = \zeta(1-2r)$; $H(r,n) = 0$ for $n \in \mathbb{N}$ such that $(-1)^r n \equiv 2,3 \bmod 4$; and, for $n \in \mathbb{N}$ such that $(-1)^r n \equiv 0,1 \bmod 4$, as

$$H(r,n) = L(1-r, \chi_D) \sum_{d|f} \mu(d)\chi_D(d) d^{r-1} \sigma_{2r-1}(f/d),$$

where D is the fundamental discriminant of $\mathbb{Q}(\sqrt{(-1)^r n})$, $(-1)^r n = Df^2$ for $f \in \mathbb{N}$, μ is the Möbius function, and $\chi_D : \mathbb{Z} \to \{-1,0,1\}$ is the Kronecker symbol. The L-function $L(s,\chi_D) = \sum_{n\in\mathbb{N}} \frac{\chi_D(n)}{n^s}$ is defined by analytic continuation, and its special values are given by twisted Bernoulli numbers B_{k,χ_D}

$$\sum_{a=1}^{|D|} \chi_D(a) \frac{te^{at}}{e^{|D|t}-1} = \sum_{k=0}^{\infty} B_{k,\chi_D} \frac{t^k}{k!} \quad ([1, \text{p. } 53]),$$

$$L(1-k, \chi_D) = -\frac{B_{k,\chi_D}}{k}, \text{ for k } \in \mathbb{N} \quad ([1, \text{p. } 152]).$$

For $\ell \in \mathbb{N}$, let $V_\ell : \mathrm{J}_{k,m} \to \mathrm{J}_{k,m\ell}$ and $U_\ell : \mathrm{J}_{k,m} \to \mathrm{J}_{k,m\ell^2}$ be the commuting family of index raising operators from [9, p. 41]. In particular, for $\phi \in \mathrm{J}_{k,m}$,

$$(\phi|V_2)(\tau,z) = 2^{k-1}\phi(2\tau, 2z) + \frac{1}{2}\Big(\phi\Big(\frac{\tau+1}{2}, z\Big) + \phi\Big(\frac{\tau}{2}, z\Big)\Big),$$

$$(\phi|U_2)(\tau,z) = 2^k\, \phi(\tau, 2z).$$

The lower bound $\dim M_4(\Gamma_0'(4)) \geqslant 4$ will by proven by constructing four linearly independent Gritsenko lifts of Jacobi forms. Enough information has already been presented to define the

Jacobi forms that we will need and to compute their initial Fourier expansions. We have normalized their constant terms to be 1.

$$\varphi_0 = E_{4,1} = 1 + \left(y^2 + 56y + 126 + 56y^{-1} + y^{-2}\right)q + \left(126y^2 + 576y + 756 + 576y^{-1} + 126y^{-2}\right)q^2 + \cdots$$

$$\varphi_1 = \frac{1}{16}E_{4,1}|U_2 = 1 + \left(y^4 + 56y^2 + 126 + 56y^{-2} + y^{-4}\right)q + \left(126y^4 + 576y^2 + 756 + 576y^{-2} + 126y^{-4}\right)q^2 + \cdots$$

$$\varphi_2 = \frac{1}{9}E_{4,1}|V_2 = 1 + \left(14y^2 + 64y + 84 + 64y^{-1} + 14y^{-2}\right)q + \left(y^4 + 64y^3 + 280y^2 + 448y + 574 + 448y^{-1} + \cdots + y^{-4}\right)q^2 + \cdots$$

$$\varphi_3 = \frac{1}{81}E_{4,1}|V_2|V_2 = 1 + \left(\frac{1}{9}y^4 + \frac{64}{9}y^3 + \frac{280}{9}y^2 + \frac{448}{9}y + \frac{574}{9} + \frac{448}{9}y^{-1} + \cdots + \frac{1}{9}y^{-4}\right)q + \left(\frac{64}{9}y^5 + \frac{686}{9}y^4 + \frac{448}{3}y^3 + 320y^2 + \frac{896}{3}y + \frac{1372}{3} + \frac{896}{3}y^{-1} + \cdots + \frac{64}{9}y^{-5}\right)q^2 + \cdots$$

A.2 | Proofs

Let $\mathrm{J}^{\mathrm{mero}}_{k,m}$ be the $\mathbb{C}$-vector space of meromorphic functions on $\mathcal{H}_1 \times \mathbb{C}$ spanned by a/b such that $a \in \mathrm{J}_{k_1,m_1}$, $b \in \mathrm{J}_{k_2,m_2} \setminus \{0\}$, and $k_1 - k_2 = k$, $m_1 - m_2 = m$; define $M^{\mathrm{mero}}_k(\Gamma)$ similarly.

Lemma A.1. *A basis for* $\mathrm{J}_{4,0}$ *is* G_4. *A basis for* $\mathrm{J}_{4,1}$ *is* φ_0. *A basis for* $\mathrm{J}_{4,2}$ *is* φ_2. *A basis for* $\mathrm{J}_{4,4}$ *is* φ_1, φ_3. *We have* $\varphi_2^2/G_4 \in \mathrm{J}^{\mathrm{mero}}_{4,4} \setminus \mathrm{J}_{4,4}$.

Proof. From [9, pp. 103–105], we have $\dim \mathrm{J}_{4,m} = 1, 1, 2$ for $m = 1, 2, 4$. The Fourier expansions show the linear independence. Assume $\varphi_2^2/G_4 \in \mathrm{J}_{4,4}$, then its Fourier expansion would be given by the quotient of the series for φ_2^2 by $G_4 = \frac{1}{240} + q + 9q^2 + 28q^3 + \cdots$. The formal series for $\varphi_2^2/(240G_4)$ begins:

$$1+\left(28y^2 + 128y - 72 + 128y^{-1} + 28y^{-2}\right)q + (198y^4 + 1920y^3 + 288y^2 - 17280y + 31908 - 17280y^{-1} + \cdots + 198y^{-4})q^2 + \cdots .$$

However, by the Fourier expansions, this is not in the span of φ_1 and φ_3. □

Remark. Let $\vartheta(\tau, z) = \sum_{n\in\mathbb{Z}}(-1)^n q^{(2n+1)^2/8} y^{(2n+1)/2}$ define the odd Jacobi theta function; then $\vartheta^8 \in \mathrm{J}_{4,4}$, and we may check $\vartheta^8 = \frac{9}{8}(\varphi_1 - \varphi_3)$.

Each Siegel modular form $f \in M_k(\Gamma_0'(N))$ has a unique Fourier–Jacobi expansion

$$f\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \sum_{m=0}^{\infty} \phi_m(\tau, z)e(m\omega) \tag{A.1}$$

for which $\phi_m \in \mathrm{J}_{k,m}$; to see this, use $\Gamma_0'(N) \cap Q = \mathrm{Sp}(4, \mathbb{Z}) \cap Q$ and [9, Theorem 6.1]. Setting $\xi = e(\omega)$, we write this more briefly as $f = \sum_{m=0}^{\infty} \phi_m \xi^m$. The following theorem [10] will allow us to obtain paramodular forms as Gritsenko lifts of the Jacobi forms φ_j. In this theorem, $c(0,0;\phi)$ is the constant term of the Fourier expansion of the Jacobi form ϕ.

Theorem A.2. *Let $k, N \in \mathbb{N}$. For $\phi \in \mathrm{J}_{k,N}$, we have $c(0,0;\phi) = 0$ unless $k \geqslant 4$ is even. An injective linear map* $\mathrm{Grit} : \mathrm{J}_{k,N} \to M_k(K(N))$ *is defined by*

$$\mathrm{Grit}(\phi) = c(0,0;\phi)G_k + \sum_{m \in \mathbb{N}} \phi | V_m \, \xi^{Nm}.$$

Definition A.3. Let $g_j = \mathrm{Grit}(\varphi_j) \in M_4(K(N_j))$ for $N_0 = 1, N_1 = 4, N_2 = 2$, and $N_3 = 4$.

Lemma A.4. *The elements $g_0, g_1, g_2, g_3 \in M_4(\Gamma_0'(4))$ are linearly independent. Each element of* $\mathrm{Span}(g_0, g_1, g_2, g_3)$ *is determined by its Fourier–Jacobi coefficients through index 4. The elements $g_1, g_2, g_3 \in M_4(M(4))$ are linearly independent.*

Proof. Since $\Gamma_0'(4) \subseteq K(N)$ for $N = 1, 2, 4$, we have $g_0, g_1, g_2, g_3 \in M_4(\Gamma_0'(4))$. By Theorem A.2, their Fourier–Jacobi expansions through index four are

$$\begin{aligned} g_0 &= G_4 + \varphi_0 \xi + 9\varphi_2 \xi^2 + \varphi_0 | V_3 \xi^3 + \varphi_0 | V_4 \xi^4 + \cdots \\ g_1 &= G_4 + 0\xi + 0\xi^2 + 0\xi^3 + \varphi_1 \xi^4 + \cdots \\ g_2 &= G_4 + 0\xi + \varphi_2 \xi^2 + 0\xi^3 + 9\varphi_3 \xi^4 + \cdots \\ g_3 &= G_4 + 0\xi + 0\xi^2 + 0\xi^3 + \varphi_3 \xi^4 + \cdots . \end{aligned} \tag{A.2}$$

We first show that the subspace of $\mathrm{Span}(g_0, g_1, g_2, g_3)$ whose Fourier–Jacobi coefficients of index 0, 1, and 2 vanish is the one-dimensional space spanned by $g_1 - g_3$; this subspace defined by vanishing conditions is well defined because Fourier–Jacobi expansions are unique. Let $f = \sum_j c_j g_j \in \mathrm{Span}(g_0, g_1, g_2, g_3)$ for $c_j \in \mathbb{C}$. The vanishing of the Jacobi coefficient of index zero gives $c_0 + c_1 + c_2 + c_3 = 0$; of index 1, $c_0 = 0$; and of index 2, $9c_0 + c_2 = 0$. Hence, we have $c_0 = c_2 = 0$, $c_3 = -c_1$, and $f = \sum_j c_j g_j = c_1(g_1 - g_3) = c_1(\varphi_1 - \varphi_3)\xi^4 + \cdots$. Since φ_1 and φ_3 are linearly independent by Lemma A.1, if we additionally demand that the fourth Jacobi coefficient $c_1(\varphi_1 - \varphi_3)$ vanishes, then $c_1 = 0$ and $f = 0$. Hence, $\mathrm{Span}(g_0, g_1, g_2, g_3)$ is determined by the Fourier–Jacobi coefficients through index 4.

On the other hand, $f = 0$ implies that the Fourier–Jacobi expansion of f vanishes though index 4, so that $c_0 + c_1 + c_2 + c_3 = 0$, $c_0 = 0$, $9c_0 + c_2 = 0$, and $c_1 = 0$, implying $c_0 = c_1 = c_2 = c_3 = 0$. Thus, the g_j are linearly independent. Since $M(4) \subseteq K(N)$ for $N = 2, 4$, we have $g_1, g_2, g_3 \in M_4(M(4))$. We have already seen their linear independence. □

Remark. In terms of the global paramodular newform theory of [26], the level one Eisenstein series g_0 is a newform for $K(1) = \mathrm{Sp}(4, \mathbb{Z})$, and g_2 is the oldform above g_0 in $K(2)$, and g_1, g_3 are the oldforms above g_0 in $K(4)$.

Lemma A.5. *Products from $M_4(M(4))$ span a six-dimensional space in $M_8(M(4))$.*

Proof. Since $\dim M_4(M(4)) = 3$, Lemma A.4 shows that g_1, g_2, g_3 is a basis. Thus, we need to show that the six products g_1^2, $g_1 g_2$, $g_1 g_3$, g_2^2, $g_2 g_3$, and g_3^2 are linearly independent in $M_8(M(4))$.

Multiplying the expansions (A.2), their Fourier–Jacobi expansions through index 4 are

$$\begin{aligned} g_1^2 &= G_4^2 + 2G_4\varphi_1\xi^4 + \cdots, \\ g_1g_2 &= G_4^2 + G_4\varphi_2\xi^2 + (9G_4\varphi_3 + G_4\varphi_1)\xi^4 + \cdots, \\ g_1g_3 &= G_4^2 + (G_4\varphi_3 + G_4\varphi_1)\xi^4 + \cdots, \\ g_2^2 &= G_4^2 + 2G_4\varphi_2\xi^2 + (18G_4\varphi_3 + \varphi_2^2)\xi^4 + \cdots, \\ g_2g_3 &= G_4^2 + G_4\varphi_2\xi^2 + 10G_4\varphi_3\xi^4 + \cdots, \\ g_3^2 &= G_4^2 + 2G_4\varphi_3\xi^4 + \cdots. \end{aligned}$$

To prove linear independence, let $\sum_{1\leqslant i\leqslant j\leqslant 3} c_{ij}g_ig_j = 0$ for some $c_{ij} \in \mathbb{C}$. Using the linear independence of φ_1, φ_3, and φ_2^2/G_4 from Lemma A.1, the vanishing of the Fourier–Jacobi coefficients of indices 0, 2, and 4 implies $c_{11} = c_{23} + c_{33}$, $c_{12} = -c_{23}$, $c_{13} = -c_{23} - 2c_{33}$, and $c_{22} = 0$, and

$$\sum_{1\leqslant i\leqslant j\leqslant 3} c_{ij}g_ig_j = (g_1 - g_3)\big(c_{23}(g_1 - g_2) + c_{33}(g_1 - g_3)\big).$$

By Lemma A.4, $g_1 - g_2$ and $g_1 - g_3$ are linearly independent, so we obtain $c_{23} = c_{33} = 0$. □

The following proof is the most interesting. It leverages dimensions of spaces of higher weight to deduce the dimension of a space of lower weight.

Proposition A.6. *We have* $\dim M_4(\Gamma_0'(4)) = 4$.

Proof. We know $g_0, g_1, g_2, g_3 \in M_4(\Gamma_0'(4))$ are linearly independent by Lemma A.4. Suppose by way of contradiction that an $f \in M_4(\Gamma_0'(4))$ exists with f, g_0, g_1, g_2, g_3 linearly independent. Using the known dimension $\dim M_8(M(4)) = 8$, let $h_1, \ldots, h_8$ be a basis of $M_8(M(4))$. The 14 elements $fg_1, fg_2, fg_3, g_0g_1, g_0g_2, g_0g_3, h_1, \ldots, h_8$, are in $M_8(\Gamma_0'(4))$, which is known to be 12-dimensional, so that there must be at least two linearly independent relations

$$fg_4 + g_0g_5 + h_0 = 0; \quad fg_4' + g_0g_5' + h_0' = 0, \tag{A.3}$$

for some $g_4, g_5, g_4', g_5' \in \operatorname{Span}(g_1, g_2, g_3) = M_4(M(4))$ and some $h_0, h_0' \in M_8(M(4))$.

We will show that g_4 is not identically zero. If g_4 and g_5 were both trivial, then h_0 would also be trivial, contradicting that the relations (A.3) have rank two. If $g_4 \equiv 0$ and $g_5 \not\equiv 0$, then

$$g_0 = -\frac{h_0}{g_5} \in M_4^{\text{mero}}(M(4)) \cap M_4\big(\Gamma_0'(4)\big) = M_4(M(4)) = \operatorname{Span}(g_1, g_2, g_3),$$

contradicting the linear independence of g_0, g_1, g_2, g_3. We will refer to this argument, that a meromorphic form for $M(4)$ that is also a holomorphic form for $\Gamma_0'(4)$ must be a holomorphic form for $M(4)$, as the integral closure argument. The principle is general: For congruence subgroups $\Gamma_1 \subseteq \Gamma_2$ of $\mathrm{Sp}(2n, \mathbb{Q})$, we have $M_k^{\text{mero}}(\Gamma_2) \cap M_k(\Gamma_1) = M_k(\Gamma_2)$. To prove this, take $f \in M_k^{\text{mero}}(\Gamma_2) \cap M_k(\Gamma_1)$ and $\gamma \in \Gamma_2$. We have $f|\gamma = f$ on some dense open subset of $\mathcal{H}_n$. Since $f \in M_k(\Gamma_1)$ is holomorphic, we have $f|\gamma = f$ on $\mathcal{H}_n$ and $f \in M_k(\Gamma_2)$. Similarly to g_4, we have g_4' not identically zero. Since f, g_1, g_2, g_3 are linearly independent, the integral closure argument also shows that g_5, g_5' are not identically zero.

We will show that $g_4g_5' - g_4'g_5$ is identically zero. If not then

$$\begin{pmatrix} f \\ g_0 \end{pmatrix} = \frac{1}{g_4g_5' - g_4'g_5} \begin{pmatrix} g_5' & -g_5 \\ -g_4' & g_4 \end{pmatrix} \begin{pmatrix} -h_0 \\ -h_0' \end{pmatrix} \in M_4(M(4)) \times M_4(M(4))$$

by the integral closure argument. However, the linear dependence of f and g_0 on g_1, g_2, g_3 contradicts the assumption that f, g_0, g_1, g_2, g_3 are linearly independent.

We use Lemma A.5. Since $g_4g_5' = g_4'g_5$ for nontrivial $g_4, g_5, g_4', g_5' \in M_4(M(4))$ and the six products g_ig_j, for $1 \leqslant i \leqslant j \leqslant 3$, are linearly independent, it follows that there exists a unit $\alpha \in \mathbb{C}^*$ such that $g_4 = \alpha g_4'$ and $g_5 = \alpha g_5'$, or $g_4 = \alpha g_5$ and $g_4' = \alpha g_5'$. In the first case, the two linear relations (A.3) become

$$\alpha f g_4' + \alpha g_0 g_5' + h_0 = 0; \quad f g_4' + g_0 g_5' + h_0' = 0,$$

so that $h_0 = \alpha h_0'$ and the relations are not linearly independent. In the second case, we obtain

$$\alpha f g_5 + g_0 g_5 + h_0 = 0; \quad \alpha f g_5' + g_0 g_5' + h_0' = 0,$$

and $\alpha f = -g_0 - h_0/g_5 \in M_4(M(4))$ by the integral closure argument, contradicting the linear independence of f, g_1, g_2, g_3. Thus, no $f \in M_4(\Gamma_0'(4))$ with f, g_0, g_1, g_2, g_3 linearly independent can exist. □

Proposition A.7. *We have* $\dim M_2(\Gamma_0'(4)) = 0$.

Proof. Take $f \in M_2(\Gamma_0'(4))$ with Fourier–Jacobi expansion $f = \sum_{m=0}^{\infty} \phi_m \xi^m$ for $\phi_m \in \mathrm{J}_{2,m}$. We have $\dim \mathrm{J}_{2,m} \leqslant 0$ for $m \leqslant 2$ by the corollary on [9, p. 103]. Therefore, $f = \sum_{m=3}^{\infty} \phi_m \xi^m$ has order at least index 3 and $f^2 \in M_4(\Gamma_0'(4)) = \mathrm{Span}(g_0, g_1, g_2, g_3)$ has order at least index 6. By Lemma A.4, this span is determined by the Fourier–Jacobi coefficients of index through 4; thus $f^2 = 0$ and $f = 0$. □

APPENDIX B: TABLES

B.1 | History of dimension formulas

TABLE B.1 History of dimension formulas for $M_k(\Gamma)$ and $S_k(\Gamma)$ for some Γ. Earlier references appear left of later (relevant) references in the reference column.

Γ	Weight	Reference
$\mathrm{Sp}(4,\mathbb{Z})$	$k \geqslant 0$	[20, Theorem 2], [11, Theorem 6-2]
$\Gamma(2)$	$k \geqslant 0$	[20, Theorem 2], [40, p. 882]
$K(2)$	$k = 1$	[18, Theorem 6.1]
	$k = 2$	[15, Section 1]
	$k = 3$	[18, Theorem 2.1]
	$k = 4$	[18, Section 2.4]
	$k \geqslant 5$	[16, Theorem 4]

(Continues)

TABLE B.1 (Continued)

Γ	Weight	Reference
$\Gamma_0(2)$	$k = 1$	[18, Theorem 6.1]
	$k = 2$	[15, Section 1]
	$k = 3$	[18, Theorem 2.2]
	$k = 4$	[41, Corollary 4.12], [18, Section 2.4]
	$k \geqslant 5$	[14, 41, Corollary 4.12], [45, Theorem 7.4]
$\Gamma_0'(2)$	$k = 1$	[18, Theorem 6.1]
	$k = 2$	[15, Section 1]
	$k = 3$	[18, Theorem 2.4]
	$k = 4$	[18, Section 2.4]
	$k \geqslant 5$	[14, 46, Theorem A.1]
$B(2)$	$k = 1$	[18, Theorem 6.1]
	$k = 2$	[15, Section 1]
	$k = 3$	[18, Theorem 2.3]
	$k = 4$	[18, Section 2.4]
	$k \geqslant 5$	[14, 46, Theorem A.2]
$K(4)$	$k \geqslant 0$	[25, Theorem 1.1]
$\Gamma_0(4)$	$k \geqslant 0$	[42, Proposition 5.4]
	$k \geqslant 5$	[38, Theorem 3.5]
$\Gamma_0^*(4)$, $\Gamma_0'(4)$, $M(4)$	$k \geqslant 0$	Tables B.2 and B.3

B.2 | Dimension formulas for all weights

TABLE B.2 Dimension formulas for $M_k(\Gamma)$. The second column indicates those cases that follow directly from [20, Theorem 2]. The last column gives references for some other places where these formulas appear in the literature.

Γ	Igusa	$\sum_{k=0}^{\infty} \dim M_k(\Gamma) t^k$	Reference
$\Gamma(2)$	•	$\frac{(1+t^2)(1+t^4)(1+t^5)}{(1-t^2)^4}$	[40, p. 883]
$\mathrm{Sp}(4,\mathbb{Z})$	•	$\frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$	[20, p. 402]
$K(2)$		$\frac{(1+t^{10})(1+t^{12})(1+t^{11})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})}$	[13, Proposition 2]
$K(4)$	•	$\frac{(1+t^{12})(1+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{17})}{(1-t^4)^2(1-t^6)(1-t^{12})}$	[19, p. 121]
$\Gamma_0(2)$	•	$\frac{1+t^{19}}{(1-t^2)(1-t^4)^2(1-t^6)}$	[17, Theorem A,C]
$\Gamma_0(4)$	•	$\frac{1+t^4+t^{11}+t^{15}}{(1-t^2)^3(1-t^6)}$	[42, Proposition 5.4]
$\Gamma_0^*(4)$	•	$\frac{(1+t^4+t^6+t^{10})(1+t^5)}{(1-t^2)^3(1-t^6)}$	

(Continues)

TABLE B.2 (Continued)

Γ	Igusa	$\sum_{k=0}^{\infty} \dim M_k(\Gamma) t^k$	Reference
$\Gamma_0'(2)$	•	$\frac{(1+t^6+t^8+t^{10}+t^{12}+t^{18})(1+t^{11})}{(1-t^4)^2(1-t^6)(1-t^{12})}$	[13, Proposition 2]
$\Gamma_0'(4)$		$\frac{1+2t^4+4t^6+t^7+5t^8+2t^9+4t^{10}+5t^{11}+5t^{12}+4t^{13}+2t^{14}+5t^{15}+t^{16}+4t^{17}+2t^{19}+t^{23}}{(1-t^4)^2(1-t^6)^2}$	
$M(4)$	•	$\frac{(1+t^4)(1+2t^6+t^7+3t^8+t^9+t^{10}+2t^{11}+t^{12}+t^{13}+2t^{14}+t^{15}+t^{16}+3t^{17}+t^{18}+2t^{19}+t^{25})}{(1-t^4)^2(1-t^6)(1-t^{12})}$	
$B(2)$	•	$\frac{(1+t^6)(1+t^{11})}{(1-t^2)(1-t^4)^3}$	[17, Theorem B,C]

We remark that all the numerator polynomials in Table B.2 are palindromic. By [39, Theorem 4.4], this is related to the graded algebra $\bigoplus_{k\geqslant 0} M_k(\Gamma)$ being a Gorenstein ring. (For the question of being Cohen-Macaulay, see [8, 43, 44].)

TABLE B.3 Dimension formulas for $S_k(\Gamma)$. The second column indicates those cases that follow directly from [20, Theorem 2], together with the codimension formulas given in Table 8. The last column gives references for some other places where these formulas appear in the literature.

Γ	Igusa	$\sum_{k=0}^{\infty} \dim S_k(\Gamma) t^k$	Reference
$\Gamma(2)$	•	$\frac{t^5(1+5t+t^2+4t^3+t^4-5t^5+t^6)}{(1-t^2)^4}$	[40, p. 882]
$\mathrm{Sp}(4,\mathbb{Z})$	•	$\frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} - \frac{1}{(1-t^4)(1-t^6)}$	[20, Theorem 3]
$K(2)$		$\frac{t^8(1+t^{12})(1+t^2+t^3+t^4-t^{12}+t^{13})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})}$	[16, Theorem 4]
$K(4)$	•	$\frac{t^7(1+t+t^2+2t^3+t^4+2t^5+t^9+t^{10}+2t^{11}+t^{12}+t^{13}+t^{14}+t^{16}-t^{21}+t^{22})}{(1-t^4)^2(1-t^6)(1-t^{12})}$	[25, Theorem 2]
$\Gamma_0(2)$	•	$\frac{t^6(1+t^2-t^8+t^{13})}{(1-t^2)(1-t^4)^2(1-t^6)}$	[17, Theorem A,C]
$\Gamma_0(4)$	•	$\frac{t^6(3+t^4+t^5-2t^6+t^9)}{(1-t^2)^3(1-t^6)}$	
$\Gamma_0^*(4)$	•	$\frac{t^5(1+3t+t^3+t^4+2t^5+t^6-t^7-t^9+t^{10})}{(1-t^2)^3(1-t^6)}$	
$\Gamma_0'(2)$	•	$\frac{t^8(1+t^2+t^3+t^4-t^5-t^6+t^7+2t^8+t^{11}-t^{14}+t^{15}+t^{16}-t^{17}-t^{18}+t^{19})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$	[46, Theorem A.1]
$\Gamma_0'(4)$		$\frac{t^7(1+3t+2t^2+9t^3+5t^4+13t^5+4t^6+6t^7+5t^8+4t^{10}-3t^{11}+2t^{12}-2t^{13}-2t^{15}+t^{16})}{(1-t^4)^2(1-t^6)^2}$	
$M(4)$	•	$\frac{t^7(1+2t+2t^3+3t^4+4t^5-t^6+4t^8+5t^9+3t^{12}+2t^{13}-2t^{15}+2t^{16}+t^{17}-t^{18}-2t^{19}+t^{20})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$	
$B(2)$	•	$\frac{t^6(1+t^2-t^4+t^5+t^6-t^7-t^8+t^9)}{(1-t^2)^2(1-t^4)^2}$	[17, Theorem B,C]

TABLE B.4 Dimension formulas for cusp forms of Saito–Kurokawa type.

Γ	$\sum_{k=0}^{\infty} \dim S_k^{(\mathrm{P})}(\Gamma) t^k$
$\Gamma(2)$	$\frac{t^5(1+t+t^2)(1+4t+10t^3-5t^4+10t^5)}{(1-t^4)(1-t^6)}$
$\mathrm{Sp}(4,\mathbb{Z})$	$\frac{t^{10}}{(1-t^2)(1-t^6)}$
$\mathrm{K}(2)$	$\frac{t^8(1+t^2+t^3+t^4)}{(1-t^4)(1-t^6)}$
$\mathrm{K}(4)$	$\frac{t^7(1+t+t^2+2t^3+t^4+2t^5)}{(1-t^4)(1-t^6)}$
$\Gamma_0(2)$	$\frac{t^6(1+t^2+2t^4)}{(1-t^2)(1-t^6)}$
$\Gamma_0(4)$	$\frac{t^6(3+3t^2+4t^4)}{(1-t^2)(1-t^6)}$
$\Gamma_0^*(4)$	$\frac{t^5(1-t+t^2)(1+4t+5t^2+4t^3)}{(1-t^2)(1-t^6)}$
$\Gamma_0'(2)$	$\frac{t^8(1+t+t^2)(1-t+2t^2)}{(1-t^4)(1-t^6)}$
$\Gamma_0'(4)$	$\frac{t^7(1+2t+t^2+4t^3+2t^4+4t^5)}{(1-t^4)(1-t^6)}$
$M(4)$	$\frac{t^7(1+t+t^2)(1+t-t^2+3t^3)}{(1-t^4)(1-t^6)}$
$B(2)$	$\frac{t^6(1+t+t^2)(1-t+3t^2-2t^3+3t^4)}{(1-t^4)(1-t^6)}$

TABLE B.5 Dimension formulas for cusp forms of general type.

Γ	$\sum_{k=0}^{\infty} \dim S_k^{(G)}(\Gamma) t^k$
$\Gamma(2)$	$\frac{t^8(1+t+t^2)(10-t+12t^2-5t^3+2t^4+13t^5-16t^6+t^7)}{(1-t^2)^2(1-t^4)(1-t^6)}$
$\mathrm{Sp}(4,\mathbb{Z})$	$\frac{t^{20}(1+t^2+t^4-t^{12}-t^{14}+t^{15})}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$
$\mathrm{K}(2)$	$\frac{t^{16}(1+t^3+t^4+t^7+t^8-2t^9-2t^{10}+t^{11})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$
$\mathrm{K}(4)$	$\frac{t^{11}(1+t+t^3+t^4+2t^5+2t^8+2t^9+t^{12}+t^{13}-2t^{14}-3t^{15}+t^{16})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$
$\Gamma_0(2)$	$\frac{t^{12}(2+2t^2-t^4-2t^6+t^7)}{(1-t^2)(1-t^4)^2(1-t^6)}$
$\Gamma_0(4)$	$\frac{t^8(3+t^3+3t^4-4t^6+t^7)}{(1-t^2)^3(1-t^6)}$
$\Gamma_0^*(4)$	$\frac{t^8(4+3t+t^2+t^3+4t^4-t^5-5t^6+t^7)}{(1-t^2)^3(1-t^6)}$
$\Gamma_0'(2)$	$\frac{t^{12}(1+t^2+t^3+2t^4+t^7+t^8+2t^{11}+t^{12}-2t^{13}-3t^{14}+t^{15})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$
$\Gamma_0'(4)$	$\frac{t^8(1+t+5t^2+4t^3+11t^4+6t^5+12t^6+8t^7+8t^8+5t^9-t^{10}+t^{11}-6t^{12}-2t^{13}-6t^{14}+t^{15})}{(1-t^4)^2(1-t^6)^2}$
$M(4)$	$\frac{t^{10}(1+2t+4t^2+t^3+3t^4+4t^5+5t^6+4t^9+4t^{10}-t^{12}+3t^{13}+t^{14}-3t^{15}-5t^{16}+t^{17})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$
$B(2)$	$\frac{t^{10}(1+t+t^2)(1-t+4t^2-2t^3+t^4+3t^5-5t^6+t^7)}{(1-t^2)(1-t^4)^2(1-t^6)}$

B.3 | Dimensions for low weights

TABLE B.6 Dimensions for low weights: All modular forms.

Γ	$\dim M_k(\Gamma)$ for $k = \cdots$																			
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
$\Gamma(2)$	0	5	0	15	1	35	5	69	15	121	35	195	69	295	121	425	195	589	295	791
$\mathrm{Sp}(4,\mathbb{Z})$	0	0	0	1	0	1	0	1	0	2	0	3	0	2	0	4	0	4	0	5
$K(2)$	0	0	0	1	0	1	0	2	0	2	1	5	0	3	1	7	1	7	2	10
$K(4)$	0	0	0	2	0	2	1	4	1	5	3	10	3	9	6	17	7	19	12	27
$\Gamma_0(2)$	0	1	0	3	0	4	0	7	0	9	0	14	0	17	0	24	0	29	1	38
$\Gamma_0(4)$	0	3	0	7	0	14	0	24	0	38	1	57	3	81	7	111	14	148	24	192
$\Gamma_0^*(4)$	0	3	0	7	1	15	3	27	7	45	15	71	27	105	45	149	71	205	105	273
$\Gamma_0'(2)$	0	0	0	2	0	2	0	4	0	5	1	10	0	9	2	17	2	19	4	26
$\Gamma_0'(4)$	0	0	0	4	0	6	1	12	2	20	7	36	10	46	22	75	32	98	50	133
$M(4)$	0	0	0	3	0	3	1	8	1	10	5	21	5	23	13	41	16	49	28	71
$B(2)$	0	1	0	4	0	5	0	11	0	14	1	24	1	30	4	45	5	55	11	76

TABLE B.7 Dimensions for low weights: All cusp forms.

Γ	$\dim S_k(\Gamma)$ for $k = \cdots$																			
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
$\Gamma(2)$	0	0	0	0	1	5	5	24	15	61	35	120	69	205	121	320	195	469	295	656
$\mathrm{Sp}(4,\mathbb{Z})$	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2	0	3
$K(2)$	0	0	0	0	0	0	0	1	0	1	1	2	0	2	1	4	1	4	2	7
$K(4)$	0	0	0	0	0	0	1	1	1	2	3	4	3	5	6	10	7	12	12	19
$\Gamma_0(2)$	0	0	0	0	0	1	0	2	0	4	0	7	0	10	0	15	0	20	1	27
$\Gamma_0(4)$	0	0	0	0	0	3	0	9	0	19	1	34	3	54	7	80	14	113	24	153
$\Gamma_0^*(4)$	0	0	0	0	1	3	3	10	7	23	15	44	27	73	45	112	71	163	105	226
$\Gamma_0'(2)$	0	0	0	0	0	0	0	1	0	2	1	4	0	5	2	10	2	12	4	18
$\Gamma_0'(4)$	0	0	0	0	0	0	1	3	2	9	7	19	10	30	22	53	32	74	50	106
$M(4)$	0	0	0	0	0	0	1	2	1	4	5	10	5	14	13	27	16	35	28	54
$B(2)$	0	0	0	0	0	1	0	3	0	6	1	12	1	18	4	29	5	39	11	56

TABLE B.8 Dimensions for low weights: Cusp forms of Saito–Kurokawa type.

	$\dim S_k^{(\mathrm{P})}(\Gamma)$ for $k = \cdots$																			
Γ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\Gamma(2)$	0	0	0	0	1	5	5	14	6	20	11	29	11	34	16	44	17	49	21	58
$\mathrm{Sp}(4,\mathbb{Z})$	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2	0	2
$K(2)$	0	0	0	0	0	0	0	1	0	1	1	2	0	2	1	3	1	3	1	4
$K(4)$	0	0	0	0	0	0	1	1	1	2	2	3	2	3	3	5	3	5	4	6
$\Gamma_0(2)$	0	0	0	0	0	1	0	2	0	4	0	5	0	6	0	8	0	9	0	10
$\Gamma_0(4)$	0	0	0	0	0	3	0	6	0	10	0	13	0	16	0	20	0	23	0	26
$\Gamma_0^*(4)$	0	0	0	0	1	3	3	6	4	10	5	13	7	16	8	20	9	23	11	26
$\Gamma_0'(2)$	0	0	0	0	0	0	0	1	0	2	1	3	0	3	1	5	1	5	1	6
$\Gamma_0'(4)$	0	0	0	0	0	0	1	2	1	4	3	6	2	6	4	10	4	10	5	12
$M(4)$	0	0	0	0	0	0	1	2	1	3	3	5	2	5	4	8	4	8	5	10
$B(2)$	0	0	0	0	0	1	0	3	0	5	1	7	0	8	1	11	1	12	1	14

TABLE B.9 Dimensions for low weights: Cusp forms of general type.

	$\dim S_k^{(\mathrm{G})}(\Gamma)$ for $k = \cdots$																			
Γ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\Gamma(2)$	0	0	0	0	0	0	0	10	9	41	24	91	58	171	105	276	178	420	274	598
$\mathrm{Sp}(4,\mathbb{Z})$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$K(2)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	3
$K(4)$	0	0	0	0	0	0	0	0	0	0	1	1	1	2	3	5	4	7	8	13
$\Gamma_0(2)$	0	0	0	0	0	0	0	0	0	0	0	2	0	4	0	7	0	11	1	17
$\Gamma_0(4)$	0	0	0	0	0	0	0	3	0	9	1	21	3	38	7	60	14	90	24	127
$\Gamma_0^*(4)$	0	0	0	0	0	0	0	4	3	13	10	31	20	57	37	92	62	140	94	200
$\Gamma_0'(2)$	0	0	0	0	0	0	0	0	0	0	0	1	0	2	1	5	1	7	3	12
$\Gamma_0'(4)$	0	0	0	0	0	0	0	1	1	5	4	13	8	24	18	43	28	64	45	94
$M(4)$	0	0	0	0	0	0	0	0	0	1	2	5	3	9	9	19	12	27	23	44
$B(2)$	0	0	0	0	0	0	0	0	0	1	0	5	1	10	3	18	4	27	10	42

B.4 | Number of automorphic representations

TABLE B.10 Number of automorphic representations: Saito–Kurokawa type.

Ω	$\sum_{k=0}^{\infty} s_k^{(\mathrm{P})}(\Omega)t^k$	Ω	$\sum_{k=0}^{\infty} s_k^{(\mathrm{P})}(\Omega)t^k$	Ω	$\sum_{k=0}^{\infty} s_k^{(\mathrm{P})}(\Omega)t^k$
IIb	$\dfrac{t^{10}}{(1-t^2)(1-t^6)}$	VIb	$\dfrac{t^6+t^8-t^{12}}{(1-t^4)(1-t^6)}$	XIb	$\dfrac{t^7}{(1-t^2)(1-t^6)}$
Vb	$\dfrac{t^8}{(1-t^4)(1-t^6)}$	VIc	$\dfrac{t^{11}}{(1-t^4)(1-t^6)}$	Va*	$\dfrac{t^5}{(1-t^4)(1-t^6)}$

TABLE B.11 Number of automorphic representations: general type.

Ω	$\sum_{k=0}^{\infty} s_k^{(G)}(\Omega) t^k$
I	$\frac{t^{20}(1+t^2+t^4-t^{12}-t^{14}+t^{15})}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$
IIa	$\frac{t^{16}(1+t^2+t^3-t^4-t^6)}{(1-t^4)^2(1-t^5)(1-t^6)}$
IIIa+VIa/b	$\frac{t^{12}(1+2t^2+2t^4-t^5+2t^6-2t^7+t^8-2t^9+t^{10}-2t^{11}+t^{14}+t^{16}+t^{17}-t^{18})}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})}$
IVa	$\frac{t^{10}(1+t^2+t^3+t^4-t^8+t^9+2t^{10}+2t^{11}+t^{12}+t^{13}-t^{14}-t^{15}-t^{16}-t^{17}+t^{20})}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})}$
Va/a*	$\frac{t^{15}(1+t^2-t^5-t^7+t^{10})}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})}$
VII+VIIIa/b	$\frac{t^{10}(1+t^2-t^6+t^7)}{(1-t^4)^2(1-t^6)^2}$
IXa	$\frac{t^8(1+t^{11})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$
X	$\frac{t^{11}(1+t^8+t^9-t^{12})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$
XIa	$\frac{t^{12}(1+t^3+t^4-t^7-t^8+t^{11})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$
sc(16)	$\frac{t^9}{(1-t^2)(1-t^4)^2(1-t^5)}$

TABLE B.12 Number of automorphic representations for low weights.

		$s_k^{(P)}(\Omega)$ or $s_k^{(G)}(\Omega)$ for $k = \cdots$																			
	Ω	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
(P)	IIb	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2	0	2
	Vb	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	1	0	1	0	2
	VIb	0	0	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0	2	0	2
	VIc	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	1	0
	XIb	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2	0	2	0	3	0
	Va*	0	0	0	0	1	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0
(G)	I	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	IIa	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1
	IIIa+VIa/b	0	0	0	0	0	0	0	0	0	0	0	1	0	2	0	3	0	5	0	6
	IVa	0	0	0	0	0	0	0	0	0	1	0	1	1	2	1	2	2	3	4	6
	Va/a*	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0
	VII+VIIIa/b	0	0	0	0	0	0	0	0	0	1	0	1	0	2	0	3	1	5	0	5
	IXa	0	0	0	0	0	0	0	1	0	1	0	2	0	3	0	4	0	5	1	8
	X	0	0	0	0	0	0	0	0	0	0	1	0	1	0	2	0	3	0	5	1
	XIa	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	3	1	4	1	5
	sc(16)	0	0	0	0	0	0	0	0	1	0	1	0	3	1	3	1	6	3	7	3

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