A decomposition of the spaces $S_k(\Gamma_0(N))$ in degree 2 and the construction of hypercuspidal modular forms

Brooks Roberts *
Ralf Schmidt †

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1 Introduction

In this paper we shall construct cuspidal Siegel modular forms $F \in S_k(\Gamma_0(N))$ of degree 2 with the following property. There exists a prime number p such that, if

$$F(\tau, z, \tau') = \sum_{m=0}^{\infty} f_m(\tau, z) e^{2\pi i m \tau'}$$

is the Fourier-Jacobi expansion of F, then $f_m = 0$ whenever $p|m$. We call such modular forms *p-hypercuspidal of degree* 1 (p-hypercuspidal of degree l would mean $f_m = 0$ whenever $p^l|m$). The existence of such modular forms is not obvious. For example, one can show that if $p \nmid N$

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or p divides N only once, then such modular forms do not exist. One can also show that cusp forms with respect to paramodular groups, as defined in [RS1], are not p-hypercuspidal; this follows from the analogous local statement, Proposition 3.4.2 in [RS2].

The examples we shall give are for $\Gamma_0(p^2M)$ with $p \nmid M$. More precisely, we shall show that every cusp form $F \in S_k(\Gamma_0(M))$ gives rise to a p-hypercuspidal modular form $\tilde{F} \in S_k(\Gamma_0(p^2M))$ in a way that preserves Hecke-eigenvalues at all good places. See Theorem 3.11 for the precise statement.

A central role in our argument will be played by a certain linear operator μ_p on the space $S_k(\Gamma_0(N))$. This endomorphism is only defined if $p^2|N$, but assuming this is the case, μ_p has rather nice properties. Amongst others, μ_p is diagonalizable and admits only the four possible eigenvalues $p(p+1)$, p, $2p$ and 0. Consequently, we obtain a direct sum decomposition

$$S_k(\Gamma_0(N)) = S_k(\Gamma_0(N))_{p(p+1)} \oplus S_k(\Gamma_0(N))_p \oplus S_k(\Gamma_0(N))_{2p} \oplus S_k(\Gamma_0(N))_0, \tag{1}$$

where $S_k(\Gamma_0(N))_i$ denotes the i-eigenspace of μ_p. In Sect. 3.3 we shall give several characterizations of the eigenspaces, one of them in terms of Fourier coefficients. The decomposition (1) is not new, but is a special case of the direct sum decomposition obtained in [Sa], §3, via "twisting operators". See also [Yo].

In this work we are particularly interested in $S_k(\Gamma_0(N))_0$, the kernel of μ_p, since it turns out that this space consists precisely of the p-hypercuspidal modular forms of degree 1. Thus, given $F \in S_k(\Gamma_0(M))$ and $p \nmid M$, our goal is to find an $\tilde{F} \in S_k(\Gamma_0(p^2M))$ with $\mu_p\tilde{F} = 0$. Our strategy will be to translate modular forms into automorphic representations and solve the analogous local problem.

Thus let L be a $\mathfrak{p}$-adic field with ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p}$ of $\mathfrak{o}$, and let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, L)$. Since we are considering modular forms of haupttype only, we shall assume that π has trivial central character. We denote by $\mathrm{Si}(\mathfrak{p}^n)$, $n \geq 0$, the local analogues of the Hecke subgroups $\Gamma_0(N)$. The local analogues of the spaces $S_k(\Gamma_0(N))$ are the finite-dimensional spaces $V_0(n) = \{v \in V : \pi(g)v = v \text{ for all } g \in \mathrm{Si}(\mathfrak{p}^n)\}$. Provided that $n \geq 2$ we shall define an endomorphism $\mu : V_0(n) \to V_0(n)$ which is analogous, and in fact compatible, with the global endomorphism μ_p of $S_k(\Gamma_0(N))$. This endomorphism is diagonalizable and admits only $q(q+1)$, q, $2q$ and 0 as its eigenvalues, where $q = \#\mathfrak{o}/\mathfrak{p}$. Hence, we get a similar decomposition as in (1) for the spaces $V_0(n)$. Since $\mathrm{Si}(\mathfrak{p}^0) = \mathrm{GSp}(4, \mathfrak{o})$, the representation π is spherical if and only if $V_0(0) \neq 0$. In this case $\dim V_0(0) = 1$, and the local analogue of our global problem consists in finding a vector $\tilde{v} \in V_0(2)$ such that $\mu\tilde{v} = 0$.

For this purpose we shall determine the dimensions of the μ-eigenspaces in $V_0(2)$ for each Iwahori-spherical, irreducible, admissible representation (π, V) of $\mathrm{GSp}(4, L)$ with trivial central character and each of the four possible eigenvalues. It turns out that this space can be at most 12-dimensional, and in this case the dimensions of the eigenspaces, in the order $q(q+1)$, q, $2q$ and 0, are 4, 4, 3 and 1. Our method to determine these dimensions consists in finding an explicit basis for $V_0(2)$ and computing the resulting 12×12 matrix of the μ operator. Hence, at least for this type of representation (type I representations in the terminology of [RS2]), the kernel of μ on $V_0(2)$ is indeed non-trivial, and is in fact one-dimensional.

More calculations and some additional arguments give the dimensions of the μ-eigenspaces at level $\mathfrak{p}^2$ for each of the Iwahori-spherical representations. The results are summarized in Table 1; see also Theorem 2.11. It turns out that not every spherical representation admits a non-trivial μ-kernel at level $\mathfrak{p}^2$; those of type IIIb, IVd and VId do not. However, by the main result of [PS], these representations are not relevant for holomorphic Siegel modular forms. More precisely, the only spherical representations that can occur in an automorphic representation generated by an element of $S_k(\Gamma_0(N))$, $k \geq 3$, are of type I or IIb. As in the type I case, spherical representations of type IIb also have a one-dimensional kernel of μ at level $\mathfrak{p}^2$. Hence, in each relevant spherical representation we can find an essentially unique vector $\tilde{v}$ in the kernel of μ at level $\mathfrak{p}^2$. If π comes from a modular form F, then, replacing the spherical vector by $\tilde{v}$, we obtain the desired cusp form $\tilde{F}$ for which $\mu_p\tilde{F} = 0$. Since we are moving to another vector within the same automorphic representation, Hecke eigenvalues at primes away from p are not affected.

This work is divided into two parts. Part one is the local part, in which we shall define the local version of the μ operator, prove its basic properties, and carry out the required calculations at level $\mathfrak{p}^2$ in Iwahori-spherical representations. Part two contains the global theory. We shall define the μ_p operator on the spaces $S_k(\Gamma_0(N))$, prove the existence of the decomposition (1), and finally apply our local results to the construction of hypercuspidal Siegel modular forms. We caution that we shall use two different, but of course isomorphic, versions of the symplectic group; in part one we find it convenient to work with the "symmetric" version, while in part two we shall switch to the "official" version.

2 Local theory

All through this section let L be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$ be the ring of integers of F and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. We fix a generator ϖ of $\mathfrak{p}$. We shall work with the algebraic L-group

$$\mathrm{GSp}(4) = \{g \in \mathrm{GL}(4) : \, {}^tgJg = \lambda(g)J \text{ for some scalar } \lambda(g)\}, \qquad J = \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}. \tag{2}$$

We shall sometimes abbreviate GSp(4) by G. The homomorphism $\lambda : \mathrm{GSp}(4) \to \mathrm{GL}(1)$ is called the multiplier homomorphism. Its kernel is the symplectic group Sp(4). Note that this is the "symmetric" version of GSp(4), which allows for the following choices of standard parabolic subgroups. As a standard minimal parabolic subgroup we choose

$$B = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ && * & * \\ &&& * \end{bmatrix}.$$

The Siegel parabolic subgroup P and the Kingen parabolic subgroup Q are defined as

$$P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ && * & * \\ && * & * \end{bmatrix}, \qquad Q = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ &&& * \end{bmatrix}.$$

(In the global part of this paper we shall switch to the "official" version of GSp(4), where J is replaced by $\begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix}$.) The *Siegel congruence subgroup* of level $\mathfrak{p}^n$ is

$$\mathrm{Si}(\mathfrak{p}^n) = \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \end{bmatrix}. \tag{3}$$

If (π, V) is a smooth representation of $\mathrm{GSp}(4, F)$, we denote by $V_0(n)$ the space of vectors $v \in V$ for which $\pi(g)v = v$ for all $g \in \mathrm{Si}(\mathfrak{p}^n)$.

2.1 The endomorphism μ of $V_0(n)$

Let (π, V) be a smooth representation of $\mathrm{GSp}(4, L)$ for which the center acts trivially. For $n \geq 0$ let $V_0(n)$ be the subspace of V consisting of $\mathrm{Si}(\mathfrak{p}^n)$ invariant vectors. For $v \in V_0(n)$ consider the summation

$$v' = \sum_{z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v. \tag{4}$$

It is easily checked that v' is invariant under all elements of the form

$$\begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p} & \mathfrak{o} \end{bmatrix}, \tag{5}$$

provided that $n \geq 2$. Hence, if we restore the $\mathrm{GL}(2,\mathfrak{o})$ invariance on the Siegel Levi, we obtain a new element of $V_0(n)$. In other words, for $n \geq 2$ there is an endomorphism $\mu:\ V_0(n) \to V_0(n)$ given by

$$\mu(v) = \sum_{A\in\mathrm{SL}(2,\mathfrak{o})/\left[\begin{smallmatrix}\mathfrak{o} & \mathfrak{o}\\ \mathfrak{p} & \mathfrak{o}\end{smallmatrix}\right]} \sum_{z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} A & \\ & A' \end{bmatrix} \begin{bmatrix} 1 & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v. \tag{6}$$

An explicit formula is

$$\mu(v) = \sum_{x,z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & z\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix})v + \sum_{z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v. \tag{7}$$

Alternatively,

$$\mu(v) = \sum_{x,z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & xz\varpi^{-1} & x^2z\varpi^{-1} \\ & 1 & z\varpi^{-1} & xz\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix})v + \sum_{z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v. \tag{8}$$

In particular, μ has a formula given completely in terms of the unipotent radical of the Siegel parabolic subgroup. In addition to μ we introduce a simple level raising operator $\beta:\ V_0(n)\to V_0(n+1)$, given by

$$\beta v=\pi(\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix})v\qquad (v\in V_0(n)).\tag{9}$$

Trivially, β is injective. The following result shows, amongst other things, that the image of β can be characterized in terms of eigenvalues of the μ operator.

2.1 Proposition. *Let (π,V) be an admissible representation of $\mathrm{GSp}(4,L)$ for which the center acts trivially. Let $n\geq 2$. Consider the endomorphism μ of $V_0(n)$ defined above.*

i) μ is diagonalizable.

ii) The only possible eigenvalues of μ on $V_0(n)$ are 0, q, $2q$ and $q(q+1)$.

iii) For $v\in V_0(n)$ we have $\mu v=q(q+1)v$ if and only if $v\in\beta(V_0(n-1))$. Consequently, at the minimal Siegel level, the only possible eigenvalues of μ are 0, q and $2q$.

iv) Let $v\in V_0(n)$. If $\mu v=qv$, then

$$\sum_{y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&y\varpi^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix})v\neq 0,\quad \textit{but}\quad \sum_{x,y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&y\varpi^{-1}\\&1&x\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})v=0.$$

v) Let $v\in V_0(n)$. If $\mu v=2qv$, then

$$\sum_{y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&y\varpi^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix})v\neq 0,\quad \textit{but}\quad \sum_{x,y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&x\varpi^{-1}&y\varpi^{-1}\\&1&&x\varpi^{-1}\\&&1&\\&&&1\end{bmatrix})v=0.$$

vi) Let $v\in V_0(n)$. We have $\mu v=0$ if and only if

$$\sum_{y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&y\varpi^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix})v=0.$$

Proof: i) Let $\langle\ ,\ \rangle$ be an inner product on V invariant under the compact subgroup

$$\begin{bmatrix}1&&\mathfrak{p}^{-1}&\mathfrak{p}^{-1}\\&1&\mathfrak{p}^{-1}&\mathfrak{p}^{-1}\\&&1&\\&&&1\end{bmatrix}.$$

It is immediate from (8) that $\langle \mu v, v' \rangle = \langle v, \mu v' \rangle$ for $v, v' \in V_0(n)$. Hence, μ is self-adjoint and therefore diagonalizable.

ii), iii) It is obvious that $\mu v = q(q+1)v$ for $v \in \beta(V_0(n-1))$. We will now prove ii) and the other direction of iii). Assume that $v \in V_0(n)$ and $\mu(v) = cv$ for some $c \in \mathbb{C}$. By (8), this means

$$cv = \sum_{x,z \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & xz\varpi^{-1} & x^2z\varpi^{-1} \\ & 1 & z\varpi^{-1} & xz\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix})v + \sum_{z \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v.$$

Let v' be defined as in (4). Multiplying the above equation with

$$\begin{bmatrix} 1 & & & y\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

and summing over $y \in \mathfrak{o}/\mathfrak{p}$ we obtain

$$cv' = \sum_{x,z \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & xz\varpi^{-1} & x^2z\varpi^{-1} \\ & 1 & z\varpi^{-1} & xz\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix})v' + qv'.$$

Hence

$$\begin{aligned}(c-q)v' &= \sum_{x,z \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & xz\varpi^{-1} & \\ & 1 & z\varpi^{-1} & xz\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix})v' \\ &= qv' + \sum_{z \in (\mathfrak{o}/\mathfrak{p})^\times} \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & xz\varpi^{-1} & \\ & 1 & z\varpi^{-1} & xz\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix})v',\end{aligned}$$

and therefore

$$(c-2q)v' = \sum_{z \in (\mathfrak{o}/\mathfrak{p})^\times} \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & x\varpi^{-1} & \\ & 1 & z\varpi^{-1} & x\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix})v'. \tag{10}$$

The vector v' already has the invariance (5). If $c \neq 2q$, then we conclude from (10) that v' is invariant under

$$\begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p} & \mathfrak{o} \end{bmatrix}.$$

In this case we get from (10)

$$(c-2q)v' = q \sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})v',$$

from which it follows that

$$(c-q)v' = q \sum_{z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})v'. \tag{11}$$

If in addition $c \neq q$, we conclude that v' is invariant under

$$\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-1}&\mathfrak{p}^{-1}\\ \mathfrak{p}&\mathfrak{o}&\mathfrak{p}^{-1}&\mathfrak{p}^{-1}\\ \mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}&\mathfrak{o}\end{bmatrix}. \tag{12}$$

From the definition (6) we see that μv is invariant under

$$\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-1}&\mathfrak{p}^{-1}\\ \mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-1}&\mathfrak{p}^{-1}\\ \mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\end{bmatrix}.$$

Consequently $\mu v = \beta v_1$ for some $v_1 \in V_0(n-1)$. Hence $cv = \beta v_1$. If $c \neq 0$ it follows that $v \in \beta(V_0(n-1))$, and therefore $\mu v = q(q+1)v$. This proves ii) and iii).

iv) follows from (11).

v) Let $\mu v = 2qv$. From (10) we get

$$\sum_{x,z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&&x\varpi^{-1}&\\&1&z\varpi^{-1}&x\varpi^{-1}\\&&1&\\&&&1\end{bmatrix})v' = \sum_{x\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&&x\varpi^{-1}&\\&1&&x\varpi^{-1}\\&&1&\\&&&1\end{bmatrix})v'.$$

If we define

$$v'' = \sum_{x\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&&x\varpi^{-1}&\\&1&&x\varpi^{-1}\\&&1&\\&&&1\end{bmatrix})v' = \sum_{x,y\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&&x\varpi^{-1}&y\varpi^{-1}\\&1&&x\varpi^{-1}\\&&1&\\&&&1\end{bmatrix})v,$$

this can be rewritten as

$$\sum_{z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})v'' = v''. \tag{13}$$

Hence v'' is invariant under

$$\begin{bmatrix}1&&&\\&1&\mathfrak{p}^{-1}&\\&&1&\\&&&1\end{bmatrix}.$$

But then (13) becomes $qv''=v''$, which implies $v''=0$. This is the assertion.

vi) We have to show that $\mu v=0$ if and only if $v'=0$. It is clear from (6) that $v'=0$ implies $\mu v=0$. Assume conversely that $\mu v=0$. As above we conclude that v' is invariant under all elements of the form (12). Hence

$$\begin{aligned}
q^2v' &= \sum_{x,y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&y\varpi^{-1}&\\&1&x\varpi^{-1}&y\varpi^{-1}\\&&1&\\&&&1\end{bmatrix})v'\\
&= \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&y\varpi^{-1}&z\varpi^{-1}\\&1&x\varpi^{-1}&y\varpi^{-1}\\&&1&\\&&&1\end{bmatrix})v\\
&= \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}\varpi^{-1}&&&\\&\varpi^{-1}&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&y&z\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v\\
&= \beta\sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&y&z\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v.
\end{aligned}$$

It follows that $v'\in V_0(n+1)$, and actually, since v' has the invariance (12), that $v'\in V_0(n)$. All we need, however, is the invariance of v' under the $\mathrm{GL}(2,\mathfrak{o})$ block in the Siegel Levi. It implies, by (6), that $\mu v=(q+1)v'$. But $\mu v=0$ by assumption, hence $v'=0$. ∎

Our goal in the following sections is to compute the dimensions of the eigenspaces of μ on $V_0(2)$ for the Iwahori-spherical representations of $\mathrm{GSp}(4,L)$.

2.2 Double coset representatives

We start by computing representatives for certain double coset spaces. This will already give the dimension of $V_0(2)$ for a number of Iwahori-spherical representations.

2.2 Lemma. *For any $n\geq1$, the following is a complete and minimal system of representatives for $\mathrm{GSp}(4,\mathfrak{o})/\mathrm{Si}(\mathfrak{p}^n)$ (here the residue characteristic of L can be arbitrary).*

$$\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x&y&&1\end{bmatrix},\qquad x,y,z\in\mathfrak{o}/\mathfrak{p}^n,\ x,y,z\equiv0\mod\mathfrak{p},\tag{14}$$

$$s_2\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x&y&&1\end{bmatrix},\qquad x,y,z\in\mathfrak{o}/\mathfrak{p}^n,\ x,y\equiv 0\mod\mathfrak{p},\tag{15}$$

$$s_1s_2\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x&y&&1\end{bmatrix},\qquad x,y,z\in\mathfrak{o}/\mathfrak{p}^n,\ x\equiv 0\mod\mathfrak{p},\tag{16}$$

$$s_2s_1s_2\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x&y&&1\end{bmatrix},\qquad x,y,z\in\mathfrak{o}/\mathfrak{p}^n.\tag{17}$$

In particular, $\#\mathrm{GSp}(4,\mathfrak{o})/\mathrm{Si}(\mathfrak{p}^n)=q^{3n-3}(q+1)(q^2+1)$.

Proof: See [RS2], Lemma 5.1.1. ■

2.3 Lemma. *i) A complete and minimal system of representatives for the double cosets* $B(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$ *is given by the following* 12 *elements.*

$$1,\qquad s_2,\qquad s_1s_2,\qquad s_2s_1s_2,$$

$$\begin{bmatrix}1&&&\\&1&&\\&\varpi&1&\\&&&1\end{bmatrix},\quad\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix},\quad\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\&\varpi&&1\end{bmatrix},\quad\begin{bmatrix}1&&&\\&1&&\\&\varpi&1&\\\varpi&&&1\end{bmatrix},$$

$$\begin{bmatrix}1&&&\\\varpi&1&&\\&&1&\\&&-\varpi&1\end{bmatrix}s_2,\quad\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix}s_2,\quad\begin{bmatrix}1&&&\\&1&&\\&\varpi&1&\\&&&1\end{bmatrix}s_1s_2,\qquad X,$$

where

$$X=\begin{bmatrix}1&&&\\&1&&\\&u\varpi&1&\\\varpi&&&1\end{bmatrix},\qquad u\in\mathfrak{o}^\times\setminus\mathfrak{o}^{\times 2},$$

if the residue characteristic of L *is odd, and*

$$X=\begin{bmatrix}1&&&\\&1&&\\\varpi&\varpi&1&\\&\varpi&&1\end{bmatrix}$$

if the residue characteristic of L *is even.*

ii) A complete and minimal system of representatives for the double cosets $P(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$ *is given by the following* 7 *elements.*

$$1,\qquad s_2,\qquad s_2s_1s_2,$$

$$\begin{bmatrix} 1&&&\\ &1&&\\ &\varpi&1&\\ &&&1 \end{bmatrix}, \qquad \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix}, \qquad \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \varpi&&&1 \end{bmatrix} s_2, \qquad X,$$

where

$$X = \begin{bmatrix} 1&&&\\ &1&&\\ &-u\varpi&1&\\ \varpi&&&1 \end{bmatrix}, \qquad u \in \mathfrak{o}^\times \setminus \mathfrak{o}^{\times 2},$$

if the residue characteristic of L is odd, and

$$X = \begin{bmatrix} 1&&&\\ &1&&\\ &\varpi&1&\\ \varpi&&&1 \end{bmatrix}$$

if the residue characteristic of L is even.

iii) A complete and minimal system of representatives for the double cosets $Q(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$ is given by the following 4 elements.

$$1, \qquad s_1s_2, \qquad \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \varpi&&&1 \end{bmatrix}, \qquad \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix}.$$

Proof: We first indicate how to check that no two of the 12 elements listed define the same double coset in $B(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$. As an example, let us show that

$$g_1 = \begin{bmatrix} 1&&&\\ &1&&\\ &\varpi&1&\\ &&&1 \end{bmatrix} \qquad \text{and} \qquad g_2 = \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix}$$

define different double cosets. Thus assume there is an equality $bg_1k = g_2$ with $b \in B(F)$ and $k \in \mathrm{Si}(\mathfrak{p}^2)$. This implies $b \in B(\mathfrak{o})$. Conjugating this equality with $\mathrm{diag}(\varpi, \varpi, 1, 1)$, all elements remain in $\mathrm{GSp}(4, \mathfrak{o})$. Reducing the new identity mod $\mathfrak{p}$, we obtain a matrix identity over the residue field of the form

$$\begin{bmatrix} *&*&&\\ &*&&\\ &&*&*\\ &&&* \end{bmatrix} \begin{bmatrix} 1&&&\\ &1&&\\ &1&1&\\ &&&1 \end{bmatrix} \begin{bmatrix} *&*&&\\ *&*&&\\ &&*&*\\ &&*&* \end{bmatrix} = \begin{bmatrix} 1&&&\\ &1&&\\ 1&&1&\\ &1&&1 \end{bmatrix}.$$

It is easy to see that this is impossible. Other cases are treated similarly, with only slight modifications. — We now show that every double coset in $B(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$ is represented by one of the 12 elements listed. By Lemma 2.2, the cosets $\mathrm{GSp}(4,\mathfrak{o})/\mathrm{Si}(\mathfrak{p}^2)$ are represented by

$$\begin{bmatrix}1&&&\\&1&&\\y\varpi&z\varpi&1&\\x\varpi&y\varpi&&1\end{bmatrix},\qquad x,y,z\in\mathfrak{o}/\mathfrak{p},\tag{18}$$

$$s_2\begin{bmatrix}1&&&\\&1&&\\y\varpi&z&1&\\x\varpi&y\varpi&&1\end{bmatrix},\qquad x,y\in\mathfrak{o}/\mathfrak{p},\ z\in\mathfrak{o}/\mathfrak{p}^2,\tag{19}$$

$$s_1s_2\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x\varpi&y&&1\end{bmatrix},\qquad x\in\mathfrak{o}/\mathfrak{p},\ y,z\in\mathfrak{o}/\mathfrak{p}^2,\tag{20}$$

$$s_2s_1s_2\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x&y&&1\end{bmatrix},\qquad x,y,z\in\mathfrak{o}/\mathfrak{p}^2.\tag{21}$$

Elements of type (21) are obviously all equivalent to $s_2s_1s_2$ in $B(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$. Elements of type (20) are equivalent to

$$s_1s_2\begin{bmatrix}1&&&\\&1&&\\&&1&\\x\varpi&&&1\end{bmatrix}=\begin{bmatrix}1&&&\\&1&&\\&x\varpi&1&\\&&&1\end{bmatrix}s_1s_2,$$

and hence, since we can conjugate by diagonal matrices with units on the diagonal, to one of the 12 elements listed. Elements of type (19) are equivalent to

$$s_2\begin{bmatrix}1&&&\\&1&&\\y\varpi&&1&\\x\varpi&y\varpi&&1\end{bmatrix}=\begin{bmatrix}1&&&\\y\varpi&1&&\\&&1&\\x\varpi&&-y\varpi&1\end{bmatrix}s_2,$$

and therefore, after conjugation with suitable unit diagonal matrices, to s_2 or

$$\begin{bmatrix}1&&&\\\varpi&1&&\\&&1&\\&&-\varpi&1\end{bmatrix}s_2\quad\text{or}\quad\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix}s_2\quad\text{or}\quad\begin{bmatrix}1&&&\\\varpi&1&&\\&&1&\\\varpi&&-\varpi&1\end{bmatrix}s_2$$

The last two matrices are actually equivalent because of the relation

$$\begin{bmatrix}1&&1&\\&1&\varpi&1\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix}s_2\begin{bmatrix}1&1&&\\&1&&\\&&1&-1\\&&&1\end{bmatrix}=\begin{bmatrix}1&&&\\\varpi&1&&\\&&1&\\\varpi&&-\varpi&1\end{bmatrix}s_2.\tag{22}$$

Hence, each matrix of type (19) is equivalent to one of the 12 elements listed. Finally, we have to deal with matrices of type (18). First assume that $x=0$. In this case, a matrix of type (18) is equivalent to

$$1 \qquad \text{or} \qquad \begin{bmatrix} 1&&&\\ &1&&\\ &\varpi&1&\\ &&&1 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&\varpi&1&\\ &\varpi&&1 \end{bmatrix}.$$

If the residue characteristic is odd, and only then, are the last two matrices equivalent, because of the identity

$$\begin{bmatrix} 1&-\frac{1}{2}&&\\ &1&&\\ &&1&\frac{1}{2}\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix} \begin{bmatrix} 1&\frac{1}{2}&&\\ &1&&\\ &&1&-\frac{1}{2}\\ &&&1 \end{bmatrix} = \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&\varpi&1&\\ &\varpi&&1 \end{bmatrix}. \tag{23}$$

This shows that a matrix of type (18) with $x=0$ is equivalent to one of the 12 elements listed. Now consider a matrix of type (18) with $x \notin \mathfrak{p}$. After a conjugation with a suitable unit diagonal matrix we can bring it into the form

$$\begin{bmatrix} 1&&&\\ &1&&\\ \varpi&z\varpi&1&\\ \varpi&\varpi&&1 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1&&&\\ &1&&\\ &z\varpi&1&\\ \varpi&&&1 \end{bmatrix}$$

with $z \in \mathfrak{o}/\mathfrak{p}$. Since

$$\begin{bmatrix} 1&1&&\\ &1&&\\ &&1&-1\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&z\varpi&1&\\ \varpi&\varpi&&1 \end{bmatrix} \begin{bmatrix} 1&-1&&\\ &1&&\\ &&1&1\\ &&&1 \end{bmatrix} = \begin{bmatrix} 1&&&\\ &1&&\\ &(z-1)\varpi&1&\\ \varpi&&&1 \end{bmatrix},$$

we may actually assume it is of the second form. If $z=0$, this matrix is in our list. Assume that z is a unit. Conjugation with diagonal matrices allows us to multiply z by an element of $\mathfrak{o}^{\times 2}$. If the residue characteristic of L is odd, then $(\mathfrak{o}^\times : \mathfrak{o}^{\times 2}) = 2$, and we may assume $z=1$ or $z=u$. Hence the proof is complete in this case. If the residue characteristic is even, then $\mathfrak{o}^\times = \mathfrak{o}^{\times 2}(1+\mathfrak{p})$. The identity

$$\begin{bmatrix} 1&&&\\ &1&\frac{-b}{1+bz\varpi}&\\ &&1&\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&&&\\ &1&&\\ &z\varpi&1&\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&&&\\ &1+bz\varpi&\frac{b}{1+bz\varpi}&\\ &&\frac{1}{1+bz\varpi}&\\ &&&1 \end{bmatrix} = \begin{bmatrix} 1&&&\\ &1&&\\ &z\varpi(1+bz\varpi)&1&\\ &&&1 \end{bmatrix}$$

shows that we are allowed to multiply z by elements of $1+\mathfrak{p}$. We may therefore assume $z=1$, and the proof is complete.

ii) Using the same method as in i), it is easy to check that no two of the 7 elements listed define the same double coset in $P(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$. It remains to show that each of the 12 elements in

i) defines the same double coset in $P(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$ as one of the 7 elements in ii). Since we are now able to multiply from the left by s_1, this is easy to see for most of the elements in i); in the following we shall only treat the non-obvious cases. If the residue characteristic of L is even, then the identity

$$\begin{bmatrix}1&&&\\1&1&&\\&&1&\\&&-1&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\\varpi&\varpi&1&\\&\varpi&&1\end{bmatrix}\begin{bmatrix}1&&&\\-1&1&&\\&&1&\\&&1&1\end{bmatrix}=\begin{bmatrix}1&&&\\&1&&\\&\varpi&1&\\-\varpi&&&1\end{bmatrix}$$

shows that the element X in i) is equivalent to the element X in ii); see the end of the proof of i). Now assume that the residue characteristic of L is odd. First assume that $-1\in\mathfrak{o}^{\times2}$. Let $v\in\mathfrak{o}^\times$ with $v^2=-1$, and let $A=\begin{bmatrix}v&1\\-\frac{v}{2}&\frac12\end{bmatrix}$. We have $A\,{}^tA=\begin{bmatrix}&1\\1&\end{bmatrix}$, from which it follows that

$$\begin{bmatrix}A'&\\&A\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&\varpi&1&\\\varpi&&&1\end{bmatrix}\begin{bmatrix}A'^{-1}&\\&A^{-1}\end{bmatrix}=\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\&\varpi&&1\end{bmatrix}.$$

Now assume that $-1\notin\mathfrak{o}^{\times2}$, and let $A=\begin{bmatrix}1&1\\-\frac12&\frac12\end{bmatrix}$. Then $A\begin{bmatrix}-1&\\&1\end{bmatrix}{}^tA=\begin{bmatrix}&1\\1&\end{bmatrix}$, from which it follows that

$$\begin{bmatrix}A'&\\&A\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&-\varpi&1&\\\varpi&&&1\end{bmatrix}\begin{bmatrix}A'^{-1}&\\&A^{-1}\end{bmatrix}=\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\&\varpi&&1\end{bmatrix}.$$

In any case we conclude that the three matrices

$$\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\&\varpi&&1\end{bmatrix},\quad\begin{bmatrix}1&&&\\&1&&\\&\varpi&1&\\\varpi&&&1\end{bmatrix},\quad\begin{bmatrix}1&&&\\&1&&\\&u\varpi&1&\\\varpi&&&1\end{bmatrix}$$

reduce to the two matrices

$$\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\&\varpi&&1\end{bmatrix},\quad\begin{bmatrix}1&&&\\&1&&\\&-u\varpi&1&\\\varpi&&&1\end{bmatrix}.$$

iii) Knowing the result of i), this is an easy exercise. ■

As an immediate consequence of this result we obtain the dimension of the space $V_0(2)$ of $\mathrm{Si}(\mathfrak{p}^2)$ invariant vectors in certain induced representations. Let χ_1, χ_2 and σ be unramified characters of $L^\times$, and let V be the standard space of the induced representation $\chi_1\times\chi_2\rtimes\sigma$. Then Lemma 2.3 i) implies that $\dim V_0(2)=12$. Similarly, let χ and σ be unramified characters of $L^\times$, and let V be the standard space of the Siegel induced representation $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$. Then Lemma 2.3 ii) implies that $\dim V_0(2) = 7$. Finally, if V is the standard space of the Klingen induced representation $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GL}(2)}$, then Lemma 2.3 iii) implies that $\dim V_0(2) = 4$. These results will be refined in Proposition 2.7 below.

2.3 The matrix of μ at level $\mathfrak{p}^2$

In this section we shall compute the eigenvalues of μ on the space $V_0(2)$ for the Iwahori-spherical representations V mentioned at the end of the previous section.

Let us denote the elements from Lemma 2.3 i) by $g_1, \ldots, g_{12}$. Now consider an induced representation $\chi_1 \times \chi_2 \rtimes \sigma$, where χ_1, χ_2, σ are unramified characters of $L^\times$. Functions f in the standard space V of this induced representation have the transformation property

$$f(\begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix} g) = |a^2b||c|^{-3/2}\chi_1(a)\chi_2(b)\sigma(c)f(g).$$

For $i = 1, \ldots, 12$ let f_i be the unique $\mathrm{Si}(\mathfrak{p}^2)$-invariant function in V such that $f_i(g_i) = 1$ and $f_i(g_j) = 0$ for $j \neq i$. Then $f_1, \ldots, f_{12}$ are a basis of $V_0(2)$.

2.4 Lemma. *Let notations be as above. If the residue characteristic of L is odd and $-1 \notin \mathfrak{o}^{\times 2}$, then the matrix of the endomorphism μ of $V_0(2)$ with respect to the basis $f_1, \ldots, f_{12}$ is given by*

$$\begin{bmatrix}
q(q+1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2q & 0 & 0 & (q-1)\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2q & 0 & 0 & (1-q^{-1})\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q+1 & 0 & 0 & 0 & 0 & 0 & (1-q^{-1})\alpha & (1-q^{-1})\beta & 0 \\
0 & q^2\beta^{-1} & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^2\alpha^{-1} & 0 & 0 & q^2 & 0 & 0 & q(q-1)\beta^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2q & \frac{(q-1)^2}{2} & 0 & q(q-1)\beta^{-1} & 0 & \frac{(q-1)^2}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & q-1 & \frac{q^2-1}{2} & 0 & q^2\beta^{-1} & q^2\alpha^{-1} & \frac{(q-1)^2}{2} \\
0 & 0 & 0 & 0 & 0 & (q-1)\beta & 0 & 0 & 2q & 0 & 0 & 0 \\
0 & 0 & 0 & q^2\alpha^{-1} & 0 & 0 & (1-q^{-1})\beta & \frac{q-1}{2}\beta & 0 & 2q-1 & 0 & \frac{q-3+2q^{-1}}{2}\beta \\
0 & 0 & 0 & q\beta^{-1} & 0 & 0 & 0 & \frac{1-q^{-1}}{2}\alpha & 0 & 0 & 2q-1 & \frac{1-q^{-1}}{2}\alpha \\
0 & 0 & 0 & 0 & 0 & 0 & q-1 & \frac{(q-1)^2}{2} & 0 & q(q-2)\beta^{-1} & q^2\alpha^{-1} & \frac{q^2+3}{2}
\end{bmatrix}$$

If the residue characteristic of L is odd and $-1 \in \mathfrak{o}^{\times 2}$, then the matrix is given by

$$\begin{bmatrix}
q(q{+}1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2q & 0 & 0 & (q{-}1)\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2q & 0 & 0 & (1{-}q^{-1})\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q{+}1 & 0 & 0 & 0 & 0 & 0 & (1{-}q^{-1})\alpha & (1{-}q^{-1})\beta & 0 \\
0 & q^2\beta^{-1} & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^2\alpha^{-1} & 0 & 0 & q^2 & 0 & 0 & q(q{-}1)\beta^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2q & \frac{(q-1)^2}{2} & 0 & q(q{-}1)\beta^{-1} & 0 & \frac{(q-1)^2}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & q{-}1 & \frac{q^2+3}{2} & 0 & q(q{-}2)\beta^{-1} & q^2\alpha^{-1} & \frac{(q-1)^2}{2} \\
0 & 0 & 0 & 0 & 0 & (q{-}1)\beta & 0 & 0 & 2q & 0 & 0 & 0 \\
0 & 0 & 0 & q^2\alpha^{-1} & 0 & 0 & (1{-}q^{-1})\beta & \frac{q-3+2q^{-1}}{2}\beta & 0 & 2q{-}1 & 0 & \frac{q-1}{2}\beta \\
0 & 0 & 0 & q\beta^{-1} & 0 & 0 & 0 & \frac{1-q^{-1}}{2}\alpha & 0 & 0 & 2q{-}1 & \frac{1-q^{-1}}{2}\alpha \\
0 & 0 & 0 & 0 & 0 & 0 & q{-}1 & \frac{(q-1)^2}{2} & 0 & q^2\beta^{-1} & q^2\alpha^{-1} & \frac{q^2-1}{2}
\end{bmatrix}$$

If the residue characteristic of L is even, then the matrix is given by

$$\begin{bmatrix}
q(q{+}1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2q & 0 & 0 & (q{-}1)\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2q & 0 & 0 & (1{-}q^{-1})\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q{+}1 & 0 & 0 & 0 & 0 & 0 & (1{-}q^{-1})\alpha & (1{-}q^{-1})\beta & 0 \\
0 & q^2\beta^{-1} & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^2\alpha^{-1} & 0 & 0 & q^2 & 0 & 0 & q(q{-}1)\beta^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q{+}1 & q(q{-}1) & 0 & 0 & 0 & q{-}1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & q^2{-}q{+}1 & 0 & q(q{-}1)\beta^{-1} & q^2\alpha^{-1} & q{-}2 \\
0 & 0 & 0 & 0 & 0 & (q{-}1)\beta & 0 & 0 & 2q & 0 & 0 & 0 \\
0 & 0 & 0 & q^2\alpha^{-1} & 0 & 0 & 0 & q^{-1}(q{-}1)^2\beta & 0 & 2q{-}1 & 0 & (1{-}q^{-1})\beta \\
0 & 0 & 0 & q\beta^{-1} & 0 & 0 & 0 & (1{-}q^{-1})\alpha & 0 & 0 & 2q{-}1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & q(q{-}2) & 0 & q^2\beta^{-1} & 0 & 2q{-}1
\end{bmatrix}$$

Here, we abbreviated $\alpha = \chi_1(\varpi)$ and $\beta = \chi_2(\varpi)$.

The **proof** consists of a long calculation, which we postpone until Sect. 2.5.

Next we treat the case of a Siegel induced representation $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$, where χ and σ are unramified characters of $L^\times$. Let $g_1, \ldots, g_7$ be the representatives for the double coset space $P(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$ listed in Lemma 2.3 ii). Let f_i be the unique $\mathrm{Si}(\mathfrak{p}^2)$-invariant function in the standard space V of $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$ with $f_i(g_i) = 1$ and $f_i(g_j) = 0$ for $j \neq i$. Then $f_1, \ldots, f_7$ are a basis of $V_0(2)$.

2.5 Lemma. *Let notations be as above. The matrix of the endomorphism μ of $V_0(2)$ with* *respect to the basis $f_1, \ldots, f_7$ is given by*

$$\begin{bmatrix} q(q+1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2q & 0 & \frac{q-1}{q^{1/2}}\alpha & 0 & 0 & 0 \\ 0 & 0 & q+1 & 0 & 0 & \frac{q^2-1}{q^{3/2}}\alpha & 0 \\ 0 & q^{5/2}\alpha^{-1} & 0 & q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(q+1)^2}{2} & q^{3/2}(q-1)\alpha^{-1} & \frac{(q-1)^2}{2} \\ 0 & 0 & q^{3/2}\alpha^{-1} & 0 & \frac{q-1}{2q^{1/2}}\alpha & 2q-1 & \frac{q-1}{2q^{1/2}}\alpha \\ 0 & 0 & 0 & 0 & \frac{q^2-1}{2} & q^{3/2}(q+1)\alpha^{-1} & \frac{q^2-1}{2} \end{bmatrix}$$

if the residue characteristic of L is odd (independent of -1 being a square or not), and by

$$\begin{bmatrix} q(q+1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2q & 0 & \frac{q-1}{q^{1/2}}\alpha & 0 & 0 & 0 \\ 0 & 0 & q+1 & 0 & 0 & \frac{q^2-1}{q^{3/2}}\alpha & 0 \\ 0 & q^{5/2}\alpha^{-1} & 0 & q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q+1 & 0 & q^2-1 \\ 0 & 0 & q^{3/2}\alpha^{-1} & 0 & 0 & 2q-1 & \frac{q-1}{q^{1/2}}\alpha \\ 0 & 0 & 0 & 0 & 1 & q^{5/2}\alpha^{-1} & q^2-1 \end{bmatrix}$$

if the residue characteristic of L is even. In both cases we abbreviated $\alpha = \chi(\varpi)$.

Proof: The calculations are similar, but easier, as in Lemma 2.4. We omit the details except for one useful matrix identity:

$$\begin{bmatrix} 1 & \frac{1}{2} & & \\ 1 & -\frac{1}{2} & & \\ & & -1 & \frac{1}{2} \\ & & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -\varpi & 1 & \\ \varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & & \\ 1 & -\frac{1}{2} & & \\ & & -1 & \frac{1}{2} \\ & & 1 & \frac{1}{2} \end{bmatrix}.$$

Alternatively, the result can be deduced from Lemma 2.4, observing that $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma \hookrightarrow \chi\nu^{-1/2} \times \chi\nu^{1/2} \rtimes \sigma$. ∎

Finally, we treat the case of a Klingen induced representation $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GL}(2)}$, where χ and σ are unramified characters of $L^\times$. Let $g_1, \ldots, g_4$ be the representatives for the double coset space $P(L)\backslash G(L)/\mathrm{Si}(\mathfrak{p}^2)$ listed in Lemma 2.3 ii). Let f_i be the unique $\mathrm{Si}(\mathfrak{p}^2)$-invariant function in the standard space V of $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$ with $f_i(g_i) = 1$ and $f_i(g_j) = 0$ for $j \neq i$. Then $f_1, \ldots, f_4$ are a basis of $V_0(2)$.

2.6 Lemma. *Let notations be as above. The matrix of the endomorphism μ of $V_0(2)$ with respect to the basis $f_1,\ldots,f_4$ is given by*

$$\begin{bmatrix} q(q+1) & 0 & 0 & 0 \\ 0 & 2q & (1-q^{-1})\chi(\varpi) & 0 \\ 0 & q^2\chi(\varpi^{-1}) & q^2 & q-1 \\ 0 & 0 & q(q-1) & 2q \end{bmatrix}$$

(independent of the residue characteristic).

Proof: Easy calculation. ■

2.7 Proposition. *The following table lists, for certain induced representations V, the dimension of the space $V_0(2)$ of $\mathrm{Si}(\mathfrak{p}^2)$ invariant vectors, together with the dimensions of the eigenspaces of the operator μ on $V_0(2)$ with respect to the four possible eigenvalues $q(q+1)$, q, $2q$ and 0.*

V	$V_0(2)$	$q(q+1)$	q	$2q$	0
$\chi_1\times\chi_2\rtimes\sigma$	12	4	4	3	1
$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	7	3	2	1	1
$\chi\rtimes\sigma 1_{\mathrm{GL}(2)}$	4	2	1	1	0

Here, χ_1, χ_2, χ and σ are unramified characters of $L^\times$. (This result is independent of the residue characteristic of L.)

Proof: This follows by determining the eigenvalues of the matrices computed in the previous lemmas. ■

2.4 Eigenvalues of μ at level $\mathfrak{p}^2$ for Iwahori-spherical representations

Let (π,V) be a smooth representation of $\mathrm{GSp}(4,L)$ for which the center acts trivially. In Sect. 3.2 we introduced the simple level raising operator $V_0(n)\to V_0(n+1)$. We define an additional level raising operator $\alpha_2:\ V_0(n)\to V_0(n+2)$ by

$$\alpha_2 v = \sum_{A\in\mathrm{SL}(2,\mathfrak{o})/\left[\begin{smallmatrix}\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}&\mathfrak{o}\end{smallmatrix}\right]} \pi(\begin{bmatrix} A & \\ & A'\end{bmatrix}\begin{bmatrix}\varpi^{-1} &&& \\ & 1 && \\ && 1 & \\ &&& \varpi\end{bmatrix})v \qquad (v\in V_0(n),\ n\geq 0). \tag{24}$$

Observe that α_2 skips one level. An explicit formula is

$$\alpha_2 v = \pi(\begin{bmatrix}\varpi^{-1} &&& \\ & 1 && \\ && 1 & \\ &&& \varpi\end{bmatrix})v + \sum_{x\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1 & x && \\ & 1 && \\ && 1 & -x \\ &&& 1\end{bmatrix}\begin{bmatrix}1 &&& \\ & \varpi^{-1} && \\ && \varpi & \\ &&& 1\end{bmatrix})v. \tag{25}$$

2.8 Lemma. *Let (π, V) be an unramified representation of type V or VI. Let $v_0 \in V$ be a non-zero* $\mathrm{GSp}(4, \mathfrak{o})$ *invariant vector. Then the four vectors*

$$v_0, \qquad \beta v_0, \qquad \beta^2 v_0, \qquad \alpha_2 v_0 \tag{26}$$

are linearly independent. In particular, $\dim(V_0(2)) \geq 4$.

Proof: Let ξ be an unramified quadratic character of $L^\times$, which may be trivial. Let V be the standard space of the representation $\nu^{-1/2}\xi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \xi\nu^{1/2}\sigma^{-1}$. Then $L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$, which is of type Vd if ξ is non-trivial and of type VId if ξ is trivial, can be realized as a subrepresentation of V. Hence, it is enough to show that the four vectors (26) are linearly independent when $v_0 \in V$ is the spherical vector. Each one of these four vectors lies in $V_0(2)$. Let $f_1, \ldots, f_7$ be the basis of $V_0(2)$ defined before Lemma 2.5. Straightforward calculations show that in this basis the vectors (26) are given by the columns of the matrix

$$\begin{bmatrix} 1 & q\xi(\varpi)\sigma(\varpi) & q^2 & q(q+1)\xi(\varpi) \\ 1 & \sigma(\varpi) & 1 & (q+1)\xi(\varpi) \\ 1 & q^{-1}\xi(\varpi)\sigma(\varpi) & q^{-2} & (q^{-1}+1)\xi(\varpi) \\ 1 & q\xi(\varpi)\sigma(\varpi) & q\xi(\varpi) & q(1+\xi(\varpi)) \\ 1 & q\xi(\varpi)\sigma(\varpi) & 1 & 2q\xi(\varpi)+q-1 \\ 1 & \sigma(\varpi) & q^{-1}\xi(\varpi) & 1+\xi(\varpi) \\ 1 & q\xi(\varpi)\sigma(\varpi) & 1 & q+1 \end{bmatrix}$$

if the residue characteristic of L is odd, and by the columns of the matrix

$$\begin{bmatrix} 1 & q\xi(\varpi)\sigma(\varpi) & q^2 & q(q+1)\xi(\varpi) \\ 1 & \sigma(\varpi) & 1 & (q+1)\xi(\varpi) \\ 1 & q^{-1}\xi(\varpi)\sigma(\varpi) & q^{-2} & (q^{-1}+1)\xi(\varpi) \\ 1 & q\xi(\varpi)\sigma(\varpi) & q\xi(\varpi) & q(1+\xi(\varpi)) \\ 1 & q\xi(\varpi)\sigma(\varpi) & 1 & q(q+1)\xi(\varpi) \\ 1 & \sigma(\varpi) & q^{-1}\xi(\varpi) & 1+\xi(\varpi) \\ 1 & q\xi(\varpi)\sigma(\varpi) & 1 & q(1+\xi(\varpi)) \end{bmatrix}$$

if the residue characteristic of L is even (only the $(5,4)$ and the $(7,4)$ coefficient are different). In any case this matrix has rank 4. This proves the lemma. ■

2.9 Lemma. *Let V be the standard space of an induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ with unramified characters χ_1, χ_2 and σ such that $\chi_1\chi_2\sigma^2 = 1$. Let $\alpha = \chi_1(\varpi)$ and $\beta = \chi_2(\varpi)$. Then the matrix of the Atkin-Lehner involution u_2 on the 12-dimensional space $V_0(2)$ with* *respect to the basis* $f_1,\ldots,f_{12}$ *introduced before Lemma 2.4 is given by*

$$\begin{bmatrix} 0&0&0&q^3\alpha^{-1}\beta^{-1}&0&0&0&0&0&0&0&0\\ 0&0&q\alpha^{-1}\beta&0&0&0&0&0&0&0&0&0\\ 0&q^{-1}\alpha\beta^{-1}&0&0&0&0&0&0&0&0&0&0\\ q^{-3}\alpha\beta&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&q^2\alpha^{-1}&0\\ 0&0&0&0&0&0&0&0&0&q\beta^{-1}&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&q^{-1}\beta&0&0&0&0&0&0\\ 0&0&0&0&q^{-2}\alpha&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

Let V *be the standard space of an induced representation* $\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$ *with unramified characters* χ *and* σ *such that* $\chi^2\sigma^2=1$. *Let* $\alpha=\chi(\varpi)$. *Then the matrix of the Atkin-Lehner involution* u_2 *on the 7-dimensional space* $V_0(2)$ *with respect to the basis* $f_1,\ldots,f_7$ *introduced before Lemma 2.5 is given by*

$$\begin{bmatrix} 0&0&q^3\alpha^{-2}&0&0&0&0\\ 0&1&0&0&0&0&0\\ q^{-3}\alpha^2&0&0&0&0&0&0\\ 0&0&0&0&0&q^{3/2}\alpha^{-1}&0\\ 0&0&0&0&1&0&0\\ 0&0&0&q^{-3/2}\alpha&0&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix}.$$

Let V *be the standard space of an induced representation* $\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$ *with unramified characters* χ *and* σ *such that* $\chi\sigma^2=1$. *Let* $\alpha=\chi(\varpi)$. *Then the matrix of the Atkin-Lehner involution* u_2 *on the 4-dimensional space* $V_0(2)$ *with respect to the basis* $f_1,\ldots,f_4$ *introduced before Lemma 2.6 is given by*

$$\begin{bmatrix} 0&q^2\alpha^{-1}&0&0\\ q^{-2}\alpha&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}.$$

(These results are independent of the residue characteristic.)

Proof: These are easy calculations. ∎

2.10 Lemma. *Let* (π,V) *be an Iwahori-spherical, irreducible, admissible representation of* $\mathrm{GSp}(4,L)$ *with trivial central character. Then the space* $V_0(2)$ *consists not exclusively of* q*-eigenvectors, and also not exclusively of* $2q$*-eigenvectors, for the* μ *operator.*

Proof: We can realize V as a subspace of the standard space W of an induced representation of the form $\chi_1\times\chi_2\rtimes\sigma$ with χ_1, χ_2 and σ unramified characters of $L^\times$. Let $i=q$ or $i=2q$. Let $V_0(2)_i$ be the i-eigenspace for μ on $V_0(2)$, and let $W_0(2)_i$ be the i-eigenspace for μ on $W_0(2)$.

Evidently, $V_0(2)_i \subset W_0(2)_i$. Now assume that $V_0(2) = V_0(2)_i$. Since $V_0(2)$ is Atkin–Lehner invariant, this implies that there exists a $w \in W_0(2)_i$ which is an Atkin–Lehner eigenvector. Then $w \in W_0(2)_i \cap u_2 W_0(2)_i$. But a calculation using the matrices from Lemma 2.4 and Lemma 2.9 shows that

$$W_0(2)_i \cap u_2 W_0(2)_i = 0.$$

This contradiction proves the lemma. ■

2.11 Theorem. *Table 1 below lists for $n = 0, 1, 2$ the dimensions of the spaces $V_0(n)$ of vectors invariant under the groups $\mathrm{Si}(\mathfrak{p}^n)$ for each Iwahori-spherical, irreducible, admissible representation (π, V) of $\mathrm{GSp}(4, L)$ with trivial central character. The signs under each dimension indicate Atkin–Lehner eigenvalues.*[1] *In addition, the last four columns show the dimensions of the eigenspaces of the μ operator on the space $V_0(2)$ for each of the four possible eigenvalues $q(q+1)$, q, $2q$ and 0.*

Proof: The dimensions and Atkin-Lehner eigenvalues for $V_0(0)$ and $V_0(1)$ have been recorded in [Sch1], Sect. 1.3, and [RS2], Table A.15. In this proof we shall be concerned with the $V_0(2)$ column. The dimensions for groups I, IIb and IIIb follows immediately from Proposition 2.7. The dimensions of the Atkin–Lehner eigenspaces for these representations can be determined from the matrices occurring in Lemma 2.9. The entries for IIa and IIIa are then obtained by subtracting the numbers for IIb resp. IIIb from the numbers for the full induced representation in which they occur. Similarly, the μ-eigenvalues for group IV are easily obtained from the way the full induced representation $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$ decomposes (see [RS2], (2.9)), since the numbers for the trivial representation are obvious.

By Lemma 2.8, the dimension of $V_0(2)$ for the representation of type VId is at least 4. On the other hand, $\mathrm{VIc} + \mathrm{VId} = 1_{L^\times} \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$, and the dimensions for this induced representation are given in Proposition 2.7. This explains the VIc and the VId row. The rest of group VI follows from the way the full induced representation $\nu \times 1_{L^\times} \rtimes \nu^{-1/2}\sigma$ decomposes; see [RS2], (2.11).

In the rest of this proof we shall explain the dimensions and eigenvalues at level $\mathfrak{p}^2$ for group V. First of all, the multiplicity of the eigenvalue $q(q+1)$ in each case equals the dimension of $V_0(1)$, by Proposition 2.1 iii). Second, observe that the dimensions and eigenvalues for Vb and Vc coincide, since these two representations differ by an unramified, quadratic twist. Since the kernel of μ on the $\mathrm{Si}(\mathfrak{p}^2)$-space of the full induced representation is only one-dimensional by Proposition 2.7, it follows that Vb and Vc must have a zero entry in the last column. But $\mathrm{Vb} + \mathrm{Vd} = \nu^{1/2}\xi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2}\sigma$, and the eigenvalues for this degenerate principal series representation are given in Proposition 2.7. It follows that Vd has a 1 in the last column, from which it follows in turn that Va has a 0 in the last column.

Next, we shall prove that the total $\mathrm{Si}(\mathfrak{p}^2)$ dimension for Vb is ≥ 3 by realizing Vb as a subrepresentation of $\nu^{1/2}\xi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2}$ (we may assume $\sigma = 1$) and applying enough linear operators

[1] The Atkin–Lehner eigenvalues listed in this table are correct if one assumes that

- in group II, where the central character is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial.
- in groups IV, V and VI, where the central character is σ^2, the character σ itself is trivial.

If these assumptions are not met, then one has to interchange the plus and minus signs in the $V_0(1)$ column.

Table 1: Eigenvalues of μ on level $\mathfrak{p}^2$ vectors in Iwahori-spherical representations

		representation	$V_0(0)$	$V_0(1)$	$V_0(2)$	$V_0(2)$ $q(q+1)$	$V_0(2)$ q	$V_0(2)$ $2q$	$V_0(2)$ 0
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	1 $+$	4 $++--$	12 $+: 8$ $-: 4$	4	4	3	1
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	0	1 $-$	5 $+++$ $--$	1	2	2	0
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	1 $+$	3 $++-$	7 $++++$ $+--$	3	2	1	1
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	0	2 $+-$	8 $++++$ $+---$	2	3	2	1
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$	1 $+$	2 $+-$	4 $+++-$	2	1	1	0
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	0	0	2 $+-$	0	1	1	0
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	0	2 $+-$	6 $++++$ $--$	2	2	1	1
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	0	1 $-$	3 $++-$	1	1	1	0
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(4)}$	1 $+$	1 $+$	1 $+$	1	0	0	0
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	0	0	2 $+-$	0	1	1	0
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	0	1 $+$	3 $++-$	1	1	1	0
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	0	1 $-$	3 $++-$	1	1	1	0
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	1 $+$	2 $+-$	4 $+++-$	2	1	0	1
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	0	1 $-$	5 $+++$ $--$	1	2	2	0
	b	$\tau(T, \nu^{-1/2}\sigma)$	0	1 $+$	3 $++-$	1	1	0	1
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	0	0	0	0	0	0	0
	d	$L(\nu, 1_{F^*} \rtimes \nu^{-1/2}\sigma)$	1 $+$	2 $+-$	4 $+++-$	2	1	1	0

to the essentially unique $\mathrm{Si}(\mathfrak{p})$-invariant vector in the subspace realizing Vb. The only problem is to identify this $\mathrm{Si}(\mathfrak{p})$-invariant vector $\tilde{h}$, but this problem was solved in [Sch2]. In the basis $f_1, \ldots f_7$ introduced before Lemma 2.5 it is given by the transpose of

$$(-q^2(q+1), (1-q)q, q+1, -q^2(q+1), -q^2(q+1), (1-q)q, -q^2(q+1)).$$

A computation using Lemmas 2.5 and 2.9 shows that $\tilde{h}$, $\mu\tilde{h}$ and $u_2\tilde{h}$ are linearly independent. Hence the $\mathrm{Si}(\mathfrak{p}^2)$ dimensions for Vb and Vc are at least three. Since $\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2} = \mathrm{Vb} + \mathrm{Vd}$, it follows from Proposition 2.7 that they are exactly three, and that the dimension for VId is 4. Again from Proposition 2.7 it follows that the $\mathrm{Si}(\mathfrak{p}^2)$ dimension for Va is 2.

Combining the latter fact with Lemma 2.10 shows that Va has one q-eigenvector and one $2q$-eigenvector. Hence we are done with Va. The representation Vb cannot have 2 linearly independent q-eigenvectors, since the same would then be true for Vc, and the total q-dimensions in group V would exceed 4. For the same reason Vb cannot have two linearly independent $2q$-eigenvectors. It follows that the q-dimensions and the $2q$-dimensions for Vb,c are 1. This finally implies that Vd has one q-eigenvector but no $2q$-eigenvector.

The Atkin-Lehner eigenvalues for group V and level $\mathfrak{p}^2$ are now easily computed in the induced models, since, as explained above, the spaces of $\mathrm{Si}(\mathfrak{p}^2)$-vectors in appropriate subrepresentations are explicitly known (at least for Vb,c,d; the Atkin-Lehner eigenvalues for Va are then obtained by subtracting). ■

2.5 Proof of Lemma 2.4

2.12 Lemma. *Using the notations explained before Lemma 2.4, we have, for any f in $V_0(2)$, the following formulas in the case of odd residue characteristic.*

i)

$$\sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} f(\begin{bmatrix}1&&&\\-z&1&&\\&&1&\\\varpi&&z&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&-z^{-1}\varpi&1&\\&&&1\end{bmatrix})$$

$$= f(g_7) + \frac{q-1}{2}f(g_8) + \frac{q-1}{2}f(g_{12}) - f(\begin{bmatrix}1&&&\\&1&&\\&-\varpi&1&\\\varpi&&&1\end{bmatrix}).$$

ii)

$$\sum_{z\in\mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\\varpi^{-1}&\varpi^{-1}&1&\\z\varpi^{-1}&\varpi^{-1}&&1\end{bmatrix}s_2s_1s_2) = q^{-3}\chi_1(\varpi)\chi_2(\varpi)f(g_7) + q^{-2}\chi_1(\varpi)f(g_{10})$$

$$+ q^{-3}\chi_1(\varpi)\chi_2(\varpi)\frac{q-1}{2}(f(g_8)+f(g_{12})) - q^{-3}\chi_1(\varpi)\chi_2(\varpi)f(\begin{bmatrix}1&&&\\&1&&\\&-\varpi&1&\\\varpi&&&1\end{bmatrix}).$$

In the case of even residue characteristic, one has to replace g_{12} and the matrix occurring on the right side of each formula by g_8, and replace g_7 by g_{12}.

Proof: We shall restrict to the case of odd residue characteristic; the even case is treated similarly.

i) For $z \notin 1+\mathfrak{p}$ we use the identity

$$\begin{bmatrix}1&&&\\-z&1&&\\&&1&\\\varpi&&z&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&-z^{-1}\varpi&1&\\&&&1\end{bmatrix}$$
$$=\frac{1}{1-z}\begin{bmatrix}1&1&&\\&1-z&&\\&&1&-1\\&&&1-z\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&-\varpi z^{-1}&1&\\\varpi&&&1\end{bmatrix}\begin{bmatrix}1&-1&&\\-z&1&&\\&&1&1\\&&z&1\end{bmatrix},$$

while for $z=1$ we use the identity

$$\begin{bmatrix}1&&&\\-1&1&&\\&&1&\\\varpi&&1&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&-\varpi&1&\\&&&1\end{bmatrix}=\begin{bmatrix}1&-\frac12&&\\&1&&\\&&-1&-\frac12\\&&&-1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\&\varpi&&1\end{bmatrix}\begin{bmatrix}\frac12&\frac12&&\\-1&1&&\\&&-\frac12&\frac12\\&&-1&-1\end{bmatrix}.$$

We obtain

$$\sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} f(\begin{bmatrix}1&&&\\-z&1&&\\&&1&\\\varpi&&z&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&-z^{-1}\varpi&1&\\&&&1\end{bmatrix})$$
$$=f(\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\&\varpi&&1\end{bmatrix})+\sum_{\substack{z\in(\mathfrak{o}/\mathfrak{p})^\times\\ z\notin 1+\mathfrak{p}}} f(\begin{bmatrix}1&&&\\&1&&\\&-\varpi z^{-1}&1&\\\varpi&&&1\end{bmatrix})$$
$$=f(g_7)+\sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} f(\begin{bmatrix}1&&&\\&1&&\\&-\varpi z^{-1}&1&\\\varpi&&&1\end{bmatrix})-f(\begin{bmatrix}1&&&\\&1&&\\&-\varpi&1&\\\varpi&&&1\end{bmatrix})$$
$$=f(g_7)+\frac{q-1}{2}f(g_8)+\frac{q-1}{2}f(g_{12})-f(\begin{bmatrix}1&&&\\&1&&\\&-\varpi&1&\\\varpi&&&1\end{bmatrix}).$$

ii) Using the identities

$$\begin{bmatrix}1&&&\\&1&&\\\varpi^{-1}&\varpi^{-1}&1&\\&\varpi^{-1}&&1\end{bmatrix}s_2s_1s_2=\begin{bmatrix}\varpi&\varpi&-1&\\&-\varpi&&1\\&&-\varpi^{-1}&\varpi^{-1}\\&&&\varpi^{-1}\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\\varpi&\varpi&1&\\&\varpi&&1\end{bmatrix}\begin{bmatrix}1&&&\\&-1&&\\&&-1&\\&&&1\end{bmatrix}\tag{27}$$

and (23), the term for $z=0$ equals $q^{-3}\chi_1(\varpi)\chi_2(\varpi)f(g_7)$. Using the identity

$$\begin{bmatrix}1&&&\\&1&&\\\varpi^{-1}&\varpi^{-1}&1&\\\varpi^{-1}&\varpi^{-1}&&1\end{bmatrix}s_2s_1s_2=\begin{bmatrix}-\varpi&-1&&1\\&1&&\\&&-1&\varpi^{-1}\\&&&\varpi^{-1}\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix}s_2\begin{bmatrix}&-1&&\\-1&1&&\\&&&1\\&&1&1\end{bmatrix},$$

the term for $z=1$ equals $q^{-2}\chi_1(\varpi)f(g_{10})$. For $z\notin\mathfrak{p}$ and $z\notin 1+\mathfrak{p}$ we use the identity

$$\begin{aligned}&\begin{bmatrix}1&&&\\&1&&\\\varpi^{-1}&\varpi^{-1}&1&\\z\varpi^{-1}&\varpi^{-1}&&1\end{bmatrix}s_2s_1s_2\\&=\begin{bmatrix}\varpi&-\varpi&\frac{1}{z-1}&-1\\&\varpi z&\frac{-z}{z-1}&\\&&-\varpi^{-1}&-\varpi^{-1}\\&&&-\varpi^{-1}z\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&(z-1)\varpi&1&\\\varpi&&&1\end{bmatrix}\begin{bmatrix}&z^{-1}&&\\\frac{1}{z-1}&\frac{-z^{-1}}{z-1}&&\\&&&z^{-1}-1\\&&-1&-z^{-1}\end{bmatrix},\end{aligned}$$

and the assertion follows easily. ■

Proof of Lemma 2.4. The (i,j)-entry of the matrix of μ on $V_0(2)$ is given by

$$(\mu f_j)(g_i)=\sum_{x,z\in\mathfrak{o}/\mathfrak{p}}f_j(g_i\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})+\sum_{z\in\mathfrak{o}/\mathfrak{p}}f_j(g_i\begin{bmatrix}1&&&z\varpi^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}).$$

The of these numbers is easy to compute. For example, it is immediate that

$$(\mu f)(1)=q(q+1)f(1)$$

for any f, giving the first row of the matrix. The main tool for the remaining cases is the matrix identity

$$\begin{bmatrix}1&\\x&1\end{bmatrix}=\begin{bmatrix}1&x^{-1}\\&1\end{bmatrix}\begin{bmatrix}-x^{-1}&\\&-x\end{bmatrix}\begin{bmatrix}&1\\-1&\end{bmatrix}\begin{bmatrix}1&x^{-1}\\&1\end{bmatrix}.\tag{28}$$

The most difficult cases are $(\mu f)(g_8)$ and $(\mu f)(g_{12})$, which are treated similarly. As an example, we shall compute $(\mu f)(g_8)=A+B$, where

$$A=\sum_{x,z\in\mathfrak{o}/\mathfrak{p}}f(\begin{bmatrix}1&&&\\&1&&\\&\varpi&1&\\\varpi&&&1\end{bmatrix}\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})$$

and

$$B = \sum_{z \in \mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ \varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}).$$

We shall carry out the calculations only in the case of odd residue characteristic; the even case requires slight modifications, but uses the same matrix identities. We begin by calculating B, which is easier. If $z \notin -1 + \mathfrak{p}$, then

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ \varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{z+1} & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & z+1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ \frac{\varpi}{z+1} & & & 1 \end{bmatrix},$$

while if $z = -1$, then

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ \varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & z\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} -\varpi^{-1} & & & -1 \\ & 1 & & \\ & & 1 & \\ & & & -\varpi \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1.$$

It follows easily from these identities that

$$B = q^2 \chi_1(\varpi)^{-1} f(g_{11}) + \frac{q-1}{2} f(g_8) + \frac{q-1}{2} f(g_{12}).$$

Next we compute

$$A = \sum_{x,z \in \mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi x & \varpi x^2 & 1 & \\ \varpi & \varpi x & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & z\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}).$$

It is necessary to distinguish two cases.

<u>Case I:</u> $-1 \notin (\mathfrak{o}/\mathfrak{p})^2$

We write $A = A_1 + A_2$, where $A_1 = \sum_{z \in \mathfrak{o}/\mathfrak{p}} \ldots$ and $A_2 = \sum_{x \in (\mathfrak{o}/\mathfrak{p})^\times} \sum_{z \in \mathfrak{o}/\mathfrak{p}} \ldots$. It is not hard to compute that

$$A_1 = q\chi_2(\varpi)^{-1} f(g_{10}) + \frac{q-1}{2} f(g_8) + \frac{q-1}{2} f(g_{12}).$$

To compute A_2, we use the matrix identity

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi x & \varpi x^2 & 1 & \\ \varpi & \varpi x & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & z\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & \frac{x\varpi^{-1}}{1+x^2} & \frac{x^{-1}}{1+x^2} & \\ & & x^{-1}\varpi & \\ & & & \frac{1}{1+x^2} \end{bmatrix}$$

$$\begin{bmatrix}1&&&\\-\varpi&1&&\\&&1&\\\varpi&&\varpi&1\end{bmatrix}s_2\begin{bmatrix}1&&&\\-x^2&1&x^2(z(1+x^2)+1)\varpi^{-1}&\\&&1&\\&&x^2&1\end{bmatrix}\begin{bmatrix}1&&&\\&-x(1+x^2)&&\\&&-x^{-1}&\\&&&1+x^2\end{bmatrix}$$

If z runs through $\mathfrak{o}/\mathfrak{p}$, then $x^2(z(1+x^2)+1)$ does as well. Hence

$$A_2=q\chi_2(\varpi)^{-1}\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}\sum_{z\in\mathfrak{o}/\mathfrak{p}}f(\begin{bmatrix}1&&&\\-\varpi&1&&\\&&1&\\\varpi&&\varpi&1\end{bmatrix}s_2\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})\tag{29}$$

The matrix identity

$$\begin{bmatrix}1&&&\\-\varpi&1&&\\&&1&\\\varpi&&\varpi&1\end{bmatrix}=\begin{bmatrix}1&&-1&\\&1&\varpi&-1\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix}\begin{bmatrix}1&&1&\\&1&&1\\&&1&\\&&&1\end{bmatrix}\tag{30}$$

shows that

$$\begin{aligned}A_2&=q\chi_2(\varpi)^{-1}\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}f(g_{10})+q\chi_2(\varpi)^{-1}\sum_{x,z\in(\mathfrak{o}/\mathfrak{p})^\times}f(\begin{bmatrix}1&&&\\\varpi&1&&\\&&1&\\\varpi&&-\varpi&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z\varpi^{-1}&1&\\&&&1\end{bmatrix}s_2)\\&=q(q-1)\chi_2(\varpi)^{-1}f(g_{10})\\&\quad+q\chi_2(\varpi)^{-1}\sum_{x,z\in(\mathfrak{o}/\mathfrak{p})^\times}f(\begin{bmatrix}1&&&\\\varpi&1&&\\&&1&\\\varpi&&-\varpi&1\end{bmatrix}\begin{bmatrix}1&&&\\&-z^{-1}\varpi&&\\&&-z\varpi^{-1}&\\&&&1\end{bmatrix}s_2\begin{bmatrix}1&&&\\&1&z^{-1}\varpi&\\&&1&\\&&&1\end{bmatrix}s_2)\\&=q(q-1)\chi_2(\varpi)^{-1}f(g_{10})+(q-1)\sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times}f(\begin{bmatrix}1&&&\\-z&1&&\\&&1&\\\varpi&&z&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&-z^{-1}\varpi&1&\\&&&1\end{bmatrix}).\end{aligned}$$

By Lemma 2.12 we get

$$A_2=q(q-1)\chi_2(\varpi)^{-1}f(g_{10})+(q-1)\big(f(g_7)+\frac{q-1}{2}f(g_8)+\frac{q-1}{2}f(g_{12})-f(g_{12})\big)$$

Adding A_1 and A_2, it follows that

$$A=(q-1)f(g_7)+\frac{q(q-1)}{2}f(g_8)+q^2\chi_2(\varpi)^{-1}f(g_{10})+\frac{(q-2)(q-1)}{2}f(g_{12}).$$

Adding A and B, we get

$$(\mu f)(g_8)=(q-1)f(g_7)+\frac{q^2-1}{2}f(g_8)+q^2\chi_2(\varpi)^{-1}f(g_{10})+q^2\chi_1(\varpi)^{-1}f(g_{11})+\frac{(q-1)^2}{2}f(g_{12}).$$

This gives the eigth row in our matrix for the case that $-1 \notin (\mathfrak{o}/\mathfrak{p})^2$.

Case II: $-1 \in (\mathfrak{o}/\mathfrak{p})^2$

Let $x_0 \in \mathfrak{o}^\times$ such that $x_0^2 = -1$. We have $A = A_1 + A_2$ with

$$A_1 = \sum_{\substack{x,z\in\mathfrak{o}/\mathfrak{p}\\ x\neq\pm x_0}} f(\begin{bmatrix}1&&&\\&1&&\\ \varpi x&\varpi x^2&1&\\ \varpi&\varpi x&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&\varpi&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix}).$$

and

$$A_2 = 2\sum_{z\in\mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\ \varpi x_0&&1&\\ \varpi&\varpi x_0&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix}).$$

Up to the point (29), the calculation of A_1 proceeds like the calculation of A in Case I. We get $A_1 = A_{11} + A_{12}$ with

$$A_{11} = q\chi_2(\varpi)^{-1}\sum_{z\in\mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\ \varpi&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z\varpi^{-1}&1&\\&&&1\end{bmatrix}s_2)$$

and

$$A_{12} = q\chi_2(\varpi)^{-1}\sum_{\substack{x\in(\mathfrak{o}/\mathfrak{p})^\times\\ x\neq\pm x_0}}\sum_{z\in\mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix}1&&&\\ \varpi&1&&\\&&1&\\ \varpi&&-\varpi&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z\varpi^{-1}&1&\\&&&1\end{bmatrix}s_2).$$

Hence A_{11} coincides with the previous A_1, and A_{12} is $\frac{q-3}{q-1}$ times the previous A_2. Thus

$$A_{11} = q\chi_2(\varpi)^{-1}f(g_{10}) + \frac{q-1}{2}f(g_8) + \frac{q-1}{2}f(g_{12})$$

and

$$A_{12} = \frac{q-3}{q-1}\Big(q(q-1)\chi_2(\varpi)^{-1}f(g_{10}) + (q-1)\big(f(g_7) + \frac{q-1}{2}f(g_8) + \frac{q-1}{2}f(g_{12}) - f(g_8)\big)\Big)$$

Note that the last term, which comes from Lemma 2.12, is $f(g_8)$ and not $f(g_{12})$, since -1 is now a square. Combining we get

$$A_1 = (q-3)f(g_7) + \frac{q^2-5q+8}{2}f(g_8) + q(q-2)\chi_2(\varpi)^{-1}f(g_{10}) + \frac{(q-1)(q-2)}{2}f(g_{12}).$$

Finally,

$$A_2 = 2\sum_{z\in\mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\ \varpi&&1&\\ \varpi&\varpi&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix}).$$

We use the identity

$$\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\\varpi&\varpi&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix}$$
$$=\frac{1}{z+1}\begin{bmatrix}1&-1&&\\&z+1&z\varpi^{-1}&z\varpi^{-1}\\&&1&1\\&&&z+1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&-\varpi&1&\\\varpi&&&1\end{bmatrix}\begin{bmatrix}1&1&&\\-z&1&&\\&&1&-1\\&&z&1\end{bmatrix}$$

for $z\notin -1+\mathfrak{p}$, and

$$\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\\varpi&\varpi&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&-\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix}=\begin{bmatrix}1&&&\\&1&-\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\&\varpi&&1\end{bmatrix}\begin{bmatrix}1&&&\\1&1&&\\&&1&\\&&-1&1\end{bmatrix}$$

for $z=-1$, and get

$$A_2=2f(g_7)+2(q-1)f(g_8).$$

Adding A_1 and A_2 gives

$$A=(q-1)f(g_7)+\frac{q^2-q+4}{2}f(g_8)+q(q-2)\chi_2(\varpi)^{-1}f(g_{10})+\frac{(q-1)(q-2)}{2}f(g_{12}).$$

Adding A and B, we get

$$(\mu f)(g_8)=(q-1)f(g_7)+\frac{q^2+3}{2}f(g_8)+q(q-2)\chi_2(\varpi)^{-1}f(g_{10})+q^2\chi_1(\varpi)^{-1}f(g_{11})+\frac{(q-1)^2}{2}f(g_{12}).$$

This gives the eigth row of our matrix in the second case. — Another complicated case, and the last one we shall treat in this proof, is $(\mu f)(g_{10})=A+B$ with

$$A=\sum_{x,z\in\mathfrak{o}/\mathfrak{p}}f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix}s_2\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})$$

and

$$B=\sum_{z\in\mathfrak{o}/\mathfrak{p}}f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix}s_2\begin{bmatrix}1&&&z\varpi^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}).$$

We write A as A_1+A_2, where $A_1=\sum_{z\in\mathfrak{o}/\mathfrak{p}}\ldots$ and $A_2=\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}\sum_{z\in\mathfrak{o}/p}\ldots$. Straightforward calculations using only (28) show that

$$B=q^2\chi_1(\varpi)^{-1}f(g_4)+(q-1)f(g_{10})$$

and
$$A_1 = f(g_{10}) + q^{-1}\chi_2(\varpi)\frac{q-1}{2}f(g_8) + q^{-1}\chi_2(\varpi)\frac{q-1}{2}f(g_{12})$$
(again assuming odd residue characteristic; in the even case replace g_{12} by g_8). In order to compute A_2 we use the identity
$$\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix}s_2\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix}=\begin{bmatrix}-x^{-1}\varpi^{-1}&-\varpi^{-1}&&x\\&-x&&\\&&x^{-1}&-1\\&&&x\varpi\end{bmatrix}$$
$$=\begin{bmatrix}1&&&\\&1&&\\\varpi^{-1}x^{-1}&\varpi^{-1}&1&\\\varpi^{-1}(z+x^{-2})&\varpi^{-1}x^{-1}&&1\end{bmatrix}s_2s_1s_2\begin{bmatrix}-1&&&\\-x^{-1}&-1&&\\&&1&\\&&-x^{-1}&1\end{bmatrix}.$$
It shows that
$$A_2 = q^2(q-1)\chi_1(\varpi)^{-1}\sum_{z\in\mathfrak{o}/\mathfrak{p}}f(\begin{bmatrix}1&&&\\&1&&\\\varpi^{-1}&\varpi^{-1}&1&\\z\varpi^{-1}&\varpi^{-1}&&1\end{bmatrix}s_2s_1s_2).$$
By Lemma 2.12 ii),
$$\begin{aligned}A_2 = {}& q^{-1}(q-1)\chi_2(\varpi)f(g_7) + (q-1)f(g_{10}) + q^{-1}\frac{(q-1)^2}{2}\chi_2(\varpi)(f(g_8)+f(g_{12}))\\ &- q^{-1}(q-1)\chi_2(\varpi)f(\begin{bmatrix}1&&&\\&1&&\\&-\varpi&1&\\\varpi&&&1\end{bmatrix}).\end{aligned}$$
Hence, in <u>Case I</u> (i.e., $-1\notin\mathfrak{o}/\mathfrak{p})^2$),
$$\begin{aligned}A_2 = {}& q^{-1}(q-1)\chi_2(\varpi)f(g_7) + (q-1)f(g_{10}) + q^{-1}\frac{(q-1)^2}{2}\chi_2(\varpi)f(g_8)\\ &+\frac{q^{-1}(q-1)(q-3)}{2}\chi_2(\varpi)f(g_{12}),\end{aligned}$$
and in <u>Case II</u>,
$$\begin{aligned}A_2 = {}& q^{-1}(q-1)\chi_2(\varpi)f(g_7) + (q-1)f(g_{10}) + \frac{q^{-1}(q-1)(q-3)}{2}\chi_2(\varpi)f(g_8)\\ &+q^{-1}\frac{(q-1)^2}{2}\chi_2(\varpi)f(g_{12}).\end{aligned}$$
Therefore, in <u>Case I</u>,
$$A = q^{-1}(q-1)\chi_2(\varpi)f(g_7) + \frac{q-1}{2}\chi_2(\varpi)f(g_8) + qf(g_{10}) + \frac{q^{-1}(q-1)(q-2)}{2}\chi_2(\varpi)f(g_{12}),$$

and in Case II,

$$A = q^{-1}(q-1)\chi_2(\varpi)f(g_7) + \frac{q^{-1}(q-1)(q-2)}{2}\chi_2(\varpi)f(g_8) + qf(g_{10}) + \frac{q-1}{2}\chi_2(\varpi)f(g_{12}).$$

Adding A and B, we get in Case I

$$\begin{aligned}(\mu f)(g_{10}) &= q^2\chi_1(\varpi)^{-1}f(g_4) + q^{-1}(q-1)\chi_2(\varpi)f(g_7)\\ &\qquad + \frac{q-1}{2}\chi_2(\varpi)f(g_8) + (2q-1)f(g_{10}) + \frac{q^{-1}(q-1)(q-2)}{2}\chi_2(\varpi)f(g_{12}),\end{aligned}$$

and in Case II

$$\begin{aligned}(\mu f)(g_{10}) &= q^2\chi_1(\varpi)^{-1}f(g_4) + q^{-1}(q-1)\chi_2(\varpi)f(g_7)\\ &\qquad + \frac{q^{-1}(q-1)(q-2)}{2}\chi_2(\varpi)f(g_8) + (2q-1)f(g_{10}) + \frac{q-1}{2}\chi_2(\varpi)f(g_{12}).\end{aligned}$$

This gives the tenth row of our matrix. ■

3 Global theory

For the global theory of Siegel modular forms it is more convenient to work with the "official" version of $G = \mathrm{GSp}(4)$ instead of the one defined in (2). Hence, from now on we shall use

$$\mathrm{GSp}(4) = \{g \in \mathrm{GL}(4) :\ {}^tg\begin{bmatrix} & 1_2\\ -1_2 & \end{bmatrix}g = \lambda(g)\begin{bmatrix} & 1_2\\ -1_2 & \end{bmatrix} \text{ for some scalar } \lambda(g)\}. \qquad (31)$$

An isomorphism between this group and the one defined in (2) is provided by switching the first two rows and the first two columns, i.e., by conjugation with the matrix

$$\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1\end{bmatrix}.$$

The minimal parabolic subgroup B, the Siegel parabolic subgroup P and the Klingen parabolic subgroup Q then take the following shapes,

$$B = \begin{bmatrix} * & & * & *\\ * & * & * & *\\ & & * & *\\ & & * & *\end{bmatrix}, \qquad P = \begin{bmatrix} * & * & * & *\\ * & * & * & *\\ & & * & *\\ & & * & *\end{bmatrix}, \qquad Q = \begin{bmatrix} * & & * & *\\ * & * & * & *\\ * & & * & *\\ & & & *\end{bmatrix}.$$

For a positive integer N we define as usual

$$\Gamma_0(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}\\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}\\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z}\\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z}\end{bmatrix}.$$

3.1 Modular forms

Let $\mathbb{H}_2$ be the Siegel upper half plane of degree 2, i.e.,

$$\mathbb{H}_2 = \{\begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix},\ \tau, z, \tau' \in \mathbb{C},\ \mathrm{im}(\tau) > 0, \mathrm{im}(\tau') > 0, \mathrm{im}(\tau)\mathrm{im}(\tau') - \mathrm{im}(z)^2 > 0\}.$$

The group $G(\mathbb{R})^+ = \{g \in \mathrm{GSp}(4, \mathbb{R}) :\ \lambda(g) > 0\}$ acts on $\mathbb{H}_2$ by linear fractional transformations

$$Z \mapsto g\langle Z\rangle = (AZ + B)(CZ + D)^{-1}, \qquad g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G(\mathbb{R})^+.$$

We define the usual modular factor

$$j(g, Z) = \det(CZ + D) \qquad \text{for } Z \in \mathbb{H}_2 \text{ and } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G(\mathbb{R})^+.$$

Let k be a positive integer. The weight-k slash operator $\big|_k$, or simply $\big|$, defines an action of $G(\mathbb{R})^+$ on functions $F :\ \mathbb{H}_2 \to \mathbb{C}$ via the formula

$$(F\big|g)(Z) = \lambda(g)^k j(g, Z)^{-k} F(g\langle Z\rangle) \qquad \text{for } g \in G(\mathbb{R})^+.$$

Note that there are different normalizations for the slash operator in the literature; our choice of factor $\lambda(g)^k = \det(g)^{k/2}$ ensures that the center of $G(\mathbb{R})^+$ acts trivially. Let Γ be a congruence subgroup of $\mathrm{Sp}(4, \mathbb{Q})$. A *modular form* (always of degree 2) of weight k with respect to Γ is a holomorphic function F on $\mathbb{H}_2$ such that $F\big|\gamma = F$ for all $\gamma \in \Gamma$. We denote the space of such modular forms by $M_k(\Gamma)$, and the subspace of cusp forms by $S_k(\Gamma)$. An element $F \in M_k(\Gamma_0(N))$ has a Fourier expansion of the form

$$F(Z) = \sum_T c(T) e^{2\pi i\, \mathrm{tr}(TZ)},$$

where T runs over positive semidefinite symmetric matrices of the form $\begin{bmatrix} n & r/2 \\ r/2 & m \end{bmatrix}$ with integers n, r, m. If we write $c(n, r, m)$ for $c(T)$ and (τ, z, τ') for $Z = \begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix} \in \mathbb{H}_2$, then the Fourier expansion reads

$$F(\tau, z, \tau') = \sum_{n,r,m\geq 0} c(n, r, m) e^{2\pi i(n\tau + rz + m\tau')}.$$

It can be rewritten in the form of a Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m=0}^{\infty} f_m(\tau, z) e^{2\pi i m\tau'}, \tag{32}$$

where $f_m(\tau, z) = \sum_{n,r} c(n, r, m) e^{2\pi i(n\tau + rz)}$ is a Jacobi form of index m with respect to a congruence subgroup of the full Jacobi modular group. Later we will require the following symmetry properties of Fourier coefficients.

$$c(T) = \det(A)^k c({}^tATA) \qquad (A \in \mathrm{SL}(2, \mathbb{Z})) \tag{33}$$

$$c(n, r, m) = (-1)^k c(m, r, n) \tag{34}$$

$$c(n, r, m) = c(n,\ r + 2\lambda n,\ m + \lambda r + \lambda^2 n) \qquad (\lambda \in \mathbb{Z}) \tag{35}$$

$$c(n, r, m) = c(n + \lambda r + \lambda^2 m,\ r + 2\lambda m,\ m) \qquad (\lambda \in \mathbb{Z}) \tag{36}$$

Modular forms as functions on the adele group

Let $\mathbb{A}$ be the ring of adeles of the number field $\mathbb{Q}$. Let $N = \prod p^{n_p}$ be the prime decomposition of the positive integer N. For each prime number p let

$$K_p = \mathrm{Si}(\mathfrak{p}^{n_p}) = \mathrm{GSp}(4, \mathbb{Z}_p) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{n_p} & \mathfrak{o} \end{bmatrix} \qquad (\mathfrak{p} = p\mathbb{Z}_p),$$

the local Siegel congruence subgroup of level $\mathfrak{p}^{n_p}$. Note that $K_p = \mathrm{GSp}(4, \mathbb{Z}_p)$ for almost all p. Then $K = \prod_{p<\infty} K_p$ is an open subgroup of $\mathrm{GSp}(4, \mathbb{A}_f)$, and

$$\Gamma_0(N) = \mathrm{GSp}(4, \mathbb{Q})^+ \cap K. \tag{37}$$

Strong approximation for $\mathrm{Sp}(4)$ implies that $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+K$. This allows us to attach to a given $F \in M_k(\Gamma_0(N))$ an adelic function $\Phi: G(\mathbb{A}) \to \mathbb{C}$ in the following way. Decomposing a given $g \in G(\mathbb{A})$ as $g = \rho h \kappa$ with $\rho \in G(\mathbb{Q})$, $h \in G(\mathbb{R})^+$ and $\kappa \in K$, we define

$$\Phi(g) = (F\big|_k h)(I), \qquad g = \rho h \kappa \text{ with } \rho \in G(\mathbb{Q}),\ h \in G(\mathbb{R})^+,\ \kappa \in K. \tag{38}$$

Here I is the element $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ of $\mathbb{H}_2$. In view of (37), the function Φ is well-defined. It has the invariance properties

$$\Phi(\rho g \kappa z) = \Phi(g) \qquad \text{for all } g \in G(\mathbb{A}),\ \rho \in G(\mathbb{Q}),\ \kappa \in K,\ z \in Z(\mathbb{A}),$$

where Z is the center of $\mathrm{GSp}(4)$. In fact, Φ is an automorphic form on $\mathrm{PGSp}(4, \mathbb{A})$. One can show that Φ is a cuspidal automorphic form if and only if $F \in S_k(\Gamma_0(N))$. Assuming this is the case, we consider the cuspidal automorphic representation $\pi = \pi_F$ generated by Φ. This representation may not be irreducible, but it always decomposes as a finite direct sum $\pi = \oplus_i \pi_i$ with irreducible, cuspidal, automorphic representations π_i.

3.2 Definition and basic properties of μ_p

In this section we shall introduce a linear operator μ_p on $M_k(\Gamma_0(N))$, $p^2|N$, which is analogous, and in fact compatible, with the local μ operator defined in Sect. 2.1. Let $F \in M_k(\Gamma_0(N))$ and p be a prime number with $p^2|N$. We consider the summation

$$F' = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} F\Big| \begin{bmatrix} 1 & & & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{39}$$

It is easily checked that this function is invariant under the group

$$\mathrm{Sp}(4, \mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} \end{bmatrix}. \tag{40}$$

If we restore the invariance under elements $\begin{bmatrix} A & \\ & {}^tA \end{bmatrix}$, $A \in \mathrm{SL}(2,\mathbb{Z})$, we obtain a new element of $M_k(\Gamma_0(N))$. Therefore, we let

$$\mu_p F := \sum_{A \in \left[\begin{smallmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right] \backslash \mathrm{SL}(2,\mathbb{Z})} \; \sum_{u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}. \tag{41}$$

This defines an endomorphism μ_p of $M_k(\Gamma_0(N))$ and, via restriction, an endomorphism μ_p of $S_k(\Gamma_0(N))$. An explicit formula is

$$\mu_p F = \sum_{t,u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & up^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ t & 1 & & \\ & & 1 & -t \\ & & & 1 \end{bmatrix} + \sum_{u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{42}$$

Alternatively,

$$\mu_p F = \sum_{t,u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & up^{-1} & tup^{-1} \\ & 1 & tup^{-1} & t^2up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} + \sum_{u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{43}$$

In particular, μ_p has a definition in terms of elements of the unipotent radical of the Siegel parabolic subgroup.

3.1 Proposition. *Let N be a positive integer and p a prime number with $p^2|N$.*

- *i) The endomorphism μ_p of $M_k(\Gamma_0(N))$ is diagonalizable.*
- *ii) The only possible eigenvalues of μ_p on $M_k(\Gamma_0(N))$ are $p(p+1)$, $2p$, p and 0. Hence*

$$M_k(\Gamma_0(N)) = M_k(\Gamma_0(N))_{p(p+1)} \oplus M_k(\Gamma_0(N))_{2p} \oplus M_k(\Gamma_0(N))_p \oplus M_k(\Gamma_0(N))_0. \tag{44}$$

Here, $M_k(\Gamma_0(N))_i$ denotes the i-eigenspace of μ_p on $M_k(\Gamma_0(N))$. A similar decomposition holds for cusp forms,

$$S_k(\Gamma_0(N)) = S_k(\Gamma_0(N))_{p(p+1)} \oplus S_k(\Gamma_0(N))_{2p} \oplus S_k(\Gamma_0(N))_p \oplus S_k(\Gamma_0(N))_0. \tag{45}$$

This decomposition is orthogonal with respect to the Petersson inner product.

For the following statements let $F \in M_k(\Gamma_0(N))$.

- *iii) We have $F \in M_k(\Gamma_0(N))_{p(p+1)}$ if and only if F is invariant under the group*

$$\mathrm{Sp}(4,\mathbb{Q}) \cap \begin{bmatrix} 1 & & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ & 1 & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{46}$$

iv) If $F \in M_k(\Gamma_0(N))_{2p}$, then

$$\sum_{t,u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & & tp^{-1} \\ & 1 & tp^{-1} & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} = 0. \tag{47}$$

v) If $F \in M_k(\Gamma_0(N))_p$, then

$$\sum_{s,u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & sp^{-1} & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} = 0. \tag{48}$$

vi) We have $\mu_p F = 0$ if and only if

$$\sum_{u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} = 0. \tag{49}$$

vii) If (47) holds, then $F \in M_k(\Gamma_0(N))_{2p} \oplus M_k(\Gamma_0(N))_0$.

viii) If (48) holds, then $F \in M_k(\Gamma_0(N))_p \oplus M_k(\Gamma_0(N))_0$.

Proof: The proofs of i) through vi) are very similar to the proofs of the analogous local statements in Proposition 2.1. However, for the sake of clarity we shall repeat them.

i) Starting from any inner product on $M_k(\Gamma_0(N))$, we can, by summation, easily construct, an inner product that is invariant under the group

$$\mathrm{Sp}(4, \mathbb{Q}) \cap \begin{bmatrix} 1 & & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ & 1 & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Using (43), it is easily checked that μ_p is self-adjoint with respect to this inner product. Therefore, μ_p is diagonalizable. (On $S_k(\Gamma_0(N))$ we could have used the Petersson inner product.)

We will now prove ii), iii), iv) and v). Assume that $\mu_p F = cF$ for some $c \in \mathbb{C}$. Let F' be defined as in (39). From (43) we get

$$cF - F' = \sum_{t,u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & up^{-1} & tup^{-1} \\ & 1 & tup^{-1} & t^2up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Applying the summation that defines F' to both sides of this equation, we get

$$(c-p)F' = \sum_{t,u\in\mathbb{Z}/p\mathbb{Z}} F'\Big|\begin{bmatrix}1&&up^{-1}&tup^{-1}\\&1&tup^{-1}&\\&&1&\\&&&1\end{bmatrix},$$

and hence

$$(c-2p)F' = \sum_{t\in\mathbb{Z}/p\mathbb{Z}}\ \sum_{u\in(\mathbb{Z}/p\mathbb{Z})^\times} F'\Big|\begin{bmatrix}1&&up^{-1}&tp^{-1}\\&1&tp^{-1}&\\&&1&\\&&&1\end{bmatrix}. \tag{50}$$

If we abbreviate

$$F'' := \sum_{t,u\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix}1&&&tp^{-1}\\&1&tp^{-1}&up^{-1}\\&&1&\\&&&1\end{bmatrix},$$

this can be written as

$$(c-2p)F' = \sum_{s\in\mathbb{Z}/p\mathbb{Z}} F''\Big|\begin{bmatrix}1&&sp^{-1}&\\&1&&\\&&1&\\&&&1\end{bmatrix} - F''. \tag{51}$$

If $c=2p$, then it follows that F'' is invariant under

$$\begin{bmatrix}1&&p^{-1}&\\&1&&\\&&1&\\&&&1\end{bmatrix}.$$

Hence the right side of (51) equals $(p-1)F''$. It follows that $F''=0$ if $c=2p$. This proves iv). Now assume that $c\neq 2p$. Then it follows from (50) that F' is invariant under

$$\mathrm{Sp}(4,\mathbb{Q})\cap\begin{bmatrix}1&&&p^{-1}\mathbb{Z}\\&1&p^{-1}\mathbb{Z}&\\&&1&\\&&&1\end{bmatrix}$$

and hence, again from (50),

$$(c-2p)F' = p\sum_{u\in(\mathbb{Z}/p\mathbb{Z})^\times} F'\Big|\begin{bmatrix}1&&up^{-1}&\\&1&&\\&&1&\\&&&1\end{bmatrix}.$$

Adding pF' to both sides, we get

$$(c-p)F' = p \sum_{u \in \mathbb{Z}/p\mathbb{Z}} F' \Big| \begin{bmatrix} 1 & & up^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{52}$$

For $c = p$ this proves v). Assume that $c \neq 2p$ and $c \neq p$. Then it follows from (52) that F' is invariant under

$$\mathrm{Sp}(4,\mathbb{Q}) \cap \begin{bmatrix} 1 & & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ & 1 & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{53}$$

By (41), the same is then true for $\mu_p F = cF$. If $c \neq 0$, it follows that F is invariant under the group (53). But in this case $\mu_p F = p(p+1)F$ by (43). This proves ii) and iii).

vi) It is clear from the definition of μ_p that (49) implies $\mu_p F = 0$. Assume conversely that $\mu_p F = 0$. Let F' be as in (39). We just proved that F' is invariant under the group (53). Consequently,

$$p^2 F' = \sum_{s,t \in \mathbb{Z}/p\mathbb{Z}} F' \Big| \begin{bmatrix} 1 & & sp^{-1} & tp^{-1} \\ & 1 & tp^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \sum_{s,t,u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & sp^{-1} & tp^{-1} \\ & 1 & tp^{-1} & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

This last expression is clearly invariant under $\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}$ with $A \in \mathrm{SL}(2,\mathbb{Z})$. By (41), $\mu_p F = (p+1)F'$. Thus $\mu_p F = 0$ implies $F' = 0$, as asserted.

vii) Write $F = F_{p(p+1)} + F_{2p} + F_p + F_0$ with F_i in the i-eigenspace of μ_p. Applying

$$\sum_{s,t,u \in \mathbb{Z}/p\mathbb{Z}} \ldots \Big| \begin{bmatrix} 1 & & sp^{-1} & tp^{-1} \\ & 1 & tp^{-1} & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

to both sides, it follows by iii), iv) and v) that $F_{p(p+1)} = 0$. It is therefore enough to show that if $F \in M_k(\Gamma_0(N))_p$ and (47) holds, then $F = 0$. The condition (47) means that the function F'' occurring in (51) is zero. With $c = p$ in (51) it follows that $F' = 0$. Hence F is in the p-eigenspace and in the kernel of μ_p, and therefore zero.

viii) For the proof of the last statement we shall use Lemma 3.5 below (which will follow from Fourier coefficient considerations). Again, write $F = F_{p(p+1)} + F_{2p} + F_p + F_0$ with F_i in the i-eigenspace of μ_p. As in the proof of vii) we conclude that $F_{p(p+1)} = 0$. It is therefore enough to show that if $F \in M_k(\Gamma_0(N))_{2p}$ and (48) holds, then $F = 0$. This is the statement of Lemma 3.5. ∎

The operator μ_p defined by (41) and the local operator μ defined by (6) are compatible, in the following sense. Let $p^2|N$ and $F\in S_k(\Gamma_0(N))$, and assume that the associated adelic function Φ defined in (38) generates an irreducible, cuspidal, automorphic representation π of $\mathrm{GSp}(4,\mathbb{A})$. We can write π as a restricted tensor product $\otimes\pi_v$ with irreducible, admissible representations (π_v,V_v) of $\mathrm{GSp}(4,\mathbb{Q}_v)$. Let us assume in addition that Φ corresponds to a pure tensor $\otimes f_v$, where $f_v\in V_v$. Then, if p^n is the exact power of p dividing N, we have $f_v\in V_{v,0}(n)$, the subspace of vectors in V_v invariant under the local congruence subgroup $\mathrm{Si}(\mathfrak{p}^n)$; here $\mathfrak{p}=p\mathbb{Z}_p$. The local and global μ operators are compatible in the sense that the cusp form μ_pF corresponds to the pure tensor

$$(\mu f_p)\cdot\bigotimes_{v\neq p}f_v, \tag{54}$$

where μ is the endomorphism of $V_{v,0}(n)$ defined by (6). In other words, in order to find the tensor corresponding to μ_pF, we replace in the tensor corresponding to F the p-component f_p by μf_p. This follows from a straightforward calculation: If we define an operator μ on adelic functions by right translating with the p-adic elements occurring in (8), then $\mu_pF=\mu\Phi$.

Fourier coefficients and μ_p

We shall now compute μ_p in terms of Fourier coefficients. First we explain our conventions about the Legendre symbol. Let p be an odd prime number and $x\in\mathbb{Z}$ with $p\nmid x$. We define, as usual,

$$\left(\frac{x}{p}\right)=\begin{cases}1 & \text{if } x \text{ is a square mod } p,\\ -1 & \text{if } x \text{ is a non-square mod } p.\end{cases}$$

For $p=2$ we define $\left(\frac{x}{p}\right)$ only for $x\in 4\mathbb{Z}+1$. We set

$$\left(\frac{x}{2}\right)=\begin{cases}1 & \text{if } x\in 8\mathbb{Z}+1,\\ -1 & \text{if } x\in 8\mathbb{Z}+5.\end{cases}$$

3.2 Lemma. *Let n,r,m be integers and p a prime number. The congruence*

$$n+rt+mt^2\equiv 0 \mod p \tag{55}$$

has

- *p solutions mod p, if $p|m$, $p|r$ and $p|n$;*
- *two solutions mod p, if $p\nmid m$, $p\nmid r^2-4mn$ and $\left(\frac{r^2-4mn}{p}\right)=1$;*
- *one solution mod p, if*
  - *$p\nmid m$, $p|r^2-4mn$, or if*
  - *$p|m$, $p\nmid r$;*
- *no solution, if*
  - *$p\nmid m$, $p\nmid r^2-4mn$ and $\left(\frac{r^2-4mn}{p}\right)=-1$, or if*

– $p|m$, $p|r$, $p\nmid n$.

Proof: First assume that p is odd. If $p|m$, the congruence (55) becomes a linear congruence, for which the number of solutions is obvious. If $p\nmid m$, (55) is equivalent to

$$(2mt+r)^2 \equiv r^2-4mn \mod p.$$

From this the lemma follows easily. – Now assume that $p=2$. In this case it is trivial to check that (55) has

- two solutions mod p, if n is even and $r+m$ is even;
- one solution mod p, if $r+m$ is odd;
- no solution, if n is odd and $r+m$ is even.

By our conventions about the Legendre symbol explained above, these are the same conditions as the one listed in the lemma. ∎

For the next lemma we use the following notation. Let n,r,m be integers and p a prime number. If p is odd, then

$$\mathrm{rank}_p(\begin{bmatrix}2n & r\\ r & 2m\end{bmatrix})$$

means the rank of the matrix $\begin{bmatrix}2n & r\\ r & 2m\end{bmatrix}$ after reduction mod p. If $p=2$, then we define

$$\mathrm{rank}_2(\begin{bmatrix}2n & r\\ r & 2m\end{bmatrix}) = \begin{cases} 0 & \text{if } 2|n,\ 2|r,\ 2|m,\\ 1 & \text{if } 2|r, \text{ but 2 divides not both of } n \text{ and } m,\\ 2 & \text{if } 2\nmid r.\end{cases}$$

3.3 Lemma. *Let N be a positive integer and p a prime with $p^2|N$. Let $F\in M_k(\Gamma_0(N))$ with Fourier expansion $F(\tau,z,\tau')=\sum_{n,r,m}c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$. Then*

$$\begin{aligned}(\mu_pF)(\tau,z,\tau') &= p(p+1)\sum_{\substack{n,r,m\\ p|m,\,p|r,\,p|n}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}\\ &+2p\sum_{\substack{n,r,m\\ p\nmid m,\,p\nmid r^2-4mn\\ \left(\frac{r^2-4mn}{p}\right)=1}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')} + 2p\sum_{\substack{n,r,m\\ p|m,\,p\nmid r}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}\\ &+p\sum_{\substack{n,r,m\\ p\nmid m,\,p|r^2-4mn}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')} + p\sum_{\substack{n,r,m\\ p|m,\,p|r,\,p\nmid n}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}. \qquad (56)\end{aligned}$$

Alternatively,

$$(\mu_pF)(Z) = p(p+1)\sum_{\substack{T\\ \mathrm{rank}_p(2T)=0}} c(T)e^{2\pi i\,\mathrm{tr}(TZ)}$$

$$+ 2p \sum_{\substack{T \\ \mathrm{rank}_p(2T)=2 \\ \left(\frac{-\det(2T)}{p}\right)=1}} c(T)e^{2\pi i \mathrm{tr}(TZ)}$$
$$+ p \sum_{\substack{T \\ \mathrm{rank}_p(2T)=1}} c(T)e^{2\pi i \mathrm{tr}(TZ)}. \tag{57}$$

Proof: It is easy to check that (56) and (57) are equivalent; we shall prove (56). We have $(\mu_p F)(\tau, z, \tau') = A + B$ with

$$A = \sum_{t,u \in \mathbb{Z}/p\mathbb{Z}} (F\big| \begin{bmatrix} 1 & & up^{-1} & tup^{-1} \\ & 1 & tup^{-1} & t^2up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix})(\tau, z, \tau')$$

and

$$B = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} (F\big| \begin{bmatrix} 1 & & & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix})(\tau, z, \tau').$$

We compute

$$\begin{aligned} A &= \sum_{t,u \in \mathbb{Z}/p\mathbb{Z}} (F\big| \begin{bmatrix} 1 & & up^{-1} & tup^{-1} \\ & 1 & tup^{-1} & t^2up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix})(\tau, z, \tau') \\ &= \sum_{t,u \in \mathbb{Z}/p\mathbb{Z}} F(\tau + up^{-1}, z + tup^{-1}, \tau' + t^2up^{-1}) \\ &= \sum_{n,r,m} \sum_{t,u \in \mathbb{Z}/p\mathbb{Z}} c(n,r,m)e^{2\pi i(n\tau + rz + m\tau')}e^{2\pi i(n+rt+mt^2)up^{-1}}. \end{aligned}$$

By Lemma 3.2,

$$\begin{aligned} A = p^2 \sum_{\substack{n,r,m \\ p|m,\, p|r,\, p|n}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')} &+ 2p \sum_{\substack{n,r,m \\ p\nmid m,\, p\nmid r^2-4mn \\ \left(\frac{r^2-4mn}{p}\right)=1}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')} \\ + p \sum_{\substack{n,r,m \\ p|m,\, p\nmid r}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')} &+ p \sum_{\substack{n,r,m \\ p\nmid m,\, p|r^2-4mn}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}. \end{aligned}$$

Furthermore,

$$B = p \sum_{\substack{n,r,m \\ p|m}} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$$

$$= p \sum_{\substack{n,r,m \\ p|m,\, p|r}} c(n,r,m) e^{2\pi i(n\tau + rz + m\tau')} + p \sum_{\substack{n,r,m \\ p|m,\, p\nmid r}} c(n,r,m) e^{2\pi i(n\tau + rz + m\tau')}.$$

Adding A and B gives the result. ■

Lemma 3.3 shows again that μ_p has only the four possible eigenvalues $p(p+1)$, $2p$, p and 0. Moreover, we note the following consequence.

3.4 Proposition. *Let N be a positive integer and p a prime number with $p^2|N$. Let $F \in M_k(\Gamma_0(N))$ have Fourier expansion $F(\tau,z,\tau') = \sum_{n,r,m} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$. Write $F = F_{p(p+1)} + F_{2p} + F_p + F_0$ with $F_i \in M_k(\Gamma_0(N))_i$ according to the decomposition (44). Then*

$$\begin{aligned}
F_{p(p+1)} &= \sum_{\substack{n,r,m \\ p|m,\, p|r,\, p|n}} c(n,r,m) e^{2\pi i(n\tau + rz + m\tau')} = \sum_{\substack{T \\ \mathrm{rank}_p(2T)=0}} c(T) e^{2\pi i \mathrm{tr}(TZ)}, \\
F_{2p} &= \sum_{\substack{n,r,m \\ p\nmid m,\, p\nmid r^2-4mn \\ \left(\frac{r^2-4mn}{p}\right)=1}} c(n,r,m) e^{2\pi i(n\tau + rz + m\tau')} + \sum_{\substack{n,r,m \\ p|m,\, p\nmid r}} c(n,r,m) e^{2\pi i(n\tau + rz + m\tau')} \\
&= \sum_{\substack{T \\ \mathrm{rank}_p(2T)=2 \\ \left(\frac{-\det(2T)}{p}\right)=1}} c(T) e^{2\pi i \mathrm{tr}(TZ)}, \\
F_p &= \sum_{\substack{n,r,m \\ p\nmid m,\, p|r^2-4mn}} c(n,r,m) e^{2\pi i(n\tau + rz + m\tau')} + \sum_{\substack{n,r,m \\ p|m,\, p|r,\, p\nmid n}} c(n,r,m) e^{2\pi i(n\tau + rz + m\tau')} \\
&= \sum_{\substack{T \\ \mathrm{rank}_p(2T)=1}} c(T) e^{2\pi i \mathrm{tr}(TZ)}, \\
F_0 &= \sum_{\substack{n,r,m \\ p\nmid m,\, p\nmid r^2-4mn \\ \left(\frac{r^2-4mn}{p}\right)=-1}} c(n,r,m) e^{2\pi i(n\tau + rz + m\tau')} = \sum_{\substack{T \\ \mathrm{rank}_p(2T)=2 \\ \left(\frac{-\det(2T)}{p}\right)=-1}} c(T) e^{2\pi i \mathrm{tr}(TZ)}.
\end{aligned}$$

In particular, these four functions are elements of $M_k(\Gamma_0(N))$. The same statements are true with $S_k(\Gamma_0(N))$ instead of $M_k(\Gamma_0(N))$.

We shall now prove Lemma 3.5, which was used in the proof of Proposition 3.1.

3.5 Lemma. *Let $F \in M_k(\Gamma_0(N))_{2p}$ such that*

$$\sum_{s,u \in \mathbb{Z}/p\mathbb{Z}} F \Big| \begin{bmatrix} 1 & & sp^{-1} & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} = 0. \tag{58}$$

Then $F = 0$.

Proof: Let $F(\tau, z, \tau') = \sum_{n,r,m} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$ be the Fourier expansion of F. By Proposition 3.1 iv), we have

$$c(n,r,m) = 0 \qquad \text{for } p|r \text{ and } p|m.$$

Equation (58) is equivalent to

$$c(n,r,m) = 0 \qquad \text{for } p|n \text{ and } p|m,$$

Observing (34), it follows that $c(n,r,m) = 0$ whenever two of the numbers n, r or m are divisible by p. We claim that $c(n,r,m) = 0$ if $p|m$ or $p|n$. We just saw that $c(n,r,m) = 0$ if $p|n$ and $p|m$. To prove the claim, we may, by (34), assume that $p \nmid n$ and $p|m$. If $p|r$, then $c(n,r,m) = 0$ since p divides two of the numbers n, r or m. If $p \nmid r$, we consider a transformation of the form (36) with $\lambda r \equiv -n \bmod p$. Then

$$c(n,r,m) = c(n + \lambda r + \lambda^2 m, r + 2\lambda m, m) = 0,$$

since $p|n + \lambda r + \lambda^2 m$ and $p|m$. This proves our claim that $c(n,r,m) = 0$ if $p|m$ or $p|n$. – Now assume that there exists n, r, m such that $c(n,r,m) \neq 0$; we will obtain a contradiction. By Lemma 3.3 and by what we just proved, we have $p \nmid m$, $p \nmid r^2 - 4mn$, and $\left(\frac{r^2-4mn}{p}\right) = 1$. By Lemma 3.2, there exists $\lambda \in \mathbb{Z}$ such that $n + \lambda r + \lambda^2 m \equiv 0 \bmod p$. By (36) it follows that $c(n,r,m) = 0$, a contradiction. ∎

3.3 Characterizations of the eigenspaces

In the following we shall give various characterizations of the eigenspaces occurring in the decomposition (44) resp. (45).

The $p(p+1)$ eigenspace

Let N be a positive integer and p a prime number. There is a simple level raising operator

$$\beta_p : \; M_k(\Gamma_0(N)) \longrightarrow M_k(\Gamma_0(Np))$$

given by

$$\beta_p F = F\Big| \begin{bmatrix} p &&& \\ & p && \\ && 1 & \\ &&& 1 \end{bmatrix}.$$

Restriction to cusp forms gives a linear map $\beta_p : \; S_k(\Gamma_0(N)) \to S_k(\Gamma_0(Np))$. Evidently, β_p is injective. Modular forms in the image of β should be considered "old", but there are many more modular forms that should be viewed as oldforms.

3.6 Proposition. *Let N be a positive integer and p a prime number with $p^2|N$. The following statements are equivalent for a non-zero $F \in M_k(\Gamma_0(N))$.*

i) F is in the image of $\beta_p : \; M_k(\Gamma_0(Np^{-1})) \to M_k(\Gamma_0(N))$.

ii) F is invariant under the congruence subgroup

$$\mathrm{Sp}(4,\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

iii) If $F(\tau,z,\tau') = \sum_{n,r,m} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$ is the Fourier expansion of F, then

$$c(n,r,m) \neq 0 \quad \Longrightarrow \quad p|n,\ p|r,\ p|m.$$

iv) $\mu_p F = p(p+1)F$.

Proof: The equivalence of i), ii) and iii) is an easy exercise. The equivalence of iii) and iv) follows from Lemma 3.3. ∎

The $2p$ eigenspace

3.7 Proposition. *Let N be a positive integer and p a prime number with $p^2|N$. The following statements are equivalent for a non-zero $F \in M_k(\Gamma_0(N))$ with Fourier expansion $F(\tau,z,\tau') = \sum_{n,r,m} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$.*

i)

$$\sum_{t,u\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} 1 & & & tp^{-1} \\ & 1 & tp^{-1} & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} = 0, \qquad \textit{but} \qquad \sum_{u\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} 1 & & & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \neq 0.$$

ii) $c(n,r,m) = 0$ if $p|m$ and $p|r$, but there exists a non-zero $c(n,r,m)$ with $p|m$ and $p\nmid r$.

iii) $c(n,r,m) \neq 0$ implies

- *$p\nmid m$, $p \nmid r^2-4mn$ and $\left(\frac{r^2-4mn}{p}\right) = 1$, or*
- *$p|m$ and $p\nmid r$.*

iv) $c(T) \neq 0$ implies $\mathrm{rank}_p(2T) = 2$ and $\left(\frac{-\det(2T)}{p}\right) = 1$.

v) $\mu_p F = 2pF$.

Proof: The equivalence of i) and ii) follows from a straightforward calculation. The equivalence of iii), iv) and v) follows from Lemma 3.3. The equivalence of i) and v) follows from Proposition 3.1. ∎

The p eigenspace

3.8 Proposition. *Let N be a positive integer and p a prime number with $p^2|N$. The following statements are equivalent for a non-zero $F \in M_k(\Gamma_0(N))$ with Fourier expansion $F(\tau, z, \tau') = \sum_{n,r,m} c(n, r, m)e^{2\pi i(n\tau+rz+m\tau')}$.*

i)

$$\sum_{s,u\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} 1 & & sp^{-1} & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} = 0, \qquad \text{but} \qquad \sum_{u\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} 1 & & & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \neq 0.$$

ii) $c(n, r, m) = 0$ *if* $p|m$ *and* $p|n$*, but there exists a non-zero* $c(n, r, m)$ *with* $p|m$ *and* $p \nmid n$.

iii) $c(n, r, m) \neq 0$ *implies*

- $p \nmid m$ *and* $p|r^2 - 4mn$*, or*
- $p|m$*,* $p|r$ *and* $p \nmid n$.

iv) $c(T) \neq 0$ *implies* $\mathrm{rank}_p(2T) = 1$.

v) $\mu_p F = pF$.

Proof: The equivalence of i) and ii) follows from a straightforward calculation. The equivalence of iii), iv) and v) follows from Lemma 3.3. The equivalence of i) and v) follows from Proposition 3.1. ∎

The kernel of μ_p

3.9 Proposition. *Let N be a positive integer and p a prime number with $p^2|N$. The following statements are equivalent for a non-zero $F \in M_k(\Gamma_0(N))$ with Fourier expansion $F(\tau, z, \tau') = \sum_{n,r,m} c(n, r, m)e^{2\pi i(n\tau+rz+m\tau')}$.*

i)

$$\sum_{u\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} 1 & & & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} = 0.$$

ii)

$$\sum_{s,u\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} 1 & & sp^{-1} & \\ & 1 & & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} = 0 \qquad \text{and} \qquad \sum_{t,u\in\mathbb{Z}/p\mathbb{Z}} F\Big|\begin{bmatrix} 1 & & & tp^{-1} \\ & 1 & tp^{-1} & up^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} = 0.$$

iii) $c(n, r, m) = 0$ *if two of the numbers* n*,* r *and* m *are divisible by* p.

iv) $c(n,r,m) \neq 0$ *implies* $p \nmid m$.

v) $c(n,r,m) \neq 0$ *implies* $p \nmid m$, $p \nmid r^2 - 4mn$ *and* $\left(\frac{r^2-4mn}{p}\right) = -1$.

vi) $c(T) \neq 0$ *implies* $\mathrm{rank}_p(2T) = 2$ *and* $\left(\frac{-\det(2T)}{p}\right) = -1$.

vii) $\mu_p F = 0$.

Proof: The equivalence of i) and iv) follows from a straightforward calculation. Observing (34), the equivalence of ii) and iii) follows also from an obvious calculation. The equivalence of v), vi) and vii) follows from Lemma 3.3. Clearly, i) implies ii). If ii) holds, then, by Proposition 3.1 vii) and viii), $F \in \ker(\mu_p)$. Thus ii) implies vii). The equivalence of i) and vii) was stated in Proposition 3.1 vi). ■

3.4 Hypercuspidal modular forms

The following definition can be made for modular forms with respect to any congruence subgroup Γ that contains

$$\begin{bmatrix} 1 &&& \\ & 1 && \mathbb{Z} \\ && 1 & \\ &&& 1 \end{bmatrix}.$$

3.10 Definition. *Let p be a prime number. The modular form $F \in M_k(\Gamma)$ is called p-hypercuspidal if there exists an integer $l \geq 1$ such that*

$$\sum_{u \in \mathbb{Z}/p^l\mathbb{Z}} F\big|_k \begin{bmatrix} 1 &&& \\ & 1 && up^{-l} \\ && 1 & \\ &&& 1 \end{bmatrix} = 0.$$

If l is minimal with this property, then we say that F is hypercuspidal of degree l.

A straightforward calculation shows that if F has the Fourier-Jacobi expansion (32), then

$$\sum_{u \in \mathbb{Z}/p^l\mathbb{Z}} (F\big|_k \begin{bmatrix} 1 &&& \\ & 1 && up^{-l} \\ && 1 & \\ &&& 1 \end{bmatrix})(\tau, z, \tau') = p^l \sum_{m=0}^{\infty} f_{mp^l}(\tau, z) e^{2\pi i m p^l \tau'}. \tag{59}$$

Hence we see that F *is p-hypercuspidal if and only if there exists an integer $l \geq 1$ such that $f_{mp^l} = 0$ for all $m \geq 0$.* Another equivalent condition is that the Fourier coefficients $c(n,r,m)$ are zero whenever $p^l | m$.

Hypercuspidal modular forms are not easy to construct. For example, it can be shown that cusp forms with respect to paramodular groups are never hypercuspidal. Also, if $p^2 \nmid N$, then a non-zero $F \in S_k(\Gamma_0(N))$ can be shown to be not hypercuspidal. The strongest form of hypercuspidality is that of degree 1. By Proposition 3.9, a non-zero $F \in M_k(\Gamma_0(N))$, $p^2|N$, is hypercuspidal of degree 1 if and only if $\mu_p F = 0$.

3.11 Theorem. *Let N and k be positive integers, $k \geq 3$, and $F \in S_k(\Gamma_0(N))$ be non-zero. Let T be a set (finite or infinite) of primes not dividing N, such that for each $v \in T$ the function F is an eigenfunction for the action of the local Hecke algebra $\mathcal{H}_v$. Then, given a prime $p \nmid N$, there exists $\tilde{F} \in S_k(\Gamma_0(p^2N))$ such that $\tilde{F}$ is p-hypercuspidal of degree 1, and such that $\tilde{F}$ is an eigenfunction for the action of the local Hecke algebra $\mathcal{H}_v$ with the same Hecke eigenvalues as F for each $v \in T$ different from p.*

Proof: The idea is to locally replace the spherical vector at the place p with a hypercuspidal vector. Let Φ be the adelic function attached to F, and let $\pi = \oplus\pi_i$ be the cuspidal representation of $G(\mathbb{A})$ generated by Φ, as explained at the end of Sect. 3.1. We write each of the irreducible components π_i as a restricted tensor product $\pi_i = \otimes_v \pi_{i,v}$. Our hypotheses imply that $\pi_{i,v}$ is spherical for each prime $v \nmid N$, and that

$$\pi_{i,v} \cong \pi_{j,v} \qquad \text{for all } v \in T \text{ and all } i, j.$$

In fact, if $\Phi = \sum \Phi_i$ with Φ_i in the space of π_i, and if Φ_i is written as a sum of pure tensors $\otimes f_v$, then we may assume that f_v is the spherical vector in $\pi_{i,v}$ for each $v \nmid N$. By the main theorem of [PS], each of the local representations $\pi_{i,v}$, $v \nmid N$, is of type I or IIb. A look at Table 1 shows that spherical type I or type IIb representations contain a Siegel vector $\tilde{f}_v$ at level $\mathfrak{p}^2$ that is in the kernel of μ; in fact, the space of such vectors is one-dimensional. Now we replace in the pure tensors $\otimes f_v$ the local vector f_p with $\tilde{f}_p$. The resulting adelic function corresponds to a cusp form $\tilde{F} \in S_k(\Gamma_0(p^2N))$. Since the local and global μ operators are compatible, see (54), we have $\mu_p\tilde{F} = 0$, as desired. Since we did not change the automorphic representations involved, but merely specific vectors in these representations, the function $\tilde{F}$ has the same Hecke properties as F away from the place p where we made the change. ∎

Remarks:

i) The hypothesis $k \geq 3$ in Theorem 3.11 is necessary. There exist elements of spaces $S_2(\Gamma_0(N))$ that generate certain CAP representations with local components of type VId. Table 1 shows that we would not be able to find elements in the kernel of μ at level $\mathfrak{p}^2$ for these representations.

ii) The strong multiplicity one conjecture is expected to hold for non-Saito–Kurokawa cusp forms of weight $k \geq 3$. Assuming this is the case, let $F \in S_k(\Gamma_0(N))$, $k \geq 3$, be an eigenform for almost all Hecke operators. We assume that F is not Saito–Kurokawa, which is equivalent to the degree-4 L-function of F not having a pole at $s = 3/2$. Then the cusp form $\tilde{F}$ in Theorem 3.11 is unique up to multiples. For, by strong multiplicity one, any other such cusp form has to lie in the same automorphic representation as $\tilde{F}$. The uniqueness then follows from local uniqueness, meaning the one-dimensionality of the kernel of μ on Siegel vectors of level $\mathfrak{p}^2$; see Table 1.

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