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# Ramanujan-type results for Siegel cusp forms of degree 2

Ameya Pitale[1] and Ralf Schmidt[2]

[1]*Department of Mathematics, University of Oklahoma, Norman, OK 73019*
*e-mail: ameya@math.ou.edu*

[2]*Department of Mathematics, University of Oklahoma, Norman, OK 73019*
*e-mail: rschmidt@math.ou.edu*

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**Abstract.** A result of Chai–Faltings on Satake parameters of Siegel cusp forms together with the classification of unitary, unramified, irreducible, admissible representations of $\mathrm{GSp}_4$ over a $p$-adic field, imply that the local components of the automorphic representation of $\mathrm{GSp}_4$ attached to a cuspidal Siegel eigenform of degree 2 must lie in certain families. Applications include estimates on Hecke eigenvalues, an improved domain of convergence of the standard $L$-function, and a new characterization of the Maaß space.

## 1. Introduction

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$, and let $\pi \cong \otimes'_v \pi_v$ be an irreducible, cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. One form of the Ramanujan conjecture for the symplectic group states that either each local component $\pi_v$ is tempered, or $\pi$ is nearly equivalent to a representation globally induced from a proper parabolic subgroup. In the terminology of [20], the second alternative means that $\pi$ is CAP with respect to one of the proper parabolic subgroups. If $\pi$ is obtained from a holomorphic Siegel modular form $F$ of degree 2 for the full modular group $\mathrm{Sp}_4(\mathbb{Z})$, then $\pi$ is CAP if and only if it is CAP with respect to the Siegel parabolic subgroup $P$ (the parabolic subgroup with abelian radical). This happens if and only if $F$ is a classical Saito–Kurokawa lifting. Therefore, we call an irreducible, cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ that is CAP with respect to $P$ a *Saito–Kurokawa representation*.

For classical cusp forms $F$ the Ramanujan conjecture is equivalent to the following statement. Suppose $F$ is an eigenform for all Hecke operators

$T(m)$, and let $\mu(m)$ be the corresponding eigenvalue. Then

$$|\mu(m)| \ll_\varepsilon m^{k-3/2+\varepsilon} \quad (\varepsilon > 0), \tag{1.1}$$

provided that $F$ is not a Saito–Kurokawa lifting[1] (see [16], [23]). Note that having (1.1) for only $m = p$, where $p$ is a prime number, is not sufficient to imply that the local representation is tempered. Several results in the direction of the Ramanujan conjecture are available in the literature. We mention the work [16], where it is proved that

$$|\mu(m)| \ll_\varepsilon m^{k-1/2+\varepsilon} \quad (\varepsilon > 0). \tag{1.2}$$

Note that this result holds for arbitrary eigen-cusp forms, Saito–Kurokawa or not. A related class of results estimates the value $|\mu(p)|$ for $p$ a prime number. The best available result in the literature for arbitrary eigen-cusp forms of degree 2 seems to be

$$|\mu(p)| \leq p^{k-1} + 2p^{k-3/2} + p^{k-2}; \tag{1.3}$$

see [11]. In the present paper we will reprove the result (1.3). We will also prove

$$|\mu(m)| \ll_\varepsilon m^{k-1+\varepsilon} \quad (\varepsilon > 0). \tag{1.4}$$

Since Saito–Kurokawa forms do not allow for smaller exponents than $k - 1$, this is the best possible estimate for arbitrary eigen-cusp forms of degree 2. Note that the estimate (1.3) alone does not imply (1.4), since the local Hecke algebra at $p$ is not generated by $T(p)$ alone. It was shown in [17] and [34] that the estimate (1.4) would also follow from expected growth properties of Fourier coefficients. Note that our results hold for cusp forms with respect to $\Gamma_0(N)$.

Our method consists in combining a strong result of Chai–Faltings with the classification of the unitary, unramified, irreducible, admissible representations of $\mathrm{GSp}_4(\mathbb{Q}_p)$. The result of Chai–Faltings says that, for each prime $p$ not dividing the level, the product of (a suitable element in the Weyl group orbit of) the Satake $p$-parameters of an eigen-cusp form of degree $n$ and weight $k > n$ has absolute value 1; see [6, p. 267]. Let $\pi \cong \otimes' \pi_p$ be a cuspidal, automorphic representation attached to a holomorphic modular form with respect to $\Gamma_0(N)$. It turns out that the Chai–Faltings condition forces $\pi_p$ to be in one of three families. We call these families [T], for tempered representations, [C], for a certain class of complementary series representations, and [SK], for Saito–Kurokawa type representations. This is our Theorem 3.2. Since it implies restrictions on the individual Satake parameters, we can get

[1] The notation means that for each $\varepsilon > 0$ there exists a constant $C$, depending on $\varepsilon$, such that $|\mu(m)| < Cm^{k-3/2+\varepsilon}$ for all $m > 0$.

better estimates on various objects that can be written in terms of these Satake parameters.

The main point of this paper, however, is not the representation-theoretic reinterpretation of the Chai–Faltings result, but the following applications. First, we derive a general formula for the Hecke eigenvalues $\mu(m)$ and combine it with the estimates from Theorem 3.2 in order to re-prove the estimate (1.3) and obtain (1.4). The results will be valid for cusp forms of weight $k > 2$ and with respect to $\Gamma_0(N)$. Next, for the same class of modular forms, we shall improve on the domain of convergence for the degree-5 standard $L$-functions. The current estimates imply convergence for $\mathrm{Re}(s) > 2$ (see [7], [32]); we will prove convergence for $\mathrm{Re}(s) > 3/2$. It follows as a corollary that the only CAP representations that can come from holomorphic modular forms of weight $k > 2$ are of Saito–Kurokawa type.

Our third application concerns characterizations of Saito–Kurokawa lifts in terms of their Fourier coefficients. One knows that a cusp form $F$ is a Saito–Kurokawa lift from a even-integer weight cusp form if and only if the Fourier coefficients of $F$ satisfy the Maaß conditions. For each prime number $p$ we will define a similar, but weaker, condition, called the $p$-Maaß condition. Using Theorem 3.2 we prove that $F$ lies in the Maaß space if and only if $F$ satisfies the $p$-Maaß conditions for almost all primes $p$. In other words, the Maaß condition obeys a local-global principle.

The final application of Theorem 3.2 presented in this paper consists in giving evidence for a conjectural dimension formula for paramodular newforms due to IBUKIYAMA. We shall also explain the role Theorem 3.2 plays in a result on hypercuspidal modular forms obtained in [28].

This paper is organized as follows. In Section 2., we define Siegel cusp forms of degree 2 and recall the construction of the corresponding global and $p$-adic representations. In Section 3., we recall the classification of irreducible, unramified, unitarizable representations of $\mathrm{GSp}_4(\mathbb{Q}_p)$ due to Rodier; see [29] or [30]. We then state and prove Theorem 3.2 using the result of Chai and Faltings. In Section 4., we present the applications of Theorem 3.2.

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## 2. Siegel cusp forms and the corresponding *p*-adic representations

Let $G = \mathrm{GSp}_4$ be the group of symplectic similitudes, defined by

$$G := \{g \in \mathrm{GL}_4 : \ {}^t gJg = \lambda(g)J, \text{ some } \lambda(g) \in \mathrm{GL}_1\} \qquad (2.5)$$

where $J$ is the symplectic form given by the matrix $J = \left[\begin{smallmatrix} & 1_2 \\ -1_2 & \end{smallmatrix}\right]$. The map $\lambda : G \to \mathrm{GL}_1$ is the *multiplier homomorphism*. Its kernel is by definition the symplectic group $\mathrm{Sp}_4$. The group $G^+(\mathbb{R}) := \{g \in \mathrm{GSp}_4(\mathbb{R}) :$

$\lambda(g) > 0\}$ acts on the Siegel upper half plane $\mathbb{H}_2 := \{Z \in \mathrm{Mat}_2(\mathbb{C}) : {}^tZ = Z,\ \mathrm{Im}(Z) > 0\}$ by linear fractional transformations in the familiar way,

$$g\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \quad \text{for } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G^+(\mathbb{R}) \text{ and } Z \in \mathbb{H}_2.$$

For a positiver integer $N$ we let $\Gamma_0(N) = \{[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in \mathrm{Sp}_4(\mathbb{Z}) :\ C \equiv 0 \mod N\}$. The space $M_k(\Gamma_0(N))$ of Siegel modular forms of weight $k$ with respect to $\Gamma_0(N)$ is the space of holomorphic $\mathbb{C}$-valued functions $F$ on $\mathbb{H}_2$ that satisfy $(F|_k\gamma)(Z) = F(Z)$ for all $\gamma \in \Gamma_0(N)$. Here,

$$(F|_kg)(Z) := \lambda(g)^k j(g,Z)^{-k} F(g\langle Z\rangle) \quad \text{for } g \in G^+(\mathbb{R}),\ Z \in \mathbb{H}_2, \tag{2.6}$$

where $j(g,Z) := \det(CZ + D)$ is the automorphy factor. We denote by $S_k(\Gamma_0(N))$ the subspace of cusp forms.

Let $G_p = \mathrm{GSp}_4(\mathbb{Q}_p)$ and $K_p = \mathrm{GSp}_4(\mathbb{Z}_p)$. The local Hecke algebra $\mathcal{H}_p = \mathcal{H}(G_p, K_p)$ is the convolution algebra of left and right $K_p$-invariant functions $G_p \to \mathbb{C}$. We recall how $\mathcal{H}_p$, for $p \nmid N$, acts on the spaces $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ (see [3], Lemma 3.3.1 and Lemma 3.3.2). Let $T$ be the characteristic function of the double coset $K_pgK_p$, and let $K_pgK_p = \bigsqcup K_pg_i$, $g_i \in G_p$, be a disjoint decomposition. The representatives $g_i$ may be chosen to be from $\mathrm{GSp}_4(\mathbb{Z}[p^{-1}])$ and to be upper triangular with only $p$-powers on the diagonal. Then

$$TF = \sum \det(g_i)^{(k-3)/2} F|g_i \tag{2.7}$$

for $F \in M_k(\Gamma_0(N))$ or $F \in S_k(\Gamma_0(N))$ (the determinant factor is there to be consistent with [1, p. 62]). Note that, since we are only considering modular forms of haupttype, the characteristic function of $p^{\pm r}K_p$ acts by multiplication with $p^{2r(k-3)}$. Hence, what is relevant for us is only the action of the subalgebra $\mathcal{H}_p^+$ spanned by the characteristic functions of double cosets represented by diagonal elements whose entries are of the form $p^i$ with $i \geq 0$. If $p \neq p'$, then the actions of $\mathcal{H}_p$ and $\mathcal{H}_{p'}$ commute.

For any prime $p$ and non-negative integer $r$ let $T(p^r)$ be the characteristic function of $\{g \in G_p :\ \lambda(g) \in p^r\mathbb{Z}_p^\times\}$. Further, let $\Delta(p^r)$ be the characteristic function of $p^rK_p$. Then $\Delta(p^r)$ is invertible, its inverse being $\Delta(p^{-r})$. The $\mathbb{C}$-algebra $\mathcal{H}_p^+$ is generated by $\Delta(p)$, $T(p)$ and $T(p^2)$. The following formal identity captures the multiplicative structure of this algebra (see [2], Proposition 3.35).

$$\sum_{j=0}^{\infty} T(p^j)X^j = \frac{1 - p^2\Delta(p)X^2}{\begin{array}{c}1 - T(p)X + (T(p)^2 - T(p^2) - p^2\Delta(p))X^2 \\ -p^3\Delta(p)T(p)X^3 + p^6\Delta(p)^2X^4\end{array}}. \tag{2.8}$$

For $p \nmid N$ we obtain Hecke operators $T(p^r)$ on $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$. Since $\Delta(p)$ acts by multiplication with $p^{2k-6}$, we obtain the formal identity

$$\sum_{j=0}^{\infty} T(p^j)X^j = \frac{1 - p^{2k-4}X^2}{\begin{array}{c}1 - T(p)X + (T(p)^2 - T(p^2) - p^{2k-4})X^2 \\ -p^{2k-3}T(p)X^3 + p^{4k-6}X^4\end{array}} \tag{2.9}$$

in the endomorphism algebra of $M_k(\Gamma_0(N))$ or $S_k(\Gamma_0(N))$. For a positive integer $m$ with $(m, N) = 1$ we let $T(m) = \prod_{p|m} T(p^{r_p})$, where $m = \prod p^{r_p}$ is the prime factorization of $m$.

We say that $F \in S_k(\Gamma_0(N))$ is an *eigenform* if it is an eigenvector for all $T(m)$, $(m, N) = 1$. This is equivalent to saying that $F$ is an eigenvector for all $T \in \mathcal{H}_p$ (or $T \in \mathcal{H}_p^+$), for all $p \nmid N$. For such an eigenform $F$ we denote by $\mu(m)$ its Hecke eigenvalues, defined by $T(m)F = \mu(m)F$. The space $S_k(\Gamma_0(N))$ has a basis consisting of eigenforms.

Assuming that $F \in S_k(\Gamma_0(N))$ is a Hecke eigenform, we will attach to it, for each prime number $p \nmid N$, an unramified, irreducible, admissible representation $\pi_{F,p}$ of $\mathrm{GSp}_4(\mathbb{Q}_p)$. For $\alpha \geq 0$ let $K_0(p^\alpha) = \{[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in \mathrm{GSp}_4(\mathbb{Z}_p) : C \equiv 0 \mod p^\alpha\mathbb{Z}_p\}$. Using strong approximation for $\mathrm{Sp}_4(\mathbb{A})$ (see [18]) one obtains the decomposition

$$G(\mathbb{A}) = G(\mathbb{Q})G^+(\mathbb{R}) \prod_{p<\infty} K_0(p^{\alpha_p}) \quad \Big(N = \prod p^{\alpha_p}\Big). \tag{2.10}$$

Given $g \in G(\mathbb{A})$, using (2.10), we can write $g = g_{\mathbb{Q}} g_\infty k_0$, where $g_{\mathbb{Q}} \in G(\mathbb{Q})$, $g_\infty \in G^+(\mathbb{R})$ and $k_0 \in K_0(N) := \prod_{p<\infty} K_0(p^{\alpha_p})$. Now define $\Phi_F : G(\mathbb{A}) \to \mathbb{C}$ by the formula

$$\Phi_F(g) := (F|_k g_\infty)(I_2), \quad I_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}. \tag{2.11}$$

Since $\Gamma_0(N) = G(\mathbb{Q}) \cap G^+(\mathbb{R})K_0(N)$ and $F \in S_k(\Gamma_0(N))$, this is well-defined. It can be shown (see [3, p. 186]) that the function $\Phi_F$ is a cuspidal automorphic form. Let $V_F$ be the space of all right translates of $\Phi_F$. The group $G(\mathbb{A})$ acts on $V_F$ by right translation. This representation may not be irreducible, but $V_F$ will decompose into a direct sum of finitely many irreducible, cuspidal, automorphic representations of $\mathrm{GSp}_4(\mathbb{A})$. Let $\pi$ be one of these irreducible components, and write $\pi$ as a restricted tensor product $\pi \cong \otimes_p' \pi_{F,p}$, where $\pi_{F,p}$ is an irreducible, admissible, unitarizable representation of $G(\mathbb{Q}_p)$. Since $\Phi_F$ is $G(\mathbb{Z}_p)$-invariant for all $p \nmid N$, the representation $\pi_{F,p}$ has, for such $p$, a non-zero, essentially unique $\mathrm{GSp}_4(\mathbb{Z}_p)$-invariant vector. The same calculations as in [3] show that, for $p \nmid N$, the equivalence class of $\pi_p$ depends only on $F$ and not on the chosen global irreducible component $\pi$.

As an unramified representation, $\pi_p$ is the unique spherical constituent of a representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ parabolically induced from an unramified character of the Borel subgroup; see [5]. In the notation of [30], $\pi_p$ is isomorphic to the Langlands quotient of an induced representation of the form $\chi_1\times\chi_2\rtimes\sigma$, where $\chi_1$, $\chi_2$ and $\sigma$ are unramified characters of $\mathbb{Q}_p^\times$ (see also the next section). The numbers

$$a_{p,1} := \chi_1(p), \quad a_{p,2} := \chi_2(p), \tag{2.12}$$

or more precisely, their Weyl group orbit, are called the *Satake $p$-parameters* of the eigenform $F$. Together with $a_{p,0} := p^{k-3/2}\sigma(p)$ they determine the equivalence class of $\pi_p$. Note that $a_{p,1}a_{p,2}a_{p,0}^2 = p^{2k-3}$, since $\pi_{F,p}$ has trivial central character. Hence, $a_{p,1}$ and $a_{p,2}$ detemine $\pi_{F,p}$ only up to an unramified, quadratic twist.[2] By [1, p. 62, 69], Satake parameters and Hecke eigenvalues are related by the formula

$$\begin{aligned}1-\mu(p)T+(\mu(p)^2-\mu(p^2)-p^{2k-4})T^2-\mu(p)p^{2k-3}T^3+p^{4k-6}T^4\\ =(1-a_{p,0}T)(1-a_{p,0}a_{p,1}T)(1-a_{p,0}a_{p,2}T)(1-a_{p,0}a_{p,1}a_{p,2}T).\end{aligned} \tag{2.13}$$

The incomplete degree-4 (spin) $L$-function of the eigenform $F$ is

$$L(s,F)=\prod_{p<\infty}\frac{1}{\begin{array}{c}(1-a_{p,0}p^{-s})(1-a_{p,0}a_{p,1}p^{-s})(1-a_{p,0}a_{p,2}p^{-s})\\ (1-a_{p,0}a_{p,1}a_{p,2}p^{-s})\end{array}},$$

and the incomplete degree-5 (standard) $L$-function of $F$ is

$$L_{\mathrm{St}}(s,F)=\prod_{p<\infty}\frac{1}{\begin{array}{c}(1-a_{p,1}p^{-s})(1-a_{p,2}p^{-s})(1-p^{-s})\\ (1-a_{p,2}^{-1}p^{-s})(1-a_{p,1}^{-1}p^{-s})\end{array}}.$$

Note that $L(s,F)$ satisfies a functional equation with respect to $s\mapsto 2k-2-s$, while $L_{\mathrm{St}}(s,F)$ satisfies a functional equation with respect to $s\mapsto 1-s$.

## 3. The representation-theoretic reinterpretation of a result of Chai–Faltings

In order to study the representations $\pi_{F,p}$ considered in the previous section, we will first review the classification of irreducible, unramified, unitariz-

[2] Note that sometimes, as in [3, Lemma 10], $a_{p,0}$ is counted amongst the set of Satake parameters, but we prefer not to do so. Also, we caution that our (classical) terminology for Satake parameters differs slightly from the one used in [5]: The semisimple conjugacy class in the $L$-group $\mathrm{GSp}_4(\mathbb{C})$ corresponding to the spherical constituent of the representation $\chi_1\times\chi_2\rtimes\sigma$ is represented by $\mathrm{diag}(\sigma(p),\chi_1(p)\sigma(p),\chi_1(p)\chi_2(p)\sigma(p),\chi_2(p)\sigma(p))$.

able representations of $G_p = \mathrm{GSp}_4(\mathbb{Q}_p)$ supported on the Borel subgroup. We will then combine this classification with a result of Chai–Faltings on the Satake $p$-parameters of Siegel cusp forms to show that the local representations $\pi_{F,p}$ lie in one of only three possible families; see Theorem 3.2.

We shall employ the notations of [30] for representations of $G_p$. For unramified characters $\chi_1, \chi_2, \sigma$ of $\mathbb{Q}_p^\times$ (i.e., trivial on units of $\mathbb{Z}_p$) we denote by $\chi_1 \times \chi_2 \rtimes \sigma$ the representation of $G_p$ induced from the character

$$\begin{bmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & \lambda a_1^{-1} & \\ & & * & \lambda a_2^{-1} \end{bmatrix} \mapsto \chi_1(a_1)\chi_2(a_2)\sigma(\lambda) \tag{3.14}$$

of the Borel subgroup. The central character of this representation is given by $\chi_1\chi_2\sigma^2$. Provided that $e(\chi_1) \geq e(\chi_2) > 0$, where $e(\chi_i)$ denotes the real number with $|\chi_i(x)| = |x|^{e(\chi_i)}$, let $L(\chi_1 \times \chi_2 \rtimes \sigma)$ be the unique irreducible quotient (Langlands quotient) of $\chi_1 \times \chi_2 \rtimes \sigma$.

If $\pi$ is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\sigma$ is a character of $\mathbb{Q}_p^\times$, let $\pi \rtimes \sigma$ be the representation of $G_p$ induced from the representation

$$\begin{bmatrix} A & * \\ & u\,{}^tA^{-1} \end{bmatrix} \mapsto \sigma(u)\pi(A) \tag{3.15}$$

of the Siegel parabolic subgroup. Assuming that $\pi$ is essentially square-integrable, there exists a unique real number $e(\pi)$ such that $|\ |^{-e(\pi)}\pi$ is unitarizable. If $e(\pi) > 0$, the induced representation $\pi \rtimes \sigma$ has a unique Langlands quotient, denoted by $L(\pi, \sigma)$.

Finally, assume that $\chi$ is an unramified character of $\mathbb{Q}_p^\times$ and $\sigma$ is a representation of $\mathrm{GSp}_2(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)$. Then $\chi \rtimes \sigma$ denotes the representation of $G_p$ induced from the representation

$$\begin{bmatrix} u & * & * & * \\ & a & * & b \\ & & (ad-bc)u^{-1} & \\ & c & * & d \end{bmatrix} \longmapsto \chi(u)\sigma\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.16}$$

of the Klingen parabolic subgroup. If $e(\chi) > 0$ there is a unique Langlands quotient denoted by $L(\chi, \sigma)$.

We denote by $\chi_1 \times \chi_2$ the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ induced from the character $\left[\begin{smallmatrix} a & * \\ & d \end{smallmatrix}\right] \mapsto \chi_1(a)\chi_2(d)$ of the standard Borel subgroup. Note that if $\mathrm{GL}_2$ is considered as the group $\mathrm{GSp}_2$, then $\chi_1 \rtimes \chi_2 = \chi_1\chi_2 \times \chi_2$.

As in [30], we denote by $\nu$ the normalized absolute value $\mathbb{Q}_p^\times \to \mathbb{C}^\times$. We now state the classification of irreducible, unramified, unitarizable representations of $G_p$ as given in [29], but using the notation of [30]. In the following proposition, roman capital letters ("Type IIb") refer to the classification of irreducible, admissible representations of $\mathrm{GSp}_4(\mathbb{Q}_p)$ used in [27].

**Proposition 3.1.** *Let $\chi_1$, $\chi_2$, $\chi$ and $\sigma$ be unramified characters of $\mathbb{Q}_p^\times$.*

1. *(Type I) The induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq \nu^{\pm 1}, \chi_2 \neq \nu^{\pm 1}$ and $\chi_1 \neq \nu^{\pm 1}\chi_2^{\pm 1}$. If irreducible, this representation is unitary if and only if one of the following holds.*
   (a) $|\chi_1| = |\chi_2| = |\sigma| = 1$, *the tempered case;*
   (b) $\chi_1 = \nu^{\beta}\chi, \chi_2 = \nu^{\beta}\chi^{-1}$ *and* $e(\sigma) = -\beta$ *with* $|\chi| = 1$, $\chi^2 \neq 1$ *and* $0 < \beta < 1/2$*;*
   (c) $\chi_1 = \nu^{\beta}, \chi_2 = \chi$ *and* $e(\sigma) = -\beta/2$ *with* $|\chi| = 1$, $\chi \neq 1$ *and* $0 < \beta < 1$*;*
   (d) $\chi_1 = \nu^{\beta_1}\chi, \chi_2 = \nu^{\beta_2}\chi$ *and* $e(\sigma) = -(\beta_1 + \beta_2)/2$ *with* $\chi^2 = 1$, $0 \leq \beta_2 \leq \beta_1$, $0 < \beta_1 < 1$ *and* $\beta_1 + \beta_2 < 1$.
2. *(Type IIb) $\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$, the unramified, irreducible constituent of $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$, with $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$, is unitarizable if and only if $\sigma$ is unitary and one of the following holds:*
   (a) $|\chi| = 1$*;*
   (b) $\chi = \xi\nu^{e(\chi)}$, $\xi^2 = 1$ *and* $0 < e(\chi) < 1/2$.
3. *(Type IIIb) $\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$, the unramified, irreducible constituent of $\chi \times \nu \rtimes \nu^{-1/2}\sigma$, with $\chi \notin \{\mathbf{1}, \nu^{\pm 2}\}$, is unitarizable if and only if $|\chi| = |\sigma| = 1$.*
4. *(Type IVd) $\sigma \mathbf{1}_{\mathrm{GSp}(4)}$, the unramified, irreducible constituent of $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$ is unitarizable if and only if $|\sigma| = 1$.*
5. *(Type Vd) $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma)$, the unramified, irreducible constituent of $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$, with $\xi_0^2 = 1, \xi_0 \neq 1$, is unitarizable if and only if $|\sigma| = 1$.*
6. *(Type VId) $L(\nu, \mathbf{1}_{\mathbb{Q}_p^\times} \rtimes \nu^{-1/2}\sigma)$, the unramified, irreducible constituent of $\nu \times \mathbf{1}_{\mathbb{Q}_p^\times} \rtimes \nu^{-1/2}\sigma$ is unitarizable if and only if $|\sigma| = 1$.*

*Each unitarizable, unramified, irreducible, admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ is isomorphic to exactly one representation in the above list.*

Now let $F \in S_k(\Gamma_0(N))$ be a Hecke eigenform. In Section 2. we constructed, for each prime $p \nmid N$, a corresponding unitarizable, unramified, irreducible, admissible representation $\pi_{F,p}$ of $\mathrm{GSp}_4(\mathbb{Q}_p)$. Hence, $\pi_{F,p}$ is one of the representations listed in Proposition 3.1.

Let $\pi \cong \otimes' \pi_{F,p}$ be one of the irreducible, cuspidal, automorphic representations considered in Section 2.. The Ramanujan conjecture for $\mathrm{GSp}_4$ predicts that either $\pi$ is a CAP representation, or $\pi_{F,p}$ is tempered for each place $p$ (case $(1)(a)$ in Proposition 3.1). One can show (see our Corollary 4.5 further below) that if $\pi$ is CAP and $k > 2$, then $\pi$ must be CAP to the Siegel parabolic subgroup, i.e., $\pi$ is a Saito–Kurokawa representation. In this case, $\pi_{F,p}$ is of type (2) in Proposition 3.1 (see [21] or [31, Lemma 2.2]). The following theorem, which is our main result, states that $\pi_{F,p}$ is either in one of the two families of representations predicted by the Ramanujan conjecture, or is a certain kind of complementary series representation.

**Theorem 3.2.** *Let $N$ and $k$ be positive integers with $k > 2$. Let $F \in S_k(\Gamma_0(N))$ be a Hecke eigenform. For $p \nmid N$ let $\pi_{F,p}$ be the corresponding local representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ constructed in Section 2.. Then $\pi_{F,p}$ can only be one of the following.*

(T) $\chi_1 \times \chi_2 \rtimes \sigma$ *irreducible with* $|\chi_1| = |\chi_2| = |\sigma| = 1$ *(the tempered case); or*

(C) $\chi_1 \times \chi_2 \rtimes \sigma$ *irreducible with* $\chi_1 = \nu^\beta\chi, \chi_2 = \nu^\beta\chi^{-1}$, $|\chi| = 1$, $e(\sigma) = -\beta$ *with* $0 < \beta < 1/2$ *(the complementary series case); or*

(SK) $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$*, the spherical constituent of* $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$*, with* $|\chi| = 1$ *(the Saito–Kurokawa case).*

*The characters $\chi_1, \chi_2, \chi$ and $\sigma$ above are unramified.*

The proof involves a key result due to Chai and Faltings [6, p. 267], which puts certain restrictions on the possible Satake parameters of the $p$-adic representation corresponding to a Siegel cusp form. Roughly speaking, the result says that the product of the Satake parameters of a cuspidal Siegel eigenform has absolute value 1, but we have to consider the action of the Weyl group. The Weyl group orbit of $(a_1, a_2) = (\chi_1(p), \chi_2(p))$ is given by the 8 elements

$$\{(a_1, a_2), (a_1^{-1}, a_2), (a_1, a_2^{-1}), (a_1^{-1}, a_2^{-1}), (a_2, a_1), (a_2^{-1}, a_1), (a_2, a_1^{-1}), (a_2^{-1}, a_1^{-1})\}. \tag{3.17}$$

In [6, p. 267], Chai and Faltings proved the following result[3] (see also [10, p. 107]).

**Proposition 3.3 (Chai–Faltings).** *Let $N$ and $k$ be positive integers with $k > 2$. Let $F \in S_k(\Gamma_0(N))$ be a Hecke eigenform. For $p \nmid N$ let $\pi_{F,p}$ be the corresponding local representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ constructed in Section 2.. There is at least one element in the Weyl group orbit (3.17) of the Satake parameters $(\chi_1(p), \chi_2(p))$ of $\pi_{F,p}$ such that the product of the Satake parameters has absolute value* 1. □

We note that Chai and Faltings proved the above result in greater generality, namely, for Satake parameters of local representations of $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$, $n > 0$, obtained from Siegel modular forms of degree $n$ and weight $k > n$. We have only stated the $\mathrm{GSp}_4$ case here.

[3]We would like to thank G. Faltings for clarifying to us that Proposition 3.3 is really the precise statement resulting from the algebro-geometric methods of [6].

*Proof of Theorem* 3.2. The proof consists of analyzing the representations listed in Proposition 3.1 and discarding those that do not satisfy Proposition 3.3. As in the comments above we will use the notation $a_1 := \chi_1(p)$ and $a_2 := \chi_2(p)$.

1. Type $(1)$ representations from Proposition 3.1:
   (a) Type $(1)(a)$: We have $|a_1| = |a_2| = 1$. Hence Proposition 3.3 is **satisfied** by all elements in the Weyl group orbit (3.17).
   (b) Type $(1)(b)$: We have $|a_1| = p^{-\beta} = |a_2|$ with $0 < \beta < 1/2$. We can then see that the element $(a_1, a_2^{-1})$ in the Weyl group orbit (3.17) **satisfies** Proposition 3.3.
   (c) Type $(1)(c)$: We have $|a_1| = p^{-\beta}$ and $|a_2| = 1$ with $0 < \beta < 1$. Hence for all elements in the Weyl group orbit (3.17) the absolute value of the product is either $p^{\beta}$ or $p^{-\beta}$. This does **not satisfy** Proposition 3.3 since $\beta \neq 0$.
   (d) Type $(1)(d)$: We have $|a_1| = p^{-\beta_1}$ and $|a_2| = p^{-\beta_2}$, with $0 \le \beta_2 \le \beta_1, 0 < \beta_1 < 1, \beta_1 + \beta_2 < 1$. Hence for all elements in the Weyl group orbit (3.17) the absolute value of the product is either $p^{\pm(\beta_1+\beta_2)}$ or $p^{\pm(\beta_1-\beta_2)}$. So, we can conclude that Proposition 3.3 is **satisfied** only if $0 < \beta_1 = \beta_2 < 1/2$.
2. Type $(2)$: We have $|a_1| = p^{-e(\chi)-1/2}$ and $|a_2| = p^{-e(\chi)+1/2}$ with $0 \le e(\chi) < 1/2$. Hence for all elements in the Weyl group orbit (3.17) the absolute value of the product is either $p^{\pm 2e(\chi)}$ or $p^{\pm 1}$. Thus the condition from Proposition 3.3 is **satisfied** only if $e(\chi) = 0$, i.e., $|\chi| = 1$.
3. Type $(3)$: We have $|a_1| = 1$ and $|a_2| = p^{-1}$. Hence for all elements in the Weyl group orbit (3.17) the absolute value of the product is $p^{\pm 1}$. Therefore Proposition 3.3 is **not satisfied**.
4. Type $(4)$: We have $|a_1| = p^{-2}$ and $|a_2| = p^{-1}$. Hence for all elements in the Weyl group orbit (3.17) the absolute value of the product is either $p^{\pm 1}$ or $p^{\pm 3}$. Thus we conclude that Proposition 3.3 is **not satisfied**.
5. Type $(5)$: We have $|a_1| = p^{-1}$ and $|a_2| = 1$. Hence for all elements in the Weyl group orbit (3.17) the absolute value of the product is $p^{\pm 1}$. Therefore the condition from Proposition 3.3 is **not satisfied**.
6. Type $(6)$: We have $|a_1| = p^{-1}$ and $|a_2| = 1$. Hence for all elements in the Weyl group orbit (3.17) the absolute value of the product is $p^{\pm 1}$, and we conclude that Proposition 3.3 is **not satisfied**.

Putting together all the above we obtain the statement of Theorem 3.2. $\square$

We remark that if Theorem 3.2 could be proved independently, it could be used to obtain the Ramanujan conjecture for elliptic cusp form of weight $2k$ with odd $k$. For let $f \in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$ be an eigenform, and let

$F \in S_{k+1}(\mathrm{Sp}(4,\mathbb{Z}))$ be its Saito–Kurokawa lift. If $\pi_F \cong \otimes'_p \pi_{F,p}$ is the corresponding cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, then it is known (see [21] or [31]) that for all primes $p$ the local representation $\pi_{F,p}$ is of type $(2)$ in Proposition 3.1, and hence of type (SK) in Theorem 3.2. The unitarity of the character $\chi$ translates into the Ramanujan estimate for $f$.

## 4. Applications

We will apply Theorem 3.2 to obtain several new results.

- We will obtain new estimates for the Hecke eigenvalues for a degree 2 Siegel cusp form $F \in S_k(\Gamma_0(N))$.
- We will show that the degree 5 standard $L$-function $L_{\mathrm{St}}(s,F)$ of $F$ converges on the half plane $\mathrm{Re}(s) > 3/2$.
- We will show that the condition for $F$ to be in the Maaß space satisfies a certain local-global principle.
- We will give evidence for a conjectural dimension formula for paramodular newforms due to IBUKIYAMA.

### 4.1 Estimates on Hecke eigenvalues

Recall the definition of the Hecke operator $T(m)$, $(m,N)=1$, on the space $S_k(\Gamma_0(N))$ given in section 2.. Let $F$ be an eigenform with eigenvalues $\mu(m)$, $(m,N)=1$. We will now list the known estimates for the $\mu(m)$.

1. [15, p. 33], [35] For the full modular group we have the so-called *trivial estimate*: For $n > 1$ and any $m > 1$,

$$|\mu(m)| < m^{nk/2-n(n+1)/2} d(m) \tag{4.18}$$

where $d(m)$ is the number of elements in $\mathrm{Sp}(2n,\mathbb{Z})\backslash\{g \in M(2n,\mathbb{Z}) : {}^t g J_n g = mJ_n\}$. In particular, for a prime $p$ we have

$$|\mu(p)| < 2^n p^{nk/2}. \tag{4.19}$$

2. [7, p. 235] For $n > 1$ and any prime $p \nmid N$, we have

$$|\mu(p)| \le 2^n p^{\frac{nk}{2} - \frac{n(n+1)}{12}}. \tag{4.20}$$

3. [7, p. 235] For $n = 2^r$, $r \geq 1$ and any prime $p \nmid N$, we have

$$|\mu(p)| \le 2^n p^{\frac{nk}{2} - \frac{n(n+1)}{8}}. \tag{4.21}$$

4. [7, p. 236] For $n = 2$ and any prime $p \nmid N$, we have

$$|\mu(p)| \le 4p^{k-1}. \tag{4.22}$$

5. [11, p. 107] For $n \geq 1$, $k \geq n+1$ and any prime $p \nmid N$, we have

$$|\mu(p)| \leq \sum_{v=0}^{n} p^{e(v)} m(n-v,v), \tag{4.23}$$

where $e(v) = (v(v+1) + (n-v)(n-v+1) + 2n(k-n-1))/4$ and

$$m(n-v,v) = \prod_{t=1}^{v} \left( \frac{p^{n+1-t}-1}{p^t-1} \right) \text{ if } v \geq 1, \quad m(n,0) = 1.$$

(This result actually holds for principal congruence subgroups.) These estimates are better than $(1), (2), (3), (4)$ above. In particular, for $n = 2$ this implies

$$|\mu(p)| \leq p^{k-1} + 2p^{k-3/2} + p^{k-2} < 4p^{k-1} \tag{4.24}$$

6. [16, p. 641] For the full modular group and $n = 2$ and any integer $m$, we have

$$\mu(m) \ll_\epsilon m^{k-1/2+\epsilon} \quad (\epsilon > 0). \tag{4.25}$$

Here "$f \ll g$" means that there exists a constant $C$, independent of the argument of $f$ and $g$, such that $|f(.)| \leq C|g(.)|$. It seems that this is the best estimate known till now for $\mu(m)$ for any value of $m$ and not just $m = p$, a prime number.

In [17, p. 86] and [34, p. 388], the authors have obtained estimates for $|\mu(m)|$ under certain assumptions on the growth of the Fourier coefficients of the Siegel cusp form of degree 2. Unfortunately, the estimates for the Fourier coefficients are not yet available.

Using our Theorem 3.2, we can reprove (4.24) and obtain better estimates for $\mu(m)$ for all $m > 1$. First, we give a precise formula for the eigenvalue $\mu(p^r)$ for $r \geq 1$ and then use the formula to get the estimate for $\mu(m)$ for all $m$.

Let $\pi_{F,p}$ be the local representation of $G_p = \mathrm{GSp}_4(\mathbb{Q}_p)$ corresponding to $F$ as constructed in Section 2.. Then $\pi_{F,p}$ is the unique spherical constituent of a representation of the form $\chi_1 \times \chi_2 \rtimes \sigma$, where $\chi_1, \chi_2$ and $\sigma$ are unramified characters of $\mathbb{Q}_p^\times$. Let us abbreviate $a := \sigma(p)$ and $b := \chi_1(p)\sigma(p)$.

**Proposition 4.1.** *Let $N$ and $k$ be positive integers with $k > 2$. Let $F \in S_k(\Gamma_0(N))$ be a Hecke eigenform with eigenvalues $\mu(m)$ for all $m > 0$, $(m, N) = 1$. For any prime $p \nmid N$ and integer $r \geq 1$ we have*

$$\frac{\mu(p^r)}{(p^r)^{(k-3/2)}} = A_{a,b}(r) + (1 - 1/p) \sum_{t=1}^{[r/2]} A_{a,b}(r-2t) \tag{4.26}$$

*where*

$$A_{a,b}(j) = \left(\sum_{u=0}^{j} a^{j-u}b^u\right)\left(\sum_{u=0}^{j}(ab)^{-u}\right). \tag{4.27}$$

*Proof.* Note that $\chi_1(p)\chi_2(p)\sigma(p) = a^{-1}$ and $\chi_2(p)\sigma(p) = b^{-1}$, since $\pi_{F,p}$ has trivial central character. From (2.13) and (2.9) we therefore get

$$\sum_{r=0}^{\infty}\mu(p^r)X^r$$
$$= \frac{1-p^{2k-4}X^2}{(1-p^{k-\frac{3}{2}}aX)(1-p^{k-\frac{3}{2}}a^{-1}X)(1-p^{k-\frac{3}{2}}bX)(1-p^{k-\frac{3}{2}}b^{-1}X)}. \tag{4.28}$$

A change of variables $T = p^{k-3/2}X$ gives

$$\sum_{r=0}^{\infty}\frac{\mu(p^r)T^r}{(p^{k-\frac{3}{2}})^r} = \frac{1-p^{-1}T^2}{(1-aT)(1-a^{-1}T)(1-bT)(1-b^{-1}T)}. \tag{4.29}$$

We define $a(p^r)$ via the formal power series

$$\sum_{r=0}^{\infty}\frac{a(p^r)T^r}{(p^{k-\frac{3}{2}})^r} = \frac{1}{(1-aT)(1-a^{-1}T)(1-bT)(1-a^{-1}T)}, \tag{4.30}$$

so that

$$\frac{\mu(p^r)}{(p^{k-\frac{3}{2}})^r} = \begin{cases} \dfrac{a(p^r)}{(p^{k-\frac{3}{2}})^r} - \dfrac{1}{p}\dfrac{a(p^{r-2})}{(p^{k-\frac{3}{2}})^{r-2}} & \text{if } r \geq 2, \\ \dfrac{a(p^r)}{(p^{k-\frac{3}{2}})^r} & \text{if } r = 1. \end{cases} \tag{4.31}$$

Let us now consider the case where $ab^{\pm 1} \neq 1, a \neq \pm 1$ and $b \neq \pm 1$. The proof for the remaining cases follows in a similar manner with slight modifications.

Using partial fractions one can rewrite (4.30) as

$$\sum_{r=0}^{\infty}\frac{a(p^r)T^r}{(p^{k-\frac{3}{2}})^r}$$
$$= \frac{1}{(a-b)(a-b^{-1})}\left(\frac{a^4}{(a^2-1)(1-aT)} - \frac{1}{(a^2-1)(1-a^{-1}T)}\right.$$
$$\left. + \frac{a}{b^{-1}(b^{-2}-1)(1-bT)} + \frac{a}{b(b^2-1)(1-b^{-1}T)}\right). \tag{4.32}$$

Using geometric series and comparing coefficients of $T^r$ we get

$$\frac{a(p^r)}{(p^{k-\frac{3}{2}})^r} = \frac{a^{r+4} - a^{-r} - a(a^2-1)(b^{r+1}+b^{r-1}+\cdots+b^{-(r-1)}+b^{-(r+1)})}{(a-b)(a-b^{-1})(a^2-1)}. \tag{4.33}$$

Let us analyze the $r=1$ case first. We have

$$\begin{aligned}\frac{\mu(p)}{p^{k-3/2}} = \frac{a(p)}{p^{k-3/2}} &= \frac{a^5 - a^{-1} - a(a^2-1)(b^2+1+b^{-2})}{(a-b)(a-b^{-1})(a^2-1)} \\ &= \frac{(a+b)(a+b^{-1})}{a} \\ &= A_{a,b}(1),\end{aligned}$$

as required. Now let us assume that $r \geq 2$. (4.31) gives us that $\mu(p^r)/(p^{k-3/2})^r$ is equal to

$$\frac{\begin{array}{c}a^{r+4} - \frac{a^{r+2}}{p} - a^{-r} + \frac{a^{-r+2}}{p} - a(a^2-1) \\ \left[\left(1-\frac{1}{p}\right)(b^{r-1}+\cdots+b^{-(r-1)}) + b^{r+1} + b^{-(r+1)}\right]\end{array}}{(a-b)(a-b^{-1})(a^2-1)}. \tag{4.34}$$

Note that $a^{r+4} - \frac{a^{r+2}}{p} - a^{-r} + \frac{a^{-r+2}}{p} = (a^{r+2} + a^{-r})(a^2-1) + (1-1/p)a^{-r+2}(a^{2r}-1)$. Using this, the above expression can be rewritten as $A+B$, where

$$A = \frac{a^{2r+2} + 1 - (b^{r+1} + b^{-(r+1)})a^{r+1}}{a^r(a-b)(a-b^{-1})}$$

and

$$B = \left(1-\frac{1}{p}\right)\frac{(a^{2r}+\cdots+a^2) - a^{r+1}(b^{r-1}+\cdots+b^{-(r-1)})}{a^r(a-b)(a-b^{-1})}.$$

It is easy to see that $A = A_{a,b}(r)$. As for $B$, we compute

$$\begin{aligned}B &= \left(1-\frac{1}{p}\right)\frac{\sum_{t=1}^{\left[\frac{r}{2}\right]}\left(a^{2r+2-2t} - a^{r+1}(b^{r+1-2t} + b^{-(r+1-2t)}) + a^{2t}\right)}{a^r(a-b)(a-b^{-1})} \\ &= \left(1-\frac{1}{p}\right)\frac{\sum_{t=1}^{\left[\frac{r}{2}\right]}(a^{2t}(a^{r+1-2t} - b^{r+1-2t})(a^{r+1-2t} - b^{-(r+1-2t)}))}{a^r(a-b)(a-b^{-1})} \\ &= \left(1-\frac{1}{p}\right)\sum_{t=1}^{[r/2]} A_{a,b}(r-2t).\end{aligned}$$

This proves the desired formula.

Let us note that if $a = \pm 1$ or $b = \pm 1$ or $ab^{\pm 1} = 1$ then we get a different formula in (4.32) but the calculation proceeds along similar lines as above to give the result of the proposition. □

**Remark.** For the special cases $r = 1$ and $r = 2$ we get

$$\mu(p) = p^{k-3/2}(a + a^{-1} + b + b^{-1}),$$

$$\mu(p^2) = p^{2(k-3/2)} \left( a^2 + a^{-2} + (a + a^{-1})(b + b^{-1}) + b^2 + b^{-2} + 2 - \frac{1}{p} \right).$$

Observing that $a$ and $b$ can only assume the values allowed by Theorem 3.2, it is easy to derive the estimates

$$|\mu(p)| \le p^{k-1}(1 + 2p^{-1/2} + p^{-1}), \tag{4.35}$$

$$|\mu(p^2)| \le p^{2k-2}(1 + 2p^{-1/2} + 4p^{-1} + 2p^{-3/2} + 2p^{-2}). \tag{4.36}$$

In general we have the following statement.

**Theorem 4.2.** *Let $N$ and $k$ be positive integers with $k > 2$. Let $F \in S_k(\Gamma_0(N))$ be a Hecke eigenform with eigenvalues $\mu(m)$ for all $m > 0$, $(m, N) = 1$. For any prime $p \nmid N$ and integer $r \ge 1$ we have*

$$|\mu(p^r)| \le 36 p^{r(k-1)}. \tag{4.37}$$

*Furthermore, given any $\varepsilon > 0$,*

$$\mu(m) \ll_\varepsilon m^{k-1+\epsilon} \text{ for all } m > 0,\ (m, N) = 1. \tag{4.38}$$

*Proof.* Let $\pi_{F,p}$ be the local representation of $G_p = \mathrm{GSp}_4(\mathbb{Q}_p)$ corresponding to $F$ as constructed in Section 2.. From Theorem 3.2 we know that $\pi_{F,p}$ is of Type (T), (C) or (SK).

First suppose that $\pi_{F,p}$ is of Type (T). Then the numbers $a = \sigma(p)$ and $b = \chi_1(p)\sigma(p)$ in Proposition 4.1 satisfy $|a| = |b| = 1$. Hence we have the estimate $|A_{a,b}(j)| \le (j + 1)^2$. This implies that

$$\begin{aligned} |\mu(p^r)| &\le p^{(k-3/2)r} \left( (r + 1)^2 + (1 - 1/p) \sum_{t=1}^{[r/2]} (r + 1 - 2t)^2 \right) \\ &\le p^{(k-3/2)r} \left( (r + 1)^2 + \frac{r}{2}(r - 1)^2 \right). \end{aligned}$$

It is an exercise to show that $(r + 1)^2 + \frac{r}{2}(r - 1)^2 \le 18p^{r/2}$ for all primes $p$ and all $r \ge 1$. In particular, we obtain the desired estimate.

Next suppose that $\pi_{F,p}$ is of Type (C) or (SK). Hence $a = p^\beta \sigma_0$ and $b = \chi_0 \sigma_0$ where $\sigma_0^2 = 1, |\chi_0| = 1$ and $0 < \beta \le 1/2$. Type (C) corresponds to $0<\beta<1/2$ and Type (SK) corresponds $\beta=1/2$. From Proposition 4.1 we obtain the equality

$$\begin{aligned}&\mu(p^r)\\&=(\sigma_0p^{k-3/2})^r\\&\quad\times\left((p^\beta)^r\left|\sum_{u=0}^{r}(p^\beta\chi_0)^{-u}\right|^2+(1-1/p)\sum_{t=1}^{[r/2]}(p^\beta)^{r-2t}\left|\sum_{u=0}^{r-2t}(p^\beta\chi_0)^{-u}\right|^2\right).\end{aligned}$$

Note that $(p^\beta)^j\big|\sum_{u=0}^{j}(p^\beta\chi_0)^{-u}\big|^2$ is a sum of terms of the form $\chi_0^l+\overline{\chi}_0^l$ (for $0\le l\le j$) with positive coefficients (written in terms of $p^\beta$). Hence we get the maximum of $(p^\beta)^j\big|\sum_{u=0}^{j}(p^\beta\chi_0)^{-u}\big|^2$ when $\chi_0=1$. Since $A_{p^\beta,1}(j)=(p^{\beta(j/2)}+p^{\beta(j/2-1)}+\cdots+p^{\beta(1-j/2)}+p^{\beta(-j/2)})^2$ and $x^\beta+x^{-\beta}$ is, for $x>1$, an increasing function of $0<\beta\le1/2$, we conclude that the maximum of $A_{p^\beta,1}(j)$ is attained at $\beta=1/2$.

Now we will use (4.34) to estimate the eigenvalue in the case $a=p^{1/2}$ and $b=1$. We have

$$\begin{aligned}|\mu(p^r)|&\le(p^r)^{k-3/2}\left|\frac{p^{r/2+2}-p^{r/2}-p^{1/2}(p-1)\left((1-1/p)r+2\right)}{(p^{1/2}-1)^2(p-1)}\right|\\&\le(p^r)^{k-3/2}\frac{p^{r/2}(p+1)+p^{1/2}\left((1-1/p)r+2\right)}{(p^{1/2}-1)^2}.\end{aligned}$$

It is easily checked that

$$\frac{p+1}{(p^{1/2}-1)^2}\le18\quad\text{and}\quad\frac{p^{1/2}(r+2)}{(p^{1/2}-1)^2}\le18p^{r/2}$$

for all primes $p$ and all $r\ge1$. It follows that

$$|\mu(p^r)|\le36(p^r)^{k-1}.$$

This completes the proof of (4.37).

Finally, we will prove (4.38). For this, let $p_1,p_2,p_3,\ldots$ be the sequence of all prime numbers not diving $N$, written in increasing order, and define $m_l:=p_1p_2\ldots p_l$. Now fix $\epsilon>0$. We can find $l_\epsilon>0$ such that for all $l>l_\epsilon$ we have $p_l^\epsilon\ge36$. Let

$$C_\epsilon:=\text{Max}\left\{\frac{36^l}{m_l^\epsilon}\ \text{ for }\ l\le l_\epsilon\right\}. \tag{4.39}$$

With this choice of $C_\epsilon$ we have $36^l\le C_\epsilon m_l^\epsilon$ for all $l>0$, and therefore

$$|\mu(m_l)|=\prod_{i=1}^{l}|\mu(p_i)|\le36^l\prod_{i=1}^{l}p_i^{k-1}\le C_\epsilon m_l^\epsilon m_l^{k-1}=C_\epsilon m_l^{k-1+\epsilon}. \tag{4.40}$$

For a general $m > 1$ let $l(m)$ be the number of distinct prime factors of $m$. Let $m = \prod_{i=1}^{l(m)} q_i^{r_i}$ with the primes $q_i$ are written in increasing order. Clearly, $q_i \geq p_i$. Hence,

$$|\mu(m)| = \prod_{i=1}^{l(m)} |\mu(q_i^{r_i})| \leq 36^{l(m)} m^{k-1} \leq C_\epsilon m_{l(m)}^{\epsilon} m^{k-1} \leq C_\epsilon m^{k-1+\epsilon}, \tag{4.41}$$

as required. □

**Corollary 4.3.** *Let $N$ and $k$ be positive integers with $k > 2$. Let $F \in S_k(\Gamma_0(N))$ be a Hecke eigenform and suppose that for a prime $p \nmid N$ the local representation $\pi_{F,p}$ is of Type (C) or (SK), i.e., $\pi_{F,p}$ is the unique spherical constituent of a representation of the form $\nu^\beta\chi \times \nu^\beta\chi^{-1} \rtimes \nu^{-\beta}\sigma$, where $|\chi| = 1$, $0 < \beta \leq 1/2$ and $\sigma_0 := \sigma(p) = \pm 1$. Then the eigenvalue $\mu(p^r)$ is positive for every even integer $r$ and has the same sign as $\sigma_0$ for every odd integer $r$.*

*Proof.* From Proposition 4.1 we have, with $\chi_0 := \chi(p)$,

$$\begin{aligned}&\mu(p^r)\\ &= (\sigma_0 p^{k-3/2})^r\\ &\quad\left((p^\beta)^r \left|\sum_{u=0}^{r}(p^\beta\chi_0)^{-u}\right|^2 + (1-1/p)\sum_{t=1}^{[r/2]}(p^\beta)^{r-2t}\left|\sum_{u=0}^{r-2t}(p^\beta\chi_0)^{-u}\right|^2\right),\end{aligned}$$

from which the corollary follows. □

**Remarks.**

1. The constant 36 in Theorem 4.2 is not very sharp and could be improved by further analysis of the terms involved in the formula for $\mu(p^r)$.
2. The statement of the corollary is not necessarily true if $\pi_{F,p}$ is of Type (T). For example, if $p = 2$ or 3 and $a = b = i$ then $\mu(p^2) = (p^2)^{k-3/2}(1 - 4/p) < 0$.
3. In [16, p. 641], Kohnen has proved that the abscissa of convergence $s_0$ of the spinor zeta function of $F$ satisfies $s_0 \leq k$. If one assumes that the first Fourier–Jacobi coefficient of $F$ is non-zero then one can conclude that $\mu(m) \ll_\epsilon m^{k-1+\epsilon}$ for every $\epsilon > 0$. But the non-vanishing of the first Fourier–Jacobi coefficient is not yet known for a general Siegel modular form.
4. Note that (4.38) cannot be obtained from (4.24) alone. This is because the Hecke algebra at $p$ has three generators : $T(p)$, $T(p^2)$ and $\Delta(p)$. Hence, for any $r \geq 1$, the Hecke operator $T(p^r)$ can be written as a polynomial of

the three generators with coefficients depending on $r$. The main problem is that it is difficult to control the size of these coefficients and hence get good estimates for $\mu(p^r)$.

### 4.2 Domain of convergence of the standard $L$-function

Let $\pi \cong \otimes_p' \pi_p$ be any irreducible cuspidal automorphic representation of $\mathrm{GSp}_{2n}(\mathbb{A})$ and $S$ be a finite set of primes such that if $p \notin S$ then $\pi_p$ is unramified. Then $\pi_p$ is the unique spherical constituent of the representation $\chi_{p,1} \times \cdots \times \chi_{p,n} \rtimes \sigma_p$, where $\chi_{p,1}, \ldots, \chi_{p,n}$ and $\sigma_p$ are unramified characters of $\mathbb{Q}_p^\times$. Here the notation is an obvious generalization of the notation given in Section 2. for the $\mathrm{GSp}_4$ case. Let $b_{p,1} := \chi_{p,1}(p), \ldots, b_{p,n} := \chi_{p,n}(p)$ be the Satake parameters of $\pi_p$. Then the partial Langlands $L$-function attached to the $(2n+1)$-dimensional representation of the $L$-group ${}^LG$ is given by

$$L_{\mathrm{St}}^S(s,\pi) = \prod_{p \notin S} (1-p^{-s})^{-1} \left( \prod_{i=1}^{n} (1 - b_{p,i} p^{-s})^{-1} (1 - b_{p,i}^{-1} p^{-s})^{-1} \right). \tag{4.42}$$

From [4], one knows that $L_{\mathrm{St}}^S(s,\pi)$, which is convergent in some right half plane, has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation with respect to $s \mapsto 1-s$. The following is known about the domain of convergence of $L_{\mathrm{St}}^S(s,\pi)$ for an irreducible, cuspidal, automorphic representation $\pi$ of $\mathrm{GSp}_{2n}(\mathbb{A})$.

1. [4] If $\pi$ is obtained from a holomorphic Siegel cusp form of degree $n$, then $L_{\mathrm{St}}^S(s,\pi)$ converges for $\mathrm{Re}(s) > n+1$.
2. [7, p. 236] For any irreducible, cuspidal, automorphic representation $\pi$ (not necessarily coming from a holomorphic modular form) $L_{\mathrm{St}}^S(s,\pi)$ converges for
$$\mathrm{Re}(s) > \begin{cases} \frac{2}{3}n + 1, & \text{for all } n; \\ \frac{n}{2} + 1, & \text{for } n = 2^r, r \geq 1. \end{cases} \tag{4.43}$$
In [7], the authors also constructed an irreducible cuspidal automorphic representation using a result of Rallis [25] for which the partial $L$-function does not converge at $n/2+1$, showing that this is the best possible bound in general.
3. [32, Theorem A] If $\pi$ is obtained from a holomorphic Siegel cusp form of degree $n$, then $L_{\mathrm{St}}^S(s,\pi)$ converges for $\mathrm{Re}(s) > n/2 + 1$.

In the next theorem, we will show that if we consider only those representations $\pi$ of $\mathrm{GSp}_4(\mathbb{A})$ that are obtained from Siegel cusp forms then (4.43) can be improved significantly.

**Theorem 4.4.** *Let $N$ and $k$ be positive integers with $k > 2$. Let $F \in S_k(\Gamma_0(N))$ be a Hecke eigenform. Let $\pi$ be any of the irreducible, cuspidal, automorphic representations of $\mathrm{GSp}_4(\mathbb{A})$ obtained from $F$; see Section 2.. Let $S$ be the set $\{\infty\} \cup \{p : \ p|N\}$. Then the degree-$5$ standard $L$-function $L^S_{\mathrm{St}}(s,\pi)$ converges absolutely and uniformly for*

$$\mathrm{Re}(s) > \frac{3}{2}. \tag{4.44}$$

*Proof.* From the definition of the standard $L$-function we have that $L^S_{\mathrm{St}}(s,\pi_F)$ is equal to

$$\prod_p (1 - b_{p,1}p^{-s})^{-1} \prod_p (1 - b_{p,2}p^{-s})^{-1}$$
$$\prod_p (1 - b_{p,1}^{-1}p^{-s})^{-1} \prod_p (1 - b_{p,2}^{-1}p^{-s})^{-1}, \tag{4.45}$$

up to a zeta factor which is convergent for $\mathrm{Re}(s) > 1$. From Theorem 3.2, we know that $p^{-1/2} \leq |b_{p,1}| \leq p^{1/2}$ and $p^{-1/2} \leq |b_{p,2}| \leq p^{1/2}$ for all $p \nmid N$. Hence, each of the four products is absolutely convergent for $\mathrm{Re}(s) > 3/2$. $\square$

**Corollary 4.5.** *Let $N$, $k$, $F$ and $\pi$ be as in Theorem 4.4. Then $\pi$ is not a CAP representation with respect to $B$, the minimal parabolic subgroup, or $Q$, the Klingen parabolic subgroup.*

*Proof.* By Theorem 4.4, the standard $L$-function $L^S_{\mathrm{St}}(s,\pi)$ has no pole at $s = 2$. The argument in the proof of Theorem 4.4 shows that the same is true for the twisted $L$-functions $L^S_{\mathrm{St}}(s,\pi,\chi)$, where $\chi$ is an arbitrary quadratic Hecke character. By [33, Theorem A], this implies that $\pi$ is not CAP with respect to $B$ or $Q$. $\square$

**Remark.** We indicate an alternative proof of Corollary 4.5. By [33, Theorem A], all CAP representations with respect to $B$ or $Q$ are obtained via theta liftings (with similitudes) from an anisotropic $\mathrm{GO}_4$ to $\mathrm{GSp}_4$. Computing local theta liftings at the archimedean place shows that the only weights accessible using these theta liftings are $k = 1$ and $k = 2$. Hence, the condition $k > 2$ implies that $\pi$ cannot be CAP with respect to $B$ or $Q$.

One can show, however, that certain CAP representations with respect to $B$ contain holomorphic modular forms of weight $k = 2$. Hence, the condition $k > 2$ in Corollary 4.5 is essential. Recently, IBUKIYAMA and SKORUPPA have shown that the spaces $S_1(\Gamma_0(N))$ are zero for any $N \geq 1$; see [24].

## 4.3 Maaß space

In this section let $\Gamma = \mathrm{Sp}(4,\mathbb{Z})$. We know (see [8, Sect. 2.6]) that $F \in S_k(\Gamma)$, $k$ even, is a classical Saito–Kurokawa lift from a weight $2k-2$ elliptic cusp form with respect to $\mathrm{SL}(2,\mathbb{Z})$ if and only if $F$ lies in the Maaß space. The Maaß space is defined as follows. Let

$$F\begin{bmatrix}\tau & z\\ z & \tau'\end{bmatrix} = \sum_{n,r,m} A(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}, \qquad \begin{bmatrix}\tau & z\\ z & \tau'\end{bmatrix} \in \mathbb{H}_2,$$

be the Fourier expansion of $F$. Then $F$ lies in the Maaß space if and only if

$$A(n,r,m) = \sum_{d|(n,r,m)} d^{k-1}A\left(\frac{nm}{d^2},\frac{r}{d},1\right) \qquad \text{for all } \ n,m,r \in \mathbb{Z}. \tag{4.46}$$

We will say that $F$ lies in the Maaß $p$-space for some prime number $p$ if the Fourier coefficients satisfy

$$\begin{aligned} &A(np,r,m) + p^{k-1}A\left(\frac{n}{p},\frac{r}{p},m\right)\\ &\quad = p^{k-1}A\left(n,\frac{r}{p},\frac{m}{p}\right) + A(n,r,mp) \text{ for } n,r,m \in \mathbb{Z}. \end{aligned} \tag{4.47}$$

Here, we understand that $A(\alpha,\beta,\gamma) = 0$ if one of $\alpha,\beta,\gamma$ is not an integer. If $F$ lies in the Maaß space, then it lies in the Maaß $p$-space for every prime $p$; this follows by substituting (4.46) into (4.47). Note that, from the above equations, it is not clear that the converse is true.

**Theorem 4.6.** *Let $F \in S_k(\Gamma)$. If $F$ lies in the Maaß $p$-space for almost all primes $p$ then $F$ lies in the Maaß space, i.e., $F$ is a Saito–Kurokawa lift.*

The proof requires some facts from the theory of paramodular representations, which we recall first. For a positive integer $N$, let $\Gamma^{\mathrm{para}}(N)$ be the paramodular group of level $N$, as defined in [26, Sect. 1]. Let $p$ be a prime number. There exist linear operators $\theta_p$ and $\theta'_p$ from $S_k(\Gamma)$ to $S_k(\Gamma^{\mathrm{para}}(p))$, given by the summation formulas (10) and (16) of [26]. The condition (4.47) is equivalent to $\theta_pF = \theta'_pF$; see [26, Sect. 5].

*Proof of Theorem* 4.6. We can write $F$ as a sum of cusp forms $F_i$, each of which (or rather the associated adelic function $\Phi_i$) lies in an irreducible space of cuspidal, automorphic forms $V_i$ on $\mathrm{GSp}_4(\mathbb{A})$. By definition, the (adelic analogues of the) operators $\theta_p$ and $\theta'_p$ preserve the space $V_i$. Hence, the hypothesis $\theta_pF = \theta'_pF$ implies $\theta_pF_i = \theta'_pF_i$ for each $i$. We may therefore assume that $F$ lies in an irreducible space of cuspidal, automorphic forms.

Let $S$ be a finite set of places, including the archimedean place, such that $F$ satisfies the $p$-Maaß conditions (4.47) for all primes $p \notin S$. By [9, Theorem 1] or [19, p. 324, 338], it is enough to show that the incomplete degree-4 Andrianov $L$-function $L^S(s,F)$ has a pole at $s=k$.

Let $\pi = \otimes \pi_p$ be the decomposition of the irreducible representation generated by $F$ into local components. Let $p \notin S$. Since $\theta_p F = \theta'_p F$, the spherical vector $f$ in the local representation $\pi_p$ satisfies $\theta f = \theta' f$; here, $\theta$ and $\theta'$ are the local level raising operators on paramodular vectors in smooth representations of $\mathrm{GSp}_4(\mathbb{Q}_p)$ defined in section 3.2 of [27]. By Proposition 5.5.13 of [27], $\pi_p$ is one of the representations listed under (2), (4), (5) or (6) of our Proposition 3.1. By Theorem 3.2, types (4), (5) and (6) are not allowed, and hence $\pi_p$ is of type (SK). In the terminology of [27], this is the IIb case. More precisely, by Proposition 5.5.13 of [27], we obtain $\pi_p \cong \chi_p \mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi_p^{-1}$ with a unitary character $\chi_p$ of $\mathbb{Q}_p^\times$. The degree-4 $L$-factor of this representation is

$$((1-\chi_p(p)p^{-s})(1-\chi_p(p)^{-1}p^{-s})(1-p^{1/2-s})(1-p^{-1/2-s}))^{-1},$$

and consequently, the global (incomplete) $L$-function of $\pi$ is

$$\zeta^S(s-1/2)\zeta^S(s+1/2)\prod_{p\notin S}((1-\chi_p(p)p^{-s})(1-\chi_p(p)^{-1}p^{-s}))^{-1}.$$

Since $\zeta(s-1/2)$ has a pole at $s=3/2$ and the other products converge absolutely at $s=3/2$, it follows that the global $L$-function has a pole at $s=3/2$. Note, however, that this $L$-function is normalized so that it has a functional equation with respect to $s \mapsto 1-s$. To obtain the Andrianov $L$-function $L(s,F)$ defined in Sect. 2., we have to shift the argument by $k-3/2$. Hence the Andrianov $L$-function has indeed a pole at $s=k$. $\square$

Note that, the characterization of the Maaß space in terms of Maaß $p$-conditions for all primes is already done in [12] using different methods than above. Recently, in [13], it has been shown that a Siegel modular form $F$ of weight $k$ is in the Maaß space if and only if $F$ lies in the Maaß $p$-space for a finite number of primes depending on the weight $k$.

### 4.4 Further applications

In this section let $\Gamma = \mathrm{Sp}(4,\mathbb{Z})$.

#### Dimension formulas

Let $p$ be a prime and $\Gamma^{\mathrm{para}}(p)$ the paramodular group of level $p$, as in the previous section. In [14], IBUKIYAMA has defined a space of *newforms*

$S_k^{\rm new}(\Gamma^{\rm para}(p))\subset S_k(\Gamma^{\rm para}(p))$. The definition coincides with the one given in [26], namely, $S_k^{\rm new}(\Gamma^{\rm para}(p))$ is the orthogonal complement of the space spanned by the images of the two level raising operators

$$\theta_p,\ \theta_p':\ S_k(\Gamma)\longrightarrow S_k(\Gamma^{\rm para}(p))$$

considered in the proof of Theorem 4.6 above. The following conjectural dimension formulas were given in [14]. If $k>2$ is even, then

$$\dim S_k^{\rm new}(\Gamma^{\rm para}(p)) \\ =\dim S_k(\Gamma^{\rm para}(p))-2\dim S_k(\Gamma)+\dim S_{2k-2}({\rm SL}(2,\mathbb{Z})), \quad (4.48)$$

and if $k$ is odd, then

$$\dim S_k^{\rm new}(\Gamma^{\rm para}(p))=\dim S_k(\Gamma^{\rm para}(p))-2\dim S_k(\Gamma). \quad (4.49)$$

We shall explain why these formulas are very plausible in view of our Theorem 3.2. The following table is a small excerpt from Table A.13 in [27]. It shows, for each of the spherical, irreducible, admissible representations $(\pi,V)$ of ${\rm GSp}_4(\mathbb{Q}_p)$ the dimension of the space of fixed vectors under ${\rm GSp}_4(\mathbb{Z}_p)$ (this dimension is always 1, of course) and under $K(p)$, the local paramodular group of level $p$.

| type | $\dim V^{{\rm GSp}_4(\mathbb{Z}_p)}$ | $\dim V^{K(p)}$ |
|---|---|---|
| I | 1 | 2 |
| IIb | 1 | 1 |
| IIIb | 1 | 2 |
| IVd | 1 | 1 |
| Vd | 1 | 0 |
| VId | 1 | 1 |

The level raising operators $\theta_p$ and $\theta_p'$ have compatible local versions, and computations show that in each case $V^{K(p)}$ is spanned by the image of $V^{{\rm GSp}_4(\mathbb{Z}_p)}$ under both $\theta_p$ and $\theta_p'$ (see [27], Theorem 5.6.1 iv)). By Theorem 3.2, only representations of type I and IIb can occur in cuspidal, automorphic representations coming from $S_k(\Gamma)$, $k>2$. Consider first the case that $k$ is odd. Then $S_k(\Gamma)$ does not contain any Saito–Kurokawa liftings, and consequently, the local Saito–Kurokawa liftings (type IIb) should also not occur. Since only type I is relevant, it follows that for each local spherical vector we have two linearly independent local oldforms at level $p$. Thus, the dimension of $S_k^{\rm old}(\Gamma^{\rm para}(p))$ is twice that of $S_k(\Gamma)$. This is the statement of equation (4.49).

Now assume that $k$ is even. Then $S_k(\Gamma)$ contains the Maaß space, which is an isomorphic image of $S_{2k-2}(\mathrm{SL}(2,\mathbb{Z}))$. Representations corresponding to cusp forms in the Maaß space have local components of type IIb everywhere. If we subtract $2\dim S_k(\Gamma)$ from $\dim S_k(\Gamma^{\mathrm{para}}(p))$, we subtract too much for each IIb case. This explains the correctional term $\dim S_{2k-2}(\mathrm{SL}(2,\mathbb{Z}))$ in (4.48).

These arguments do not quite prove the formulas (4.48) and (4.49), since we still have to assume a mild version of the Ramanujan conjecture saying that representations of type IIb can only come from Saito–Kurokawa liftings. However, representations of type IVd, Vd or VId, which would invalidate the dimension formulas, cannot occur by Theorem 3.2.

### Hypercuspidal modular forms

Finally, we explain the role Theorem 3.2 plays in a result on hypercuspidal modular forms obtained in [28]. For $F \in S_k(\Gamma_0(N))$ let

$$F(\tau, z, \tau') = \sum_{m=1}^{\infty} f_m(\tau, z) e^{2\pi i m \tau'}$$

be the usual Fourier-Jacobi expansion of $F$. Here, $f_m$ is a Jacobi form of index $m$. Given a prime number $p$, we say that $F$ is *$p$-hypercuspidal of degree* 1, if $f_m = 0$ whenever $p|m$. As explained in [28], such modular forms are not easy to construct. However, we have the following result.

**Theorem 4.7.** *Let $F \in S_k(\Gamma)$, $k > 2$, be an eigenform. For any prime number $p$, there exists a cusp form $F' \in S_k(\Gamma_0(p^2))$ that is $p$-hypercuspidal of degree 1 and has the same Hecke eigenvalues $\lambda(m)$ as $F$ for each $m$ not divisible by $p$.*

The proof consists in exchanging the spherical vector in the $p$-component of a cuspidal, automorphic representation generated by $F$ for a local hypercuspidal vector; see [28] for details. One has to make sure, however, that such local hypercuspidal vectors exist. Local computations show that this is not the case for representations of type IIIb, IVd and VId. But, by our Theorem 3.2, these types of representations do not occur.

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