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Bessel models for $\mathrm{GSp}(4)$: Siegel vectors of square-free level

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ABSTRACT

We determine test vectors and explicit formulas for all Bessel models for those Iwahori-spherical representations of GSp_4 over a p -adic field that have non-zero vectors fixed under the Siegel congruence subgroup.

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1. Introduction

For various classical groups, Bessel models of local or global representations have proven to be a useful substitute for the frequently missing Whittaker model. The uniqueness of Bessel models in the local non-archimedean case has now been established in a wide variety of cases; see [1,3]. In this work we are only concerned with the group GSp_4 over a p -adic field F . For this group uniqueness of Bessel models was proven as early as 1973, at least for representations with trivial central character; see [4,5]. A proof of uniqueness for all non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}_4(F)$ can be found in [10].

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In contrast to Whittaker models, which are essentially independent of any choices made, Bessel models depend on some arithmetic data. In the case of GSp_4 , part of this data is a choice of non-degenerate symmetric 2×2 -matrix S over the field F . The discriminant \mathbf{d} of this matrix determines a quadratic extension L of F ; this extension may be isomorphic to $F \oplus F$, which will be referred to as the *split case*. The second ingredient entering into the definition of a Bessel model is a character Λ of the multiplicative group L^\times .

Now let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}_4(F)$. Given S and Λ , the representation π may or may not have a Bessel model with respect to this data. In the case of GSp_4 , it is possible to precisely say which representations have which Bessel models; see [8,10]. In particular, every irreducible, admissible representation has a Bessel model for an appropriate choice of S and Λ .

Given π , S and Λ , it is one thing to know that a Bessel model exists, but it is another to identify a good *test vector*. By definition, a test vector is a vector in the space of π on which the relevant Bessel functional is non-zero; note that this is a well-defined notion by the uniqueness of Bessel functionals. Equivalently, in the actual Bessel model consisting of functions B on the group with the Bessel transformation property, B is a test vector if and only if $B(1)$ is non-zero. In this paper, we will identify test vectors for a class of representations that is relevant for the theory of Siegel modular forms of degree 2. In addition, we shall give explicit formulas for the corresponding Bessel functions. See [6,11] for the Steinberg case and [14] for the spherical case.

More precisely, the classes of representations we consider are those that have a non-zero vector invariant under P_1 , the Siegel (Γ_0 -type) congruence subgroup of level \mathfrak{p} . These representations appear as local components of global automorphic representations generated by Siegel modular eigenforms of degree 2 with respect to the congruence subgroup $\Gamma_0(N)$, where N is a square-free positive integer. They fall naturally into thirteen classes, only four of which consist of generic representations (meaning representations that admit a Whittaker model); see our Table 1 below.

Of the thirteen classes, six are actually spherical, meaning they have a non-zero vector invariant under the maximal compact subgroup $\mathrm{GSp}_4(\mathfrak{o})$ (here, \mathfrak{o} is the ring of integers of our local field F). Sugano [14] has given test vectors and explicit formulas in the spherical cases. Our main focus, therefore, is on the seven classes consisting of non-spherical representations with non-zero P_1 -fixed vectors. Of those, five classes have a one-dimensional space of P_1 -fixed vectors, and two classes have a two-dimensional space of P_1 -fixed vectors. The one-dimensional cases require a slightly different treatment from the two-dimensional cases. In all cases, our main tool will be two Hecke operators $T_{1,0}$ and $T_{0,1}$, coming from the double cosets

$$P_1 \operatorname{diag}(\varpi, \varpi, 1, 1)P_1 \quad \text{and} \quad P_1 \operatorname{diag}(\varpi^2, \varpi, 1, \varpi)P_1.$$

Here, ϖ is a generator of the maximal ideal of \mathfrak{o} . Evidently, these operators act on the spaces of P_1 -invariant vectors. In Section 3 we give their eigenvalues for all of our seven

classes of representations. The results are contained in Table 3. In the one-dimensional cases, trivially, the unique P_1 -invariant vector is a common eigenvector for both $T_{1,0}$ and $T_{0,1}$. In the two-dimensional cases, it turns out there is a nice basis consisting of common eigenvectors for $T_{1,0}$ and $T_{0,1}$.

In Section 5 we will apply the two Hecke operators to P_1 -invariant Bessel functions B and evaluate at certain elements of $\mathrm{GSp}_4(F)$. Assuming that B is an eigenfunction, this leads to several formulas relating the values of B at various elements of the group; see Lemma 5.1 as an example for this kind of result. The calculations are all based on a GL_2 integration formula, which we establish in Lemma 4.3.

The P_1 -invariant element B in π seems, naively, the natural test vector candidate. But, if Λ is ramified, then considering the intersection of P_1 with the Bessel subgroup, it is easy to see that B is clearly not a test vector. Hence, we are led to consider $\pi(h(0, m))B(1) = B(h(0, m))$, where

$$h(0, m) = \mathrm{diag}(\varpi^{2m}, \varpi^m, 1, \varpi^m).$$

In Section 6 we use some of the formulas from the Hecke operator computations to establish a generating series for the values of B at these diagonal elements; see Proposition 6.1. It turns out that there is one “initial element” $h(0, m_0)$, where m_0 is the conductor of the Bessel character Λ ; see (31) for the precise definition. If $B(h(0, m_0)) = 0$, then B is zero on *all* diagonal elements.

A generating series like in Proposition 6.1 is precisely the kind of result that will be useful in global applications. However, it still has to be established that $B(h(0, m_0)) \neq 0$; this is the test vector problem, and it is not trivial since B may vanish on all diagonal elements and yet be non-zero. It turns out that, in almost all cases, $B(h(0, m_0)) \neq 0$ as expected; see our main results Theorem 8.2 and Theorem 9.3. Note that this implies that $\pi(h(0, m_0))B$ is a test vector.

However, there is *one* exceptional case, occurring only for split Bessel models of representations of type IIa, and then only for a certain unramified Bessel character Λ . In this very special case, our Bessel function B is non-zero not at the identity, as expected, but at a certain other element; see Theorem 8.2(i) for the precise statement.

To handle this exceptional case, and one other split case for VIa type representations, we require an additional tool besides our Hecke operators. This additional tool are zeta integrals, which are closely related to split Bessel models. We also require part of the theory of paramodular vectors from [9]. Roughly speaking, we take paramodular vectors and “Siegelize” them to obtain P_1 -invariant vectors. Calculations with zeta integrals then establish the desired non-vanishing at specific elements. This is the topic of Section 7.

The final Sections 8 and 9 contain our main results. Theorem 8.2 exhibits test vectors in all one-dimensional cases, and Theorem 9.3 exhibits test vectors in all two-dimensional cases. As mentioned above, explicit formulas for these vectors can be found in Proposition 6.1. We mention that, evidently, explicit formulas imply uniqueness of Bessel models.

Hence, as a by-product of our calculations, we reprove the uniqueness of these models for the representations under consideration.

Let us remark here that the proofs of several of the lemmas in this paper are very long but not conceptually very deep. Hence, we have omitted the proofs of these results. We direct the reader to [7] for a longer version of this article which contains all the details of the proofs.

2. Basic facts and definitions

Let F be a non-archimedean local field of characteristic zero, \mathfrak{o} its ring of integers, \mathfrak{p} the maximal ideal of \mathfrak{o} , and ϖ a generator of \mathfrak{p} . We fix a non-trivial character ψ of F such that ψ is trivial on \mathfrak{o} but non-trivial on \mathfrak{p}^{-1} . We let v be the normalized valuation map on F .

As in [2] we fix three elements $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in F such that $\mathbf{d} = \mathbf{b}^2 - 4\mathbf{a}\mathbf{c} \neq 0$. Let

$$S = \begin{bmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} \end{bmatrix}, \quad \xi = \begin{bmatrix} \frac{\mathbf{b}}{2} & \mathbf{c} \\ -\mathbf{a} & \frac{-\mathbf{b}}{2} \end{bmatrix}. \tag{1}$$

Then $F(\xi) = F + F\xi$ is a two-dimensional F -algebra. If \mathbf{d} is not a square in F^\times , then $F(\xi)$ is isomorphic to the field $L = F(\sqrt{\mathbf{d}})$ via the map $x + y\xi \mapsto x + y\frac{\sqrt{\mathbf{d}}}{2}$. If \mathbf{d} is a square in F^\times , then $F(\xi)$ is isomorphic to $L = F \oplus F$ via $x + y\xi \mapsto (x + y\frac{\sqrt{\mathbf{d}}}{2}, x - y\frac{\sqrt{\mathbf{d}}}{2})$. Let $z \mapsto \bar{z}$ be the obvious involution on L whose fixed point set is F . The determinant map on $F(\xi)$ corresponds to the norm map on L , defined by $N(z) = z\bar{z}$. Let

$$T(F) = \{g \in \mathrm{GL}_2(F): {}^t g S g = \det(g) S\}. \tag{2}$$

One can check that $T(F) = F(\xi)^\times$, so that $T(F) \cong L^\times$ via the isomorphism $F(\xi) \cong L$. We define the Legendre symbol as

$$\left(\frac{L}{\mathfrak{p}}\right) = \begin{cases} -1 & \text{if } L/F \text{ is an unramified field extension,} \\ 0 & \text{if } L/F \text{ is a ramified field extension,} \\ 1 & \text{if } L = F \oplus F. \end{cases} \tag{3}$$

These three cases are referred to as the *inert case*, *ramified case*, and *split case*, respectively. If L is a field, then let \mathfrak{o}_L be its ring of integers and \mathfrak{p}_L be the maximal ideal of \mathfrak{o}_L . If $L = F \oplus F$, then let $\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o}$. Let ϖ_L be a uniformizer in \mathfrak{o}_L if L is a field, and set $\varpi_L = (\varpi, 1)$ if L is not a field. In the field case let v_L be the normalized valuation on L . Except in Section 7, where we consider certain split cases, we will make the following *standard assumptions*,

- $\mathbf{a}, \mathbf{b} \in \mathfrak{o}$ and $\mathbf{c} \in \mathfrak{o}^\times$.
 - If $\mathbf{d} \notin F^{\times 2}$, then \mathbf{d} is a generator of the discriminant of L/F .
 - If $\mathbf{d} \in F^{\times 2}$, then $\mathbf{d} \in \mathfrak{o}^\times$.
- (4)

Under these assumptions, one can check that the group $T(\mathfrak{o}) := T(F) \cap \mathrm{GL}_2(\mathfrak{o})$ is isomorphic to \mathfrak{o}_L^\times via the isomorphism $T(F) \cong L^\times$. One can prove (see our remarks further below) that a given matrix S can always be transformed into one satisfying the standard assumptions.

Under our assumptions (4) it makes sense to consider the quadratic equation $\mathbf{c}u^2 + \mathbf{b}u + \mathbf{a} = 0$ over the residue class field $\mathfrak{o}/\mathfrak{p}$. The number of solutions of this equation is $(\frac{L}{\mathfrak{p}}) + 1$. In the ramified case we will fix an element $u_0 \in \mathfrak{o}$ and in the split case we will fix two mod \mathfrak{p} inequivalent elements $u_1, u_2 \in \mathfrak{o}$ such that

$$\mathbf{c}u_i^2 + \mathbf{b}u_i + \mathbf{a} \in \mathfrak{p}, \quad i = 0, 1, 2. \tag{5}$$

2.1. Groups

We define the group GSp_4 , considered as an algebraic F -group, using the symplectic form $J = \begin{bmatrix} & & & 1 \\ & & & \\ & & -1 & \\ & & & \end{bmatrix}$. Hence, $\mathrm{GSp}_4(F) = \{g \in \mathrm{GL}_4(F) : {}^t g J g = \mu(g) J\}$, where the scalar $\mu(g) \in F^\times$ is called the *multiplier* of g . Let Z be the center of GSp_4 . Let B denote the standard Borel subgroup of GSp_4 , consisting of matrices that become upper triangular when switching the first two rows and first two columns. Let P be the Siegel parabolic subgroup (abelian radical) containing B , and Q be the Klingen parabolic subgroup (non-abelian radical) containing B . Let $K = \mathrm{GSp}_4(\mathfrak{o})$ be the standard maximal compact subgroup of $\mathrm{GSp}_4(F)$. The parahoric subgroups corresponding to B , P and Q are the *Iwahori subgroup* I , the *Siegel congruence subgroup* P_1 , and the *Klingen congruence subgroup* P_2 , given by

$$I = K \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix},$$

$$P_1 = K \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{bmatrix}, \quad P_2 = K \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \tag{6}$$

We will also have occasion to consider, for a non-negative integer n , the *paramodular group* of level \mathfrak{p}^n , defined as

$$K^{\mathrm{para}}(\mathfrak{p}^n) = \left\{ g \in \mathrm{GSp}_4(F) : g \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{o} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}, \det(g) \in \mathfrak{o}^\times \right\}. \tag{7}$$

We will abbreviate $K^{\mathrm{para}}(\mathfrak{p})$ by P_{02} , which is a maximal compact subgroup of $\mathrm{GSp}_4(F)$ not conjugate to K .

The eight-element Weyl group W of GSp_4 , defined in the usual way as the normalizer modulo the centralizer of the subgroup of diagonal matrices, is generated by the images of

$$s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}. \tag{8}$$

2.2. Bessel models

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the matrix S and the torus $T(F)$ be as above. We consider $T(F)$ a subgroup of $\mathrm{GSp}_4(F)$ via

$$T(F) \ni g \mapsto \begin{bmatrix} g & \\ & \det(g)^t g^{-1} \end{bmatrix} \in \mathrm{GSp}_4(F). \tag{9}$$

Let $U(F)$ be the unipotent radical of the Siegel parabolic subgroup P and $R(F) = T(F)U(F)$. We call $R(F)$ the *Bessel subgroup* of $\mathrm{GSp}_4(F)$ (with respect to the given data $\mathbf{a}, \mathbf{b}, \mathbf{c}$). Let $\theta : U(F) \rightarrow \mathbb{C}^\times$ be the character given by

$$\theta \left(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \right) = \psi(\mathrm{tr}(SX)), \tag{10}$$

where ψ is our fixed character of F of conductor \mathfrak{o} . We have $\theta(t^{-1}ut) = \theta(u)$ for all $u \in U(F)$ and $t \in T(F)$. Hence, if Λ is any character of $T(F)$, then the map $tu \mapsto \Lambda(t)\theta(u)$ defines a character of $R(F)$. We denote this character by $\Lambda \otimes \theta$. Let $\mathcal{S}(\Lambda, \theta)$ be the space of all locally constant functions $B : \mathrm{GSp}_4(F) \rightarrow \mathbb{C}$ with the *Bessel transformation property*

$$B(rg) = (\Lambda \otimes \theta)(r)B(g) \quad \text{for all } r \in R(F) \text{ and } g \in \mathrm{GSp}_4(F). \tag{11}$$

Our main object of investigation is the subspace $\mathcal{S}(\Lambda, \theta, P_1)$ consisting of functions that are right invariant under P_1 . The group $\mathrm{GSp}_4(F)$ acts on $\mathcal{S}(\Lambda, \theta)$ by right translation. If an irreducible, admissible representation (π, V) of $\mathrm{GSp}_4(F)$ is isomorphic to a subrepresentation of $\mathcal{S}(\Lambda, \theta)$, then this realization of π is called a (Λ, θ) -*Bessel model*. It is known by [5] for trivial central character, and by [10] for all non-supercuspidal representations (or all representations if the Bessel model is split), that such a model, if it exists, is unique; we denote it by $\mathcal{B}_{\Lambda, \theta}(\pi)$. Since the Bessel subgroup contains the center, an obvious necessary condition for existence is $\Lambda(z) = \omega_\pi(z)$ for all $z \in F^\times$, where ω_π is the central character of π .

2.3. Change of models

Of course, Bessel models can be defined with respect to any non-degenerate symmetric matrix S , not necessarily subject to the conditions (4) we imposed on \mathbf{a} , \mathbf{b} , \mathbf{c} . Since our calculations and explicit formulas will assume these conditions, we shall briefly describe how to switch to more general Bessel models. Hence, let λ be in F^\times and A be in $\mathrm{GL}_2(F)$, and let $S' = \lambda {}^tASA$. Replacing S by S' in the definitions (2) and (10), we obtain the group $T'(F)$ and the character θ' of $U(F)$. There is an isomorphism $T'(F) \rightarrow T(F)$ given by $t \mapsto AtA^{-1}$. Let λ' be the character of $T'(F)$ given by $\lambda'(t) = \lambda(AtA^{-1})$. For $B \in \mathcal{B}_{\lambda, \theta}(\pi)$, let $B'(g) = B\left(\begin{bmatrix} A & \\ & \lambda^{-1}tA^{-1} \end{bmatrix}g\right)$, for $g \in \mathrm{GSp}_4(F)$. It is easily verified that B' has the (λ', θ') -Bessel transformation property, and that the map $B \mapsto B'$ provides an isomorphism $\mathcal{B}_{\lambda, \theta}(\pi) \cong \mathcal{B}_{\lambda', \theta'}(\pi)$.

Let $S = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{bmatrix}$ be the usual matrix such that $\mathbf{d} = \mathbf{b}^2 - 4\mathbf{a}\mathbf{c}$ is a square in F^\times , and let

$$S' = \begin{bmatrix} & 1/2 \\ 1/2 & \end{bmatrix}. \tag{12}$$

Then we can take $\lambda = 1$ and $A = \frac{1}{\sqrt{\mathbf{d}}} \begin{bmatrix} 1 & -2\mathbf{c} \\ -\frac{1}{2\mathbf{c}}(\mathbf{b} - \sqrt{\mathbf{d}}) & \mathbf{b} + \sqrt{\mathbf{d}} \end{bmatrix}$ above to get $S' = {}^tASA$. The torus $T'(F)$ is explicitly given by

$$T'(F) = \left\{ \begin{bmatrix} a & \\ & d \end{bmatrix} : a, d \in F^\times \right\}. \tag{13}$$

If $B \in \mathcal{S}(A, \theta, P_1)$ and B is a $T_{1,0}$ eigenfunction (see (16)) with non-zero eigenvalue then an explicit computation shows that

$$B'(1) \neq 0 \iff B(1) \neq 0. \tag{14}$$

See [7] for details.

One can prove (see [7]) that for arbitrary $S = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{bmatrix}$ there exist $\lambda \in F^\times$ and $A \in \mathrm{GL}_2(F)$ such that for the transformed matrix $S' = \lambda {}^tASA$ the standard assumptions (4) are satisfied. Hence, there is no restriction of generality in making these assumptions.

2.4. The Iwahori-spherical representations of $\mathrm{GSp}_4(F)$ and their Bessel models

Table 1 below is a reproduction of Table A.15 of [9]. It lists all the irreducible, admissible representations of $\mathrm{GSp}_4(F)$ that have a non-zero fixed vector under the Iwahori subgroup I , using notation borrowed from [12]. In Table 1, all characters are assumed to be unramified. Further, Table 1 lists the dimensions of the spaces of fixed vectors under the various parahoric subgroups; every parahoric subgroup is conjugate to exactly one of K , P_{02} , P_2 , P_1 or I . In this paper we are interested in the representations that have a

Table 1

The Iwahori-spherical representations of $\mathrm{GSp}_4(F)$ and the dimensions of their spaces of fixed vectors under the parahoric subgroups. Also listed are the conductor $a(\pi)$ and the value of the ε -factor at $1/2$. The symbol ν stands for the absolute value on F^\times , normalized such that $\nu(\varpi) = q^{-1}$, and ξ stands for the non-trivial, unramified, quadratic character of F^\times .

	π	$a(\pi)$	$\varepsilon(1/2, \pi)$	K	P_{02}	P_2	P_1	I
I	$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	0	1	1	2 +-	4	4 ++ --	8 ++++ ----
II	a $\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	1	$-(\sigma\chi)(\varpi)$	0	1	2	1	4 +---
	b $\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	0	1	1	1 +	2	3 ++-	4 ++++
III	a $\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	2	1	0	0	1	2	4 +---
	b $\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$	0	1	1	2 +-	3	2 +-	4 ++++
IV	a $\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	3	$-\sigma(\varpi)$	0	0	0	0	1 -
	b $L(\nu^2, \nu^{-1} \sigma \mathrm{St}_{\mathrm{GSp}(2)})$	2	1	0	0	1	2	3 +--
	c $L(\nu^{3/2} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2} \sigma)$	1	$-\sigma(\varpi)$	0	1	2	1	3 +--
	d $\sigma \mathbf{1}_{\mathrm{GSp}(4)}$	0	1	1	1 +	1	1 +	1 +
V	a $\delta([\xi, \nu\xi], \nu^{-1/2} \sigma)$	2	-1	0	0	1	0	2 +-
	b $L(\nu^{1/2} \xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2} \sigma)$	1	$\sigma(\varpi)$	0	1 +	1	1 +	2 ++
	c $L(\nu^{1/2} \xi \mathrm{St}_{\mathrm{GL}(2)}, \xi \nu^{-1/2} \sigma)$	1	$-\sigma(\varpi)$	0	1	1	1	2 --
	d $L(\nu\xi, \xi \rtimes \nu^{-1/2} \sigma)$	0	1	1	0	1	2 +-	2 +-
VI	a $\tau(S, \nu^{-1/2} \sigma)$	2	1	0	0	1	1	3 +--
	b $\tau(T, \nu^{-1/2} \sigma)$	2	1	0	0	0	1 +	1 +
	c $L(\nu^{1/2} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2} \sigma)$	1	$-\sigma(\varpi)$	0	1	1	0	1 -
	d $L(\nu, \mathbf{1}_{F^\times} \rtimes \nu^{-1/2} \sigma)$	0	1	1	1 +	2	2 +-	3 +--

non-zero P_1 -invariant vector. Except for the one-dimensional representations, these are the following:

- I, IIb, IIIb, IVd, Vd and VI d. These are the *spherical* representations, meaning they have a K -invariant vector.
- IIa, IVc, Vb, VIa and VIb. These are non-spherical representations that have a one-dimensional space of P_1 -fixed vectors. Note that Vc is simply a twist of Vb by the character ξ , so we will not list it separately.
- IIIa and IVb. These are the non-spherical representations that have a two-dimensional space of P_1 -fixed vectors.

Let π be any irreducible, admissible representation of $\mathrm{GSp}_4(F)$. Let S be a matrix as in (1), and θ the associated character of $U(F)$; see (10). Given this data, one may ask for which characters Λ of the torus $T(F)$, defined in (2) and embedded into $\mathrm{GSp}_4(F)$ via (9), the representation π admits a (Λ, θ) -Bessel model. This question can be answered, based

Table 2

The Bessel models of the irreducible, admissible representations of $\mathrm{GSp}_4(F)$ that can be obtained via induction from the Borel subgroup. The symbol N stands for the norm map from $L^\times \cong T(F)$ to F^\times .

Representation		(Λ, θ) -Bessel functional exists exactly for ...	
		$L = F \oplus F$	L/F a field extension
I	$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	all Λ	all Λ
II	a $\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	all Λ	$\Lambda \neq (\chi\sigma) \circ N$
	b $\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	$\Lambda = (\chi\sigma) \circ N$	$\Lambda = (\chi\sigma) \circ N$
III	a $\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	all Λ	all Λ
	b $\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$	$\Lambda(\mathrm{diag}(a, b, b, a)) = \chi(a)\sigma(ab)$ or $\chi(b)\sigma(ab)$	–
IV	a $\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	all Λ	$\Lambda \neq \sigma \circ N$
	b $L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	$\Lambda = \sigma \circ N$	$\Lambda = \sigma \circ N$
	c $L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\Lambda(\mathrm{diag}(a, b, b, a)) = \nu(ab^{-1})\sigma(ab)$ or $\nu(a^{-1}b)\sigma(ab)$	–
	d $\sigma \mathbf{1}_{\mathrm{GSp}(4)}$	–	–
V	a $\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	all Λ	$\Lambda \neq \sigma \circ N, \Lambda \neq (\xi\sigma) \circ N$
	b $L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\Lambda = \sigma \circ N$	$\Lambda = \sigma \circ N, \Lambda \neq (\xi\sigma) \circ N$
	c $L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$\Lambda = (\xi\sigma) \circ N$	$\Lambda \neq \sigma \circ N, \Lambda = (\xi\sigma) \circ N$
	d $L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	–	$\Lambda = \sigma \circ N, \Lambda = (\xi\sigma) \circ N$
VI	a $\tau(S, \nu^{-1/2}\sigma)$	all Λ	$\Lambda \neq \sigma \circ N$
	b $\tau(T, \nu^{-1/2}\sigma)$	–	$\Lambda = \sigma \circ N$
	c $L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\Lambda = \sigma \circ N$	–
	d $L(\nu, \mathbf{1}_{F^\times} \rtimes \nu^{-1/2}\sigma)$	$\Lambda = \sigma \circ N$	–

on results from [8] and [10]. We have listed the data relevant for our current purposes in Table 2 (reproduced from [10]). Note that, in this table, the characters $\chi, \chi_1, \chi_2, \sigma, \xi$ are not necessarily assumed to be unramified, i.e., these results hold for all Borel-induced representations. For the split case $L = F \oplus F$ in Table 2, it is assumed that the matrix S is the one in (12). The resulting torus $T(F)$ is given in (13); embedded into $\mathrm{GSp}_4(F)$ it consists of all matrices $\mathrm{diag}(a, b, b, a)$ with a, b in F^\times . We refer to [10] for additional details and results.

3. Hecke eigenvalues

For integers l and m , let

$$h(l, m) = \begin{bmatrix} \varpi^{l+2m} & & & \\ & \varpi^{l+m} & & \\ & & 1 & \\ & & & \varpi^m \end{bmatrix}. \tag{15}$$

Let (π, V) be a smooth representation of $\mathrm{GSp}_4(F)$ for which $Z(\mathfrak{o})$ acts trivially. We define two endomorphisms of V by the formulas

$$T_{1,0}v = \frac{1}{\mathrm{vol}(P_1)} \int_{P_1 h(1,0) P_1} \pi(g)v dg, \tag{16}$$

Table 3

Eigenvalues of the operators $T_{1,0}$, $T_{0,1}$ and η on spaces of P_1 -invariant vectors in irreducible representations.

Type	dim	$T_{1,0}$	$T_{0,1}$	η
IIa	1	$\alpha\gamma q$	$\alpha^2\gamma^2(\alpha + \alpha^{-1})q^{3/2}$	$-\alpha\gamma$
IIIa	2	$\alpha\gamma q, \gamma q$	$\alpha\gamma^2(\alpha q + 1)q, \alpha\gamma^2(\alpha^{-1}q + 1)q$	$\pm\sqrt{\alpha}\gamma$
IVb	2	$\gamma, \gamma q^2$	$\gamma^2(q + 1), \gamma^2q(q^3 + 1)$	$\pm\gamma$
IVc	1	γq	$\gamma^2(q^3 + 1)$	$-\gamma$
Vb	1	$-\gamma q$	$-\gamma^2q(q + 1)$	γ
VIa	1	γq	$\gamma^2q(q + 1)$	$-\gamma$
VIb	1	γq	$\gamma^2q(q + 1)$	γ

Table 4

Notation for Satake parameters for those Iwahori-spherical representations listed in Table 3. The “Restrictions” column reflects the fact that certain characters χ are not allowed in type IIa and IIIa representations. The last column shows the central character of the representation.

Type	Representation	Parameters	Restrictions	C.c.
IIa	$\chi\text{St}_{\text{GL}(2)} \rtimes \sigma$	$\alpha = \chi(\varpi), \gamma = \sigma(\varpi)$	$\alpha^2 \neq q^{\pm 1}, \alpha \neq q^{\pm 3/2}$	$\chi^2\sigma^2$
IIIa	$\chi \rtimes \sigma\text{St}_{\text{GSp}(2)}$	$\alpha = \chi(\varpi), \gamma = \sigma(\varpi)$	$\alpha \neq 1, \alpha \neq q^{\pm 2}$	$\chi\sigma^2$
IVb	$L(\nu^2, \nu^{-1}\sigma\text{St}_{\text{GSp}(2)})$	$\gamma = \sigma(\varpi)$		σ^2
IVc	$L(\nu^{3/2}\xi\text{St}_{\text{GL}(2)}, \nu^{-3/2}\sigma)$	$\gamma = \sigma(\varpi)$		σ^2
Vb	$L(\nu^{1/2}\xi\text{St}_{\text{GL}(2)}, \nu^{-1/2}\sigma)$	$\gamma = \sigma(\varpi)$		σ^2
VIa	$\tau(S, \nu^{-1/2}\sigma)$	$\gamma = \sigma(\varpi)$		σ^2
VIb	$\tau(T, \nu^{-1/2}\sigma)$	$\gamma = \sigma(\varpi)$		σ^2

$$T_{0,1}v = \frac{1}{\text{vol}(P_1)} \int_{P_1 h(0,1)P_1} \pi(g)v dg \tag{17}$$

and the Atkin–Lehner element

$$\eta = \begin{bmatrix} & & -1 \\ & 1 & \\ -\varpi & \varpi & \end{bmatrix} = s_2 s_1 s_2 \begin{bmatrix} \varpi & & \\ & -\varpi & \\ & & 1 \\ & & & -1 \end{bmatrix}. \tag{18}$$

Evidently, $T_{1,0}$ and $T_{0,1}$ induce endomorphisms of the subspace of V consisting of P_1 -invariant vectors. Table 3 gives the eigenvalues of $T_{1,0}$ and $T_{0,1}$ on the space of P_1 -invariant vectors for the representations in Table 1 which have non-zero P_1 -invariant vectors, but no non-zero K -invariant vectors. We will not give all the details of the eigenvalue calculation, since the method is similar to the one employed in [13]. See [7] for some of the details. Table 4 explains the notation.

We make one more comment on the eigenvalues in Table 3, concerning the representations where the space of P_1 invariant vectors is two-dimensional. One can verify that the operators $T_{1,0}$ and $T_{0,1}$ commute, so that there exists a basis of common eigenvectors. The ordering for types IIIa and IVb in Table 3 is such that the first eigenvalue for $T_{1,0}$ corresponds to the first eigenvalue for $T_{0,1}$, and the second eigenvalue for $T_{1,0}$ corresponds to the second eigenvalue for $T_{0,1}$.

4. Double cosets, an integration formula and automatic vanishing

Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be as in Section 2, subject to the conditions (4). Let $T(F)$ be the subgroup of $GL_2(F)$ defined in (2). By [14, Lemma 2-4], there is a disjoint decomposition

$$GL_2(F) = \bigsqcup_{m=0}^{\infty} T(F) \begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} GL_2(\mathfrak{o}) \tag{19}$$

(here it is important that our assumptions (4) on $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are in force; for example, (19) would obviously be wrong for $\mathfrak{a} = \mathfrak{c} = 0$). The following two lemmas provide further decompositions for the group $GL_2(\mathfrak{o})$. We will use the notations

$$\Gamma_0(\mathfrak{p}) = GL_2(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{bmatrix} \quad \text{and} \quad \Gamma^0(\mathfrak{p}) = GL_2(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} \\ & \mathfrak{o} \end{bmatrix}. \tag{20}$$

Lemma 4.1.

(i) *In the inert case $(\frac{L}{\mathfrak{p}}) = -1$,*

$$GL_2(\mathfrak{o}) = T(\mathfrak{o}) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma_0(\mathfrak{p}) = T(\mathfrak{o}) \Gamma^0(\mathfrak{p}). \tag{21}$$

(ii) *In the ramified case $(\frac{L}{\mathfrak{p}}) = 0$,*

$$\begin{aligned} GL_2(\mathfrak{o}) &= T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma_0(\mathfrak{p}) \\ &= T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma^0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \Gamma^0(\mathfrak{p}), \end{aligned} \tag{22}$$

with u_0 as in (5). We have $T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) = \begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p})$.

(iii) *In the split case $(\frac{L}{\mathfrak{p}}) = 1$,*

$$\begin{aligned} GL_2(\mathfrak{o}) &= T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_1 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_2 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma_0(\mathfrak{p}) \\ &= T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_1 & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma^0(\mathfrak{p}) \\ &\quad \sqcup T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_2 & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma^0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \Gamma^0(\mathfrak{p}), \end{aligned} \tag{23}$$

with u_1, u_2 as in (5). We have $T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_i & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) = \begin{bmatrix} 1 & \\ u_i & 1 \end{bmatrix} \Gamma_0(\mathfrak{p})$ for $i = 1, 2$.

(iv) Let m be a positive integer, and

$$T(\mathfrak{o})_m := \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} T(\mathfrak{o}) \begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} \cap \mathrm{GL}_2(\mathfrak{o}). \tag{24}$$

Then

$$\begin{aligned} \mathrm{GL}_2(\mathfrak{o}) &= T(\mathfrak{o})_m \Gamma_0(\mathfrak{p}) \sqcup T(\mathfrak{o})_m \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma_0(\mathfrak{p}) \\ &= T(\mathfrak{o})_m \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma^0(\mathfrak{p}) \sqcup T(\mathfrak{o})_m \Gamma^0(\mathfrak{p}). \end{aligned} \tag{25}$$

We have $T(\mathfrak{o})_m \Gamma_0(\mathfrak{p}) = \Gamma_0(\mathfrak{p})$.

Proof. The lemma follows by using standard representatives for $\mathrm{GL}_2(\mathfrak{o})/\Gamma_0(\mathfrak{p})$ given by $\begin{bmatrix} 1 & \\ & u \end{bmatrix}$ and $\begin{bmatrix} 1 & \\ u & 1 \end{bmatrix}$, $u \in \mathfrak{o}/\mathfrak{p}$, and the definition of $T(\mathfrak{o})$. See [7]. \square

We turn to double coset decompositions for GSp_4 . For $l, m \in \mathbb{Z}$, let $h(l, m)$ be as in (15). Using (19) and the Iwasawa decomposition, it is easy to see that

$$\mathrm{GSp}_4(F) = \bigsqcup_{\substack{l, m \in \mathbb{Z} \\ m \geq 0}} R(F)h(l, m)K; \tag{26}$$

cf. (3.4.2) of [2]. Using Lemma 4.1, the right coset representatives for K/P_1 from [9, Lemma 5.1.1], and employing the useful identity

$$\begin{bmatrix} 1 & \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} -z^{-1} & \\ & -z \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ & 1 \end{bmatrix}, \tag{27}$$

which holds for $z \in F^\times$, we get the following proposition (see [7] for details).

Proposition 4.2.

(i) For $m > 0$,

$$\begin{aligned} R(F)h(l, m)K &= R(F)h(l, m)P_1 \sqcup R(F)h(l, m)s_2P_1 \\ &\sqcup R(F)h(l, m)s_1s_2P_1 \sqcup R(F)h(l, m)s_2s_1s_2P_1. \end{aligned} \tag{28}$$

(ii) In the inert case $(\frac{l}{\mathfrak{p}}) = -1$,

$$R(F)h(l, 0)K = R(F)h(l, 0)P_1 \sqcup R(F)h(l, 0)s_2P_1 \sqcup R(F)h(l, 0)s_2s_1s_2P_1.$$

(iii) In the ramified case $(\frac{L}{\mathfrak{p}}) = 0$,

$$R(F)h(l, 0)K = R(F)h(l, 0)P_1 \sqcup R(F)h(l, 0)s_2P_1 \sqcup R(F)h(l, 0)s_2s_1s_2P_1 \\ \sqcup R(F)h(l, 0)\hat{u}_0s_1s_2P_1,$$

where

$$\hat{u}_0 = \begin{bmatrix} 1 & & & \\ u_0 & 1 & & \\ & & 1 & -u_0 \\ & & & 1 \end{bmatrix}. \tag{29}$$

(iv) In the split case $(\frac{L}{\mathfrak{p}}) = 1$,

$$R(F)h(l, 0)K = R(F)h(l, 0)P_1 \sqcup R(F)h(l, 0)s_2P_1 \sqcup R(F)h(l, 0)s_2s_1s_2P_1 \\ \sqcup R(F)h(l, 0)\hat{u}_1s_1s_2P_1 \sqcup R(F)h(l, 0)\hat{u}_2s_1s_2P_1,$$

where, for $i = 1, 2$,

$$\hat{u}_i = \begin{bmatrix} 1 & & & \\ u_i & 1 & & \\ & & 1 & -u_i \\ & & & 1 \end{bmatrix}. \tag{30}$$

4.1. An integration formula

The integration formula on $GL_2(\mathfrak{o})$ presented in the following lemma will be used in many of our Hecke operator calculations. Let Λ be a character of $L^\times \cong T(F)$. Let

$$m_0 = \min\{m \geq 0: \Lambda|_{(1+\mathfrak{p}^m\mathfrak{o}_L) \cap \mathfrak{o}_L^\times} = 1\}. \tag{31}$$

Note that, only in the inert case, m_0 is the usual conductor of the character Λ .

Lemma 4.3. *Let Λ be a character of $L^\times \cong T(F)$ which is trivial on \mathfrak{o}^\times , and let m_0 be as in (31). Let m be a non-negative integer. Let $f : GL_2(\mathfrak{o}) \rightarrow \mathbb{C}$ be a function with the property*

$$f(tg) = \Lambda\left(\begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} t \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix}\right) f(g) \quad \text{for all } t \in T(\mathfrak{o})_m.$$

Let the Haar measure on $GL_2(\mathfrak{o})$ be normalized such that the total volume is 1.

(i) Assume that f is right invariant under $\Gamma_0(\mathfrak{p})$. Then

$$\int_{\mathrm{GL}_2(\mathfrak{o})} f(g) dg = \begin{cases} 0 & \text{if } m < m_0, \\ \frac{1}{q+1}(f(1) + qf([\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}])) & \text{if } m \geq \max(m_0, 1), \\ f([\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}])) & \text{if } m = m_0 = 0, \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \frac{1}{q+1}(f([\begin{smallmatrix} 1 & \\ u_0 & 1 \end{smallmatrix}])) + qf([\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}])) & \text{if } m = m_0 = 0, \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \frac{1}{q+1}(f([\begin{smallmatrix} 1 & \\ u_1 & 1 \end{smallmatrix}])) + f([\begin{smallmatrix} 1 & \\ u_2 & 1 \end{smallmatrix}])) + (q-1)f([\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}])) & \text{if } m = m_0 = 0, \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

(ii) Assume that f is right invariant under $\Gamma^0(\mathfrak{p})$. Then

$$\int_{\mathrm{GL}_2(\mathfrak{o})} f(g) dg = \begin{cases} 0 & \text{if } m < m_0, \\ \frac{1}{q+1}(f([\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}])) + qf(1) & \text{if } m \geq \max(m_0, 1), \\ f(1) & \text{if } m = m_0 = 0, \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \frac{1}{q+1}(f([\begin{smallmatrix} 1 & \\ u_0 & 1 \end{smallmatrix}]][\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}])) + qf(1) & \text{if } m = m_0 = 0, \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \frac{1}{q+1}(f([\begin{smallmatrix} 1 & \\ u_1 & 1 \end{smallmatrix}]][\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}])) + f([\begin{smallmatrix} 1 & \\ u_2 & 1 \end{smallmatrix}]][\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}])) + (q-1)f(1) & \text{if } m = m_0 = 0, \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

Proof. The lemma follows from Lemma 4.1. \square

4.2. Automatic vanishing

Many of the cosets from Proposition 4.2 cannot be in the support of a Bessel function. The following lemma exhibits several cases of automatic vanishing.

Lemma 4.4. Let Λ be a character of $L^\times \cong T(F)$ which is trivial on \mathfrak{o}^\times , and let m_0 be as in (31). Let $B \in \mathcal{S}(\Lambda, \theta, P_1)$, and let l and m be integers. Then:

(i)

$$\begin{aligned} B(h(l, m)) &= B(h(l, m)s_2) = 0 \quad \text{if } l < 0, \\ B(h(l, m)s_1s_2) &= B(h(l, m)s_2s_1s_2) = 0 \quad \text{if } l < -1. \end{aligned}$$

(ii)

$$\begin{aligned} B(h(l, m)) &= B(h(l, m)s_2s_1s_2) = 0 \quad \text{for any } l \text{ and } 0 \leq m < m_0, \\ B(h(l, m)s_2) &= 0 \quad \text{for any } l \text{ and } 0 \leq m < m_0 - 1. \end{aligned}$$

(iii) Assume that $m_0 > 0$. Then

$$B(h(l, 0)\hat{u}_i s_1 s_2) = 0$$

with $i = 0$ in the ramified case and $i \in \{1, 2\}$ in the split case.

(iv) In the ramified case,

$$B(h(l, 0)\hat{u}_0 s_1 s_2) = 0 \quad \text{for } l < -1.$$

(v) In the split case,

$$B(h(l, 0)\hat{u}_i s_1 s_2) = 0 \quad \text{for } l < 0.$$

Proof. These are standard arguments using the P_1 -invariance of B and the definition of m_0 . \square

5. The Hecke operators

In the following lemmas we will give the value of $T_{1,0}B$ and $T_{0,1}B$, where $B \in \mathcal{S}(\Lambda, \theta, P_1)$, on various double coset representatives from Proposition 4.2. The main tools are the integration formulas in Lemma 4.3 and the useful matrix identity (27). The proofs are long and tedious and we will not present them here. See [7] for details.

Lemma 5.1. *Let $B \in \mathcal{S}(\Lambda, \theta, P_1)$, and let l and m be non-negative integers. Let $h(l, m)$ be as in (15). Then the following formulas hold.*

(i)

$$(T_{1,0}B)(h(l, m)) = q^3 B(h(l + 1, m)).$$

(ii)

$$(T_{1,0}B)(h(l, m)s_2) = q^2(q - 1)B(h(l + 1, m)) + \begin{cases} -qB(h(l - 1, m + 1)s_1 s_2) & \text{if } m < m_0, \\ q\Lambda(\varpi)B(h(l + 1, m - 1)s_2) \\ \quad + q(q - 1)B(h(l - 1, m + 1)s_1 s_2) & \text{if } m \geq \max(m_0, 1), \\ q^2 B(h(l - 1, 1)s_1 s_2) & \text{if } m = m_0 = 0, \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ q\Lambda(\varpi_L)B(h(l, 0)\hat{u}_0 s_1 s_2) \\ \quad + q(q - 1)B(h(l - 1, 1)s_1 s_2) & \text{if } m = m_0 = 0, \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ q(\Lambda(\varpi, 1)B(h(l, 0)\hat{u}_2 s_1 s_2) \\ \quad + \Lambda(1, \varpi)B(h(l, 0)\hat{u}_1 s_1 s_2)) \\ \quad + q(q - 2)B(h(l - 1, 1)s_1 s_2) & \text{if } m = m_0 = 0, \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

(iii)

$$(T_{1,0}B)(h(l, m)s_1s_2) = q^2(q - 1)B(h(l + 1, m))$$

$$+ \begin{cases} \begin{aligned} & -q\Lambda(\varpi)B(h(l + 1, m - 1)s_2) && \text{if } m < m_0, \\ & q^2B(h(l - 1, m + 1)s_1s_2) && \text{if } m \geq \max(m_0, 1), \\ & q(q + 1)B(h(l - 1, 1)s_1s_2) \\ & \quad - q\Lambda(\varpi)B(h(l + 1, -1)s_2) && \text{if } m = m_0 = 0, \left(\frac{l}{p}\right) = -1, \\ & q\Lambda(\varpi_L)B(h(l, 0)\hat{u}_0s_1s_2) \\ & \quad + q^2B(h(l - 1, 1)s_1s_2) \\ & \quad - q\Lambda(\varpi)B(h(l + 1, -1)s_2) && \text{if } m = m_0 = 0, \left(\frac{l}{p}\right) = 0, \\ & q\Lambda(1, \varpi)B(h(l, 0)\hat{u}_1s_1s_2) \\ & \quad + q\Lambda(\varpi, 1)B(h(l, 0)\hat{u}_2s_1s_2) \\ & \quad + q(q - 1)B(h(l - 1, 1)s_1s_2) \\ & \quad - q\Lambda(\varpi)B(h(l + 1, -1)s_2) && \text{if } m = m_0 = 0, \left(\frac{l}{p}\right) = 1. \end{aligned} \end{cases}$$

(iv)

$$(T_{1,0}B)(h(l, m)s_2s_1s_2) = q^2(q - 1)B(h(l + 1, m)) + \Lambda(\varpi)B(h(l - 1, m)s_2s_1s_2)$$

$$+ \begin{cases} \begin{aligned} & 0 && \text{if } m < m_0, \\ & q(q - 1)B(h(l - 1, m + 1)s_1s_2) \\ & \quad + (q - 1)\Lambda(\varpi)B(h(l + 1, m - 1)s_2) && \text{if } m \geq \max(m_0, 1), \\ & (q^2 - 1)B(h(l - 1, 1)s_1s_2) && \text{if } m = m_0 = 0, \left(\frac{l}{p}\right) = -1, \\ & (q - 1)\Lambda(\varpi_L)B(h(l, 0)\hat{u}_0s_1s_2) \\ & \quad + q(q - 1)B(h(l - 1, 1)s_1s_2) && \text{if } m = m_0 = 0, \left(\frac{l}{p}\right) = 0, \\ & (q - 1)(\Lambda(1, \varpi)B(h(l, 0)\hat{u}_1s_1s_2) \\ & \quad + \Lambda(\varpi, 1)B(h(l, 0)\hat{u}_2s_1s_2)) \\ & \quad + (q - 1)^2B(h(l - 1, 1)s_1s_2) && \text{if } m = m_0 = 0, \left(\frac{l}{p}\right) = 1. \end{aligned} \end{cases}$$

Lemma 5.2. Let Λ be an unramified character of L^\times , and let $l \geq -1$ be an integer. Then:

(i) In the ramified case $\left(\frac{l}{p}\right) = 0$, for all integers $l \geq -1$,

$$(T_{1,0}B)(h(l, 0)\hat{u}_0s_1s_2) = \begin{cases} -q^2B(1) & \text{if } l = -1, \\ q^2(q - 1)B(h(l + 1, 0)) + q^2B(h(l - 1, 1)s_1s_2) & \text{if } l \geq 0. \end{cases}$$

(ii) In the split case $\left(\frac{l}{p}\right) = 1$, for all integers $l \geq 0$,

$$(T_{1,0}B)(h(l, 0)\hat{u}_1s_1s_2) = q^2(q - 1)B(h(l + 1, 0)) + q(q - 1)B(h(l - 1, 1)s_1s_2) + q\Lambda(1, \varpi)B(h(l, 0)\hat{u}_1s_1s_2),$$

$$(T_{1,0}B)(h(l, 0)\hat{u}_2s_1s_2) = q^2(q - 1)B(h(l + 1, 0)) + q(q - 1)B(h(l - 1, 1)s_1s_2) + q\Lambda(\varpi, 1)B(h(l, 0)\hat{u}_2s_1s_2).$$

Lemma 5.3. *Let $B \in \mathcal{S}(\Lambda, \theta, P_1)$, and let l and m be non-negative integers. Let $h(l, m)$ be as in (15). Then the following formulas hold.*

(i)

$$(T_{0,1}B)(h(l, m)) = \begin{cases} 0 & \text{if } m < m_0, \\ \begin{aligned} & q^3\Lambda(\varpi)B(h(l + 2, m - 1)) \\ & + q^4B(h(l, m + 1)) \end{aligned} & \text{if } m \geq \max(m_0, 1), \\ q^3(q + 1)B(h(l, 1)) & \text{if } m = m_0 = 0, \left(\frac{L}{p}\right) = -1, \\ \begin{aligned} & q^3\Lambda(\varpi_L)B(h(l + 1, 0)) + q^4B(h(l, 1)) \\ & + q^3(\Lambda(\varpi, 1) + \Lambda(1, \varpi))B(h(l + 1, 0)) \end{aligned} & \text{if } m = m_0 = 0, \left(\frac{L}{p}\right) = 0, \\ \begin{aligned} & q^3(\Lambda(\varpi, 1) + \Lambda(1, \varpi))B(h(l + 1, 0)) \\ & + q^3(q - 1)B(h(l, 1)) \end{aligned} & \text{if } m = m_0 = 0, \left(\frac{L}{p}\right) = 1. \end{cases}$$

(ii)

$$(T_{0,1}B)(h(l, m)s_2) = \begin{cases} \begin{aligned} & -q^3B(h(l, m + 1)s_1s_2) \\ & - \Lambda(\varpi)B(h(l - 2, m + 1)s_1s_2) \end{aligned} & \text{if } m < m_0, \\ \begin{aligned} & q^3\Lambda(\varpi)B(h(l + 2, m - 1)s_2) \\ & + \Lambda(\varpi)^2B(h(l, m - 1)s_2) \\ & + (q - 1)\Lambda(\varpi)B(h(l - 2, m + 1)s_1s_2) \\ & + q^3(q - 1)B(h(l, m + 1)s_1s_2) \end{aligned} & \text{if } m \geq \max(m_0, 1), \\ q^4B(h(l, 1)s_1s_2) + q\Lambda(\varpi)B(h(l - 2, 1)s_1s_2) & \text{if } m = m_0 = 0, \left(\frac{L}{p}\right) = -1, \\ \begin{aligned} & q^3\Lambda(\varpi_L)B(h(l + 1, 0)\hat{u}_0s_1s_2) \\ & + \Lambda(\varpi)\Lambda(\varpi_L)B(h(l - 1, 0)\hat{u}_0s_1s_2) \\ & + (q - 1)\Lambda(\varpi)B(h(l - 2, 1)s_1s_2) \\ & + q^3(q - 1)B(h(l, 1)s_1s_2) \end{aligned} & \text{if } m = m_0 = 0, \left(\frac{L}{p}\right) = 0, \\ \begin{aligned} & q^3(\Lambda(\varpi, 1)B(h(l + 1, 0)\hat{u}_2s_1s_2) \\ & + \Lambda(1, \varpi)B(h(l + 1, 0)\hat{u}_1s_1s_2)) \\ & + \Lambda(\varpi)(\Lambda(\varpi, 1)B(h(l - 1, 0)\hat{u}_2s_1s_2) \\ & + \Lambda(1, \varpi)B(h(l - 1, 0)\hat{u}_1s_1s_2)) \\ & + q^3(q - 2)B(h(l, 1)s_1s_2) \\ & + (q - 2)\Lambda(\varpi)B(h(l - 2, 1)s_1s_2) \end{aligned} & \text{if } m = m_0 = 0, \left(\frac{L}{p}\right) = 1 \end{cases}$$

$$+ q^2(q - 1)B(h(l, m + 1)) + \left\{ \begin{aligned} & \left(\frac{L}{p}\right)q\Lambda(\varpi)B(1), & \text{if } l = m = 0, \\ & 0 & \text{if } l = 0, m \geq 1, \\ & q(q - 1)\Lambda(\varpi)B(h(l, m)) & \text{if } l \geq 1 \end{aligned} \right\}.$$

Lemma 5.4. *Let Λ be an unramified character of L^\times , and let $l \geq -1$ be an integer. Then:*

(i) *In the ramified case $(\frac{L}{\mathfrak{p}}) = 0$, for all integers $l \geq -1$,*

$$(T_{0,1}B)(h(l, 0)\hat{u}_0s_1s_2) = \begin{cases} q^4B(h(-1, 1)s_1s_2) - q^2\Lambda(\varpi_L)B(1) & \text{if } l = -1, \\ \begin{aligned} & q^4B(h(0, 1)s_1s_2) \\ & + q^2(q-1)\Lambda(\varpi_L)B(h(1, 0)) - q\Lambda(\varpi)B(1) \end{aligned} & \text{if } l = 0, \\ \begin{aligned} & q^4B(h(l, 1)s_1s_2) + q\Lambda(\varpi)B(h(l-2, 1)s_1s_2) \\ & + q^2(q-1)\Lambda(\varpi_L)B(h(l+1, 0)) \\ & + \Lambda(\varpi)q(q-1)B(h(l, 0)) \end{aligned} & \text{if } l \geq 1. \end{cases}$$

(ii) *In the split case $(\frac{L}{\mathfrak{p}}) = 1$, for all integers $l \geq 0$,*

$$(T_{0,1}B)(h(l, 0)\hat{u}_1s_1s_2) = \begin{cases} \begin{aligned} & q^3\Lambda(1, \varpi)B(h(1, 0)\hat{u}_1s_1s_2) + q^3(q-1)B(h(0, 1)s_1s_2) \\ & + q^2(q-1)\Lambda(\varpi, 1)B(h(1, 0)) \end{aligned} & \text{if } l = 0, \\ \begin{aligned} & q^3\Lambda(1, \varpi)B(h(l+1, 0)\hat{u}_1s_1s_2) + q^3(q-1)B(h(l, 1)s_1s_2) \\ & + q^2(q-1)\Lambda(\varpi, 1)B(h(l+1, 0)) \\ & + \Lambda(\varpi)q(q-1)B(h(l, 0)) \\ & + \Lambda(\varpi)\Lambda(1, \varpi)B(h(l-1, 0)\hat{u}_1s_1s_2) \\ & + (q-1)\Lambda(\varpi)B(h(l-2, 1)s_1s_2) \end{aligned} & \text{if } l \geq 1, \end{cases}$$

$$(T_{0,1}B)(h(l, 0)\hat{u}_2s_1s_2) = \begin{cases} \begin{aligned} & q^3\Lambda(\varpi, 1)B(h(1, 0)\hat{u}_2s_1s_2) + q^3(q-1)B(h(0, 1)s_1s_2) \\ & + q^2(q-1)\Lambda(1, \varpi)B(h(1, 0)) \end{aligned} & \text{if } l = 0, \\ \begin{aligned} & q^3\Lambda(\varpi, 1)B(h(l+1, 0)\hat{u}_2s_1s_2) + q^3(q-1)B(h(l, 1)s_1s_2) \\ & + q^2(q-1)\Lambda(1, \varpi)B(h(l+1, 0)) \\ & + \Lambda(\varpi)q(q-1)B(h(l, 0)) \\ & + \Lambda(\varpi)\Lambda(\varpi, 1)B(h(l-1, 0)\hat{u}_2s_1s_2) \\ & + (q-1)\Lambda(\varpi)B(h(l-2, 1)s_1s_2) \end{aligned} & \text{if } l \geq 1. \end{cases}$$

Lemma 5.5. *Let $B \in \mathcal{S}(\Lambda, \theta, P_1)$ be such that $T_{1,0}B = \lambda B$ and $T_{0,1}B = \mu B$. Let l and m be non-negative integers.*

- *For $m \geq \max(m_0, 1)$, or for $m \geq m_0 - 1$ if $m_0 > 0$, we have*

$$\begin{aligned} & \lambda\Lambda(\varpi)B(h(l, m)s_2) - \mu qB(h(l+1, m)s_2) + \lambda q^3B(h(l+2, m)s_2) \\ & = q^5(q-1)B(h(l+3, m)) - q^3(q-1)B(h(l+1, m+1)). \end{aligned} \tag{32}$$

- For $m \geq \max(m_0, 1)$, we have

$$\begin{aligned} &\mu B(h(0, m)s_2) - q^2 \lambda B(h(1, m)s_2) - \Lambda(\varpi)^2 B(h(0, m - 1)s_2) \\ &= q^2(q - 1)B(h(0, m + 1)) - q^4(q - 1)B(h(2, m)). \end{aligned} \tag{33}$$

- For $m_0 > 0$, we have

$$\begin{aligned} &\mu B(h(0, m_0)s_2) - q^2 \lambda B(h(1, m_0)s_2) - \Lambda(\varpi)^2 B(h(0, m_0 - 1)s_2) \\ &= q^{-2}(q - 1)(\mu - \lambda^2)B(h(0, m_0)). \end{aligned} \tag{34}$$

- For $l \geq 0, m_0 > 0$, we have

$$\begin{aligned} &\lambda \Lambda(\varpi)B(h(l, m_0 - 1)s_2) - \mu q B(h(l + 1, m_0 - 1)s_2) + \lambda q^3 B(h(l + 2, m_0 - 1)s_2) \\ &= -\lambda(q - 1)B(h(l, m_0)). \end{aligned} \tag{35}$$

Proof. (32) and (33) follow from Lemma 5.3(ii) and Lemma 5.1(ii). (34) follows from (33), Lemma 5.1(i) and Lemma 5.3(i). (35) follows from (32), the identity $\lambda B(h(l, m_0)) = q^3 B(h(l + 1, m_0))$ and some automatic vanishing. \square

6. The main tower

Again we consider the matrix S and the associated character θ of $U(F)$; see (10). As usual, the assumptions (4) are in force. Let Λ be a character of $T(F)$ and define the non-negative integer m_0 as in (31). Let B be a function in the space $\mathcal{S}(\Lambda, \theta, P_1)$. We refer to the values of B at the elements $h(l, m)$, defined in (15), as the *main tower* (in view of Proposition 4.2, there is also an s_2 -tower etc.). Note that $B(h(l, m)) = 0$ for $l < 0$ or $0 \leq m < m_0$ by Lemma 4.4. The following result relates the values of $B(h(l, m))$ to $B(h(0, m_0))$, only assuming that B is an eigenfunction for $T_{1,0}$ and $T_{0,1}$.

Proposition 6.1. *Let $B \in \mathcal{S}(\Lambda, \theta, P_1)$ be an eigenfunction for $T_{1,0}$ and $T_{0,1}$ with eigenvalues λ and μ , respectively. Then*

$$B(h(l + 1, m)) = \lambda q^{-3} B(h(l, m)) \quad \text{for all } l \geq 0 \text{ and } m \geq 0. \tag{36}$$

Furthermore, for any $l \geq 0$, there is a formal identity

$$Y^{-m_0} \sum_{m=m_0}^{\infty} B(h(l, m)) Y^m = \frac{1 - \kappa q^{-4} Y}{1 - \mu q^{-4} Y + \lambda^2 q^{-7} \Lambda(\varpi) Y^2} B(h(l, m_0)),$$

where

$$\kappa = \begin{cases} 0 & \text{if } m_0 > 0, \\ (q + 1)^{-1}\mu & \text{if } m_0 = 0 \text{ and } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \Lambda(\varpi_L)\lambda & \text{if } m_0 = 0 \text{ and } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ (q - 1)^{-1}(q\lambda(\Lambda(\varpi, 1) + \Lambda(1, \varpi)) - \mu) & \text{if } m_0 = 0 \text{ and } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

Proof. The relation (36) is immediate from Lemma 5.1(i). Combining (36) with Lemma 5.3(i), we get, for $l \geq 0$ and $m \geq m_0$,

$$q^4 B(h(l, m + 2)) - \mu B(h(l, m + 1)) + \lambda^2 q^{-3} \Lambda(\varpi) B(h(l, m)) = 0. \tag{37}$$

We multiply this by Y^{m+2} and apply $\sum_{m=m_0}^\infty$ to both sides, arriving at the formal identity

$$\sum_{m=m_0}^\infty B(h(l, m)) Y^m = \frac{(q^4 - \mu Y) B(h(l, m_0)) + q^4 Y B(h(l, m_0 + 1))}{q^4 - \mu Y + \lambda^2 q^{-3} \Lambda(\varpi) Y^2} Y^{m_0}.$$

Setting $m = m_0$ in Lemma 5.3(i) and using (36) provides a relation between $B(h(l, m_0))$ and $B(h(l, m_0 + 1))$. Substituting this relation, we obtain the asserted formula. \square

Corollary 6.2. *Let $B \in \mathcal{S}(\Lambda, \theta, P_1)$ be an eigenfunction for $T_{1,0}$ and $T_{0,1}$. Thus, if $B(h(0, m_0)) = 0$, then $B(h(l, m)) = 0$ for all l, m .*

7. Generic representations and split Bessel models

We recall some basic facts about generic representations of $\mathrm{GSp}_4(F)$, and refer to Section 2.6 of [9] for details. We denote by N the unipotent radical of the Borel subgroup B , and for c_1, c_2 in F^\times , consider the character ψ_{c_1, c_2} of $N(F)$ given in Section 2.1 of [9]. An irreducible, admissible representation π of $\mathrm{GSp}_4(F)$ is called *generic* if $\mathrm{Hom}_{N(F)}(\pi, \psi_{c_1, c_2}) \neq 0$. In this case there is an associated Whittaker model $\mathcal{W}(\pi, \psi_{c_1, c_2})$ consisting of functions $\mathrm{GSp}_4(F) \rightarrow \mathbb{C}$ that transform on the left according to ψ_{c_1, c_2} . For $W \in \mathcal{W}(\pi, \psi_{c_1, c_2})$, there is an associated zeta integral

$$Z(s, W) = \int_{F^\times} \int_F W \left(\begin{bmatrix} a & & & \\ & a & & \\ x & & 1 & \\ & & & 1 \end{bmatrix} \right) |a|^{s-3/2} dx d^\times a. \tag{38}$$

This integral is convergent for $\mathrm{Re}(s) > s_0$, where s_0 is independent of W ; see [9, Proposition 2.6.3]. Moreover, there exists an L -factor of the form $L(s, \pi) = Q(q^{-s})^{-1}$, $Q(X) \in \mathbb{C}[X]$, $Q(0) = 1$, such that

$$\frac{Z(s, W)}{L(s, \pi)} \in \mathbb{C}[q^{-s}, q^s] \quad \text{for all } W \in \mathcal{W}(\pi, \psi_{c_1, c_2}). \tag{39}$$

We consider split Bessel models with respect to the quadratic form S defined in (12). Let θ be the corresponding character of $U(F)$. The resulting group $T(F)$, defined in (2), is a split torus. We think of $T(F)$ embedded into $\mathrm{GSp}_4(F)$ as all matrices of the form $\mathrm{diag}(a, b, b, a)$ with $a, b \in F^\times$. We write a character Λ of $T(F)$ as a function $\Lambda(a, b)$. Consider the functional f_s on the ψ_{c_1, c_2} -Whittaker model of π given by

$$f_s(W) = \frac{Z(s, \pi(s_2)(W))}{L(s, \pi)}. \tag{40}$$

By analytic continuation and the defining property of $L(s, \pi)$, this is a well-defined and non-zero functional on π for any value of s . A direct computation shows that the functional f_s is a split (Λ, θ) -Bessel functional with respect to the character Λ given by $\Lambda(\mathrm{diag}(a, b, b, a)) = |a^{-1}b|^{-s+1/2}$. If we assume that π has trivial central character, then any unramified character of $T(F)$ is of this form for an appropriate s .

7.1. Zeta integrals of Siegel vectors

Let (π, V) be an irreducible, admissible, generic representation of $\mathrm{GSp}_4(F)$ with unramified central character. We assume that $V = \mathcal{W}(\pi, \psi_{c_1, c_2})$ is the Whittaker model with respect to the character ψ_{c_1, c_2} of $N(F)$. For what follows we will assume that $c_1, c_2 \in \mathfrak{o}^\times$. Recall the Siegel congruence subgroup P_1 and the Klingen congruence subgroup P_2 defined in (6).

Lemma 7.1. *Let (π, V) be as above, and let W be an element of $V = \mathcal{W}(\pi, \psi_{c_1, c_2})$.*

(i) *If W is P_1 -invariant, then*

$$Z(s, \pi(s_2)W) = \int_{F^\times} W \left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 \right) |a|^{s-3/2} d^\times a.$$

(ii) *If W is P_2 -invariant, then*

$$Z(s, W) = \int_{F^\times} W \left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) |a|^{s-3/2} d^\times a.$$

Proof. (ii) is a special case of Lemma 4.1.1 of [9], and (i) is proved similarly. \square

Let $W \in V$ be a P_2 -invariant vector. Then

$$T_{\mathrm{Si}}W := \sum_{g \in \mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} \pi \left(\begin{bmatrix} g & \\ & {}_t g^{-1} \end{bmatrix} \right) W$$

is P_1 -invariant. Using Lemma 7.1, the standard representatives for $GL_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})$ and (27), one calculates that

$$Z(s, \pi(s_2)(T_{Si}W)) = q \int_{F^\times} W \left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 \right) |a|^{s-3/2} d^\times a + Z(s, W). \tag{41}$$

7.2. The II_a case

Lemma 7.2. *Let $\pi = \chi \text{St}_{GL(2)} \rtimes \sigma$ be a representation of type II_a with unramified χ and σ . Let $\alpha = \chi(\varpi)$ and $\gamma = \sigma(\varpi)$. We assume that $\alpha^2 \gamma^2 = 1$, i.e., that π has trivial central character. Let W_0 be a non-zero $K^{\text{para}}(\mathfrak{p})$ -invariant vector in the ψ_{c_1, c_2} -Whittaker model of π normalized such that $Z(s, W_0) = L(s, \pi)$ (see Corollary 7.2.6 of [9]). Let $\omega = -\alpha\gamma$ be the eigenvalue of the Atkin–Lehner element η on W_0 .*

(i) For any $s \in \mathbb{C}$,

$$Z(s, \pi(s_2)(T_{Si}W_0)) = (\omega q^{s-1/2} + 1)L(s, \pi).$$

(ii) If $\omega q^{s-1/2} + 1 = 0$, then

$$Z(s, T_{Si}W_0) = (1 - q^{-1})^{-1}.$$

Proof. We may assume that s is in the region of convergence. Since the element $t_1 = h(1, 0)^{-1} s_1 s_2 s_1 h(1, 0)$ lies in the paramodular group $K^{\text{para}}(\mathfrak{p})$ we have, for any g in $GSp_4(F)$, the relation $\omega W_0(g s_2 s_1) = W_0(gh(1, 0))$. Substituting this into (41), we obtain

$$\begin{aligned} Z(s, \pi(s_2)(T_{Si}W_0)) &= \omega q \int_{F^\times} W_0 \left(\begin{bmatrix} a\varpi & & & \\ & a\varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) |a|^{s-3/2} d^\times a + Z(s, W_0) \\ &= (\omega q^{s-1/2} + 1)L(s, \pi). \end{aligned}$$

This proves part (i) of the lemma. The proof of part (ii) is a long computation and we do not present it here. See [7] for details. \square

For the following proposition we continue to assume that $\pi = \chi \text{St}_{GL(2)} \rtimes \sigma$ is a representation of type II_a with unramified χ and σ , but we will drop the condition that π has trivial central character. If the non-zero vector W spans the space of P_1 -invariant vectors, then we still have $\eta W = \omega W$ with $\omega = -\alpha\gamma$, but this constant is no longer necessarily ± 1 . We consider split Bessel models with respect to S as in (12). We want the character Λ of $T(F)$ to coincide on the center of $GSp_4(F)$ with the central character

of π , i.e., $\Lambda(a, a) = (\chi\sigma)^2(a)$. Since $\Lambda(\varpi, 1)\Lambda(1, \varpi) = \omega^2$, we have $\Lambda(1, \varpi) = -\omega$ if and only if $\Lambda(\varpi, 1) = -\omega$.

Proposition 7.3. *Assume that $\pi = \chi\text{St}_{\text{GL}(2)} \rtimes \sigma$ is a representation of type IIa with unramified χ and σ . Let S, θ and $T(F)$ be as above. Let Λ be an unramified character of $T(F)$ such that $\Lambda(a, a) = (\chi\sigma)^2(a)$. Let B be a non-zero vector in the (Λ, θ) -Bessel model of π spanning the one-dimensional space of P_1 -invariant vectors. Then we have*

$$\Lambda(\varpi, 1) \neq -\omega \neq \Lambda(1, \varpi) \iff B(1) \neq 0.$$

Proof. After twisting by an unramified character, we may assume that π has trivial central character. Let W_0 be a non-zero vector in the ψ_{c_1, c_2} -Whittaker model of π spanning the one-dimensional space of P_{02} -invariant vectors. By Lemma 7.2, the vector $T_{S_i}W_0$ is non-zero, and hence spans the one-dimensional space of P_1 -invariant vectors. Let f_s be as in (40). Note that $\Lambda(\varpi, 1) = -\omega = \Lambda(1, \varpi)$ if and only if $\omega q^{s-1/2} + 1 = 0$. Then, by (i) of Lemma 7.2, we have $f_s(T_{S_i}W_0) \neq 0$ if and only if $\Lambda(\varpi, 1) \neq -\omega \neq \Lambda(1, \varpi)$. The proposition now follows from the fact that $f_s(T_{S_i}W_0) \neq 0$ is equivalent to $B(1) \neq 0$ by uniqueness of Bessel models. \square

7.3. The VIa case

Let $\pi = \tau(S, \nu^{-1/2}\sigma)$ be the representation of type VIa. We assume that σ is unramified and that π has trivial central character, i.e., $\sigma^2 = 1$. This is a representation of conductor 2. By Table A.8 of [9],

$$L(s, \pi) = L(s, \nu^{1/2}\sigma)^2 = \frac{1}{(1 - \sigma(\varpi)q^{-1/2-s})^2}. \tag{42}$$

We assume that π is given in its ψ_{c_1, c_2} -Whittaker model. Let W_0 be the paramodular newform, i.e., a non-zero element invariant under the paramodular group of level \mathfrak{p}^2 ; see (7). By Theorem 7.5.4 of [9], we can normalize W_0 such that $Z(s, W_0) = L(s, \pi)$. We let

$$W' := \sum_{x, y, z \in \mathfrak{o}/\mathfrak{p}} \pi \left(\begin{bmatrix} 1 & y\varpi & & \\ & 1 & & \\ & x\varpi & 1 & \\ x\varpi & z\varpi & -y\varpi & 1 \end{bmatrix} \right) W_0. \tag{43}$$

This is a P_2 -invariant vector; it was called the *shadow of the newform* in Section 7.4 of [9]. By Proposition 7.4.8 of [9],

$$Z(s, W') = (1 - q^{-1})L(s, \pi). \tag{44}$$

Lemma 7.4. For any complex number s ,

$$\frac{Z(s, \pi(s_2)(T_{S_i}W'))}{L(s, \pi)} = 2(q - 1)\sigma(\varpi)q^{-1/2+s}. \tag{45}$$

Proof. We may assume that s is in the region of convergence. By (41) and (44),

$$\begin{aligned} Z(s, \pi(s_2)(T_{S_i}W')) &= q \int_{F^\times} W' \left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 \right) |a|^{s-3/2} d^\times a \\ &\quad + (1 - q^{-1})L(s, \pi). \end{aligned} \tag{46}$$

To evaluate the integral in this equation, we compute the zeta integral $Z(s, \pi(s_2 s_1)W')$ in two different ways – first, by using the local functional equation for $Z(s, W)$, and second, by using a direct computation and the fact that the representation has no $K^{\text{para}}(\mathfrak{p})$ -invariant vectors. The result is that

$$\int_{F^\times} W' \left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 \right) |a|^{s-3/2} d^\times a = (q - 1)q^{2s-1}(L(s, \pi) - 1). \tag{47}$$

From (46) and (47), we now get

$$Z(s, \pi(s_2)(T_{S_i}W')) = (q - 1)q^{2s}(L(s, \pi) - 1) + (1 - q^{-1})L(s, \pi). \tag{48}$$

Using the explicit form (42) of $L(s, \pi)$, the assertion follows. See [7] for further details. \square

Proposition 7.5. Assume that $\pi = \tau(S, \nu^{-1/2}\sigma)$ is a representation of type VIa with unramified σ . Let S be as in (12). Let Λ be an unramified character of $T(F)$ such that $\Lambda(a, a) = \sigma^2(a)$. Let B be a non-zero vector in the (Λ, θ) -Bessel model of π spanning the one-dimensional space of P_1 -invariant vectors. Then $B(1) \neq 0$.

Proof. After twisting by an unramified character, we may assume that π has trivial central character. Let W_0 be a non-zero vector in the ψ_{c_1, c_2} -Whittaker model of π spanning the one-dimensional space of $K^{\text{para}}(\mathfrak{p}^2)$ -invariant vectors, and let W' be the P_2 -invariant vector defined in (43). By Lemma 7.4, the vector $T_{S_i}W'$ is non-zero, and hence spans the one-dimensional space of P_1 -invariant vectors. Let f_s be as in (40). By Lemma 7.4, we have $f_s(T_{S_i}W') \neq 0$. By uniqueness of Bessel models, this is equivalent to $B(1) \neq 0$. \square

8. The one-dimensional cases

In this section we will identify good test vectors for those irreducible, admissible representations of $\mathrm{GSp}_4(F)$ which are not spherical, but possess a one-dimensional space of P_1 -invariant vectors. A look at Table 1 shows that these are the Iwahori-spherical representations of type IIa, IVc, Vb, VIa and VIb (Vc is a twist of Vb and is not counted separately). This entire section assumes that the elements \mathbf{a} , \mathbf{b} , \mathbf{c} satisfy the conditions (4). The matrix S , the group $T(F) \cong L^\times$ and the character θ have the usual meaning.

Lemma 8.1. *Let Λ be a character of L^\times , and m_0 as in (31).*

(i) *Assume that $B \in \mathcal{S}(\Lambda, \theta, P_1)$ satisfies*

$$T_{1,0}B = \lambda B, \quad T_{0,1}B = \mu B, \quad \eta B = \omega B$$

with complex numbers λ, μ, ω satisfying $\lambda = -q\omega$.

- *If $m_0 = 0$, $(\frac{L}{\mathfrak{p}}) = 0$ and $\Lambda(\varpi_L) = -\omega$, then $B = 0$.*
- *If $m_0 = 0$, $(\frac{L}{\mathfrak{p}}) = 1$, $\Lambda(1, \varpi) = -\omega = \Lambda(\varpi, 1)$ and $B(1) = 0$, then*

$$B = 0 \iff B(\hat{u}_1 s_1 s_2) = 0 \iff B(\hat{u}_2 s_1 s_2) = 0.$$

- *Otherwise,*

$$B = 0 \iff B(h(0, m_0)) = 0.$$

(ii) *Assume that $B \in \mathcal{S}(\Lambda, \theta, P_1)$ satisfies $\eta B = \omega B$ and*

$$\sum_{c \in \mathfrak{o}/\mathfrak{p}} \pi \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ c & & 1 & \\ & & & 1 \end{bmatrix} \right) B + \pi(s_2)B = 0. \tag{49}$$

Then

$$B = 0 \iff B(h(0, m_0)) = 0.$$

Proof. (i) From Lemma 5.1 and the Atkin–Lehner relation $\eta B = \omega B$, we get

$$\lambda B(h(l, m)) = q^3 B(h(l + 1, m)), \quad \text{for } l, m \geq 0, \tag{50}$$

$$(q - 1)B(h(l, m)) = q^2 B(h(l, m)s_2) + q\omega B(h(l + 1, m - 1)s_2) \quad \text{for } l \geq 0, m \geq \max(m_0, 1). \tag{51}$$

Assume that $m_0 > 0$ and that $B(h(0, m_0)) = 0$. Then, by Corollary 6.2, the values $B(h(l, m))$ are zero for all l and m . Considering (51) with $l = 0$, (51) with $l = 1$, Lemma 5.5 with $l = 0$, $m = m_0 - 1$, (33) and Lemma 5.3(ii) with $l = 0$, $m = m_0 - 1$ we get a homogeneous system of 5 linear equations with a non-singular matrix in the variables

$$B(h(0, m_0 - 1)s_2), \quad B(h(1, m_0 - 1)s_2), \quad B(h(2, m_0 - 1)s_2), \\ B(h(0, m_0)s_2), \quad B(h(1, m_0)s_2).$$

Hence, the above values are equal to zero. Lemma 5.5 then implies that $B(h(l, m_0 - 1)s_2) = 0$ for all $l \geq 0$. Using (51), it follows that $B(h(l, m)s_2) = 0$ for all $l \geq 0$ and $m \geq m_0 - 1$. The Atkin–Lehner relations now imply that the main tower, s_2 -tower, s_1s_2 -tower and $s_2s_1s_2$ -tower are all zero. By Proposition 4.2 and Lemma 4.4, the function B is zero. This proves (i) of the lemma for $m_0 > 0$.

Now assume that $m_0 = 0$. Using Proposition 6.1, Lemma 4.4, (51), and the Atkin–Lehner relations, we can show the following: If $B(1) = 0$ and $B(h(l, 0)s_2) = 0$ for all $l \geq 0$, then $B(h(l, m)w) = 0$ for all $l \in \mathbb{Z}$, all $m \geq 0$, and all $w \in \{1, s_2, s_2s_1s_2\}$, as well as $B(h(l, m)s_1s_2) = 0$ for all $l \in \mathbb{Z}$ and all $m \geq 1$.

Assume that $(\frac{L}{p}) = -1$ (the inert case). Then, from Lemma 5.1(ii), (50), Atkin–Lehner relations and using $\lambda = -q\omega$, we get $q(q + 1)B(h(l, 0)s_2) = (q - 1)B(h(l, 0))$ for $l \geq 0$. Now, from the double coset representatives in Proposition 4.2, it follows that $B = 0$ if $B(1) = 0$. The other cases are similar. See [7] for details.

(ii) Evaluating (49) at $h(l, m)s_2$, we obtain

$$qB(h(l, m)s_2) = -B(h(l, m)) \tag{52}$$

for all $l, m \geq 0$. Let $i = 0$ in the ramified case and $i = 1$ or $i = 2$ in the split case. Evaluating (49) at $h(l, 0)\hat{u}_i s_1 s_2 s_1 s_2$ leads to

$$qB(h(l, 0)s_2 s_1 s_2) = -B(h(l, 0)\hat{u}_i s_1 s_2) \tag{53}$$

for $l \geq -1$; observe here the defining property (5) of the elements u_i . Assume that $B(h(0, m_0))$, and therefore, by Corollary 6.2, the whole main tower, is zero. Then, by (52) and Lemma 4.4, the entire s_2 -tower is also zero. By the Atkin–Lehner relations, the s_1s_2 and $s_2s_1s_2$ towers are also zero. In the inert case, in view of the double cosets given in Proposition 4.2, and (v) of Lemma 4.4, it follows that $B = 0$. Taking into account (53), the same conclusion holds in the ramified and split cases. \square

The condition $\lambda = -q\omega$ in (i) of the above lemma is satisfied by the representations of type IIa, IVc, Vb and VIa. The representations of type VIIb satisfy $\lambda = q\omega$. These representations have a one-dimensional space of P_1 -invariant vectors, but no non-zero P_2 -invariant vectors. Hence, if a non-zero P_1 -invariant vector B is made P_2 -invariant by

summation, the result is zero – using standard representatives, we get (49) in (ii) of the above lemma.

8.1. The main result

The following theorem, which is the main result of this section, identifies good test vectors for those irreducible, admissible, infinite-dimensional representations of $\mathrm{GSp}_4(F)$ that have a one-dimensional space of P_1 -invariant vectors.

Theorem 8.2. *Let π be an irreducible, admissible, Iwahori-spherical representation of $\mathrm{GSp}_4(F)$ that is not spherical but has a one-dimensional space of P_1 -invariant vectors. Let S be the matrix defined in (1), with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ subject to the conditions (4). Let $T(F)$ be the group defined in (2). Let θ be the character of $U(F)$ defined in (10), and let Λ be a character of $T(F) \cong L^\times$. Let m_0 be as in (31). We assume that π admits a (Λ, θ) -Bessel model. Let B be an element in this Bessel model spanning the space of P_1 -invariant vectors.*

- (i) *Assume that π is of type IIa. Then $B(h(0, m_0)) \neq 0$, except in the split case with $\Lambda(1, \varpi) = -\omega = \Lambda(\varpi, 1)$ and $m_0 = 0$. In this latter case $B(1) = 0$, but $B(\hat{u}_i s_1 s_2) \neq 0$ for $i = 1, 2$. Here, the elements \hat{u}_i are defined in (30).*
- (ii) *If π is of type IVc, Vb, VIa or VIb, then $B(h(0, m_0)) \neq 0$.*

Proof. Let B be a non-zero P_1 -invariant vector in the (Λ, θ) -Bessel model of π . Then B is an element of the space $\mathcal{S}(\Lambda, \theta, P_1)$. Since the space of P_1 -invariant vectors in π is one-dimensional, B is an eigenvector for $T_{1,0}, T_{0,1}$ and η ; let λ, μ and ω be the respective eigenvalues. Lemma 8.1 proves our assertions in case $m_0 > 0$. For the rest of the proof we will therefore assume that $m_0 = 0$, meaning that Λ is unramified.

(i) In this case we have $\lambda = -q\omega$. If we are not in the split case, or if we are in the split case and $\Lambda(1, \varpi) \neq -\omega$ then our assertion follows from Lemma 8.1(i). Assume that we are in the split case and that $\Lambda(1, \varpi) = -\omega = \Lambda(\varpi, 1)$. Then $B(1) = 0$ by Proposition 7.3 and (14). Hence our assertion follows from Lemma 8.1(i).

(ii) If π is of type VIb, then B satisfies (49) and the assertion follows from Lemma 8.1(ii). Assume that π is of type IVc, Vb or VIa. If we are not in the split case, our assertions follow from Lemma 8.1(i). Assume we are in the split case, and that π is of type IVc or Vb. Then, by Table 2 and Table 3, we have $\Lambda(1, \varpi) \neq -\omega$. Hence our assertions follow from Lemma 8.1(i). Assume we are in the split case, and that π is of type VIa. Then our assertion follows from Proposition 7.5 and (14). \square

9. The two-dimensional cases

In this section, let π be an irreducible, admissible representation of $\mathrm{GSp}_4(F)$ of type IIIa or IVb. In both cases the space of P_1 -invariant vectors is two-dimensional.

9.1. The IIIa case

Let B_1 and B_2 be common eigenvectors for $T_{1,0}$ and $T_{0,1}$ in the IIIa case. Then, one can choose the normalizations such that

$$\begin{aligned} T_{1,0} B_1 &= \alpha\gamma q B_1, & T_{1,0} B_2 &= \gamma q B_2, \\ T_{0,1} B_1 &= \alpha\gamma^2(\alpha q + 1)q B_1, & T_{0,1} B_2 &= \alpha\gamma^2(\alpha^{-1}q + 1)q B_2, \\ \eta B_1 &= \alpha\gamma B_2, & \eta B_2 &= \gamma B_1. \end{aligned} \tag{54}$$

Note that $\alpha \neq 1$ in the IIIa case; see Table 4.

Lemma 9.1. *Let Λ be a character of L^\times , and m_0 as in (31). Assume that $B_1, B_2 \in \mathcal{S}(\Lambda, \theta, P_1)$ satisfy (54) with $\alpha \neq 1$. Assume also that $\Lambda(\varpi) = \alpha\gamma^2$. Then*

$$B_1 = 0 \iff B_1(h(0, m_0)) = 0.$$

Proof. Using the Atkin–Lehner relations (54) and Lemma 5.1, we get for any $l \geq -1$ and $m \geq 0$,

$$\gamma B_1(h(l, m)s_2s_1s_2) = q^2 B_1(h(l + 1, m)s_2s_1s_2), \tag{55}$$

and, for $l \geq 0$ and $m \geq \max(m_0, 1)$, the three equations

$$0 = -\alpha\gamma q B_1(h(l, m)s_1s_2) + q^2 B_1(h(l - 1, m + 1)s_1s_2) + \alpha\gamma(q - 1)B_1(h(l, m)), \tag{56}$$

$$0 = -\gamma q B_1(h(l + 1, m - 1)s_2) + q^2 B_1(h(l, m)s_2) + q^2(q - 1)B_1(h(l, m)s_2s_1s_2), \tag{57}$$

$$0 = q^3 B_1(h(l, m)s_2s_1s_2) + q^2 B_1(h(l, m)s_2) + q B_1(h(l, m)s_1s_2) + B_1(h(l, m)). \tag{58}$$

We will first consider the case $\mathbf{m}_0 > \mathbf{0}$. Assume that $B_1(h(0, m_0)) = 0$; we will show that $B_1 = B_2 = 0$. Note that, by the Atkin–Lehner relations, we only need to show that $B_1 = 0$. By Corollary 6.2, we know that $B_1(h(l, m)) = 0$ for all $l, m \geq 0$. Considering (58) with $l = 0$ and $m = m_0$, Lemma 5.3(ii) with $B = B_2$, $l = 0$ and $m = m_0 - 1$, (35) applied to B_1 , (34) applied to B_1 , Lemma 5.3(ii) with $B = B_2$, $l = 0$ and $m = m_0$, Lemma 5.3(ii) with $B = B_1$, $l = 0$ and $m = m_0 - 1$, (35) applied to B_2 and Lemma 5.3(ii) with $B = B_1$, $l = 0$ and $m = m_0$, we get a homogeneous system of 8 linear equations in the 7 variables

$$\begin{aligned} B_1(h(0, m_0 - 1)s_2), & B_1(h(0, m_0)s_2), & B_1(h(1, m_0)s_2), & B_1(h(0, m_0)s_1s_2), \\ B_1(h(1, m_0)s_1s_2), & B_1(h(-1, m_0)s_1s_2), & B_1(h(0, m_0)s_2s_1s_2). \end{aligned} \tag{59}$$

For any α , either the set of first 7 equations or the set of last 7 equations has a non-singular matrix. Hence, all the values in (59) are zero. Now (32) and (57) imply that $B_1(h(l, m)s_2) = 0$ for all $l \geq 0$, $m = m_0 - 1$ or $m = m_0$. Using (55) and (58), we

get $B_1(h(l, m_0)s_2s_1s_2) = B_1(h(l, m_0)s_1s_2) = 0$ for $l \geq -1$. Now, using (56), (57), (58), induction, Proposition 4.2 and the automatic vanishing from Lemma 4.4, it follows that $B_1 = 0$. This concludes our proof in case $m_0 > 0$.

We next consider the case $\mathbf{m}_0 = \mathbf{0}$. Assume that $B_1(1) = 0$; we will show that $B_1 = 0$. By Corollary 6.2, we know that $B_1(h(l, m)) = 0$ for all $l, m \geq 0$. Suppose we can show that B_1 vanishes on all the double coset representatives in Proposition 4.2 that have $m = 0$ and that $B_1(h(l, 1)s_1s_2) = 0$ for all $l \geq -1$. Using induction and (56), we get $B_1(h(l, m)s_1s_2) = 0$ for all $l \geq -1$ and $m \geq 0$. Now, induction and (57), (58), gives us $B_1 = 0$, and hence, $B_2 = 0$.

We will give the proof of the inert case $(\frac{L}{p}) = -1$ here. The other cases are similar (see [7]). Using Lemma 5.1(ii), we get, for $l \geq 0$,

$$\alpha\gamma q B_1(h(l, 0)s_2) = q^2 B_1(h(l - 1, 1)s_1s_2). \tag{60}$$

Using (55) and Lemma 5.1(iv), we get, for $l \geq 0$,

$$\alpha\gamma q B_1(h(l, 0)s_2s_1s_2) = -(q + 1)B_1(h(l - 1, 1)s_1s_2). \tag{61}$$

Hence, from (55), (60) and (61), we get, for $l \geq 0$,

$$\gamma B_1(h(l, 0)s_2) = q^2 B_1(h(l + 1, 0)s_2). \tag{62}$$

Using (60), (62) and Lemma 5.3(ii), we get

$$\alpha\gamma^2(\alpha q + 1)q B_1(s_2) = q^4 B_1(h(0, 1)s_1s_2) = \alpha\gamma q^3 B_1(h(1, 0)s_2) = \alpha\gamma^2 q B_1(s_2). \tag{63}$$

Since $\alpha \neq 0$, we get $B_1(s_2) = 0$. Now (60), (61) and (62) imply that B_1 vanishes on all the double coset representatives in Proposition 4.2 that have $m = 0$. Hence $B_1 = 0$, as claimed. \square

9.2. The IVb case

Let B_1 and B_2 be common eigenvectors for $T_{1,0}$ and $T_{0,1}$ in the IVb case. We can choose the normalizations such that

$$\begin{aligned} T_{1,0} B_1 &= \gamma B_1, & T_{1,0} B_2 &= \gamma q^2 B_2, \\ T_{0,1} B_1 &= \gamma^2(q + 1) B_1, & T_{0,1} B_2 &= \gamma^2 q(q^3 + 1) B_2, \\ \eta B_1 &= \gamma B_2, & \eta B_2 &= \gamma B_1. \end{aligned} \tag{64}$$

Recall that $\gamma = \sigma(\varpi)$, where σ is an unramified character. From Table 2, a (Λ, θ) -Bessel model exists if and only if $\Lambda = \sigma \circ N_{L/F}$. In particular, the number m_0 defined in (31) must be zero, i.e., Λ must be unramified. The central character condition is equivalent to $\Lambda(\varpi) = \gamma^2$. Moreover, in the ramified case $(\frac{L}{p}) = 0$, evaluating at ϖ_L , we

get $\Lambda(\varpi_L) = \gamma$, and in the split case $(\frac{L}{p}) = 1$, evaluating at $(\varpi, 1)$ and $(1, \varpi)$, we get $\Lambda(\varpi, 1) = \Lambda(1, \varpi) = \gamma$.

Lemma 9.2. *Let Λ be an unramified character of L^\times satisfying $\Lambda(\varpi) = \gamma^2$. If $(\frac{L}{p}) = 0$, assume that $\Lambda(\varpi_L) = \gamma$, and if $(\frac{L}{p}) = 1$, assume that $\Lambda(\varpi, 1) = \Lambda(1, \varpi) = \gamma$. Assume that $B_1, B_2 \in \mathcal{S}(\Lambda, \theta, P_1)$ satisfy (64). Then*

$$B_1 = 0 \iff B_1(1) = 0 \iff B_2(1) = 0 \iff B_2 = 0.$$

Proof. The proof is similar to that of Lemma 9.1, $m_0 = 0$ case, and is therefore omitted. See [7] for details. \square

9.3. The main result

The following theorem identifies good test vectors for those irreducible, admissible representations of $\mathrm{GSp}_4(F)$ that are not spherical but have a two-dimensional space of P_1 -invariant vectors. Recall from Table 1 that these are precisely the Iwahori-spherical representations of type IIIa and IVb.

Theorem 9.3. *Let π be an irreducible, admissible, Iwahori-spherical representation of $\mathrm{GSp}_4(F)$ that is not spherical but has a two-dimensional space of P_1 -invariant vectors. Let S be the matrix defined in (1), with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ subject to the conditions (4). Let $T(F)$ be the group defined in (2). Let θ be the character of $U(F)$ defined in (10), and let Λ be a character of $T(F) \cong L^\times$. Let m_0 be as in (31). We assume that π admits a (Λ, θ) -Bessel model. Then the space of P_1 -invariant vectors is spanned by common eigenvectors for the Hecke operators $T_{1,0}$ and $T_{0,1}$, and if B is any such eigenvector, then $B(h(0, m_0)) \neq 0$.*

Proof. Assume first that π is of type IIIa. Then π has a Bessel model with respect to any Λ ; see Table 2. Let B_1 and B_2 be the P_1 -invariant vectors which are common eigenvectors for $T_{1,0}$ and $T_{0,1}$, as in (54). Then $B_1(h(0, m_0)) \neq 0$ by Lemma 9.1. If we replace γ by $\alpha^{-1}\gamma$ and then α by α^{-1} in Eqs. (54), then the roles of B_1 and B_2 get reversed. This symmetry shows that also $B_2(h(0, m_0)) \neq 0$.

Now assume that π is the representation $L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$ of type IVb. Then, by Table 2, we must have $\Lambda = \sigma \circ N_{L/F}$. In particular, Λ is unramified and satisfies the hypotheses of Lemma 9.2. Let B_1 and B_2 be the P_1 -invariant vectors which are common eigenvectors for $T_{1,0}$ and $T_{0,1}$, as in (64). Then $B_1(1) \neq 0$ and $B_2(1) \neq 0$ by Lemma 9.2. \square

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