Bessel models for $\mathrm{GSp}(4)$: Siegel vectors of square-free level

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Abstract. We determine test vectors and explicit formulas for all Bessel models for those Iwahori-spherical representations of GSp_4 over a p-adic field that have non-zero vectors fixed under the Siegel congruence subgroup.

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Introduction

For various classical groups, Bessel models of local or global representations have proven to be a useful substitute for the frequently missing Whittaker model. The uniqueness of Bessel models in the local non-archimedean case has now been established in a wide variety of cases; see [1], [3]. In this work we are only concerned with the group GSp_4 over a p-adic field F. For this group uniqueness of Bessel models was proven as early as 1973, at least for representations with trivial central character; see [5], [6]. A proof of uniqueness for all non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}_4(F)$ can be found in [10].

In contrast to Whittaker models, which are essentially independent of any choices made, Bessel models depend on some arithmetic data. In the case of GSp_4, part of this data is a choice of non-degenerate symmetric 2×2-matrix S over the field F. The discriminant $\mathbf{d}$ of this matrix determines a quadratic extension L of F; this extension may be isomorphic to $F\oplus F$, which will be referred to as the *split case*. The second ingredient entering into the definition of a Bessel model is a character Λ of the multiplicative group $L^\times$.

Now let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}_4(F)$. Given S and Λ, the representation π may or may not have a Bessel model with respect to this data. In the case of

GSp_4, it is possible to precisely say which representations have which Bessel models; see [9], [10]. In particular, every irreducible, admissible representation has a Bessel model for an appropriate choice of S and Λ.

Given π, S and Λ, it is one thing to know that a Bessel model exists, but it is another to identify a good *test vector*. By definition, a test vector is a vector in the space of π on which the relevant Bessel functional is non-zero; note that this is a well-defined notion by the uniqueness of Bessel functionals. Equivalently, in the actual Bessel model consisting of functions B on the group with the Bessel transformation property, B is a test vector if and only if $B(1)$ is non-zero. In this paper, we will identify test vectors for a class of representations that is relevant for the theory of Siegel modular forms of degree 2. In addition, we shall give explicit formulas for the corresponding Bessel functions. See [7], [12] for the Steinberg case and [16] for the spherical case.

More precisely, the class of representations we consider are those that have a non-zero vector invariant under P_1, the Siegel (Γ_0-type) congruence subgroup of level $\mathfrak{p}$. These representations appear as local components of global automorphic representations generated by Siegel modular eigenforms of degree 2 with respect to the congruence subgroup $\Gamma_0(N)$, where N is a square-free positive integer. They fall naturally into thirteen classes, only four of which consist of generic representations (meaning representations that admit a Whittaker model); see our Table 1 below.

Of the thirteen classes, six are actually spherical, meaning they have a non-zero vector invariant under the maximal compact subgroup $\mathrm{GSp}_4(\mathfrak{o})$ (here, $\mathfrak{o}$ is the ring of integers of our local field F). Sugano [16] has given test vectors and explicit formulas in the spherical cases. Our main focus, therefore, is on the seven classes consisting of non-spherical representations with non-zero P_1-fixed vectors. Of those, five classes have a one-dimensional space of P_1-fixed vectors, and two classes have a two-dimensional space of P_1-fixed vectors. The one-dimensional cases require a slightly different treatment from the two-dimensional cases. In all cases, our main tool will be two Hecke operators $T_{1,0}$ and $T_{0,1}$, coming from the double cosets

$$P_1\,\mathrm{diag}(\varpi,\varpi,1,1)P_1 \qquad \text{and} \qquad P_1\,\mathrm{diag}(\varpi^2,\varpi,1,\varpi)P_1.$$

Here, ϖ is a generator of the maximal ideal of $\mathfrak{o}$. Evidently, these operators act on the spaces of P_1-invariant vectors. In Sect. 2 we will calculate their eigenvalues for all of our seven classes of representations. This has nothing to do with Bessel models; we will perform the calculations in standard parabolically induced representations. The results are contained in Table 4. In the one-dimensional cases, trivially, the unique P_1-invariant vector is a common eigenvector for both $T_{1,0}$ and $T_{0,1}$. In the two-dimensional cases, it turns out there is a nice basis consisting of common eigenvectors for $T_{1,0}$ and $T_{0,1}$.

In Sects. 4 and 5 we will apply the two Hecke operators to P_1-invariant Bessel functions B and evaluate at certain elements of $\mathrm{GSp}_4(F)$. Assuming that B is an eigenfunction, this leads to several formulas relating the values of B at various elements of the group; see Lemma 4.1 as an example for this kind of result. The calculations are all based on a GL_2 integration formula, which we establish in Lemma 3.5.

The P_1-invariant element B in π seems, naively, the natural test vector candidate. But, if Λ is ramified, then considering the intersection of P_1 with the Bessel subgroup, it is easy to see that B is clearly not a test vector. Hence, we are led to consider $\pi(h(0,m))B(1) = B(h(0,m))$, where

$$h(0,m) = \mathrm{diag}(\varpi^{2m},\varpi^m,1,\varpi^m).$$

In Sect. 6 we use some of the formulas from the Hecke operator computations to establish a generating series for the values of B at these diagonal elements; see Proposition 6.1. It turns out that there is one "initial element" $h(0, m_0)$, where m_0 is the conductor of the Bessel character Λ; see (65) for the precise definition. If $B(h(0, m_0)) = 0$, then B is zero on *all* diagonal elements.

A generating series like in Proposition 6.1 is precisely the kind of result that will be useful in global applications. However, it still has to be established that $B(h(0, m_0)) \neq 0$; this is the test vector problem, and it is not trivial since B may vanish on all diagonal elements and yet be non-zero. It turns out that, in almost all cases, $B(h(0, m_0)) \neq 0$ as expected; see our main results Theorem 8.3 and Theorem 9.3. Note that this implies that $\pi(h(0, m_0))B$ is a test vector.

However, there is *one* exceptional case, occurring only for split Bessel models of representations of type IIa, and then only for a certain unramified Bessel character Λ. In this very special case, our Bessel function B is non-zero not at the identity, as expected, but at a certain other element; see Theorem 8.3 i) for the precise statement.

To handle this exceptional case, and one other split case for VIa type representations, we require an additional tool besides our Hecke operators. This additional tool are zeta integrals, which are closely related to split Bessel models. We also require part of the theory of paramodular vectors from [11]. Roughly speaking, we take paramodular vectors and "Siegelize" them to obtain P_1-invariant vectors. Calculations with zeta integrals then establish the desired non-vanishing at specific elements. This is the topic of Sect. 7.

The final Sects. 8 and 9 contain our main results. Theorem 8.3 exhibits test vectors in all one-dimensional cases, and Theorem 9.3 exhibits test vectors in all two-dimensional cases. As mentioned above, explicit formulas for these vectors can be found in Proposition 6.1. We mention that, evidently, explicit formulas imply uniqueness of Bessel models. Hence, as a by-product of our calculations, we reprove the uniqueness of these models for the representations under consideration.

1 Basic facts and definitions

Let F be a non-archimedean local field of characteristic zero, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and ϖ a generator of $\mathfrak{p}$. We fix a non-trivial character ψ of F such that ψ is trivial on $\mathfrak{o}$ but non-trivial on $\mathfrak{p}^{-1}$. We let v be the normalized valuation map on F.

As in [2] we fix three elements $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in F such that $\mathbf{d} = \mathbf{b}^2 - 4\mathbf{ac} \neq 0$. Let

$$S = \begin{bmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} \end{bmatrix}, \qquad \xi = \begin{bmatrix} \frac{\mathbf{b}}{2} & \mathbf{c} \\ -\mathbf{a} & \frac{-\mathbf{b}}{2} \end{bmatrix}. \tag{1}$$

Then $F(\xi) = F + F\xi$ is a two-dimensional F-algebra. If $\mathbf{d}$ is not a square in $F^\times$, then $F(\xi)$ is isomorphic to the field $L = F(\sqrt{\mathbf{d}})$ via the map $x + y\xi \mapsto x + y\frac{\sqrt{\mathbf{d}}}{2}$. If $\mathbf{d}$ is a square in $F^\times$, then $F(\xi)$ is isomorphic to $L = F \oplus F$ via $x + y\xi \mapsto (x + y\frac{\sqrt{\mathbf{d}}}{2}, x - y\frac{\sqrt{\mathbf{d}}}{2})$. Let $z \mapsto \bar{z}$ be the obvious involution on L whose fixed point set is F. The determinant map on $F(\xi)$ corresponds to the norm map on L, defined by $N(z) = z\bar{z}$. Let

$$T(F) = \{g \in \mathrm{GL}_2(F) : {}^tgSg = \det(g)S\}. \tag{2}$$

One can check that $T(F) = F(\xi)^\times$, so that $T(F) \cong L^\times$ via the isomorphism $F(\xi) \cong L$. We define the Legendre symbol as

$$\left(\frac{L}{\mathfrak{p}}\right) = \begin{cases} -1 & \text{if } L/F \text{ is an unramified field extension,} \\ 0 & \text{if } L/F \text{ is a ramified field extension,} \\ 1 & \text{if } L = F \oplus F. \end{cases} \tag{3}$$

These three cases are referred to as the *inert case*, *ramified case*, and *split case*, respectively. If L is a field, then let $\mathfrak{o}_L$ be its ring of integers and $\mathfrak{p}_L$ be the maximal ideal of $\mathfrak{o}_L$. If $L = F \oplus F$, then let $\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o}$. Let ϖ_L be a uniformizer in $\mathfrak{o}_L$ if L is a field, and set $\varpi_L = (\varpi, 1)$ if L is not a field. In the field case let v_L be the normalized valuation on L. We fix the following ideal in $\mathfrak{o}_L$,

$$\mathfrak{P} := \mathfrak{p}\mathfrak{o}_L = \begin{cases} \mathfrak{p}_L & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \mathfrak{p}_L^2 & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \mathfrak{p} \oplus \mathfrak{p} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases} \tag{4}$$

Note that $\mathfrak{P}$ is prime only if $\left(\frac{L}{\mathfrak{p}}\right) = -1$. We have $\mathfrak{P}^n \cap \mathfrak{o} = \mathfrak{p}^n$ for all $n \geq 0$. Except in Sect. 7, where we consider certain split cases, we will make the following *standard assumptions*,

- $\mathbf{a}, \mathbf{b} \in \mathfrak{o}$ and $\mathbf{c} \in \mathfrak{o}^\times$.
- If $\mathbf{d} \notin F^{\times 2}$, then $\mathbf{d}$ is a generator of the discriminant of L/F. (5)
- If $\mathbf{d} \in F^{\times 2}$, then $\mathbf{d} \in \mathfrak{o}^\times$.

Under these assumptions, the group $T(\mathfrak{o}) := T(F) \cap \mathrm{GL}_2(\mathfrak{o})$ is isomorphic to $\mathfrak{o}_L^\times$ via the isomorphism $T(F) \cong L^\times$. Furthermore, if we define elements α and ξ_0 of L by

$$\alpha = \begin{cases} \dfrac{\mathbf{b} + \sqrt{\mathbf{d}}}{2\mathbf{c}} & \text{if } L \text{ is a field,} \\ \left(\dfrac{\mathbf{b} + \sqrt{\mathbf{d}}}{2\mathbf{c}}, \dfrac{\mathbf{b} - \sqrt{\mathbf{d}}}{2\mathbf{c}}\right) & \text{if } L = F \oplus F. \end{cases} \tag{6}$$

$$\xi_0 = \begin{cases} \dfrac{-\mathbf{b} + \sqrt{\mathbf{d}}}{2} & \text{if } L \text{ is a field,} \\ \left(\dfrac{-\mathbf{b} + \sqrt{\mathbf{d}}}{2}, \dfrac{-\mathbf{b} - \sqrt{\mathbf{d}}}{2}\right) & \text{if } L = F \oplus F, \end{cases} \tag{7}$$

then, by Lemma 3.1.1 of [8],

$$\mathfrak{o}_L = \mathfrak{o} + \mathfrak{o}\alpha = \mathfrak{o} + \mathfrak{o}\xi_0, \tag{8}$$

Under our assumptions (5) it makes sense to consider the quadratic equation $\mathbf{c}u^2 + \mathbf{b}u + \mathbf{a} = 0$ over the residue class field $\mathfrak{o}/\mathfrak{p}$. The number of solutions of this equation is $\left(\frac{L}{\mathfrak{p}}\right) + 1$. In the ramified case we will fix an element $u_0 \in \mathfrak{o}$ such that

$$\mathbf{c}u_0^2 + \mathbf{b}u_0 + \mathbf{a} \in \mathfrak{p} \qquad \text{(ramified case),} \tag{9}$$

and in the split case we will fix two mod $\mathfrak{p}$ inequivalent elements $u_1, u_2 \in \mathfrak{o}$ such that

$$\mathbf{c}u_i^2 + \mathbf{b}u_i + \mathbf{a} \in \mathfrak{p}, \qquad i = 1, 2 \qquad \text{(split case).} \tag{10}$$

Note that $\mathbf{c}(X+\frac{\mathbf{b}+\sqrt{\mathbf{d}}}{2\mathbf{c}})(X+\frac{\mathbf{b}-\sqrt{\mathbf{d}}}{2\mathbf{c}})=\mathbf{c}X^2+\mathbf{b}X+\mathbf{a}$. We claim that we can arrange the u_i such that

$$v(\mathbf{c}u_i^2+\mathbf{b}u_i+\mathbf{a})=1. \tag{11}$$

Indeed, in the ramified case this is automatic; see Lemma 3.1.1 of [8]. In the split case, let $\tilde{u}_1=-\frac{\mathbf{b}+\sqrt{\mathbf{d}}}{2\mathbf{c}}$ and $\tilde{u}_2=-\frac{\mathbf{b}-\sqrt{\mathbf{d}}}{2\mathbf{c}}$. Let $f(X)=\mathbf{c}X^2+\mathbf{b}X+\mathbf{a}-\varpi$. Then $f'(X)=2\mathbf{c}X+\mathbf{b}$, and

$$|f(\tilde{u}_i)|<|f'(\tilde{u}_i)|^2,\qquad i=1,2.$$

By Hensel's Lemma (see [4], II.2, Proposition 2), there exists $u_i\in\mathfrak{o}$ such that $f(u_i)=0$ and $u_i\equiv\tilde{u}_i \bmod \mathfrak{p}$. This proves our claim.

For later use we note that the elements u_i in (9) and (10) have the property

$$\begin{bmatrix}1&\\-u_i&1\end{bmatrix}T(\mathfrak{o})\begin{bmatrix}1&\\u_i&1\end{bmatrix}\subset\Gamma_0(\mathfrak{p}):=\mathrm{GL}_2(\mathfrak{o})\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}&\mathfrak{o}\end{bmatrix}. \tag{12}$$

Groups

We define the group GSp_4, considered as an algebraic F-group, using the symplectic form

$$J=\begin{bmatrix}&1_2\\-1_2&\end{bmatrix}.$$

Hence, $\mathrm{GSp}_4(F)=\{g\in\mathrm{GL}_4(F):\ {}^tgJg=\mu(g)J\}$, where the scalar $\mu(g)\in F^\times$ is called the *multiplier* of g. Let Z be the center of GSp_4. The Borel subgroup B, Siegel parabolic subgroup P, and Klingen parabolic subgroup Q consist of matrices in GSp_4 of the form

$$B=\begin{bmatrix}*&&*&*\\ *&*&*&*\\ &&*&*\\ &&&*\end{bmatrix},\qquad P=\begin{bmatrix}*&*&*&*\\ *&*&*&*\\ &&*&*\\ &&*&*\end{bmatrix},\qquad Q=\begin{bmatrix}*&&*&*\\ *&*&*&*\\ *&&*&*\\ &&&*\end{bmatrix}, \tag{13}$$

respectively. Let $K=\mathrm{GSp}_4(\mathfrak{o})$ be the standard maximal compact subgroup of $\mathrm{GSp}_4(F)$. The parahoric subgroups corresponding to B, P and Q are the *Iwahori subgroup* I, the *Siegel congruence subgroup* P_1, and the *Klingen congruence subgroup* P_2, given by

$$I=K\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}&\mathfrak{o}\end{bmatrix},\quad P_1=K\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\end{bmatrix},\quad P_2=K\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{o}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}&\mathfrak{o}\end{bmatrix}. \tag{14}$$

We will also have occasion to consider, for a non-negative integer n, the *paramodular group* of level $\mathfrak{p}^n$, defined as

$$K^{\mathrm{para}}(\mathfrak{p}^n)=\{g\in\mathrm{GSp}_4(F):\ g\in\begin{bmatrix}\mathfrak{o}&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\\\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-n}\\\mathfrak{o}&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}\end{bmatrix},\ \det(g)\in\mathfrak{o}^\times\}. \tag{15}$$

We will abbreviate $K^{\rm para}(\mathfrak{p})$ by P_{02}, which is a maximal compact subgroup of $\mathrm{GSp}_4(F)$ not conjugate to K.

The eight-element Weyl group W of GSp_4, defined in the usual way as the normalizer modulo the centralizer of the subgroup of diagonal matrices, is generated by the images of

$$s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \qquad \text{and} \qquad s_2 = \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}. \tag{16}$$

Throughout this work dg denotes the Haar measure on $\mathrm{GSp}_4(F)$ which gives K volume 1. In Sect. 2 we will also use the Haar measure d^Ig which gives the Iwahori subgroup I volume 1. These two measures are related by

$$d^Ig = [K:I]\,dg = (1+q+q^2+q^3)(1+q)\,dg. \tag{17}$$

Bessel models

Let $\mathbf{a},\mathbf{b},\mathbf{c}$, the matrix S and the torus $T(F)$ be as above. We consider $T(F)$ a subgroup of $\mathrm{GSp}_4(F)$ via

$$T(F) \ni g \longmapsto \begin{bmatrix} g & \\ & \det(g)\,{}^tg^{-1} \end{bmatrix} \in \mathrm{GSp}_4(F). \tag{18}$$

Let

$$U(F) = \{\begin{bmatrix} 1_2 & X \\ & 1_2 \end{bmatrix} \in \mathrm{GSp}_4(F) :\ {}^tX = X\}$$

and $R(F) = T(F)U(F)$. We call $R(F)$ the *Bessel subgroup* of $\mathrm{GSp}_4(F)$ (with respect to the given data $\mathbf{a},\mathbf{b},\mathbf{c}$). Let $\theta :\ U(F) \to \mathbb{C}^\times$ be the character given by

$$\theta(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}) = \psi(\mathrm{tr}(SX)), \tag{19}$$

where ψ is our fixed character of F of conductor $\mathfrak{o}$. Explicitly,

$$\theta(\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix}) = \psi(\mathbf{a}x + \mathbf{b}y + \mathbf{c}z). \tag{20}$$

We have $\theta(t^{-1}ut) = \theta(u)$ for all $u \in U(F)$ and $t \in T(F)$. Hence, if Λ is any character of $T(F)$, then the map $tu \mapsto \Lambda(t)\theta(u)$ defines a character of $R(F)$. We denote this character by $\Lambda \otimes \theta$. Let $\mathcal{S}(\Lambda,\theta)$ be the space of all locally constant functions $B :\ \mathrm{GSp}_4(F) \to \mathbb{C}$ with the *Bessel transformation property*

$$B(rg) = (\Lambda\otimes\theta)(r)B(g) \qquad \text{for all } r \in R(F) \text{ and } g \in \mathrm{GSp}_4(F). \tag{21}$$

Our main object of investigation is the subspace $\mathcal{S}(\Lambda,\theta,P_1)$ consisting of functions that are right invariant under P_1. The group $\mathrm{GSp}_4(F)$ acts on $\mathcal{S}(\Lambda,\theta)$ by right translation. If an irreducible, admissible representation (π, V) of $\mathrm{GSp}_4(F)$ is isomorphic to a subrepresentation of $\mathcal{S}(\Lambda,\theta)$, then this realization of π is called a (Λ,θ)-*Bessel model*. It is known by [6] for trivial central character, and by [10] for all non-supercuspidal representations (or all representations if the Bessel model is split), that such a model, if it exists, is unique; we denote it by $\mathcal{B}_{\Lambda,\theta}(\pi)$. Since the Bessel subgroup contains the center, an obvious necessary condition for existence is

$$\Lambda(z) = \omega_\pi(z) \qquad \text{for all } z \in F^\times, \tag{22}$$

where ω_π is the central character of π.

Change of models

Of course, Bessel models can be defined with respect to any non-degenerate symmetric matrix S, not necessarily subject to the conditions (5) we imposed on $\mathbf{a},\mathbf{b},\mathbf{c}$. Since our calculations and explicit formulas will assume these conditions, we shall briefly describe how to switch to more general Bessel models. Hence, let λ be in $F^\times$ and A be in $\mathrm{GL}_2(F)$, and let $S' = \lambda\,{}^tASA$. Replacing S by S' in the definitions (2) and (19), we obtain the group $T'(F)$ and the character θ' of $U(F)$. There is an isomorphism $T'(F) \to T(F)$ given by $t \mapsto AtA^{-1}$. Let Λ' be the character of $T'(F)$ given by $\Lambda'(t) = \Lambda(AtA^{-1})$. For $B \in \mathcal{B}_{\Lambda,\theta}(\pi)$, let

$$B'(g) = B(\begin{bmatrix} A & \\ & \lambda^{-1}\,{}^tA^{-1}\end{bmatrix} g), \qquad g \in \mathrm{GSp}_4(F). \tag{23}$$

It is easily verified that B' has the (Λ',θ')-Bessel transformation property, and that the map $B \mapsto B'$ provides an isomorphism $\mathcal{B}_{\Lambda,\theta}(\pi) \cong \mathcal{B}_{\Lambda',\theta'}(\pi)$.

Let us make a specific change of models in the split case more explicit. Hence, assume that $\mathbf{d} = \mathbf{b}^2 - 4\mathbf{ac}$ is a square in $F^\times$. Let $S = \begin{bmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c}\end{bmatrix}$ be the usual matrix, and let

$$S' = \begin{bmatrix} & 1/2 \\ 1/2 & \end{bmatrix}. \tag{24}$$

Let $T(F)$ be as in (2), and let $T'(F)$ be the analogously defined torus for the matrix S'. Explicitly,

$$T'(F) = \{\begin{bmatrix} a & \\ & d\end{bmatrix} :\ a, d \in F^\times\}. \tag{25}$$

Let

$$A = \frac{1}{\sqrt{\mathbf{d}}}\begin{bmatrix} 1 & -2\mathbf{c} \\ -\frac{1}{2\mathbf{c}}(\mathbf{b}-\sqrt{\mathbf{d}}) & \mathbf{b}+\sqrt{\mathbf{d}}\end{bmatrix}. \tag{26}$$

Then $S' = {}^tASA$, and

$$T'(F) = A^{-1}T(F)A. \tag{27}$$

Coming back to arbitrary S, the following lemma shows that there is no restriction of generality in assuming that the elements $\mathbf{a},\mathbf{b},\mathbf{c}$ satisfy the standard assumptions (5).

1.1 Lemma. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in F$ *be such that* $\mathbf{d} = \mathbf{b}^2 - 4\mathbf{ac}$ *is non-zero. Set* $S = \begin{bmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} \end{bmatrix}$. *Then there exist* $\lambda \in F^\times$ *and* $A \in \mathrm{GL}_2(F)$ *such that the elements* $\mathbf{a}', \mathbf{b}', \mathbf{c}'$, *defined by*

$$\begin{bmatrix} \mathbf{a}' & \frac{\mathbf{b}'}{2} \\ \frac{\mathbf{b}'}{2} & \mathbf{c}' \end{bmatrix} = \lambda\, {}^tASA,$$

satisfy the standard assumptions (5).

Proof. First assume the split case, i.e., $\mathbf{d} \in F^{\times 2}$. Then, as shown above, there exists $A \in \mathrm{GL}_2(F)$ such that $S' = {}^tASA$ is the matrix in (24). A further transformation using $A = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ leads to $S'' = \begin{bmatrix} & 1/2 \\ 1/2 & 1 \end{bmatrix}$. Hence, the standard assumptions are satisfied for S''.

From now on assume that $\mathbf{d} \notin F^{\times 2}$, so that $L = F(\sqrt{\mathbf{d}})$ is a field. Since every quadratic form can be diagonalized, a suitable transformation yields a matrix

$$S' = \begin{bmatrix} a & \\ & c \end{bmatrix}. \tag{28}$$

For transparency, consider first the case of odd residual characteristic. Since we are allowed to scale and permute the diagonal elements, we may assume that in (28) we have $c = 1$ and $v(a) \in \{0, 1\}$. If $v(a) = 0$, then $d = -4ac$ is a unit, and L is the unramified quadratic extension of F. Hence, the standard assumptions (5) are satisfied in this case. If $v(a) = 1$, then $L = F(\sqrt{d})$ is a ramified quadratic extension of F. The exponent of the discriminant in this case is 1, so that again (5) is satisfied.

Now assume that F has even residual characteristic. Again, we may assume that $c = 1$. We will modify a below to have a certain valuation. In any case $L = F(\sqrt{-4a}) = F(\sqrt{-a})$. Using some algebraic number theory, one can prove that there are two possibilities for the quadratic extension L of F:

(A) The exponent of the discriminant of L/F equals $2j$ for some $j \in \{0, 1, \ldots, v(2)\}$. In this case $L = F(\sqrt{u})$ for a unit u which may be chosen from $1 + \varpi^{2(v(2)-j)}\mathfrak{o}$.

(B) The exponent of the discriminant of L/F equals $2v(2) + 1$. In this case $L = F(\sqrt{\varpi u})$ for some unit u.

Assume we are in case (A). Since $F(\sqrt{-a}) = F(\sqrt{u})$, we have $a = -y^2 u$ for some $y \in F^\times$. We may multiply a by squares, and may hence assume that $a = -\varpi^{2(j-v(2))}u$. Then

$$\begin{bmatrix} 1 & \varpi^{j-v(2)} \\ & 1 \end{bmatrix} \begin{bmatrix} a & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi^{j-v(2)} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}' & \frac{\mathbf{b}'}{2} \\ \frac{\mathbf{b}'}{2} & 1 \end{bmatrix} \tag{29}$$

with $\mathbf{a}' = \varpi^{2(j-v(2))}(1-u) \in \mathfrak{o}$ and $\mathbf{b}' = 2\varpi^{j-v(2)} \in \mathfrak{o}$. The valuation of the discriminant of the matrix in (29) is $2j$, so that the standard assumptions (5) are satisfied for this matrix.

Finally, assume we are in case (B). Then the valuation of a is odd, and we may assume that $v(a) = 1$. The valuation of the discriminant of the matrix in (28) is then $2v(2) + 1$, so that the assumptions (5) are satisfied without further transformation. ∎

The Iwahori-spherical representations of $\mathrm{GSp}_4(F)$ and their Bessel models

Table 1 below is a reproduction of Table A.15 of [11]. It lists all the irreducible, admissible representations of $\mathrm{GSp}_4(F)$ that have a non-zero fixed vector under the Iwahori subgroup I, using notation borrowed from [13]. As is well-known, any such representation is a subquotient of a representation parabolically induced from an unramified character of the Borel subgroup. Hence, in Table 1, all characters are assumed to be unramified. The symbol ν stands for the absolute value on $F^\times$, normalized such that $\nu(\varpi) = q^{-1}$, and ξ stands for the non-trivial, unramified, quadratic character of $F^\times$.

Also listed in Table 1 are the conductor $a(\pi)$ and the value of the ε-factor at $1/2$ for each representation π (we assume that the additive character that enters into the definition of the ε-factor is our fixed character ψ of conductor $\mathfrak{o}$). Note that the full ε-factor is given by

$$\varepsilon(s,\pi) = \varepsilon(1/2,\pi)q^{-a(\pi)(s-1/2)}. \tag{30}$$

Further, Table 1 lists the dimensions of the spaces of fixed vectors under the various parahoric subgroups; every parahoric subgroup is conjugate to exactly one of K, P_{02}, P_2, P_1 or I. In this paper we are interested in the representations that have a non-zero P_1-invariant vector. Except for the one-dimensional representations, these are the following:

- I, IIb, IIIb, IVd, Vd and VId. These are the *spherical* representations, meaning they have a K-invariant vector.
- IIa, IVc, Vb, VIa and VIb. These are non-spherical representations that have a one-dimensional space of P_1-fixed vectors. Note that Vc is simply a twist of Vb by the character ξ, so we will not list it separately.
- IIIa and IVb. These are the non-spherical representations that have a two-dimensional space of P_1-fixed vectors.

Let π be any irreducible, admissible representation of $\mathrm{GSp}_4(F)$. Let S be a matrix as in (1), and θ the associated character of $U(F)$; see (19). Given this data, one may ask for which characters Λ of the torus $T(F)$, defined in (2) and embedded into $\mathrm{GSp}_4(F)$ via (18), the representation π admits a (Λ,θ)-Bessel model. This question can be answered, based on results from [9] and [10]. We have listed the data relevant for our current purposes in Table 2 (reproduced from [10]). Note that, in this table, the characters $\chi,\chi_1,\chi_2,\sigma,\xi$ are not necessarily assumed to be unramified, i.e., these results hold for all Borel-induced representations. For the split case $L = F\oplus F$ in Table 2, it is assumed that the matrix S is the one in (24). The resulting torus $T(F)$ is given in (25); embedded into $\mathrm{GSp}_4(F)$ it consists of all matrices $\mathrm{diag}(a,b,b,a)$ with a,b in $F^\times$. We refer to [10] for additional details and results.

2 Hecke eigenvalues

For integers l and m, let

$$h(l,m) = \begin{bmatrix} \varpi^{l+2m} &&& \\ & \varpi^{l+m} && \\ && 1 & \\ &&& \varpi^m \end{bmatrix}. \tag{31}$$

Table 1: The Iwahori-spherical representations of $\mathrm{GSp}_4(F)$ and the dimensions of their spaces of fixed vectors under the parahoric subgroups. The symbols K, P_{02}, P_2, P_1 and I stand for $\mathrm{GSp}_4(\mathfrak{o})$, the paramodular group of level $\mathfrak{p}$, the Klingen congruence subgroup of level $\mathfrak{p}$, the Siegel congruence subgroup of level $\mathfrak{p}$, and the Iwahori subgroup, respectively. Also listed are the conductor $a(\pi)$ and the value of the ε-factor at $1/2$.

		π	$a(\pi)$	$\varepsilon(1/2,\pi)$	K	P_{02}	P_2	P_1	I
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	0	1	1	$\underset{+-}{2}$	4	$\underset{\substack{++\\--}}{4}$	$\underset{\substack{++++\\----}}{8}$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	1	$-(\sigma\chi)(\varpi)$	0	$\underset{-}{1}$	2	$\underset{-}{1}$	$\underset{+---}{4}$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	0	1	1	$\underset{+}{1}$	2	$\underset{++-}{3}$	$\underset{+++-}{4}$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	2	1	0	0	1	$\underset{+-}{2}$	$\underset{++--}{4}$
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	0	1	1	$\underset{+-}{2}$	3	$\underset{+-}{2}$	$\underset{++--}{4}$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	3	$-\sigma(\varpi)$	0	0	0	0	$\underset{-}{1}$
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	2	1	0	0	1	$\underset{+-}{2}$	$\underset{++-}{3}$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	2	$\underset{-}{1}$	$\underset{+--}{3}$
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	0	1	1	$\underset{+}{1}$	1	$\underset{+}{1}$	$\underset{+}{1}$
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	2	-1	0	0	1	0	$\underset{+-}{2}$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	1	$\sigma(\varpi)$	0	$\underset{+}{1}$	1	$\underset{+}{1}$	$\underset{++}{2}$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	1	$\underset{-}{1}$	$\underset{--}{2}$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	0	1	1	0	1	$\underset{+-}{2}$	$\underset{+-}{2}$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	2	1	0	0	1	$\underset{-}{1}$	$\underset{+--}{3}$
	b	$\tau(T, \nu^{-1/2}\sigma)$	2	1	0	0	0	$\underset{+}{1}$	$\underset{+}{1}$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	1	0	$\underset{-}{1}$
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	0	1	1	$\underset{+}{1}$	2	$\underset{+-}{2}$	$\underset{++-}{3}$

Table 2: The Bessel models of the irreducible, admissible representations of $\mathrm{GSp}_4(F)$ that can be obtained via induction from the Borel subgroup. The symbol N stands for the norm map from $L^\times \cong T(F)$ to $F^\times$.

		representation	(Λ,θ)-Bessel functional exists exactly for … $L = F \oplus F$	(Λ,θ)-Bessel functional exists exactly for … L/F a field extension
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	all Λ	all Λ
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	all Λ	$\Lambda \neq (\chi\sigma) \circ N$
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	$\Lambda = (\chi\sigma) \circ N$	$\Lambda = (\chi\sigma) \circ N$
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	all Λ	all Λ
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$	$\Lambda(\mathrm{diag}(a,b,b,a)) =$ $\chi(a)\sigma(ab)$ or $\chi(b)\sigma(ab)$	—
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	all Λ	$\Lambda \neq \sigma \circ N$
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	$\Lambda = \sigma \circ N$	$\Lambda = \sigma \circ N$
	c	$L(\nu^{3/2} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\Lambda(\mathrm{diag}(a,b,b,a)) =$ $\nu(ab^{-1})\sigma(ab)$ or $\nu(a^{-1}b)\sigma(ab)$	—
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(4)}$	—	—
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	all Λ	$\Lambda \neq \sigma \circ N,\ \Lambda \neq (\xi\sigma) \circ N$
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\Lambda = \sigma \circ N$	$\Lambda = \sigma \circ N,\ \Lambda \neq (\xi\sigma) \circ N$
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$\Lambda = (\xi\sigma) \circ N$	$\Lambda \neq \sigma \circ N,\ \Lambda = (\xi\sigma) \circ N$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	—	$\Lambda = \sigma \circ N,\ \Lambda = (\xi\sigma) \circ N$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	all Λ	$\Lambda \neq \sigma \circ N$
	b	$\tau(T, \nu^{-1/2}\sigma)$	—	$\Lambda = \sigma \circ N$
	c	$L(\nu^{1/2} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\Lambda = \sigma \circ N$	—
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	$\Lambda = \sigma \circ N$	—

Let (π, V) be a smooth representation of $\mathrm{GSp}_4(F)$ for which $Z(\mathfrak{o})$ acts trivially. We define two endomorphisms of V by the formulas

$$T_{1,0}v = \frac{1}{\mathrm{vol}(P_1)} \int\limits_{P_1 h(1,0) P_1} \pi(g) v \, dg, \tag{32}$$

$$T_{0,1}v = \frac{1}{\mathrm{vol}(P_1)} \int\limits_{P_1 h(0,1) P_1} \pi(g) v \, dg. \tag{33}$$

Evidently, $T_{1,0}$ and $T_{0,1}$ induce endomorphisms of the subspace of V consisting of P_1-invariant vectors. Our goal in this section is to determine the eigenvalues of $T_{1,0}$ and $T_{0,1}$ on the space of P_1-invariant vectors for the representations in Table 1 which have non-zero P_1-invariant vectors, but no non-zero K-invariant vectors. The main results are listed in Table 4. This section has nothing to do with Bessel models.

We will not give all the details of the eigenvalue calculation, since the method is similar to the one employed in [14]. Essentially, the idea is to express $T_{1,0}$ and $T_{0,1}$ in terms of the standard generators of the Iwahori-Hecke algebra, and do explicit calculations in induced models. We will therefore briefly recall the structure of the Iwahori-Hecke algebra $\mathcal{I}$, which is the convolution algebra of compactly supported left and right I-invariant functions on $\mathrm{GSp}_4(F)$. Explicitly, for T and T' in $\mathcal{I}$, their product is given by

$$(T \cdot T')(x) = \int\limits_{\mathrm{GSp}_4(F)} T(xy^{-1}) T'(y) \, d^I y.$$

Here, $d^I y$ is the Haar measure on $\mathrm{GSp}_4(F)$ which gives I volume 1; see (17). The characteristic function of I is the identity element of $\mathcal{I}$; we denote it by e. The group I is normalized by the *Atkin-Lehner element*

$$\eta = \begin{bmatrix} &&&-1\\ &&1&\\ &\varpi&&\\ -\varpi&&& \end{bmatrix} = s_2 s_1 s_2 \begin{bmatrix} \varpi&&&\\ &-\varpi&&\\ &&1&\\ &&&-1 \end{bmatrix}. \tag{34}$$

The characteristic function of ηI is an element of $\mathcal{I}$, which we denote again by η. In addition to the usual Weyl group elements s_1 and s_2 defined in (16), we let $s_0 = \eta s_2 \eta^{-1}$. For $j = 0, 1, 2$ let e_i be the characteristic function of $I s_i I$. Then $\mathcal{I}$ is generated by e_0, e_1, e_2 and η, and we have the following relations.

- $e_i^2 = (q-1)e_i + qe$ for $i = 0, 1, 2$.
- $\eta e_0 \eta^{-1} = e_2, \quad \eta e_1 \eta^{-1} = e_1, \quad \eta e_2 \eta^{-1} = e_0.$
- $e_0 e_1 e_0 e_1 = e_1 e_0 e_1 e_0, \quad e_1 e_2 e_1 e_2 = e_2 e_1 e_2 e_1, \quad e_0 e_2 = e_2 e_0.$

If (π, V) is a smooth representation of $\mathrm{GSp}_4(F)$, let V^I be its subspace of I-invariant vectors. The Iwahori-Hecke algebra $\mathcal{I}$ acts on V^I by

$$Tv = \int\limits_{\mathrm{GSp}_4(F)} T(x) \pi(x) v \, d^I x.$$

Let $d_1 := (q+1)^{-1}(e+e_1)$. Then d_1 acts on V^I as the projection to the subspace of P_1-invariant vectors. It is not difficult to see that, as endomorphisms of V^I, the operators defined in (32) and (33) can be expressed as elements of the Iwahori-Hecke algebra as follows,

$$T_{1,0} = d_1 e_2 e_1 e_2 \eta d_1, \tag{35}$$
$$T_{0,1} = (1+q^{-1}) d_1 e_2 e_1 e_0 e_1 d_1. \tag{36}$$

The next step is to calculate the action of the generators e_0, e_1, e_2 and η on the space of I-invariant vectors in certain induced representations. Hence, let χ_1, χ_2, σ be unramified characters of $F^\times$, and let V be the standard space of the induced representation $\chi_1 \times \chi_2 \rtimes \sigma$. Let W be the eight-element Weyl group, generated by s_1 and s_2. The eight-dimensional space V^I has the basis f_w, $w \in W$, where f_w is the unique I-invariant function with $f_w(w) = 1$ and $f_w(w') = 0$ for $w' \in W$, $w' \neq w$. It is convenient to order the basis as follows:

$$f_e, \quad f_1, \quad f_2, \quad f_{21}, \quad f_{121}, \quad f_{12}, \quad f_{1212}, \quad f_{212}, \tag{37}$$

where we have abbreviated $f_1 = f_{s_1}$ and so on. Having fixed this basis, the operators e_0, e_1, e_2 and η on V^I become 8×8–matrices. These are given in the following lemma.

2.1 Lemma. *Let notations be as above. With respect to the basis (37) of V^I, the action of the elements e_1 and e_2 on V^I is given by the following matrices.*

$$\pi(e_1) = \begin{pmatrix} 0 & q &&&&&& \\ 1 & q-1 &&&&&& \\ && 0 & q &&&& \\ && 1 & q-1 &&&& \\ &&&& q-1 & 1 && \\ &&&& q & 0 && \\ &&&&&& q-1 & 1 \\ &&&&&& q & 0 \end{pmatrix}, \quad \pi(e_2) = \begin{pmatrix} 0&0&q&0&0&0&0&0 \\ 0&0&0&0&0&q&0&0 \\ 1&0&q-1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&q \\ 0&0&0&0&0&0&q&0 \\ 0&1&0&0&0&q-1&0&0 \\ 0&0&0&0&1&0&q-1&0 \\ 0&0&0&1&0&0&0&q-1 \end{pmatrix}$$

The action of η is given by

$$\pi(\eta) = \begin{pmatrix} &&&&&&& \gamma q^{3/2} \\ &&&&&& \gamma q^{3/2} & \\ &&&&& \alpha\gamma q^{1/2} && \\ &&&& \alpha\gamma q^{1/2} &&& \\ &&& \beta\gamma q^{-1/2} &&&& \\ && \beta\gamma q^{-1/2} &&&&& \\ & \alpha\beta\gamma q^{-3/2} &&&&&& \\ \alpha\beta\gamma q^{-3/2} &&&&&&& \end{pmatrix}.$$

The action of e_0 is given by the matrix $\pi(\eta)\pi(e_2)\pi(\eta)^{-1}$.

This lemma follows easily from calculations in the standard model V of $\chi_1 \times \chi_2 \rtimes \sigma$. A basis for the four-dimensional space V^{P_1} is given by

$$f_e + f_1, \qquad f_2 + f_{21}, \qquad f_{121} + f_{12}, \qquad f_{1212} + f_{212}. \tag{38}$$

For χ and σ characters of $F^\times$, there are exact sequences

$$0 \longrightarrow \chi 1_{\mathrm{GL}(2)} \rtimes \sigma \longrightarrow \nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma \longrightarrow \chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma \longrightarrow 0$$

and

$$0 \longrightarrow \chi \rtimes \sigma 1_{\mathrm{GSp}(2)} \longrightarrow \nu^{-1} \times \chi \rtimes \nu^{1/2}\sigma \longrightarrow \chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)} \longrightarrow 0.$$

We assume that χ and σ are unramified, and consider the corresponding sequences of P_1-fixed vectors,

$$0 \longrightarrow \underbrace{(\chi 1_{\mathrm{GL}(2)} \rtimes \sigma)^{P_1}}_{3-\mathrm{dim.}} \longrightarrow \underbrace{(\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma)^{P_1}}_{4-\mathrm{dim.}} \longrightarrow \underbrace{(\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma)^{P_1}}_{1-\mathrm{dim.}} \longrightarrow 0$$

and

$$0 \longrightarrow \underbrace{(\chi \rtimes \sigma 1_{\mathrm{GSp}(2)})^{P_1}}_{2-\mathrm{dim.}} \longrightarrow \underbrace{(\nu^{-1} \times \chi \rtimes \nu^{1/2}\sigma)^{P_1}}_{4-\mathrm{dim.}} \longrightarrow \underbrace{(\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)})^{P_1}}_{2-\mathrm{dim.}} \longrightarrow 0.$$

Using (38), one can show that a basis for the three-dimensional space $(\chi 1_{\mathrm{GL}(2)} \rtimes \sigma)^{P_1}$ is given by

$$f_e + f_1, \qquad f_2 + f_{21} + f_{121} + f_{12}, \qquad f_{1212} + f_{212}, \tag{39}$$

and a basis for the two-dimensional space $(\chi \rtimes \sigma 1_{\mathrm{GSp}(2)})^{P_1}$ is given by

$$f_e + f_1 + f_2 + f_{21}, \qquad f_{121} + f_{12} + f_{1212} + f_{212}. \tag{40}$$

Combining (35), (36), Lemma 2.1 and (38), it is easy to calculate the matrices of $T_{1,0}$ and $T_{0,1}$ on the space $(\chi_1 \times \chi_2 \rtimes \sigma)^{P_1}$. Using (39) and (40), one can similarly calculate the matrices of these operators on the spaces $(\chi 1_{\mathrm{GL}(2)} \rtimes \sigma)^{P_1}$ and $(\chi \rtimes \sigma 1_{\mathrm{GSp}(2)})^{P_1}$. The results of these calculations are listed in Table 3.

The main result of this section is Table 4; the accompanying Table 5 explains the notation. For the most part, the eigenvalues in this table can easily be determined from the matrices given in Table 3, together with well-known information on how Borel-induced representations decompose. For the latter, see Sect. 2.2 of [11]. (The Atkin-Lehner eigenvalues in Table 4 can also be read off Table A.15 of [11].). The only necessary additional ingredient are the following two pieces of information.

- Consider the representation $L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ of type Vb, which can be realized as a subrepresentation of $\nu^{1/2}\xi 1_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2}$. Here, ξ is the non-trivial, unramified, quadratic character of $F^\times$. The functions f in this model satisfy the transformation property

  $$f(\begin{bmatrix} A & * \\ & u\,{}^tA^{-1} \end{bmatrix} g) = \sigma(u)\,\xi(u^{-1}\det(A))|u^{-1}\det(A)|^2 f(g).$$

  Then a specific function f spanning the one-dimensional space $L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)^{P_1}$ is given by

  $$f(1) = -q^2(q+1), \qquad f(s_2) = -q(q-1), \qquad f(s_2 s_1 s_2) = q+1.$$

Table 3: Matrices of the operators $T_{1,0}$ and $T_{0,1}$ when acting on the four-dimensional space $(\chi_1\times\chi_2\rtimes\sigma)^{P_1}$ with respect to the basis (38), and on the three-dimensional space $(\chi 1_{\mathrm{GL}(2)}\rtimes\sigma)^{P_1}$ with respect to the basis (39), and on the two-dimensional space $(\chi\rtimes\sigma 1_{\mathrm{GSp}(2)})^{P_1}$ with respect to the basis (40). In the first case, the notations are such that $\alpha=\chi_1(\varpi)$, $\beta=\chi_2(\varpi)$ and $\gamma=\sigma(\varpi)$. In the other cases, we used the abbreviations $\alpha=\chi(\varpi)$ and $\gamma=\sigma(\varpi)$.

operator	matrix on $(\chi_1\times\chi_2\rtimes\sigma)^{P_1}$
$T_{1,0}$	$\begin{bmatrix}\alpha\beta\gamma q^{3/2} & & & \\ \alpha\beta\gamma(q-1)q^{1/2} & \beta\gamma q^{3/2} & & \\ \alpha\beta\gamma(q-1)q^{1/2} & \beta\gamma(q-1)q^{1/2} & \alpha\gamma q^{3/2} & \\ \alpha\beta\gamma(q-1)q^{1/2} & \beta\gamma(q-1)q^{1/2} & \alpha\gamma(q-1)q^{1/2} & \gamma q^{3/2}\end{bmatrix}$
$T_{0,1}$	$\alpha\beta\gamma^2\begin{bmatrix}(\alpha+\beta)q^2 & & & \\ (\alpha+1)q(q-1) & (\alpha^{-1}+\beta)q^2 & & \\ (\beta+q)(q-1) & (\alpha^{-1}+\beta)q(q-1) & (\alpha+\beta^{-1})q^2 & \\ (\alpha+\beta)(q-1) & (\beta q+1)(q-1) & (\alpha+1)q(q-1) & (\alpha^{-1}+\beta^{-1})q^2\end{bmatrix}$
	matrix on $(\chi 1_{\mathrm{GL}(2)}\rtimes\sigma)^{P_1}$
$T_{1,0}$	$\begin{bmatrix}\alpha^2\gamma q^{3/2} & & \\ \alpha^2\gamma(q-1)q^{1/2} & \alpha\gamma q^2 & \\ \alpha^2\gamma(q-1)q^{1/2} & \alpha\gamma(q^2-1) & \gamma q^{3/2}\end{bmatrix}$
$T_{0,1}$	$\alpha^2\gamma^2\begin{bmatrix}\alpha(q+1)q^{3/2} & & \\ (\alpha q^{1/2}+q)(q-1) & (\alpha+\alpha^{-1})q^{5/2} & \\ \alpha(q^2-1)q^{-1/2} & (\alpha q^{1/2}+1)(q^2-1) & \alpha^{-1}(q+1)q^{3/2}\end{bmatrix}$
	matrix on $(\chi\rtimes\sigma 1_{\mathrm{GSp}(2)})^{P_1}$
$T_{1,0}$	$\begin{bmatrix}\alpha\gamma q^2 & \\ \alpha\gamma(q^2-1) & \gamma q^2\end{bmatrix}$
$T_{0,1}$	$\alpha\gamma^2\begin{bmatrix}(\alpha+q)q^2 & \\ (\alpha+1)(q^2-1) & (\alpha^{-1}+q)q^2\end{bmatrix}$

Table 4: Eigenvalues of the operators $T_{1,0}$, $T_{0,1}$ and η on spaces of P_1-invariant vectors in irreducible representations. Listed are those Iwahori-spherical representations which have non-zero P_1-invariant vectors, but no non-zero K-invariant vectors.

type	dim	$T_{1,0}$	$T_{0,1}$	η
IIa	1	$\alpha\gamma q$	$\alpha^2\gamma^2(\alpha+\alpha^{-1})q^{3/2}$	$-\alpha\gamma$
IIIa	2	$\alpha\gamma q,\ \gamma q$	$\alpha\gamma^2(\alpha q+1)q,\ \alpha\gamma^2(\alpha^{-1}q+1)q$	$\pm\sqrt{\alpha}\gamma$
IVb	2	$\gamma,\ \gamma q^2$	$\gamma^2(q+1),\ \gamma^2 q(q^3+1)$	$\pm\gamma$
IVc	1	γq	$\gamma^2(q^3+1)$	$-\gamma$
Vb	1	$-\gamma q$	$-\gamma^2 q(q+1)$	γ
VIa	1	γq	$\gamma^2 q(q+1)$	$-\gamma$
VIb	1	γq	$\gamma^2 q(q+1)$	γ

Table 5: Notation for Satake parameters for those Iwahori-spherical representations listed in Table 4. The "restrictions" column reflects the fact that certain characters χ are not allowed in type IIa and IIIa representations. The last column shows the central character of the representation.

type	representation	parameters	restrictions	c.c.
IIa	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$\alpha=\chi(\varpi),\ \gamma=\sigma(\varpi)$	$\alpha^2\neq q^{\pm1},\ \alpha\neq q^{\pm3/2}$	$\chi^2\sigma^2$
IIIa	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\alpha=\chi(\varpi),\ \gamma=\sigma(\varpi)$	$\alpha\neq1,\ \alpha\neq q^{\pm2}$	$\chi\sigma^2$
IVb	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\gamma=\sigma(\varpi)$		σ^2
IVc	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$\gamma=\sigma(\varpi)$		σ^2
Vb	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\gamma=\sigma(\varpi)$		σ^2
VIa	$\tau(S,\nu^{-1/2}\sigma)$	$\gamma=\sigma(\varpi)$		σ^2
VIb	$\tau(T,\nu^{-1/2}\sigma)$	$\gamma=\sigma(\varpi)$		σ^2

- Consider the representation $\tau(T,\nu^{-1/2}\sigma)$ of type VIb, realized as a subrepresentation of $\nu^{1/2}1_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma$. The functions f in this model satisfy the transformation property

$$f(\begin{bmatrix}A&*\\&u\,{}^tA^{-1}\end{bmatrix}g)=\sigma(u)\,|u^{-1}\det(A)|^2f(g).$$

Then a specific function f spanning the one-dimensional space $\tau(T,\nu^{-1/2}\sigma)^{P_1}$ is given by

$$f(1)=q^2,\qquad f(s_2)=-q,\qquad f(s_2s_1s_2)=1.$$

This was calculated in [15], Corollary 2.6 (note that there it was assumed that $\sigma=1$, but the formulas hold for non-trivial σ as well, which can easily be seen by twisting).

We make one more comment on the eigenvalues in Table 4, concerning the representations where the space of P_1 invariant vectors is two-dimensional. We will prove below that the operators $T_{1,0}$ and $T_{0,1}$ commute, so that there exists a basis of common eigenvectors. The ordering for types IIIa and IVb in Table 4 is such that *the first eigenvalue for $T_{1,0}$ corresponds to the first eigenvalue for $T_{0,1}$, and the second eigenvalue for $T_{1,0}$ corresponds to the second eigenvalue for $T_{0,1}$*. This follows from the following two facts.

- Consider the two 4×4 matrices of $T_{1,0}$ and $T_{0,1}$ on the space $(\chi\times\nu\rtimes\nu^{-1/2}\sigma)^{P_1}$. The representation $\chi\times\nu\rtimes\nu^{-1/2}\sigma$ contains $\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$ (type IIIa) as a subrepresentation. Let

$$v_1=\begin{bmatrix}q^2(\alpha-1)\\ \alpha q(q-1)\\ 1-\alpha q\\ 0\end{bmatrix},\qquad v_2=\begin{bmatrix}0\\ (1-\alpha q)q\\ q-1\\ \alpha-1\end{bmatrix}. \tag{41}$$

  Then

$$\begin{aligned} T_{1,0}\,v_1&=\alpha\gamma q\,v_1, & T_{1,0}\,v_2&=\gamma q\,v_2,\\ T_{0,1}\,v_1&=\alpha\gamma^2(\alpha q+1)q\,v_1, & T_{0,1}\,v_2&=\alpha\gamma^2(\alpha^{-1}q+1)q\,v_2,\\ \eta\,v_1&=\alpha\gamma\,v_2, & \eta\,v_2&=\gamma\,v_1. \end{aligned} \tag{42}$$

- Consider the two 3×3 matrices of $T_{1,0}$ and $T_{0,1}$ on the space $(\nu^{3/2}1_{\mathrm{GL}(2)}\rtimes\nu^{-3/2}\sigma)^{P_1}$. The representation $\nu^{3/2}1_{\mathrm{GL}(2)}\rtimes\nu^{-3/2}\sigma$ contains $L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$ (type IVb) as a subrepresentation. Let

$$v_1=\begin{bmatrix}(q+1)q^3\\ -q^2\\ 0\end{bmatrix},\qquad v_2=\begin{bmatrix}0\\ -q^2\\ q+1\end{bmatrix} \tag{43}$$

  Then

$$\begin{aligned} T_{1,0}\,v_1&=\gamma\,v_1, & T_{1,0}\,v_2&=\gamma q^2\,v_2,\\ T_{0,1}\,v_1&=\gamma^2(q+1)\,v_1, & T_{0,1}\,v_2&=\gamma^2 q(q^3+1)\,v_2,\\ \eta\,v_1&=\gamma\,v_2, & \eta\,v_2&=\gamma\,v_1. \end{aligned} \tag{44}$$

Commuting lemma

While the Hecke operators $T_{1,0}$ and $T_{0,1}$ do not commute as elements of the Iwahori-Hecke algebra, they do commute as endomorphisms on spaces of P_1-invariant vectors. We do not really need this fact in general, as it follows from the above calculations for the cases of interest to us. However, we include the lemma below for completeness. It makes use of the double coset decompositions (91) and (94) further below.

2.2 Lemma. *Let (π,V) be a smooth representation of $\mathrm{GSp}_4(F)$ for which the center acts trivially. Then the operators $T_{1,0}$ and $T_{0,1}$ commute as endomorphisms of the space of P_1-invariant vectors.*

Proof. Let $v\in V$ be P_1-invariant. We have $T_{1,0}T_{0,1}v=A+B$ with

$$A=\sum_{\substack{x,y,z,u,y'\in\mathfrak{o}/\mathfrak{p}\\ x'\in\mathfrak{o}/\mathfrak{p}^2}}\pi(\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}$$

$$\begin{bmatrix}1&u&&\\&1&&\\&&1&\\&&-u&1\end{bmatrix}\begin{bmatrix}1&&x'&y'\\&1&y'&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})v,$$

$$B=\sum_{\substack{x,y,z,y'\in\mathfrak{o}/\mathfrak{p}\\ z'\in\mathfrak{o}/\mathfrak{p}^2}}\pi(\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}$$
$$\begin{bmatrix}1&&&y'\\&1&y'&z'\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})v,$$

and $T_{0,1}T_{1,0}v=C+D$ with

$$C=\sum_{\substack{x,y,z,u,y'\in\mathfrak{o}/\mathfrak{p}\\ x'\in\mathfrak{o}/\mathfrak{p}^2}}\pi(\begin{bmatrix}1&u&&\\&1&&\\&&1&\\&&-u&1\end{bmatrix}\begin{bmatrix}1&&x'&y'\\&1&y'&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}$$
$$\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v,$$

$$D=\sum_{\substack{x,y,z,y'\in\mathfrak{o}/\mathfrak{p}\\ z'\in\mathfrak{o}/\mathfrak{p}^2}}\pi(\begin{bmatrix}1&&&y'\\&1&y'&z'\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}$$
$$\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v.$$

It is straightforward to show that

$$A=C=\sum_{\substack{u,z\in\mathfrak{o}/\mathfrak{p}\\ y\in\mathfrak{o}/\mathfrak{p}^2\\ x\in\mathfrak{o}/\mathfrak{p}^3}}\pi(\begin{bmatrix}1&u&&\\&1&&\\&&1&\\&&-u&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^3&&&\\&\varpi^2&&\\&&1&\\&&&\varpi\end{bmatrix})v$$

and

$$B=D=\sum_{\substack{x\in\mathfrak{o}/\mathfrak{p}\\ y\in\mathfrak{o}/\mathfrak{p}^2\\ z\in\mathfrak{o}/\mathfrak{p}^3}}\pi(\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi^3&&\\&&\varpi&\\&&&1\end{bmatrix})v.$$

This concludes the proof. ■

3 Double cosets, an integration formula and automatic vanishing

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be as in Sect. 1, subject to the conditions (5). Let $T(F)$ be the subgroup of $\mathrm{GL}_2(F)$ defined in (2). By [16], Lemma 2-4, there is a disjoint decomposition

$$\mathrm{GL}_2(F) = \bigsqcup_{m=0}^{\infty} T(F) \begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} \mathrm{GL}_2(\mathfrak{o}) \tag{45}$$

(here it is important that our assumptions (5) on $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are in force; for example, (45) would obviously be wrong for $\mathbf{a} = \mathbf{c} = 0$). The following two lemmas provide further decompositions for the group $\mathrm{GL}_2(\mathfrak{o})$. We will use the notations

$$\Gamma_0(\mathfrak{p}) = \mathrm{GL}_2(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{bmatrix} \qquad \text{and} \qquad \Gamma^0(\mathfrak{p}) = \mathrm{GL}_2(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} \end{bmatrix}. \tag{46}$$

Recall that, since $T(F) = F(\xi)^\times$ with ξ as in (1), the group $T(F)$ consists of all matrices

$$g_{x,y} = \begin{bmatrix} x + y\frac{b}{2} & cy \\ -ay & x - y\frac{b}{2} \end{bmatrix}, \qquad \det(g) = x^2 - \frac{1}{4}y^2(b^2 - 4ac) \neq 0, \tag{47}$$

and that $T(\mathfrak{o}) = T(F) \cap \mathrm{GL}_2(\mathfrak{o})$.

3.1 Lemma. *i) In the inert case* $\left(\frac{L}{\mathfrak{p}}\right) = -1$,

$$\begin{aligned} \mathrm{GL}_2(\mathfrak{o}) &= T(\mathfrak{o}) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma_0(\mathfrak{p}) \\ &= T(\mathfrak{o}) \Gamma^0(\mathfrak{p}). \end{aligned} \tag{48}$$

ii) In the ramified case $\left(\frac{L}{\mathfrak{p}}\right) = 0$,

$$\begin{aligned} \mathrm{GL}_2(\mathfrak{o}) &= T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma_0(\mathfrak{p}) \\ &= T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma^0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \Gamma^0(\mathfrak{p}), \end{aligned} \tag{49}$$

with u_0 *as in (9). We have* $T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) = \begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p})$.

iii) In the split case $\left(\frac{L}{\mathfrak{p}}\right) = 1$,

$$\begin{aligned} \mathrm{GL}_2(\mathfrak{o}) &= T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_1 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_2 & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma_0(\mathfrak{p}) \\ &= T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_1 & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma^0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_2 & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \Gamma^0(\mathfrak{p}) \sqcup T(\mathfrak{o}) \Gamma^0(\mathfrak{p}). \end{aligned} \tag{50}$$

with u_1, u_2 *as in (10). We have* $T(\mathfrak{o}) \begin{bmatrix} 1 & \\ u_i & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) = \begin{bmatrix} 1 & \\ u_i & 1 \end{bmatrix} \Gamma_0(\mathfrak{p})$ *for* $i = 1, 2$.

Proof. We will only prove the assertions made for $\Gamma_0(\mathfrak{p})$, since those for $\Gamma^0(\mathfrak{p})$ follow by multiplying from the right by $\begin{bmatrix} & 1\\ 1 & \end{bmatrix}$. Using standard representatives for $\mathrm{GL}_2(\mathfrak{o})/\Gamma_0(\mathfrak{p})$, we have

$$\mathrm{GL}_2(\mathfrak{o}) = \bigcup_{u\in\mathfrak{o}/\mathfrak{p}} T(\mathfrak{o})\begin{bmatrix}1 & \\ u & 1\end{bmatrix}\Gamma_0(\mathfrak{p}) \ \cup\ T(\mathfrak{o})\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\Gamma_0(\mathfrak{p}).$$

If we set $x = -\frac{\mathbf{b}}{2} - u\mathbf{c}$ and $y = 1$ in (47), then

$$g_{x,y}\begin{bmatrix}1 & \\ u & 1\end{bmatrix} = \begin{bmatrix} & 1\\ 1 & \end{bmatrix}\gamma \qquad \text{with } \gamma = \begin{bmatrix}-\mathbf{c}u^2 - \mathbf{b}u - \mathbf{a} & -\mathbf{b} - \mathbf{c}\mathbf{u}\\ & \mathbf{c}\end{bmatrix}. \tag{51}$$

In the inert case, $\mathbf{c}u^2 + \mathbf{b}u + \mathbf{a}$ is a unit for all $u \in \mathfrak{o}$, so that $\gamma \in \Gamma_0(\mathfrak{p})$. Hence, in this case, there is only one double coset. In the ramified and split case, the same identity shows that the double cosets on the right side of (49) and (50) exhaust $\mathrm{GL}_2(\mathfrak{o})$. The disjointness is straightforward to verify. The remaining assertions follow from (12). ■

3.2 Lemma. *Let m be a positive integer, and*

$$T(\mathfrak{o})_m := \begin{bmatrix}\varpi^{-m} & \\ & 1\end{bmatrix} T(\mathfrak{o}) \begin{bmatrix}\varpi^{m} & \\ & 1\end{bmatrix} \cap \mathrm{GL}_2(\mathfrak{o}). \tag{52}$$

Then

$$\begin{aligned}\mathrm{GL}_2(\mathfrak{o}) &= T(\mathfrak{o})_m\Gamma_0(\mathfrak{p}) \sqcup T(\mathfrak{o})_m\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\Gamma_0(\mathfrak{p})\\ &= T(\mathfrak{o})_m\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\Gamma^0(\mathfrak{p}) \sqcup T(\mathfrak{o})_m\Gamma^0(\mathfrak{p}).\end{aligned} \tag{53}$$

We have $T(\mathfrak{o})_m\Gamma_0(\mathfrak{p}) = \Gamma_0(\mathfrak{p})$.

Proof. Since the claim regarding $\Gamma_0(\mathfrak{p})$ follows from the claim regarding $\Gamma^0(\mathfrak{p})$ by multiplying from the right by $\begin{bmatrix} & 1\\ 1 & \end{bmatrix}$, we will only prove the $\Gamma^0(\mathfrak{p})$ statement. Using standard representatives for $\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})$, we have

$$\mathrm{GL}_2(\mathfrak{o}) = \bigcup_{u\in\mathfrak{o}/\mathfrak{p}} T(\mathfrak{o})_m\begin{bmatrix}1 & u\\ & 1\end{bmatrix}\Gamma^0(\mathfrak{p}) \ \cup\ T(\mathfrak{o})_m\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\Gamma^0(\mathfrak{p}).$$

If we set $x = \varpi^m u\frac{\mathbf{b}}{2} + \mathbf{c}$ and $y = -\varpi^m u$ in (47), then

$$\underbrace{\begin{bmatrix}\varpi^{-m} & \\ & 1\end{bmatrix} g_{x,y}\begin{bmatrix}\varpi^{m} & \\ & 1\end{bmatrix}}_{\in\mathrm{GL}_2(\mathfrak{o})}\begin{bmatrix}1 & u\\ & 1\end{bmatrix} = \gamma \qquad \text{with } \gamma = \begin{bmatrix}\mathbf{c} & \\ \varpi^{2m}\mathbf{a}u & \mathbf{c} + \mathbf{b}\varpi^m u + \mathbf{a}\varpi^{2m}u^2\end{bmatrix}.$$

Since $\mathbf{c}$ is a unit and $m > 0$, the matrix γ is in $\Gamma^0(\mathfrak{p})$. This shows that the double cosets on the right side of (53) exhaust all of $\mathrm{GL}_2(\mathfrak{o})$. The disjointness is easy to verify. The last statement is obvious. ∎

We turn to double coset decompositions for GSp_4. Let $h(l,m)$ be as in (31). Using (45) and the Iwasawa decomposition, it is easy to see that

$$\mathrm{GSp}_4(F) = \bigsqcup_{\substack{l,m\in\mathbb{Z}\\ m\geq 0}} R(F)h(l,m)K; \tag{54}$$

cf. (3.4.2) of [2]. In view of the Bessel transformation property (21), it follows that a *spherical Bessel function* B is determined by its values on elements $h(l,m)$ with l,m integers, m non-negative. This was used by Sugano, who gave the values $B(h(l,m))$ in terms of a generating function; see [16], Sect. 2-4, and the summary in [2], Sect. (3.6).

We will be interested in Bessel functions that are right invariant under the Siegel congruence subgroup P_1 defined in (14), and thus require a refinement of the decomposition (54). Our strategy will be to decompose each double coset $R(F)h(l,m)K$ occurring in (54) into double cosets of the form $R(F)h(l,m)kP_1$ with some k in K. We start with the following decomposition from [11], Lemma 5.1.1,

$$\begin{aligned}K = P_1 \sqcup \bigsqcup_{z\in\mathfrak{o}/\mathfrak{p}} s_2 \begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix} P_1 \\ \sqcup \bigsqcup_{y,z\in\mathfrak{o}/\mathfrak{p}} s_1s_2 \begin{bmatrix}1&&&\\&1&&\\&y&1&\\y&z&&1\end{bmatrix} P_1 \sqcup \bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}} s_2s_1s_2 \begin{bmatrix}1&&&\\&1&&\\x&y&1&\\y&z&&1\end{bmatrix} P_1.\end{aligned} \tag{55}$$

It implies that

$$\begin{aligned}R(F)h(l,m)K = R(F)h(l,m)P_1 \cup R(F)h(l,m)s_2s_1s_2P_1 \\ \cup \bigcup_{z\in\mathfrak{o}/\mathfrak{p}} R(F)h(l,m)s_2 \begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix} P_1 \\ \cup \bigcup_{y,z\in\mathfrak{o}/\mathfrak{p}} R(F)h(l,m)s_1s_2 \begin{bmatrix}1&&&\\&1&&\\&y&1&\\y&z&&1\end{bmatrix} P_1.\end{aligned} \tag{56}$$

For a unit z, employing the useful identity

$$\begin{bmatrix}1&\\z&1\end{bmatrix} = \begin{bmatrix}1&z^{-1}\\&1\end{bmatrix}\begin{bmatrix}-z^{-1}&\\&-z\end{bmatrix}\begin{bmatrix}&1\\-1&\end{bmatrix}\begin{bmatrix}1&z^{-1}\\&1\end{bmatrix}, \tag{57}$$

we see that

$$R(F)h(l,m)s_2\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix}P_1=R(F)h(l,m)s_2s_1s_2P_1.$$

For a unit z and arbitrary y in $\mathfrak{o}$, the identity

$$\begin{bmatrix}1&&&\\&1&&\\&y&1&\\y&z&&1\end{bmatrix}=\begin{bmatrix}1&&&\\-yz^{-1}&1&&z^{-1}\\-y^2z^{-1}&&1&yz^{-1}\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&&&1\\&&1&\\&-1&&\end{bmatrix}\begin{bmatrix}1&&&\\-y&-z&&-1\\&&1&-yz^{-1}\\&&&-z^{-1}\end{bmatrix}$$

shows that

$$R(F)h(l,m)s_1s_2\begin{bmatrix}1&&&\\&1&&\\&y&1&\\y&z&&1\end{bmatrix}P_1=R(F)h(l,m)s_2s_1s_2P_1.$$

Hence, from (56),

$$\begin{aligned}R(F)h(l,m)K&=R(F)h(l,m)P_1\ \cup\ R(F)h(l,m)s_2s_1s_2P_1\\&\cup\ R(F)h(l,m)s_2P_1\cup\bigcup_{u\in\mathfrak{o}/\mathfrak{p}}R(F)h(l,m)\begin{bmatrix}1&&&\\u&1&&\\&&1&-u\\&&&1\end{bmatrix}s_1s_2\,P_1.\end{aligned}\tag{58}$$

For the remaining double cosets in (58) we will use the following lemma.

3.3 Lemma. *Let l,m be integers. The following are equivalent for $U,V\in\mathrm{GL}_2(\mathfrak{o})$.*

i)

$$R(F)h(l,m)\begin{bmatrix}U&\\&{}^tU^{-1}\end{bmatrix}s_1s_2\,P_1=R(F)h(l,m)\begin{bmatrix}V&\\&{}^tV^{-1}\end{bmatrix}s_1s_2\,P_1.\tag{59}$$

ii) There exist $g\in T(F)$ and $\gamma\in\Gamma_0(\mathfrak{p})$ such that

$$\begin{bmatrix}\varpi^{-m}&\\&1\end{bmatrix}g\begin{bmatrix}\varpi^{m}&\\&1\end{bmatrix}U=V\gamma.\tag{60}$$

Proof. This follows by considering the upper left 2×2 block. ∎

In view of Lemmas 3.1 and 3.2, we now obtain the following result. The asserted disjointness is easy to verify in each case.

3.4 Proposition. *We have*

$$\mathrm{GSp}_4(F)=\bigsqcup_{\substack{l,m\in\mathbb{Z}\\m\geq0}}R(F)h(l,m)K,\tag{61}$$

where, for $m > 0$,

$$R(F)h(l,m)K = R(F)h(l,m)P_1 \sqcup R(F)h(l,m)s_2P_1 \\ \sqcup R(F)h(l,m)s_1s_2P_1 \sqcup R(F)h(l,m)s_2s_1s_2\,P_1, \tag{62}$$

and where the double cosets for $m = 0$ are given as follows.

i) In the inert case $\left(\frac{L}{\mathfrak{p}}\right) = -1$,

$$R(F)h(l,0)K = R(F)h(l,0)P_1 \sqcup R(F)h(l,0)s_2P_1 \sqcup R(F)h(l,0)s_2s_1s_2P_1.$$

ii) In the ramified case $\left(\frac{L}{\mathfrak{p}}\right) = 0$,

$$R(F)h(l,0)K = R(F)h(l,0)P_1 \sqcup R(F)h(l,0)s_2P_1 \sqcup R(F)h(l,0)s_2s_1s_2P_1 \\ \sqcup R(F)h(l,0)\hat{u}_0s_1s_2\,P_1,$$

where

$$\hat{u}_0 = \begin{bmatrix} 1 &&& \\ u_0 & 1 && \\ && 1 & -u_0 \\ &&& 1 \end{bmatrix}, \tag{63}$$

and u_0 as in (9).

iii) In the split case $\left(\frac{L}{\mathfrak{p}}\right) = 1$,

$$R(F)h(l,0)K = R(F)h(l,0)P_1 \sqcup R(F)h(l,0)s_2P_1 \sqcup R(F)h(l,0)s_2s_1s_2P_1 \\ \sqcup R(F)h(l,0)\hat{u}_1s_1s_2\,P_1 \sqcup R(F)h(l,0)\hat{u}_2s_1s_2\,P_1,$$

where, for $i = 1, 2$,

$$\hat{u}_i = \begin{bmatrix} 1 &&& \\ u_i & 1 && \\ && 1 & -u_i \\ &&& 1 \end{bmatrix}, \tag{64}$$

and u_i as in (10).

An integration formula

The integration formula on $\mathrm{GL}_2(\mathfrak{o})$ presented in the following lemma will be used in many of our Hecke operator calculations. Let Λ be a character of $L^\times \cong T(F)$. Let

$$m_0 = \min\{m \geq 0 \,:\, \Lambda\big|_{(1+\mathfrak{P}^m)\cap\mathfrak{o}_L^\times} = 1\}. \tag{65}$$

where $\mathfrak{P}$ is the ideal defined in (4). Note that only in the inert case is m_0 the usual conductor of the character Λ. For a non-negative integer m, let $T(\mathfrak{o})_m$ be as defined in (52). Using $\mathfrak{o}_L = \mathfrak{o} + \mathfrak{o}\xi_0$, see (8), it is easy to see that

$$\begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} T(\mathfrak{o})_m \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} = \mathfrak{o}^\times((1+\mathfrak{P}^m)\cap\mathfrak{o}_L^\times) \tag{66}$$

via the isomorphism $\mathfrak{o}_L^\times \cong T(\mathfrak{o})$.

3.5 Lemma. *Let Λ be a character of $L^\times \cong T(F)$ which is trivial on $\mathfrak{o}^\times$, and let m_0 be as in (65). Let m be a non-negative integer. Let $f: \mathrm{GL}_2(\mathfrak{o}) \to \mathbb{C}$ be a function with the property*

$$f(tg) = \Lambda(\begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} t \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix}) f(g) \qquad \text{for all } t \in T(\mathfrak{o})_m.$$

Let the Haar measure on $\mathrm{GL}_2(\mathfrak{o})$ be normalized such that the total volume is 1.

i) Assume that f is right invariant under $\Gamma_0(\mathfrak{p})$. Then

$$\int\limits_{\mathrm{GL}_2(\mathfrak{o})} f(g)\,dg = \begin{cases} 0 & \text{if } m < m_0, \\ \frac{1}{q+1}(f(1) + qf(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix})) & \text{if } m \geq \max(m_0, 1), \\ f(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \frac{1}{q+1}(f(\begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix}) + qf(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix})) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \frac{1}{q+1}(f(\begin{bmatrix} 1 & \\ u_1 & 1 \end{bmatrix}) + f(\begin{bmatrix} 1 & \\ u_2 & 1 \end{bmatrix}) + (q-1)f(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix})) & \\ & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

ii) Assume that f is right invariant under $\Gamma^0(\mathfrak{p})$. Then

$$\int\limits_{\mathrm{GL}_2(\mathfrak{o})} f(g)\,dg = \begin{cases} 0 & \text{if } m < m_0, \\ \frac{1}{q+1}(f(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}) + qf(1)) & \text{if } m \geq \max(m_0, 1), \\ f(1) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \frac{1}{q+1}(f(\begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}) + qf(1)) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \frac{1}{q+1}(f(\begin{bmatrix} 1 & \\ u_1 & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}) + f(\begin{bmatrix} 1 & \\ u_2 & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}) + (q-1)f(1)) & \\ & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

Proof. ii) follows from i) by considering the function $g \mapsto f(g\begin{bmatrix} & 1 \\ 1 & \end{bmatrix})$. We will prove i). If $m < m_0$, then, using (66), an inner integral over $T(\mathfrak{o})_m$ shows that $\int_{\mathrm{GL}_2(\mathfrak{o})} f(g)\,dg = 0$. Assume that $m \geq m_0$ and $m > 0$. Then, again by (66), f is left invariant under $T(\mathfrak{o})_m$. Hence, by Lemma 3.2,

$$\int\limits_{\mathrm{GL}_2(\mathfrak{o})} f(g)\,dg = \int\limits_{T(\mathfrak{o})_m \Gamma_0(\mathfrak{p})} f(g)\,dg + \int\limits_{T(\mathfrak{o})_m \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right] \Gamma_0(\mathfrak{p})} f(g)\,dg$$

$$= \mathrm{vol}\big(T(\mathfrak{o})_m\Gamma_0(\mathfrak{p})\big)f(1) + \mathrm{vol}\big(T(\mathfrak{o})_m\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\Gamma_0(\mathfrak{p})\big)f(\begin{bmatrix} & 1\\ 1 & \end{bmatrix}).$$

Again by Lemma 3.2, the first volume equals $\frac{1}{q+1}$ and the second volume equals $\frac{q}{q+1}$. This concludes the proof for $m > 0$. The argument for $m = 0$ is analogous, using Lemma 3.1 instead of Lemma 3.2. ∎

Automatic vanishing

Many of the cosets from Proposition 3.4 cannot be in the support of a Bessel function. The following lemma exhibits several cases of automatic vanishing.

3.6 Lemma. *Let Λ be a character of $L^\times \cong T(F)$ which is trivial on $\mathfrak{o}^\times$, and let m_0 be as in (65). Let $B \in \mathcal{S}(\Lambda,\theta,P_1)$, and let l and m be integers. Then:*

i)
$$B(h(l,m)) = B(h(l,m)s_2) = 0 \qquad \text{if } l < 0.$$

ii)
$$B(h(l,m)) = B(h(l,m)s_2s_1s_2) = 0 \qquad \text{for any } l \text{ and } 0 \le m < m_0.$$

iii)
$$B(h(l,m)s_2) = 0 \qquad \text{for any } l \text{ and } 0 \le m < m_0 - 1.$$

iv)
$$B(h(l,m)s_1s_2) = B(h(l,m)s_2s_1s_2) = 0 \qquad \text{if } l < -1.$$

v) Assume that $m_0 > 0$. Then
$$B(h(l,0)\hat{u}_is_1s_2) = 0$$
with $i = 0$ in the ramified case and $i \in \{1,2\}$ in the split case.

vi) In the ramified case,
$$B(h(l,0)\hat{u}_0s_1s_2) = 0 \qquad \text{for } l < -1.$$

vii) In the split case,
$$B(h(l,0)\hat{u}_is_1s_2) = 0 \qquad \text{for } l < 0.$$

Proof. i) follows by integrating the identity

$$B(h(l,m)) = B(h(l,m)\begin{bmatrix}1&&&\\&1&&z\\&&1&\\&&&1\end{bmatrix}) = \psi(\mathbf{c}z\varpi^l)B(h(l,m)), \qquad z \in \mathfrak{o},$$

over z in $\mathfrak{o}$; similarly with an additional s_2.

ii) follows by integrating the identities

$$B(h(l,m)) = B(h(l,m)\begin{bmatrix} g & \\ & \det(g)\,{}^tg^{-1}\end{bmatrix}), \qquad g \in \mathrm{GL}_2(\mathfrak{o}),$$

and

$$B(h(l,m)s_2s_1s_2) = B(h(l,m)\begin{bmatrix} g & \\ & \det(g)\,{}^tg^{-1}\end{bmatrix}s_2s_1s_2), \qquad g \in \mathrm{GL}_2(\mathfrak{o}),$$

over g in $\mathrm{GL}_2(\mathfrak{o})$ and observing Lemma 3.5.

iii) follows by integrating the identity

$$B(h(l,m)s_2) = B(h(l,m)\begin{bmatrix} g & \\ & \det(g)\,{}^tg^{-1}\end{bmatrix}s_2), \qquad g \in \Gamma^0(\mathfrak{p}),$$

over the group

$$T(\mathfrak{o})'_m := \begin{bmatrix} \varpi^{-m} & \\ & 1\end{bmatrix} T(\mathfrak{o}) \begin{bmatrix} \varpi^{m} & \\ & 1\end{bmatrix} \cap \Gamma^0(\mathfrak{p}), \tag{67}$$

observing that

$$\begin{bmatrix} \varpi^{m} & \\ & 1\end{bmatrix} T(\mathfrak{o})'_m \begin{bmatrix} \varpi^{-m} & \\ & 1\end{bmatrix} = \mathfrak{o}^\times((1+\mathfrak{P}^{m+1}) \cap \mathfrak{o}_L^\times) \tag{68}$$

via the isomorphism $\mathfrak{o}_L^\times \cong T(\mathfrak{o})$; cf. (66).

iv) follows from integrating the identity

$$B(h(l,m)s_1s_2) = B(h(l,m)s_1s_2\begin{bmatrix} 1 &&& \\ & 1 && \\ z && 1 & \\ &&& 1\end{bmatrix}) = \psi(\mathbf{c}z\varpi^l)B(h(l,m)s_1s_2), \qquad z \in \mathfrak{p},$$

over z in $\mathfrak{p}$.

v) It follows from (12) that

$$B(h(l,0)\hat{u}_is_1s_2) = \Lambda(g)B(h(l,0)\hat{u}_is_1s_2)$$

for all $g \in T(\mathfrak{o})$. Hence, if Λ is not trivial on $T(\mathfrak{o})$, then $B(h(l,0)\hat{u}_is_1s_2) = 0$.

vi) follows from integrating the identity

$$\begin{aligned} B(h(l,0)\hat{u}_0s_1s_2) &= B(h(l,0)\hat{u}_0s_1s_2\begin{bmatrix} 1 &&& \\ & 1 && x \\ && 1 & \\ &&& 1\end{bmatrix}) \\ &= \psi((\mathbf{c}u_0^2+\mathbf{b}u_0+\mathbf{a})x\varpi^l)B(h(l,0)\hat{u}_0s_1s_2), \qquad x \in \mathfrak{o}, \end{aligned}$$

over x in $\mathfrak{o}$ and observing (11).

vii) follows from integrating the identity

$$B(h(l,0)\hat{u}_is_1s_2) = B(h(l,0)\hat{u}_is_1s_2\begin{bmatrix} 1 &&& \\ -x & 1 && \\ && 1 & x \\ &&& 1\end{bmatrix})$$

$$= \psi((\mathbf{b} + 2\mathbf{c}u_i)x\varpi^l)B(h(l,0)\hat{u}_i s_1 s_2)$$
$$= \psi(\pm\sqrt{\mathbf{d}}\,x\varpi^l)B(h(l,0)\hat{u}_i s_1 s_2), \qquad x \in \mathfrak{o},$$

over x in $\mathfrak{o}$, observing that $\mathbf{d}$ is a unit in the split case. ∎

3.7 Lemma. *Let Λ be an unramified character of $L^\times$. Let B be an element of $\mathcal{S}(\Lambda, \theta, P_1)$. Let l be any integer.*

i) In the ramified case $\left(\frac{L}{\mathfrak{p}}\right) = 0$ the following formulas hold.

$$\Lambda(\varpi_L)B(h(l,0)\hat{u}_0 s_1 s_2) = B(h(l,0)\hat{u}_0 s_2 h(1,0)), \tag{69}$$
$$\Lambda(\varpi_L)B(h(l+1,0)\hat{u}_0 s_1 s_2) = B(h(l,0)\hat{u}_0 s_2 s_1 h(0,1)), \tag{70}$$
$$\Lambda(\varpi_L)B(h(l+1,0)) = B(h(l,0)\hat{u}_0 s_1 h(0,1)), \tag{71}$$
$$\Lambda(\varpi)\Lambda(\varpi_L)B(h(l-1,0)\hat{u}_0 s_1 s_2) = B(h(l,0)\hat{u}_0 s_2 h(0,1)). \tag{72}$$

For all $g \in \mathrm{GSp}_4(F)$,

$$B(h(l,0)\hat{u}_0 \begin{bmatrix} 1 & & x & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} g) = B(h(l,0)\hat{u}_0 g) \qquad \text{for } l \geq -1 \text{ and } x, y \in \mathfrak{o}. \tag{73}$$

If $\eta B = \omega B$, then

$$\Lambda(\varpi_L)B(h(l,0)\hat{u}_0 s_1 s_2) = \omega B(h(l,0)\hat{u}_0 s_1 s_2). \tag{74}$$

ii) In the split case $\left(\frac{L}{\mathfrak{p}}\right) = 1$, after possibly interchanging u_1 and u_2, the following formulas hold.

$$\Lambda(\varpi,1)B(h(l,0)\hat{u}_2 s_1 s_2) = B(h(l,0)\hat{u}_1 s_2 h(1,0)), \tag{75}$$
$$\Lambda(1,\varpi)B(h(l,0)\hat{u}_1 s_1 s_2) = B(h(l,0)\hat{u}_2 s_2 h(1,0)), \tag{76}$$
$$\Lambda(\varpi,1)B(h(l+1,0)\hat{u}_2 s_1 s_2) = B(h(l,0)\hat{u}_1 s_2 s_1 h(0,1)), \tag{77}$$
$$\Lambda(1,\varpi)B(h(l+1,0)\hat{u}_1 s_1 s_2) = B(h(l,0)\hat{u}_2 s_2 s_1 h(0,1)), \tag{78}$$
$$\Lambda(1,\varpi)B(h(l+1,0)) = B(h(l,0)\hat{u}_2 s_1 h(0,1)), \tag{79}$$
$$\Lambda(\varpi,1)B(h(l+1,0)) = B(h(l,0)\hat{u}_1 s_1 h(0,1)), \tag{80}$$
$$\Lambda(\varpi)\Lambda(\varpi,1)B(h(l-1,0)\hat{u}_2 s_1 s_2) = B(h(l,0)\hat{u}_1 s_2 h(0,1)), \tag{81}$$
$$\Lambda(\varpi)\Lambda(1,\varpi)B(h(l-1,0)\hat{u}_1 s_1 s_2) = B(h(l,0)\hat{u}_2 s_2 h(0,1)). \tag{82}$$

If $\eta B = \omega B$, then

$$\Lambda(\varpi,1)B(h(l,0)\hat{u}_2 s_1 s_2) = \omega B(h(l,0)\hat{u}_1 s_1 s_2), \tag{83}$$
$$\Lambda(1,\varpi)B(h(l,0)\hat{u}_1 s_1 s_2) = \omega B(h(l,0)\hat{u}_2 s_1 s_2). \tag{84}$$

Proof. i) Setting $x = \mathbf{c}u_0 + \frac{\mathbf{b}}{2}$ and $y = 1$ in (47), we obtain the identity

$$g_{x,y}\begin{bmatrix}1&\\u_0&1\end{bmatrix}\begin{bmatrix}&1\\1&\end{bmatrix} = \begin{bmatrix}1&\\u_0&1\end{bmatrix}\begin{bmatrix}1&\\&\varpi\end{bmatrix}k, \tag{85}$$

where

$$k = \begin{bmatrix}\mathbf{c} & \mathbf{b}+2\mathbf{c}u_0\\ 0 & -\varpi^{-1}(\mathbf{c}u_0^2+\mathbf{b}u_0+\mathbf{a})\end{bmatrix} \in \begin{bmatrix}\mathfrak{o}^\times & \mathfrak{p}\\ \mathfrak{p} & \mathfrak{o}^\times\end{bmatrix};$$

observe here (11). Note that the element $t = g_{x,y}$ satisfies $\Lambda(t) = \Lambda(\varpi_L)$; this follows from the identity

$$\Big(\mathbf{c}u_0 + \frac{\mathbf{b}}{2} + \frac{\sqrt{\mathbf{d}}}{2}\Big)\Big(\mathbf{c}u_0 + \frac{\mathbf{b}}{2} - \frac{\sqrt{\mathbf{d}}}{2}\Big) = \mathbf{c}\big(\mathbf{c}u_0^2 + \mathbf{b}u_0 + \mathbf{a}\big).$$

together with (11). It follows from (85) that, for any $l \in \mathbb{Z}$,

$$th(l,0)\hat{u}_0 s_1 s_2 = h(l,0)\hat{u}_0 s_2 h(1,0)k_1 \tag{86}$$

with

$$k_1 \in \begin{bmatrix}\mathfrak{o}^\times&&&\mathfrak{p}\\&\mathfrak{o}^\times&\mathfrak{p}&\\&\mathfrak{p}&\mathfrak{o}^\times&\\\mathfrak{p}&&&\mathfrak{o}^\times\end{bmatrix}.$$

Evaluating B on both sides of (86), we obtain (69). Multiplying (86) from the right with $\mathrm{diag}(1,\varpi,\varpi,1)$ and evaluating B on both sides, we obtain (70). Multiplying (86) from the right by s_2, we get

$$th(l,0)\hat{u}_0 s_1 = h(l,0)\hat{u}_0\begin{bmatrix}1&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}k_2 \tag{87}$$

with

$$k_2 \in \begin{bmatrix}\mathfrak{o}^\times&\mathfrak{p}&&\\\mathfrak{p}&\mathfrak{o}^\times&&\\&&\mathfrak{o}^\times&\mathfrak{p}\\&&\mathfrak{p}&\mathfrak{o}^\times\end{bmatrix}.$$

Multiplying (87) from the right with $h(0,1)$ and evaluating B on both sides, we obtain (71). Multiplying (86) from the right by $\mathrm{diag}(\varpi,1,1,\varpi)$ and evaluating B on both sides, we obtain (72). Using (11) and $\mathbf{b} - 2\mathbf{c}u_0 \in \mathfrak{p}$ we obtain (73). Finally, (74) follows from (69).

ii) Setting $x = \mathbf{c}u_1 + \frac{\mathbf{b}}{2}$ and $y = 1$ in (47), we obtain the identity

$$g_{x,y}\begin{bmatrix}1&\\u_2&1\end{bmatrix}\begin{bmatrix}&1\\1&\end{bmatrix} = \begin{bmatrix}1&\\u_1&1\end{bmatrix}\begin{bmatrix}1&\\&\varpi\end{bmatrix}k, \tag{88}$$

where

$$k = \begin{bmatrix}\mathbf{c} & \mathbf{b}+\mathbf{c}(u_1+u_2)\\ 0 & -\varpi^{-1}(\mathbf{c}u_1^2+\mathbf{b}u_1+\mathbf{a})\end{bmatrix} \in \begin{bmatrix}\mathfrak{o}^\times & \mathfrak{p}\\ \mathfrak{p} & \mathfrak{o}^\times\end{bmatrix};$$

observe here (11). Note that the element $t = g_{x,y}$ satisfies

$$\Lambda(t) = \Lambda\Big(x + y\frac{\sqrt{\mathbf{d}}}{2}, x - y\frac{\sqrt{\mathbf{d}}}{2}\Big) = \Lambda(u_1 + \alpha, u_1 + \bar{\alpha}).$$

By (11), one of $u_1 + \alpha, u_1 + \bar{\alpha}$ is a unit and the other has valuation one. After possibly interchanging u_1 and u_2, we may assume that $\Lambda(t) = \Lambda(\varpi, 1)$. It follows from (88) that, for any $l \in \mathbb{Z}$,

$$th(l,0)\hat{u}_2 s_1 = h(l,0)\hat{u}_1 \mathrm{diag}(1, \varpi, \varpi, 1)k_1 \tag{89}$$

and

$$th(l,0)\hat{u}_2 s_1 s_2 = h(l,0)\hat{u}_1 s_2 h(1,0)k_2 \tag{90}$$

with

$$k_1 \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p} & & \\ \mathfrak{p} & \mathfrak{o}^\times & & \\ & & \mathfrak{o}^\times & \mathfrak{p} \\ & & \mathfrak{p} & \mathfrak{o}^\times \end{bmatrix}, \qquad k_2 \in \begin{bmatrix} \mathfrak{o}^\times & & & \mathfrak{p} \\ & \mathfrak{o}^\times & \mathfrak{p} & \\ & \mathfrak{p} & \mathfrak{o}^\times & \\ \mathfrak{p} & & & \mathfrak{o}^\times \end{bmatrix}.$$

Evaluating B on both sides of (90), we obtain (75), and similarly (76). Multiplying (90) from the right by $s_1\mathrm{diag}(\varpi, 1, 1, \varpi)$ and evaluating B on both sides, we obtain (77), and similarly (78). Multiplying (89) from the right by $h(0, 1)$ and evaluating B on both sides, we obtain (79), and similarly (80). Multiplying (90) from the right by $\mathrm{diag}(\varpi, 1, 1, \varpi)$ and evaluating B on both sides, we obtain (81), and similarly (82). (83) and (84) follow from (75) and (76). ■

4 The Hecke operator $T_{1,0}$

In this section we calculate the action of the Hecke operator $T_{1,0}$ defined in (32) on an element B of the space $\mathcal{S}(\Lambda, \theta, P_1)$; see (21). If (π, V) is any smooth representation of $\mathrm{GSp}_4(F)$, and if $v \in V$ is any P_1-invariant vector, one can easily verify the explicit formula

$$T_{1,0}v = \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v. \tag{91}$$

4.1 Lemma. *Let $B \in \mathcal{S}(\Lambda, \theta, P_1)$, and let l and m be non-negative integers. Let $h(l, m)$ be as in (31). Then the following formulas hold.*

i)

$$(T_{1,0}B)(h(l,m)) = q^3 B(h(l+1,m)).$$

ii)

$$(T_{1,0}B)(h(l,m)s_2) = q^2(q-1)B(h(l+1,m))$$

$$+\begin{cases} -qB(h(l-1,m+1)s_1s_2) & \text{if } m < m_0, \\ q\Lambda(\varpi)B(h(l+1,m-1)s_2) + q(q-1)B(h(l-1,m+1)s_1s_2) & \text{if } m \geq \max(m_0,1), \\ q^2B(h(l-1,1)s_1s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ q\Lambda(\varpi_L)B(h(l,0)\hat{u}_0s_1s_2) + q(q-1)B(h(l-1,1)s_1s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ q\big(\Lambda(\varpi,1)B(h(l,0)\hat{u}_2s_1s_2) + \Lambda(1,\varpi)B(h(l,0)\hat{u}_1s_1s_2)\big) & \\ \quad +q(q-2)B(h(l-1,1)s_1s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

iii)

$$(T_{1,0}B)(h(l,m)s_1s_2) = q^2(q-1)B(h(l+1,m))$$
$$+\begin{cases} -q\Lambda(\varpi)B(h(l+1,m-1)s_2) & \text{if } m < m_0, \\ q^2B(h(l-1,m+1)s_1s_2) & \text{if } m \geq \max(m_0,1), \\ q(q+1)B(h(l-1,1)s_1s_2) - q\Lambda(\varpi)B(h(l+1,-1)s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ q\Lambda(\varpi_L)B(h(l,0)\hat{u}_0s_1s_2) + q^2B(h(l-1,1)s_1s_2) & \\ \quad -q\Lambda(\varpi)B(h(l+1,-1)s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ q\Lambda(1,\varpi)B(h(l,0)\hat{u}_1s_1s_2) + q\Lambda(\varpi,1)B(h(l,0)\hat{u}_2s_1s_2) + q(q-1)B(h(l-1,1)s_1s_2) & \\ \quad -q\Lambda(\varpi)B(h(l+1,-1)s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

iv)

$$(T_{1,0}B)(h(l,m)s_2s_1s_2) = q^2(q-1)B(h(l+1,m)) + \Lambda(\varpi)B(h(l-1,m)s_2s_1s_2)$$
$$+\begin{cases} 0 & \text{if } m < m_0, \\ q(q-1)B(h(l-1,m+1)s_1s_2) + (q-1)\Lambda(\varpi)B(h(l+1,m-1)s_2) & \\ & \text{if } m \geq \max(m_0,1), \\ (q^2-1)B(h(l-1,1)s_1s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ (q-1)\Lambda(\varpi_L)B(h(l,0)\hat{u}_0s_1s_2) + q(q-1)B(h(l-1,1)s_1s_2) & \\ & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ (q-1)\big(\Lambda(1,\varpi)B(h(l,0)\hat{u}_1s_1s_2) + \Lambda(\varpi,1)B(h(l,0)\hat{u}_2s_1s_2)\big) & \\ \quad +(q-1)^2B(h(l-1,1)s_1s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

Proof. i) is immediate from (91).

ii) By (91),

$$(T_{1,0}B)(h(l,m)s_2) = q \sum_{x,y\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&&&\\y&1&&\\x&&1&-y\\&&&1\end{bmatrix}s_2h(1,0)) = A+B,$$

where A consists of all the terms with $x\in(\mathfrak{o}/\mathfrak{p})^\times$, and B consists of all the terms with $x=0$. Using (57), we have

$$A = q\sum_{\substack{x\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}}} B(h(l,m)\begin{bmatrix}1&&&\\y&1&&\\&&1&-y\\&&&1\end{bmatrix}\begin{bmatrix}1&&x^{-1}&\\&1&&\\&&1&\\&&&1\end{bmatrix}h(1,0)) = q^2(q-1)B(h(l+1,m)).$$

Adding and subtracting $qB(h(l,m)s_1s_2h(1,0))$ to the expression for B, we can rewrite it as

$$\begin{aligned}B &= q\sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma_0(\mathfrak{p})} B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_2h(1,0)) - qB(h(l,m)s_1s_2h(1,0))\\ &= q(q+1)\int\limits_{\mathrm{GL}_2(\mathfrak{o})} B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_2h(1,0))\,dg - qB(h(l,m)s_1s_2h(1,0)).\end{aligned}$$

Applying Lemma 3.5 i) to the function $f(g) = B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_2h(1,0))$, we see that the first term equals

$$\begin{cases}0 & \text{if } m<m_0,\\ qB(h(l,m)s_2h(1,0)) + q^2B(h(l,m)s_1s_2h(1,0)) & \text{if } m\ge\max(m_0,1),\\ q(q+1)B(h(l,0)s_1s_2h(1,0)) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=-1,\\ qB(h(l,0)\hat u_0s_2h(1,0)) + q^2B(h(l,0)s_1s_2h(1,0)) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=0,\\ q\sum\limits_{i=1,2}B(h(l,0)\hat u_is_2h(1,0)) + q(q-1)B(h(l,0)s_1s_2h(1,0)) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=1.\end{cases}$$

Observing $s_1s_2h(1,0) = h(-1,1)s_1s_2$ and Lemma 3.7, the assertion follows.

iii) By (91), we have

$$(T_{1,0}B)(h(l,m)s_1s_2) = \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)s_1s_2\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$

$$= q\sum_{x,y\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&y&&\\&1&&\\&&1&\\&x&-y&1\end{bmatrix}s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$
$$= A_1 + A_2,$$

where A_1 consists of those terms with $x=0$, and A_2 consists of those terms where x is a unit. Using Lemma 3.5, we calculate

$$A_1 = q\sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&y&&\\&1&&\\&&1&\\&&-y&1\end{bmatrix}s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$
$$= q\sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_1s_2h(1,0)) - qB(h(l,m)s_2h(1,0))$$
$$= q(q+1)\int\limits_{\mathrm{GL}_2(\mathfrak{o})} B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_1s_2h(1,0))\,dg - q\Lambda(\varpi)B(h(l+1,m-1)s_2)$$
$$= \begin{cases} -q\Lambda(\varpi)B(h(l+1,m-1)s_2) & \text{if } m<m_0,\\ q^2B(h(l-1,m+1)s_1s_2) & \text{if } m\geq\max(m_0,1),\\ q(q+1)B(h(l-1,1)s_1s_2) - q\Lambda(\varpi)B(h(l+1,-1)s_2) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=-1,\\ qB(h(l,0)\hat{u}_0s_2h(1,0)) + q^2B(h(l-1,1)s_1s_2) & \\ \qquad -q\Lambda(\varpi)B(h(l+1,-1)s_2) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=0,\\ q\sum\limits_{i=1,2}B(h(l,0)\hat{u}_is_2h(1,0)) + q(q-1)B(h(l-1,1)s_1s_2) & \\ \qquad -q\Lambda(\varpi)B(h(l+1,-1)s_2) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=1,\end{cases}$$

By Lemma 3.7, the terms $B(h(l,0)\hat{u}_0s_2h(1,0))$ and $\sum_{i=1,2}B(h(l,0)\hat{u}_is_2h(1,0))$ can be rewritten to give the corresponding terms in the asserted formula. Using (57), we have

$$A_2 = q\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\x\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&y&&\\&1&&\\&&1&\\&&-y&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&x^{-1}\\&&1&\\&&&1\end{bmatrix}s_1\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$
$$= q^2(q-1)B(h(l+1,m)).$$

This concludes the proof of iii).

iv) By (91), we have

$$(T_{1,0}B)(h(l,m)s_2s_1s_2) = \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&&&\\&1&&\\x&y&1&\\y&z&&1\end{bmatrix}s_2s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}).$$

Let A_1 be the part where x is a unit, and let A_2 be the part where $x=0$. Using

$$\begin{bmatrix}1&&&\\&1&&\\x&y&1&\\y&z&&1\end{bmatrix} = \begin{bmatrix}1&-yx^{-1}&x^{-1}&\\&1&&\\&&1&\\&&yx^{-1}&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&y&1&\\y&z-y^2x^{-1}&&1\end{bmatrix}s_2\begin{bmatrix}-x&&-1&\\&1&&\\&&-x^{-1}&\\&&&1\end{bmatrix}, \tag{92}$$

we have

$$\begin{aligned}A_1 &= \sum_{\substack{y,z\in\mathfrak{o}/\mathfrak{p}\\x\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&-yx^{-1}&x^{-1}&\\&1&&\\&&1&\\&&yx^{-1}&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&y&1&\\y&z-y^2x^{-1}&&1\end{bmatrix}s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})\\
&= \sum_{\substack{y,z\in\mathfrak{o}/\mathfrak{p}\\x\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&-yx^{-1}&&\\&1&&\\&&1&\\&&yx^{-1}&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&y&1&\\y&z&&1\end{bmatrix}s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}).\end{aligned}$$

Let A_{11} be the part where y is a unit, and let A_{12} be the part where $y=0$. Using the identity

$$\begin{bmatrix}1&&&\\&1&&\\&y&1&\\y&z&&1\end{bmatrix} = \begin{bmatrix}1&-y^{-1}z&&y^{-1}\\&1&y^{-1}&\\&&1&\\&&y^{-1}z&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix}\begin{bmatrix}&&&-1\\&&-1&\\&1&&\\1&&&\end{bmatrix}\begin{bmatrix}y&&&1\\&y&1&\\&&y^{-1}&\\&&&y^{-1}\end{bmatrix}, \tag{93}$$

we calculate

$$\begin{aligned}A_{11} &= \sum_{\substack{z\in\mathfrak{o}/\mathfrak{p}\\x,y\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&-yx^{-1}&&\\&1&&\\&&1&\\&&yx^{-1}&1\end{bmatrix}\begin{bmatrix}1&-y^{-1}z&&y^{-1}\\&1&y^{-1}&\\&&1&\\&&y^{-1}z&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix}\\
&\qquad s_1s_2s_1\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})\\
&= \sum_{\substack{z\in\mathfrak{o}/\mathfrak{p}\\x,y\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&x-y^{-1}z&&\\&1&&\\&&1&\\&&-x+y^{-1}z&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix}s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})\end{aligned}$$

$$= \sum_{\substack{x,z\in\mathfrak{o}/\mathfrak{p}\\ y\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1 & x-y^{-1}z & & \\ & 1 & & \\ & & 1 & \\ & & -x+y^{-1}z & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix})$$

$$- \sum_{\substack{z\in\mathfrak{o}/\mathfrak{p}\\ y\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1 & -y^{-1}z & & \\ & 1 & & \\ & & 1 & \\ & & y^{-1}z & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix})$$

$$= \sum_{\substack{x,z\in\mathfrak{o}/\mathfrak{p}\\ y\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix})$$

$$- \sum_{y,z\in(\mathfrak{o}/\mathfrak{p})^\times} B(h(l,m)\begin{bmatrix}1 & -y^{-1}z & & \\ & 1 & & \\ & & 1 & \\ & & y^{-1}z & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix})$$

$$- \sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times} B(h(l,m)s_1s_2h(1,0))$$

$$= (q-1)\sum_{x,z\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix})$$

$$- \sum_{y,z\in(\mathfrak{o}/\mathfrak{p})^\times} B(h(l,m)\begin{bmatrix}1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix})$$

$$- (q-1)B(h(l,m)s_1s_2h(1,0))$$

$$= (q-1)\sum_{\substack{x\in\mathfrak{o}/\mathfrak{p}\\ z\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix})$$

$$+ (q-1)\sum_{x\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix})$$

$$- \sum_{y,z\in(\mathfrak{o}/\mathfrak{p})^\times} B(h(l,m)\begin{bmatrix}1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & z & & 1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix})$$

$$- (q-1)B(h(l,m)s_1s_2h(1,0)).$$

Let A_{111} be the first sum, A_{112} be the second sum, and A_{113} be the third sum. We have

$$\begin{aligned}
A_{111} &= \sum_{\substack{x\in\mathfrak{o}/\mathfrak{p}\\ z\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&x&&\\&1&&\\&&1&\\&&-x&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix} s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})\\
&= \sum_{\substack{x\in\mathfrak{o}/\mathfrak{p}\\ z\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&x&&\\&1&&\\&&1&\\&&-x&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&z^{-1}\\&&1&\\&&&1\end{bmatrix} s_1s_2s_1s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})\\
&= q(q-1)B(h(l+1,m)).
\end{aligned}$$

Similary,

$$A_{113} = (q-1)^2B(h(l+1,m)).$$

Hence,

$$(q-1)A_{111} - A_{113} = (q-1)^3B(h(l+1,m)).$$

By Lemma 3.5,

$$\begin{aligned}
A_{112} &= \sum_{x\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&x&&\\&1&&\\&&1&\\&&-x&1\end{bmatrix} s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})\\
&= \sum_{g\in \mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_1s_2h(1,0)) - B(h(l,m)s_2h(1,0))\\
&= (q+1)\int\limits_{\mathrm{GL}_2(\mathfrak{o})} B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_1s_2h(1,0))\,dg - \Lambda(\varpi)B(h(l+1,m-1)s_2)\\
&= \begin{cases}
-\Lambda(\varpi)B(h(l+1,m-1)s_2) & \text{if } m<m_0,\\
qB(h(l,m)s_1s_2h(1,0)) & \text{if } m\ge \max(m_0,1),\\
(q+1)B(h(l,m)s_1s_2h(1,0)) - \Lambda(\varpi)B(h(l+1,m-1)s_2) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=-1,\\
B(h(l,m)\hat{u}_0s_2h(1,0)) + qB(h(l,m)s_1s_2h(1,0)) - \Lambda(\varpi)B(h(l+1,m-1)s_2) & \\
\qquad & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=0,\\
\sum\limits_{i=1,2} B(h(l,m)\hat{u}_is_2h(1,0)) + (q-1)B(h(l,m)s_1s_2h(1,0)) - \Lambda(\varpi)B(h(l+1,m-1)s_2) & \\
\qquad & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=1.
\end{cases}
\end{aligned}$$

Next,

$$A_{12} = (q-1)\sum_{z\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix}s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$

$$= (q-1)\sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} B(h(l,m)\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix}s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$

$$+ (q-1)B(h(l,m)s_1s_2h(1,0))$$

$$= (q-1)^2B(h(l+1,m)) + (q-1)B(h(l,m)s_1s_2h(1,0)).$$

Next we calculate

$$A_2 = \sum_{y,z\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&&&\\&1&&\\&y&1&\\y&z&&1\end{bmatrix}s_2s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}).$$

Let A_{21} be the part where y is a unit, and let A_{22} be the part where $y=0$. Using (93), we have

$$A_{21} = \sum_{\substack{z\in\mathfrak{o}/\mathfrak{p}\\ y\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&-y^{-1}z&&y^{-1}\\&1&y^{-1}&\\&&1&\\&&y^{-1}z&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$

$$= q(q-1)B(h(l+1,m)).$$

Finally,

$$A_{22} = \sum_{z\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z&&1\end{bmatrix}s_2s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$

$$= \sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} B(h(l,m)\begin{bmatrix}1&&&\\&1&&z^{-1}\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1s_2s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$

$$+ B(h(l,m)s_2s_1s_2h(1,0))$$

$$= \sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} B(h(l,m)s_2h(1,0)) + \Lambda(\varpi)B(h(l-1,m)s_2s_1s_2)$$

$$= (q-1)\Lambda(\varpi)B(h(l+1,m-1)s_2) + \Lambda(\varpi)B(h(l-1,m)s_2s_1s_2).$$

Hence

$$A_2 = q(q-1)B(h(l+1,m)) + (q-1)\Lambda(\varpi)B(h(l+1,m-1)s_2) + \Lambda(\varpi)B(h(l-1,m)s_2s_1s_2).$$

Summarizing everything and using Lemma 3.7, we get the asserted formula. ∎

4.2 Lemma. *Let Λ be an unramified character of $L^\times$. Then*

i) In the ramified case $\left(\frac{L}{\mathfrak{p}}\right) = 0$, for all integers $l \geq -1$,

$$(T_{1,0}B)(h(l,0)\hat{u}_0 s_1 s_2) = \begin{cases} -q^2 B(1) & \text{if } l = -1, \\ q^2(q-1)B(h(l+1,0)) + q^2 B(h(l-1,1)s_1 s_2) & \text{if } l \geq 0. \end{cases}$$

ii) In the split case $\left(\frac{L}{\mathfrak{p}}\right) = 1$, for all integers $l \geq 0$,

$$\begin{aligned}(T_{1,0}B)(h(l,0)\hat{u}_1 s_1 s_2) = q^2(q-1)B(h(l+1,0)) + q(q-1)B(h(l-1,1)s_1 s_2)\\ + q\Lambda(1,\varpi)B(h(l,0)\hat{u}_1 s_1 s_2),\\ (T_{1,0}B)(h(l,0)\hat{u}_2 s_1 s_2) = q^2(q-1)B(h(l+1,0)) + q(q-1)B(h(l-1,1)s_1 s_2)\\ + q\Lambda(\varpi,1)B(h(l,0)\hat{u}_2 s_1 s_2).\end{aligned}$$

Here, the u_i's are labeled as in ii) of Lemma 3.7.

Proof. i) Using (91), we calculate

$$\begin{aligned}(T_{1,0}B)(h(l,0)\hat{u}_0 s_1 s_2) &= \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_0 s_1 s_2 \begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})\\ &\overset{(73)}{=} q\sum_{x,y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_0 s_1 s_2 \begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix} h(1,0))\\ &= A_1 + A_2,\end{aligned}$$

where A_1 consists of those terms where x is a unit, and A_2 consists of those terms where $x = 0$. Using (57), we first calculate

$$\begin{aligned}A_1 &= q\sum_{\substack{x\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&\\&&1&\\&x&&1\end{bmatrix} s_1 s_2 \begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix} h(1,0))\\ &= q\sum_{\substack{x\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&x\\&&1&\\&&&1\end{bmatrix} s_1 s_2 s_1 \begin{bmatrix}1&&&\\&1&&x^{-1}\\&&1&\\&&&1\end{bmatrix} s_1 s_2 \begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix} h(1,0))\\ &= q\sum_{\substack{x\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}}} \psi(\mathbf{c}x\varpi^l) B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&\\&&1&\\&-x^{-1}&&1\end{bmatrix}\begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix} h(1,0))\end{aligned}$$

$$= q \sum_{\substack{x\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}}} \psi(\mathbf{c}x\varpi^l)B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&x^{-1}y^2&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&x^{-1}y&&\\&1&&\\&&1&\\&-x^{-1}&-x^{-1}y&1\end{bmatrix}h(1,0))$$

$$\overset{(73)}{=} q \sum_{\substack{x\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}}} \psi(\mathbf{c}x\varpi^l)B(h(l,0)\hat{u}_0h(1,0))$$

$$= q^2 \sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times} \psi(\mathbf{c}x\varpi^l)B(h(l+1,0))$$

$$= \begin{cases} -q^2B(1) & \text{if } l=-1,\\ q^2(q-1)B(h(l+1,0)) & \text{if } l\geq 0.\end{cases}$$

Next, using Lemma 3.5 ii),

$$A_2 = q\sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_0\begin{bmatrix}1&y&&\\&1&&\\&&1&\\&&-y&1\end{bmatrix}s_1s_2h(1,0))$$

$$= q\sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0)\hat{u}_0\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_1s_2h(1,0)) - qB(h(l,0)\hat{u}_0s_2h(1,0))$$

$$= q^2B(h(l,0)s_1s_2h(1,0))$$

$$= q^2B(h(l-1,1)s_1s_2).$$

By Lemma 3.6, this is zero for $l=-1$. This concludes the proof of i).

ii) is proved similarly; at some point one uses Lemma 3.7 ii). ■

5 The Hecke operator $T_{0,1}$

This section is analogous to the previous one. We will calculate the action of the Hecke operator $T_{0,1}$ defined in (33) on an element B of the space $\mathcal{S}(\Lambda,\theta,P_1)$; see (21). It is easy to verify the explicit formula

$$T_{0,1}v = \sum_{\substack{u,y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} \pi(\begin{bmatrix}1&u&&\\&1&&\\&&1&\\&&-u&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})v$$
$$+\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in\mathfrak{o}/\mathfrak{p}^2}} \pi(\begin{bmatrix}1&&&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})v \tag{94}$$

which holds for any P_1-invariant vector v in a smooth representation (π,V) of $\mathrm{GSp}_4(F)$.

5.1 Lemma. *Let $B \in \mathcal{S}(\Lambda,\theta,P_1)$, and let l and m be non-negative integers. Let $h(l,m)$ be as in (31). Then the following formulas hold.*

i)

$$(T_{0,1}B)(h(l,m)) = \begin{cases} 0 & \text{if } m < m_0, \\ q^3\Lambda(\varpi)B(h(l+2,m-1)) + q^4B(h(l,m+1)) & \\ & \text{if } m \geq \max(m_0,1), \\ q^3(q+1)B(h(l,1)) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ q^3\Lambda(\varpi_L)B(h(l+1,0)) + q^4B(h(l,1)) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ q^3(\Lambda(\varpi,1)+\Lambda(1,\varpi))B(h(l+1,0)) + q^3(q-1)B(h(l,1)) & \\ & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

ii)

$$\begin{aligned} &(T_{0,1}B)(h(l,m)s_2) \\ &= \begin{cases} -q^3B(h(l,m+1)s_1s_2) - \Lambda(\varpi)B(h(l-2,m+1)s_1s_2) & \text{if } m < m_0, \\ q^3\Lambda(\varpi)B(h(l+2,m-1)s_2) + \Lambda(\varpi)^2B(h(l,m-1)s_2) & \\ \quad +(q-1)\Lambda(\varpi)B(h(l-2,m+1)s_1s_2) + q^3(q-1)B(h(l,m+1)s_1s_2) & \\ & \text{if } m \geq \max(m_0,1), \\ q^4B(h(l,1)s_1s_2) + q\Lambda(\varpi)B(h(l-2,1)s_1s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ q^3\Lambda(\varpi_L)B(h(l+1,0)\hat{u}_0s_1s_2) + \Lambda(\varpi)\Lambda(\varpi_L)B(h(l-1,0)\hat{u}_0s_1s_2) & \\ \quad +(q-1)\Lambda(\varpi)B(h(l-2,1)s_1s_2) + q^3(q-1)B(h(l,1)s_1s_2) & \\ & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ q^3(\Lambda(\varpi,1)B(h(l+1,0)\hat{u}_2s_1s_2) + \Lambda(1,\varpi)B(h(l+1,0)\hat{u}_1s_1s_2)) & \\ \quad +\Lambda(\varpi)\big(\Lambda(\varpi,1)B(h(l-1,0)\hat{u}_2s_1s_2) + \Lambda(1,\varpi)B(h(l-1,0)\hat{u}_1s_1s_2)\big) & \\ \quad +q^3(q-2)B(h(l,1)s_1s_2) + (q-2)\Lambda(\varpi)B(h(l-2,1)s_1s_2) & \\ & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases} \\ &\quad + q^2(q-1)B(h(l,m+1)) + \left\{ \begin{array}{ll} \left(\dfrac{L}{\mathfrak{p}}\right)q\Lambda(\varpi)B(1), & \text{if } l = m = 0, \\ 0 & \text{if } l = 0,\ m \geq 1, \\ q(q-1)\Lambda(\varpi)B(h(l,m)) & \text{if } l \geq 1. \end{array} \right\}. \end{aligned}$$

Proof. i) By (94), we have

$$\begin{aligned}
(T_{0,1}B)(h(l,m)) &= \sum_{\substack{u,y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,m)\begin{bmatrix}1&u&&\\&1&&\\&&1&\\&&-u&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})\\
&\quad+\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,m)\begin{bmatrix}1&&&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})\\
&= q^3\sum_{u\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&u&&\\&1&&\\&&1&\\&&-u&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})\\
&\quad+q^3B(h(l,m)\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})\\
&= q^3\sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}h(0,1)).
\end{aligned}$$

Hence, by Lemma 3.5 and observing $s_1h(0,1)=\varpi h(2,-1)s_1$,

$$(T_{0,1}B)(h(l,m))=\begin{cases}0 & \text{if } m<m_0,\\
q^3\Lambda(\varpi)B(h(l+2,m-1))+q^4B(h(l,m+1)) & \text{if } m\geq\max(m_0,1),\\
q^3(q+1)B(h(l,1)) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=-1,\\
q^3B(h(l,0)\hat{u}_0s_1h(0,1))+q^4B(h(l,1)) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=0,\\
q^3\displaystyle\sum_{i=1,2}B(h(l,0)\hat{u}_is_1h(0,1))+q^3(q-1)B(h(l,1)) & \\
& \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=1.\end{cases}$$

The asserted formula is obtained after using Lemma 3.7 to rewrite the terms containing $\hat{u}_i$.

ii) By (94), we have $(T_{0,1}B)(h(l,m)s_2s_1)=A_1+A_2$ with

$$A_1=\sum_{\substack{u,y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,m)s_2s_1\begin{bmatrix}1&u&&\\&1&&\\&&1&\\&&-u&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

and

$$A_2 = \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,m)s_2s_1 \begin{bmatrix}1&&&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}).$$

We have

$$\begin{aligned}
A_1 &= q^3\sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&&&\\y&1&&\\&&1&-y\\&&&1\end{bmatrix}s_2\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})\\
&= q^3\sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma_0(\mathfrak{p})} B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_2\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})\\
&\qquad - q^3B(h(l,m)s_1s_2\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})\\
&= q^3(q+1)\int\limits_{\mathrm{GL}_2(\mathfrak{o})} B(h(l,m)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix}s_2\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})\,dg - q^3B(h(l,m+1)s_1s_2).
\end{aligned}$$

By Lemma 3.5 i), and Lemma 3.7, we have

$$A_1 = \begin{cases}
-q^3B(h(l,m+1)s_1s_2) & \text{if } m<m_0,\\
q^3\Lambda(\varpi)B(h(l+2,m-1)s_2)+q^3(q-1)B(h(l,m+1)s_1s_2) & \text{if } m\ge\max(m_0,1),\\
q^4B(h(l,1)s_1s_2) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=-1,\\
q^3\Lambda(\varpi_L)B(h(l+1,0)\hat{u}_0s_1s_2)+q^3(q-1)B(h(l,1)s_1s_2) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=0,\\
q^3(\Lambda(\varpi,1)B(h(l+1,0)\hat{u}_2s_1s_2)+\Lambda(1,\varpi)B(h(l+1,0)\hat{u}_1s_1s_2)) & \\
\qquad +q^3(q-2)B(h(l,1)s_1s_2) & \text{if } m=m_0=0,\ \left(\frac{L}{\mathfrak{p}}\right)=1.
\end{cases}$$

Next,

$$A_2 = \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,m)\begin{bmatrix}1&&&\\y&1&&\\z&&1&-y\\&&&1\end{bmatrix}s_2\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}) = A_{21}+A_{22}+A_{23},$$

where A_{21} consists of those terms where z is a unit, A_{22} consists of those terms where the valuation of z is 1, and A_{23} consists of the terms where $z = 0$. Using (57), we calculate

$$\begin{aligned} A_{21} &= \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in(\mathfrak{o}/\mathfrak{p}^2)^\times}} B(h(l,m)\begin{bmatrix}1&&&\\y&1&&\\&&1&-y\\&&&1\end{bmatrix}\begin{bmatrix}1&&z^{-1}&\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})\\ &= q^2(q-1)B(h(l,m+1)). \end{aligned}$$

By the same identity,

$$\begin{aligned} A_{22} &= \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&&&\\y&1&&\\z\varpi&&1&-y\\&&&1\end{bmatrix}s_2\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})\\ &= \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&&&\\y&1&&\\&&1&-y\\&&&1\end{bmatrix}\begin{bmatrix}1&&z^{-1}\varpi^{-1}&\\&1&&\\&&1&\\&&&1\end{bmatrix}\\ &\qquad\begin{bmatrix}-z^{-1}\varpi^{-1}&&&\\&1&&\\&&-z\varpi&\\&&&1\end{bmatrix}s_2\begin{bmatrix}1&&z^{-1}\varpi^{-1}&\\&1&&\\&&1&\\&&&1\end{bmatrix}s_2\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})\\ &= \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,m)\begin{bmatrix}1&&z^{-1}\varpi^{-1}&yz^{-1}\varpi^{-1}\\&1&yz^{-1}\varpi^{-1}&y^2z^{-1}\varpi^{-1}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y&1&&\\&&1&-y\\&&&1\end{bmatrix})\\ &= \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in(\mathfrak{o}/\mathfrak{p})^\times}} \psi\big(z\varpi^{l-1}(\mathbf{c}y^2+\mathbf{b}y\varpi^m+\mathbf{a}\varpi^{2m})\big)B(h(l,m))\\ &= \begin{cases} q(q-1)\Lambda(\varpi)B(h(l,m)) & \text{if } l\geq 1,\\ 0 & \text{if } l=0,\ m\geq 1,\\ -q\Lambda(\varpi)B(1) & \text{if } l=m=0,\ \big(\frac{L}{\mathfrak{p}}\big)=-1,\\ 0 & \text{if } l=m=0,\ \big(\frac{L}{\mathfrak{p}}\big)=0,\\ q\Lambda(\varpi)B(1) & \text{if } l=m=0,\ \big(\frac{L}{\mathfrak{p}}\big)=1. \end{cases} \end{aligned}$$

Finally, we calculate

$$A_{23} = \sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,m)\begin{bmatrix}1&&&\\y&1&&\\&&1&-y\\&&&1\end{bmatrix}s_2\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$= \sum_{g\in \mathrm{GL}_2(\mathfrak{o})/\Gamma_0(\mathfrak{p})} B(h(l,m)\begin{bmatrix} g & \\ & \det(g)\,{}^tg^{-1}\end{bmatrix} s_2h(0,1)) - B(h(l,m)s_1s_2h(0,1))$$
$$= (q+1)\int\limits_{\mathrm{GL}_2(\mathfrak{o})} B(h(l,m)\begin{bmatrix} g & \\ & \det(g)\,{}^tg^{-1}\end{bmatrix} s_2h(0,1))\,dg - \Lambda(\varpi)B(h(l-2,m+1)s_1s_2).$$

By Lemma 3.5 i), and Lemma 3.7, we have

$$A_{23} = \begin{cases} -\Lambda(\varpi)B(h(l-2,m+1)s_1s_2) & \text{if } m < m_0,\\ \Lambda(\varpi)^2B(h(l,m-1)s_2) + (q-1)\Lambda(\varpi)B(h(l-2,m+1)s_1s_2) & \\ & \text{if } m \geq \max(m_0,1),\\ q\Lambda(\varpi)B(h(l-2,1)s_1s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = -1,\\ \Lambda(\varpi)\Lambda(\varpi_L)B(h(l-1,0)\hat{u}_0s_1s_2) + (q-1)\Lambda(\varpi)B(h(l-2,1)s_1s_2) & \\ & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 0,\\ \Lambda(\varpi)(\Lambda(\varpi,1)B(h(l-1,0)\hat{u}_2s_1s_2) + \Lambda(1,\varpi)B(h(l-1,0)\hat{u}_1s_1s_2)) & \\ \quad +(q-2)\Lambda(\varpi)B(h(l-2,1)s_1s_2) & \text{if } m = m_0 = 0,\ \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

Summarizing everything, we obtain the asserted formula. ∎

5.2 Lemma. *Let $B \in \mathcal{S}(\Lambda,\theta,P_1)$ be such that $T_{1,0}B = \lambda B$ and $T_{0,1}B = \mu B$. Let l and m be non-negative integers. Then*

$$\begin{aligned}\lambda\Lambda(\varpi)B(h(l,m)s_2) - \mu qB(h(l+1,m)s_2) + \lambda q^3B(h(l+2,m)s_2)\\ = q^5(q-1)B(h(l+3,m)) - q^3(q-1)B(h(l+1,m+1))\end{aligned} \tag{95}$$

for $m \geq \max(m_0,1)$, or for $m \geq m_0-1$ if $m_0 > 0$.

Proof. Assume that $l \geq 0$ and $m \geq \max(m_0,1)$. Then, replacing l by $l+1$ in ii) of Lemma 5.1,

$$\begin{aligned}\mu B(h(l+1,m)s_2) = q^3\Lambda(\varpi)B(h(l+3,m-1)s_2) + q^3(q-1)B(h(l+1,m+1)s_1s_2)\\ + \Lambda(\varpi)^2B(h(l+1,m-1)s_2) + (q-1)\Lambda(\varpi)B(h(l-1,m+1)s_1s_2)\\ + q^2(q-1)B(h(l+1,m+1)) + q(q-1)\Lambda(\varpi)B(h(l+1,m)).\end{aligned}$$

Recall from Lemma 4.1 ii) that

$$\begin{aligned}\lambda B(h(l,m)s_2) = q^2(q-1)B(h(l+1,m)) + q\Lambda(\varpi)B(h(l+1,m-1)s_2)\\ + q(q-1)B(h(l-1,m+1)s_1s_2)\end{aligned} \tag{96}$$

for $l \geq 0$ and $m \geq \max(m_0,1)$. Substituting, we obtain (95) for $l \geq 0$ and $m \geq \max(m_0,1)$. Assume now that $m_0 > 0$. Setting $m = m_0-1$ and replacing l by $l+1$ in ii) of Lemma 5.1, we get

$$\mu B(h(l+1,m_0-1)s_2) = -q^3B(h(l+1,m_0)s_1s_2) - \Lambda(\varpi)B(h(l-1,m_0)s_1s_2)$$

$$+ q^2(q-1)B(h(l+1,m_0)) \tag{97}$$

for $l \geq 0$; observe here Lemma 3.6. By Lemma 4.1 ii),

$$\lambda B(h(l,m_0-1)s_2) = -qB(h(l-1,m_0)s_1s_2) \tag{98}$$

for $l \geq 0$. Substituting (98) into (97), we obtain (95) for $l \geq 0$ and $m = m_0 - 1 \geq 0$. ∎

From ii) of Lemma 5.1 we have for $l = 0$ and $m \geq \max(m_0, 1)$,

$$\begin{aligned}\mu B(h(0,m)s_2) &= q^3\Lambda(\varpi)B(h(2,m-1)s_2) + q^3(q-1)B(h(0,m+1)s_1s_2)\\ &\quad + \Lambda(\varpi)^2B(h(0,m-1)s_2) + q^2(q-1)B(h(0,m+1)).\end{aligned}$$

Substituting (96) (for $l = 1$) into this, we get

$$\begin{aligned}&\mu B(h(0,m)s_2) - q^2\lambda B(h(1,m)s_2) - \Lambda(\varpi)^2B(h(0,m-1)s_2)\\ &\quad = q^2(q-1)B(h(0,m+1)) - q^4(q-1)B(h(2,m))\end{aligned} \tag{99}$$

for $m \geq \max(m_0, 1)$. We will use (99) in the proof of Lemma 8.1. For $m_0 > 0$ and $m = m_0$, a variant of this identity is as follows. By Lemma 4.1 i) and Lemma 5.1 i), we have $\lambda B(h(l,m_0)) = q^3B(h(l+1,m_0))$ and $\mu B(h(l,m_0)) = q^4B(h(l,m_0+1))$ for $l \geq 0$. Hence, from (99) we get

$$\begin{aligned}&\mu B(h(0,m_0)s_2) - q^2\lambda B(h(1,m_0)s_2) - \Lambda(\varpi)^2B(h(0,m_0-1)s_2)\\ &\quad = q^{-2}(q-1)(\mu-\lambda^2)B(h(0,m_0))\end{aligned} \tag{100}$$

for $m_0 > 0$. We will use (100) while considering the case IIIa.

We will also write down an alternative version of (95) for $m_0 > 0$ and $m = m_0 - 1$. Using $\lambda B(h(l,m_0)) = q^3B(h(l+1,m_0))$ and some automatic vanishing, we have

$$\begin{aligned}&\lambda\Lambda(\varpi)B(h(l,m_0-1)s_2) - \mu qB(h(l+1,m_0-1)s_2) + \lambda q^3B(h(l+2,m_0-1)s_2)\\ &\quad = -\lambda(q-1)B(h(l,m_0))\end{aligned} \tag{101}$$

for $l \geq 0$ (and $m_0 > 0$). We will use (101) while considering the case IIIa.

5.3 Lemma. *Let Λ be an unramified character of $L^\times$. Then*

i) In the ramified case $\left(\frac{L}{\mathfrak{p}}\right) = 0$, for all integers $l \geq -1$,

$$(T_{0,1}B)(h(l,0)\hat{u}_0s_1s_2) = \begin{cases} q^4B(h(-1,1)s_1s_2) - q^2\Lambda(\varpi_L)B(1) & \text{if } l=-1,\\ q^4B(h(0,1)s_1s_2) & \\ \quad +q^2(q-1)\Lambda(\varpi_L)B(h(1,0)) - q\Lambda(\varpi)B(1) & \text{if } l=0,\\ q^4B(h(l,1)s_1s_2) + q\Lambda(\varpi)B(h(l-2,1)s_1s_2) & \\ \quad +q^2(q-1)\Lambda(\varpi_L)B(h(l+1,0)) & \\ \quad +\Lambda(\varpi)q(q-1)B(h(l,0))) & \text{if } l\geq 1.\end{cases}$$

ii) In the split case $\left(\frac{L}{\mathfrak{p}}\right)=1$*, for all integers* $l\geq 0$*,*

$$(T_{0,1}B)(h(l,0)\hat{u}_1s_1s_2)=\begin{cases} q^3\Lambda(1,\varpi)B(h(1,0)\hat{u}_1s_1s_2)+q^3(q-1)B(h(0,1)s_1s_2) \\ \qquad +q^2(q-1)\Lambda(\varpi,1)B(h(1,0)) & \text{if } l=0,\\ q^3\Lambda(1,\varpi)B(h(l+1,0)\hat{u}_1s_1s_2)+q^3(q-1)B(h(l,1)s_1s_2) \\ \qquad +q^2(q-1)\Lambda(\varpi,1)B(h(l+1,0)) \\ \qquad +\Lambda(\varpi)q(q-1)B(h(l,0)) \\ \qquad +\Lambda(\varpi)\Lambda(1,\varpi)B(h(l-1,0)\hat{u}_1s_1s_2) \\ \qquad +(q-1)\Lambda(\varpi)B(h(l-2,1)s_1s_2) & \text{if } l\geq 1.\end{cases}$$

$$(T_{0,1}B)(h(l,0)\hat{u}_2s_1s_2)=\begin{cases} q^3\Lambda(\varpi,1)B(h(1,0)\hat{u}_2s_1s_2)+q^3(q-1)B(h(0,1)s_1s_2) \\ \qquad +q^2(q-1)\Lambda(1,\varpi)B(h(1,0)) & \text{if } l=0,\\ q^3\Lambda(\varpi,1)B(h(l+1,0)\hat{u}_2s_1s_2)+q^3(q-1)B(h(l,1)s_1s_2) \\ \qquad +q^2(q-1)\Lambda(1,\varpi)B(h(l+1,0)) \\ \qquad +\Lambda(\varpi)q(q-1)B(h(l,0)) \\ \qquad +\Lambda(\varpi)\Lambda(\varpi,1)B(h(l-1,0)\hat{u}_2s_1s_2) \\ \qquad +(q-1)\Lambda(\varpi)B(h(l-2,1)s_1s_2) & \text{if } l\geq 1.\end{cases}$$

Here, the u_i*'s are labeled as in ii) of Lemma 3.7.*

Proof. i) Using (94), we calculate

$$(T_{0,1}B)(h(l,0)\hat{u}_0s_1s_2)=A_1+A_2,$$

where

$$A_1=\sum_{\substack{u,y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}}B(h(l,0)\hat{u}_0s_1s_2\begin{bmatrix}1&u&&\\&1&&\\&&1&\\&&-u&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

and

$$A_2=\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in\mathfrak{o}/\mathfrak{p}^2}}B(h(l,0)\hat{u}_0s_1s_2\begin{bmatrix}1&&&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}).$$

Let A_{11} be the part of A_1 where u is a unit, and let A_{12} be the part where $u=0$. Using (57), we calculate

$$A_{11}=\sum_{\substack{u\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}}B(h(l,0)\hat{u}_0\begin{bmatrix}1&&&\\&1&&\\&u&1&\\u&&&1\end{bmatrix}s_1s_2\begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$= \sum_{\substack{u\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&u^{-1}\\&1&u^{-1}&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}-u^{-1}&&&\\&-u^{-1}&&\\&&-u&\\&&&-u\end{bmatrix} s_2s_1s_2$$

$$\begin{bmatrix}1&&&u^{-1}\\&1&u^{-1}&\\&&1&\\&&&1\end{bmatrix} s_1s_2 \begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$\overset{(73)}{=} \sum_{\substack{u\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_0 s_1s_2s_1 \begin{bmatrix}1&&&\\u&1&&\\&&1&-u\\&&&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$= \sum_{\substack{u\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_0 s_1s_2s_1 \begin{bmatrix}1&&x&ux+y\\&1&ux+y&u^2x+2uy\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$\overset{(73)}{=} \sum_{\substack{u\in(\mathfrak{o}/\mathfrak{p})^\times\\ y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_0 s_1s_2s_1 \begin{bmatrix}1&&&ux+y\\&1&ux+y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$= q^2(q-1)\sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&y&&\\&1&&\\&&1&\\&&-y&1\end{bmatrix} s_1s_2s_1 \begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$= q^2(q-1)\sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0)\hat{u}_0 \begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix} s_1s_2s_1h(0,1))$$
$$\quad - q^2(q-1)B(h(l,0)\hat{u}_0s_2s_1h(0,1))$$

$$= q^2(q-1)\sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0) \begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix} s_1s_2s_1h(0,1))$$
$$\quad - q^2(q-1)B(h(l,0)\hat{u}_0s_2s_1h(0,1)).$$

By Lemma 3.5 ii),

$$\begin{aligned}A_{11} &= q^3(q-1)B(h(l,0)s_1s_2s_1h(0,1))\\ &= q^3(q-1)B(h(l,1)s_1s_2).\end{aligned}$$

Next,

$$\begin{aligned} A_{12} &= \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_0 s_1 s_2 \begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}) \\ &= \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&\\&&1&\\&x&&1\end{bmatrix} s_1 s_2 \begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}). \end{aligned}$$

Let A_{121} be the part where x is a unit, A_{122} be the part where x has valuation 1, and A_{123} be the part where $x=0$. We have

$$\begin{aligned} A_{121} = &\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p}^2)^\times}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&x^{-1}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&-x^{-1}&&\\&&1&\\&&&-x\end{bmatrix} s_1 s_2 s_1 \begin{bmatrix}1&&&\\&1&&x^{-1}\\&&1&\\&&&1\end{bmatrix} \\ &\qquad s_1 s_2 \begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}) \\ = &\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p}^2)^\times}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&x^{-1}\\&&1&\\&&&1\end{bmatrix} s_1 \begin{bmatrix}1&&&\\&1&&\\-x&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix} h(0,1)) \\ = &\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p}^2)^\times}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&x^{-1}\\&&1&\\&&&1\end{bmatrix} s_1 \begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\xy&1&&xy^2\\-x&&1&-xy\\&&&1\end{bmatrix} h(0,1)) \\ \overset{(73)}{=} &\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p}^2)^\times}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&x\\&&1&\\&&&1\end{bmatrix} s_1 h(0,1)) \\ = &\, q \sum_{x\in(\mathfrak{o}/\mathfrak{p}^2)^\times} \psi(\mathbf{c}x\varpi^l) B(h(l,0)\hat{u}_0 s_1 h(0,1)) \\ \overset{(71)}{=} &\begin{cases} -q^2\Lambda(\varpi_L)B(1) & \text{if } l=-1, \\ q^2(q-1)\Lambda(\varpi_L)B(h(l+1,0)) & \text{if } l\geq 0. \end{cases} \end{aligned}$$

Next,

$$A_{122} = \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&\\&&1&\\&x\varpi&&1\end{bmatrix} s_1 s_2 \begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$
\begin{aligned}
&= \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&x^{-1}\varpi^{-1}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&-x^{-1}\varpi^{-1}&&\\&&1&\\&&&-x\varpi\end{bmatrix}\\
&\qquad s_1s_2s_1\begin{bmatrix}1&&&\\&1&&x^{-1}\varpi^{-1}\\&&1&\\&&&1\end{bmatrix} s_1s_2\begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})\\
&= \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&x^{-1}\varpi^{-1}\\&&1&\\&&&1\end{bmatrix}\\
&\qquad s_1\begin{bmatrix}1&&&\\&1&&\\-x\varpi&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&-x^{-1}\varpi^{-1}y\\&1&-x^{-1}\varpi^{-1}y&\\&&1&\\&&&1\end{bmatrix})\\
&= \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&&\\&1&&x^{-1}\varpi^{-1}\\&&1&\\&&&1\end{bmatrix}\\
&\qquad s_1\begin{bmatrix}1&&&-x^{-1}\varpi^{-1}y\\&1&-x^{-1}\varpi^{-1}y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\-y&1&&x^{-1}\varpi^{-1}y^2\\-x\varpi&&1&y\\&&&1\end{bmatrix})\\
&= \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&&x^{-1}\varpi^{-1}y^2&x^{-1}\varpi^{-1}y\\&1&x^{-1}\varpi^{-1}y&x^{-1}\varpi^{-1}\\&&1&\\&&&1\end{bmatrix})\\
&= \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} \psi\big(\mathbf{a}x\varpi^{l-1}y^2+\mathbf{b}x\varpi^{l-1}(u_0y^2+y)+\mathbf{c}x\varpi^{l-1}(u_0^2y^2+2u_0y+1)\big)B(h(l,0)).
\end{aligned}
$$

By Lemma 3.6, this is zero if $l=-1$. Assume that $l\geq 0$. Then, by (9) and $\mathbf{b}+2\mathbf{c}u_0\in\mathfrak{p}$,

$$
A_{122} = \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} \psi(\mathbf{c}x\varpi^{l-1})B(h(l,0)).
$$

Hence,

$$
A_{122} = \begin{cases} 0 & \text{if } l=-1,\\ -\Lambda(\varpi)qB(1) & \text{if } l=0,\\ \Lambda(\varpi)q(q-1)B(h(l,0)) & \text{if } l\geq 1.\end{cases}
$$

Next,

$$A_{123} = \sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_0 s_1 s_2 \begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$
$$= \sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&y&&\\&1&&\\&&1&\\&&-y&1\end{bmatrix} s_1 s_2 h(0,1))$$
$$= \sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0)\hat{u}_0 \begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix} s_1 s_2 h(0,1)) - B(h(l,0)\hat{u}_0 s_2 h(0,1))$$
$$= \sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix} s_1 s_2 h(0,1)) - B(h(l,0)\hat{u}_0 s_2 h(0,1)).$$

By Lemma 3.5 ii),

$$A_{123} = qB(h(l,0)s_1 s_2 h(0,1))$$
$$= q\Lambda(\varpi)B(h(l-2,1)s_1 s_2).$$

Note that, by Lemma 3.6, this is zero if $l=-1$ or $l=0$. Next we calculate

$$A_2 = \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_0 s_1 s_2 \begin{bmatrix}1&&&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$
$$\overset{(73)}{=} q^2 \sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_0 \begin{bmatrix}1&y&&\\&1&&\\&&1&\\&&-y&1\end{bmatrix} s_1 s_2 \begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$
$$= q^2 \sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0)\hat{u}_0 \begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix} s_1 s_2 \begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$
$$- q^2 B(h(l,0)\hat{u}_0 s_1 s_1 s_2 \begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$
$$= q^2 \sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0)\begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix} s_1 s_2 \begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$- q^2 B(h(l,0)\hat{u}_0 s_2 \begin{bmatrix} \varpi &&& \\ & \varpi^2 && \\ && \varpi & \\ &&& 1 \end{bmatrix}).$$

By Lemma 3.5,

$$A_2 = q^3 B(h(l,1)s_1 s_2).$$

Summarizing everything, we obtain the asserted formulas.

ii) Using (94), we calculate

$$(T_{0,1}B)(h(l,0)\hat{u}_1 s_1 s_2) = A_1 + A_2,$$

where

$$A_1 = \sum_{\substack{u,y \in \mathfrak{o}/\mathfrak{p} \\ x \in \mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1 s_1 s_2 \begin{bmatrix} 1 & u && \\ & 1 && \\ && 1 & \\ && -u & 1 \end{bmatrix} \begin{bmatrix} 1 && x & y \\ & 1 & y & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi^2 &&& \\ & \varpi && \\ && 1 & \\ &&& \varpi \end{bmatrix})$$

and

$$A_2 = \sum_{\substack{y \in \mathfrak{o}/\mathfrak{p} \\ z \in \mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1 s_1 s_2 \begin{bmatrix} 1 &&& y \\ & 1 & y & z \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & \varpi^2 && \\ && \varpi & \\ &&& 1 \end{bmatrix}).$$

Let A_{11} be the part of A_1 where u is a unit, and let A_{12} be the part where $u = 0$. Using (57), we calculate

$$A_{11} = \sum_{\substack{u \in (\mathfrak{o}/\mathfrak{p})^\times \\ y \in \mathfrak{o}/\mathfrak{p} \\ x \in \mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1 \begin{bmatrix} 1 &&& \\ & 1 && \\ & u & 1 & \\ u &&& 1 \end{bmatrix} s_1 s_2 \begin{bmatrix} 1 && x & y \\ & 1 & y & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi^2 &&& \\ & \varpi && \\ && 1 & \\ &&& \varpi \end{bmatrix})$$

$$= \sum_{\substack{u \in (\mathfrak{o}/\mathfrak{p})^\times \\ y \in \mathfrak{o}/\mathfrak{p} \\ x \in \mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1 \begin{bmatrix} 1 &&& u^{-1} \\ & 1 & u^{-1} & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} -u^{-1} &&& \\ & -u^{-1} && \\ && -u & \\ &&& -u \end{bmatrix} s_2 s_1 s_2$$

$$\begin{bmatrix} 1 &&& u^{-1} \\ & 1 & u^{-1} & \\ && 1 & \\ &&& 1 \end{bmatrix} s_1 s_2 \begin{bmatrix} 1 && x & y \\ & 1 & y & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi^2 &&& \\ & \varpi && \\ && 1 & \\ &&& \varpi \end{bmatrix})$$

$$= \sum_{\substack{u \in (\mathfrak{o}/\mathfrak{p})^\times \\ y \in \mathfrak{o}/\mathfrak{p} \\ x \in \mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1 s_1 s_2 s_1 \begin{bmatrix} 1 &&& \\ u & 1 && \\ && 1 & -u \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 && x & y \\ & 1 & y & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi^2 &&& \\ & \varpi && \\ && 1 & \\ &&& \varpi \end{bmatrix})$$

$$= \sum_{\substack{u\in(\mathfrak{o}/\mathfrak{p})^\times \\ y\in\mathfrak{o}/\mathfrak{p} \\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1 s_1 s_2 s_1 \begin{bmatrix} 1 & & x & ux+y \\ & 1 & ux+y & u^2x+2uy \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix})$$

$$= \sum_{\substack{u\in(\mathfrak{o}/\mathfrak{p})^\times \\ y\in\mathfrak{o}/\mathfrak{p} \\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1 s_1 s_2 s_1 \begin{bmatrix} 1 & & & ux+y \\ & 1 & ux+y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix})$$

$$= q^2(q-1) \sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_1 \begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix})$$

$$= q^2(q-1) \sum_{g\in \mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0)\hat{u}_1 \begin{bmatrix} g & \\ & {}^tg^{-1} \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix})$$

$$- q^2(q-1)B(h(l,0)\hat{u}_1 s_1 s_1 s_2 s_1 \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix})$$

$$= q^2(q-1) \sum_{g\in \mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0) \begin{bmatrix} g & \\ & {}^tg^{-1} \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix})$$

$$- q^2(q-1)B(h(l,0)\hat{u}_1 s_2 s_1 \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}).$$

By Lemma 3.5 ii) and Lemma 3.7 ii),

$$A_{11} = q^2(q-1)\Big(\sum_{i=1,2} B(h(l,0)\hat{u}_i s_2 s_1 h(0,1)) + (q-1)B(h(l,0)s_1 s_2 s_1 h(0,1))\Big)$$

$$- q^2(q-1)B(h(l,0)\hat{u}_1 s_2 s_1 \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}),$$

$$= q^2(q-1)\Big(B(h(l,0)\hat{u}_2 s_2 s_1 h(0,1)) + (q-1)B(h(l,1)s_1 s_2 s_1)\Big)$$

$$= q^2(q-1)\Big(\Lambda(1,\varpi)B(h(l+1,0)\hat{u}_1 s_1 s_2) + (q-1)B(h(l,1)s_1 s_2)\Big).$$

Next,

$$A_{12} = \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1 s_1 s_2 \begin{bmatrix}1&&x&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$
$$= \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1 \begin{bmatrix}1&&&\\&1&&\\&&1&\\&x&&1\end{bmatrix} s_1 s_2 \begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}).$$

Let A_{121} be the part where x is a unit, A_{122} be the part where x has valuation 1, and A_{123} be the part where $x=0$. As in the ramified case, one calculates

$$A_{121} = q^2(q-1)B(h(l,0)\hat{u}_1 s_1 h(0,1))$$
$$\overset{(80)}{=} q^2(q-1)\Lambda(\varpi,1)B(h(l+1,0)).$$

Also as in the ramified case, we have

$$A_{122} = \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} \psi\big(\mathbf{a}x\varpi^{l-1}y^2 + \mathbf{b}x\varpi^{l-1}(u_1y^2+y) + \mathbf{c}x\varpi^{l-1}(u_1^2y^2+2u_1y+1)\big)B(h(l,0)).$$

By (10) and $\mathbf{b}+\mathbf{c}(u_1+u_2)\in\mathfrak{p}$,

$$A_{122} = \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} \psi\big(\mathbf{b}x\varpi^{l-1}y + \mathbf{c}x\varpi^{l-1}(2u_1y+1)\big)B(h(l,0))$$
$$= \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} \psi\big(x\varpi^{l-1}y(\mathbf{b}+2u_1\mathbf{c}) + \mathbf{c}x\varpi^{l-1}\big)B(h(l,0))$$
$$= \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} \psi\big(x\varpi^{l-1}y(\mathbf{b}+(u_1+u_2)\mathbf{c}) + x\varpi^{l-1}y(u_1-u_2)\mathbf{c} + \mathbf{c}x\varpi^{l-1}\big)B(h(l,0))$$
$$= \Lambda(\varpi)\sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ x\in(\mathfrak{o}/\mathfrak{p})^\times}} \psi\big(x\varpi^{l-1}y(u_1-u_2)\mathbf{c} + \mathbf{c}x\varpi^{l-1}\big)B(h(l,0)).$$

Since u_1 and u_2 are modulo $\mathfrak{p}$ different, it follows that

$$A_{122} = \begin{cases}0 & \text{if } l=0,\\ \Lambda(\varpi)q(q-1)B(h(l,0)) & \text{if } l\geq 1.\end{cases}$$

Next,

$$A_{123} = \sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_1 s_1 s_2 \begin{bmatrix}1&&&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$= \sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_1 \begin{bmatrix}1&y&&\\&1&&\\&&1&\\&&-y&1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$= \sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0)\hat{u}_1 \begin{bmatrix}g&\\&{}^tg^{-1}\end{bmatrix} s_1s_2h(0,1)) - B(h(l,0)\hat{u}_1s_1s_1s_2h(0,1))$$

$$= \sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0) \begin{bmatrix}g&\\&{}^tg^{-1}\end{bmatrix} s_1s_2h(0,1)) - B(h(l,0)\hat{u}_1s_2h(0,1)).$$

By Lemma 3.5 ii),

$$\begin{aligned} A_{123} &= \sum_{i=1,2} B(h(l,0)\hat{u}_is_2h(0,1)) + (q-1)B(h(l,0)s_1s_2h(0,1)) - B(h(l,0)\hat{u}_1s_2h(0,1)) \\ &= B(h(l,0)\hat{u}_2s_2h(0,1)) + (q-1)\Lambda(\varpi)B(h(l-2,1)s_1s_2) \\ &\overset{(82)}{=} \Lambda(\varpi)\Lambda(1,\varpi)B(h(l-1,0)\hat{u}_1s_1s_2) + (q-1)\Lambda(\varpi)B(h(l-2,1)s_1s_2) \\ &= \begin{cases} 0 & \text{if } l=0, \\ \Lambda(\varpi)\Lambda(1,\varpi)B(h(l-1,0)\hat{u}_1s_1s_2) + (q-1)\Lambda(\varpi)B(h(l-2,1)s_1s_2) & \text{if } l\geq 1. \end{cases} \end{aligned}$$

For the last equality we used Lemma 3.6. Next we calculate

$$A_2 = \sum_{\substack{y\in\mathfrak{o}/\mathfrak{p}\\ z\in\mathfrak{o}/\mathfrak{p}^2}} B(h(l,0)\hat{u}_1s_1s_2 \begin{bmatrix}1&&&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix} \begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$\overset{(73)}{=} q^2 \sum_{y\in\mathfrak{o}/\mathfrak{p}} B(h(l,0)\hat{u}_1 \begin{bmatrix}1&y&&\\&1&&\\&&1&\\&&-y&1\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$= q^2 \sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0)\hat{u}_1 \begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$- q^2B(h(l,0)\hat{u}_1s_1s_1s_2 \begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$= q^2 \sum_{g\in\mathrm{GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} B(h(l,0) \begin{bmatrix}g&\\&\det(g)\,{}^tg^{-1}\end{bmatrix} s_1s_2 \begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$- q^2 B(h(l,0)\hat{u}_1 s_2 \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}).$$

By Lemma 3.5 ii),

$$\begin{aligned} A_2 &= q^2 \sum_{i=1,2} B(h(l,0)\hat{u}_i s_2 \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}) + q^2(q-1)B(h(l,0)s_1 s_2 \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}) \\ &\quad - q^2 B(h(l,0)\hat{u}_1 s_2 \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}) \\ &= q^2 B(h(l,0)\hat{u}_2 s_2 s_1 h(0,1)) + q^2(q-1)B(h(l,1)s_1 s_2) \\ &\overset{(78)}{=} q^2 \Lambda(1,\varpi) B(h(l+1,0)\hat{u}_1 s_1 s_2) + q^2(q-1)B(h(l,1)s_1 s_2). \end{aligned}$$

Summarizing everything, we obtain the asserted formulas. ■

6 The main tower

Again we consider the matrix S and the associated character θ of $U(F)$; see (19). As usual, the assumptions (5) are in force. Let Λ be a character of $T(F)$ and define the non-negative integer m_0 as in (65). Let B be a function in the space $\mathcal{S}(\Lambda,\theta,P_1)$. We refer to the values of B at the elements $h(l,m)$, defined in (31), as the *main tower* (in view of Proposition 3.4, there is also an s_2-tower etc). Note that $B(h(l,m)) = 0$ for $l < 0$ or $0 \le m < m_0$ by Lemma 3.6. The following result relates the values of $B(h(l,m))$ to $B(h(0,m_0))$, only assuming that B is an eigenfunction for $T_{1,0}$ and $T_{0,1}$.

6.1 Proposition. *Let $B \in \mathcal{S}(\Lambda,\theta,P_1)$ be an eigenfunction for $T_{1,0}$ and $T_{0,1}$ with eigenvalues λ and μ, respectively. Then*

$$B(h(l+1,m)) = \lambda q^{-3} B(h(l,m)) \qquad \text{for all } l \ge 0 \text{ and } m \ge 0. \tag{102}$$

Furthermore, for any $l \ge 0$, there is a formal identity

$$Y^{-m_0} \sum_{m=m_0}^{\infty} B(h(l,m))Y^m = \frac{1-\kappa q^{-4}Y}{1-\mu q^{-4}Y + \lambda^2 q^{-7}\Lambda(\varpi)Y^2} B(h(l,m_0)),$$

where

$$\kappa = \begin{cases} 0 & \text{if } m_0 > 0, \\ (q+1)^{-1}\mu & \text{if } m_0 = 0 \text{ and } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \Lambda(\varpi_L)\lambda & \text{if } m_0 = 0 \text{ and } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ (q-1)^{-1}(q\lambda(\Lambda(\varpi,1)+\Lambda(1,\varpi)) - \mu) & \text{if } m_0 = 0 \text{ and } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}$$

Proof. The relation (102) is immediate from Lemma 4.1 i). Combining (102) with Lemma 5.1 i), we get, for $l \geq 0$ and $m \geq m_0$,

$$q^4 B(h(l,m+2)) - \mu B(h(l,m+1)) + \lambda^2 q^{-3}\Lambda(\varpi)B(h(l,m)) = 0. \tag{103}$$

We multiply this by Y^{m+2} and apply $\sum_{m=m_0}^{\infty}$ to both sides, arriving at the formal identity

$$\sum_{m=m_0}^{\infty} B(h(l,m))Y^m = \frac{(q^4-\mu Y)B(h(l,m_0)) + q^4 Y B(h(l,m_0+1))}{q^4 - \mu Y + \lambda^2 q^{-3}\Lambda(\varpi)Y^2} Y^{m_0}.$$

Setting $m = m_0$ in Lemma 5.1 i) and using (102) provides a relation between $B(h(l,m_0))$ and $B(h(l,m_0+1))$. Substituting this relation, we obtain the asserted formula. ∎

6.2 Corollary. *Let $B \in \mathcal{S}(\Lambda,\theta,P_1)$ be an eigenfunction for $T_{1,0}$ and $T_{0,1}$. If $B(h(0,m_0)) = 0$, then $B(h(l,m)) = 0$ for all l, m.*

7 Generic representations and split Bessel models

We recall some basic facts about generic representations of $\mathrm{GSp}_4(F)$, and refer to Sect. 2.6 of [11] for details. We denote by N the unipotent radical of the Borel subgroup B; see (13). For c_1, c_2 in $F^\times$, consider the character ψ_{c_1,c_2} of $N(F)$ given by

$$\psi_{c_1,c_2}(\begin{bmatrix}1&&y&*\\x&1&*&*\\&&1&-x\\&&&1\end{bmatrix}) = \psi(c_1x + c_2y).$$

An irreducible, admissible representation π of $\mathrm{GSp}_4(F)$ is called *generic* if $\mathrm{Hom}_{N(F)}(\pi,\psi_{c_1,c_2}) \neq 0$. In this case there is an associated Whittaker model $\mathcal{W}(\pi,\psi_{c_1,c_2})$ consisting of functions $\mathrm{GSp}_4(F) \to \mathbb{C}$ that transform on the left according to ψ_{c_1,c_2}. For $W \in \mathcal{W}(\pi,\psi_{c_1,c_2})$, there is an associated zeta integral

$$Z(s,W) = \int\limits_{F^\times}\int\limits_{F} W(\begin{bmatrix}a&&&\\&a&&\\x&&1&\\&&&1\end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a. \tag{104}$$

This integral is convergent for $\mathrm{Re}(s) > s_0$, where s_0 is independent of W; see [11], Proposition 2.6.3. More precisely, the integral converges to an element of $\mathbb{C}(q^{-s})$, and therefore has meromorphic continuation to all of $\mathbb{C}$. Moreover, there exists an L-factor of the form

$$L(s,\pi) = \frac{1}{Q(q^{-s})}, \qquad Q(X) \in \mathbb{C}[X],\ Q(0) = 1,$$

such that

$$\frac{Z(s,W)}{L(s,\pi)} \in \mathbb{C}[q^{-s},q^s] \qquad \text{for all } W \in \mathcal{W}(\pi,\psi_{c_1,c_2}). \tag{105}$$

Consider the functional f_s on $\mathcal{W}(\pi,\psi_{c_1,c_2})$ given by

$$f_s(W)=\frac{Z(s,\pi(s_2)W)}{L(s,\pi)}, \tag{106}$$

with s_2 as in (16). By (105), this functional is defined for *any* value of s. It is straightforward to verify that f_s has the properties

$$f_s(\pi(\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix})W)=\psi(c_1y)f_s(W). \tag{107}$$

for all $x,y,z\in F$, and

$$f_s(\pi(\begin{bmatrix}1&&&\\&a&&\\&&a&\\&&&1\end{bmatrix})W)=|a|^{-s+1/2}f_s(W). \tag{108}$$

for all $a\in F^\times$.

Next we explain the connection with split Bessel models. Let $\mathbf{a},\mathbf{b},\mathbf{c}\in F$ and define the matrix S as in (1). We assume that we are in the split case, i.e., that the quantity $\mathbf{d}=\mathbf{b}^2-4\mathbf{ac}$ lies in $F^{\times 2}$. In this section it will *not* be necessary to make our usual assumptions $\mathbf{a},\mathbf{b}\in\mathfrak{o}$ and $\mathbf{c},\mathbf{d}\in\mathfrak{o}^\times$ (see Sect. 1), and we will not make these assumptions. Since we are in the split case, there exists a matrix A such that

$${}^tASA=\begin{bmatrix}&1/2\\1/2&\end{bmatrix}.$$

After a change of models, see (23), we may assume that S has this special form. In other words, we will assume that $\mathbf{a}=\mathbf{c}=0$ and $\mathbf{b}=1$. In this case the group $T(F)$ defined in (2) is given by

$$T(F)=\{\begin{bmatrix}a&\\&b\end{bmatrix}:\ a,b\in F^\times\}. \tag{109}$$

The embedding of this group into $\mathrm{GSp}_4(F)$ consists of all matrices of the form $\mathrm{diag}(a,b,b,a)$ with a,b in $F^\times$. Hence, equations (107) and (108) show that f_s is a Bessel functional with respect to the matrix $S=\begin{bmatrix}&1/2\\1/2&\end{bmatrix}$ and a certain unramified character Λ. In fact, since s in (108) is arbitrary, it follows that the generic representation π admits a split Bessel model with respect to any unramified character Λ of $T(F)$ that satisfies the central character condition $\Lambda\big|_{F^\times}=\omega_\pi$. With just slightly more effort one can show that generic representations have split Bessel models with respect to *any* character Λ of $T(F)$; see Lemma 3.16 of [7].

Zeta integrals of Siegel vectors

Let (π,V) be an irreducible, admissible, generic representation of $\mathrm{GSp}_4(F)$ with unramified central character. We assume that $V=\mathcal{W}(\pi,\psi_{c_1,c_2})$ is the Whittaker model with respect to the character ψ_{c_1,c_2} of $N(F)$. For what follows we will assume that $c_1,c_2\in\mathfrak{o}^\times$. Recall the Siegel congruence subgroup P_1 and the Klingen congruence subgroup P_2 defined in (14).

7.1 Lemma. *Let (π, V) be as above, and let W be an element of $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$.*

i) If W is P_1-invariant, then

$$Z(s, \pi(s_2)W) = \int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2)|a|^{s-3/2}\, d^\times a.$$

ii) If W is P_2-invariant, then

$$Z(s, W) = \int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a.$$

Proof. ii) is a special case of Lemma 4.1.1 of [11], and i) is proved similarly. ∎

Let $W \in V$ be a P_2-invariant vector. Then

$$T_{\rm Si}W := \sum_{g \in {\rm GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} \pi(\begin{bmatrix} g & \\ & {}^tg^{-1} \end{bmatrix})W$$

is P_1-invariant. We calculate, using Lemma 7.1,

$$\begin{aligned} Z(s, \pi(s_2)(T_{\rm Si}W)) &= \int\limits_{F^\times} (T_{\rm Si}W)(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2)|a|^{s-3/2}\, d^\times a \\ &= \sum_{g \in {\rm GL}_2(\mathfrak{o})/\Gamma^0(\mathfrak{p})} \int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2 \begin{bmatrix} g & \\ & {}^tg^{-1} \end{bmatrix})|a|^{s-3/2}\, d^\times a \\ &= A + B, \end{aligned}$$

where

$$A = \sum_{y \in \mathfrak{o}/\mathfrak{p}} \int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2 \begin{bmatrix} 1 & y && \\ & 1 && \\ && 1 & \\ && -y & 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a$$

and

$$B = \int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2 s_1)|a|^{s-3/2}\, d^\times a.$$

Using (57), we have

$$\begin{aligned} A = &\sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times}\int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix} s_2 \begin{bmatrix} 1 &&& \\ y & 1 && \\ && 1 & -y \\ &&& 1\end{bmatrix} s_1)|a|^{s-3/2}\,d^\times a \\ &+\int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix} s_2)|a|^{s-3/2}\,d^\times a \\ = &\sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times}\int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix} s_2 s_1)|a|^{s-3/2}\,d^\times a \\ &+\int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix})|a|^{s-3/2}\,d^\times a \\ = &(q-1)\int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix} s_2 s_1)|a|^{s-3/2}\,d^\times a + Z(s,W). \end{aligned}$$

Thus we obtain the formula

$$Z(s,\pi(s_2)(T_{\rm Si}W)) = q\int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix} s_2 s_1)|a|^{s-3/2}\,d^\times a + Z(s,W). \tag{110}$$

The IIa case

Now assume that $\pi = \chi{\rm St}_{{\rm GL}(2)} \rtimes \sigma$ is a representation of type IIa with unramified χ and σ. Let $\alpha = \chi(\varpi)$ and $\gamma = \sigma(\varpi)$. We assume that $\alpha^2\gamma^2 = 1$, i.e., that π has trivial central character. Recall the paramodular group $P_{02} = K^{\rm para}(\mathfrak{p})$ defined in (15). By Corollary 7.2.6 of [11], the space of $K^{\rm para}(\mathfrak{p})$-invariant vectors in π is one-dimensional. Moreover, if W_0 is a vector in the ψ_{c_1,c_2}-Whittaker model of π spanning this one-dimensional space, then $W_0(1) \neq 0$, and if we normalize W_0 by setting

$$W_0(1) = \frac{1}{1-q^{-1}}, \tag{111}$$

then

$$Z(s,W_0) = L(s,\pi) = L(s,\sigma)L(s,\sigma^{-1})L(s,\nu^{1/2}\chi\sigma). \tag{112}$$

By Table A.15 of [11], the Atkin-Lehner element η defined in (34) acts on W_0 and on the one-dimensional space of P_1-invariant vectors with the same eigenvalue $\omega = -\alpha\gamma$ (see also our Table

4). The lemmas below will make use of the following two general identities. Let

$$t_1 = \begin{bmatrix} 1 &&& \\ &&& \varpi^{-1} \\ && 1 & \\ & -\varpi && \end{bmatrix} = h(1,0)^{-1} s_1 s_2 s_1 h(1,0). \tag{113}$$

This element lies in the paramodular group $K^{\rm para}(\mathfrak{p})$. Therefore, for any g in $\mathrm{GSp}_4(F)$,

$$\begin{aligned} \omega W_0(gs_2s_1) &= W_0(gs_2s_1\eta) \\ &= W_0(gs_2s_1s_2s_1s_2h(1,0)) \\ &= W_0(gs_1s_2s_1h(1,0)) \\ &= W_0(gh(1,0)t_1) \\ &= W_0(gh(1,0)) \end{aligned} \tag{114}$$

and

$$\begin{aligned} \omega W_0(gs_1) &= W_0(gs_1\eta) \\ &= W_0(gs_1s_2s_1s_2h(1,0)) \\ &= W_0(gs_2s_1s_2s_1h(1,0)) \\ &= W_0(gs_2h(1,0)t_1) \\ &= W_0(gs_2h(1,0)) \\ &= W_0(g\begin{bmatrix} 1 &&& \\ & \varpi && \\ && \varpi & \\ &&& 1 \end{bmatrix}). \end{aligned} \tag{115}$$

7.2 Lemma. *Let $\pi = \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ be a representation of type IIa with unramified χ and σ. Let $\alpha = \chi(\varpi)$ and $\gamma = \sigma(\varpi)$. We assume that $\alpha^2\gamma^2 = 1$, i.e., that π has trivial central character. Let W_0 be a non-zero $K^{\rm para}(\mathfrak{p})$-invariant vector in π normalized such that $Z(s, W_0) = L(s, \pi)$. Let $\omega = -\alpha\gamma$ be the eigenvalue of the Atkin-Lehner element η on W_0.*

i) For any $s \in \mathbb{C}$,

$$Z(s, \pi(s_2)(T_{\rm Si}W_0)) = (\omega q^{s-1/2} + 1)L(s, \pi)$$

ii) If $\omega q^{s-1/2} + 1 = 0$, then

$$Z(s, T_{\rm Si}W_0) = \frac{1}{1 - q^{-1}}.$$

Proof. i) We may assume that s is in the region of convergence. Substituting the relation (114) into (110), we obtain

$$Z(s, \pi(s_2)(T_{\rm Si}W_0)) = \omega q \int\limits_{F^\times} W_0(\begin{bmatrix} a\varpi &&& \\ & a\varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a + Z(s, W_0)$$

$$= \omega q^{s-1/2} \int\limits_{F^\times} W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\,d^\times a + Z(s, W_0)$$
$$= (\omega q^{s-1/2} + 1) Z(s, W_0)$$
$$= (\omega q^{s-1/2} + 1) L(s, \pi). \tag{116}$$

This proves i).

ii) For the following calculation, we assume again that s is in the region of convergence. By the definitions involved, $Z(s, T_{\rm Si} W_0) = A + B$, where

$$A = \sum_{y \in \mathfrak{o}/\mathfrak{p}} \int\limits_{F^\times} \int\limits_F W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 & y && \\ & 1 && \\ && 1 & \\ && -y & 1 \end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a$$

and

$$B = \int\limits_{F^\times} \int\limits_F W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\,dx\,d^\times a.$$

Using (57), we calculate

$$A = \sum_{y \in (\mathfrak{o}/\mathfrak{p})^\times} \int\limits_{F^\times} \int\limits_F W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ y & 1 && \\ && 1 & -y \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\,dx\,d^\times a$$
$$+ \int\limits_{F^\times} \int\limits_F W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a$$
$$= \sum_{y \in (\mathfrak{o}/\mathfrak{p})^\times} \int\limits_{F^\times} \int\limits_F W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\,dx\,d^\times a$$
$$+ Z(s, W_0)$$
$$= (q-1) \int\limits_{F^\times} \int\limits_F W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\,dx\,d^\times a + Z(s, W_0).$$

Hence,

$$Z(s, T_{\rm Si} W_0) = q \int\limits_{F^\times} \int\limits_F W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\,dx\,d^\times a + Z(s, W_0).$$

Now

$$\int\limits_{F^\times}\int\limits_{F} W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\,dx\,d^\times a = C + D$$

with

$$C = \int\limits_{F^\times}\int\limits_{\mathfrak{p}} W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\,dx\,d^\times a$$

and

$$D = \int\limits_{F^\times}\int\limits_{\substack{F\\ v(x)\le 0}} W_0(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\,dx\,d^\times a.$$

By Lemma 7.3 below,

$$C = q^{-s-2}(q^{-s} - (\gamma+\gamma^{-1}))Z(s, W_0).$$

We calculate

$$\begin{aligned}
D &= \int\limits_{F^\times}\int\limits_{\substack{F\\ v(x)\le 0}} W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_1 \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & x && 1 \end{bmatrix})|a|^{s-3/2}\,dx \\
&= \int\limits_{F^\times}\int\limits_{\substack{F\\ v(x)\le 0}} W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_1 \begin{bmatrix} 1 &&& \\ & 1 && x^{-1} \\ && 1 & \\ &&& 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & -x^{-1} && \\ && 1 & \\ &&& -x \end{bmatrix} s_1s_2s_1)|a|^{s-3/2}\,dx \\
&= \int\limits_{F^\times}\int\limits_{\substack{F\\ v(x)\le 0}} \psi(c_2ax^{-1})W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_1 \begin{bmatrix} 1 &&& \\ & x^{-1} && \\ && 1 & \\ &&& x \end{bmatrix} s_1s_2s_1)|a|^{s-3/2}\,dx \\
&= \int\limits_{F^\times}\int\limits_{\substack{F\\ v(x)\le 0}} \psi(c_2ax^{-1})W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_1 \begin{bmatrix} 1 &&& \\ & x^{-1}\varpi && \\ && 1 & \\ &&& x\varpi^{-1} \end{bmatrix} t_1)|a|^{s-3/2}\,dx \\
&= \int\limits_{F^\times}\int\limits_{\substack{F\\ v(x)\le 0}} \psi(c_2ax^{-1})W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} x^{-1}\varpi &&& \\ & 1 && \\ && x\varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\,dx.
\end{aligned}$$

Now, for any $y \in \mathfrak{p}$,

$$
\begin{aligned}
&W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} x^{-1}\varpi &&& \\ & 1 && \\ && x\varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1) \\
&= W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} x^{-1}\varpi &&& \\ & 1 && \\ && x\varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1 \begin{bmatrix} 1 & y && \\ & 1 && \\ && 1 & \\ && -y & 1 \end{bmatrix}) \\
&= W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} x^{-1}\varpi &&& \\ & 1 && \\ && x\varpi^{-1} & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ y & 1 && \\ && 1 & -y \\ &&& 1 \end{bmatrix} s_1) \\
&= \psi(c_1 y x\varpi^{-1}) W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} x^{-1}\varpi &&& \\ & 1 && \\ && x\varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1).
\end{aligned}
$$

Hence this expression is zero if $v(x) < 0$. It follows that

$$
\begin{aligned}
D &= \int\limits_{F^\times} \int\limits_{\mathfrak{o}^\times} \psi(c_2 a x^{-1}) W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} x^{-1}\varpi &&& \\ & 1 && \\ && x\varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1) |a|^{s-3/2} \, dx \, d^\times a \\
&= \int\limits_{F^\times} \int\limits_{\mathfrak{o}^\times} \psi(c_2 a x) W_0(\begin{bmatrix} a\varpi &&& \\ & a && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1) |a|^{s-3/2} \, dx \, d^\times a \\
&= -q^{-1} \int\limits_{\substack{F^\times \\ v(a)=-1}} W_0(\begin{bmatrix} a\varpi &&& \\ & a && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1) |a|^{s-3/2} \, d^\times a \\
&\quad + (1-q^{-1}) \int\limits_{\substack{F^\times \\ v(a)\geq 0}} W_0(\begin{bmatrix} a\varpi &&& \\ & a && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1) |a|^{s-3/2} \, d^\times a \\
&= -q^{s-5/2}(1-q^{-1}) W_0(\begin{bmatrix} 1 &&& \\ & \varpi^{-1} && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (1-q^{-1}) \int\limits_{\substack{F^\times \\ v(a)\geq 0}} W_0(\begin{bmatrix} a\varpi &&& \\ & a && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\, d^\times a \\
&\overset{(115)}{=} -\omega q^{s-5/2}(1-q^{-1})W_0(1) \\
&\quad + \omega(1-q^{-1}) \int\limits_{\substack{F^\times \\ v(a)\geq 0}} W_0(\begin{bmatrix} a\varpi &&& \\ & a\varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a \\
&= -\omega q^{s-5/2}(1-q^{-1})W_0(1) \\
&\quad + \omega q^{s-3/2}(1-q^{-1}) \int\limits_{\substack{F^\times \\ v(a)\geq 1}} W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a \\
&= -\omega q^{s-5/2}(1-q^{-1})W_0(1) \\
&\quad + \omega q^{s-3/2}(1-q^{-1})\Big(Z(s,W_0) - \int\limits_{\mathfrak{o}^\times} W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a\Big) \\
&= -\omega q^{s-5/2}(1-q^{-1})W_0(1) \\
&\quad + \omega q^{s-3/2}(1-q^{-1})\big(Z(s,W_0) - (1-q^{-1})W_0(1)\big) \\
&= \omega q^{s-3/2}(1-q^{-1})\Big(- q^{-1}W_0(1) + Z(s,W_0) - (1-q^{-1})W_0(1)\Big) \\
&= \omega q^{s-3/2}(1-q^{-1})(Z(s,W_0) - W_0(1)).
\end{aligned}
$$

To summarize,

$$
C + D = q^{-s-2}(q^{-s} - (\gamma+\gamma^{-1}))Z(s,W_0) + \omega q^{s-3/2}(1-q^{-1})(Z(s,W_0) - W_0(1)).
$$

Hence,

$$
\begin{aligned}
Z(s, T_{\mathrm{Si}}W_0) &= q(C+D) + Z(s,W_0) \\
&= q^{-s-1}(q^{-s} - (\gamma+\gamma^{-1}))Z(s,W_0) + \omega q^{s-1/2}(1-q^{-1})(Z(s,W_0) - W_0(1)) + Z(s,W_0) \\
&= q^{-1}(1-\gamma q^{-s})(1-\gamma^{-1}q^{-s})Z(s,W_0) + \omega q^{s-1/2}(1-q^{-1})(Z(s,W_0) - W_0(1)) \\
&\qquad\qquad + (1-q^{-1})Z(s,W_0) \\
&= q^{-1}(1-\gamma q^{-s})(1-\gamma^{-1}q^{-s})L(s,\sigma)L(s,\sigma^{-1})L(s,\nu^{1/2}\chi\sigma)(1-q^{-1})W_0(1) \\
&\quad + (\omega q^{s-1/2}+1)(1-q^{-1})Z(s,W_0) - \omega q^{s-1/2}(1-q^{-1})W_0(1) \\
&= q^{-1}L(s,\nu^{1/2}\chi\sigma)(1-q^{-1})W_0(1) \\
&\quad + (\omega q^{s-1/2}+1)(1-q^{-1})L(s,\sigma)L(s,\sigma^{-1})L(s,\nu^{1/2}\chi\sigma)(1-q^{-1})W_0(1) \\
&\quad - \omega q^{s-1/2}(1-q^{-1})W_0(1)
\end{aligned}
$$

$$= \frac{-q^{s-1/2}\omega}{1+\omega q^{-s-1/2}}(1-q^{-1})W_0(1)$$
$$+ \omega q^{s-1/2}(1+\omega q^{-s+1/2})(1-q^{-1})L(s,\sigma)L(s,\sigma^{-1})L(s,\nu^{1/2}\chi\sigma)(1-q^{-1})W_0(1)$$
$$= \frac{-q^{s-1/2}\omega}{1+\omega q^{-s-1/2}}\big(1-(1+\omega q^{-s+1/2})(1-q^{-1})L(s,\sigma)L(s,\sigma^{-1})\big)(1-q^{-1})W_0(1).$$

By analytic continuation, this last identity holds for all values of $\mathbb{C}$ for which both sides are defined. Hence, if $1+\omega q^{-s+1/2}=0$, then

$$Z(s,T_{\rm Si}W_0) = \frac{-q^{s-1/2}\omega}{1+\omega q^{-s-1/2}}(1-q^{-1})W_0(1) = W_0(1).$$

This concludes the proof. ∎

The previous proof made use of the following result.

7.3 Lemma. *Let the hypotheses be as in Lemma 7.2. Then*

$$\int\limits_{F^\times} W_0(\begin{bmatrix} a&&&\\&a&&\\&&1&\\&&&1\end{bmatrix} s_1)|a|^{s-3/2}\,d^\times a = q^{-s-1}(q^{-s}-(\gamma+\gamma^{-1}))Z(s,W_0).$$

Proof. Let A be the left hand side of the asserted formula. By (115),

$$A = \omega\int\limits_{F^\times} W_0(\begin{bmatrix} a&&&\\&a\varpi&&\\&&\varpi&\\&&&1\end{bmatrix})|a|^{s-3/2}\,d^\times a.$$

Since, for $y\in\mathfrak{o}$,

$$W_0(\begin{bmatrix} a&&&\\&a\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}) = W_0(\begin{bmatrix} a&&&\\&a\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}\begin{bmatrix} 1&&y&\\&1&&\\&&1&\\&&&1\end{bmatrix})$$
$$= \psi(c_2ya\varpi^{-1})W_0(\begin{bmatrix} a&&&\\&a\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}),$$

it follows that

$$A = \omega\int\limits_{\substack{F^\times\\ v(a)\geq 1}} W_0(\begin{bmatrix} a&&&\\&a\varpi&&\\&&\varpi&\\&&&1\end{bmatrix})|a|^{s-3/2}\,d^\times a$$

$$= \omega \sum_{n=1}^{\infty} \int_{\varpi^n \mathfrak{o}^\times} W_0(\begin{bmatrix} a &&& \\ & a\varpi && \\ && \varpi & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a$$

$$= \omega \sum_{n=1}^{\infty} \int_{\mathfrak{o}^\times} W_0(\begin{bmatrix} \varpi^n &&& \\ & \varpi^{n+1} && \\ && \varpi & \\ &&& 1 \end{bmatrix})|\varpi^n|^{s-3/2}\, d^\times a.$$

With

$$c_{i,j} = W_0(\begin{bmatrix} \varpi^{i+j} &&& \\ & \varpi^{2i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix}),$$

we get

$$\begin{aligned} A &= \omega \sum_{n=1}^{\infty} \int_{\mathfrak{o}^\times} c_{1,n-1}|\varpi^n|^{s-3/2}\, d^\times a \\ &= \omega \sum_{n=0}^{\infty} \int_{\mathfrak{o}^\times} c_{1,n}|\varpi^{n+1}|^{s-3/2}\, d^\times a \\ &= \omega q^{-s+3/2} \sum_{n=0}^{\infty} \int_{\mathfrak{o}^\times} c_{1,n}|\varpi^n|^{s-3/2}\, d^\times a. \end{aligned}$$

By Lemma 7.2.4 of [11],

$$c_{1,n} = q^{-4}(\mu c_{0,n} + \omega c_{0,n-1}) \qquad \text{for all } n \geq 0,$$

where

$$\mu = q^{3/2}(\gamma + \gamma^{-1})\alpha\gamma = q^{3/2}(\alpha + \alpha^{-1}).$$

Hence

$$\begin{aligned} A &= \omega q^{-s+3/2} \sum_{n=0}^{\infty} \int_{\mathfrak{o}^\times} q^{-4}(\mu c_{0,n} + \omega c_{0,n-1})|\varpi^n|^{s-3/2}\, d^\times a \\ &= \omega\mu q^{-s-5/2} \sum_{n=0}^{\infty} \int_{\mathfrak{o}^\times} c_{0,n}|\varpi^n|^{s-3/2}\, d^\times a + q^{-s-5/2} \sum_{n=1}^{\infty} \int_{\mathfrak{o}^\times} c_{0,n-1}|\varpi^n|^{s-3/2}\, d^\times a \\ &= \omega\mu q^{-s-5/2} \sum_{n=0}^{\infty} \int_{\mathfrak{o}^\times} c_{0,n}|\varpi^n|^{s-3/2}\, d^\times a + q^{-2s-1} \sum_{n=0}^{\infty} \int_{\mathfrak{o}^\times} c_{0,n}|\varpi^n|^{s-3/2}\, d^\times a \\ &= (\omega\mu q^{-s-5/2} + q^{-2s-1}) \sum_{n=0}^{\infty} \int_{\mathfrak{o}^\times} W_0(\begin{bmatrix} \varpi^n &&& \\ & \varpi^n && \\ && 1 & \\ &&& 1 \end{bmatrix})|\varpi^n|^{s-3/2}\, d^\times a \end{aligned}$$

$$= (\omega\mu q^{-s-5/2} + q^{-2s-1}) \int\limits_{F^\times} W_0(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a$$
$$= (\omega\mu q^{-s-5/2} + q^{-2s-1}) Z(s, W_0).$$

Since $\omega\mu = (-\alpha\gamma)q^{3/2}(\gamma+\gamma^{-1})\alpha\gamma = -q^{3/2}(\gamma+\gamma^{-1})$, this concludes the proof. ■

For the following proposition we continue to assume that $\pi = \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ is a representation of type IIa with unramified χ and σ, but we will drop the condition that π has trivial central character. As before, let $\alpha = \chi(\varpi)$ and $\gamma = \sigma(\varpi)$. If the non-zero vector W spans the space of P_1-invariant vectors, then we still have $\eta W = \omega W$ with $\omega = -\alpha\gamma$, but this constant is no longer necessarily ± 1. We consider split Bessel models with respect to the quadratic form

$$S = \begin{bmatrix} & 1/2 \\ 1/2 & \end{bmatrix}. \tag{117}$$

Let θ be the corresponding character of $U(F)$; see (20). The resulting group $T(F)$, defined in (2), is a split torus and is explicitly given in (109). We think of $T(F)$ embedded into $\mathrm{GSp}_4(F)$ as all matrices of the form $\mathrm{diag}(a,b,b,a)$ with $a, b \in F^\times$. We write a character Λ of $T(F)$ as a function $\Lambda(a,b)$. We want all such Λ to coincide on the center of $\mathrm{GSp}_4(F)$ with the central character of π, i.e., $\Lambda(a,a) = (\chi\sigma)^2(a)$. Since $\Lambda(\varpi,1)\Lambda(1,\varpi) = \omega^2$, we have $\Lambda(1,\varpi) = -\omega$ if and only if $\Lambda(\varpi,1) = -\omega$.

7.4 Proposition. *Assume that $\pi = \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ is a representation of type IIa with unramified χ and σ. Let S, θ and $T(F) \cong F^\times \times F^\times$ be as above. Let Λ be an unramified character of $T(F)$ such that $\Lambda(a,a) = (\chi\sigma)^2(a)$. Let B be a non-zero vector in the (Λ,θ)-Bessel model of π spanning the one-dimensional space of P_1-invariant vectors.*

i) *Assume that $\Lambda(\varpi,1) \neq -\omega \neq \Lambda(1,\varpi)$. Then $B(1) \neq 0$.*

ii) *Assume that $\Lambda(\varpi,1) = -\omega = \Lambda(1,\varpi)$. Then $B(1) = 0$ and $B(s_2) \neq 0$.*

Proof. After twisting by an unramified character, we may assume that π has trivial central character. Let W_0 be a non-zero vector in the ψ_{c_1,c_2}-Whittaker model of π spanning the one-dimensional space of P_{02}-invariant vectors. By Lemma 7.2, the vector $T_{\mathrm{Si}}W_0$ is non-zero, and hence spans the one-dimensional space of P_1-invariant vectors.

Consider the functional f_s on the ψ_{c_1,c_2}-Whittaker model of π given by

$$f_s(W) = \frac{Z(s, \pi(s_2)(W))}{L(s,\pi)}.$$

By analytic continuation and the defining property of $L(s,\pi)$, this is a well-defined and non-zero functional on π for *any* value of s. By (107) and (108), the functional f_s is a split (Λ,θ)-Bessel functional with respect to the character Λ given by

$$\Lambda(\mathrm{diag}(a,b,b,a)) = |a^{-1}b|^{-s+1/2}.$$

Under our trivial central character hypothesis, any unramified character of $T(F)$ is of this form for an appropriate s. Note that

$$\Lambda(\varpi,1)=-\omega=\Lambda(1,\varpi)\qquad\Longleftrightarrow\qquad \omega q^{s-1/2}+1=0.$$

Assume that $\Lambda(\varpi,1)\neq-\omega\neq\Lambda(1,\varpi)$. Then, by i) of Lemma 7.2, we have $f_s(T_{\rm Si}W_0)\neq 0$. This is equivalent to $B(1)\neq 0$ by uniqueness of Bessel models. Assume that $\Lambda(\varpi,1)=-\omega=\Lambda(1,\varpi)$. Then, by i) and ii) of Lemma 7.2, we have $f_s(T_{\rm Si}W_0)=0$ and $f_s(\pi(s_2)T_{\rm Si}W_0)\neq 0$. This is equivalent to $B(1)=0$ and $B(s_2)\neq 0$ by uniqueness of Bessel models. ∎

The VIa case

Let $\pi=\tau(S,\nu^{-1/2}\sigma)$ be the representation of type VIa. We assume that σ is unramified and that π has trivial central character, i.e., $\sigma^2=1$. This is a representation of conductor 2; by Table 1 and (30),

$$\varepsilon(s,\pi)=q^{-2(s-1/2)}. \tag{118}$$

By Table A.8 of [11],

$$L(s,\pi)=L(s,\nu^{1/2}\sigma)^2=\frac{1}{(1-\sigma(\varpi)q^{-1/2-s})^2}. \tag{119}$$

We assume that π is given in its ψ_{c_1,c_2}-Whittaker model. Let W_0 be the paramodular newform, i.e., a non-zero element invariant under the paramodular group of level $\mathfrak{p}^2$; see (15). By Theorem 7.5.4 of [11], we can normalize W_0 such that

$$Z(s,W_0)=L(s,\pi).$$

We let

$$W':=\sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&y\varpi&&\\&1&&\\&x\varpi&1&\\x\varpi&z\varpi&-y\varpi&1\end{bmatrix})W_0. \tag{120}$$

This is a P_2-invariant vector; it was called the *shadow of the newform* in Sect. 7.4 of [11]. By Proposition 7.4.8 of [11],

$$Z(s,W')=(1-q^{-1})L(s,\pi). \tag{121}$$

For the next lemma let t_1 be the element defined in (113).

7.5 Lemma. *For all $g\in{\rm GSp}_4(F)$,*

$$\sum_{u\in\mathfrak{o}/\mathfrak{p}}W'(g\begin{bmatrix}1&&&\\&1&&u\varpi^{-1}\\&&1&\\&&&1\end{bmatrix})=-W'(gt_1).$$

Proof. By Lemma 3.3.1 of [11],

$$K^{\rm para}(\mathfrak{p}) = \bigsqcup_{u\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1&&&\\&1&&u\varpi^{-1}\\&&1&\\&&&1\end{bmatrix} P_2 \sqcup t_1P_2.$$

Paramodularizing W' gives zero, since π has no $K^{\rm para}(\mathfrak{p})$-invariant vectors. The assertion follows. ∎

7.6 Lemma. *For any complex number s,*

$$\frac{Z(s,\pi(s_2)(T_{\rm Si}W'))}{L(s,\pi)} = 2(q-1)\sigma(\varpi)q^{-1/2+s}. \tag{122}$$

Proof. We may assume that s is in the region of convergence. By (110) and (121),

$$Z(s,\pi(s_2)(T_{\rm Si}W')) = q\int\limits_{F^\times} W'(\begin{bmatrix}a&&&\\&a&&\\&&1&\\&&&1\end{bmatrix}s_2s_1)|a|^{s-3/2}\,d^\times a + (1-q^{-1})L(s,\pi). \tag{123}$$

To evaluate the integral in this equation, we will compute the zeta integral $Z(s,\pi(s_2s_1)W')$ in two different ways. First, we will employ the local functional equation

$$\frac{Z(1-s,\pi(s_2^{-1}s_1s_2)W)}{L(1-s,\pi)} = \varepsilon(s,\pi)\frac{Z(s,W)}{L(s,\pi)}; \tag{124}$$

see equation (2.62) of [11]. Applied to $W=\pi(s_2s_1)W'$, it shows that

$$\begin{aligned} Z(s,\pi(s_2s_1)W') &= \frac{L(s,\pi)}{L(1-s,\pi)\varepsilon(s,\pi)}Z(1-s,\pi(s_2^{-1}s_1s_2s_2s_1)W')\\ &= \frac{L(s,\pi)}{L(1-s,\pi)\varepsilon(s,\pi)}Z(1-s,W')\\ &\overset{(121)}{=} (1-q^{-1})\frac{L(s,\pi)}{\varepsilon(s,\pi)}. \end{aligned} \tag{125}$$

Second, we will compute $Z(s,\pi(s_2s_1)W')$ using the defining formula. Let us set

$$g = \begin{bmatrix}a\varpi&&&\\&a&&\\&&\varpi^{-1}&\\&&&1\end{bmatrix}s_1$$

in Lemma 7.5. We get

$$W'(\begin{bmatrix}a&&&\\&a&&\\&&1&\\&&&1\end{bmatrix}s_2s_1) = \begin{cases}0 & \text{if } v(a)<-1,\\ -qW'(\begin{bmatrix}a\varpi&&&\\&a&&\\&&\varpi^{-1}&\\&&&1\end{bmatrix}s_1) & \text{if } v(a)\ge -1.\end{cases} \tag{126}$$

The useful identity (57) shows that

$$W'(\begin{bmatrix} a &&& \\ & a && \\ x && 1 & \\ &&& 1 \end{bmatrix} s_2s_1) = \begin{cases} 0 & \text{if } v(x) < -1, \\ \psi(c_2ax^{-1})W'(\begin{bmatrix} a\varpi &&& \\ & a && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1) & \text{if } v(x) = -1, \\ W'(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2s_1) & \text{if } v(x) \geq 0. \end{cases}$$

Hence, using the definition (104),

$$\begin{aligned} Z(s, \pi(s_2s_1)W') &= \int\limits_{F^\times} \int\limits_{\varpi^{-1}\mathfrak{o}^\times} \psi(c_2ax^{-1})W'(\begin{bmatrix} a\varpi &&& \\ & a && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\, dx\, d^\times a \\ &\quad + \int\limits_{F^\times} W'(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2s_1)|a|^{s-3/2}\, d^\times a \\ &= -\int\limits_{\varpi^{-2}\mathfrak{o}^\times} W'(\begin{bmatrix} a\varpi &&& \\ & a && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\, d^\times a \\ &\quad + (q-1) \int\limits_{\substack{F^\times \\ v(a) \geq -1}} W'(\begin{bmatrix} a\varpi &&& \\ & a && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1)|a|^{s-3/2}\, d^\times a \\ &\quad + \int\limits_{F^\times} W'(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2s_1)|a|^{s-3/2}\, d^\times a \\ &\overset{(126)}{=} -\int\limits_{\varpi^{-2}\mathfrak{o}^\times} W'(\begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-2} && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1)|\varpi^{-2}|^{s-3/2}\, d^\times a \\ &\quad - q^{-1}(q-1) \int\limits_{\substack{F^\times \\ v(a) \geq -1}} W'(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2s_1)|a|^{s-3/2}\, d^\times a \end{aligned}$$

$$+ \int\limits_{F^\times} W'(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2 s_1)|a|^{s-3/2}\, d^\times a$$

$$= -(1-q^{-1})W'(\begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-2} && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1)q^{2s-3}$$

$$+ q^{-1} \int\limits_{F^\times} W'(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2 s_1)|a|^{s-3/2}\, d^\times a.$$

From this and (125), we get

$$(1-q^{-1})\frac{L(s,\pi)}{\varepsilon(s,\pi)} = -(1-q^{-1})W'(\begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-2} && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1)q^{2s-3}$$

$$+ q^{-1} \int\limits_{F^\times} W'(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2 s_1)|a|^{s-3/2}\, d^\times a.$$

By (118) and (119), the left hand side is of the form

$$\sum_{n=-2}^{\infty} c_n (q^{-s})^n, \qquad c_n \in \mathbb{C}.$$

By (126), the integral on the right hand side is a power series of the form

$$\sum_{n=-1}^{\infty} d_n (q^{-s})^n, \qquad d_n \in \mathbb{C}.$$

Hence, we can compare coefficients, and obtain

$$W'(\begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-2} && \\ && \varpi^{-1} & \\ &&& 1 \end{bmatrix} s_1) = -q^2, \tag{127}$$

as well as

$$\int\limits_{F^\times} W'(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2 s_1)|a|^{s-3/2}\, d^\times a = (q-1)q^{2s-1}(L(s,\pi)-1). \tag{128}$$

From (128) and (123), we now get

$$Z(s,\pi(s_2)(T_{\rm Si}W'))=(q-1)q^{2s}(L(s,\pi)-1)+(1-q^{-1})L(s,\pi). \tag{129}$$

Using the explicit form (119) of $L(s,\pi)$, we get the assertion. ∎

For the main result of this subsection we continue to assume that $\pi=\tau(S,\nu^{-1/2}\sigma)$ is a representation of type VIa with unramified σ, but will drop the condition that π has trivial central character. We consider split Bessel models with respect to the quadratic form S given in (117). Let θ be the corresponding character of $U(F)$, and $T(F)$ the resulting split torus. As before, we write a character Λ of $T(F)$ as a function $\Lambda(a,b)$. We require such Λ to coincide on the center of ${\rm GSp}_4(F)$ with the central character of π, i.e., $\Lambda(a,a)=\sigma^2(a)$.

7.7 Proposition. *Assume that $\pi=\tau(S,\nu^{-1/2}\sigma)$ is a representation of type VIa with unramified σ. Let S, θ and $T(F)\cong F^\times\times F^\times$ be as above. Let Λ be an unramified character of $T(F)$ such that $\Lambda(a,a)=\sigma^2(a)$. Let B be a non-zero vector in the (Λ,θ)-Bessel model of π spanning the one-dimensional space of P_1-invariant vectors. Then $B(1)\neq 0$.*

Proof. After twisting by an unramified character, we may assume that π has trivial central character. Let W_0 be a non-zero vector in the ψ_{c_1,c_2}-Whittaker model of π spanning the one-dimensional space of $K^{\rm para}(\mathfrak{p}^2)$-invariant vectors, and let W' be the P_2-invariant vector defined in (120). By Lemma 7.6, the vector $T_{\rm Si}W'$ is non-zero, and hence spans the one-dimensional space of P_1-invariant vectors.

Consider the functional f_s on the ψ_{c_1,c_2}-Whittaker model of π given by

$$f_s(W)=\frac{Z(s,\pi(s_2)(W))}{L(s,\pi)}.$$

By analytic continuation and the defining property of $L(s,\pi)$, this is a well-defined and non-zero functional on π for *any* value of s. By (107) and (108), the functional f_s is a split (Λ,θ)-Bessel functional with respect to the character Λ given by $\Lambda({\rm diag}(a,b,b,a))=|a^{-1}b|^{-s+1/2}$. Note that, under our trivial central character hypothesis, any unramified character of $T(F)$ is of this form. By Lemma 7.6, we have $f_s(T_{\rm Si}W')\neq 0$. This is equivalent to $B(1)\neq 0$. ∎

Transformation to standard assumptions

In this section we have worked with the matrix S given in (117). Let us rename this matrix S', and let $S=\begin{bmatrix}\mathbf{a}&\frac{\mathbf{b}}{2}\\ \frac{\mathbf{b}}{2}&\mathbf{c}\end{bmatrix}$. Here, $\mathbf{a},\mathbf{b},\mathbf{c}$ are elements of $F^\times$ subject to our standard assumptions (5). We assume that we are in the split case, so that $\mathbf{d}=\mathbf{b}^2-4\mathbf{ac}$ is a square in $\mathfrak{o}^\times$. We shall describe explicitly how to relate Bessel functions with respect to S and S'. Let

$$A=\frac{1}{\sqrt{\mathbf{d}}}\begin{bmatrix}1&-2\mathbf{c}\\ -\frac{1}{2\mathbf{c}}(\mathbf{b}-\sqrt{\mathbf{d}})&\mathbf{b}+\sqrt{\mathbf{d}}\end{bmatrix}.$$

Note that $\det(A)=2/\sqrt{\mathbf{d}}$ and

$${}^tA^{-1}=\begin{bmatrix}\frac{1}{2}(\mathbf{b}+\sqrt{\mathbf{d}})&\frac{1}{4\mathbf{c}}(\mathbf{b}-\sqrt{\mathbf{d}})\\ \mathbf{c}&\frac{1}{2}\end{bmatrix}.$$

We have $S' = {}^tASA$. Let $T(F) = \{g \in \mathrm{GL}_2(F) : {}^tgSg = \det(g)S\}$ as usual, and

$$T'(F) = \{g \in \mathrm{GL}_2(F) : {}^tgS'g = \det(g)S'\} = \{\begin{bmatrix} a & \\ & d \end{bmatrix} : a, d \in F^\times\}. \tag{130}$$

Then $T'(F) = A^{-1}T(F)A$. Let θ be the character of $U(F)$ as in (19), and let θ' be defined analogously. Let Λ be a character of $T(F)$, and let Λ' be the character of $T'(F)$ given by $\Lambda'(t) = \Lambda(AtA^{-1})$. If B is a function with the (Λ, θ)-Bessel transformation property, then the function B' defined by

$$B'(g) = B(\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix} g), \qquad g \in \mathrm{GSp}_4(F), \tag{131}$$

has the (Λ', θ')-Bessel transformation property.

In the case of odd residual characteristic, we have

$$A = \frac{1}{\sqrt{\mathbf{d}}} \begin{bmatrix} 1 & \\ -\frac{1}{2\mathbf{c}}(\mathbf{b} + \sqrt{\mathbf{d}}) & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} 1 & \\ \frac{\mathbf{c}}{\sqrt{\mathbf{d}}} & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{\mathbf{d}}}{\mathbf{c}} & \\ & -\frac{\mathbf{c}}{\sqrt{\mathbf{d}}} \end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o}).$$

In the case of even residual characteristic, let $g_{x,y}$ be as in (47), and set $x = 1 - \sqrt{d}/4$ and $y = 1/2$. Then

$$\begin{aligned} g_{x,y}A &= \frac{1}{\sqrt{\mathbf{d}}} \begin{bmatrix} 1 & \mathbf{c}(\sqrt{\mathbf{d}} - 2) \\ -\frac{1}{2\mathbf{c}}(\mathbf{b} - \sqrt{\mathbf{d}}) & \mathbf{ac} + \mathbf{b} + \sqrt{\mathbf{d}} - \frac{1}{4}(\mathbf{b} + \sqrt{\mathbf{d}})^2 \end{bmatrix} \\ &= \frac{1}{\sqrt{\mathbf{d}}} \begin{bmatrix} 1 & \\ -\frac{1}{2\mathbf{c}}(\mathbf{b} + \sqrt{\mathbf{d}}) & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} 1 & \\ \frac{\mathbf{c}}{\sqrt{\mathbf{d}}} & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{\mathbf{d}}}{\mathbf{c}} & \\ & \mathbf{c}(\sqrt{\mathbf{d}} - 2) \end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o}). \end{aligned}$$

If we assume that $B \in \mathcal{S}(\Lambda, \theta, P_1)$, in either case, evaluating (131) at $g = 1$, we see that $B'(1)$ differs from

$$B(\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})$$

by a non-zero constant. Therefore, if B is a $T_{1,0}$ eigenfunction with non-zero eigenvalue, then, by i) of Lemma 4.1,

$$B'(1) \neq 0 \qquad \Longleftrightarrow \qquad B(1) \neq 0. \tag{132}$$

Also, evaluating (131) at $g = s_2$, we see that $B'(s_2)$ differs from

$$B(\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \hat{u}_1 s_1 s_2) \qquad \text{or} \qquad B(\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \hat{u}_2 s_1 s_2)$$

by a non-zero constant. Observe here that, by the comments right after (11), either u_1 or u_2 is congruent to $-\frac{1}{2\mathbf{c}}(\mathbf{b} + \sqrt{\mathbf{d}})$ mod $\mathfrak{p}$.

8 The one-dimensional cases

In this section we will identify good test vectors for those irreducible, admissible representations of $\mathrm{GSp}_4(F)$ which are not spherical, but possess a one-dimensional space of P_1-invariant vectors. A look at Table 1 shows that these are the Iwahori-spherical representations of type IIa, IVc, Vb, VIa and VIb (Vc is a twist of Vb and is not counted separately). It turns out that, in almost all cases where the Bessel character is unramified, the P_1-invariant vector is a test vector. However, there is one exception, which only occurs for IIa, and only in the split case. Recall from Sect. 1 that, in the split case, $T(F) \cong L^\times \cong F^\times \times F^\times$. Hence, a character Λ of $T(F)$ is really a pair of characters Λ_1, Λ_2 of $F^\times$, and we shall write $\Lambda(a,b) = \Lambda_1(a)\Lambda_2(b)$. The exception occurs when Λ_1 and Λ_2 are equal, unramified, and take the value $-\omega$ on a uniformizer, where ω is the eigenvalue of the Atkin-Lehner element η, given in (34), on the space of P_1-invariant vectors. Hence, the exceptional case is that

$$\Lambda \text{ is unramified and } \Lambda(1,\varpi) = -\omega = \Lambda(\varpi,1). \tag{133}$$

Observe that, from considering the central character, $\Lambda(\varpi,\varpi) = \omega^2$, so that $\Lambda(1,\varpi) = -\omega$ if and only if $\Lambda(\varpi,1) = -\omega$.

This entire section assumes that the elements $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfy the conditions (5). The matrix S, the group $T(F) \cong L^\times$ and the character θ have the usual meaning. The following lemma will have a hypothesis that $\lambda = -q\omega$, where λ is the $T_{1,0}$-eigenvalue and ω is the Atkin-Lehner eigenvalue of some P_1-invariant Bessel function B. Looking at the values in Table 4, we find that this hypothesis is satisfied for representations of type IIa, IVc, Vb and VIa.

8.1 Lemma. *Let Λ be a character of $L^\times$, and m_0 as in (65). Assume that $B \in \mathcal{S}(\Lambda,\theta,P_1)$ satisfies*

$$T_{1,0}B = \lambda B, \qquad T_{0,1}B = \mu B, \qquad \eta B = \omega B$$

with complex numbers λ, μ, ω. Assume that $\lambda = -q\omega$.

i) Assume that $m_0 > 0$. Then

$$B = 0 \qquad \Longleftrightarrow \qquad B(h(0,m_0)) = 0.$$

ii) Assume that $m_0 = 0$ and $\left(\frac{L}{\mathfrak{p}}\right) = -1$. Then

$$B = 0 \qquad \Longleftrightarrow \qquad B(1) = 0.$$

iii) Assume that $m_0 = 0$ and $\left(\frac{L}{\mathfrak{p}}\right) = 0$.

- *If $\Lambda(\varpi_L) = -\omega$, then $B = 0$.*
- *If $\Lambda(\varpi_L) = \omega$, then*

$$B = 0 \qquad \Longleftrightarrow \qquad B(1) = 0.$$

iv) Assume that $m_0 = 0$ and $\left(\frac{L}{\mathfrak{p}}\right) = 1$.

- *Assume that Λ is not the exceptional character (133). Then*

$$B = 0 \qquad \Longleftrightarrow \qquad B(1) = 0.$$

- *Assume that* Λ *is the exceptional character (133) and that* $B(1) = 0$*. Then*

$$B = 0 \qquad \Longleftrightarrow \qquad B(\hat{u}_1 s_1 s_2) = 0 \qquad \Longleftrightarrow \qquad B(\hat{u}_2 s_1 s_2) = 0.$$

Proof. First observe that $\eta B = \omega B$ implies $\Lambda(\varpi) = \omega^2$; this will be used without comment in the following calculations. From Lemma 4.1 we have the following formulas for $l \geq 0$ and $m \geq \max(m_0, 1)$,

$$\lambda B(h(l,m)) = q^3 B(h(l+1,m)), \tag{134}$$

$$\lambda B(h(l,m)s_2) = q\Lambda(\varpi)B(h(l+1,m-1)s_2) + q(q-1)B(h(l-1,m+1)s_1s_2) + q^2(q-1)B(h(l+1,m)). \tag{135}$$

Evaluating the relation $\eta B = \omega B$ at specific elements, we get

$$B(h(l,m)) = \omega B(h(l-1,m)s_2s_1s_2), \tag{136}$$

$$\omega B(h(l,m)s_2) = B(h(l-1,m+1)s_1s_2) \tag{137}$$

for all integers l and m. Substituting these into (135) gives

$$(q-1)B(h(l,m)) = q^2 B(h(l,m)s_2) + q\omega B(h(l+1,m-1)s_2) \tag{138}$$

for $l \geq 0$ and $m \geq \max(m_0, 1)$.

i) Assume that $m_0 > 0$, and consider the following equations,

$$q^2 B(h(0,m_0)s_2) + q\omega B(h(1,m_0-1)s_2) = (q-1)B(h(0,m_0)), \tag{139}$$

$$q^2 B(h(1,m_0)s_2) + q\omega B(h(2,m_0-1)s_2) = (q-1)B(h(1,m_0)), \tag{140}$$

$$\omega^3 B(h(0,m_0-1)s_2) + \mu B(h(1,m_0-1)s_2) + q^3\omega B(h(2,m_0-1)s_2) = q^2(q-1)B(h(1,m_0)), \tag{141}$$

$$\mu B(h(0,m_0)s_2) + q^3\omega B(h(1,m_0)s_2) - \omega^4 B(h(0,m_0-1)s_2) = q^2(q-1)B(h(0,m_0+1)) - q^4(q-1)B(h(2,m_0)), \tag{142}$$

$$\mu B(h(0,m_0-1)s_2) + q^3\omega B(h(1,m_0-1)s_2) = q^2(q-1)B(h(0,m_0)). \tag{143}$$

Here, (139) and (140) are special cases of (138). Equation (141) follows from Lemma 5.2, combined with Lemma 3.6. Equation (142) follows from (99). Equation (143) follows from Lemma 5.1 ii), combined with (137). In matrix form, we can write these equations as

$$\begin{bmatrix} & q\omega & & q^2 & \\ & & q\omega & & q^2 \\ \omega^3 & \mu & q^3\omega & & \\ -\omega^4 & & & \mu & q^3\omega \\ \mu & q^3\omega & & & \end{bmatrix} \begin{bmatrix} B(h(0,m_0-1)s_2) \\ B(h(1,m_0-1)s_2) \\ B(h(2,m_0-1)s_2) \\ B(h(0,m_0)s_2) \\ B(h(1,m_0)s_2) \end{bmatrix} = v, \tag{144}$$

where v is a vector whose entries are linear combinations in the values $B(h(l,m))$ for various l and m. Now assume that $B(h(0,m_0)) = 0$. Then, by Proposition 6.1, the values $B(h(l,m))$ are zero for all l and m, so that the vector v is zero. The matrix on the left hand side of

(144) is easily seen to be invertible, regardless of the value of μ. This implies in particular that $B(h(0,m_0-1)s_2) = B(h(1,m_0-1)s_2) = 0$. Lemma 5.2 then implies that $B(h(l,m_0-1)s_2) = 0$ for all $l \geq 0$. Using (138), it follows that $B(h(l,m)s_2) = 0$ for all $l \geq 0$ and $m \geq m_0 - 1$. In fact, this is true for all $l \in \mathbb{Z}$ by Lemma 3.6. Equations (136) and (137) now imply that the main tower, s_2-tower, s_1s_2-tower and $s_2s_1s_2$-tower are all zero. By Proposition 3.4 and Lemma 3.6 v), the function B is zero. This proves i).

ii), iii) and iv) From now on assume that $m_0 = 0$. We first claim that

If $B(1) = 0$ and $B(h(l,0)s_2) = 0$ for all $l \geq 0$, then $B(h(l,m)w) = 0$ for all $l \in \mathbb{Z}$, all $m \geq 0$, and all $w \in \{1, s_2, s_2s_1s_2\}$, as well as $B(h(l,m)s_1s_2) = 0$ for all $l \in \mathbb{Z}$ and all $m \geq 1$. (145)

To see this, assume that $B(1) = 0$ and $B(h(l,0)s_2) = 0$ for all $l \geq 0$. Then, by Proposition 6.1, the values $B(h(l,m))$ are zero for all $l \in \mathbb{Z}$ and $m \geq 0$. Using $B(h(l,0)s_2) = 0$ for all $l \geq 0$, it follows inductively from (138) that $B(h(l,m)s_2) = 0$ for all $l \geq 0$ and $m \geq 0$. Using the Atkin-Lehner relations (136) and (137), as well as Lemma 3.6, it follows that $B(h(l,m)w) = 0$ for all $l \in \mathbb{Z}$, all $m \geq 0$, and all $w \in \{1, s_2, s_2s_1s_2\}$, as well as $B(h(l,m)s_1s_2) = 0$ for all $l \in \mathbb{Z}$ and all $m \geq 1$. This proves our claim.

ii) Assume that $\left(\frac{L}{\mathfrak{p}}\right) = -1$ (the inert case). Then, from Lemma 4.1 ii),

$$\begin{aligned}\lambda B(h(l,0)s_2) &= q^2(q-1)B(h(l+1,0)) + q^2 B(h(l-1,1)s_1s_2)\\ &= q^2(q-1)B(h(l+1,0)) + q^2\omega B(h(l,0)s_2)\end{aligned}$$

for $l \geq 0$. Hence, since $\lambda = -q\omega$, and using (134),

$$q(q+1)B(h(l,0)s_2) = (q-1)B(h(l,0)) \tag{146}$$

for $l \geq 0$. In view of (145) and the double coset representatives in Proposition 3.4, it follows that $B = 0$ if $B(1) = 0$.

iii) Assume that $\left(\frac{L}{\mathfrak{p}}\right) = 0$ (the ramified case). By Lemma 4.1 ii) and (137),

$$\begin{aligned}\lambda B(h(l,0)s_2) = q^2(q-1)&B(h(l+1,0))\\ &+ q\Lambda(\varpi_L)B(h(l,0)\hat{u}_0s_1s_2) + q(q-1)\omega B(h(l,0)s_2).\end{aligned}$$

Using $\lambda = -q\omega$ and (134),

$$q\Lambda(\varpi_L)B(h(l,0)\hat{u}_0s_1s_2) = \omega(q-1)B(h(l,0)) - q^2\omega B(h(l,0)s_2). \tag{147}$$

By Lemma 5.3 and (134), (137), we also have

$$\mu B(h(-1,0)\hat{u}_0s_1s_2) = q^4\omega B(s_2) - q^2\Lambda(\varpi_L)B(1) \tag{148}$$

and

$$\mu B(\hat{u}_0s_1s_2) = q^4\omega B(h(1,0)s_2) - \omega\big((q-1)\Lambda(\varpi_L) + \omega q\big)B(1) \tag{149}$$

Recall that $\Lambda(\varpi_L)^2 = \Lambda(\varpi) = \omega^2$, so that $\Lambda(\varpi_L) = \pm\omega$. Assume first that $\Lambda(\varpi_L) = -\omega$. By Lemma 4.2, together with $\lambda = -q\omega$ and (134),

$$q\omega B(h(l,0)\hat{u}_0s_1s_2) = \omega(q-1)B(h(l,0)) - q^2\omega B(h(l,0)s_2) \tag{150}$$

for $l \geq 0$. Equations (147) and (150) imply that $B(h(l,0)\hat{u}_0 s_1 s_2) = 0$ and

$$q^2 B(h(l,0)s_2) = (q-1)B(h(l,0)) \tag{151}$$

for $l \geq 0$. Equations (149) and (151), together with (134), imply that $B(1) = 0$. Hence, by (148) and Lemma 3.6 vi), the values $B(h(l,0)\hat{u}_0 s_1 s_2)$ are zero for all integers l. By (145) and the double coset representatives in Proposition 3.4, it follows that $B = 0$.

Now assume that $\Lambda(\varpi_L) = \omega$. By Lemma 5.3 and (134), (137),

$$\begin{aligned}\mu B(h(l,0)\hat{u}_0 s_1 s_2) = q^4 \omega B(h(l+1,0)s_2) + q\Lambda(\varpi)\omega B(h(l-1,0)s_2) \\ + \omega(q\omega - \Lambda(\varpi_L))(q-1)B(h(l,0))\end{aligned} \tag{152}$$

for $l \geq 1$. Assume that $B(1)$, and therefore the main tower, is zero. Then, from (147) and (152), we get the recursive relation

$$\omega^3 B(h(l-1,0)s_2) + \mu B(h(l,0)s_2) + q^3 \omega B(h(l+1,0)s_2) = 0 \tag{153}$$

for $l \geq 1$. From Lemma 4.2 we get $B(h(-1,0)\hat{u}_0 s_1 s_2) = 0$, and then $B(s_2) = 0$ by (148). It follows from (147) that $B(\hat{u}_0 s_1 s_2) = 0$, and then $B(h(1,0)s_2) = 0$ by (149). It now follows from (153) that

$$B(h(l,0)s_2) = 0 \qquad \text{for all } l \geq 0.$$

By (147),

$$B(h(l,0)\hat{u}_0 s_1 s_2) = 0 \qquad \text{for all } l \geq -1.$$

In view of (145) and the double coset representatives in Proposition 3.4, it follows that $B = 0$.

iv) Assume that $\left(\frac{L}{\mathfrak{p}}\right) = 1$ (the split case). By Lemma 4.1 ii) and (137),

$$\begin{aligned}-\omega B(h(l,0)s_2) = q(q-1)B(h(l+1,0)) + (q-2)\omega B(h(l,0)s_2) \\ + \Lambda(\varpi,1)B(h(l,0)\hat{u}_2 s_1 s_2) + \Lambda(1,\varpi)B(h(l,0)\hat{u}_1 s_1 s_2).\end{aligned}$$

By Lemma 4.2 and (137),

$$\begin{aligned}-(\omega + \Lambda(1,\varpi))B(h(l,0)\hat{u}_1 s_1 s_2) &= -(\omega + \Lambda(\varpi,1))B(h(l,0)\hat{u}_2 s_1 s_2) \\ &= q(q-1)B(h(l+1,0)) + (q-1)\omega B(h(l,0)s_2).\end{aligned} \tag{154}$$

Assume that $\Lambda(1,\varpi) = -\omega = \Lambda(\varpi,1)$ and $B(1) = 0$. Then, from (154),

$$B(h(l,0)s_2) = 0 \tag{155}$$

for all $l \geq 0$. Using this, Lemma 5.3 ii) and (137), we get

$$\omega^3 B(h(l-1,0)\hat{u}_1 s_1 s_2) + \mu B(h(l,0)\hat{u}_1 s_1 s_2) + q^3 \omega B(h(l+1,0)\hat{u}_1 s_1 s_2) = 0 \tag{156}$$

for all $l \geq 0$. Observe that $B(h(-1,0)\hat{u}_1 s_1 s_2) = 0$ by Lemma 3.6 vii). Hence, if $B(\hat{u}_1 s_1 s_2) = 0$, then $B(h(l,0)\hat{u}_1 s_1 s_2) = 0$ for all $l \geq 0$. Then also $B(h(l,0)\hat{u}_2 s_1 s_2) = 0$ for all $l \geq 0$ by (83). By (145) and the double coset representatives in Proposition 3.4, it follows that $B = 0$.

Assume that $\omega+\Lambda(1,\varpi)\neq 0\neq\omega+\Lambda(\varpi,1)$. Then, from (154), for $l\geq 0$,

$$B(h(l,0)\hat{u}_1s_1s_2)=\frac{-1}{\omega+\Lambda(1,\varpi)}\Big(q(q-1)B(h(l+1,0))+(q-1)\omega B(h(l,0)s_2)\Big). \tag{157}$$

and

$$B(h(l,0)\hat{u}_2s_1s_2)=\frac{-1}{\omega+\Lambda(\varpi,1)}\Big(q(q-1)B(h(l+1,0))+(q-1)\omega B(h(l,0)s_2)\Big). \tag{158}$$

Assume that $B(1)$, and therefore the main tower, is zero. By Lemma 5.3 ii) and Lemma 3.6, for all integers $l\geq 0$,

$$\begin{aligned}\mu B(h(l,0)\hat{u}_1s_1s_2)=q^3\Lambda(1,\varpi)&B(h(l+1,0)\hat{u}_1s_1s_2)+q^3(q-1)\omega B(h(l+1,0)s_2)\\&+\Lambda(\varpi)\Lambda(1,\varpi)B(h(l-1,0)\hat{u}_1s_1s_2)+(q-1)\omega^3B(h(l-1,0)s_2)\end{aligned}$$

Substituting (157) into this, we get

$$\omega^3B(h(l-1,0)s_2)+\mu B(h(l,0)s_2)+q^3\omega B(h(l+1,0)s_2)=0. \tag{159}$$

Using this equation for $l=0$ and $l=1$, equation (138) for $l=0,1$ and $m=1$, and (99) for $m=1$, we have

$$\begin{aligned}\mu B(s_2)+q^3\omega B(h(1,0)s_2)&=0,\\ \omega^3B(s_2)+\mu B(h(1,0)s_2)+q^3\omega B(h(2,0)s_2)&=0,\\ qB(h(0,1)s_2)+\omega B(h(1,0)s_2)&=0,\\ qB(h(1,1)s_2)+\omega B(h(2,0)s_2)&=0,\\ \mu B(h(0,1)s_2)+q^3\omega B(h(1,1)s_2)-\omega^4B(s_2)&=0.\end{aligned}$$

The determinant of the resulting matrix is $q^7\omega^6(1-q)$. Since this determinant is non-zero, it follows that $B(h(l,0)s_2)=0$ for all $l\geq 0$. By (157), this implies $B(h(l,0)\hat{u}_is_1s_2)=0$ for all $l\geq 0$. In view of (145) and the double coset representatives in Proposition 3.4, it follows that $B=0$. This concludes the proof. ∎

The VIb case

The lemma we just proved does not apply to Iwahori-spherical representations of type VIb, since these representations satisfy $\lambda=q\omega$ (and not $\lambda=-q\omega$). But in some sense the VIb case is easiest, because, by Table 1, this representation has a one-dimensional space of P_1-invariant vectors, but no non-zero P_2-invariant vectors (see (14) for the definition of the Klingen congruence subgroup P_2). Hence, if a non-zero P_1-invariant vector B is made P_2-invariant by summation, the result is zero; this is an additional relation which was unavailable for the representations considered in Lemma 8.1. It is easy to see that this Klingenization process is given by the summation on the left hand side of the relation

$$\sum_{\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in\mathrm{GL}_2(\mathfrak{o})/\Gamma_0(\mathfrak{p})}\pi(\begin{bmatrix}a&&b&\\&1&&\\c&&d&\\&&&1\end{bmatrix})B=0.$$

Using standard representatives, this relation can be rewritten as

$$\sum_{c\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&\\&1&&\\c&&1&\\&&&1\end{bmatrix})B+\pi(s_2)B=0. \tag{160}$$

8.2 Lemma. *Let Λ be a character of $L^\times$, and m_0 as in (65). Assume that $B\in\mathcal{S}(\Lambda,\theta,P_1)$ satisfies $\eta B=\omega B$ with a complex number ω. Assume also that (160) is satisfied. Then*

$$B=0\qquad\Longleftrightarrow\qquad B(h(0,m_0))=0.$$

Proof. Evaluating the relation $\eta B=\omega B$ at specific elements, we get

$$B(h(l,m))=\omega B(h(l-1,m)s_2s_1s_2), \tag{161}$$
$$\omega B(h(l,m)s_2)=B(h(l-1,m+1)s_1s_2) \tag{162}$$

for all integers l and m. Evaluating (160) at $h(l,m)s_2$, we obtain

$$qB(h(l,m)s_2)=-B(h(l,m)) \tag{163}$$

for all $l,m\geq 0$. Let $i=0$ in the ramified case and $i=1$ or $i=2$ in the split case. Evaluating (160) at $h(l,0)\hat{u}_is_1s_2s_1s_2$ leads to

$$qB(h(l,0)s_2s_1s_2)=-B(h(l,0)\hat{u}_is_1s_2) \tag{164}$$

for $l\geq -1$; observe here the defining property (10) of the elements u_i. Assume that $B(h(0,m_0))$, and therefore, by Proposition 6.1, the whole main tower, is zero. Then, by (163) and Lemma 3.6, the entire s_2-tower is also zero. By (161) and (162), the s_1s_2 and $s_2s_1s_2$ towers are also zero. In the inert case, in view of the double cosets given in Proposition 3.4, and v) of Lemma 3.6, it follows that $B=0$. Taking into account (164), the same conclusion holds in the ramified and split cases. ∎

The main result

The following theorem, which is the main result of this section, identifies good test vectors for those irreducible, admissible, infinite-dimensional representations of $\mathrm{GSp}_4(F)$ that have a one-dimensional space of P_1-invariant vectors. The remaining non-spherical representations with non-zero P_1-fixed vectors will be treated in the next section.

8.3 Theorem. *Let π be an irreducible, admissible, Iwahori-spherical representation of $\mathrm{GSp}_4(F)$ that is not spherical but has a one-dimensional space of P_1-invariant vectors. Let S be the matrix defined in (1), with $\mathbf{a},\mathbf{b},\mathbf{c}$ subject to the conditions (5). Let $T(F)$ be the group defined in (2). Let θ be the character of $U(F)$ defined in (19), and let Λ be a character of $T(F)\cong L^\times$. Let m_0 be as in (65). We assume that π admits a (Λ,θ)-Bessel model. Let B be an element in this Bessel model spanning the space of P_1-invariant vectors.*

i) Assume that π is of type IIa. Then $B(h(0,m_0)) \neq 0$, except in the split case with Λ being the exceptional character (133). In this latter case $B(1)=0$, but $B(\hat{u}_i s_1 s_2) \neq 0$ for $i=1,2$. Here, the elements $\hat{u}_i$ are defined in (64).

ii) If π is of type IVc, Vb, VIa or VIb, then $B(h(0,m_0)) \neq 0$.

Proof. Let B be a non-zero P_1-invariant vector in the (Λ,θ)-Bessel model of π. Then B is an element of the space $\mathcal{S}(\Lambda,\theta,P_1)$. Since the space of P_1-invariant vectors in π is one-dimensional, B is an eigenvector for $T_{1,0}$, $T_{0,1}$ and η; let λ, μ and ω be the respective eigenvalues. Lemma 8.1 i) and Lemma 8.2 proves our assertions in case $m_0 > 0$. For the rest of the proof we will therefore assume that $m_0 = 0$, meaning that Λ is unramified.

i) If we are not in the split case, or if we are in the split case and Λ is not the exceptional character (133), then our assertion follows from Lemma 8.1. Assume that we are in the split case and that Λ is the exceptional character (133). Then $B(1)=0$ by Proposition 7.4 and (132). Hence our assertion follows from Lemma 8.1 iv).

ii) If π is of type VIb, the assertion follows from Lemma 8.2. Assume that π is of type IVc, Vb or VIa. If we are not in the split case, our assertions follow from Lemma 8.1. Assume we are in the split case, and that π is of type IVc or Vb. Then, by Table 2 and Table 4, the character Λ is *not* the exceptional character (133). Hence our assertions follow from Lemma 8.1 iv). Assume we are in the split case, and that π is of type VIa. Then our assertion follows from Proposition 7.7 and (132). ∎

9 The two-dimensional cases

Let π be an irreducible, admissible representation of $\mathrm{GSp}_4(F)$ which is not spherical but has non-zero P_1-invariant vectors. In the previous section we treated the cases for which the space of P_1-invariant vectors is one-dimensional. In this section we will consider the cases where this space is more than one-dimensional. A look at Table 1 shows that these are precisely the representations of type IIIa and IVb, and in both cases the space of P_1-invariant vectors is two-dimensional. The lack of one-dimensionality makes it necessary to identify candidates for test vectors; clearly, not every vector in a two-dimensional space can be a test vector. From Lemma 2.2, we know that the two Hecke operators $T_{1,0}$ and $T_{0,1}$ commute, and a natural approach is to work with the common eigenvectors. At the end of Sect. 2 we have identified these common eigenvectors v_1 and v_2 explicitly in the induced models, for both the IIIa and the IVb case. By (42) and (44), the Atkin-Lehner involution η essentially takes one of these eigenvectors to the other, which is another indication that v_1 and v_2 are a natural basis for the space of P_1-invariant vectors.

The IIIa case

Let B_1 and B_2 be common eigenvectors for $T_{1,0}$ and $T_{0,1}$ in the IIIa case. Then, by (42), one can choose the normalizations such that

$$
\begin{aligned}
T_{1,0}\,B_1 &= \alpha\gamma q\,B_1, & T_{1,0}\,B_2 &= \gamma q\,B_2,\\
T_{0,1}\,B_1 &= \alpha\gamma^2(\alpha q+1)q\,B_1, & T_{0,1}\,B_2 &= \alpha\gamma^2(\alpha^{-1}q+1)q\,B_2,
\end{aligned}
\tag{165}
$$

$$\eta\, B_1 = \alpha\gamma\, B_2, \qquad\qquad \eta\, B_2 = \gamma\, B_1.$$

Note that $\alpha \neq 1$ in the IIIa case; see Table 5.

9.1 Lemma. *Let Λ be a character of $L^\times$, and m_0 as in (65). Assume that $B_1, B_2 \in \mathcal{S}(\Lambda, \theta, P_1)$ satisfy (165) with $\alpha \neq 1$. Assume also that $\Lambda(\varpi) = \alpha\gamma^2$. Then*

$$B_1 = 0 \qquad \Longleftrightarrow \qquad B_1(h(0, m_0)) = 0.$$

Proof. Evaluating the Atkin-Lehner relations at specific elements, we get for all integers l, m,

$$\begin{aligned} B_2(h(l,m)) &= \gamma B_1(h(l-1,m)s_2s_1s_2),\\ \alpha\gamma B_2(h(l,m)s_2) &= B_1(h(l-1,m+1)s_1s_2),\\ B_2(h(l,m)s_1s_2) &= \gamma B_1(h(l+1,m-1)s_2),\\ \alpha\gamma B_2(h(l,m)s_2s_1s_2) &= B_1(h(l+1,m)). \end{aligned}$$

By Lemma 4.1,

$$\alpha\gamma q B_1(h(l,m)) = q^3 B_1(h(l+1,m)), \tag{166}$$
$$\gamma q B_2(h(l,m)) = q^3 B_2(h(l+1,m)). \tag{167}$$

for $l, m \geq 0$. Also by Lemma 4.1, for $l \geq 0$ and $m \geq \max(m_0, 1)$,

$$\begin{aligned} \alpha\gamma q B_1(h(l,m)s_2) = q\Lambda(\varpi)B_1(h(l+1,m-1)s_2) + q(q-1)B_1(h(l-1,m+1)s_1s_2)\\ + q^2(q-1)B_1(h(l+1,m)), \end{aligned} \tag{168}$$
$$\alpha\gamma q B_1(h(l,m)s_1s_2) = q^2 B_1(h(l-1,m+1)s_1s_2) + q^2(q-1)B_1(h(l+1,m)), \tag{169}$$
$$\gamma q B_2(h(l,m)s_1s_2) = q^2 B_2(h(l-1,m+1)s_1s_2) + q^2(q-1)B_2(h(l+1,m)). \tag{170}$$

Using (166), (167) and the Atkin-Lehner conditions, we first get, for $l, m \geq 0$,

$$B_1(h(l,m)s_2s_1s_2) = q^{-2} B_2(h(l,m)), \tag{171}$$
$$B_2(h(l,m)s_2s_1s_2) = q^{-2} B_1(h(l,m)). \tag{172}$$

Using the Atkin-Lehner conditions and (171), we get for any $l \geq -1$ and $m \geq 0$,

$$\gamma B_1(h(l,m)s_2s_1s_2) = q^2 B_1(h(l+1,m)s_2s_1s_2). \tag{173}$$

We can rewrite (168) to (170) as follows,

$$\begin{aligned} \alpha\gamma q B_1(h(l,m)s_2) = q\alpha\gamma^2 B_1(h(l+1,m-1)s_2) + q(q-1)B_1(h(l-1,m+1)s_1s_2)\\ + \alpha\gamma(q-1)B_1(h(l,m)), \end{aligned} \tag{174}$$
$$\alpha\gamma q B_1(h(l,m)s_1s_2) = q^2 B_1(h(l-1,m+1)s_1s_2) + \alpha\gamma(q-1)B_1(h(l,m)), \tag{175}$$
$$\gamma^2 q B_1(h(l+1,m-1)s_2) = q^2\gamma B_1(h(l,m)s_2) + q^2(q-1)\gamma B_1(h(l,m)s_2s_1s_2). \tag{176}$$

Let us use (175) and (176) to solve for $B_1(h(l-1,m+1)s_1s_2)$ and $B_1(h(l+1,m-1)s_2)$ and substitute the value into (174). We get, for $l \geq 0$ and $m \geq \max(m_0, 1)$, the three equations

$$0 = -\alpha\gamma q B_1(h(l,m)s_1s_2) + q^2 B_1(h(l-1,m+1)s_1s_2) + \alpha\gamma(q-1)B_1(h(l,m)), \quad (177)$$

$$0 = -\gamma q B_1(h(l+1,m-1)s_2) + q^2 B_1(h(l,m)s_2) + q^2(q-1)B_1(h(l,m)s_2s_1s_2), \quad (178)$$

$$0 = q^3 B_1(h(l,m)s_2s_1s_2) + q^2 B_1(h(l,m)s_2) + qB_1(h(l,m)s_1s_2) + B_1(h(l,m)). \quad (179)$$

We will first consider the case $\mathbf{m_0 > 0}$. Assume that $B_1(h(0,m_0)) = 0$; we will show that $B_1 = B_2 = 0$. Note that, by the Atkin-Lehner relations, we only need to show that $B_1 = 0$. By Proposition 6.1, we know that $B_1(h(l,m)) = 0$ for all $l, m \geq 0$. Note that Lemma 5.1 i) implies that $B_2(h(l,m)) = 0$ for any $m < m_0$. Substituting $l = 0$ and $m = m_0$ in (179), we get

$$q^3 B_1(h(0,m_0)s_2s_1s_2) + q^2 B_1(h(0,m_0)s_2) + qB_1(h(0,m_0)s_1s_2) = -B_1(h(0,m_0)). \quad (180)$$

Substituting $B = B_2$, $l = 0$ and $m = m_0 - 1$ in Lemma 5.1 ii), using the Atkin-Lehner conditions, as well as (171) and (178), we get

$$\gamma(q\alpha^{-1} + 1)B_1(h(-1,m_0)s_1s_2) + q^3 B_1(h(0,m_0)s_2) = 0. \quad (181)$$

From (101), applied to B_1, and using (173) and (178), we get

$$\begin{aligned}&\alpha\gamma^2 B_1(h(0,m_0-1)s_2) - (\alpha q+1)q^2 B_1(h(0,m_0)s_2) + q^4\gamma^{-1}B_1(h(1,m_0)s_2)\\ &- \alpha q^3(q-1)B_1(h(0,m_0)s_2s_1s_2) = -(q-1)B_1(h(0,m_0)).\end{aligned} \quad (182)$$

From (100), applied to B_1, we get

$$\begin{aligned}&\gamma(\alpha q+1)qB_1(h(0,m_0)s_2) - q^3 B_1(h(1,m_0)s_2) - \alpha\gamma^3 B_1(h(0,m_0-1)s_2)\\ &\quad = q^{-1}(q-1)\gamma B_1(h(0,m_0)).\end{aligned} \quad (183)$$

From Lemma 5.1 ii), applied to both B_1 and B_2 with $l = 0$ and $m = m_0$, and using the relation $B_2(h(0,m_0+1)) = \mu_2 q^{-4} B_2(h(0,m_0))$ from Lemma 5.1 i), as well as (173), (177) and (178), we get

$$\begin{aligned}&-\alpha\gamma^2(\alpha q+1)qB_1(h(0,m_0)s_2) + q^4\alpha\gamma B_1(h(1,m_0)s_2) + \alpha\gamma q^2(q-1)B_1(h(1,m_0)s_1s_2)\\ &\quad + q^2(q-1)\alpha\gamma^2 B_1(h(0,m_0)s_2s_1s_2) + \alpha^2\gamma^4 B_1(h(0,m_0-1)s_2)\\ &\qquad = -q^2(q-1)B_1(h(0,m_0+1)) + q(q-1)^2\alpha\gamma B_1(h(1,m_0)),\end{aligned} \quad (184)$$

$$\begin{aligned}&-(q+\alpha)\gamma B_1(h(0,m_0)s_1s_2) + q^3 B_1(h(1,m_0)s_1s_2) + q^3(q-1)B_1(h(1,m_0)s_2)\\ &\quad + \alpha\gamma^2 B_1(h(-1,m_0)s_1s_2) + \gamma(q+\alpha)q(q-1)B_1(h(0,m_0)s_2s_1s_2)\\ &\quad = -(q+\alpha)\gamma(q-1)q^{-1}B_1(h(0,m_0)).\end{aligned} \quad (185)$$

Lemma 5.1 ii), applied to B_1 and with $l = 0$ and $m = m_0 - 1$, gives

$$\alpha\gamma^2(\alpha q+1)B_1(h(0,m_0-1)s_2) + q^2 B_1(h(0,m_0)s_1s_2) = q(q-1)B_1(h(0,m_0)). \quad (186)$$

From (101), applied to B_2, we get

$$\alpha\gamma^2 B_1(h(-1,m_0)s_1s_2) - \gamma(q+\alpha)qB_1(h(0,m_0)s_1s_2) + q^3 B_1(h(1,m_0)s_1s_2)$$

$$= -q^2(q-1)\alpha\gamma B_1(h(0,m_0)s_2s_1s_2). \tag{187}$$

Now, (181) to (187) give us 8 linear equations in the 7 variables

$$B_1(h(0,m_0-1)s_2),\ B_1(h(0,m_0)s_2),\ B_1(h(1,m_0)s_2),\ B_1(h(0,m_0)s_1s_2),$$
$$B_1(h(1,m_0)s_1s_2),\ B_1(h(-1,m_0)s_1s_2),\ B_1(h(0,m_0)s_2s_1s_2).$$

Let v be the column vector whose coordinates are the above 7 variables, in this order. If we consider the 7 equations (180) to (183) and (185) to (187), we get a matrix equation $A_1v = v_1$. If we consider the 7 equations (181) to (187), we get another matrix equation $A_2v = v_2$. Here, both v_1, v_2 are column vectors whose entries all depend only on $B_1(h(l,m))$. Now

$$\det(A_1) = -\alpha^3\gamma^6q^9(q-1)^3\big(\alpha(q^2+2q-1)+q(q-1)\big),$$
$$\det(A_2) = -\alpha^3\gamma^6q^{11}(q-1)^3\big(q\alpha^3+(q^2+1)\alpha^2-(q^2+2q-1)\alpha-q\big).$$

Note that, for any α, either $\det(A_1)$ or $\det(A_2)$ are non-zero. Hence, since $B_1(h(l,m)) = 0$ for all l,m, every entry of the column vector v is zero. Now (95) and (178) imply that

$$B_1(h(l,m)s_2) = 0 \qquad \text{for } l \geq 0,\ m = m_0-1 \text{ or } m = m_0.$$

Using (173), we get

$$B_1(h(l,m_0)s_2s_1s_2) = 0 \qquad \text{for } l \geq -1.$$

Using that $v = 0$ and (179), we get

$$B_1(h(l,m_0)s_1s_2) = 0 \qquad \text{for } l \geq -1.$$

Now, using (177) and induction, we get

$$B_1(h(l,m)s_1s_2) = 0 \qquad \text{for } l \geq -1,\ m \geq m_0.$$

Using (178), (179) and induction, we get

$$B_1(h(l,m)s_2) = 0 \qquad \text{for } l \geq 0,\ m \geq m_0-1$$
$$B_1(h(l,m)s_2s_1s_2) = 0 \qquad \text{for } l \geq -1,\ m \geq m_0.$$

In view of the double cosets in Proposition 3.4 and the automatic vanishing from Lemma 3.6, it follows that $B_1 = 0$. This concludes our proof in case $m_0 > 0$.

We next consider the case $\mathbf{m_0 = 0}$. Assume that $B_1(1) = 0$; we will show that $B_1 = 0$. By Proposition 6.1, we know that $B_1(h(l,m)) = 0$ for all $l,m \geq 0$. Suppose we can show that B_1 vanishes on all the double coset representatives in Proposition 3.4 that have $m = 0$ and that $B_1(h(l,1)s_1s_2) = 0$ for all $l \geq -1$. Using induction and (177), we get $B_1(h(l,m)s_1s_2) = 0$ for all $l \geq -1$ and $m \geq 0$. Now, induction and (178), (179), we get that $B_1 \equiv 0$, and hence, $B_2 \equiv 0$.

Let us first assume the inert case $\left(\frac{L}{\mathfrak{p}}\right) = -1$. Using Lemma 4.1 ii), we get, for $l \geq 0$,

$$\alpha\gamma q B_1(h(l,0)s_2) = q^2B_1(h(l-1,1)s_1s_2). \tag{188}$$

Using (173) and Lemma 4.1 iv), we get, for $l \geq 0$,

$$\alpha\gamma q B_1(h(l,0)s_2s_1s_2) = -(q+1)B_1(h(l-1,1)s_1s_2). \tag{189}$$

Hence, from (173), (188) and (189), we get, for $l \geq 0$,

$$\gamma B_1(h(l,0)s_2) = q^2 B_1(h(l+1,0)s_2). \tag{190}$$

Using (188), (190) and Lemma 5.1 ii), we get

$$\alpha\gamma^2(\alpha q+1)qB_1(s_2) = q^4 B_1(h(0,1)s_1s_2) = \alpha\gamma q^3 B_1(h(1,0)s_2) = \alpha\gamma^2 q B_1(s_2). \tag{191}$$

Since $\alpha \neq 0$, we get $B_1(s_2) = 0$. Now (188), (189) and (190) imply that B_1 vanishes on all the double coset representatives in Proposition 3.4 that have $m = 0$. Hence $B_1 = 0$, as claimed.

Next, assume the ramified case $\left(\frac{L}{\mathfrak{p}}\right) = 0$. By Lemma 4.1 ii), for $l \geq 0$,

$$\alpha\gamma q B_1(h(l,0)s_2) = q\Lambda(\varpi_L)B_1(h(l,0)\hat{u}_0s_1s_2) + q(q-1)B_1(h(l-1,1)s_1s_2). \tag{192}$$

By Lemma 4.1 iv) and (173), for $l \geq 0$,

$$\alpha\gamma q B_1(h(l,0)s_2s_1s_2) = -\Lambda(\varpi_L)B_1(h(l,0)\hat{u}_0s_1s_2) - qB_1(h(l-1,1)s_1s_2). \tag{193}$$

From Lemma 4.2 i), we get $B_1(h(-1,0)\hat{u}_0s_1s_2) = 0$ and

$$\alpha\gamma B_1(h(l,0)\hat{u}_0s_1s_2) = qB_1(h(l-1,1)s_1s_2) \tag{194}$$

for all $l \geq 0$. It follows from $\alpha \neq 1$ that $\Lambda(\varpi_L)(\alpha\gamma)^{-1} + 1 \neq 0$. Hence, (173), (193) and (194) imply that, for $l \geq 0$,

$$\gamma B_1(h(l-1,1)s_1s_2) = q^2 B_1(h(l,1)s_1s_2). \tag{195}$$

Setting $l = -1$ in Lemma 5.3 i) gives $B_1(h(-1,1)s_1s_2) = 0$. Therefore (195) implies that $B_1(h(l,1)s_1s_2) = 0$ for all $l \geq -1$. Now (192) to (194) imply that B_1 vanishes on all the double coset representatives in Proposition 3.4 that have $m = 0$. Hence $B_1 = 0$, as claimed.

Finally, assume the split case $\left(\frac{L}{\mathfrak{p}}\right) = 1$. By Lemma 4.1 ii), for $l \geq 0$,

$$\begin{aligned}\alpha\gamma q B_1(h(l,0)s_2) = q\Big(&\Lambda(1,\varpi)B_1(h(l,0)\hat{u}_1s_1s_2) + \Lambda(\varpi,1)B_1(h(l,0)\hat{u}_2s_1s_2)\Big)\\ &+ q(q-2)B_1(h(l-1,1)s_1s_2).\end{aligned} \tag{196}$$

By (173) and Lemma 4.1 iv), for $l \geq 0$,

$$\begin{aligned}\alpha\gamma q B_1(h(l,0)s_2s_1s_2) = -\Big(&\Lambda(1,\varpi)B_1(h(l,0)\hat{u}_1s_1s_2) + \Lambda(\varpi,1)B_1(h(l,0)\hat{u}_2s_1s_2)\Big)\\ &- (q-1)B_1(h(l-1,1)s_1s_2).\end{aligned} \tag{197}$$

Using Lemma 4.2 ii), we get for $l \geq 0$,

$$(\alpha\gamma - \Lambda(1,\varpi))B_1(h(l,0)\hat{u}_1s_1s_2) = (q-1)B_1(h(l-1,1)s_1s_2), \tag{198}$$

$$(\alpha\gamma - \Lambda(\varpi,1))B_1(h(l,0)\hat{u}_2s_1s_2) = (q-1)B_1(h(l-1,1)s_1s_2), \tag{199}$$

Since, $\Lambda(1,\varpi)\Lambda(\varpi,1) = \Lambda(\varpi,\varpi) = \alpha\gamma^2$ and $\alpha \neq 1$, we see that we cannot have $\Lambda(1,\varpi) = \Lambda(\varpi,1) = \alpha\gamma$.

Case 1: Suppose one of $\Lambda(1,\varpi)$ and $\Lambda(\varpi,1)$ is equal to $\alpha\gamma$. Without loss of generality, let us take $\Lambda(1,\varpi) = \alpha\gamma$. Then $\Lambda(\varpi,1) = \gamma \neq \alpha\gamma$. Then (198) and (199) implies that

$B_1(h(l-1,1)s_1s_2) = B_1(h(l,0)\hat{u}_2s_1s_2) = 0$ for all $l \geq 0$. Then using (173) and (197), we get for $l \geq 0$,

$$\gamma B_1(h(l,0)\hat{u}_1s_1s_2) = q^2 B_1(h(l+1,0)\hat{u}_1s_1s_2). \tag{200}$$

Also, using Lemma 5.3 ii), for $l=0$, we have

$$\gamma(\alpha q+1)B_1(\hat{u}_1s_1s_2) = q^2 B_1(h(1,0)\hat{u}_1s_1s_2). \tag{201}$$

Comparing this equation to (200) with $l=0$, we get $B_1(\hat{u}_1s_1s_2)=0$ since $\alpha \neq 0$. Now, using (200), we get $B(h(l,0)\hat{u}_1s_1s_2)=0$ for all l. Now (196) and (197) imply that B_1 vanishes on all the double coset representatives in Proposition 3.4 that have $m=0$. Hence $B_1=0$, as claimed.

Case 2: Suppose neither of $\Lambda(1,\varpi)$ and $\Lambda(\varpi,1)$ is equal to $\alpha\gamma$. Substituting (198) and (199) in (197) we get

$$\alpha\gamma q B_1(h(l,0)s_2s_1s_2) = -\frac{(q-1)\alpha\gamma^2(\alpha-1)}{(\alpha\gamma-\Lambda(1,\varpi))(\alpha\gamma-\Lambda(\varpi,1))}B_1(h(l-1,1)s_1s_2). \tag{202}$$

Since $\alpha \neq 1$, we can use (173) to get for $l \geq 0$,

$$\gamma B_1(h(l-1,1)s_1s_2) = q^2 B_1(h(l,1)s_1s_2). \tag{203}$$

From Lemma 5.3 ii), with $l=0$, we get,

$$\alpha\gamma^2(\alpha q+1)qB_1(\hat{u}_1s_1s_2) = q^3\Lambda(1,\varpi)B_1(h(1,0)\hat{u}_1s_1s_2) + q^3(q-1)B_1(h(0,1)s_1s_2). \tag{204}$$

Substitute (198) above for $l=0,1$ and simplify to get

$$\gamma(\alpha q+1)B_1(h(-1,1)s_1s_2) = q^2 B_1(h(0,1)s_1s_2). \tag{205}$$

Comparing the above to (203), with $l=0$, we get $B_1(h(-1,1)s_1s_2)=0$ since $\alpha \neq 0$. Now, using (203), we get $B(h(l,1)s_1s_2)=0$ for all l. Now (196) to (199) imply that B_1 vanishes on all the double coset representatives in Proposition 3.4 that have $m=0$. Hence $B_1=0$, as claimed. ∎

The IVb case

Let B_1 and B_2 be common eigenvectors for $T_{1,0}$ and $T_{0,1}$ in the IVb case. Then, by (44), one can choose the normalizations such that

$$\begin{aligned} T_{1,0}\,B_1 &= \gamma\,B_1, & T_{1,0}\,B_2 &= \gamma q^2\,B_2,\\ T_{0,1}\,B_1 &= \gamma^2(q+1)\,B_1, & T_{0,1}\,B_2 &= \gamma^2 q(q^3+1)\,B_2,\\ \eta\,B_1 &= \gamma\,B_2, & \eta\,B_2 &= \gamma\,B_1. \end{aligned} \tag{206}$$

Recall that $\gamma=\sigma(\varpi)$, where σ is an unramified character. From Table 2, a (Λ,θ)-Bessel model exists if and only if $\Lambda=\sigma\circ N_{L/F}$. In particular, the number m_0 defined in (65) must be zero, i.e., Λ must be unramified. The central character condition is equivalent to $\Lambda(\varpi)=\gamma^2$. Moreover, in the ramified case $\left(\frac{L}{\mathfrak{p}}\right)=0$, evaluating at ϖ_L, we get $\Lambda(\varpi_L)=\gamma$, and in the split case $\left(\frac{L}{\mathfrak{p}}\right)=1$, evaluating at $(\varpi,1)$ and $(1,\varpi)$, we get $\Lambda(\varpi,1)=\Lambda(1,\varpi)=\gamma$.

9.2 Lemma. *Let Λ be an unramified character of $L^\times$ satisfying $\Lambda(\varpi) = \gamma^2$. If $\left(\frac{L}{\mathfrak{p}}\right) = 0$, assume that $\Lambda(\varpi_L) = \gamma$, and if $\left(\frac{L}{\mathfrak{p}}\right) = 1$, assume that $\Lambda(\varpi, 1) = \Lambda(1, \varpi) = \gamma$. Assume that $B_1, B_2 \in \mathcal{S}(\Lambda, \theta, P_1)$ satisfy (206). Then*

$$B_1 = 0 \iff B_1(1) = 0 \iff B_2(1) = 0 \iff B_2 = 0.$$

Proof. It is clear from the Atkin-Lehner relations that $B_1 = 0$ if and only if $B_2 = 0$. We will prove the first equivalence above; the analogous statement for B_2 is proved similarly. From the Atkin-Lehner relations, we get for all integers l, m,

$$\begin{aligned} B_2(h(l,m)) &= \gamma B_1(h(l-1,m)s_2s_1s_2), \\ \gamma B_2(h(l,m)s_2) &= B_1(h(l-1,m+1)s_1s_2), \\ B_2(h(l,m)s_1s_2) &= \gamma B_1(h(l+1,m-1)s_2), \\ \gamma B_2(h(l,m)s_2s_1s_2) &= B_1(h(l+1,m)). \end{aligned}$$

By Lemma 4.1,

$$\gamma B_1(h(l,m)) = q^3 B_1(h(l+1,m)), \tag{207}$$

$$\gamma q^2 B_2(h(l,m)) = q^3 B_2(h(l+1,m)). \tag{208}$$

for $l, m \geq 0$. Also by Lemma 4.1, for $l \geq 0$ and $m \geq 1$,

$$\begin{aligned} \gamma B_1(h(l,m)s_2) = q\Lambda(\varpi)B_1(h(l+1,m-1)s_2) + q(q-1)B_1(h(l-1,m+1)s_1s_2) \\ + q^2(q-1)B_1(h(l+1,m)), \end{aligned} \tag{209}$$

$$\gamma B_1(h(l,m)s_1s_2) = q^2 B_1(h(l-1,m+1)s_1s_2) + q^2(q-1)B_1(h(l+1,m)), \tag{210}$$

$$\gamma q^2 B_2(h(l,m)s_1s_2) = q^2 B_2(h(l-1,m+1)s_1s_2) + q^2(q-1)B_2(h(l+1,m)). \tag{211}$$

Using (207), (208) and the Atkin-Lehner conditions, we get, for $l, m \geq 0$,

$$B_1(h(l,m)s_2s_1s_2) = q^{-1} B_2(h(l,m)), \tag{212}$$

$$B_2(h(l,m)s_2s_1s_2) = q^{-3} B_1(h(l,m)). \tag{213}$$

Using the Atkin-Lehner conditions and (212), we get for any $l \geq -1$, $m \geq 0$,

$$\gamma B_1(h(l,m)s_2s_1s_2) = qB_1(h(l+1,m)s_2s_1s_2). \tag{214}$$

We can rewrite (209) to (211) as follows,

$$\begin{aligned} \gamma B_1(h(l,m)s_2) = q\gamma^2 B_1(h(l+1,m-1)s_2) + q(q-1)B_1(h(l-1,m+1)s_1s_2) \\ + \gamma q^{-1}(q-1)B_1(h(l,m)), \end{aligned} \tag{215}$$

$$\gamma B_1(h(l,m)s_1s_2) = q^2 B_1(h(l-1,m+1)s_1s_2) + \gamma q^{-1}(q-1)B_1(h(l,m)), \tag{216}$$

$$\gamma^2 q^2 B_1(h(l+1,m-1)s_2) = q^2\gamma B_1(h(l,m)s_2) + q^2(q-1)\gamma B_1(h(l,m)s_2s_1s_2). \tag{217}$$

Let us use (216) and (217) to solve for $B_1(h(l-1,m+1)s_1s_2)$ and $B_1(h(l+1,m-1)s_2)$ and substitute the value into (215). We get, for $l \geq 0$ and $m \geq 1$, the three equations

$$0 = -\gamma B_1(h(l,m)s_1s_2) + q^2 B_1(h(l-1,m+1)s_1s_2) + \gamma q^{-1}(q-1)B_1(h(l,m)), \tag{218}$$

$$0 = -\gamma B_1(h(l+1, m-1)s_2) + B_1(h(l,m)s_2) + (q-1)B_1(h(l,m)s_2s_1s_2), \tag{219}$$

$$0 = q^3 B_1(h(l,m)s_2s_1s_2) + q^2 B_1(h(l,m)s_2) + qB_1(h(l,m)s_1s_2) + B_1(h(l,m)). \tag{220}$$

Assume that $B_1(1) = 0$; we will show that B_1 vanishes everywhere. By Proposition 6.1, we know that $B_1(h(l,m)) = 0$ for all $l, m \geq 0$. Suppose we can show that B_1 vanishes on all the double coset representatives in Proposition 3.4 that have $m = 0$ and that $B_1(h(l,1)s_1s_2) = 0$ for all l. Using induction and (218), we get $B_1(h(l,m)s_1s_2) = 0$ for all l, m. Then $B_1 = 0$ by (219), (220) and induction.

Let us first assume that $\left(\frac{L}{\mathfrak{p}}\right) = -1$. By Lemma 4.1 ii), for $l \geq 0$,

$$\gamma B_1(h(l,0)s_2) = q^2 B_1(h(l-1,1)s_1s_2). \tag{221}$$

By Lemma 4.1 iv) and (214), for $l \geq 0$,

$$\gamma B_1(h(l,0)s_2s_1s_2) = -(1+q)B_1(h(l-1,1)s_1s_2). \tag{222}$$

Hence, from (214), (221) and (222), for $l \geq 0$,

$$\gamma B_1(h(l,0)s_2) = qB_1(h(l+1,0)s_2). \tag{223}$$

Using (221), (223) and Lemma 5.1 ii), we get

$$\gamma^2(q+1)B_1(s_2) = q^4 B_1(h(0,1)s_1s_2) = \gamma q^2 B_1(h(1,0)s_2) = \gamma^2 q B_1(s_2). \tag{224}$$

Since $\gamma \neq 0$, it follows that $B_1(s_2) = 0$. Now (221), (222) and (223) implies that B_1 vanishes on all the double coset representatives in Proposition 3.4 that have $m = 0$. Hence $B_1 = 0$, as claimed.

Let us now assume that $\left(\frac{L}{\mathfrak{p}}\right) = 0$. By Lemma 4.1 ii), for $l \geq 0$,

$$\gamma B_1(h(l,0)s_2) = q\gamma B_1(h(l,0)\hat{u}_0s_1s_2) + q(q-1)B_1(h(l-1,1)s_1s_2). \tag{225}$$

By Lemma 4.1 iv) and (214), for $l \geq 0$,

$$\gamma B_1(h(l,0)s_2s_1s_2) = -\gamma B_1(h(l,0)\hat{u}_0s_1s_2) - qB_1(h(l-1,1)s_1s_2). \tag{226}$$

By Lemma 4.2 i), for all $l \geq -1$,

$$\gamma B_1(h(l,0)\hat{u}_0s_1s_2) = q^2 B_1(h(l-1,1)s_1s_2). \tag{227}$$

Using Lemma 3.6 iv), it follows that $B_1(h(-1,0)\hat{u}_0s_1s_2) = 0$. Now (214), (226) and (227) imply that, for $l \geq 0$,

$$\gamma B_1(h(l-1,1)s_1s_2) = qB_1(h(l,1)s_1s_2). \tag{228}$$

From Lemma 5.3 i), for $l = -1$, we get $B_1(h(-1,1)s_1s_2) = 0$. Then (228) implies that $B_1(h(l,1)s_1s_2) = 0$ for all $l \geq -1$. Now (225) - (227) imply that B_1 vanishes on all the double coset representatives in Proposition 3.4 that have $m = 0$. Hence $B_1 = 0$, as claimed.

Finally, we assume that $\left(\frac{L}{\mathfrak{p}}\right) = 1$. Using Lemma 4.1 ii) and $\Lambda(1, \varpi) = \Lambda(\varpi, 1) = \gamma$, we get, for $l \geq 0$,

$$\gamma B_1(h(l,0)s_2) = q\gamma\big(B_1(h(l,0)\hat{u}_1s_1s_2) + B_1(h(l,0)\hat{u}_2s_1s_2)\big)$$

$$+ q(q-2)B_1(h(l-1,1)s_1s_2). \tag{229}$$

By Lemma 4.1 iv) and (214), for $l \geq 0$,

$$\gamma B_1(h(l,0)s_2s_1s_2) = -\gamma\big(B_1(h(l,0)\hat{u}_1s_1s_2) + B_1(h(l,0)\hat{u}_2s_1s_2)\big) \\ - (q-1)B_1(h(l-1,1)s_1s_2). \tag{230}$$

By Lemma 4.2 ii), for $l \geq 0$,

$$\gamma B_1(h(l,0)\hat{u}_1s_1s_2) = -qB_1(h(l-1,1)s_1s_2), \tag{231}$$

$$\gamma B_1(h(l,0)\hat{u}_2s_1s_2) = -qB_1(h(l-1,1)s_1s_2). \tag{232}$$

Using (214), (230), (231) and (232), we get, for all $l \geq 0$,

$$\gamma B_1(h(l-1,1)s_1s_2) = qB_1(h(l,1)s_1s_2). \tag{233}$$

From Lemma 5.3 ii), with $l = 0$, we get,

$$\gamma^2(q+1)B_1(\hat{u}_1s_1s_2) = q^3\gamma B_1(h(1,0)\hat{u}_1s_1s_2) + q^3(q-1)B_1(h(0,1)s_1s_2). \tag{234}$$

Substitute (231) above for $l = 0, 1$ and simplify to get

$$\gamma(q+1)B_1(h(-1,1)s_1s_2) = q^2B_1(h(0,1)s_1s_2). \tag{235}$$

Comparing the above to (233), with $l = 0$, we get $B_1(h(-1,1)s_1s_2) = 0$. Then (233) implies that $B(h(l,1)s_1s_2) = 0$ for all $l \geq -1$. Now (229) - (232) imply that B_1 vanishes on all the double coset representatives in Proposition 3.4 that have $m = 0$. Hence $B_1 = 0$, as claimed. ∎

The main result

The following theorem identifies good test vectors for those irreducible, admissible representations of $\mathrm{GSp}_4(F)$ that are not spherical but have a two-dimensional space of P_1-invariant vectors. Recall from Table 1 that these are precisely the Iwahori-spherical representations of type IIIa and IVb.

9.3 Theorem. *Let π be an irreducible, admissible, Iwahori-spherical representation of $\mathrm{GSp}_4(F)$ that is not spherical but has a two-dimensional space of P_1-invariant vectors. Let S be the matrix defined in (1), with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ subject to the conditions (5). Let $T(F)$ be the group defined in (2). Let θ be the character of $U(F)$ defined in (19), and let Λ be a character of $T(F) \cong L^\times$. Let m_0 be as in (65). We assume that π admits a (Λ, θ)-Bessel model. Then the space of P_1-invariant vectors is spanned by common eigenvectors for the Hecke operators $T_{1,0}$ and $T_{0,1}$, and if B is any such eigenvector, then $B(h(0, m_0)) \neq 0$.*

Proof. Assume first that π is of type IIIa. Then π has a Bessel model with respect to any Λ; see Table 2. Let B_1 and B_2 be the P_1-invariant vectors which are common eigenvectors for $T_{1,0}$ and $T_{0,1}$, as in (165). Then $B_1(h(0, m_0)) \neq 0$ by Lemma 9.1. If we replace γ by $\alpha^{-1}\gamma$ and then α

by α^{-1} in the equations (165), then the roles of B_1 and B_2 get reversed. This symmetry shows that also $B_2(h(0,m_0)) \neq 0$.

Now assume that π is the representation $L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$ of type IVb. Then, by Table 2, we must have $\Lambda = \sigma \circ N_{L/F}$. In particular, Λ is unramified and satisfies the hypotheses of Lemma 9.2. Let B_1 and B_2 be the P_1-invariant vectors which are common eigenvectors for $T_{1,0}$ and $T_{0,1}$, as in (206). Then $B_1(1) \neq 0$ and $B_2(1) \neq 0$ by Lemma 9.2. ∎

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