Dimension Formulas of Siegel Modular Forms of Weight 3 and Supersingular Abelian Surfaces(Revised version)

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In this article, first we give dimension formulas of the space $S_3(\Gamma)$ of Siegel cusp forms of weight 3 of degree 2 belonging to discrete subgroups Γ of *parahoric type*, including Hecke type groups $\Gamma_0(p)$, paramodular groups $K(p)$, Klingen type groups $\Gamma_0'(p)$ and the Iwahori subgroup $B(p)$ of any prime level. As for weight $k \geq 4$, the dimension formulas for Siegel modular forms of degree two are explicitly known for many discrete subgroups and there are two well known methods to calculate these: the Riemann-Roch-Hirzebruch theorem for $k \geq 4$ and the Selberg trace formula for $k \geq 5$. Both methods does not work for weight $k = 3$ in general and there are no general ways to calculate the dimension for $k \leq 3$. Before our present work, the dimensions of $S_3(\Gamma)$ were calculated only for finitely many (conjugacy classes of) discrete subgroups by rather complicated technical numerical calculation depending heavily on each discrete group Γ (cf. e.g. Poor and Yuen [23].) In this paper, we use holomorphic Lefschetz formula (a group invariant version of Riemann-Roch-Hirzebruch). The obstruction of the cohomology in this formula does not vanish in general, but by showing that it vanishes for the above discrete groups, we obtain our dimension formula. As for paramodular groups and Hecke type groups, our formula coincides with those conjectured in [17] and [7].

Secondly, we give a geometric meaning of the dimensions of weight 3. We shall show that the dimensions of cusp forms of weight 3 is related with the geometry of principally polarized super-singular abelian surfaces in several ways. It was known that certain arithmetic invariants of the locus of super-singular abelian surfaces in the moduli of principally polarized abelian surfaces over algebraically closed field of characteristic p are related with some class numbers of the compact real form of the split symplectic group $Sp(2,\mathbb{R})$ of size four(due to Katsura, Oort and partly myself). These class numbers are nothing but the dimension of certain automorphic forms of weight 0 of the compact twist. In general, automorphic forms of the compact twist should correspond with Siegel modular forms by Langlands conjecture. A precise conjecture of this type on bijective correspondence of automorphic forms has been formulated in [14] and [13]. We have not proved this conjecture itself, but by our dimension formula, we can show at least certain dimensional equalities between these forms belonging to compact or non-compact real forms of the symplectic group. This leads to the above interpretation.

Historically the above results on dimensions and geometric interpretation are a part of the conjectures by the author in [17] first announced in *Conference on L functions* at Kyushu University. The author would like to thank Professo Takayuki Oda for asking him the possibility to use Riemann-Roch theorem at this talk. He also thanks Professor Ryuji Tsushima for giving him a kind guidance to cohomological methods with which the author was not familiar before.

1 Definition of Siegel Modular Forms

We denote by H_n the Siegel upper half space of degree n.

$$H_n = \{Z = X + iY \in M_n(\mathbb{C});\ {}^tZ = Z, Y > 0\}.$$

The symplectic group

$$Sp(n,\mathbb{R}) = \{g \in M_{2n}(\mathbb{R}); gJ^tg = J\}$$

acts on H_n is the usual way, where $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. The weight of Siegel modular forms is given by an irreducible rational represenstation (ρ, V) over $\mathbb{C}$ of $GL_n(\mathbb{C})$. We take a discrete subgroup $\Gamma \subset Sp(n,\mathbb{R})$ with $\mathrm{vol}(\Gamma\backslash H_n) < \infty$. A Siegel modular form of Γ of weight ρ is defined to be a V-valued holomorphic function F of H_n such that

$$(F|_\rho[\gamma])(Z) := \rho(CZ + D)^{-1}F(\gamma Z) = F(Z)$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, with the standard regularity condition at each cusp when $n = 1$. When $\rho = \det^k$, we say F is of weight k.

When $n = 2$, we write $\rho_{k,j} = \det^k Sym_j$, where Sym_j is the j-th symmetric tensor representation of $GL(2)$ on $\mathbb{C}^{j+1}$. We denote by $M_{k,j}(\Gamma)$ the space of Siegel modular forms of weight $\rho_{k,j}$ belonging to Γ. A function $F \in M_{k,j}(\Gamma)$ is called a cusp form if it vanishes on the boundaries of the Satake compactification and the space of such forms are denoted by $S_{k,j}(\Gamma)$.

2 Dimension Formulas

2.1 Some general remarks

First we give some general remarks on dimension formulas. For any degree n, if k is big enough, there are some theoretical way to calculate dimensions, though it is often too complicated to execute calculation. Two possible ways to calculate dimensions are
(1) Riemann-Roch-Hirzebruch and Lefschetz fixed point theorem. Here we need the assumption that $k \geq n+2$ for the vanishing of the obstructions of cohomology in general.
(2) The Selberg trace formula. Here we need the assumption that $k > 2n$ for the convergence of the kernel function of Godement.
By the method (2) and as an application of special values of zeta functions of prehomogeneous vector spaces, we can give an explicit conjectural dimension formula for $S_k(\Gamma(N))$ of any weight $k > 2n$ for the congruence subgroups $\Gamma(N)$ with $N \geq 3$ of any degree (Joint work with H. Saito, cf. [19].) This matches the known result for $n \leq 3$ by Morita, Christian, Yamazaki, Tsushima. The condition $k > 2n$ comes from the condition that the Selberg trace formula is valid, but actually if this conjecture is true for $k > 2n$, then it is automatically true for $k > n+1$ by (1).

Now if the weight is very small, i.e. if $k < n/2$, all the Siegel modular forms are so called singular modular forms and there are no cusp forms. There exists no general way to calculate the dimensions of $S_k(\Gamma)$ for $n/2+1 \leq k \leq n+1$. Now we assume that $n=2$. For $k \leq 3$, there exists no general way to calculate dimensions. In this article we treat the case $n=2$ and $k=3$ for relatively big discrete subgroups Γ of $Sp(2,\mathbb{R})$. The proof is related with some new vanishing theorem for $M_{1,j}(\Gamma)$ for $j > 0$, which was first obtained for $j=0$ in the joint work [20] with Skoruppa. This depends on a choice of Γ, and we cannot expect that it vanishes for general Γ.

2.2 Parahoric subgroups

We consider discrete subgroups Γ of parahoric type here. These groups are defined as follow. Let N be any natural number. We put

$$B(N) = Sp(2,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

and

$$\rho_N = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & N & 0 & 0 \\ N & 0 & 0 & 0 \end{pmatrix},$$

We also put

$$K(N) = Sp(2,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

$$\Gamma_0(N) = Sp(2,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

$$\Gamma_0'(N) = Sp(2,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

$$\Gamma_0''(N) = Sp(2,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} = \rho_N^{-1}\Gamma_0'(N)\rho_N$$

The group $K(N)$ is called paramodular group of level N.

When $N = p$ is a prime, a discrete group Γ such that $B(p) \subset \Gamma \subset Sp(2,\mathbb{Q})$ is either $Sp(2,\mathbb{Z})$, $\rho_p Sp(2,\mathbb{Z})\rho_p^{-1}$, $K(p)$, $\Gamma_0(p)$, $\Gamma_0'(p)$ or $\Gamma_0''(p)$ by virtue of the well-known Bruhat-Tits theory. The p-adic completion of Γ is a so-called parahoric subgroup in $Sp(2,\mathbb{Q}_p)$. Here we call such Γ a discrete subgroup of parahoric type.

2.3 Dimension formulas of weight 3

In this section we assume that p is any prime. The dimension formulas for $\dim S_k(\Gamma)$ where Γ is any discrete subgroup of parahoric type of prime level p were known for $k \geq 5$ (cf. [7], [14], [9]), all obtained by the Selberg trace formula. We give new results for $k = 3$ in this section and results for $k = 4$ in the next section. Outline of the proofs of these theorems will be given in the last section. We note that, by Freitag [4] p.155 Hilfssatz 2.1, 2.5 and Satz 2.6, cusp forms of weight 3 of Γ correspond bijectively with sections of the canonical divisor of any smooth compactification $\overline{\Gamma\backslash H_2}$ of $\Gamma\backslash H_2$. Although we do not use this fact at all in our proof, we can deduce several results on Siegel modular varieties from this fact and our new dimension formulas. For example, by Freitag loc. cit., Siegel modular varieties $\overline{\Gamma\backslash H_2}$ are not rational for any Γ such that $\dim S_3(\Gamma) > 0$, so we can give many such examples of Γ from below.

Theorem 2.1 *For paramodular groups $K(p)$, we have $\dim S_3(K(2)) = \dim S_3(K(3)) = 0$, and for $p \geq 5$ we have*

$$
\begin{aligned}
&\dim S_3(K(p)) = \\
&\quad -1 + \frac{1}{2880}(p^2-1) + \frac{1}{64}(p+1)\left(1 - \left(\frac{-1}{p}\right)\right) \\
&\quad + \frac{5}{192}(p-1)\left(1 + \left(\frac{-1}{p}\right)\right) \\
&\quad + \frac{1}{72}(p+1)\left(1 - \left(\frac{-3}{p}\right)\right) \\
&\quad + \frac{1}{36}(p-1)\left(1 + \left(\frac{-3}{p}\right)\right) \\
&\quad + \begin{cases} 2/5 & \textit{if } p \equiv 2,\ 3 \bmod 5 \\ 1/5 & \textit{if } p = 5 \\ 0 & \textit{otherwise} \end{cases} \\
&\quad + \frac{1}{8}\left(1 - \left(\frac{2}{p}\right)\right) + \begin{cases} 1/6 & \textit{if } p \equiv 5 \bmod 12 \\ 0 & \textit{otherwise} \end{cases}
\end{aligned}
$$

Numerical examples.

p	2	3	5	7	11	13	17	19	23	29	31	37
dim	0	0	0	0	0	1	1	1	1	2	2	4

Theorem 2.2 *For Hecke type groups* $\Gamma_0(p)$*, we have* $S_3(\Gamma_0(2)) = S_3(\Gamma_0(3)) = 0$*, and for* $p \geq 5$ *we have*

$$
\begin{aligned}
&\dim S_3(\Gamma_0(p)) = \\
&\quad \frac{(p+1)(p^2+1)}{2880} - \frac{7}{576}(p+1)^2 + \frac{55}{288}(p+1) \\
&\quad + \frac{1}{36}(p-23)\left(1+\left(\frac{-3}{p}\right)\right) \\
&\quad + \frac{1}{96}(2p-25)\left(1+\left(\frac{-1}{p}\right)\right) \\
&\quad - \frac{1}{12}\left(1+\left(\frac{-1}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right) + \begin{cases} -1/2 & \textit{if } p \equiv 1 \bmod 8 \\ -1/4 & \textit{if } p \equiv 3, 5 \bmod 8 \\ 0 & \textit{of } p \equiv 7 \bmod 8 \end{cases} \\
&\quad + \begin{cases} -4/5 & \textit{if } p \equiv 1 \bmod 5 \\ 0 & \textit{if } p \equiv 2,\ 3,\ 4 \bmod 5 \\ -1/5 & \textit{if } p = 5. \end{cases}
\end{aligned}
$$

p	2	3	5	7	11	13	17	19	23	29	31	37
dim	0	0	0	0	0	0	1	1	2	4	4	9

Theorem 2.3 *For the Iwahori subgroups for $B(p)$, we have* $\dim S_3(B(2)) = \dim S_3(B(3)) = 0$, *and for* $p \geq 5$, *we have*

$$\dim S_3(B(p)) =$$

$$\frac{(p+1)^2(p^2+1)}{2880} - \frac{13}{288}(p+1)^2 + \frac{1}{3}(p+1)$$

$$+\frac{5p-37}{48}\left(1+\left(\frac{-1}{p}\right)\right) + \frac{3p-29}{6}\left(1+\left(\frac{-3}{p}\right)\right)$$

$$-\frac{1}{6}\left(1+\left(\frac{-1}{3}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right)$$

$$+\begin{cases} -1 & \textit{if } p \equiv 1 \bmod 8 \\ 0 & \textit{otherwise} \end{cases}$$

$$+\begin{cases} -8/5 & \textit{if } p \equiv 1 \bmod 5 \\ 1/5 & \textit{if } p = 5 \\ 0 & \textit{otherwise} \end{cases}$$

p	2	3	5	7	11	13	17	19	23	29	31	37
dim	0	0	0	0	2	9	25	42	88	237	312	649

Theorem 2.4 *For the Klingen type discrete subgroups* $\Gamma_0'(p)$, *we have* $\dim S_3(\Gamma_0'(2)) = \dim S_3(\Gamma_0'(3)) = 0$, *and for* $p \geq 5$, *we have*

$$\dim S_3(\Gamma_0'(p)) =$$

$$\frac{(p+1)(p^2+1)}{2880} - \frac{1}{96}(p+1)^2 + \frac{43}{288}(p+1)$$

$$+\frac{p-11}{32}\left(1+\left(\frac{-1}{4}\right)\right) + \frac{p-13}{18}\left(1+\left(\frac{-3}{p}\right)\right)$$

$$+\begin{cases} -1/2 & \textit{if } p \equiv 1 \bmod 8 \\ 0 & \textit{otherwise} \end{cases}$$

$$+\begin{cases} -4/5 & \textit{if } p \equiv 1 \bmod 5 \\ 1/5 & \textit{if } p = 5 \\ 0 & \textit{otherwise} \end{cases}$$

p	2	3	5	7	11	13	17	19	23	29	31	37
dim	0	0	0	0	0	1	1	2	2	5	6	13

We define

$$\begin{aligned}&\dim S_k^{new}(B(p)) = \\ &\quad \dim S_k(B(p)) - \dim S_k(\Gamma_0(p)) \\ &\quad -2\dim S_k(\Gamma_0^{'}(p)) + 2\dim S_k(Sp(2,\mathbb{Z})) \\ &\quad +\dim S_k(K(p))\end{aligned}$$

Then it is known that $\dim S_3^{new}(B(p))$ is the dimension of the space of $F \in S_3(B(p))$ whose corresponding local representation at p is the Steinberg representation.

Theorem 2.5 *We have* $\dim S_3^{new}(B(2)) = \dim S_3^{new}(B(3)) = 0$ *and for* $p \geq 5$*, we have*

$$\begin{aligned}&\dim S_3^{new}(B(p)) = \\ &1 + \frac{1}{2880}(p-1)(p^3-1) - \frac{7}{576}(p-1)^2 \\ &-\frac{1}{32}(p-1)\left(1-\left(\frac{-1}{p}\right)\right) - \frac{1}{24}(p-1)\left(1-\left(\frac{-3}{p}\right)\right) \\ &-\frac{1}{12}\left(1-\left(\frac{-1}{p}\right)\right)\left(1-\left(\frac{-3}{p}\right)\right) \\ &-\frac{1}{9}\left(1-\left(\frac{-3}{p}\right)\right)^2 \\ &-\begin{cases} 1/5 & \textit{if } p=5 \\ 2/5 & \textit{if } p \equiv 2,3 \bmod 5 \\ 4/5 & \textit{if } p \equiv 4 \bmod 5 \\ 0 & \textit{otherwise} \end{cases} \\ &-\begin{cases} 1/2 & \textit{if } p \equiv 7 \bmod 8 \\ 0 & \textit{otherwise} \end{cases} \\ &-\begin{cases} 1/6 & \textit{if } p \equiv 11 \bmod 12 \\ 0 & \textit{otherwise} \end{cases}\end{aligned}$$

Numerical examples.

p	2	3	5	7	11	13	17	19	23	29	31
dim	0	0	0	0	2	8	23	38	83	225	298

2.4 Dimension formulas of weight 4

By combining the method (1) and (2) in Section 2, we can easily see that

Theorem 2.6 *The dimension formulas of $S_k(\Gamma)$ in [7], [14], [9] are valid also for $k=4$.*

The more precise proof will be explained in the last section. For the readers' convenience, we write explicit results for $k=4$ below. We obtain the first one from [7] and the second one from [14] by putting $k=4$ in the formula there.

For $\Gamma_0(p)$, we have $\dim S_4(\Gamma_0(2))=0$, $\dim S_4(\Gamma_0(3))=\dim S_4(\Gamma_0(5))=1$ and for any prime $p>5$, we have

$$\begin{aligned}\dim S_4(\Gamma_0(p)) &= \frac{1}{576}(p^2+1)(p+1)+\frac{7}{192}(p+1)^2-\frac{11}{288}(p+12)\\ &\quad +\frac{1}{36}(p-1)\left(\frac{-3}{p}\right)+\frac{2p-41}{96}\left(\frac{-1}{p}\right)-\frac{1}{12}\left(\frac{3}{p}\right)\\ &\quad +\frac{1}{8}\times\begin{cases}4 & \text{if } p\equiv 1 \bmod 8\\ 2 & \text{if } p\equiv 3,5 \bmod 8\\ 0 & \text{if } p\equiv 7 \bmod 8\end{cases} +\frac{1}{12}\times\begin{cases}4 & \text{if } p\equiv 1 \bmod 12\\ 2 & \text{if } p\equiv 3,5 \bmod 12\\ 0 & \text{if } p\equiv 7 \bmod 12,\end{cases}\end{aligned}$$

where $\left(\frac{-d}{p}\right)$ is the Legendre symbol.

For $K(p)$, we have $S_4(K(2))=S_4(K(3))=0$ and for any prime $p\geq 5$ we have

$$\begin{aligned}\dim S_4(K(p)) &= \frac{1}{576}(p^2+1)+\frac{p-2}{8}+\frac{1}{96}(p-12)\left(\frac{-1}{p}\right)+\frac{p}{36}\left(\frac{-3}{p}\right)\\ &\quad +\frac{1}{8}\left(\frac{2}{p}\right)+\frac{1}{12}\left(\frac{3}{p}\right)\end{aligned}$$

We have $\dim S_4(\Gamma_0'(2))=\dim S_4(\Gamma_0'(3))=0$. For any prime $p\geq 5$, we have

$$\begin{aligned}\dim S_4(\Gamma_0'(p)) &= \frac{1}{576}(p^2+1)(p+1)-\frac{5}{288}(p+1)^2+\frac{23}{288}(p+1)\\ &\quad +\frac{1}{96}(7p-41)\left(1+\left(\frac{-1}{p}\right)\right)+\frac{1}{36}(3p-7)\left(1+\left(\frac{-3}{p}\right)\right)\\ &\quad +\frac{1}{2}\times\begin{cases}1 & \text{if } p\equiv 1 \bmod 8\\ 0 & \text{otherwise}\end{cases} +\frac{1}{3}\times\begin{cases}1 & \text{if } p\equiv 1 \bmod 12\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

We have $\dim S_4(B(2)) = 0$ and $\dim S_4(B(3)) = 1$. For $p \geq 5$, we have

$$\begin{aligned}
&\dim S_4(B(p)) \\
= \; & \frac{1}{576}(p+1)^2(p^2+1) + \frac{11}{288}(p+1)^2 - \frac{1}{3}(p+1) + 1 \\
& + \frac{3}{16}(p-5)\left(1 + \left(\frac{-1}{p}\right)\right) + \frac{2}{9}(p-1)\left(1 + \left(\frac{-3}{p}\right)\right) \\
& - \frac{1}{6}\left(1 + \left(\frac{-1}{p}\right)\right)\left(1 + \left(\frac{-3}{p}\right)\right) \\
& + \begin{cases} 1 & \text{if } p \equiv 1 \bmod 8 \\ 0 & \text{otherwise} \end{cases} + \frac{2}{3} \times \begin{cases} 1 & \text{if } p \equiv 1 \bmod 12 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

We also have $\dim S_4^{new}(B(2)) = \dim S_4^{new}(B(3)) = 0$. For $p \geq 5$, we have

$$\begin{aligned}
&\dim S_4^{new}(B(p)) = \\
& \frac{1}{576}(p-1)(p^3-1) + \frac{7}{192}(p-1)^2 - \frac{1}{32}(p-1)\left(1 - \left(\frac{-1}{p}\right)\right) \\
& - \frac{1}{18}(p-1)\left(1 - \left(\frac{-3}{18}\right)\right) - \frac{1}{12}\left(1 - \left(\frac{-1}{p}\right)\right)\left(1 - \left(\frac{-3}{p}\right)\right) \\
& + \frac{1}{2} \times \begin{cases} 1 & \text{if } p \equiv 7 \bmod 8 \\ 0 & \text{otherwise} \end{cases} + \frac{1}{3} \times \begin{cases} 1 & \text{if } p \equiv 11 \bmod 12 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Numerical examples of dimensions of weight 4 is given below.

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43
$S_4(K(p))$	0	0	0	1	1	2	2	3	3	4	6	8	7	9
$S_4(\Gamma_0(p))$	0	1	1	3	7	11	20	27	41	75	90	143	185	211
$S_4(\Gamma_0'(p))$	0	0	0	1	1	5	7	10	14	34	43	79	101	118
$S_4(B(p))$	0	1	2	9	33	70	176	269	545	1350	1753	3506	5220	6297
$S_4^{new}(B(p))$	0	0	1	5	25	51	144	225	479	1211	1583	3213	4840	5859

3 Geometric Meanings of the Dimensions

An abelian surface A over an algebraically closed field k of characteristic p is said to be super-singular if A is isogenous to E^2 where E is a super-singular elliptic curve over k. We denote by $\mathcal{A}_{2,1}$ the moduli space of principally polarized abelian surfaces.

Theorem 3.1 *The number of irreducible components of the locus S of principally polarized super-singular abelian surfaces in the moduli space $A_{2,1}$ is equal to*

$$\dim S_3(K(p)) + 1.$$

Theorem 3.2 *The arithmetic genus of the locus S is equal to* $\dim S_3^{new}(B(p))$.

Remark. Each irreducible component of S is birational to $\mathbb{P}^1$ but has many singularities. The generalized arithmetic genus of a singular reducible curve was defined by J. P. Serre [24].

We obtain these theorems by applying the relation between arithmetic of quaternion hermitian lattices and geometry, due to Katsura-Oort and partly myself, and relation of dimension of Siegel modular forms and "class numbers" of quaternion hermitian lattices obtained in [8]. In next section, we explain this.

4 Arithmetic of Quaternion Hermitian Groups

Let D be the definite quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞ and O a maximal order of D. We put

$$G = \{g \in M_2(D); g^t\overline{g} = n(g)1_2, n(g) > 0\}.$$

Let G_A be the adelization of G. The local factor at a place $v \leq \infty$ is defined by

$$G_v = \{g \in M_2(D_v) : g^t\overline{g} = n(g)1_2, n(g) \in \mathbb{Q}_v^\times\}.$$

Here we understand that $\mathbb{Q}_\infty = \mathbb{R}$ and in that case we have $n(g) > 0$ automatically since D_∞ is definite. In particular, $G_\infty/center$ is a compact group isomorphic to $\{g \in M_2(\mathbb{H}); g^t\overline{g} = 1_2\}/\{\pm 1_2\}$ where $\mathbb{H}$ is the Hamilton quaternion algebra. For any prime $q \neq p$, we have $G_q \cong GSp(2,\mathbb{Q}_q) = \{g \in M_4(\mathbb{Q}_q); gJ^tg = n(g)J\}$. We put $U_q = GSp(2,\mathbb{Z}_q) = GSp(2,\mathbb{Q}_q) \cap M_4(\mathbb{Z}_q)^\times$. To define automorphic forms, we take a subgroup $U = G_\infty U_p \prod_{q \neq p} U_q$ of G_A, where we define a open compact group U_p of G_p later. The open subgroup $U \subset G_A$ plays the role of a "discrete subgroup". We take a representation $\rho : G_A \to G_\infty \to G_\infty/center \to GL(V)$. An automorphic form with respect to U of weight ρ is defined as a V-valued function of G_A such that $f(uga) = \rho(u)f(g)$ for any $u \in U$, $g \in G_A$ and $a \in G$, where G is diagonally embedded in G_A as usual. We denote the space of these functions by $M_\rho(U)$. When ρ is the trivial representation, we write $M_\rho(U) = M_0(U)$. In this case, the above definition implies that $\dim M_0(U) = \#(U\backslash G_A/G)$, which is called the class number of U.

Now by changing the coordinate a little, we take another group G_p^* isomorphic to G_p, and under this identification we take U_p as a subgroup $U_{p,0}$, $U_{p,1}$ or $U_{p,2}$ of G_p^* as follows.

$$\begin{aligned}
G_p \cong G_p^* &= \left\{ g \in M_2(D_p); g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \\
U_{p,1} &= G_p^* \cap M_2(O_p)^\times \\
U_{p,2} &= G_p^* \cap \begin{pmatrix} O_p & \pi^{-1}O_p \\ \pi O_p & O_p \end{pmatrix}^\times \\
U_{p,0} &= U_{p,1} \cap U_{p,2}.
\end{aligned}$$

The group $U_{p,0}$ is the minimal parahoric subgroup of G_p^*. We denote by U_i ($i = 0, 1, 2$) the corresponding open sugroup U of G_A defined as before by taking $U_p = U_{p,i}$. Then we have the following theorems.

Theorem 4.1 (Ib.-Katsura-Oort[10]) *The number of principal polarizations on E^2 is the class number of U_1.*

Theorem 4.2 (Katsura-Oort [11]) *The number of irreducible components of supersingular locus S in $A_{2,1}$ is the class number of U_2.*

The following conjecture was suggested in late 1980's in the discussion with Professor Ekedahl.

Conjecture 4.3 *The arithmetic genus of S is*

$$\dim M_0(U_0) - \dim M_0(U_1) - \dim M_0(U_2) + 1.$$

In this occasion, we would like to add several more remarks on known results about the arithmetic of quaternions and geometry.

(1) The configuration of S is described by inclusions of double cosets of U_0gG, U_1hG, U_2kG.

(2) For each class $U_2g_iG_A$ ($1 \leq i \leq H$), consider the finite groups $\Gamma_i = G \cap g_i^{-1}Ug_i$. Each Γ_i is the automorphism group of the "Moret-Bailey" family (over the irreducible component of S). We can determine all Γ_i by a kind of new mass formula. (cf. [15]).

(3) The curves C of genus 2 with $J(C) \cong E^2$ have models defined over $\mathbb{F}_{p^2}$. Each number of curves C of genus 3 such that $J(C) \cong E^2$ defined over

F_p or defined over F_{p^2} is counted. (cf. [12]).

(4) For $p \geq 3$, there exists a genus 3 curve C such that $\#(C(\mathbb{F}_{p^2}))$ attains Weil's maximum. (cf. [16]). It is an open problem if we can take the above C as a hyper-elliptic curve, as far as the author knows.

Now, by the general philosoply of Langlands, or Ihara's old conjecture in [22], the relation of automorphic forms of G_A should be related with automorphic forms of $Sp(2, \mathbb{Q}_A)$. But here the concrete description of corresponging discrete groups or new forms were not clear. The author gave more concrete conjectures of this type correspondence in early 1980's, aiming to generalize classical theorems of Eichler betwee $SL(2)$ and $SU(2)$. We shortly review them now. We denote by ρ_{f_1,f_2} the finite dimensional representation of $Sp(2) = \{g \in M_2(\mathbb{H}); g^t\overline{g} = 1_2\}$ corresponding to the Young diagram parameter $f_1 \geq f_2 \geq 0$ with $f_1 \equiv f_2 \bmod 2$.

Conjecture 4.4 ([14]) *For any even non-negative integer j and any integer $k \geq 3$, there should exist an isomorphism*

$$M^{new}_{\rho_{k+j-3,k-3}}(U_2) \cong S^{new}_{k,j}(K(p))$$

which preserves L functions.

Conjecture 4.5 ([13]) *There should exist an isomorphism*

$$M^{new}_{\rho_{k+j-3,k-3}}(U_0) \cong S^{new}_{k,j}(B(p)).$$

Here we do not explain the meaning of new forms in detail in general case (see the above quoted references.) But when $k = 3$ and $j = 0$, then we define $M^{new}_0(U)$ is space of automorphic forms orthogonal to the constant functions and $S^{new}_3(K(p)) = S_3(K(p))$ (i.e. there are no old forms in this case).

5 Outline of the Proofs

We use the Riemann-Roch-Hirzebruch theorem and the holomorphic Lefschetz theorem. For a discrete subgroup Γ in question, we take a torsion free normal subgroup Γ' of Γ with finite index. We denote by $\overline{X}$ the Satake compactification of $\Gamma'\backslash H_2$ and $\widetilde{X}$ a smooth toroidal compactification of $\overline{X}$. We put $D = \widetilde{X} - X$. Then D is a divisor with simple normal crossing. Let L be a holomorphic line bundle which is a natural prolongation of $\Gamma'\backslash H_2 \times \mathbb{C}$ where Γ' acts on $H_2 \times \mathbb{C}$ by $(Z, u) \to (MZ, \det(CZ + D)u)$. Let

Ω be a sheaf of holomorphic 1 forms on $\widetilde{X}$. We have $L^3 = \Omega^3 \otimes [D]$ and $\Omega^3 = L^3 \otimes [D]^{-1}$. We put $G = \Gamma/\Gamma'$ and for any right G-module M, we put $M^G = \{m \in M; m = mg\}$. We have $H^0(\widetilde{X}, \Omega^3) = S_3(\Gamma')$ and hence we have

$$H^0(\widetilde{X}, \Omega^3)^G = S_3(\Gamma).$$

So we must calculate $H^0(\widetilde{X}, \Omega^3)^G$. For any holomorphic vector bundle V on $\widetilde{X}$, we put

$$\chi(\widetilde{X}, V) = \sum_{i=0}^{3} (-1)^i \dim H^i(\widetilde{X}, V).$$

This is called the Euler-Poincare characteristic of V. The formula to give this number is called the Riemann-Roch-Hirzebruch theorem. The G-invariant version of this theorem is given by the holomorphic Lefschetz theorem. This is a formula to calculate

$$\chi_G(\widetilde{X}, V) = \sum_{i=0}^{3} (-1)^i \dim H^i(\widetilde{X}, V)^G.$$

We have $\dim H^i(\widetilde{X}, \Omega^3 \times L^{k-3})^G = \dim S_k(\Gamma)$. To calculate $\dim H^0(\widetilde{X}, V)$ for $V = \Omega^3 \otimes L^{k-3}$, we must do two things.
(1) Calculation of this altenating sum $\chi_G(\widetilde{X}, V)$.
(2) Calculation of $\dim H^i(\widetilde{X}, V)^G$ for $i \neq 0$.
There are done roughly as follows. We know already the formula for (1) for big k by Selberg trace formula, and we can use this value also for smaller k as we shall see later. As for (2), we shall show that $H^1(\widetilde{X}, \Omega^3)^G = H^2(\widetilde{X}, \Omega^3) = 0$ for our discrete subgroups in question. It is also easy to see $H^3(\widetilde{X}, \Omega^3)^G \cong \mathbb{C}$. So as a whole we have $\dim S_3(\Gamma) = \chi_G(\widetilde{X}, \Omega^3) + 1$.

We shall see these more in detail. First we explain the calculation (1). We review the holomorphic Lefschetz theorem (cf. [2]). This is given by the following formula.

$$\begin{aligned}
\sum_{i=0}^{3} (-1)^p \dim H^p(\widetilde{X}, V)^G &= \frac{1}{|G|} \sum_{g \in G} \tau(g) \\
\tau(g) &= \sum_{\alpha} \tau(g, X_\alpha^g) \\
\tau(g, X_\alpha^g) &= \left\{ \frac{ch(V|X_\alpha^g) \cdot \prod_g \mathcal{U}^\theta(N_\alpha^g(\theta)) \cdot \mathcal{T}(X_\alpha^g)}{\det(1 - g|(N_\alpha^g)^*)} \right\} [X_\alpha^g]
\end{aligned}$$

Here, for $g \in G$, $X^g \subset X$ denotes the fixed point set of g. Let X_α^g be an irreducible component of X^g. We denote by N_α^g the normal bundle of X_α^g

and by $(N^g_\alpha)^*$ the dual bundle. We write

$$V|X^g_\alpha = \sum_i a_i \otimes \chi_i \qquad a_i \in K(X^g_\alpha),\ \chi_i \in R(G),$$

where $K(X^g_\alpha)$ is the Grothendieck group of ventor bundles and $R(G)$ is the representation ring over $\mathbb{C}$. We denote by $ch(*)$ the Chern character of $*$ and $ch(V|X^g_\alpha)(g) = \sum_i \chi_i(g)ch(a_i)$. Now g acts also on the normal bundle N^g_α. Since g is of finite order, the eigenvalues of g of this action are roots of unity and we denote them by $e^{i\theta}$. We decompose N^g_α into eigenspaces and write the corresponding bundle by $N^g_\alpha(\theta)$. We decompose the total Chern class formally as follows.

$$\begin{aligned} c(N^g_\alpha(\theta)) &= 1 + c_1(N^g_\alpha(\theta)) + \cdots + c_n(N^g_\alpha(\theta)) \\ &= \prod_\beta (1 + x_\beta). \end{aligned}$$

Then we define

$$\mathcal{U}^\theta(N^g_\alpha(\theta)) = \prod_\beta \left(\frac{1 - e^{-x_\beta - i\theta}}{1 - e^{-i\theta}} \right)^{-1}.$$

We denote by $\mathcal{T}(X^g_\alpha)$ the Todd class of X^g_α and by $[X^g_\alpha]$ the fundamental class in $H_{2d}(X^g_\alpha, \mathbb{Z})$ where $d = \dim X^g_\alpha$.

When V is a line bundle as in our case, then there is no decomposition of the representation and we have just a character of g. If we take $L^{\otimes k}$, then the action is the k-th power of the action on L. So if we put $L|X^g_\alpha = a \otimes \chi$, then we have $L^{\otimes k}|X^g_\alpha = a \otimes \chi_i^k$ where a is a line bundle on X^g_α. So in this case we have $ch(L^{\otimes k}|X^g_\alpha) = ch(a)^k \chi^k(g)$. Since g is of finite order, $\chi(g)$ are roots of unity. In the holomorphic Lefschetz formula for $V = L^k - D$, the only part which depends on k is the term $ch(V|X^g_\alpha)(g)$. By the usual Rieman-Roch-Hirzebruch theorem, this part is expressed with the product of $\chi^k(g)$ and polynomials of k with coefficients which are independent of k. So we can conclude that

Lemma 5.1 *There exists a certain natural number M such that for each fixed k mod M, $\chi_G(\widetilde{X}, \Omega \otimes L^{k-3})$ is a polynomial of k with constant coefficients.*

Corollary 5.2 *If we have a formula for $S_k(\Gamma)$ for all $k >> 4$, then the same formula gives the G-invariant Euler-Poincare characteristic for any $k \geq 3$.*

In other words, for any weight k, the holomorphic Lefschetz formula gives us a formula of the "G-invariant Euler-Poincaré characteristics" $\chi_G(\widetilde{X}, \Omega \otimes$

L^{k-3}) as polynomials of k for each k mod M for some big M. On the other hand, it is known that $\chi_G(\widetilde{X}, \Omega \otimes L^{k-3}) = \dim S_k(\Gamma)$ for any $k \geq 4$. (cf. Tsushima [27]). For discrete groups Γ of parahoric type, $\dim S_k(\Gamma)$ has been calculated by Selberg trace formula in [7], [14], [9] (except for the case $\Gamma = B(3)$ or $\Gamma_0'(3)$, the case excluded in [9] by complication of calculation. But we can show $S_3(B(3)) = S_3'(\Gamma_0(3)) = 0$ by other ad hoc argument.) So the first calculation is done.

Next we explain (2) for $V = \Omega^3$. By Serre duality, we have

$$H^i(\widetilde{X}, \Omega^3)^G \cong H^{3-i}(\widetilde{X}, \mathcal{O})^G.$$

By this, we have

$$H^3(\widetilde{X}, \Omega^3)^G \cong H^0(\widetilde{X}, \mathcal{O})^G = \mathbb{C}.$$

By Dolbeault-Hodge theorem, we have

$$H^{3-i}(\widetilde{X}, \mathcal{O})^G \cong H^0(\widetilde{X}, \Omega^{3-i})^G.$$

The left hand side is $(H^{0,3-i})^G$ and the right hand side $(H^{3-i,0})^G$ is the complex conjugation. Hence we have

$$H^1(\widetilde{X}, \Omega^3)^G \cong H^0(\widetilde{X}, \Omega^2)^G \subset H^0(\widetilde{X}, \Omega^2(logD)) \cong A_{\det \cdot Sym_2}(\Gamma)$$

$$H^2(\widetilde{X}, \Omega)^G \cong H^0(\widetilde{X}, \Omega^1)^G \subset H^0(\widetilde{X}, \Omega(logD)) \cong A_{Sym_2}(\Gamma)$$

The author was informed of the fact $H^0(\widetilde{X}, \Omega^1) = 0$ first by Takayuki Oda. He explained the author that we can show this by standard theorems on cohomology and rather folklore. Here we can give an alternative proof using the following theorem by Freitag.

Theorem 5.3 (Freitag [3]) *For any congruence group Γ of $Sp(2, \mathbb{Z})$, we have $A_{Sym_2}(\Gamma) = 0$.*

Freitag's proof uses various modular embeddings of Hilbert modular forms and induce a contradiction. Anyway, we have $H^2(\widetilde{X}, \Omega^3) = 0$.

We will explain that $A_{\det \cdot Sym(2)}(\Gamma) = 0$ for our Γ in the next section. If we admit this, we can conclude as follows.

Theorem 5.4 *For a discrete group $\Gamma = K(N)$, $\Gamma_0(N)$, $\Gamma_0'(N)$, $\Gamma_0''(N)$, $B(p)$, where N is any squarefree natural number and p is any prime, we have*

$$\dim S_3(\Gamma) = \chi_G(\widetilde{X}, \Omega^3) + 1.$$

As we explained, we get explicit value of $\chi_G(\widetilde{X}, \Omega^3)$ by putting $k = 3$ in the general formula for big k in [7], [14], [9]. So we have dimension formulas of weight 3.

6 Vanishing of Weight One

Here we sketch the proof of the following theorem.

Theorem 6.1 *For any j, we have $A_{detSym_j}(\Gamma) = 0$ for $Sp(2,\mathbb{Z})$, $K(N)$, $\Gamma_0(N)$, $\Gamma_0'(N)$, $B(p)$, as far as N is a squarefree natural number and p is a prime.*

For $\Gamma = \Gamma_0(N)$ and $j = 0$, this theorem is already in the joint work with Skoruppa(cf. [20]) (though there was an error for the vanishing of Jacobi forms and the condition that N is (at least) squarefree should be added. Actually Lemma claim holds for a little more general N.) The results for $j > 0$ or other groups are new. We do not know in general if $S_{1,j}(B(N)) = 0$ for natural numbers N which are not primes.

Proof is essentially based on the following result on Jacobi forms.

Theorem 6.2 ([20]) *We have $J_{1,m}(\Gamma_0^{(1)}(N)) = 0$ if N is squarefree and m is coprime to N.*

Note that in [20] contains an error. There it was claimed that this holds for any N, but we need some conditions on N including all squarefree cases. Here $\Gamma_0^{(1)}(N)$ is the usual subgroup of $SL_2(\mathbb{Z})$. In particular, if $N = 1$ i.e. for $\Gamma_0(1) = SL_2(\mathbb{Z})$, we have $J_{1,m} = 0$ always. This was known already in Skoruppa [25].

So we review Jacobi forms here shortly. Let Γ_1 be a finite index subgroup of $SL_2(\mathbb{Z})$. A holomorphic function $f(\tau, z)$ of $H_1 \times \mathbb{C}$ is said to be a Jacobi form of weight k of index m belonging to $\Gamma_1^J = \Gamma\cdot\mathbb{Z}^2$ if it satisfies the following conditions (1) (2) (3).

(1) We have

$$f\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^m\left(\frac{cz^2}{c\tau+d}\right) f(\tau,z)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$, where we write $e^m(x) = e^{2\pi i m x}$.

(2) We have

$$f(\tau, z + \lambda\tau + \mu) = e^m(-\tau\lambda^2 - 2\lambda z) f(\tau, z)$$

for any λ, $\mu \in \mathbb{Z}$.

(3) For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we put

$$f|_{k,m}M = (c\tau+d)^{-k} e^m\left(-\frac{cz^2}{c\tau+d}\right) f\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right).$$

Then for any $M \in SL_2(\mathbb{Z})$, we have the following Fourier expansion.

$$(f|_{k,j}M)(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n,r)e(n\tau)e(rz)$$

where $c(n,r) = 0$ unless $4nm - r^2 \geq 0$.

The space of such Jacobi forms are denoted by $J_{k,m}(\Gamma_0(N)^J)$.

Now put

$$P_1 = \begin{pmatrix} \mathbb{Q} & 0 & \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 & \mathbb{Q} \end{pmatrix} \cap Sp(2,\mathbb{Q}).$$

Let Γ be a subgroup of $Sp(2,\mathbb{Q})$ such that $\Gamma \cap P_1$ contains all

$$\begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with λ, μ, $\kappa \in \mathbb{Z}$ and

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$. For any $F \in S_{k,j}(\Gamma)$, we write the Fourier-Jacobi expansion as

$$F(Z) = \sum_{m=1}^{\infty} f_m(\tau, z)e^m(\omega)$$

for $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$. Here $f_m(\tau, z)$ is a $\mathbb{C}^{j+1}$-valued function on $H_1 \times \mathbb{C}$. We denote by $\phi_m(\tau, z)$ is the last component of $f_m(\tau, z)$. We can show that $\phi_m(\tau, z)$ is a Jacobi form of weight k of index m belonging to Γ_1^J in the usual sense. This can been seen by the relation

$$\begin{aligned} f_m\left(\frac{a\tau + d}{c\tau + d}, \frac{z}{c\tau + d}\right) &= (c\tau + d)^k e^m(\frac{cz^2}{c\tau + d})\rho_j \begin{pmatrix} c\tau + d & cz \\ 0 & 1 \end{pmatrix} f_m(\tau, z) \\ f_m(\tau, z + \lambda\tau + \mu) &= e^m(-\lambda^2\tau - 2\lambda z) \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} f_m(\tau, z) \end{aligned}$$

and Koecher principle of F. In general, the other comonents are not Jacobi forms, but obtained by derivatives of Jacobi forms (cf. [21]).

So, for $\Gamma = \Gamma_0(N)$ or $B(N)$, the last component $\phi_m(\tau, z)$ of the m-th Fourier Jacobi coefficients belongs to $J_{k,m}(\Gamma_0^{(1)}(N)^J)$. For $\Gamma = \Gamma_0'(N)$, we have $\phi_m \in J_{k,m}(SL_2(\mathbb{Z}))^J$. When $\Gamma = K(N)$, since $\kappa \in N^{-1}\mathbb{Z}$, ϕ_m is not zero only when $N|m$. So if we renumber the coefficients as $\phi_{m/N}$, then $\phi_m \in J_{k,Nm}(SL_2(\mathbb{Z})^J)$.

The following lemma is obtained easily by seeing the automorphic property of Siegel modular forms.

Lemma 6.3 *Let Γ be one of the above discrete groups. If the last component of $F \in S_{k,j}(\Gamma)$ is identically zero, then F itself is identically zero.*

Since we have $J_{1,m}(SL_2(\mathbb{Z})^J) = 0$ for any m, this lemma implies immediately that $S_{1,j}(\Gamma_0'(N)) = S_{1.j}(K(N)) = S_{1,j}(Sp(2,\mathbb{Z})) = 0$ for any $j \geq 0$. Since $\Gamma_0'(N)$ and $\Gamma_0''(N)$ are conjugate, we also have $S_{1,j}(\Gamma_0''(N)) = 0$. As for the claim that $S_{1,j}(\Gamma_0(N)) = 0$ for any natural number N and $S_{1,j}(B(p)) = 0$ for any prime p with $j \geq 0$, we need more argument similar to those as in [20]. We omit the details of the proof here.

Correction: In the paper [9], there are following typos.
p.44 l.2; "$[-1, -k+1, -k+2, k-1, k-2; 6]$" should read $[-1, -k+1, -k+2, 1, k-1, k-2; 6]$.
p.44 l.10; In the right hand side of $t(\hat{\beta}_5, k)+t(\hat{\beta}_6, k)$, "$(-4/9)(1+(-1/p))[1, -1, 0; 3]$" should read $(-4/9)(1+(-3/p))[1, -1, 0; 3]$.
p.47 l.21; In the right hand side of T_6, "$[-k+1, -k+2; 2]$" should read $[-k+2, -k+1; 2]$.
p.49 l.1; In the right hand side of $H_3(U_1'(p))$, "$(p-1)(1-(-1/p))/2^6/3^2$" should read $(p-1)(1-(-1/p))/2^4/3$.
p. 75; In (5.17), "$= m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p))$" should read $m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = p + 2 + (-1/p)$. Similarly, in (5.18), "$= m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p))$" should read $m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = p + 2 + (-3/p)$.

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