

GENERATORS FOR A MODULE OF VECTOR-VALUED SIEGEL MODULAR FORMS OF DEGREE 2

Christiaan van Dorp*

September 25, 2018

Abstract

In this paper we will describe all vector-valued Siegel modular forms of degree 2 and weight $\text{Sym}^6(\text{St}) \otimes \det^k(\text{St})$ with k odd. These vector-valued forms constitute a module over the ring of classical Siegel modular forms of degree 2 and even weight and this module turns out to be free. In order to find generators, we generalize certain Rankin-Cohen differential operators on triples of classical Siegel modular forms that were first considered by Ibukiyama and we find a Rankin-Cohen bracket on vector-valued Siegel modular forms.

1 Introduction

In comparison to elliptic modular forms, Siegel modular forms and especially vector-valued Siegel modular forms are much less understood. Although the dimensions of the vector spaces of genus 2 modular forms are known due to Tsushima [18], explicit generators are unknown in almost all cases. For some instances however, such generators are known. These ‘first few’ examples are due to Satoh [17] and Ibukiyama [12, 13].

Tsushima’s dimension formula can give clues—apart from the dimensions—about the structure of the modules of forms. For instance, the dimension formula can predict relations and in those cases the modules will not be free. When no relations are predicted, the modules could be freely generated over the ring of classical modular forms of even weight. Ibukiyama has conjectured that one of the modules he studied is in fact free, but he did not prove this [13]. In this paper we will prove his hypothesis and in order to do this, we develop some methods for constructing vector-valued Siegel modular forms of genus 2. Our method allows us to compute a few eigenvalues for the Hecke operators. Our results agree with calculations done by van der Geer based on his joint work with Faber [6, 7].

1.1 Siegel modular forms

We will first introduce some notions and notation. Let V be a finite dimensional \mathbb{C} -vector space, g a positive integer and let $\rho : \text{GL}(g, \mathbb{C}) \rightarrow \text{GL}(V)$ be a representation. A Siegel modular form f of weight ρ and genus (degree) $g \geq 2$ is then a V -valued holomorphic function on the

*Korteweg- de Vries Institute for Mathematics, Universiteit van Amsterdam, Amsterdam. Current address: Theoretical Biology and Bioinformatics, Universiteit Utrecht, Utrecht.

Siegel upper half-space \mathcal{H}_g that satisfies for every element γ of the symplectic group $\Gamma_g := \mathrm{Sp}(2g, \mathbb{Z})$ the functional equation

$$f(\gamma \cdot \tau) = \rho(j(\gamma, \tau))f(\tau), \quad \tau \in \mathcal{H}_g.$$

Here we write $j(\gamma, \tau) := c\tau + d$ for the factor of automorphy and $\gamma \cdot \tau := (a\tau + b)(c\tau + d)^{-1}$, where $a, b, c, d \in \mathfrak{gl}(g, \mathbb{Z})$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g$. A Siegel modular form f has a Fourier series

$$f(\tau) = \sum_n a(n)q^n, \quad q^n := e^{2\pi i \sigma(n\tau)},$$

where the sum is taken over the set $S_g := \{n = (n_{ij}) \mid n_{ii} \in \mathbb{Z}, 2n_{ij} \in \mathbb{Z}, n_{ij} = n_{ji}\}$ of all half-integer, symmetric $g \times g$ matrices n and $\sigma(x)$ denotes the trace of a square matrix x . We will write x' for the transpose of a matrix x and when y is a square matrix of appropriate size, then $y[x] := x'yx$.

The Fourier transform a_f of a Siegel modular form $f = \sum a_f(n)q^n$ defines a function $a_f : S_g \rightarrow V$ and if $g > 1$, then $a_f(n) \neq 0 \implies n \geq 0$ (the Koecher principle). Denote by S_g^+ the subset of S_g of semi-positive matrices. A form f for which $a_f(n)$ vanishes for non-positive $n \in S_g^+$ is called a cusp form. We will denote the space of modular forms of weight ρ by $M_\rho(\Gamma_g)$ and the space of cusp forms by $S_\rho(\Gamma_g)$. The general linear group $\mathrm{GL}(g, \mathbb{Z}) \hookrightarrow \Gamma_g$, embedded by $u \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$, acts on S_g^+ by $u : n \mapsto un u'$ and the Fourier transform a_f of f behaves well under this action:

$$a_f(unu') = \rho(u)a_f(n). \quad (1)$$

From now on, we assume that $g = 2$ unless otherwise specified. For convenience, we often write square matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as $(a, b; c, d)$ and when no confusion can be possible, we will write $(n, r/2; r/2, m) \in S_2$ as (n, m, r) . The irreducible representations ρ of $\mathrm{GL}(2, \mathbb{C})$ can be characterized by their highest weight vector $(\lambda_1 \geq \lambda_2)$ and if $\lambda_2 < 0$, then $\dim_{\mathbb{C}} M_\rho(\Gamma_2) = 0$. Therefore we only have to consider ‘polynomial’ representations. If we write $j = \lambda_1 - \lambda_2$ and $k = \lambda_2$, then ρ will be isomorphic to $\mathrm{Sym}^j(\mathrm{St}) \otimes \det^k(\mathrm{St})$, where St denotes the standard representation of $\mathrm{GL}(2, \mathbb{C})$ and Sym^j and \det^k denote the j -fold symmetric product and k -th power of the determinant respectively. The representation $\mathrm{Sym}^j(\mathrm{St}) \otimes \det^k(\mathrm{St})$ is abbreviated to (j, k) and we write k to denote the weight $(0, k)$.

The structure of the ring of classical Siegel modular forms $M_* := \bigoplus_k M_k(\Gamma_2)$ of genus 2 was determined by Igusa [14]. The subring $M_*^0 := \bigoplus_{k \equiv 0(2)} M_k(\Gamma_2) \subseteq M_*$ is a polynomial ring:

$$M_*^0 = \mathbb{C}[\varphi_4, \varphi_6, \chi_{10}, \chi_{12}],$$

where φ_4 and φ_6 are Eisenstein series of weight 4 and 6 (with Fourier series normalized at $(0, 0, 0)$) and χ_{10} and χ_{12} are cusp forms of weight 10 and 12 (with Fourier series normalized at $(1, 1, 1)$). Satoh has determined the structure of the M_*^0 -module $M_{(j,*)}^i := \bigoplus_{k \equiv i(2)} M_{(j,k)}(\Gamma_2)$ for $(j, i) = (2, 0)$ and Ibukiyama did the same for $(j, i) = (2, 1), (4, 0), (4, 1)$ and $(6, 0)$. In this paper we will determine the structure of $M_{(6,*)}^1$. Our main result can be formulated as follows.

Main Theorem. *The M_*^0 -module $M_{(6,*)}^1$ is freely generated by seven elements F_k of weight $(6, k)$ with $k \in \{11, 13, 15, 17, 19, 21, 23\}$.*

We shall give the elements mentioned in the above theorem explicitly, but in order to do this we need to introduce Rankin-Cohen differential operators on Siegel modular forms.

1.2 Rankin-Cohen operators

Rankin-Cohen operators (RC-operators) send t -tuples of classical Siegel modular forms to (possibly vector-valued) Siegel modular forms. They are therefore useful when we want to find generators for modules of Siegel modular forms. RC-operators were studied in full generality by (among others) Ibukiyama, Eholzer and Choie [11, 5, 3]. The general construction of RC-operators can be quite cumbersome and therefore we only explain the genus 2 case here.

Write $H_j = \{p \in \mathbb{C}[x, y] \mid \forall \lambda \in \mathbb{C} : p(\lambda x, \lambda y) = \lambda^j p(x, y)\}$ for the space of homogeneous polynomials of degree j in two variables x and y . The representation $\rho := \text{Sym}^j(\text{St}) \otimes \det^\ell(\text{St}) : \text{GL}(2, \mathbb{C}) \rightarrow \text{GL}(H_j)$ is given explicitly by:

$$(\rho(G) \cdot p)(x, y) = \det(G)^\ell \cdot p((x, y)G), \quad G \in \text{GL}(2, \mathbb{C}), \quad p \in H_j.$$

Let $R_t = \mathbb{C}[\mathbf{r}_{ij}^s \mid 0 < i \leq j \leq 2, 0 < s \leq t]$ be the polynomial ring in the $3t$ variables \mathbf{r}_{ij}^s and consider an element P of $H_j \otimes_{\mathbb{C}} R_t$. It is convenient to write $P(\mathbf{r}_{11}^1, \dots, \mathbf{r}_{22}^t, x, y) =: P(\mathbf{r}^1, \dots, \mathbf{r}^t; v)$, where $\mathbf{r}^s = (\mathbf{r}_{11}^s, \mathbf{r}_{12}^s; \mathbf{r}_{12}^s, \mathbf{r}_{22}^s)$ is a symmetric 2×2 matrix and $v = (x, y)$. Using this notation, we can give the following definition.

Definition 1.1. An element P of $H_j \otimes_{\mathbb{C}} R_t$ is called ρ -homogeneous if

$$P(G\mathbf{r}^1 G', \dots, G\mathbf{r}^t G'; v) = \det(G)^\ell \cdot P(\mathbf{r}^1, \dots, \mathbf{r}^t; G'v)$$

for all $G \in \text{GL}(2, \mathbb{C})$.

Ibukiyama refers to certain special elements P of the space $H_j \otimes_{\mathbb{C}} R_t$ as ‘associated polynomials’, since they are ‘associated’ to other polynomials \tilde{P} called ‘pluri-harmonic’ polynomials [11]. These polynomials \tilde{P} can be constructed from P by first choosing an element $k = (k_1, \dots, k_t) \in \mathbb{Z}_{>0}^t$, which we will refer to as a *type*, and then replacing the matrices \mathbf{r}^s by $\xi^s \xi^{s'}$ with $\xi^s = (\xi_{ij}^s)$ a $2 \times 2k_s$ matrix of indeterminates:

$$P \mapsto \tilde{P} : H_j \otimes_{\mathbb{C}} R_t \rightarrow H_j \otimes_{\mathbb{C}} \mathbb{C}[\xi_{ij}^s \mid 0 < i \leq 2, 0 < j < 2k_s, 0 < s \leq t],$$

$$\tilde{P}(\xi_{11}^1, \dots, \xi_{2, 2k_t}^t, x, y) := P(\xi^1 \xi^{1'}, \dots, \xi^t \xi^{t'}; v).$$

Note that the map $P \mapsto \tilde{P}$ depends on the choice of the type k and that for $P \in H_j \otimes_{\mathbb{C}} R_t$, the polynomial \tilde{P} is not necessarily pluri-harmonic. We can now give the following definition.

Definition 1.2. An element $P \in H_j \otimes_{\mathbb{C}} R_t$ is called k -harmonic if \tilde{P} is harmonic in the sense that

$$\Delta \tilde{P} := \sum_{i,j,s} \frac{\partial^2 \tilde{P}}{(\partial \xi_{ij}^s)^2} = 0.$$

Polynomials that are ρ -homogeneous and k -harmonic can be used to define RC-operators as shown in the following theorem and therefore we will refer to these polynomials as *RC-polynomials*. The space of RC-polynomials is denoted by $\mathcal{H}_\rho(k) \subset H_j \otimes_{\mathbb{C}} R_t$. Write $\tau = (\tau_1, z; z, \tau_2)$ for an element $\tau \in \mathcal{H}_2$ and write $|k| = \sum_s k_s$.

Theorem 1.3 (Ibukiyama). *Suppose that $P \in \mathcal{H}_\rho(k)$ with $\rho = (j, \ell)$ and let f_1, \dots, f_t be classical Siegel modular forms on Γ_2 of weight k_1, \dots, k_t respectively. Write $d/d\tau^s := (\partial/\partial\tau_1^s, \frac{1}{2}\partial/\partial z^s; \frac{1}{2}\partial/\partial z^s, \partial/\partial\tau_2^s)$. The H_j -valued function*

$$\mathcal{D}[P](f_1, \dots, f_t)(\tau) := \frac{1}{(2\pi i)^{j/2+\ell}} P(d/d\tau^1, \dots, d/d\tau^t; v) f_1(\tau^1) \cdots f_t(\tau^t) \Big|_{\tau^1 = \dots = \tau^t = \tau}, \quad \tau \in \mathcal{H}_2 \quad (2)$$

is a Siegel modular form of weight $\rho \otimes \det^{|k|}(\text{St}) = (j, \ell + |k|)$ and genus 2.

A more general version of Theorem 1.3 was proven by Ibukiyama [11]. Ibukiyama and others also give explicit examples of RC-polynomials and all RC-polynomials for type of length 2 were determined explicitly by Miyawaki [15]. If the length t of the type k equals 2, then a non-zero Siegel modular form that has been constructed using a RC-operator will only have weight (j, ℓ) with ℓ odd if a classical Siegel modular form of odd weight has been used in this construction [17, 12]. The cusp form $\chi_{35} \in M_{35}(\Gamma_2)$ is such a classical Siegel modular form of odd weight, but if we were to use it in a construction with an RC-operator, the weight of the resulting Siegel modular form will have $\ell \geq 39$. The dimension of e.g. $M_{(2,21)}(\Gamma_2)$ equals 1, hence we need RC-operators with a higher type length in order to get forms of ‘low’ weight. This ‘trade-off’ causes further problems when we increase j and we will give a (partial) solution to this problem below by introducing a Rankin-Cohen bracket on vector-valued Siegel modular forms.

Ibukiyama constructed RC-polynomials of weight $(2, 1)$ and $(4, 1)$ and type length $t = 3$ in order to find generators for $M_{(2,*)}^1$ and $M_{(4,*)}^1$. We will generalize these polynomials to find generators for $M_{(6,*)}^1$, but our generalization can also be used to find other vector-valued Siegel modular forms of weight (j, ℓ) with $j \geq 2$ and $\ell \geq 15$ odd.

1.3 Acknowledgements

We would like to thank Gerard van der Geer, who stimulated the author to write this paper, and Fabien Cléry for the useful discussions and their corrections and comments.

2 Results

The Hilbert-Poincaré series for the dimensions of $M_{(6,\ell)}(\Gamma_2)$ with ℓ odd is given by (cf. [18])

$$\sum_{\ell \equiv 1(2)} \dim_{\mathbb{C}} M_{(6,\ell)}(\Gamma_2) \cdot X^{\ell} = \frac{X^{11} + X^{13} + X^{15} + X^{17} + X^{19} + X^{21} + X^{23}}{(1 - X^4)(1 - X^6)(1 - X^{10})(1 - X^{12})}. \quad (3)$$

This suggests that we should look for generators for $M_{(6,*)}^1$ of weights $(6, \ell)$ with $\ell = 11, 13, \dots, 23$. We will first give two methods for constructing forms of weight $(6, \ell)$ with $\ell = 15, 17, \dots, 23$ and $\ell = 11, 13$ respectively and then we will give explicit generators.

2.1 RC-polynomials of weight $(j, 1)$

We start with an example by Ibukiyama and Eholzer [5]. The RC-polynomials for elliptic modular forms were studied by Rankin and Cohen [4] and they can be used to construct RC-polynomials of weight $(j, 0)$. Such a genus 1 RC-polynomial $p_{j,k} \in \mathbb{C}[r_1, r_2]$ can be written as

$$p_{j,k}(r_1, r_2) = \sum_{i=0}^{j/2} (-1)^i \binom{j/2}{i} (k_1 + j/2 - 1)_i (k_2 + j/2 - 1)_{j/2-i} r_1^{j/2-i} r_2^i, \quad j \equiv 0(2). \quad (4)$$

Here we use the Pochhammer symbol $(x)_n := x(x-1)(x-2)\cdots(x-n+1)$. The corresponding RC-polynomial $P_{j,k} \in \mathcal{H}_{(j,0)}(k)$ is then given by

$$P_{j,k}(\mathbf{r}^1, \mathbf{r}^2; v) := p_{j,k}(\mathbf{r}^1[v], \mathbf{r}^2[v]),$$

and it is easy to verify that $P_{j,k}$ is indeed $(j, 0)$ -homogeneous and k -harmonic. Given a polynomial p in the variables r_1, \dots, r_t , we then denote the map that replaces the variables r_s by $\mathbf{r}^s[v]$ by Ψ :

$$\Psi : p(r_1, \dots, r_t) \mapsto p(\mathbf{r}^1[v], \dots, \mathbf{r}^t[v]) : \mathbb{C}[r_1, \dots, r_t] \longrightarrow \bigoplus_{j \geq 0} H_j \otimes_{\mathbb{C}} R_t.$$

We will now give a construction that uses elliptic (i.e. genus $g = 1$) RC-polynomials and produces RC-polynomials of weight $(j, 1)$ and type $k = (k_1, k_2, k_3)$. We first consider the cross product on the space of 2×2 symmetric matrices. Let $J = (0, 1; -1, 0)$ and define for $A = (a, b; b, c)$ and $B = (a', b'; b', c')$

$$A \times B := AJB - BJA = \begin{pmatrix} 2ab' - 2ba' & ac' - ca' \\ ac' - ca' & 2bc' - 2cb' \end{pmatrix}, \quad (5)$$

then $(GAG') \times (GBG') = \det(G) \cdot G(A \times B)G'$ for all $G \in \mathrm{GL}(2, \mathbb{C})$.

Definition 2.1. The operator $\mathbf{M}_k : \mathbb{C}[r_1, r_2, r_3] \longrightarrow \bigoplus_j H_j \otimes_{\mathbb{C}} R_3$ is defined as follows:

$$\mathbf{M}_k p = \mathbf{r}^1 \times \mathbf{r}^2[v] \Psi \left(k_3 p + r_3 \frac{\partial}{\partial r_3} p \right) - \mathbf{r}^1 \times \mathbf{r}^3[v] \Psi \left(k_2 p + r_2 \frac{\partial}{\partial r_2} p \right) + \mathbf{r}^2 \times \mathbf{r}^3[v] \Psi \left(k_1 p + r_1 \frac{\partial}{\partial r_1} p \right).$$

Proposition 2.2. Let $p = p_{j, (k_1+1, k_2+1)}$ be the elliptic RC-polynomial defined by equation (4), then $\mathbf{M}_k p$ is a RC-polynomial of weight $(j+2, 1)$ and type $k = (k_1, k_2, k_3)$.

The proof of Proposition 2.2 is elementary but very tedious. We will therefore omit the details. The operator \mathbf{M}_k can be shown to ‘commute’ with the Laplacian Δ at the cost of a shift in the type $(k_1, k_2, k_3) \mapsto (k_1+1, k_2+1, k_3+1)$. The above result then follows immediately. Note that we could also use other appropriate polynomials $p \in \mathbb{C}[r_1, r_2, r_3]$.

We now have a recipe to construct vector-valued Siegel modular forms of weight (j, ℓ) with ℓ odd. First take $f_s \in M_{k_s}(\Gamma_2)$ for $s = 1, 2, 3$, then choose an elliptic RC-polynomial $p = p_{j-2, (k_1+1, k_2+1)}$ and define $P = \mathbf{M}_{(k_1, k_2, k_3)} p \in \mathcal{H}_{(j, 1)}(k_1, k_2, k_3)$. Theorem 1.3 then tells us that the function $\mathcal{D}[P](f_1, f_2, f_3)$ is an element of $M_{(j, |k|+1)}(\Gamma_2)$. This would be of no use if the resulting Siegel modular forms vanish identically. We can show that $\mathcal{D}[P](f_1, f_2, f_3)$ is non-vanishing by computing a non-zero Fourier coefficient. If $f_s = \sum_n a_s(n)q^n$ and $\mathcal{D}[P](f_1, f_2, f_3) = \sum_n b(n)q^n$, then

$$b(n) = \sum_{\substack{(n_1, n_2, n_3) \in (\mathbb{S}_2^+)^3 \\ n_1 + n_2 + n_3 = n}} P(n_1, n_2, n_3; v) a_1(n_1) a_2(n_2) a_3(n_3). \quad (6)$$

Although this is a simple formula, the fact that we take a sum over all triples $(n_1, n_2, n_3) \in (\mathbb{S}_2^+)^3$ such that $n_1 + n_2 + n_3 = n$ can result in a computationally difficult problem, since the number of partitions $n_1 + n_2 + n_3 = n$ grows very fast as the trace $\sigma(n)$ grows.

Example 2.3. The operator \mathbf{M}_k defined in Definition 2.1 generalises the polynomials given by Ibukiyama [12, 13]. These polynomials are given by

$$\mathbf{M}_k 1 \in \mathcal{H}_{(2, 1)}(k) \quad \text{and} \quad \mathbf{M}_k((k_1+1)r_2 - (k_2+1)r_1) \in \mathcal{H}_{(4, 1)}(k)$$

with $k = (k_1, k_2, k_3)$. Ibukiyama uses these polynomials to define Rankin–Cohen brackets on triples of classical Siegel modular forms and shows that the resulting vector-valued Siegel modular forms generate the modules $M_{(2, *)}^1$ and $M_{(4, *)}^1$. The operator $\mathcal{D}[\mathbf{M}_k 1]$ can also be described

in terms of the cross product (5) and Satoh's RC-brackets [17]. Satoh uses the space of symmetric 2×2 complex matrices as a representation space for $\text{Sym}^2(\text{St})$ (where the $\text{GL}(2, \mathbb{C})$ action is given by $G : A \rightarrow GAG'$ for A a symmetric matrix and $G \in \text{GL}(2, \mathbb{C})$) and then defines

$$[f_1, f_2] := k_1 f_1 \frac{d}{d\tau} f_2 - k_2 f_2 \frac{d}{d\tau} f_1 \in M_{(2, k_1+k_2)}(\Gamma_2), \quad f_s \in M_{k_s}(\Gamma_2).$$

Let $f_s \in M_{k_s}(\Gamma_2)$ for $s = 1, 2, 3$. Then

$$F := [f_1, f_2] \times [f_1, f_3] \in M_{(2, 2k_1+k_2+k_3+1)}(\Gamma_2)$$

and unraveling the definitions easily shows that F is divisible by f_1 in the M_* -module $M_{(2,*)}$. Ibukiyama's RC-brackets $[f_1, f_2, f_3]$ (in [12]) are then given by $[f_1, f_2, f_3] := cF/f_1$ for some non-zero constant c .

2.2 A Rankin-Cohen bracket on vector-valued Siegel modular forms

Our aim is to find all Siegel modular forms of weight $(6, \ell)$ with ℓ odd. Formula (3) shows that $\dim_{\mathbb{C}} M_{(6,11)}(\Gamma_2) = 1$ and the lowest ℓ for which we can use the operator \mathbf{M}_k to construct a non-zero form of weight $(6, \ell)$ equals 15. Note that by taking $k = (4, 4, 4)$ and $p = p_{4,(5,5)}$, we can get a form $\mathcal{D}[\mathbf{M}_k p](\varphi_4, \varphi_4, \varphi_4) \in M_{(6,13)}(\Gamma_2)$, but unfortunately this form vanishes identically. This means that the above described construction with \mathbf{M}_k is not sufficient for our purpose, unless we would be able to divide by a classical Siegel modular form, but this appears to be quite difficult without prior knowledge of the quotient. Ibukiyama encountered a similar problem when he determined all modular forms of weight $(6, \ell)$ with ℓ even and he solved this by using a theta series Θ_8 of weight $(6, 8)$ (cf. [13, 9]) and a Klingen-Eisenstein series E_6 of weight $(6, 6)$ (cf. [2]). These forms can not be constructed directly by means of RC-operators. However, as we will point out below, we can use RC-operators to compute their Fourier coefficients. In this paper we have scaled Θ_8 such that the Fourier coefficient of Θ_8 at $(1, 1, 1)$ equals $x^4 y^2 + 2x^3 y^3 + x^2 y^4$. The form E_6 is scaled such that its Fourier coefficient at $(1, 0, 0)$ equals x^6 .

The forms Θ_8 and E_6 can be used to construct forms of weight $(6, 13)$ and $(6, 11)$. In order to see how this can be done, we must first give a new interpretation of the operators $\mathcal{D}[\mathbf{M}_k p]$, where p is an elliptic RC-polynomial. Choose $p = p_{j-2, (k_1+1, k_2+1)}$ and let $f_s \in M_{k_s}(\Gamma_2)$ for $s = 1, 2, 3$. Also let $q = p_{j, (k_1, k_2)}$ and define $F = \mathcal{D}[\Psi q](f_1, f_2) \in M_{(j, k_1+k_2)}(\Gamma_2)$. We can re-write $G := \mathcal{D}[\mathbf{M}_k p](f_1, f_2, f_3)$ in such a way that

$$G = c \cdot \{F, f_3\}$$

for some bilinear form $\{\cdot, \cdot\}$ on $M_{(j, k_1+k_2)}(\Gamma_2) \times M_{k_3}(\Gamma_2)$ and a constant $c \in \mathbb{C}^*$. We will give the exact description of $\{\cdot, \cdot\}$ below, but let us first state the advantage of this effort. We can replace the modular form F that was defined via a RC-operator by any other modular form of weight $(j, k_1 + k_2)$. So for instance, we can take $F = E_6$ and $f_3 = \varphi_4$ and if we then apply $\{\cdot, \cdot\}$, we get

$$\{E_6, \varphi_4\} \in M_{(6,11)}(\Gamma_2).$$

We will now give an explicit formula for $\{\cdot, \cdot\}$.

Definition 2.4. Suppose that $F \in M_{(j,k)}(\Gamma_2)$ and $\varphi \in M_{\ell}(\Gamma_2)$. Define the determinant $W(\mathbf{r})$ by

$$W(\mathbf{r}) := \begin{vmatrix} \mathbf{r}_{11} & \mathbf{r}_{12} & \mathbf{r}_{22} \\ y^2 & -xy & x^2 \\ \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{vmatrix} : H_j \rightarrow H_j \otimes_{\mathbb{C}} R_1.$$

We now define the brackets $\{\cdot, \cdot\}$ as follows.

$$\{F, \varphi\} := \frac{1}{(j-1)2\pi i} \left((k+j/2-1)W\left(\frac{d\varphi}{d\tau}\right)F - \ell\varphi W\left(\frac{d}{d\tau}\right)F \right).$$

By our considerations above, we then have the following result.

Proposition 2.5. *Take F and φ as in Definition 2.4. The form $\{F, \varphi\}$ is then a Siegel modular form of weight $(j, k + \ell + 1)$.*

In order to be able to work with the forms $\{F, \varphi\}$, we need to know how to compute Fourier coefficients. We can easily find a formula similar to (6). Write $\varphi = \sum_{n \geq 0} a(n)q^n$ and $F = \sum_{n \geq 0} b(n)q^n$ and let $\{F, \varphi\} = \sum_{n \geq 0} c(n)q^n$, then we get

$$(j-1)c(n) = \sum_{\substack{(n_1, n_2) \in (S_2^+)^2 \\ n_1 + n_2 = n}} (k+j/2-1)a(n_1)W(n_1)b(n_2) - \ell a(n_1)W(n_2)b(n_2). \quad (7)$$

Remark. A modular form that is defined using the brackets $\{\cdot, \cdot\}$ will always be a cusp form. Suppose that we use $F \in M_{(j,k)}(\Gamma_2)$ and $\varphi \in M_\ell(\Gamma_2)$ in order to construct $\{F, \varphi\} \in M_{(j,k+\ell+1)}(\Gamma_2)$, then either $k + \ell + 1$ is odd and hence $\{F, \varphi\}$ must be a cusp form, or one of the integers ℓ or k is odd, implying that F or φ is a cusp form.

We can also see directly from Formula (7) that the brackets map to $S(\Gamma_2)$ by considering the Fourier coefficients at singular indices n . Suppose that $n = n_1 + n_2$ where $n, n_1, n_2 \in S_2^+$ and n is singular, then we can find a $u \in \text{SL}(2, \mathbb{Z})$ such that $unu' = (v, 0, 0)$, $un_1u' = (v_1, 0, 0)$ and $un_2u' = (v_2, 0, 0)$. Hence, by Formula (1) we can assume without loss of generality that n is of the form $n = (v, 0, 0)$. The Fourier coefficients $b(n)$ of F for $n = (v, 0, 0)$ always have the form $\alpha(v) \cdot x^j$ for some constant $\alpha(v)$. Therefore, we get for $s = 1, 2$ that

$$W(n_s)b(n_2) = v_s \left| \begin{array}{cc} -xy & x^2 \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{array} \right| \alpha(v_2)x^j = 0.$$

This shows that $c(n)$ in Formula (7) vanishes for singular n .

2.3 Generators for $M_{(6,*)}^1$

As promised, we will now give explicit generators for $M_{(6,*)}^1$. We first use the brackets $\{\cdot, \cdot\}$ to define forms of weight (6, 11) and (6, 13):

$$F_{11} := \{E_6, \varphi_4\}/1152, \quad F_{13} := \{\Theta_8, \varphi_4\}/4$$

and for the remainder of the generators, we use the construction with \mathbf{M}_k . Choose the following elliptic RC-polynomials p_i and types k_i :

p_i	polynomial	type k_i
$p_{15}(r_1, r_2)$	$(5r_1^2 - 14r_1r_2 + 7r_2^2)/160$	(5, 4, 5)
$p_{17}(r_1, r_2)$	$(4r_1^2 - 8r_1r_2 + 3r_2^2)/192$	(5, 6, 5)
$p_{19}(r_1, r_2)$	$(22r_1^2 - 24r_1r_2 + 5r_2^2)/1920$	(4, 10, 4)
$p_{21}(r_1, r_2)$	$(22r_1^2 - 24r_1r_2 + 5r_2^2)/2880$	(4, 10, 6)
$p_{23}(r_1, r_2)$	$(13r_1^2 - 14r_1r_2 + 3r_2^2)/16$	(5, 12, 5)

k	$\lambda(2)$ on $N_{(6,k)}(\Gamma_2)$	$\lambda(2)$ on $S_{(6,k)}(\Gamma_2)$
6	$-24 \cdot (1 + 2^4)$	—
8	—	0
10	$216 \cdot (1 + 2^8)$	1680
11	—	-11616
12	$-528 \cdot (1 + 2^{10})$	$X^2 - 22368X + 57231360$
13	—	-24000
15	—	$X^2 + 68256X + 593510400$
17	—	$X^3 + 363264X^2 + 136028160X - 4603543289856000$
19	—	$X^4 + 1202400X^3 - 1311202861056X^2$ $-179858880190218240X - 1566691549034368204800$

Table 1: Eigenvalues of the Hecke operator $T(2)$ on $M_{(6,k)}(\Gamma_2)$ for some values of k . If a polynomial in X is given, the eigenvalues $\lambda(2)$ are the roots of this polynomial. The space $N_{(6,k)}(\Gamma_2)$ is the orthogonal complement of $S_{(6,k)}(\Gamma_2)$ with respect to the Petersson product.

We then write $P_i = \mathbf{M}_{k_i} p_i$ for $i = 15, 17, \dots, 23$ and define modular forms F_i as follows.

F_i	modular form	weight
F_{15}	$\mathcal{D}[P_{15}](\chi_5, \varphi_4, \chi_5)$	(6, 15)
F_{17}	$\mathcal{D}[P_{17}](\chi_5, \varphi_6, \chi_5)$	(6, 17)
F_{19}	$\mathcal{D}[P_{19}](\varphi_4, \chi_{10}, \varphi_4)$	(6, 19)
F_{21}	$\mathcal{D}[P_{21}](\varphi_4, \chi_{10}, \varphi_6)$	(6, 21)
F_{23}	$\mathcal{D}[P_{23}](\chi_5, \chi_{12}, \chi_5)$	(6, 23)

The form χ_5 denotes the square root of χ_{10} in the ring of holomorphic functions on \mathcal{H}_2 (see e.g. [8]). Our main result can now be formulated as follows:

Main Theorem. *The M_*^0 -module $M_{(6,*)}^1$ is free and can be written as the following direct sum:*

$$M_{(6,*)}^1 = \bigoplus_{i \in I} F_i \cdot M_*^0,$$

where F_i are defined above and the direct sum is taken over the set $I = \{11, 13, 15, 17, 19, 21, 23\}$.

2.4 Eigenvalues of the Hecke operators

Since we now know all modular forms of weight $(6, k)$ with $k \in \mathbb{Z}$, we can calculate the eigenvalues of the Hecke operators $T(p)$ (cf. [2, 1]). We did this for $p = 2, 3$ and some k (Table 1 and 2).

At our request, G. van der Geer computed some of these eigenvalues using a completely independent method that is based on counting points on hyperelliptic curves over finite fields [9, 6, 7]. The values listed here agree with these. Also note that the characteristic polynomials of $T(2)$ and $T(3)$ on $S_{(6,k)}(\Gamma_2)$ with $k = 12$ and 15 have the following discriminant:

k	$\Delta(\det(T(2) - X))$	$\Delta(\det(T(3) - X))$
12	$2^{10} 3^2 7^2 601$	$2^{14} 3^6 7^2 13^2 601$
15	$2^{10} 3^2 29 \cdot 83 \cdot 103$	$2^{12} 3^8 29 \cdot 53^2 83 \cdot 103$

k	$\lambda(3)$ on $S_{(6,k)}(\Gamma_2)$
8	-27000
10	-6120
11	-106488
12	$X^2 + 335664X - 14832719455680$
13	-8505000
15	$X^2 + 228022128X + 8319716602228800$
17	$X^3 + 1086146712X^2 - 341960280255362880X - 188775313801934579676864000$

Table 2: Eigenvalues of the Hecke operator $T(3)$ on $M_{(6,k)}(\Gamma_2)$ for some values of k .

This shows for $p = 2, 3$ that the eigenvalues of $T(p)$ on $S_{(6,12)}(\Gamma_2)$ and $S_{(6,15)}(\Gamma_2)$ are elements of the same quadratic number field $\mathbb{Q}(\sqrt{601})$ and $\mathbb{Q}(\sqrt{29 \cdot 83 \cdot 103})$ respectively. We also verified that the characteristic polynomials of $T(2)$ and $T(3)$ on $S_{(6,17)}(\Gamma_2)$ define the same number field.

3 Proof of the main theorem

We will only give a sketch of the full proof since it involves many elementary computations. Let U denote the Hodge bundle corresponding to the factor of automorphy j . The forms F_i defined above are sections of $\text{Sym}^6(U) \otimes L^{\otimes k_i}$, where L denotes the line bundle $\det(U)$. In order to show that the forms F_i are independent over M_*^0 , we have to show that $\chi_{140} := F_{11} \wedge F_{13} \wedge \cdots \wedge F_{23} \in L^{\otimes 140}$ is non-vanishing. The form χ_{140} is an element of $M_{140}(\Gamma_2)$ and we can compute the Fourier coefficients of this form. We will therefore prove the main theorem by computing a non-zero Fourier coefficient of χ_{140} . Write $\chi_{140} = \sum_n c(n)q^n$ and $F_i = \sum_n a_i(n)q^n$. The following formula holds for the Fourier coefficients $c(n)$:

$$c(n) = \sum_{\sum_i n_i = n} \det(a_{11}(n_1), a_{13}(n_2), \dots, a_{23}(n_7)). \quad (8)$$

As mentioned above, this can be hard to compute if $\sigma(n)$ is large. In order to find a non-zero Fourier coefficient $c(n)$ of χ_{140} , we need at least $\sigma(n) \geq 14$ since the forms F_i are cusp forms. Indeed, under the Siegel operator Φ , the forms F_i map to elliptic modular forms of odd weight. These forms all vanish.

Using algorithms provided by Resnikoff and Saldaña [16], we calculated Fourier coefficients of the classical Siegel modular forms $\varphi_4, \varphi_6, \chi_{10}$ and χ_{12} . The Fourier coefficients of χ_5 can be computed using $\chi_5^2 = \chi_{10}$.¹ We then were able to compute Fourier coefficients of the forms F_i .

In order to find Fourier coefficients of F_{11} and F_{13} , we first had to compute Fourier coefficients of E_6 and Θ_8 . The modular form $\varphi_4 E_6$ is an element of $M_{(6,10)}(\Gamma_2)$ and this space has dimension 2. Using a RC-polynomial of weight $(6, 0)$ and the modular forms φ_4 and φ_6 , we can find a modular form F_{10} in the space $M_{(6,10)}(\Gamma_2)$ (cf. Ibukiyama [13]) and this form is not an eigenform. Hence, we can find Fourier coefficients of F_{10} and $T(2)F_{10}$. For some $\alpha, \beta \in \mathbb{Q}$, we must have $\alpha F_{10} + \beta T(2)F_{10} = \varphi_4 E_6$. The form E_6 is an eigenform and by a theorem due to Arakawa [2] we know the corresponding eigenvalue of $T(2)$ (cf. Table 1). This gives a relation

¹While computing the Fourier coefficients of χ_5 we encountered an error in Table IV of [16]. The coefficients at calculation classes $(3, 3, 3)$ and $(2, 6, 0)$ should have opposite sign.

i	11	13	15	17	19	21	23
n	(1,1,0)	(1,1,1)	(2,1,0)	(2,1,0)	(2,1,1)	(2,1,1)	(2,2,1)
$a_i(n)$	0	0	0	0	1	-5	3
	-20	-2	312	0	14	-10	-37
	0	-5	0	0	36	-6	-50
	0	0	180	-300	24	-24	0
	0	5	0	0	0	-30	50
	20	2	-102	354	0	-12	37
	0	0	0	0	0	0	-3

Table 3: A few Fourier coefficients of the forms F_i . The Fourier coefficients are written as column vectors. The determinant of the above matrix occurs in the sum (8) for $c(12,8,4)$. This determinant equals $2^{14}3^55^311$.

for the Fourier coefficients of E_6 and using this relation, we were able to find α and β . This also allowed us to compute the Fourier coefficients of E_6 .

A similar method can be used to compute Fourier coefficients of Θ_8 . Again using a RC-operator of weight (6,2) and the modular forms φ_4 and φ_6 , we can find a modular form F_{12} in $S_{(6,12)}(\Gamma_2)$ (cf. Ibukiyama [13]). This form is again not an eigenform. The form $\varphi_4\Theta_8 \in S_{(6,12)}(\Gamma_2)$ must be a linear combination $\alpha F_{12} + \beta T(2)F_{12}$ and since Ibukiyama has computed a few Fourier coefficients of Θ_8 (cf. [10]), we were able to determine α and β .

We then used formulas (6) and (7) to compute Fourier coefficients of the forms F_i . A few examples are given in Table 3.

We wrote a script to compute the Fourier coefficient of χ_{140} at $n = (12,8,4)$ and found that $c(12,8,4) = -2^{18}3^75^2 \neq 0$. We checked our computations by also computing the Fourier coefficient at $(12,8,-4)$ and found that $c(12,8,-4) = c(12,8,4)$ which is in line with equation (1). This proves our main result.

References

- [1] A. N. Andrianov. Euler products corresponding to Siegel modular forms of genus 2. *Russian Math. Surveys*, 29(3):45–116, 1974.
- [2] T. Arakawa. Vector valued Siegel’s modular forms of degree two and the associated Andrianov L-functions. *Manuscripta math.*, 44:155–185, 1983.
- [3] Y. Choie and W. Eholzer. Rankin-Cohen operators for Jacobi and Siegel forms. *Journal of Number Theory*, 68:160–177, 1998.
- [4] H. Cohen. Sums involving the values at negative integers of L-functions of quadratic characters. *Mathematische Annalen*, 217:271–285, 1975.
- [5] W. Eholzer and T. Ibukiyama. Rankin-Cohen type differential operators for Siegel modular forms. *International Journal of Mathematics*, 9(4):442–463, 1998.

- [6] C. Faber and G. van der Geer. Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes, I. *Comptes Rendus, Mathématique*, 338(5):381–384, 2004.
- [7] C. Faber and G. van der Geer. Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes, II. *Comptes Rendus, Mathématique*, 338(6):467–470, 2004.
- [8] E. Freitag. *Siegelsche Modulfunctionen*, volume 254 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin Heidelberg New York, 1983.
- [9] G. van der Geer. Siegel modular forms. In *The 1-2-3 of Modular Forms*, Universitext. Springer-Verlag, Berlin Heidelberg, 2008.
- [10] T. Ibukiyama. An answer to van der Geer’s question: Construction of the vector valued Siegel cusp form of $\mathrm{Sp}(2, \mathbb{Z})$ of weight $\det 8 \mathrm{Sym} 6$. Letter to G. van der Geer.
- [11] T. Ibukiyama. On Differential operators on Automorphic Forms and Invariant Pluri-harmonic Polynomials. *Commentarii Mathematici Universitatis Sancti Pauli*, 48(1):103–117, 1999.
- [12] T. Ibukiyama. Vector valued Siegel modular forms of symmetric tensor representation of degree two. Unpublished manuscript, 2000.
- [13] T. Ibukiyama. Vector valued Siegel modular forms of $\det k \mathrm{sym}(4)$ and $\det k \mathrm{sym}(6)$: A short announcement. Unpublished manuscript, 2001.
- [14] J. Igusa. On Siegel Modular Forms of Genus Two. *American Journal of Mathematics*, 84(1):175–200, 1962.
- [15] M. Miyawaki. Explicit construction of Rankin–Cohen-type differential operators for vector-valued Siegel modular forms. *Kyushu J. Math.*, 55:369–385, 2001.
- [16] H. L. Resnikoff and R. L. Saldaña. Some properties of Fourier coefficients of Eisenstein series of degree 2. *J. Reine Angew. Math.*, 265:90–109, 1974.
- [17] T. Satoh. On Certain Vector Valued Siegel Modular forms of Degree Two. *Mathematische Annalen*, 274:335–352, 1986.
- [18] R. Tsushima. An Explicit Dimension Formula for the Spaces of Generalized Automorphic Forms with Respect to $\mathrm{Sp}(2, \mathbb{Z})$. *Proc. Japan Acad.*, 59(4):139–142, 1983.