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Dimension formulas for spaces of vector-valued Siegel cusp forms of degree two

Satoshi Wakatsuki *,[1]

Faculty of Mathematics and Physics, Institute of Science and Engineering, Kanazawa University, Kakumamachi, Kanazawa, Ishikawa, 920-1192, Japan

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ABSTRACT

We give a general arithmetic dimension formula for spaces of vector-valued Siegel cusp forms of degree two. Then, using this formula, we derive explicit dimension formulas for arithmetic subgroups of any level for each $\mathbb{Q}$-form of $Sp(2;\mathbb{R})$. Tsushima has already given the dimension formulas for some congruence subgroups of the split $\mathbb{Q}$-form in Tsushima (1983, 1997) [32,33]. We obtain an alternative proof for his results by using the Selberg trace formula and the theory of prehomogeneous vector spaces. As for the non-split $\mathbb{Q}$-forms, our results are new. We generalize the results and proofs given in Arakawa (1981) [1], Christian (1969, 1975, 1977) [5,6], Hashimoto (1983, 1984) [12,13], Morita (1974) [25] for the scalar-valued case to the vector-valued case using the Selberg trace formula.

1. Introduction

In this paper, we give explicit dimension formulas for spaces of vector-valued Siegel cusp forms of degree two with respect to the full modular groups $\Gamma(1)$ and $\Gamma^*(1)$ and the principal congruence subgroups $\Gamma(N)$ and $\Gamma^*(N)$ of all $\mathbb{Q}$-forms of $Sp(2;\mathbb{R})$ and the congruence subgroup $\Gamma_0(p)$ of the split $\mathbb{Q}$-form. The dimension formulas for the scalar-valued case are already known. Tsushima has already given the dimension formulas for the vector-valued case for such congruence subgroups of the split $\mathbb{Q}$-form by using the Riemann–Roch theorem in [32,33]. We obtain an alternative proof for his results by using the Selberg trace formula and the theory of prehomogeneous vector spaces. As for the non-split $\mathbb{Q}$-forms, our results are new.

* Fax: +81 76 264 5738.
E-mail address: wakatuki@kenroku.kanazawa-u.ac.jp.
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We generalize the results and proofs obtained by Christian [5,6], Morita [25], Arakawa [1], and Hashimoto [12,13] for the scalar-valued case to the vector-valued case using the Selberg trace formula. In particular, we obtain a general arithmetic dimension formula (Theorem 3.1), which is a generalization of [12, Theorem 5-1]. There are two problems associated with the generalization of the proofs of these theorems. First, we must prove the convergence of some infinite series, in order to transform the Godement formula into the infinite sum of the orbital integrals with dumping factors, e.g., we have to interchange the integral and the infinite sum. Next, we must explicitly calculate the orbital integrals with dumping factors. The explicit forms for the semisimple orbital integrals have been obtained by Langlands [24]. He used the limit formula for the semisimple orbital integrals. We also use the limit formula for the unipotent orbital integrals (cf. [26,3]). Furthermore, we have to carry out some calculations similar to those in [25] and [12], since we cannot directly apply the limit formula to the unipotent or quasi-unipotent orbital integrals with dumping factors.

We give a formula (Theorem 5.7) for unipotent contributions, which are concerned with zeta functions associated to symmetric matrices, by using the theory of prehomogeneous vector spaces. Then, we obtain an alternative proof using this formula for such unipotent contributions in dimension formulas. First, Morita has explicitly calculated the unipotent contributions. After that, Shintani has simplified the proof by using the theory of prehomogeneous vector spaces and obtained a formula that expresses such unipotent contributions for general degree by special values of zeta functions associated to symmetric matrices. Special values of the zeta functions have been determined by Shintani [28], Sato [27] (degree two, split case), Arakawa [1] (degree two, non-split case), and Ibukiyama and Saito [20] (general degree, split case). We generalize Shintani's formula to the vector-valued case for degree two. In order to generalize his formula, we have to prove the convergence of the zeta integrals of prehomogeneous vector spaces and explicitly calculate the integral of a certain function, which is related to the Fourier transform of the trace of irreducible rational representations. The integral is well known in the scalar-valued case, but it is nontrivial in the vector-valued case. We can calculate the integral by using the Fourier transform which was given by Godement [8].

Note that our dimension formulas are concrete. We can get concrete numerical values of dimensions by using our dimension formulas. We give some numerical tables of dimensions in Sections 6 and 7.

Our motivations are as follows. First, we use our main result for a concrete study of the Jacquet–Langlands–Ihara correspondence for $Sp(2;\mathbb{R})$. Actually Hashimoto and Ibukiyama obtained good global dimensional relations between automorphic forms of $Sp(2;\mathbb{R})$ and its compact twist in 1984 by using dimension formulas (cf. [18,15]). We generalize the dimension formula [18, Theorem 4] for the paramodular groups $K(p)$ to the vector-valued case by using our formula. Furthermore, we obtain the correspondence for the vector-valued case by comparing the dimensions. Second, we investigate dimensions of vector-valued Siegel modular forms of low weights and the surjectivity of the Witt operator by using the dimension formula for $\Gamma_e(1)$ given in [21], where $\Gamma_e(1)$ is the index two normal subgroup of $Sp(2;\mathbb{Z})$. We have explicitly calculated the dimension formula for $\Gamma_e(1)$ by using our formula (for the scalar-valued case, the dimension formula has been given by Igusa [22]). Third, we study the Shimura correspondence between Siegel cusp forms. Ibukiyama has given a conjecture for the Shimura correspondence between vector-valued Siegel cusp forms of degree two of integral weights and half-integral weights (cf. [19]); in this case, it is essential to consider vector-valued forms. Although we do not consider these topics in this paper, we study the traces of the trivial actions of the Hecke operators as the first step, which are the dimensions of spaces.

This paper is organized as follows. In Section 2, we review arithmetic groups, Siegel cusp forms, and the conjugacy classes of $Sp(2;\mathbb{R})$. In Section 3, we give the general arithmetic dimension formula (Theorem 3.1), which is one of our main results. We also give another formula (Theorem 3.2) that is a modified form of Theorem 3.1. We need it to derive the dimension formulas for the congruence subgroups $\Gamma(N)$, $\Gamma^*(N)$ ($N \geqslant 2$), and $\Gamma_0(p)$. In Section 4, we prove Theorem 3.1. In Section 5, we give the formula (Theorem 5.7) for unipotent contributions, which are concerned with zeta functions associated to symmetric matrices. In Sections 6 and 7, we give explicit dimension formulas, which are our main results, and some numerical tables of dimensions. In Appendix A, we review non-cusp forms. In Appendix B, we review a formula for elliptic contributions, which is required for the calculation of explicit dimension formulas.

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2. Preliminaries

2.1. Notation

Let $\mathbb{Z}$ denote the ring of rational integer, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the field of rational, real, and complex numbers, respectively, and i denote the complex number $\sqrt{-1}$. For $z = x + iy \in \mathbb{C}$, $|z|$ is the absolute value of z, given by $\sqrt{x^2 + y^2}$, and $\bar{z}$ is the complex conjugate of z, given by $x - iy$. For a ring R, we denote the ring of matrices of degree n over R by $M(n; R)$. Let $GL(n; R)$ denote the group of invertible matrices in $M(n; R)$, and $SL(n; R)$ denote the subgroup of matrices with determinant one in $GL(n; R)$. Further, we denote the unit matrix of $M(n; R)$ by I_n. For a matrix x, ${}^t x$ is the transpose of x. Let $SM(n; R)$ denote the totality of symmetric matrices in $M(n; R)$. If G is an algebraic group over $\mathbb{Q}$, let $G(\mathbb{Z})$, $G(\mathbb{Q})$, $G(\mathbb{R})$, and $G(\mathbb{C})$ denote the group of $\mathbb{Z}$-valued, $\mathbb{Q}$-valued, $\mathbb{R}$-valued, and $\mathbb{C}$-valued points of G, respectively. For a subgroup C of $GL(n; \mathbb{R})$, we put $\overline{C} = \{\pm I_n\} \cdot C/\{\pm I_n\}$. If H is a subgroup of a group G, let $\{g\}_H$ denote the H-conjugacy class represented by $g \in G$. Let $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ denote the diagonal matrix whose entries are given by $a_1, a_2, \ldots, a_n$. If X is a positive (resp. negative) definite symmetric matrix over $\mathbb{R}$, then we write $X > 0$ (resp. $X < 0$). We denote the gamma function by $\Gamma(s)$.

2.2. $\mathbb{Q}$-forms of $Sp(2; \mathbb{R})$

Let $Sp(2; \mathbb{R})$ be the real symplectic group of degree two, i.e.,

$$Sp(2;\mathbb{R}) = \left\{ g \in GL(4;\mathbb{R});\ g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}.$$

Let $\mathbf{B}$ be an indefinite quaternion algebra over $\mathbb{Q}$ ($\mathbf{B} \otimes_{\mathbb{Q}} \mathbb{R} \cong M(2; \mathbb{R})$), $a \mapsto a^\iota$ ($a \in \mathbf{B}$) the canonical involution of $\mathbf{B}$. We set

$$G(\mathbb{Q}) = U(2;\mathbf{B}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2;\mathbf{B});\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^\iota & c^\iota \\ b^\iota & d^\iota \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

We know that the isomorphism $\phi : G(\mathbb{R}) \to Sp(2; \mathbb{R})$ is given by

$$\phi(g) = \begin{pmatrix} a_1 & a_2 & b_2 & -b_1 \\ a_3 & a_4 & b_4 & -b_3 \\ c_3 & c_4 & d_4 & -d_3 \\ -c_1 & -c_2 & -d_2 & d_1 \end{pmatrix}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\mathbb{R}),$$

where $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$, $D = \left(\begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \end{smallmatrix}\right) \in \mathbf{B} \otimes_{\mathbb{Q}} \mathbb{R}$. By using the isomorphism ϕ, we identify $G(\mathbb{R})$ with $Sp(2; \mathbb{R})$. The realization $Sp(2; \mathbb{R})$ is used when the matrices are written down. For each subgroup H in $G(\mathbb{R})$, we identify H with $\phi(H)$. The $\mathbb{Q}$-rank of $G(\mathbb{Q})$ is one or two, depending on whether $\mathbf{B}$ is a division algebra or not. If $\mathbf{B} = M(2; \mathbb{Q})$, then $\phi(G(\mathbb{Q})) = Sp(2; \mathbb{Q})$. It is known that for each $\mathbb{Q}$-form of $Sp(2; \mathbb{R})$, there exists an indefinite quaternion algebra $\mathbf{B}$ such that the $\mathbb{Q}$-form is isomorphic to $U(2; \mathbf{B})$.

2.3. Arithmetic subgroups

Consider an indefinite quaternion algebra $\mathbf{B}$. Let Γ be an arithmetic subgroup of $G(\mathbb{Q})$, i.e., $\Gamma\ (\subset G(\mathbb{Q}))$ is commensurable with $G(\mathbb{Z})$. It is known that $G(\mathbb{Z})$ is commensurable with $U(2;\mathbf{B})_L = \{g \in U(2;\mathbf{B});\ L\cdot g = L\}$ for any lattice L in $\mathbf{B}^2$. Let $\mathfrak{O}$ be a maximal order of $\mathbf{B}$. We put $G(\mathfrak{O}) = U(2;\mathfrak{O})$. If $L = \mathfrak{O}^2$, we have $G(\mathfrak{O}) = U(2;\mathbf{B})_L$. We can fix a maximal order $\mathfrak{O}$ without loss of generality up to isomorphisms, because $\mathbf{B}$ only has a maximal order up to inner automorphisms. When $\mathbf{B} = M(2;\mathbb{Q})$, we fix $\mathfrak{O} = M(2;\mathbb{Z})$. Then, we have $\phi(G(\mathfrak{O})) = Sp(2;\mathbb{Z})$.

Next, we make an assumption for the arithmetic subgroup Γ. Let $P_0(\mathbb{Q}) = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in G(\mathbb{Q})\}$, $M_0(\mathbb{Q}) = \{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) \in G(\mathbb{Q})\}$, and $N_0(\mathbb{Q}) = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \in G(\mathbb{Q})\}$. We have $P_0(\mathbb{Q}) = M_0(\mathbb{Q})\cdot N_0(\mathbb{Q})$. Let $G(\mathbb{Q}) = \bigcup_{m=1}^{\nu_0} \Gamma h_m P_0(\mathbb{Q})$ (disjoint union) ($h_m \in G(\mathbb{Q})$, $h_1 = I_2$). If $\Gamma = G(\mathfrak{O})$, then $\nu_0 = 1$. We need the following assumption to explicitly calculate the unipotent contributions of Γ (cf. (e) Unipotent in Section 3).

Assumption 2.1. There exist $h_1, h_2, \ldots, h_{\nu_0}$ such that the equality $P_0(\mathbb{Q}) \cap (h_m^{-1}\Gamma h_m) = (M_0(\mathbb{Q}) \cap (h_m^{-1}\Gamma h_m))\cdot(N_0(\mathbb{Q}) \cap (h_m^{-1}\Gamma h_m))$ holds for each m ($1 \leqslant m \leqslant \nu_0$).

2.4. Siegel cusp forms

Let $\rho_{k,j} : GL(2;\mathbb{C}) \to GL(j+1;\mathbb{C})$ be the irreducible rational representation of the signature $(j+k, k)$ ($j,k \in \mathbb{Z}_{\geqslant 0}$), i.e., $\rho_{k,j} = \det^k \otimes Sym_j$, where Sym_j is the symmetric j-tensor representation of $GL(2;\mathbb{C})$. Let $\mathfrak{H}_2$ be the Siegel upper half-space of degree two, i.e., $\mathfrak{H}_2 = \{Z \in M(2;\mathbb{C});\ {}^tZ = Z,\ \mathrm{Im}(Z)$ is positive definite$\}$. The group $Sp(2;\mathbb{R})$ acts on $\mathfrak{H}_2$ as $g\cdot Z := (AZ+B)(CZ+D)^{-1}$ for $Z \in \mathfrak{H}_2$, $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2;\mathbb{R})$. Let χ be a one-dimensional unitary representation of Γ such that $[\Gamma : \ker(\chi)] < \infty$. Let $S_{k,j}(\Gamma,\chi)$ be the space of Siegel cusp forms of type $(\rho_{k,j},\chi,\Gamma)$, i.e., the space of holomorphic functions $f : \mathfrak{H}_2 \to \mathbb{C}^{j+1}$ satisfying (i) $f(\gamma\cdot Z) = \rho_{k,j}(CZ+D)f(Z)\chi(\gamma)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, $Z \in \mathfrak{H}_2$, and (ii) $|\rho_{k,j}(\mathrm{Im}(Z)^{1/2})f(Z)|_{\mathbb{C}^{j+1}}$ is bounded on $\mathfrak{H}_2$, where $\mathrm{Im}(Z)^{1/2} \in SM(2;\mathbb{R})$ and $(\mathrm{Im}(Z)^{1/2})^2 = \mathrm{Im}(Z)$. We call $\rho_{k,j}$ the weight of the Siegel cusp forms of $S_{k,j}(\Gamma,\chi)$. If χ is trivial, $S_{k,j}(\Gamma,\chi)$ is simply denoted by $S_{k,j}(\Gamma)$. It is known that $\dim_{\mathbb{C}} S_{k,j}(\Gamma)$ ($k \geqslant 3$) is equal to the multiplicity of the holomorphic discrete series representation of the Harish-Chandra parameter $(j+k-1, k-2)$ in the discrete spectrum of $L^2(\Gamma\backslash G(\mathbb{R}))$ (cf. Wallach [35]). Our aim is to obtain explicit formulas for $\dim_{\mathbb{C}} S_{k,j}(\Gamma,\chi)$. Note that $\dim_{\mathbb{C}} S_{k,j}(\Gamma,\chi) = 0$ if $-I_4 \in \Gamma$, $\chi(-I_4) = 1$, and j is odd.

2.5. Conjugacy classes

The representative elements of $G(\mathbb{R})$-conjugacy classes for $G(\mathbb{R})$ have been described concretely in [25] and [12]. Here, we give a list of these elements. Let

$$\alpha(\theta_1,\theta_2) = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}, \qquad k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

$$\beta(a,b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \end{pmatrix}, \qquad \delta(u,u') = \begin{pmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \varpi_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\varpi_2 = \begin{pmatrix} \Upsilon & 0 \\ 0 & {}^t\Upsilon^{-1} \end{pmatrix}, \quad \text{where } \Upsilon\ (\in GL(2;\mathbb{R})) \text{ satisfies } \Upsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} {}^t\Upsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $C(\gamma; G(\mathbb{R}))$ denote the centralizer of γ in $G(\mathbb{R})$. The representative elements are as follows:

(a) Central. $\gamma = \pm I_4$, $C(\gamma; G(\mathbb{R})) = G(\mathbb{R})$.
(b) Elliptic.
(b-1) (regular) $\gamma = \alpha(\mu, \nu)$ $(k(\mu)^2, k(\nu)^2, k(\mu)k(\nu) \neq I_2,\ k(\mu) \neq k(\nu))$, $C(\gamma; G(\mathbb{R})) \cong SO(2;\mathbb{R}) \times SO(2;\mathbb{R})$.
(b-2) $\gamma = \alpha(\mu, \mu)$ $(k(\mu)^2 \neq I_2)$, $C(\gamma; G(\mathbb{R})) \cong U(2)$.
(b-3) $\gamma = \alpha(\mu, -\mu)$ $(k(\mu)^2 \neq I_2)$, $C(\gamma; G(\mathbb{R})) \cong U(1,1)$.
(b-4) $\gamma = \alpha(\mu, 0)$ $(k(\mu)^2 \neq I_2)$, $C(\gamma; G(\mathbb{R})) \cong SO(2;\mathbb{R}) \times SL(2;\mathbb{R})$.
(b-5) $\gamma = \alpha(0, \pi)$, $C(\gamma; G(\mathbb{R})) \cong SL(2;\mathbb{R}) \times SL(2;\mathbb{R})$.
(c) Hyperbolic.
(c-1) (regular) $\gamma = \beta(a,b)$ $(a^2, b^2, ab \neq 1,\ a \neq b)$, $C(\gamma; G(\mathbb{R})) \cong \mathbb{R}^\times \times \mathbb{R}^\times$.
(c-2) $\gamma = \beta(a,a)$ $(a^2 \neq 1)$, $C(\gamma; G(\mathbb{R})) \cong GL(2;\mathbb{R})$.
(c-3) $\gamma = \pm\beta(a,1)$ $(a^2 \neq 1)$, $C(\gamma; G(\mathbb{R})) \cong \mathbb{R}^\times \times SL(2;\mathbb{R})$.
(d) Elliptic–hyperbolic.
(d-1) (regular) $\gamma = \alpha(\mu,0)\beta(1,a)$ $(k(\mu)^2 \neq I_2,\ a^2 \neq 1)$, $C(\gamma; G(\mathbb{R})) \cong \mathbb{R}^\times \times SO(2;\mathbb{R})$.
(d-2) (regular) $\gamma = \varpi_1\varpi_2\alpha(\mu,-\mu)\varpi_2^{-1}\varpi_1^{-1}\beta(a,a)$ $(k(\mu)^2 \neq I_2,\ a^2 \neq 1)$, $C(\gamma; G(\mathbb{R})) \cong \mathbb{R}_+^\times \times SO(2;\mathbb{R})$.
(e) Unipotent.
(e-1) (principal) $\gamma = \pm\delta(u,0)\varpi_1\varpi_2\delta(1,-1)\varpi_2^{-1}\varpi_1^{-1}$ $(u = \pm 1)$, $C(\gamma; G(\mathbb{R})) \cong \{\pm 1\} \times \mathbb{R}^2$.
(e-2) (subregular) $\gamma = \pm\delta(u,u)$ $(u = \pm 1)$, $C(\gamma; G(\mathbb{R})) \cong O(2;\mathbb{R}) \ltimes SM(2;\mathbb{R})$.
(e-3) (subregular) $\gamma = \pm\delta(1,-1)$, $C(\gamma; G(\mathbb{R})) \cong O(1,1;\mathbb{R}) \ltimes SM(2;\mathbb{R})$.
(e-4) (minimal) $\gamma = \pm\delta(0,\pm 1)$, $C(\gamma; G(\mathbb{R})) \cong \{\pm 1\} \times (SL(2;\mathbb{R}) \ltimes (\mathbb{R} \ltimes \mathbb{R}^2))$.
(f) Quasi-unipotent.
(f-1) $\gamma = \pm\alpha(0,\pi)\delta(0,u)$ $(u = \pm 1)$, $C(\gamma; G(\mathbb{R})) \cong \{\pm 1\} \times \mathbb{R} \times SL(2;\mathbb{R})$.
(f-2) $\gamma = \pm\alpha(0,\pi)\delta(1,u)$ $(u = \pm 1)$, $C(\gamma; G(\mathbb{R})) \cong \{\pm 1\} \times \{\pm 1\} \times \mathbb{R}^2$.
(f-3) $\gamma = \varpi_1\varpi_2\alpha(\mu,-\mu)\varpi_2^{-1}\varpi_1^{-1}\delta(u,u)$ $(k(\mu)^2 \neq I_2,\ u = \pm 1)$, $C(\gamma; G(\mathbb{R})) \cong \mathbb{R} \times SO(2;\mathbb{R})$.
(f-4) $\gamma = \pm\alpha(\mu,0)\delta(0,u)$ $(k(\mu)^2 \neq I_2,\ u = \pm 1)$, $C(\gamma; G(\mathbb{R})) \cong \{\pm 1\} \times \mathbb{R} \times SO(2;\mathbb{R})$.
(g) Hyperbolic-unipotent.
(g-1) $\gamma = \pm\beta(a,1)\delta(0,u)$ $(a^2 \neq 1,\ u = \pm 1)$, $C(\gamma; G(\mathbb{R})) \cong \{\pm 1\} \times \mathbb{R}^\times \times \mathbb{R}$.
(g-2) $\gamma = \beta(a,a^{-1})\varpi_2\delta(u,-u)\varpi_2^{-1}$ $(a^2 \neq 1,\ u = \pm 1)$, $C(\gamma; G(\mathbb{R})) \cong \mathbb{R}^\times \times \mathbb{R}$.

3. General arithmetic formula

In Section 3.1, we give the general arithmetic dimension formula (Theorem 3.1) for $\dim_{\mathbb{C}} S_{k,j}(\Gamma)$. In Section 3.2, we give a formula (Theorem 3.2) that is a modified form of Theorem 3.1. We use Theorem 3.2 to derive explicit dimension formulas for $\Gamma^*(N)$, $\Gamma(N)$ $(N \geqslant 2)$, and $\Gamma_0(p)$ in Sections 6 and 7.

3.1. *General arithmetic formula for* $\dim_{\mathbb{C}} S_{k,j}(\Gamma)$

We explain some notations used in Theorem 3.1. Let $\{\gamma\}_\Gamma$ denote the Γ-conjugacy class represented by γ. Let $C(\gamma;\Gamma) = C(\gamma; G(\mathbb{R})) \cap \Gamma$. Let γ be an element of Γ, which is $G(\mathbb{R})$-conjugate to one of (a) central, (b) elliptic, (f) quasi-unipotent, and (e) unipotent elements, except for the principal unipotent elements and the elements $G(\mathbb{Q})$-conjugate to $\pm\left(\begin{smallmatrix} I_2 & S \\ 0 & I_2 \end{smallmatrix}\right)$, $\det(S) < 0$, $-\det(S) \notin (\mathbb{Q}^\times)^2$. The contributions of such γs appear in the dimension formula. For each γ, we will later define a closed connected normal subgroup $C_0(\gamma; G(\mathbb{R}))$ of $C(\gamma; G(\mathbb{R}))$ and a certain integral $J_0(\gamma; s)$ with a parameter s. We set $C_0(\gamma;\Gamma) = C_0(\gamma; G(\mathbb{R})) \cap \Gamma$ and $J_0(\gamma) = J_0(\gamma; 0)$. We will later fix a Haar measure on $C_0(\gamma; G(\mathbb{R}))$ for each γ. The subgroup $C_0(\gamma; G(\mathbb{R}))$ has the following three properties: (i) $C_0(\gamma; G(\mathbb{R}))$ does not have compact semi-direct factors, (ii) $\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma; G(\mathbb{R}))) < +\infty$, and (iii) $[C(\gamma;\Gamma) : C_0(\gamma;\Gamma)] < +\infty$. For each γ, we set

$$[\gamma]_\Gamma = \bigl\{\gamma' \in \Gamma;\ \gamma_s = \gamma'_s,\ C_0\bigl(\gamma'; G(\mathbb{R})\bigr) = C_0\bigl(\gamma; G(\mathbb{R})\bigr),\ \text{and } C\bigl(\gamma'; G(\mathbb{R})\bigr) \cong C\bigl(\gamma; G(\mathbb{R})\bigr)\bigr\},$$

where γ_s (resp. γ'_s) is the semisimple factor of the Jordan decomposition of γ (resp. γ'). We call the set $[\gamma]_\Gamma$ the family represented by γ. Note that $C_0(\gamma';\Gamma) = C_0(\gamma;\Gamma)$ for any $\gamma' \in [\gamma]_\Gamma$. Let $\sim$

denote the equivalence relation defined by Γ-conjugations for each family $[\gamma]_\Gamma$ of (e). Let $[\gamma]_\Gamma/\sim$ be a complete system of representative elements of the equivalence classes in $[\gamma]_\Gamma$. We set $c_{k,j} = 2^{-6}\pi^{-3}(k-2)(j+k-1)(j+2k-3)$. Let $Z(\Gamma)$ be the center of Γ and $\sharp(Z(\Gamma))$ the order of $Z(\Gamma)$. For a subgroup C of $Sp(2;\mathbb{R})$, we set $\bar{C} = \{\pm I_4\}\cdot C/\{\pm I_4\}$.

Theorem 3.1. *If $k \geqslant 5$ and Γ satisfies Assumption* 2.1, *then we have*

$$\begin{aligned}
&\dim_{\mathbb{C}} S_{k,j}(\Gamma) \\
&\quad = \frac{c_{k,j}}{\sharp(Z(\Gamma))}\sum_{\{\gamma\}_\Gamma}\frac{\mathrm{vol}(\bar{C}_0(\gamma;\Gamma)\backslash\bar{C}_0(\gamma;G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]}J_0(\gamma) \\
&\qquad + \frac{c_{k,j}}{\sharp(Z(\Gamma))}\sum_{[\gamma]_\Gamma}\frac{\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]}\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma}J_0(\gamma';s) \\
&\qquad + \frac{c_{k,j}}{\sharp(Z(\Gamma))}\sum_{[\gamma]_\Gamma}\mathrm{vol}\big(C_0(\gamma;\Gamma)\backslash C_0\big(\gamma;G(\mathbb{R})\big)\big)\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma/\sim}\frac{J_0(\gamma';s)}{[\bar{C}(\gamma';\Gamma):\bar{C}_0(\gamma';\Gamma)]},
\end{aligned}$$

where in the first term, $\{\gamma\}_\Gamma$ runs over the set of Γ-conjugacy classes of (a) *central and* (b) *elliptic elements in Γ; in the second term, $[\gamma]_\Gamma$ runs over a complete system of representative elements of Γ-conjugacy classes of families of* (f) *quasi-unipotent elements*; *and in the third term, $[\gamma]_\Gamma$ runs over a complete system of representative elements of Γ-conjugacy classes of families of* (e) *unipotent elements, except for the principal unipotent elements and the elements $G(\mathbb{Q})$-conjugate to $\pm\left(\begin{smallmatrix} I_2 & S \\ 0 & I_2\end{smallmatrix}\right)$, $\det(S)<0$, $-\det(S)\notin(\mathbb{Q}^\times)^2$.*

Next, we provide the definitions and evaluations for $C_0(\gamma;G(\mathbb{R}))$, $J_0(\gamma;s)$, and the limits in Theorem 3.1. We set

$$H_\gamma^{k,j}(Z) = \mathrm{tr}\left[\rho_{k,j}(CZ+D)^{-1}\rho_{k,j}\left(\frac{\gamma\cdot Z-\bar{Z}}{2i}\right)^{-1}\rho_{k,j}(Y)\right],$$

$$Z = \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2\end{pmatrix}, \qquad X = \begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2\end{pmatrix}, \qquad Y = \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2\end{pmatrix},$$

$$dZ = \det(Y)^{-3}\,dX\,dY, \qquad dX = dx_1\,dx_{12}\,dx_2, \qquad dY = dy_1\,dy_{12}\,dy_2,$$

for $Z = X+iY\in\mathfrak{H}_2$, $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in G(\mathbb{R})$, where dx_* and dy_* are the Lebesgue measures on $\mathbb{R}$. The function $H_g^{k,j}(iI_2)$ ($g\in G(\mathbb{R})$) is called the spherical trace function (cf. [36, Chapter 6]). We define the integral $J_0(\gamma;s)$ as

$$J_0(\gamma;s) = \int_{C_0(\gamma;G(\mathbb{R}))\backslash\mathfrak{H}_2} H_\gamma^{k,j}(\hat{Z})\,\nu(\gamma;\hat{Z},s)\,d\hat{Z},$$

where $d\hat{Z}$ is an invariant measure on $C_0(\gamma;G(\mathbb{R}))\backslash\mathfrak{H}_2$ induced from dZ and a Haar measure on $C_0(\gamma;G(\mathbb{R}))$. We will later define the function $\nu(\gamma;Z,s)$ ($Z\in\mathfrak{H}_2$, $s\in\mathbb{R}_{>0}$), which is invariant under the actions of $C_0(\gamma;G(\mathbb{R}))$ on $Z\in\mathfrak{H}_2$, for each family $[\gamma]_\Gamma$. The function $\nu(\gamma;Z,s)$ is called the dumping factor. For all γ of (a) central or (b) elliptic elements, we set $\nu(\gamma;Z,s)=1$. Hence, in these cases, $J_0(\gamma;s)$ is a constant with respect to s, which we denote simply by $J_0(\gamma)$. We note that $H_{-\gamma}^{k,j}(Z) = (-1)^j H_\gamma^{k,j}(Z)$ and $J_0(-\gamma;s) = (-1)^j J_0(\gamma;s)$.

(a) Central. $C_0(\pm\alpha(0,0); G(\mathbb{R})) = G(\mathbb{R})$.

$$J_0\big(\alpha(0,0)\big) = c_{k,j}^{-1} \times 2^{-6}\pi^{-3} \times (j+1)(k-2)(j+k-1)(j+2k-3).$$

(b) Elliptic. Let γ be an elliptic element in Γ. There exists an element $g \in G(\mathbb{R})$ such that $g^{-1}\gamma g = \alpha(\mu,\nu)$. We give an explicit form of $J_0(\alpha(\mu,\nu))$. If we change $\gamma \to \gamma^{-1}$ ($\mu \to -\mu$, $\nu \to -\nu$), then our descriptions will be the same as those in [24] and [12]. Let $-\pi < \mu, \nu \leqslant \pi$.

(b-1) $\mu \neq \pm\nu$. $\mu, \nu \neq 0, \pi$. $C_0(\alpha(\mu,\nu); G(\mathbb{R})) = \{I_4\}$.

$$J_0\big(\alpha(\mu,\nu)\big) = c_{k,j}^{-1} \times \frac{e^{i(k-2)\mu}e^{i(j+k-1)\nu} - e^{i(j+k-1)\mu}e^{i(k-2)\nu}}{(e^{i\mu}-e^{-i\mu})(e^{i\nu}-e^{-i\nu})(e^{i(\mu+\nu)/2}-e^{-i(\mu+\nu)/2})(e^{i(\mu-\nu)/2}-e^{-i(\mu-\nu)/2})}.$$

(b-2) $\mu = \nu$. $\mu, \nu \neq 0, \pi$. $C_0(\alpha(\mu,\mu); G(\mathbb{R})) = \{I_4\}$.

$$J_0\big(\alpha(\mu,\mu)\big) = c_{k,j}^{-1} \times \frac{-(j+1)e^{i(j+2k-3)\mu}}{(e^{i\mu}-e^{-i\mu})^3}.$$

(b-3) $\mu = -\nu$. $\mu, \nu \neq 0, \pi$. We set $\alpha'(\mu,-\mu) = \varpi_1\varpi_2\alpha(\mu,-\mu)\varpi_2^{-1}\varpi_1^{-1}$.

$$C_0\big(\alpha'(\mu,-\mu); G(\mathbb{R})\big) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix};\ ad-bc=1 \right\} \cong SL(2;\mathbb{R}).$$

For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in SL(2;\mathbb{R})$, we take the Haar measure $d\alpha = 2v^{-3}\,du\,dv\,d\theta$ on $C_0(\alpha'(\mu,-\mu); G(\mathbb{R}))$.

$$J_0\big(\alpha'(\mu,-\mu)\big) = c_{k,j}^{-1} \times \frac{-(j+2k-3)(e^{i(j+1)\mu}-e^{-i(j+1)\mu})}{2^2\pi^2(e^{i\mu}-e^{-i\mu})^3}.$$

(b-4) $\nu = 0$. $\mu \neq 0, \pi$.

$$C_0\big(\alpha(\mu,0); G(\mathbb{R})\big) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix};\ ad-bc=1 \right\} \cong SL(2;\mathbb{R}).$$

We take the measure $d\alpha$ on $C_0(\alpha(\mu,0); G(\mathbb{R}))$ (cf. (b-3)).

$$J_0\big(\alpha(\mu,0)\big) = c_{k,j}^{-1} \times \frac{-(j+k-1)e^{i(k-2)\mu} + (k-2)e^{i(j+k-1)\mu}}{2^3\pi^2(e^{i\mu}-e^{-i\mu})(e^{i\mu/2}-e^{-i\mu/2})^2}.$$

(b-5) $\nu = 0$. $\mu = \pi$.

$$C_0\big(\alpha(\pi,0); G(\mathbb{R})\big) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix};\ \begin{matrix} ad-bc=1 \\ a'd'-b'c'=1 \end{matrix} \right\} \cong SL(2;\mathbb{R}) \times SL(2;\mathbb{R}).$$

We take the measure on $C_0(\alpha(\pi,0);G(\mathbb{R}))$ by the direct product of $d\alpha$ (cf. (b-3)).

$$J_0\bigl(\alpha(\pi,0)\bigr)=c_{k,j}^{-1}\times\frac{(-1)^k(j+k-1)(k-2)\{1+(-1)^j\}}{2^7\pi^4}.$$

(e) Unipotent. The notations $P_0(\mathbb{Q})$, $M_0(\mathbb{Q})$, $N_0(\mathbb{Q})$, and h_m have been defined in Section 2.3. Let γ $(\in\Gamma)$ be an element that is $G(\mathbb{Q})$-conjugate to $\delta(S)=\left(\begin{smallmatrix}I_2 & S\\ 0 & I_2\end{smallmatrix}\right)$, where S is a non-degenerate symmetric matrix over $\mathbb{Q}$. We easily observe that $\{\gamma\}_\Gamma$ has a non-empty intersection with $h_mN_0(\mathbb{Q})h_m^{-1}$ for a certain m. Here, we replace h_m with g. Let L be a lattice in $SM(2;\mathbb{R})$, which satisfies

$$\bigl\{\delta(T);\ T\in L\bigr\}=N_0(\mathbb{R})\cap g^{-1}\Gamma g.$$

The lattice L has a $\mathbb{Q}$-structure. We have $gP_0(\mathbb{Q})g^{-1}=(gM_0(\mathbb{Q})g^{-1})\cdot(gN_0(\mathbb{Q})g^{-1})$, $gM_0(\mathbb{R})g^{-1}\cong GL(2;\mathbb{R})$, and $gN_0(\mathbb{R})g^{-1}\cong SM(2;\mathbb{R})$. By Assumption 2.1, the equality $gP_0(\mathbb{Q})g^{-1}\cap\Gamma=(gM_0(\mathbb{Q})g^{-1}\cap\Gamma)\cdot(gN_0(\mathbb{Q})g^{-1}\cap\Gamma)$ holds.

(e-2) Consider the element $\gamma=g\delta(S)g^{-1}\in\Gamma$ for the case where $S>0$ or $S<0$. Here, we write $S>0$ if S is positive definite and $S<0$ if S is negative definite. We set

$$C_0\bigl(\gamma;G(\mathbb{R})\bigr)=g\bigl\{\delta(X);\ X\in SM(2;\mathbb{R})\bigr\}g^{-1}$$

and $v(\gamma;Z,s)=\det(\mathrm{Im}(g^{-1}\cdot Z))^{-s}$. As a coordinate of $C_0(g^{-1}\gamma g;G(\mathbb{R}))\backslash\mathfrak{H}_2$, we fix $\{iY\in\mathfrak{H}_2;\ Y>0\}$. We take the Haar measure on $C_0(\gamma;G(\mathbb{R}))$ such that $d\hat{Z}$ is transformed to $(\det Y)^{-3}\,dY$ by the g-conjugation. An evaluated form of $J_0(\gamma;s)$ is given by

$$J_0(\gamma;s)=\left\{c_{k,j}^{-1}\times\frac{(j+1)}{2^3\pi^2}+o(s)\right\}\times\frac{e^{\pm\pi i(-3-2s)/2}}{(\det S)^{s+3/2}},$$

where the sign is $+$ (resp. $-$) if $S<0$ (resp. $S>0$), $o(s)$ is a function such that $o(s)\to 0$ $(s\to+0)$ and $o(s)$ is independent of S. The family $[\gamma]_\Gamma$ is given by

$$[\gamma]_\Gamma=g\bigl\{\delta(T);\ T\in L,\ T>0\text{ or }T<0\bigr\}g^{-1}.$$

We identify $gM_0(\mathbb{R})g^{-1}$ with $GL(2;\mathbb{R})$ under an isomorphism. Let $\tilde{\Gamma}=gM_0(\mathbb{Q})g^{-1}\cap\Gamma$, $GL_+(2;\mathbb{R})=\{g\in GL(2;\mathbb{R});\ \det(g)>0\}$, and $\tilde{\Gamma}_+=\tilde{\Gamma}\cap GL_+(2;\mathbb{R})$. From the argument given in [25, p. 242], we obtain

$$\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma/\sim}\frac{J_0(\gamma';s)}{[\bar{C}(\gamma';\Gamma):\bar{C}_0(\gamma';\Gamma)]}=c_{k,j}^{-1}\times\frac{(j+1)}{2^2\cdot\pi}\times\frac{1}{[\tilde{\Gamma}:\tilde{\Gamma}_+]}\times\frac{\mathrm{vol}(\tilde{\Gamma}_+\backslash\mathfrak{H}_1)}{\mathrm{vol}(L)},$$

where $\mathfrak{H}_1$ is the upper half-plane $\{z\in\mathbb{C};\ \mathrm{Im}(z)>0\}$, the measure on $\mathfrak{H}_1$ is given by $y^{-2}\,dx\,dy$ for $z=x+iy$, and $\mathrm{vol}(L)=\int_{L\backslash SM(2;\mathbb{R})}dX$. The value $\mathrm{vol}(\tilde{\Gamma}_+\backslash\mathfrak{H}_1)$ comes from the residues of the zeta functions associated to symmetric matrices of degree two (cf. Section 5). For the calculation of the residues, we refer to [28, Theorem 2], [1, Proposition 1], [12, Proposition 5-1], and [27, Theorem 1].

(e-3) Consider the element $\gamma=g\delta(S)g^{-1}\in\Gamma$ for the case where S is indefinite and $\det(S)\neq 0$. If $-\det(S)\notin(\mathbb{Q}^\times)^2$, then the contribution of γ vanishes (cf. Section 4.13). Hence, we consider only the case $-\det(S)\in(\mathbb{Q}^\times)^2$. This case occurs only if $G(\mathbb{Q})$ is split. Let $-\det(S)\in(\mathbb{Q}^\times)^2$.

Then, we set

$$C_0\big(\gamma;G(\mathbb{R})\big)=g\big\{\delta(X);\ X\in SM(2;\mathbb{R})\big\}g^{-1}.$$

We set $v(\gamma;Z,s)=(y_1^{*-1}\det(\mathrm{Im}(g^{-1}\cdot Z)))^{-s}$ for $y_1^*\leqslant y_2^*$ and $v(\gamma;Z,s)=(y_2^{*-1}\det(\mathrm{Im}(g^{-1}\cdot Z)))^{-s}$ for $y_2^*\leqslant y_1^*$, where $\mathrm{Im}(g^{-1}\cdot Z)=\left(\begin{smallmatrix} y_1^* & y_{12}^* \\ y_{12}^* & y_2^*\end{smallmatrix}\right)$. As a coordinate of $C_0(g^{-1}\gamma g;G(\mathbb{R}))\backslash\mathfrak{H}_2$, we fix $\{iY\in\mathfrak{H}_2;\ Y>0\}$. We take the Haar measure on $C_0(\gamma;G(\mathbb{R}))$ such that $d\hat{Z}$ is transformed to $(\det Y)^{-3}\,dY$ by the g-conjugation. An evaluated form of $J_0(\gamma;s)$ is given by

$$J_0(\gamma;s)=\left\{-c_{k,j}^{-1}\times\frac{(j+1)}{2^3\pi^2}+o(s)\right\}\times\frac{1}{|\det S|^{3/2}},$$

where $o(s)$ is a function such that $o(s)\to 0$ $(s\to +0)$ and $o(s)$ is independent of S. We set $L'=\{T\in L;\ T$ is indefinite, $-\det(T)\in(\mathbb{Q}^\times)^2\}$. The family $[\gamma]_\Gamma$ is given by

$$[\gamma]_\Gamma=g\big\{\delta(T);\ T\in L'\big\}g^{-1}.$$

Let β_u $(1\leqslant u\leqslant t)$ be an element in $SL(2;\mathbb{Q})$ such that $\{\beta_u\cdot\infty;\ 1\leqslant u\leqslant t\}$ is a complete system of $\tilde{\Gamma}_+$-inequivalent cusps for $\tilde{\Gamma}_+\backslash\mathfrak{H}_1$. Let $L'/{\sim'}$ denote a complete system of representative elements of $\tilde{\Gamma}_+$-orbits in L'. By [12, Lemma 5-2], there exist positive rational numbers c_u and d_u such that

$$L'/{\sim'}=\bigcup_{u=1}^{t}\left\{\beta_u\begin{pmatrix}0 & s_{12}\\ s_{12} & s_2\end{pmatrix}{}^t\beta_u\in L';\ s_{12}\in L_{1,u},\ s_2\in L_{2,u}(s_{12})\right\}\quad\text{(disjoint union)},$$

where $L_{1,u}=\{d_u n;\ n\in\mathbb{Z}_{>0}\}$, $L_{2,u}(s_{12})$ is a finite subset depending on s_{12} in $\mathbb{Q}$, and $\sharp(L_{2,u}(d_u n))=c_u n$. Then, we have

$$\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma/\sim}\frac{J_0(\gamma';s)}{[\bar{C}(\gamma';\Gamma):\bar{C}_0(\gamma';\Gamma)]}=-c_{k,j}^{-1}\times\frac{(j+1)}{2^4\cdot 3}\times\frac{1}{[\tilde{\Gamma}:\tilde{\Gamma}_+]}\times\sum_{u=1}^{t}\frac{c_u}{d_u^3}.$$

If $\Gamma=Sp(2;\mathbb{Z})$, we have $t=1$, $d_u=1$, and $c_u=2$.

(e-4) Let γ be an element of Γ, which is $G(\mathbb{R})$-conjugate to $\delta(0,\pm1)$. This case occurs only if $G(\mathbb{Q})$ is split. There exist $g\in G(\mathbb{Q})$ and $\lambda\in\mathbb{Q}$ $(\lambda\neq 0)$ such that $\gamma=g\delta(0,\lambda)g^{-1}$. We set

$$C_0\big(\gamma;G(\mathbb{R})\big)=g\left\{\begin{pmatrix}a' & 0 & b' & 0\\ 0 & 1 & 0 & 0\\ c' & 0 & d' & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & t\\ s & 1 & t & u\\ 0 & 0 & 1 & -s\\ 0 & 0 & 0 & 1\end{pmatrix};\ \begin{matrix}a'd'-b'c'=1\\ s,t,u\in\mathbb{R}\end{matrix}\right\}g^{-1}$$

and $v(\gamma;Z,s)=1$. We take the Haar measure on $C_0(\gamma;G(\mathbb{R}))$, which is transformed to $d\alpha\,ds\,dt\,du$ by the g-conjugation, where $d\alpha$ is the Haar measure on $SL(2;\mathbb{R})$ (cf. (b-3)). Then, an explicit form of $J_0(\gamma;s)$ is given by

$$J_0(\gamma;s)=-c_{k,j}^{-1}\times\frac{(j+1)(j+2k-3)}{2^5\pi^4}\times\frac{1}{|\lambda|^2}.$$

The family $[\gamma]_\Gamma$ is given by

$$[\gamma]_\Gamma = g\{\delta(0,n);\ n \in b\mathbb{Z},\ n \neq 0\}g^{-1},$$

where $b \in \mathbb{Q}$ $(b > 0)$. Hence, we have

$$\lim_{s\to+0} \sum_{\gamma' \in [\gamma]_\Gamma/\sim} \frac{J_0(\gamma'; s)}{[\bar{C}(\gamma; \Gamma) : \bar{C}_0(\gamma; \Gamma)]} = -c_{k,j}^{-1} \times \frac{(j+1)(j+2k-3)}{2^5 \cdot 3 \cdot \pi^2} \times \frac{1}{b^2}.$$

(f) Quasi-unipotent.

(f-1) Let γ be an element of Γ, which is $G(\mathbb{R})$-conjugate to $\alpha(0,\pi)\delta(0,\pm 1)$. This case occurs only if $G(\mathbb{Q})$ is split. There exist $g \in G(\mathbb{Q})$ and $\lambda \in \mathbb{Q}$ $(\lambda \neq 0)$ such that

$$\gamma = g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \lambda \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} g^{-1}.$$

We set

$$C_0(\gamma; G(\mathbb{R})) = g \left\{ \begin{pmatrix} a' & 0 & b' & 0 \\ 0 & 1 & 0 & u \\ c' & 0 & d' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};\ \begin{matrix} a'd' - b'c' = 1 \\ u \in \mathbb{R} \end{matrix} \right\} g^{-1}$$

and $v(\gamma; Z, s) = (y_1^{*-1} \det(\mathrm{Im}(g^{-1} \cdot Z)))^{-s}$, where $\mathrm{Im}(g^{-1} \cdot Z) = \left(\begin{smallmatrix} y_1^* & y_{12}^* \\ y_{12}^* & y_2^* \end{smallmatrix}\right)$. We take the Haar measure on $C_0(\gamma; G(\mathbb{R}))$, which is transformed to $d\alpha\, du$ by the g-conjugation, where $d\alpha$ is the Haar measure on $SL(2;\mathbb{R})$ (cf. (b-3)). Then, an evaluated form of $J_0(\gamma; s)$ is given by

$$J_0(\gamma; s) = \left\{ c_{k,j}^{-1} \times \frac{(-1)^{k-2}(j+k-1) - (-1)^{j+k-1}(k-2)}{2^6 \pi^3} + o(s) \right\} \times \frac{e^{\mathrm{sgn}(\lambda)\pi i(s+1)/2}}{|\lambda|^{s+1}},$$

where $o(s)$ is a function such that $o(s) \to 0$ $(s \to +0)$ and $o(s)$ is independent of λ. The family $[\gamma]_\Gamma$ is given by

$$[\gamma]_\Gamma = g \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & b(n+a) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};\ \begin{matrix} n \in \mathbb{Z}, \\ n + a \neq 0 \end{matrix} \right\} g^{-1},$$

where $a \in \mathbb{Q}$ $(0 \leqslant a < 1)$ and $b \in \mathbb{Q}$ $(b > 0)$. From the argument given in [12, p. 442], we obtain

$$\lim_{s\to+0} \sum_{\gamma' \in [\gamma]_\Gamma} J_0(\gamma'; s) = c_{k,j}^{-1} \times \frac{(-1)^{k-2}(j+k-1) - (-1)^{j+k-1}(k-2)}{2^6 \cdot \pi^2} \times \frac{-1 + i \cdot \cot^* \pi a}{b},$$

where $\cot^* \theta = \cot \theta$ $(\theta \notin \mathbb{Z}\pi)$, 0 $(\theta \in \mathbb{Z}\pi)$.

(f-2) Let γ be an element of Γ, which is $G(\mathbb{R})$-conjugate to $\alpha(0,\pi)\delta(1,\pm 1)$. This case occurs only if $G(\mathbb{Q})$ is split. There exist $g \in G(\mathbb{Q})$ and $\lambda_1, \lambda_2 \in \mathbb{Q}$ ($\lambda_1, \lambda_2 \neq 0$) such that

$$\gamma = g \begin{pmatrix} 1 & 0 & \lambda_1 & 0 \\ 0 & -1 & 0 & \lambda_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} g^{-1}.$$

We set

$$C_0\big(\gamma; G(\mathbb{R})\big) = g \left\{ \begin{pmatrix} 1 & 0 & u_1 & 0 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ;\ u_1, u_2 \in \mathbb{R} \right\} g^{-1}$$

and $v(\gamma; Z, s) = \det(\mathrm{Im}(g^{-1} \cdot Z))^{-s}$. We take the Haar measure on $C_0(\gamma; G(\mathbb{R}))$, which is transformed to $du_1\, du_2$ by the g-conjugation. Then, an evaluated form of $J_0(\gamma; s)$ is given by

$$J_0(\gamma; s) = \left\{ c_{k,j}^{-1} \times \frac{(-1)^{k-2} - (-1)^{j+k-1}}{2^4 \pi^2} + o(s) \right\} \times \frac{e^{\mathrm{sgn}(\lambda_1)\pi i(s+1)/2}}{|\lambda_1|^{s+1}} \times \frac{e^{-\mathrm{sgn}(\lambda_2)\pi i(s+1)/2}}{|\lambda_2|^{s+1}},$$

where $o(s)$ is a function such that $o(s) \to 0$ ($s \to +0$) and $o(s)$ is independent of λ_1 and λ_2. The family $[\gamma]_\Gamma$ is given by

$$[\gamma]_\Gamma = \bigcup_{t=1}^{l} gR(t)g^{-1} \quad \text{(disjoint union)},$$
$$R(t) = \left\{ \begin{pmatrix} 1 & 0 & b_1(n_1 + a_{1,t}) & 0 \\ 0 & -1 & 0 & b_2(n_2 + a_{2,t}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ;\ \begin{matrix} n_1, n_2 \in \mathbb{Z}, \\ n_1 + a_{1,t}, n_2 + a_{2,t} \neq 0 \end{matrix} \right\},$$

where $a_{1,t}, a_{2,t}, b_1, b_2 \in \mathbb{Q}$ ($0 \leqslant a_{1,t}, a_{2,t} < 1$, $b_1, b_2 > 0$). From the argument given in [12, p. 442], we obtain

$$\lim_{s \to +0} \sum_{\gamma' \in [\gamma]_\Gamma} J_0\big(\gamma'; s\big) = c_{k,j}^{-1} \times \frac{(-1)^{k-2} - (-1)^{j+k-1}}{2^4} \times \frac{1}{b_1 b_2}$$
$$\times \sum_{t=1}^{l} \big(1 - i \cdot \cot^* \pi a_{1,t}\big)\big(1 + i \cdot \cot^* \pi a_{2,t}\big),$$

where $\cot^* \theta = \cot\theta$ ($\theta \notin \mathbb{Z}\pi$), 0 ($\theta \in \mathbb{Z}\pi$).

(f-3) Let γ be an element of Γ, which is $G(\mathbb{R})$-conjugate to $\varpi_1\varpi_2\alpha(\mu,-\mu)\varpi_2^{-1}\varpi_1^{-1}\delta(u,u)$ ($k(\mu)^2 \neq I_2$, $u = \pm 1$). There exist $g \in G(\mathbb{R})$ and $\lambda \in \mathbb{Q}$ ($\lambda \neq 0$) such that

$$\gamma = g \begin{pmatrix} \cos\theta & \sin\theta & \lambda\cos\theta & \lambda\sin\theta \\ -\sin\theta & \cos\theta & -\lambda\sin\theta & \lambda\cos\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} g^{-1}.$$

We set

$$C_0(\gamma; G(\mathbb{R})) = g\left\{\begin{pmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \ u \in \mathbb{R}\right\} g^{-1}$$

and $v(\gamma; Z, s) = \det(\mathrm{Im}(g^{-1} \cdot Z))^{-s}$. We take the Haar measure on $C_0(\gamma; G(\mathbb{R}))$, which is transformed to du by the g-conjugation. Then, an evaluated form of $J_0(\gamma; s)$ is given by

$$J_0(\gamma; s) = \left\{c_{k,j}^{-1} \times \frac{-e^{i(j+1)\theta} + e^{-i(j+1)\theta}}{2\pi(e^{i\theta} - e^{-i\theta})^3} + o(s)\right\} \times \frac{e^{\mathrm{sgn}(\lambda)\pi i(2s+1)/2}}{|\lambda|^{2s+1}},$$

where $o(s)$ is a function such that $o(s) \to 0$ $(s \to +0)$ and $o(s)$ is independent of λ. The family $[\gamma]_\Gamma$ is given by

$$[\gamma]_\Gamma = g\left\{\begin{pmatrix} \cos\theta & \sin\theta & b(n+a)\cos\theta & b(n+a)\sin\theta \\ -\sin\theta & \cos\theta & -b(n+a)\sin\theta & b(n+a)\cos\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}; \ n \in \mathbb{Z}, \ n+a \neq 0\right\} g^{-1},$$

where $a \in \mathbb{Q}$ $(0 \leqslant a < 1)$ and $b \in \mathbb{Q}$ $(b > 0)$. From the argument given in [12, p. 442], we obtain

$$\lim_{s \to +0} \sum_{\gamma' \in [\gamma]_\Gamma} J_0(\gamma'; s) = c_{k,j}^{-1} \times \frac{e^{i(j+1)\theta} - e^{-i(j+1)\theta}}{2(e^{i\theta} - e^{-i\theta})^3} \times \frac{1 - i \cdot \cot^* \pi a}{b},$$

where $\cot^* \theta = \cot\theta$ $(\theta \notin \mathbb{Z}\pi)$, 0 $(\theta \in \mathbb{Z}\pi)$.

(f-4) Let γ be an element of Γ, which is $G(\mathbb{R})$-conjugate to $\alpha(\mu, 0)\delta(0, \pm 1)$ $(k(\mu)^2 \neq I_2)$. This case occurs only if $G(\mathbb{Q})$ is split. There exist $g \in G(\mathbb{R})$ and $\lambda \in \mathbb{Q}$ $(\lambda \neq 0)$ such that

$$\gamma = g\begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & \lambda \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g^{-1}.$$

We set

$$C_0(\gamma; G(\mathbb{R})) = g\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \ u \in \mathbb{R}\right\} g^{-1}$$

and $v(\gamma; Z, s) = (y_1^{*-1} \det(\mathrm{Im}(g^{-1} \cdot Z)))^{-s}$, where $\mathrm{Im}(g^{-1} \cdot Z) = \begin{pmatrix} y_1^* & y_{12}^* \\ y_{12}^* & y_2^* \end{pmatrix}$. We take the Haar measure on $C_0(\gamma; G(\mathbb{R}))$, which is transformed to du by the g-conjugation. Then, an evaluated form of $J_0(\gamma; s)$ is given by

$$J_0(\gamma; s) = \left\{c_{k,j}^{-1} \times \frac{-e^{i(k-2)\theta} + e^{i(j+k-1)\theta}}{2\pi(e^{i\theta} - e^{-i\theta})(e^{i\theta/2} - e^{-i\theta/2})^2} + o(s)\right\} \times \frac{e^{\mathrm{sgn}(\lambda)\pi i(s+1)/2}}{|\lambda|^{s+1}},$$

where $o(s)$ is a function such that $o(s) \to 0$ $(s \to +0)$ and $o(s)$ is independent of λ. The family $[\gamma]_\Gamma$ is given by

$$[\gamma]_\Gamma = g\left\{\begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & b(n+a) \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};\ n\in\mathbb{Z},\ n+a\neq 0\right\} g^{-1},$$

where $a\in\mathbb{Q}$ $(0\leqslant a<1)$ and $b\in\mathbb{Q}$ $(b>0)$. From the argument given in [12, p. 442], we obtain

$$\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma} J_0(\gamma';s) = c_{k,j}^{-1}\times\frac{e^{i(k-2)\theta}-e^{i(j+k-1)\theta}}{2(e^{i\theta}-e^{-i\theta})(e^{i\theta/2}-e^{-i\theta/2})^2}\times\frac{1-i\cdot\cot^*\pi a}{b},$$

where $\cot^*\theta=\cot\theta$ $(\theta\notin\mathbb{Z}\pi)$, 0 $(\theta\in\mathbb{Z}\pi)$.

3.2. Normal subgroups and unitary characters

Let Γ' be a normal subgroup of Γ such that $[\Gamma:\Gamma']<+\infty$, and χ be a one-dimensional unitary representation of Γ' such that $[\Gamma':\ker(\chi)]<\infty$. Using $H^{k,j}_{g^{-1}\gamma g}(Z)=H^{k,j}_\gamma(g\cdot Z)$ for any γ, g in $G(\mathbb{R})$ and the Godement formula (Theorem 4.1), we can easily modify Theorem 3.1 under some conditions.

Theorem 3.2. *We assume that $\chi(\gamma)=1$ for every unipotent element $\gamma\in\Gamma'$ and $\chi(\delta^{-1}\gamma\delta)=\chi(\gamma)$ for any $\gamma\in\Gamma'$, $\delta\in\Gamma$. If $k\geqslant 5$ and Γ satisfies Assumption* 2.1, *then we have*

$$\begin{aligned}
&\dim_{\mathbb{C}} S_{k,j}(\Gamma',\chi)\\
&\quad=\frac{c_{k,j}\cdot[\overline{\Gamma}:\overline{\Gamma'}]}{\sharp(Z(\Gamma'))}\sum_{\{\gamma\}_\Gamma}\frac{\mathrm{vol}(\bar{C}_0(\gamma;\Gamma)\backslash\bar{C}_0(\gamma;G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]}J_0(\gamma)\chi(\gamma)^{-1}\\
&\qquad+\frac{c_{k,j}\cdot[\overline{\Gamma}:\overline{\Gamma'}]}{\sharp(Z(\Gamma'))}\sum_{[\gamma]_{\Gamma'}}\frac{\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]}\chi(\gamma)^{-1}\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_{\Gamma'}}J_0(\gamma';s)\\
&\qquad+\frac{c_{k,j}\cdot[\overline{\Gamma}:\overline{\Gamma'}]}{\sharp(Z(\Gamma'))}\sum_{[\gamma]_{\Gamma'}}\mathrm{vol}\big(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G(\mathbb{R}))\big)\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_{\Gamma'}/\sim}\frac{J_0(\gamma';s)}{[\bar{C}(\gamma';\Gamma):\bar{C}_0(\gamma';\Gamma)]},
\end{aligned}$$

where in the first term, $\{\gamma\}_\Gamma$ runs over the set of Γ-conjugacy classes of (a) *and* (b) *in Γ'; in the second and third terms, $[\gamma]_{\Gamma'}$ runs over a complete system of representative elements of Γ-conjugacy classes of families of Γ', same as that in Theorem* 3.1. *The equivalence relation $\sim$ is defined by Γ-conjugations. In* (e) *Unipotent, for $g=h_m$ of Γ, L is a lattice that satisfies $\{\delta(T);\ T\in L\}=N_0(\mathbb{R})\cap g^{-1}\Gamma' g$ and $\tilde{\Gamma}=gM_0(\mathbb{Q})g^{-1}\cap\Gamma$.*

Note that we can assume $\Gamma'=\Gamma$ in this theorem. In the case of $\Gamma'=\Gamma$, for any character χ, it is clear that $\chi(\delta^{-1}\gamma\delta)=\chi(\gamma)$ for any γ, δ in Γ. On the other hand, there exist unitary characters χ that do not satisfy $\chi(\gamma)=1$ for every unipotent element $\gamma\in\Gamma$.

4. Proof of Theorem 3.1

4.1. Godement formula

For $Z=X+iY\in\mathfrak{H}_2$ and $\gamma=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\Gamma$, we set

$$H_\gamma^{k,j,\chi}(Z)=\mathrm{tr}\left[\rho_{k,j}(CZ+D)^{-1}\rho_{k,j}\left(\frac{\gamma\cdot Z-\overline{Z}}{2i}\right)^{-1}\rho_{k,j}(Y)\right]\chi(\gamma)^{-1}.$$

If χ is trivial, then we have $H_\gamma^{k,j}(Z) = H_\gamma^{k,j,\chi}(Z)$ (cf. Section 3). Let $k \geqslant 5$. It is known that the function $H_\gamma^{k,j,\chi}(Z)$ has the following three properties: (i) $H_\gamma^{k,j}(g \cdot Z) = H_{g^{-1}\gamma g}^{k,j}(Z)$ for any γ, g in $G(\mathbb{R})$, (ii) $|\sum_{\gamma\in\Gamma} H_\gamma^{k,j,\chi}(Z)|$ is bounded on the fundamental domain of Γ, and (iii) $\sum_{\gamma\in\Gamma} |H_\gamma^{k,j,\chi}(Z)| < +\infty$. We note that $\int_{\Gamma\backslash\mathfrak{H}_2} \sum_{\gamma\in\Gamma} |H_\gamma^{k,j,\chi}(Z)|\, dZ = +\infty$ and $\int_{\Gamma\backslash\mathfrak{H}_2} dZ < +\infty$ for any arithmetic subgroup Γ.

Godement obtained the following formula (cf. [8, Expose 10, Théorème 8]). In order to prove Theorem 3.1, we calculate the right-hand side of this formula for the trivial character χ.

Theorem 4.1 *(Godement). If $k \geqslant 5$, then we have*

$$\dim_{\mathbb{C}} S_{k,j}(\Gamma,\chi) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \int\limits_{\Gamma\backslash\mathfrak{H}_2} \sum_{\gamma\in\Gamma} H_\gamma^{k,j,\chi}(Z)\, dZ.$$

4.2. Siegel sets

We set

$$P_0(\mathbb{Q}) = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in G(\mathbb{Q}) \right\}, \qquad P_1(\mathbb{Q}) = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in G(\mathbb{Q}) \right\}.$$

If $G(\mathbb{Q})$ is $\mathbb{Q}$-split, $G(\mathbb{Q})$ has the maximal parabolic subgroups $P_0(\mathbb{Q})$ and $P_1(\mathbb{Q})$ and the Borel subgroup $P_0(\mathbb{Q}) \cap P_1(\mathbb{Q})$ up to $G(\mathbb{Q})$-conjugation. If $G(\mathbb{Q})$ is not $\mathbb{Q}$-split, $G(\mathbb{Q})$ only has the parabolic subgroup $P_0(\mathbb{Q})$ up to $G(\mathbb{Q})$-conjugation. We set $G(\mathbb{Q}) = \bigcup_{n=1}^{\nu} \Gamma g_n (P_0(\mathbb{Q}) \cap P_1(\mathbb{Q}))$ (disjoint union) if $G(\mathbb{Q})$ is $\mathbb{Q}$-split, and $G(\mathbb{Q}) = \bigcup_{n=1}^{\nu} \Gamma g_n P_0(\mathbb{Q})$ (disjoint union) if $G(\mathbb{Q})$ is not $\mathbb{Q}$-split.

It is well known that there exists a Siegel set Σ for Γ and $\{g_n\}_{n=1}^{\nu}$. We put $\Omega_2 = \{Y \in SM(2;\mathbb{R});\ Y > 0\}$. The Siegel set Σ of Γ is given by

$$\Sigma = \{Z = X + iY \in \mathfrak{H}_2;\ X \in \mathcal{W},\ Y \in R\},$$

where $\mathcal{W}$ is a compact subset of $SM(2;\mathbb{R})$, $R = \{\left(\begin{smallmatrix} 1 & 0 \\ y'_{12} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y'_1 & 0 \\ 0 & y'_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y'_{12} \\ 0 & 1 \end{smallmatrix}\right) \in \Omega_2;\ y'_{12} \in \mathcal{W}',\ \alpha \leqslant \beta y'_1 \leqslant y'_2\}$ for certain positive constants α and β and a compact subset $\mathcal{W}'$ in $\mathbb{R}$ if $G(\mathbb{Q})$ is $\mathbb{Q}$-split, and $R = \{a \in \mathbb{R};\ a > \alpha'\} \times \mathcal{W}''$ for a certain positive constant α' and a certain compact subset $\mathcal{W}''$ in $\{x \in \Omega_2;\ \det(x) = 1\}$ if $G(\mathbb{Q})$ is not $\mathbb{Q}$-split. We know that $\bigcup_{n=1}^{\nu} g_n\Sigma$ is a fundamental set of Γ and contains a fundamental domain $\mathcal{F}$ of Γ in $\mathfrak{H}_2$. We can divide the fundamental domain into $\mathcal{F} = \bigcup_{n=1}^{\nu} F_n$ (disjoint union) such that $g_n^{-1} F_n \subset \Sigma$.

4.3. Lemma for absolute values

By using Lemma 4.2, we can reduce the problems of absolute convergence to the scalar-valued case.

Lemma 4.2. *There exists a constant $c'(j)$, which depends only on j, such that*

$$\left|H_\gamma^{k,j}(Z)\right| < c'(j) \times \left|H_\gamma^{k,0}(Z)\right|.$$

The constant $c'(j)$ is independent of $\gamma \in \Gamma$ and $Z \in \mathfrak{H}_2$.

Proof. We set

$$K(g,g')=J(g^{-1},iI_2)\,\rho_{k,j}\left(\frac{g^{-1}\cdot iI_2-\overline{g'^{-1}\cdot iI_2}}{2i}\right)^{-1}{}^t\overline{J(g'^{-1},iI_2)},$$

where $J(g,Z)=\rho_{k,j}(CZ+D)^{-1}$ for $g=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in Sp(2;\mathbb{R})$, $Z\in\mathfrak{H}_2$. We can easily observe that $\mathrm{tr}(K(g^{-1}\gamma^{-1}g,I_4))=\mathrm{tr}(K(g^{-1}\gamma^{-1},g^{-1}))=H_\gamma^{k,j}(g\cdot iI_2)$ (cf. [8,24]). By using the Cartan decomposition, for any $g\in G(\mathbb{R})$, we have $g=hah'$, where $h,h'\in U(2)$ and $a=\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})$, $(a_1,a_2\in\mathbb{R}^\times)$. Then, we have

$$\left|\mathrm{tr}\left(K\left((hah')^{-1},I_4\right)\right)\right|=\left|\mathrm{tr}\left({}^t\overline{J(h,iI_2)^{-1}}\,J(h',iI_2)\,\rho_{k,j}\left(\mathrm{diag}\left((a_1+a_1^{-1})/2,\,(a_2+a_2^{-1})/2\right)\right)^{-1}\right)\right|.$$

From $J(h,iI_2)$, $J(h',iI_2)\in U(2)$ and $|a_1+a_1^{-1}|,|a_2+a_2^{-1}|\geqslant 2$, we deduce

$$\left|H_g^{k,j}(iI_2)\right|<c'(j)\times\left|\det\left(\mathrm{diag}\left((a_1+a_1^{-1})/2,\,(a_2+a_2^{-1})/2\right)\right)^{-k}\right|=c'(j)\times\left|H_g^{k,0}(iI_2)\right|,$$

where $g=hah'$. Thus, we have proved this lemma. □

4.4. Estimates of infinite series

Let $\Gamma_\infty^0=\Gamma\cap P_0(\mathbb{Q})$ and $\Gamma_\infty^1=\Gamma\cap P_1(\mathbb{Q})$. If $G(\mathbb{Q})$ is not $\mathbb{Q}$-split, then we set $P_1(\mathbb{Q})=\emptyset$ and $\Gamma_\infty^1=\emptyset$. Let Γ_{M_0} be the image of $\Gamma\cap P_0(\mathbb{Q})$ under the natural projection $P_0(\mathbb{Q})\to M_0(\mathbb{Q})=P_0(\mathbb{Q})/N_0(\mathbb{Q})$. Then, Γ_{M_0} is an arithmetic subgroup of $M_0(\mathbb{Q})$. As a generalization of [5, Satz 1], [25, Section 4], and [1, Proposition 6], we get the following.

Lemma 4.3. *Let $k\geqslant 5$ and $Z=X+iY\in\Sigma$. We have the following inequalities*:

$$\sum_{\gamma\in\Gamma_\infty^0\cap\Gamma_\infty^1}\left|H_\gamma^{k,j}(Z)\right|<C_{k,j,\Gamma,1}\times y_1y_2^2, \tag{4.1}$$

$$\sum_{\gamma\in\Gamma_\infty^0-\Gamma_\infty^1}\left|H_\gamma^{k,j}(Z)\right|<C_{k,j,\Gamma,2}\times (y_1y_2)^{3/2}, \tag{4.2}$$

$$\sum_{\gamma\in\Gamma_\infty^1-\Gamma_\infty^0}\left|H_\gamma^{k,j}(Z)\right|<C_{k,j,\Gamma,3}\times y_1^{-1}y_2^2, \tag{4.3}$$

$$\sum_{\gamma\in\Gamma-(\Gamma_\infty^0\cup\Gamma_\infty^1)}\left|H_\gamma^{k,j}(Z)\right|<C_{k,j,\Gamma,4}\times\begin{cases}y_1^{-1}y_2^{3/2}, & G(\mathbb{Q})\text{ is split},\\ 1, & G(\mathbb{Q})\text{ is not split},\end{cases} \tag{4.4}$$

where the positive constant $C_{k,j,\Gamma,l}$ $(l=1,2,3,4)$ depends only on k, j, and Γ. If $G(\mathbb{Q})$ is $\mathbb{Q}$-split, then we have $\int_\Sigma y_1^{a_1}y_2^{a_2}\,dZ<+\infty$ for $a_1+a_2<3$ and $a_2<2$. If $G(\mathbb{Q})$ is not $\mathbb{Q}$-split, then we have $\int_\Sigma (y_1y_2)^{3/2-s}\,dZ<+\infty$ for $s>0$.

Proof. By Lemma 4.2 we may assume $j=0$. By using the proof of [25, Proposition 23], we can easily prove (4.1) and (4.2) for any arithmetic subgroups. Hence, we consider (4.3) and (4.4). If $G(\mathbb{Q})$ is not split, then we can easily prove (4.4) for any arithmetic subgroups by using the proofs of [1, Lemma 7 and Proposition 6]. Hence, we have only to consider the case when $G(\mathbb{Q})$ is split. We set $\mathcal{S}_1=\{\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma;\ \det(C)\neq 0\}$, and $\mathcal{S}_2=\{\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma;\ \mathrm{rank}(C)=1\}$. We take a lattice L'' in $SM(2;\mathbb{R})$

such that $N_0'' = \{\delta(S);\ S \in L''\}$ and $\Gamma_\infty^0 \subset N_0'' \cdot \Gamma_{M_0}$. It follows from [5, Kapitel II] and [25, Proofs of Propositions 21 and 22] that

$$\sum_{\gamma\in\mathcal{S}_r} \left|H_\gamma^{k,j}(Z)\right| < \text{constant}$$

$$\times \sum_{\gamma\in(N_0''\cdot\Gamma_{M_0})\backslash\mathcal{S}_r} \sum_{U\in\Gamma_{M_0}} \left|\det\bigl(Y+U\operatorname{Im}(\gamma\cdot Z)\,{}^tU\bigr)\right|^{-k+3/2}\left|\det(CZ+D)\right|^{-k}\det(Y)^k$$

for $r=1,2$, where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$. Using [5, Hilfssatz 2 and pp. 86–87] and the method proposed by Braun [4], for $\mathcal{S}_1$, we see that $\sum_{\gamma\in\mathcal{S}_1}|H_\gamma^{k,j}(Z)| < \text{constant}\times\det(Y)^{4-k}$. Furthermore, for $\mathcal{S}_2$, we can use an argument similar to that in [5, pp. 70–86] and [5, pp. 13–18]. Therefore, we have $\sum_{\gamma\in\mathcal{S}_2}|H_\gamma^{k,j}(Z)| < \text{constant}\times y_1^{3-k}y_2^2$ and $\sum_{\gamma\in\mathcal{S}_2-\Gamma_\infty^1}|H_\gamma^{k,j}(Z)| < \text{constant}\times y_1^{-k+7/2}y_2^{3/2}$. $\square$

If $X-Y>0$ ($X,Y\in\Omega_2$), then we write $X>Y$. Let μ, ς_1, and ς_2 be arbitrary positive constants. We set $\mathfrak{H}_2(\mu,\varsigma_1,\varsigma_2) = \{X+\sqrt{-1}Y\in\mathfrak{H}_2;\ Y>\mu I_2,\ y_2\geqslant\varsigma_1 y_1\geqslant\varsigma_2|y_{12}|\}$. There exist μ, ς_1, and ς_2 such that $\Sigma\subset\mathfrak{H}_2(\mu,\varsigma_1,\varsigma_2)$. (4.5) is a generalization of [25, Proposition 24].

Lemma 4.4. *Let $k\geqslant 5$ and $Z=X+iY\in\mathfrak{H}_2(\mu,\varsigma_1,\varsigma_2)$. Then, there exist constants $C'_{j,k,\mu,\varsigma_1,\varsigma_2,\Gamma}$ and $C''_{j,k,\mu,\varsigma_1,\varsigma_2,\Gamma}$ depending only on k, j, μ, ς_1, ς_2, and Γ such that*

$$\sum_{\gamma_2\in(N_0(\mathbb{Q})\cap\Gamma)\backslash\Gamma_\infty^0}\left|\sum_{\gamma_1\in(N_0(\mathbb{Q})\cap\Gamma)} H_{\gamma_1\gamma_2}^{k,j}(Z)\right| < C'_{k,j,\mu,\varsigma_1,\varsigma_2,\Gamma}, \tag{4.5}$$

$$\sum_{\gamma_2\in(N_0(\mathbb{Q})\cap\Gamma)\backslash\Gamma_\infty^1}\left|\sum_{\gamma_1\in(N_0(\mathbb{Q})\cap\Gamma)} H_{\gamma_1\gamma_2}^{k,j}(Z)\right| < C''_{k,j,\mu,\varsigma_1,\varsigma_2,\Gamma}. \tag{4.6}$$

Proof. Let $\mathcal{L}$ be a lattice of $SM(2;\mathbb{R})$ such that $N_0(\mathbb{Q})\cap\Gamma = \{\delta(S);\ S\in\mathcal{L}\}$. We use a certain Poisson summation formula to prove this lemma. For $x\in SM(2;\mathbb{C})$, $M\in GL(2;\mathbb{C})$, by using [7, Theorem XI.2.4], we can set $\operatorname{tr}(\rho_{k,j}(xM)) = \sum_{l=1}^t a_l(M)\Delta_{m_l}(g_l x\,{}^tg_l)\det(xM)^k$, where $g_l\in GL(2;\mathbb{R})$, a_l ($l=1,\dots,t$) are polynomials for the entries of M, and Δ_{m_l} are defined in Section 5.1. Here, the degree of the polynomial Δ_{m_l} is equal to j for x. From [7, Lemma XI.2.3], classical methods, and an argument similar to that in Section 5.1, we obtain the following Poisson summation formula:

$$\sum_{S\in\mathcal{L}}\operatorname{tr}\bigl\{\rho_{k,j}\bigl(\bigl((\gamma\cdot Z-\overline{Z}+S)/2i\bigr)^{-1}Y(CZ+D)^{-1}\bigr)\bigr\}$$

$$=\sum_{T\in\mathcal{L}^*\cap\Omega_2}\sum_{l=1}^t b_l\cdot a_l(M)\det(M)^k\cdot\Delta_{m_l}\bigl(g_lT\,{}^tg_l\bigr)\det(T)^{k-3/2}\exp\bigl(2\pi i\operatorname{tr}\bigl(T(\gamma\cdot Z-\overline{Z})\bigr)\bigr),$$

where $M=Y(CZ+D)^{-1}$, $\gamma=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$, $\mathcal{L}^*$ is the dual lattice of $\mathcal{L}$, and b_l is a constant depending on l (cf. Section 5).

First, we prove (4.5). We set

$$I = \sum_{\gamma_2\in(N_0(\mathbb{Q})\cap\Gamma)\backslash\Gamma_\infty^0}\left|\sum_{\gamma_1\in N_0(\mathbb{Q})\cap\Gamma} H_{\gamma_1\gamma_2}^{k,j}(Z)\right|$$

$$= \sum_{\gamma_2\in(N_0(\mathbb{Q})\cap\Gamma)\backslash\Gamma_\infty^0}\left|\sum_{S\in\mathcal{L}}\operatorname{tr}\bigl\{\rho_{k,j}\bigl(\bigl((AZ\,{}^tA-\overline{Z}+S+S')/2i\bigr)^{-1}Y\,{}^tA\bigr)\bigr\}\right|,$$

where $\gamma_2 = \begin{pmatrix} I_2 & S' \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$. By using the Poisson summation formula, we get

$$I \leqslant \text{constant} \times \sum_{A\in\Gamma_{M_0}} \sum_{T\in\mathcal{L}^*\cap\Omega_2} \left|f(A,Y,T)\right| \det(TY)^k \exp\bigl(-2\pi\,\mathrm{tr}\bigl(T\bigl(AY\,{}^tA+Y\bigr)\bigr)\bigr),$$

where $f(A,Y,T)$ is a polynomial for the entries of A, Y, and T and its degree is j for each A, Y, and T. Since $Y > \mu I_2$, we have $\mathrm{tr}(T(AY\,{}^tA+Y)) > \mathrm{tr}(TY) + \mu\,\mathrm{tr}(TA\,{}^tA)$. We put $T = \begin{pmatrix} t_1 & t_{12} \\ t_{12} & t_2 \end{pmatrix}$. Since $T \in \mathcal{L}^* \cap \Omega_2$, there exists a positive constant ξ such that $t_1 > \xi$ and $t_2 > \xi$. Hence, we deduce $|t_{12}| < (t_1t_2)^{1/2} < \xi^{-1}|t_1t_2|$ from $\det(T) > 0$. We also deduce $y_1 > \mu$, $y_2 > \mu$, and $|y_{12}| < \mu^{-1}|y_1y_2|$ from $Y > \mu I_2$. We put $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. There exists a polynomial f_1 for $|a_{11}|$, $|a_{12}|$, $|a_{21}|$, $|a_{22}|$ such that $f(A,Y,T) < \text{constant} \times (t_1t_2y_1y_2)^j \times f_1(|a_{11}|,|a_{12}|,|a_{21}|,|a_{22}|)$. Hence, we get

$$\begin{aligned} I \leqslant \text{constant} \times &\sum_{T\in\mathcal{L}^*\cap\Omega_2} (t_1t_2y_1y_2)^j \det(TY)^k \exp\bigl(-2\pi\,\mathrm{tr}(TY)\bigr) \\ &\times \sum_{A\in\Gamma_{M_0}} f_1\bigl(|a_{11}|,|a_{12}|,|a_{21}|,|a_{22}|\bigr) \exp\bigl(-2\pi\mu\,\mathrm{tr}\bigl(TA\,{}^tA\bigr)\bigr). \end{aligned}$$

Therefore, we can reduce this proof to the proof of [25, Proposition 24].

Next, we prove (4.6). We set

$$\begin{aligned} I' &= \sum_{\gamma_2\in(N_0(\mathbb{Q})\cap\Gamma)\backslash\Gamma^1_\infty} \left| \sum_{\gamma_1\in N_0(\mathbb{Q})\cap\Gamma} H^{k,j}_{\gamma_1\gamma_2}(Z) \right| \\ &= \sum_{\gamma_2\in(N_0(\mathbb{Q})\cap\Gamma)\backslash\Gamma^1_\infty} \left| \sum_{S\in\mathcal{L}} \mathrm{tr}\bigl\{\rho_{k,j}\bigl(\bigl((\gamma_2\cdot Z - \overline{Z} + S)/2i\bigr)^{-1} Y(CZ+D)^{-1}\bigr)\bigr\} \right|. \end{aligned}$$

By using the Poisson summation formula, we get

$$\begin{aligned} I' \leqslant \text{constant} \times &\sum_{\gamma_2\in(N_0(\mathbb{Q})\cap\Gamma)\backslash\Gamma^1_\infty} \sum_{T\in\mathcal{L}^*\cap\Omega_2} \left|f'\bigl((CZ+D)^{-1},Y,T\bigr)\right| \det(TY)^k \\ &\times \left|\det(CZ+D)^{-k}\right| \exp\bigl(-2\pi\,\mathrm{tr}\bigl(T\,{}^t(C\overline{Z}+D)^{-1}Y(CZ+D)^{-1}+TY\bigr)\bigr), \end{aligned}$$

where f' is a polynomial for the entries of $(CZ+D)^{-1}$, Y, and T and its degree is j for each $(CZ+D)^{-1}$, Y, and T. We put

$$\gamma_2 = \pm \begin{pmatrix} a & 0 & b & * \\ * & 1 & * & * \\ c & 0 & d & n \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we have

$$(CZ+D)^{-1} = \pm(cz_1+d)^{-1} \begin{pmatrix} 1 & -(cz_{12}+n) \\ 0 & cz_1+d \end{pmatrix}.$$

Since $Y > \mu I_2$, we have

$$\mathrm{tr}\bigl(T\,{}^t(C\overline{Z}+D)^{-1}Y(CZ+D)^{-1}+TY\bigr) > \mu\,\mathrm{tr}\bigl(T\,{}^t(C\overline{Z}+D)^{-1}(CZ+D)^{-1}\bigr) + \mathrm{tr}(TY).$$

By direct calculation, we have

$$\begin{aligned}&\mathrm{tr}\big(T\,{}^t(C\overline{Z}+D)^{-1}(CZ+D)^{-1}\big)\\&\quad=|cz_1+d|^{-2}\times\big\{t_1-2(cx_{12}+n)t_{12}+\big(|cz_1+d|^2+|cz_{12}+n|^2\big)t_2\big\}\\&\quad=|cz_1+d|^{-2}\times\big\{c^2y_{12}^2t_2+t_2^{-1}\det(T)+t_2\big(n+cx_{12}-t_2^{-1}t_{12}\big)^2+|cz_1+d|^2t_2\big\}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}I'\leqslant\ &\mathrm{constant}\times\sum_{T\in\mathcal{L}^*\cap\Omega_2}(t_1t_2)^j(y_1y_2)^{2j}\det(TY)^k\exp\big(-2\pi\,\mathrm{tr}(TY)\big)\\&\times\sum_{g\in N''\backslash\Gamma''}\sum_{n\in\mu'\mathbb{Z}}|cz_1+d|^{-k}f_1'\left(\frac{|n+cx_{12}|}{|cz_1+d|},\frac{1}{|cz_1+d|}\right)\exp\left(-2\pi\xi\mu\frac{|n+cx_{12}-t_2^{-1}t_{12}|^2}{|cz_1+d|^2}\right),\end{aligned}$$

where $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, Γ'' is an arithmetic subgroup of $SL(2;\mathbb{Q})$, μ' and μ'' are constants, $N''=\{\left(\begin{smallmatrix}1&\mu''n\\0&1\end{smallmatrix}\right);\ n\in\mathbb{Z}\}\subset\Gamma''$, and $f_1'(a',b')$ is a polynomial for a' and b' ($\deg(f_1')\leqslant j$).

We consider the infinite series

$$I_1'=\sum_{n\in\mu'\mathbb{Z}}|n+\alpha_1|^t\exp\big(-\beta(n+\alpha_1-\alpha_2)^2\big)$$

for constants $t\in\mathbb{Z}_{\geqslant0}$, α_1, $\alpha_2\in\mathbb{R}$, and $\beta\in\mathbb{R}_{>0}$. We have

$$I_1'\leqslant\mathrm{constant}\times\sum_{l=0}^{t}|\alpha_2|^{t-l}\sum_{n\in\alpha_1-\alpha_2+\mu'\mathbb{Z}}|n|^l\exp\big(-\beta n^2\big)$$

by change of variable. Hence, we get

$$\begin{aligned}I_1'\leqslant\ &\mathrm{constant}\times\sum_{l=0}^{t}|\alpha_2|^{t-l}\\&\times\left\{\left|\frac{l}{2\beta}\right|^{l/2}\exp(-l/2)\left(2\left|\frac{l}{2\beta}\right|^{1/2}+2\mu'\right)+|\beta|^{-(l+1)/2}\int_{\mathbb{R}}|x|^l\exp\big(-x^2\big)\,dx\right\}.\end{aligned}$$

Therefore, we obtain

$$I'\leqslant\mathrm{constant}\times\sum_{T\in\mathcal{L}^*\cap\Omega_2}(t_1t_2)^{2j}(y_1y_2)^{2j}\det(TY)^k\exp\big(-2\pi\,\mathrm{tr}(TY)\big)\sum_{g\in N''\backslash\Gamma''}|cz_1+d|^{-k+1},$$

if we set $\alpha_1=cx_{12}$, $\alpha_2=t_2^{-1}t_{12}$, and $\beta=2\pi\xi\mu|cz_1+d|^{-2}$ for I_1'.

We shall explain an evaluation for $\sum_{g\in N''\backslash\Gamma''}|cz_1+d|^{-k+1}$. If $c=0$, then we have $d=\pm1$ and $|cz_1+d|=1$. We also have

$$\sum_{g\in N''\backslash\Gamma'',\, c\neq 0} |cz_1+d|^{-k+1} \leqslant \sum_{c\in\kappa_1\mathbb{Z},\, c\neq 0}\sum_{d\in\kappa_2\mathbb{Z}} \left|(d+cx_1)^2+y_1^2c^2\right|^{-(k-1)/2}$$
$$\leqslant \text{constant} \times \sum_{c\in\kappa_1\mathbb{Z},\, c\neq 0} |c|^{-k+2} \leqslant \text{constant}$$

(cf. [25, Lemma 5]). Hence, we have $\sum_{g\in N''\backslash\Gamma''} |cz_1+d|^{-k+1} \leqslant \text{constant}$. Therefore, we obtain

$$I' \leqslant \text{constant} \times \sum_{T\in\mathcal{L}^*\cap\Omega_2} (t_1t_2)^{2j}(y_1y_2)^{2j}\det(TY)^k\exp\bigl(-2\pi\,\mathrm{tr}(TY)\bigr).$$

Since $y_2 \geqslant \varsigma_1 y_1 \geqslant \varsigma_2|y_{12}|$, there exists a positive constant κ_3 such that $Y > \kappa_3\left(\begin{smallmatrix} y_1 & 0\\ 0 & y_2\end{smallmatrix}\right)$. Hence, we have $\mathrm{tr}(TY) > \kappa_3(y_1t_1+y_2t_2)$ and

$$I' \leqslant \text{constant} \times \sum_{T\in\mathcal{L}^*\cap\Omega_2} (t_1t_2y_1y_2)^{2j+k}\exp\bigl(-2\pi\kappa_3(y_1t_1+y_2t_2)\bigr).$$

Since $t_1t_2 > t_{12}^2$, we have

$$I' \leqslant \text{constant} \times \sum_{t_1\in\kappa_4\mathbb{Z}_{>0},\, t_2\in\kappa_5\mathbb{Z}_{>0}} (t_1t_2y_1y_2)^{2j+k+1}\exp\bigl(-2\pi\kappa_3(y_1t_1+y_2t_2)\bigr).$$

Hence, we have

$$\begin{aligned} I' \leqslant{} & \text{constant} \times (y_1y_2)^{2j+k+1}\exp\bigl(-\pi\kappa_3\xi(y_1+y_2)\bigr)\\ & \times \sum_{t_1\in\kappa_4\mathbb{Z}_{>0},\, t_2\in\kappa_5\mathbb{Z}_{>0}} (t_1t_2)^{2j+k+1}\exp\bigl(-\pi\kappa_3\mu(t_1+t_2)\bigr). \end{aligned}$$

Thus, we obtain (4.6). □

4.5. Interchange of the integral and the infinite sum

We set

$$\mathfrak{A}_{n,0} = \bigl(g_n^{-1}\Gamma g_n\bigr)_\infty^0 - \bigl(g_n^{-1}\Gamma g_n\bigr)_\infty^1, \qquad \mathfrak{A}_{n,1} = \bigl(g_n^{-1}\Gamma g_n\bigr)_\infty^1,$$
$$\mathfrak{A}_{n,2} = g_n^{-1}\Gamma g_n - \bigl(\bigl(g_n^{-1}\Gamma g_n\bigr)_\infty^0 \cup \bigl(g_n^{-1}\Gamma g_n\bigr)_\infty^1\bigr),$$
$$\bigl(g_n^{-1}\cdot F_n\bigr)_{0,s} = \bigl\{Z=X+iY\in g_n^{-1}\cdot F_n;\ y_1\geqslant\exp(1/s),\ y_2-y_1^{-1}y_{12}^2\geqslant\exp\bigl(1/s^2\bigr)\bigr\},$$
$$\bigl(g_n^{-1}\cdot F_n\bigr)_{1,s} = \bigl\{Z=X+iY\in g_n^{-1}\cdot F_n;\ y_2-y_1^{-1}y_{12}^2\geqslant\exp\bigl(1/s^2\bigr)\bigr\},$$
$$\mathfrak{F}_{n,0,s} = g_n^{-1}\cdot F_n - \bigl(g_n^{-1}\cdot F_n\bigr)_{0,s}, \qquad \mathfrak{F}_{n,1,s} = g_n^{-1}\cdot F_n - \bigl(g_n^{-1}\cdot F_n\bigr)_{1,s}, \qquad \mathfrak{F}_{n,2,s} = g_n^{-1}\cdot F_n.$$

Note that $g_n^{-1}\Gamma g_n$ becomes an arithmetic subgroup of $G(\mathbb{Q})$, $\mathfrak{F}_{n,2,s}$ does not depend on s, and $g_n^{-1}\cdot F_n\subset\Sigma$. The following proposition is a generalization of [25, Theorem 3].

Proposition 4.5. *If $k \geqslant 5$, then we have*

$$\int_{\Gamma\backslash\mathfrak{H}_2} \sum_{\gamma\in\Gamma} H_\gamma^{k,j}(Z)\,dZ = \sum_{n=1}^{\nu}\sum_{r=0}^{2} \lim_{s\to+0} \sum_{\gamma\in\mathfrak{A}_{n,r}} \int_{\mathfrak{F}_{n,r,s}} H_\gamma^{k,j}(Z)\,dZ.$$

Proof. Since $\int_{\Gamma\backslash\mathfrak{H}_2} |\sum_{\gamma\in\Gamma} H_\gamma^{k,j}(Z)|\,dZ < \infty$, we have

$$\int_{\Gamma\backslash\mathfrak{H}_2} \sum_{\gamma\in\Gamma} H_\gamma^{k,j}(Z)\,dZ = \sum_{n=1}^{\nu} \int_{F_n} \sum_{\gamma\in\Gamma} H_\gamma^{k,j}(Z)\,dZ = \sum_{n=1}^{\nu} \int_{g_n^{-1}\cdot F_n} \sum_{\gamma\in g_n^{-1}\Gamma g_n} H_\gamma^{k,j}(Z)\,dZ.$$

By using Lemmas 4.3 and 4.4, we have

$$\int_{g_n^{-1}\cdot F_n} \sum_{\gamma\in g_n^{-1}\Gamma g_n} H_\gamma^{k,j}(Z)\,dZ = \sum_{r=0}^{2} \int_{g_n^{-1}\cdot F_n} \sum_{\gamma\in\mathfrak{A}_{n,r}} H_\gamma^{k,j}(Z)\,dZ.$$

By Lemma 4.3, for $s > 0$, we obtain

$$\int_{\mathfrak{F}_{n,r,s}} \sum_{\gamma\in\mathfrak{A}_{n,r}} \left|H_\gamma^{k,j}(Z)\right| dZ < \infty.$$

It follows from Lemma 4.4 that

$$\begin{aligned}
\lim_{s\to+0} \int_{(g_n^{-1}\cdot F_n)_{r,s}} \left|\sum_{\gamma\in\mathfrak{A}_{n,r}} H_\gamma^{k,j}(Z)\right| dZ &\leqslant \text{constant} \times \lim_{s\to+0} \int_{(g_n^{-1}\cdot F_n)_{r,s}} dZ \\
&\leqslant \text{constant} \times \lim_{s\to+0} \int_{\exp(1/s^2)}^{\infty} y^{-3}\,dy = 0,
\end{aligned}$$

where $r = 0$ or 1. Hence, it follows from Lebesgue's convergence theorem that

$$\begin{aligned}
\int_{g_n^{-1}\cdot F_n} \sum_{\gamma\in\mathfrak{A}_{n,r}} H_\gamma^{k,j}(Z)\,dZ &= \lim_{s\to+0}\left\{ \int_{\mathfrak{F}_{n,r,s}} \sum_{\gamma\in\mathfrak{A}_{n,r}} H_\gamma^{k,j}(Z)\,dZ + \int_{(g_n^{-1}\cdot F_n)_{r,s}} \sum_{\gamma\in\mathfrak{A}_{n,r}} H_\gamma^{k,j}(Z)\,dZ \right\} \\
&= \lim_{s\to+0} \sum_{\gamma\in\mathfrak{A}_{n,r}} \int_{\mathfrak{F}_{n,r,s}} H_\gamma^{k,j}(Z)\,dZ.
\end{aligned}$$

Thus, we have proved the proposition. □

For each subset A of Γ, we can consider the value of

$$I(A) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \sum_{n=1}^{\nu}\sum_{r=0}^{2} \lim_{s\to+0} \sum_{\gamma\in g_n^{-1}Ag_n\cap\mathfrak{A}_{n,r}} \int_{\mathfrak{F}_{n,r,s}} H_\gamma^{k,j}(Z)\,dZ$$

if $I(A)$ is convergent. We call $I(A)$ the contribution of A to the dimension formula.

4.6. Semisimple contributions

From Lemma 4.2, the proof of Lemma 4.20, [25, Theorem 5], [1, Proposition 8], and [12, Section 3], we obtain the following.

Lemma 4.6. *Let $k \geqslant 5$. Let $A(ss)$ be the subset that consists of all semisimple elements of Γ. We have $\sum_{\gamma \in A(ss)} \int_{\Gamma \backslash \mathfrak{H}_2} |H_\gamma^{k,j}(Z)|\, dZ < +\infty$.*

By using Lemma 4.6, Proposition 4.5, and the Selberg trace formula, we have

$$I\big(A(ss)\big) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \sum_{\{\gamma\}_\Gamma \subset A(ss)} \mathrm{vol}\big(\overline{C}(\gamma;\Gamma)\backslash \overline{C}\big(\gamma; G(\mathbb{R})\big)\big) \int_{C(\gamma; G(\mathbb{R}))\backslash \mathfrak{H}_2} H_\gamma^{k,j}(\hat{Z})\, d\hat{Z}.$$

The semisimple orbital integrals have been explicitly given by Langlands [24]. Hence, we obtain the semisimple part of Theorem 3.1.

4.7. Vanishing case for non-semisimple contributions

Next, we consider the elements of types (e-1), (g-1), and (g-2). We prove that their contributions are zero by Morita's method [25]. If $G(\mathbb{Q})$ is not $\mathbb{Q}$-split, then Γ does not contain the elements of type (e-1) (cf. [1, Proposition 7]), and the set of (g-1) satisfies the absolute convergence, which is the same as that of Lemma 4.6 (cf. [1, Proposition 8]).

First, we consider the elements of type (e-1). Hence, we assume that $G(\mathbb{Q})$ is $\mathbb{Q}$-split. Let $A(e1)$ be the subset of all elements in Γ, which are $G(\mathbb{R})$-conjugate to the representative elements of (e-1). We have

$$I\big(A(e1)\big) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \lim_{s\to +0} \sum_{\{\gamma\}_\Gamma \subset A(e1)} \sum_{n=1}^{v} \sum_{r=0}^{2} \sum_{\delta \in g_n^{-1}\{\gamma\}_\Gamma g_n \cap \mathfrak{A}_{n,r}} \int_{\mathfrak{F}_{n,r,s}} H_\delta^{k,j}(Z)\, dZ,$$

where $\{\gamma\}_\Gamma$ runs over all Γ-conjugacy classes in Γ, which are contained in $A(e1)$. If we set $\mathfrak{B}_{n,r,\gamma} = \{\omega \in \Gamma;\ g_n^{-1}\omega^{-1}\gamma\omega g_n \in \mathfrak{A}_{n,r}\}$, then we have

$$I\big(A(e1)\big) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \lim_{s\to +0} \sum_{\{\gamma\}_\Gamma \subset A(e1)} \sum_{n=1}^{v} \sum_{r=0}^{2} \sum_{\omega \in C(\gamma;\Gamma)\backslash \mathfrak{B}_{n,r,\gamma}} \int_{\mathfrak{F}_{n,r,s}} H^{k,j}_{g_n^{-1}\omega^{-1}\gamma\omega g_n}(Z)\, dZ.$$

For each Γ-conjugacy class, we can take a representative element γ which belongs to $\Gamma \cap g_m P_0(\mathbb{Q}) g_m^{-1} \cap g_m P_1(\mathbb{Q}) g_m^{-1}$ for a certain m. We fix such γ and m. Note that $\Gamma \cap g_n P_r(\mathbb{Q}) g_n^{-1} = g_n (g_n^{-1}\Gamma g_n)_\infty^r g_n^{-1}$ $(r = 0, 1)$. By [25, Proposition 12], for any g_n, we see that $\epsilon^{-1} g_m^{-1}\gamma g_m \epsilon$ $(\epsilon \in g_m^{-1}\Gamma g_n)$ belongs to $(g_n^{-1}\Gamma g_n)_\infty^r$ if and only if ϵ belongs to $g_m^{-1}\Gamma g_n \cap P_r(\mathbb{Q})$. Hence, for any $g_n^{-1}\omega^{-1}\gamma\omega g_n \in (g_n^{-1}\Gamma g_n)_\infty^r$, we have $g_m^{-1}\omega g_n \in g_m^{-1}\Gamma g_n \cap P_r(\mathbb{Q})$. Furthermore, we find that $g_m^{-1}\omega g_n$ runs over all elements of $\bigcup_{n=1}^{v} g_m^{-1} \cdot C(\gamma;\Gamma)\backslash \Gamma \cdot g_n$ in the above sum. Hence, we can use the same argument as in [25, Proof of Theorem 6] on the basis of these facts and Lemma 4.2, if we prove Lemma 4.7, which is a generalization of [25, Lemma 13]. Therefore, we have $I(A(e1)) = 0$.

Lemma 4.7. *Let $B_{1,s} = \{X + iY \in \mathfrak{H}_2;\ y_1 > 0,\ y_2 - y_1^{-1} y_{12}^2 \geqslant \exp(1/s^2)\}$. Let s be a sufficiently small positive real number. Then, there exist positive constants c and c', which depend only on Γ, such that*

$$\bigcup_{n=1}^{v} \bigcup_{\xi \in g_m^{-1}\Gamma g_n \cap P_1(\mathbb{Q})} \xi \cdot \big(g_n^{-1} \cdot F_n\big)_{1,cs} \subset B_{1,s} \subset \bigcup_{n=1}^{v} \bigcup_{\xi \in g_m^{-1}\Gamma g_n} \xi \cdot \big(g_n^{-1} \cdot F_n\big)_{1,c's}.$$

Proof. We can easily observe that $\bigcup_{n=1}^{\nu}\bigcup_{\xi\in g_m^{-1}\Gamma g_n\cap P_1(\mathbb{Q})}\xi\cdot(g_n^{-1}\cdot F_n)_{1,cs}\subset B_{1,s}$. Hence, we have only to prove $B_{1,s}\subset\bigcup_{n=1}^{\nu}\bigcup_{\xi\in g_m^{-1}\Gamma g_n}\xi\cdot(g_n^{-1}\cdot F_n)_{1,c's}$. Let $W=U+iV\in B_{1,s}$. Then, there exists an element $\xi\in g_m^{-1}\Gamma g_n$ such that $W\in\xi\cdot(g_n^{-1}\cdot F_n)$. We take $Z=X+iY\in g_n^{-1}\cdot F_n$ such that $\xi\cdot Z=W\in B_{1,s}$. We have only to prove $Z\in(g_n^{-1}\cdot F_n)_{1,c's}$ for a constant c' depending only on $g_m^{-1}\Gamma g_n$. By the action of $\xi=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$, we have $Y\mapsto V={}^t(CZ+D)^{-1}Y(C\overline{Z}+D)^{-1}$. Hence, we have $V^{-1}=(CX+D)Y^{-1}(X\,{}^tC+{}^tD)+CY\,{}^tC$.

If $\det(C)\neq 0$, then $V^{-1}>\text{constant}\times I_2$. Hence, we get $\xi\cdot(g_n^{-1}\cdot F_n)\cap B_{1,s}=\emptyset$.

Let $\operatorname{rank}(C)=1$. If we set $\xi=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)=\left(\begin{smallmatrix}H&0\\0&{}^tH^{-1}\end{smallmatrix}\right)\left(\begin{smallmatrix}A'&B'\\C'&D'\end{smallmatrix}\right)\left(\begin{smallmatrix}P&0\\0&{}^tP^{-1}\end{smallmatrix}\right)$, $C'=\left(\begin{smallmatrix}c_1&0\\0&0\end{smallmatrix}\right)$, $D'=\left(\begin{smallmatrix}d_1&d_2\\0&d_3\end{smallmatrix}\right)$, $X'=PX\,{}^tP=\left(\begin{smallmatrix}x_1'&x_{12}'\\x_{12}'&x_2'\end{smallmatrix}\right)$, and $Y'=PY\,{}^tP=\left(\begin{smallmatrix}y_1'&y_{12}'\\y_{12}'&y_2'\end{smallmatrix}\right)$, then we get

$$V^{-1}={}^tH^{-1}\left\{\begin{pmatrix}c_1x_1'+d_1&c_1x_{12}'+d_2\\0&d_4\end{pmatrix}Y'^{-1}\begin{pmatrix}c_1x_1'+d_1&0\\c_1x_{12}'+d_2&d_4\end{pmatrix}+\begin{pmatrix}c_1^2y_1'&0\\0&0\end{pmatrix}\right\}H^{-1}.$$

Therefore, if $H\notin\{\left(\begin{smallmatrix}*&0\\ *&*\end{smallmatrix}\right)\in GL(2;\mathbb{Q})\}$, then we have $v_1\det(V)^{-1}>c''$ for a constant c'' and $\xi\cdot(g_n^{-1}\cdot F_n)\cap B_{1,s}=\emptyset$. Hence, we can take $\xi_1\in P_1(\mathbb{Q})$ and $\xi_2\in P_0(\mathbb{Q})$ such that $\xi=\xi_1\times\xi_2$ and the components of ξ_1 and ξ_2 belong to a certain lattice on $\mathbb{Q}$. We know that $\xi_1^{-1}\cdot B_{1,s}\subset B_{1,c'''s}$ for a constant c'''. We can reduce this case to the case of $C=0$.

Let $C=0$. Then, we have $V^{-1}=DY^{-1}\,{}^tD$. We set $D=\left(\begin{smallmatrix}d_1&d_2\\d_3&d_4\end{smallmatrix}\right)$. If we prove $d_3^2y_2-2d_3d_4y_{12}+d_4^2y_1>\text{constant}\times y_1$, then we have $Z=X+iY\in F_{1,c's}$. By $\left(\begin{smallmatrix}y_1&y_{12}\\y_{12}&y_2\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&0\\u&1\end{smallmatrix}\right)\left(\begin{smallmatrix}y_1&\\&y_2'\end{smallmatrix}\right)\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)$, we have $d_3^2y_2-2d_3d_4y_{12}+d_4^2y_1=y_1(d_4-ud_3)^2+d_3^2y_2'$. Since $Z\in g_n^{-1}\cdot F_n\subset\Sigma$, we easily obtain the inequality. $\square$

Since we can also prove the vanishing of the contributions for types (g-1) and (g-2) by using an argument similar to that of type (e-1), we omit the proof for types (g-1) and (g-2).

We shall explain the reason for $I(A(e1))=0$ shortly. Using the argument in [25, p. 230] and the above mentioned argument, we can express the contribution $I(A(e1))$ as

$$I\bigl(A(e1)\bigr)=\lim_{s\to+0}\sum_{\{\gamma\}_\Gamma\subset A(e1)}\int_{F_{\gamma,s}}H_\gamma^{k,j}(Z)\,dZ,$$

where $F_{\gamma,s}$ is a certain domain satisfying $\lim_{s\to+0}F_{\gamma,s}=F_\gamma$ and F_γ is the fundamental domain of the centralizer of γ. Furthermore, we have

$$\int_{F_{\gamma,s}}H_\gamma^{k,j}(Z)\,dZ=\sum_{l,m\in\mathbb{Z}_{\geqslant 0},\,l+5\leqslant m\leqslant j+k}\int_{\mathcal{D}_s}\left\{\int_{-\infty}^{\infty}\bigl(f_1(W)w+f_2(W)\bigr)^{-m}f_{l,m}(W)w^l\,dw\right\}dW,$$

where $F_{\gamma,s}\cong(-\infty,\infty)\times\mathcal{D}_s$, $dZ=dw\,dW$, f_1, f_2, and $f_{l,m}$ are polynomials of W, and $f_1(W)w+f_2(W)\neq 0$ $({}^\forall(w,W)\in(-\infty,\infty)\times\mathcal{D}_s)$. Using $\int_{-\infty}^{\infty}(f_1(W)w+f_2(W))^{-m}\,dw=0$ and the induction via

$$\int_{-\infty}^{\infty}\frac{w^l}{(aw+b)^m}\,dw=\left[-\frac{w^l}{a(m-1)(aw+b)^{m-1}}\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{lw^{l-1}}{a(m-1)(aw+b)^{m-1}}\,dw$$

(a, b are constants and $aw+b\neq 0$), we find that the contribution is zero.

4.8. Limit formulas and orbital integrals

Before we calculate the contributions for unipotent and quasi-unipotent elements, we explicitly calculate some orbital integrals. We use limit formulas for unipotent orbital integrals on real semisimple Lie groups. Barbasch, Vogan, Rossmann [26], and Božičević [3] have studied limit formulas for such orbital integrals. We need these formulas for the cases of $SL(2;\mathbb{R})$ and $Sp(2;\mathbb{R})$. As for $Sp(2;\mathbb{R})$, we use the limit formulas given in [26] and [3]. Since they gave these limit formulas on Lie algebras, we need to lift these formulas from the Lie algebras to the groups. As for the lift, we refer to [36, Chapter 8].

Let $n(u)=\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)$, $k(\theta)=\left(\begin{smallmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{smallmatrix}\right)$, $a(v)=\left(\begin{smallmatrix}v&0\\0&v^{-1}\end{smallmatrix}\right)$. We define the Haar measure on $SL(2;\mathbb{R})$ by $d\alpha=2v^{-3}\,du\,dv\,d\theta$ and $\alpha=n(u)a(v)k(\theta)$. We denote the space of $\mathbb{C}$-valued C^∞-class compactly supported functions on an analytic group H by $C^\infty_{\mathrm{com}}(H)$.

Lemma 4.8. *For $f\in C^\infty_{\mathrm{com}}(SL(2;\mathbb{R}))$, we have*

$$\lim_{\theta\to 0,\,\theta\in\mathcal{C}_{n(u)}}\left(e^{i\theta}-e^{-i\theta}\right)\int_1^\infty\int_0^\pi f\left(\left(a(v)k(\theta')\right)^{-1}k(\theta)\left(a(v)k(\theta')\right)\right)2\left(1-v^{-4}\right)v\,dv\,d\theta'$$

$$=\kappa_{n(u)}\times\int_0^{2\pi}\int_0^\infty f\left(\left(a(v)k(\theta')\right)^{-1}n(u)\left(a(v)k(\theta')\right)\right)2v^{-3}\,dv\,d\theta',$$

where $\kappa_{n(u)}=ui$, $\mathcal{C}_{n(u)}=\{\theta>0\}$ if $u>0$, and $\mathcal{C}_{n(u)}=\{\theta<0\}$ if $u<0$. On the left-hand side, the measure on $SO(2;\mathbb{R})\backslash SL(2;\mathbb{R})$ is given by $d\mu\backslash d\alpha$ where $SO(2;\mathbb{R})=\{k(\mu)\}$. On the right-hand side, the measure on $N\backslash SL(2;\mathbb{R})$ is given by $du\backslash d\alpha$, where $N=\{n(u);\ u\in\mathbb{R}\}$.

For $f\in C^\infty_{\mathrm{com}}(G(\mathbb{R}))$ and $\gamma\in G(\mathbb{R})$, we set $\Phi_f(\gamma)=\int_{C(\gamma;G(\mathbb{R}))\backslash G(\mathbb{R})}f(\hat{g}^{-1}\gamma\hat{g})\,d\hat{g}$, where $d\hat{g}$ is the invariant measure on $C(\gamma;G(\mathbb{R}))\backslash G(\mathbb{R})$, which is induced from the Haar measures on $G(\mathbb{R})$ and $C(\gamma;G(\mathbb{R}))$. For all regular elliptic elements $\alpha(\theta_1,\theta_2)$, we take the Haar measure $(2\pi)^{-2}\,dw_1\,dw_2$ on the compact Cartan subgroup $C(\alpha(\theta_1,\theta_2);G(\mathbb{R}))=\{\alpha(w_1,w_2)\}$.

Lemma 4.9. *We set*

(1)

$$\nu=\begin{pmatrix}0&0&0&0\\0&0&0&t\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad(t=\pm1),\qquad p_\nu(\theta_1,\theta_2)=(\theta_1-\theta_2)(\theta_1+\theta_2),$$

$$\mathcal{C}_\nu=\{\theta_1>\theta_2>0\}\quad(t=1),\qquad\mathcal{C}_\nu=\{\theta_1<\theta_2<0\}\quad(t=-1),$$

(2)

$$\nu=\begin{pmatrix}0&S\\0&0\end{pmatrix}\quad(S=\pm I_2),\qquad p_\nu(\theta_1,\theta_2)=\theta_1-\theta_2,$$

$$\mathcal{C}_\nu=\{\theta_1>\theta_2>0\}\quad(S=I_2),\qquad\mathcal{C}_\nu=\{\theta_1<\theta_2<0\}\quad(S=-I_2).$$

For $f\in C^\infty_{\mathrm{com}}(G(\mathbb{R}))$, we have

$$\lim_{(\theta_1,\theta_2)\to(0,0),\,(\theta_1,\theta_2)\in\mathcal{C}_\nu}p_\nu(\partial_1,\partial_2)\Delta(\theta_1,\theta_2)\Phi_f\left(\alpha(\theta_1,\theta_2)\right)=\kappa_\nu\times\Phi_f\left(\exp(\nu)\right),$$

where κ_ν is a constant, which is independent of f, $\partial_i=\partial/\partial\theta_i$ $(i=1,2)$, and $\Delta(\theta_1,\theta_2)=(e^{i\theta_1}-e^{-i\theta_1})(e^{i\theta_2}-e^{-i\theta_2})(e^{i(\theta_1+\theta_2)/2}-e^{-i(\theta_1+\theta_2)/2})(e^{i(\theta_1-\theta_2)/2}-e^{-i(\theta_1-\theta_2)/2})$.

Proof. We must prove a condition for the nilpotent elements of (1) in order to use Rossmann's limit formula (cf. [26, Section 5]). Because the case of (1) is not considered in [3] (the minimal nilpotent orbits are not Richardson). For the nilpotent element $\nu \in \mathfrak{g}_{\mathbb{R}}$ of (1), we show that $G(\mathbb{R}) \cdot \nu = (G(\mathbb{C}) \cdot \nu) \cap (\bigcap_{\mu \in \mathcal{C}'_\nu} (\mathcal{N} \cap \overline{G(\mathbb{R}) \cdot \mathbb{R}_+^\times \mu}))$, where $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$ and

$$\mathcal{C}'_\nu = \left\{ \begin{pmatrix} 0 & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & \theta_2 \\ -\theta_1 & 0 & 0 & 0 \\ 0 & -\theta_2 & 0 & 0 \end{pmatrix};\ (\theta_1, \theta_2) \in \mathcal{C}_\nu \right\},$$

under the adjoint action. We can easily find a sequence $\{\mu_l\}\,(\subset G(\mathbb{R}) \cdot \mathbb{R}_+^\times \mu)$ such that $\mu_l \to \nu$. Hence, we have only to prove that $-\nu$ does not belong to $(G(\mathbb{C}) \cdot \nu) \cap (\bigcap_{\mu \in \mathcal{C}_\nu} (\mathcal{N} \cap \overline{G(\mathbb{R}) \cdot \mathbb{R}_+^\times \mu}))$. We observe that $\mathcal{C}'_\nu = \mathcal{C}''_\nu J_2$, where $\mathcal{C}''_\nu = \{\mathrm{diag}(\theta_1, \theta_2, \theta_1, \theta_2);\ (\theta_1, \theta_2) \in \mathcal{C}_\nu\}$, $J_2 = \left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right)$, and $g\mathcal{C}'_\nu g^{-1} = g\mathcal{C}''_\nu {}^t g J_2$. Hence, $\overline{G(\mathbb{R}) \cdot \mathbb{R}_+^\times \mu}$ is contained in $\{x \in \mathfrak{g}_{\mathbb{R}};\ xJ_2^{-1}$ is half-positive definite$\}$ if $t = 1$, $\{x \in \mathfrak{g}_{\mathbb{R}};\ xJ_2^{-1}$ is half-negative definite$\}$ if $t = -1$. If $t = 1$ (resp. $t = -1$), then νJ_2^{-1} is half-positive (resp. half-negative) definite and $-\nu J_2$ is half-negative (resp. half-positive) definite. Thus, we have proved the condition for (1). For the case of (2), we can prove the condition similarly (the case of (2) is considered in [3]). $\square$

We need the following lemma to use the above mentioned limit formulas for the calculations of $J_0(\gamma; 0)$ (cf. [24, Section 6]), because the support of $H_\gamma^{k,j}(Z)$ is not compact.

Lemma 4.10. *Let γ be an element of type* (e-2), (e-4), (f-1), (f-2), (f-3) *or* (f-4) *in Γ. The integral $J_0(\gamma; 0)$ is absolutely convergent.*

Proof. For (e-2) and (e-4), we can prove the absolute convergence of $J_0(\gamma; 0)$ by using Lemmas 5.1 and 4.13. In the case of (f-1) and (f-2), we can easily obtain the absolute convergence of $J_0(\gamma; 0)$ by direct calculation (cf. Sections 4.14 and 4.15). It follows from Lemmas 4.23 and 4.25 that $\int_{\Gamma\backslash\mathfrak{H}_2} \sum_{\omega \in \{\gamma\}_\Gamma} |H_\omega^{k,j}(Z)|\, dZ < \infty$ for (f-3) and (f-4). Hence, by using the equality $\frac{\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]} J_0(\gamma; 0) = \sum_{\omega \in \{\gamma\}_\Gamma} \int_{\Gamma\backslash\mathfrak{H}_2} H_\omega^{k,j}(Z)\, dZ$, we obtain the absolute convergence of $J_0(\gamma; 0)$ for (f-3) and (f-4). $\square$

From the values of orbital integrals [25, Theorems 8, 9] ($j = 0$), we know the following.

Lemma 4.11. *Let ν be a nilpotent element in Lemma* 4.9. *We take the measures $d\hat{g}$, which are the same as those described in Section* 3. *The centralizers are given by $C(\gamma; G(\mathbb{R})) = \{\pm I_4\} \times C_0(\gamma; G(\mathbb{R}))$ in* (1) *and $C(\gamma; G(\mathbb{R})) = O(2;\mathbb{R}) \ltimes C_0(\gamma; G(\mathbb{R}))$ in* (2). *The measures on $C_0(\gamma; G(\mathbb{R}))$ have been defined in Section* 3. *We assume that the volumes of $\{\pm I_4\}$ and $O(2;\mathbb{R})$ are equal to one. In case of* (1), *we have $\kappa_\nu = -2^6 \cdot \pi^4$. In case of* (2), *we have $\kappa_\nu = 2^4\pi^2$ ($S = I_2$), $\kappa_\nu = -2^4\pi^2$ ($S = -I_2$).*

By Lemma 4.10, we can apply the limit formulas (Lemmas 4.8, 4.9, 4.11) to the calculations of the integrals $J_0(\gamma; 0)$, similar to [24, Section 5]. Thus, we obtain the following result.

Lemma 4.12. *Let γ be an element of type* (e-2), (e-4), (f-1), (f-2), (f-3), *or* (f-4) *in Γ. Then, from the limit formulas, we obtain the explicit form of $J_0(\gamma; 0)$, which has been described in Section* 3.

We note that we cannot apply the limit formula to the integral $J_0(\gamma; s)$ for (e-3) $-\det(S) \in (\mathbb{Q}^\times)^2$, because it is not an orbital integral.

4.9. Unipotent contribution of (e-4)

If $G(\mathbb{Q})$ is not split, the elements of (e-4) do not appear in Γ. By Lemma 4.2 and [25, Theorem 8], we get the following.

Lemma 4.13. *Let $k \geqslant 5$. Let $A(e4)$ be the subset of Γ, which consists of all elements of type* (e-4)*. Then, we have* $\sum_{\gamma \in g_n^{-1} A(e4) g_n} \int_{g_n^{-1} \cdot F_n} |H_\gamma^{k,j}(Z)|\, dZ < +\infty$.

From this lemma and Lemma 4.12, we can easily deduce the result for (e-4) given in Section 3.

4.10. Contributions of (e-2), (e-3), (f-1), (f-2), (f-3), and (f-4)

Let $A(**)$ be the subset of Γ, which consists of all elements of type $(*\text{-}*)$, where $(*\text{-}*)$ indicates (e-2), (e-3), (f-1), (f-2), (f-3), or (f-4). Let $A(e3)'$ be the subset of $A(e3)$, which consists of the elements $G(\mathbb{Q})$-conjugate to $\pm\delta(T)$, $\det(T) < 0$, $-\det(T) \in (\mathbb{Q}^\times)^2$. We set $A(e2)' = A(e2) \cup (A(e3) - A(e3)')$. For $\gamma = g\delta(S)g^{-1} \in A(e2)$ $(g \in G(\mathbb{Q}))$, we set

$$\mathfrak{G}_{\gamma,\Gamma} = \left\{ g\delta(T)g^{-1} \in \Gamma;\ \det(T) \neq 0,\ -\det(T) \notin \left(\mathbb{Q}^\times\right)^2 \right\}.$$

For $-\gamma = -g\delta(S)g^{-1} \in A(e2)$, we set $\mathfrak{G}_{-\gamma,\Gamma} = \{-\omega \in \Gamma;\ \omega \in \mathfrak{G}_{\gamma,\Gamma}\}$. We have a one-to-one correspondence between $\mathfrak{G}_{\gamma,\Gamma}$ and $[\gamma]_\Gamma$ for $\gamma \in A(e2)$. We require the following transformation in order to calculate the contribution of each family. Note that there exists only a finite number of Γ-conjugacy classes of families for them.

Proposition 4.14. *If $k \geqslant 5$, then we have*

$$I\big(A(e2)'\big) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \sum_{\mathfrak{G}_{\gamma,\Gamma}} \sum_{n=1}^{\nu} \int_{g_n^{-1} \cdot F_n} \sum_{\omega \in \bigcup_{\gamma' \in \mathfrak{G}_{\gamma,\Gamma}} \{\gamma'\}_\Gamma} H^{k,j}_{g_n^{-1}\omega g_n}(Z)\, dZ,$$

where $\mathfrak{G}_{\gamma,\Gamma}$ runs over subsets which correspond to a complete system of representative elements of Γ-conjugacy classes of families of (e-2)*. Let A be one of the subsets $A(e3)'$, $A(f1)$, $A(f2)$, $A(f3)$, or $A(f4)$. If $k \geqslant 5$, then we have*

$$I(A) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \sum_{[\gamma]_\Gamma} \sum_{n=1}^{\nu} \int_{g_n^{-1} \cdot F_n} \sum_{\omega \in \bigcup_{\gamma' \in [\gamma]_\Gamma} \{\gamma'\}_\Gamma} H^{k,j}_{g_n^{-1}\omega g_n}(Z)\, dZ,$$

where $[\gamma]_\Gamma$ runs over a complete system of representative elements of Γ-conjugacy classes of families which are contained in A.

Proof. We set $\mathfrak{J}_{n,\gamma} = \bigcup_{\gamma' \in \mathfrak{G}_{\gamma,\Gamma}} g_n^{-1}\{\gamma'\}_\Gamma g_n$ for $\gamma \in A(e2)$. We also set $\mathfrak{J}_{n,\gamma} = \bigcup_{\gamma' \in [\gamma]_\Gamma} g_n^{-1}\{\gamma'\}_\Gamma g_n$ for $\gamma \in A(e3)'$, $A(f1)$, $A(f2)$, $A(f3)$, or $A(f4)$. It is sufficient to prove

$$\int_{g_n^{-1} \cdot F_n} \left| \sum_{\delta \in \mathfrak{J}_{n,\gamma} \cap \mathfrak{A}_{n,r}} H_\delta^{k,j}(Z) \right| dZ < +\infty \tag{4.7}$$

for $r = 0$, 1. By using Lemmas 4.4, 4.13, 4.15, and 4.16, we get (4.7) for $A(e2)'$ and $A(e3)'$. For $A(f1)$, by using Lemmas 4.20 and 4.21, we have (4.7) if we prove $\int_{g_n^{-1} \cdot F_n} |\sum_\gamma H_\gamma^{k,j}(Z)|\, dZ <$ constant, where

γ runs the set

$$\left\{\pm\begin{pmatrix}1&0&0&*\\0&1&*&*\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\ *&-1&0&0\\0&0&1&*\\0&0&0&-1\end{pmatrix}\in g_n^{-1}A(f1)g_n\right\}.$$

We can prove this convergence by replacing $-Q(Z;\gamma)\to(2i)^{-1}s_2y_1+(2i)^{-1}s_2'y_1+y_1y_2+(2i)^{-1}y_1((c't)^2z_1-2c'tz_{12})-(i^{-1}x_{12}-(2i)^{-1}(c'tz_1+cs_{12}))^2$ and $-4^{-1}|c'tz_1+cs_{12}|^2\to|z_{12}-2^{-1}(c'tz_1+cs_{12})|^2$ in the proof of Lemma 4.16. For $A(f2)$, we can prove (4.7) by using the results in Section 4.15. For $A(f3)$, we can prove (4.7) by using Lemma 4.23, $A(f3)\cap P_1(\mathbb{Q})=\emptyset$, the coordinate given in Section 4.16, and an argument similar to that in Section 4.15. For $A(f4)$, we can prove (4.7) by using Lemma 4.25, $A(f4)\cap P_0(\mathbb{Q})=\emptyset$, the coordinate given in Section 4.17, and an argument similar to that in Section 4.15. $\square$

4.11. Convergence of unipotent terms

Before we calculate the contributions of (e-2) and (e-3), we require some lemmas for studying the convergence of some unipotent terms. In the case of non-split $\mathbb{Q}$-forms, we do not require the following lemmas. By Lemma 4.2 and the proofs of [25, Lemmas 14, 15, 16], we get the following, which is a generalization of [25, Lemmas 14, 15, 16].

Lemma 4.15. *We set*

$$\mathfrak{E}_{n,1}=g_n^{-1}\big(A(e2)\cup A(e3)\big)g_n\cap\big(\big(g_n^{-1}\Gamma g_n\big)^1_\infty-\big(g_n^{-1}\Gamma g_n\big)^0_\infty\big),$$
$$\mathfrak{E}_{n,2}=\big\{\delta(S)\in\big(g_n^{-1}\Gamma g_n\big)^0_\infty\cap\big(g_n^{-1}\Gamma g_n\big)^1_\infty;\ -\det(S)\in\big(\mathbb{Q}^\times\big)^2\big\},$$
$$\mathfrak{E}_{n,3}=g_n^{-1}A(e3)'g_n\cap\big(\big(g_n^{-1}\Gamma g_n\big)^0_\infty-\big(g_n^{-1}\Gamma g_n\big)^1_\infty\big).$$

If $k\geqslant 5$, then we have $\int_{g_n^{-1}\cdot F_n}\sum_{\gamma\in\mathfrak{E}_{n,l}}|H_\gamma^{k,j}(Z)|\,dZ<+\infty$ for $l=1,2,3$.

The following lemma is a generalization of [25, Proposition 25].

Lemma 4.16. *Let $k\geqslant 5$ and $Z=X+iY\in\mathfrak{H}_2(\mu,\varsigma_1,\varsigma_2)$. Let $a,a',a'',c,c'\in\mathbb{R}_{>0}$. Then, there exists a constant $C'''_{k,j,\mu,\varsigma_1,\varsigma_2,a,a',a'',c,c'}$ depending only on k, j, μ, ς_1, ς_2, a, a', a'', c, and c' such that*

$$\sum_{s_2'\in a'\mathbb{Z},\,|s_2'|<a''}\ \sum_{t,s_{12}\in\mathbb{Z}}\left|\sum_{s_2\in a\mathbb{Z}}H_\gamma^{k,j}(Z)\right|<C'''_{k,j,\mu,\varsigma_1,\varsigma_2,a,a',a'',c,c'},$$

where

$$\gamma=\begin{pmatrix}1&0&0&cs_{12}\\0&1&cs_{12}&s_2+s_2'\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\c't&1&0&0\\0&0&1&-c't\\0&0&0&1\end{pmatrix}.$$

Proof. The above mentioned sum is equal to

$$\begin{aligned}
&\mathrm{tr}\left[\rho_{k,j}\left(\frac{1}{2i}\left(\begin{pmatrix}1 & 0\\ c't & 1\end{pmatrix} Z \begin{pmatrix}1 & c't\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & cs_{12}\\ cs_{12} & s_2+s_2'\end{pmatrix} - \overline{Z}\right)\begin{pmatrix}1 & -c't\\ 0 & 1\end{pmatrix}\right)^{-1}\rho_{k,j}(Y)\right] \\
&\quad = \sum_{j_1,j_2\geqslant 0,\, j_1+2j_2=j} a_{j_1j_2}\frac{\det(Y)^{k+j_2}(Q(Z;\gamma)+\det(Y)-4^{-1}|c'tz_1+cs_{12}|^2)^{j_1}}{Q(Z;\gamma)^{k+j_1+j_2}} \\
&\quad = \sum_{j_1,j_2\geqslant 0,\, j_1+2j_2=j} a_{j_1j_2}\sum_{p_1,p_2,p_3\geqslant 0,\, p_1+p_2+p_3=j_1}\frac{j_1!}{p_1!p_2!p_3!}\frac{\det(Y)^{k+j_2+p_2}4^{-p_3}|c'tz_1+cs_{12}|^{2p_3}}{Q(Z;\gamma)^{k+j_1+j_2-p_1}},
\end{aligned}$$

where $Q(Z;\gamma)=(2i)^{-1}s_2y_1+(2i)^{-1}s_2'y_1+y_1y_2+(2i)^{-1}y_1((c't)^2z_1+2c'tz_{12})-(y_{12}+(2i)^{-1}(c'tz_1+cs_{12}))^2$. Hence, we have only to prove

$$I=\sum_{t,s_{12}\in\mathbb{Z}}\left|\sum_{s_2\in a\mathbb{Z}}\frac{\det(Y)^{k+p}(c'tz_1+cs_{12})^{2q}}{Q(Z;\gamma)^{k+j_2+p+q}}\right|<\text{constant}.$$

Furthermore, by using the Poisson summation formula for s_2, we have

$$\begin{aligned}
I &= \text{constant}\times\sum_{t,s_{12}\in\mathbb{Z}}\frac{\det(Y)^{k+p}|c'tz_1+cs_{12}|^{2p}}{y_1^{k+j_2+p+q}}\times\left|\sum_{m=1}^{\infty}m^{k+j_2+p+q-1}\right. \\
&\quad\left.\times\exp\left(2\pi a^{-1}mi\left\{s_2'+2iy_2+(c't)^2z_1+2c'tz_{12}-(2iy_1)^{-1}(2iy_{12}+c'tz_1+cs_{12})^2\right\}\right)\right| \\
&\leqslant \text{constant}\times\sum_{r=0}^{2p}\frac{(2p)!}{r!(2p-r)!}\sum_{t,s_{12}\in\mathbb{Z}}\sum_{m=1}^{\infty}\frac{\det(Y)^{k+p}|c'tx_1+cs_{12}|^{2q-r}m^{k+j_2+p+q-1}}{y_1^{k+j_2+p+q-r}} \\
&\quad\times\exp\left(-4\pi a^{-1}m\left\{y_2-y_1^{-1}y_{12}^2+4^{-1}(c't)^2y_1+4^{-1}y_1^{-1}(c'tx_1+cs_{12})^2\right\}\right) \\
&\leqslant \text{constant}\times\sum_{r=0}^{2p}\frac{(2p)!}{r!(2p-r)!}\sum_{t\in\mathbb{Z}}\sum_{m=1}^{\infty}\frac{\det(Y)^{k+p}m^{k+j_2+p-2+r/2}}{y_1^{k+j_2+p-1-r/2}} \\
&\quad\times\exp\left(-4\pi a^{-1}m\left(y_2-y_1^{-1}y_{12}^2\right)-\pi a^{-1}m(c't)^2y_1\right).
\end{aligned}$$

We can reduce this proof to the proof of [25, Proposition 25]. □

4.12. Unipotent contribution of (e-2)

We assume that Assumption 2.1 holds. For $\{h_m\}_{m=1}^{v_0}$ in Assumption 2.1, we may assume $\{h_m\}_{m=1}^{v_0}\subset\{g_n\}_{n=1}^{v}$ and g_n satisfies the equality of Assumption 2.1 for each n. Under Assumption 2.1, we have $(g_n^{-1}\Gamma g_n)_{M_0}=g_n^{-1}\Gamma g_n\cap M_0(\mathbb{Q})$. We use the same notations and conditions for (e-2) as those mentioned in Section 3. We treat the family $[\gamma]_\Gamma$ for $\gamma=g\delta(S)g^{-1}\in A(e2)$, where $S>0$ or $S<0$. It follows from [25, Proof of Theorem 9], Assumption 2.1, Proposition 4.14, (4.7), Lemmas 4.15 and 4.16 that

$$I\left(\bigcup_{\gamma'\in[\gamma]_\Gamma}\{\gamma'\}_\Gamma\right)=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\mathrm{vol}\left(C_0(\gamma;\Gamma)\backslash C_0\left(\gamma;G(\mathbb{R})\right)\right)\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma/\sim}\frac{J_0(\gamma';s)}{[\overline{C}(\gamma;\Gamma):\overline{C}_0(\gamma;\Gamma)]}.$$

We use the following two properties to prove this equality. Let δ be an element of $g_m^{-1}A(e2)g_m \cap (g_m^{-1}\Gamma g_m)^0_\infty$. One is that $\epsilon^{-1}\delta\epsilon$ ($\epsilon \in g_m^{-1}\Gamma g_n$) belongs to $(g_n^{-1}\Gamma g_n)^0_\infty$ if and only if ϵ belongs to $g_m^{-1}\Gamma g_n \cap P_0(\mathbb{Q})$. The other is that $\det(Y)$ is multiplied by a positive constant under the action of $g_m^{-1}\Gamma g_n \cap P_0(\mathbb{Q})$ on $\mathfrak{H}_2$. Note that there exists an element $h \in P_0(\mathbb{Q})$ such that $g_m^{-1}\Gamma g_n = g_m^{-1}\Gamma g_m h$ if $g_m^{-1}\Gamma g_n \cap P_0(\mathbb{Q}) \neq \emptyset$. Therefore, we have only to calculate the integral $J_0(\gamma; s)$. We easily get

$$J_0(\gamma; s) = \int_{Y>0} \operatorname{tr}\bigl\{\rho_{k,j}\bigl(I_2 + (2i)^{-1}Y^{-1}S\bigr)^{-1}\bigr\}\bigl(\det(Y)\bigr)^{-3-s}\,dY.$$

We consider an element $h \in GL(2;\mathbb{R})$ such that $S = \pm h\,{}^t h$. If we transform $Y \mapsto hY\,{}^t h$, then we have

$$\begin{aligned}
J_0(\gamma; s) &= \det(S)^{-s-3/2} \int_{Y>0} \operatorname{tr}\bigl\{\rho_{k,j}\bigl(I_2 \pm (2i)^{-1}Y^{-1}\bigr)^{-1}\bigr\}\bigl(\det(Y)\bigr)^{-3-s}\,dY \\
&= \det(S)^{-s-3/2} \sum_{j_1+2j_2=j,\ j_1,j_2\geqslant 0} a_{j_1,j_2} \int_{Y>0} \frac{(2\det(Y) \pm (2i)^{-1}\operatorname{tr}(Y))^{j_1}\det(Y)^{j_2+k-3-s}}{(\det(Y) \pm (2i)^{-1}\operatorname{tr}(Y) - 4^{-1})^{j_1+j_2+k}}\,dY \\
&= \det(S)^{-s-3/2} \sum_{j_1+2j_2=j,\ j_1,j_2\geqslant 0} a_{j_1,j_2} \times \sum_{p+q+r=j_1} \frac{j_1!}{p!q!r!} \times 2^{-2r} \\
&\quad \times \int_{Y>0} \frac{\det(Y)^{j_2+k-3-s+q}}{(\det(Y) \pm (2i)^{-1}\operatorname{tr}(Y) - 4^{-1})^{j_1+j_2+k-p}}\,dY,
\end{aligned}$$

where we define the constants a_{j_1,j_2} by $\operatorname{tr}(\rho_{0,j}(z)) = \sum_{j_1+2j_2=j,\ j_1,j_2\geqslant 0} a_{j_1,j_2}\operatorname{tr}(z)^{j_1}\det(z)^{j_2}$ for $z \in M(2;\mathbb{C})$. It follows from [25, Proof of Theorem 9] that

$$\begin{aligned}
J_0(\gamma; s) &= \det(S)^{-s-3/2} \times 2^{3+2s} \\
&\quad \times \sum_{j_1+2j_2=j,\ j_1,j_2\geqslant 0} a_{j_1,j_2} \sum_{p+q+r=j_1} \frac{j_1!}{p!q!r!} \times \exp\bigl(\pm(\pi/2)i(3+2s+2r)\bigr) \\
&\quad \times \frac{\Gamma(k''+1-s)\Gamma(1/2)\Gamma(k'-k''-3/2+s)\Gamma(k''+3/2-s)\Gamma(k'-k''-2+s)}{\Gamma(k')\Gamma(k'-1/2)},
\end{aligned}$$

where $k' = j_1 + j_2 + k - p$ and $k'' = j_2 + k - 3 + q$. Since $\Gamma(s)$ is continuous, we have $\Gamma(a_r \pm s) = \Gamma(a_r) + o(s)$ uniformly for a finite set $\{a_r\}_r$ ($a_r > 0$). Hence, we get

$$\begin{aligned}
J_0(\gamma; s) &= \det(S)^{-s}\exp(\pm s\pi i) \\
&\quad \times \Biggl\{2^3\det(S)^{-3/2} \sum_{j_1+2j_2=j,\ j_1,j_2\geqslant 0} a_{j_1,j_2} \sum_{p+q+r=j_1} \frac{j_1!}{p!q!r!}\exp\bigl(\pm(\pi/2)i(3+2r)\bigr) \\
&\quad \times \frac{\Gamma(k''+1)\Gamma(1/2)\Gamma(k'-k''-3/2)\Gamma(k''+3/2)\Gamma(k'-k''-2)}{\Gamma(k')\Gamma(k'-1/2)} + o(s)\Biggr\}.
\end{aligned}$$

From this we have $J_0(\gamma; s) = \{J_0(\gamma; 0) + o(s)\} \times \det(S)^{-s}\exp(\pm s\pi i)$. Thus, we obtain the result for (e-2), given in Section 3, by using Lemma 4.12.

4.13. Unipotent contribution of (e-3)

First, by [25, Proof of Theorem 9], Proposition 4.14, Lemmas 4.15 and 4.16, and the arguments in Sections 4.7 and 4.12, we find that the contribution of $A(e3)-A(e3)'$ is zero.

Next, we treat the family $[\gamma]_\Gamma$ for $\gamma=g\delta(S)g^{-1}\in A(e3)'$, where S is indefinite and $-\det(S)\in(\mathbb{Q}^\times)^2$. If $G(\mathbb{Q})$ is not split, then such elements do not appear in Γ. We may set $g_m=g$ for a certain m, and $[\gamma]_\Gamma/{\sim}=\{g_m\delta(S)g_m^{-1};\ S\in\bigcup_{u=1}^t\mathcal{L}_u\}$, where $L'_{2,u}(s_{12})$ is a certain subset of $L_{2,u}(s_{12})$ and $\mathcal{L}_u=\{\beta_u\left(\begin{smallmatrix}0&s_{12}\\ s_{12}&s_2\end{smallmatrix}\right){}^t\beta_u\in L';\ s_{12}\in L_{1,u},\ s_2\in L'_{2,u}(s_{12})\}$. It follows from Proposition 4.14 and Lemmas 4.15 and 4.16 that

$$I\Bigl(\bigcup_{\gamma'\in[\gamma]_\Gamma/\sim}\{\gamma'\}_\Gamma\Bigr)=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\sum_{n=1}^{v}\sum_{u=1}^{t}\int_{g_n^{-1}\cdot F_n}\sum_{S\in\mathcal{L}_u}\sum_{\delta'\in g_n^{-1}\{g_m\delta(S)g_m^{-1}\}_\Gamma g_n}H_{\delta'}(Z)\,dZ.$$

Hence, we have only to calculate the contribution for $u=1$. We can assume $\beta_1=I_2$ without loss of generality. Hence, we consider the contribution of $\bigcup_{S\in\mathcal{L}_1}\{g_m\delta(S)g_m^{-1}\}_\Gamma$, where $\mathcal{L}_1=\{\left(\begin{smallmatrix}0&s_{12}\\ s_{12}&s_2\end{smallmatrix}\right)\in L';\ s_{12}\in L_{1,1},\ s_2\in L'_{2,1}(s_{12})\}$. We set

$$\delta'(s_{12},s_2)=\begin{pmatrix}1&0&0&s_{12}\\0&1&s_{12}&s_2\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad\text{and}\quad\eta(w)=\begin{pmatrix}1&0&0&0\\w&1&0&0\\0&0&1&-w\\0&0&0&1\end{pmatrix}.$$

For an integer N', we set

$$\Delta'=\bigl\{\delta'(s_{12},s_2)\eta(w);\ w,s_{12}\in N'^{-1}\mathbb{Z},\ s_2\in N'^{-2}\mathbb{Z},\ w\neq0\bigr\}.$$

Then, there exists an integer N' such that Δ' contains the set of all element of $\bigcup_{n=1}^{v}g_n^{-1}\Gamma g_n$, which are of the form $\delta'(s_{12},s_2)\eta(w)$. Fix such an N'.

Lemma 4.17. *(See [25, Proposition 18].) For any $h\in\Delta'$, there exists an element $\xi\in Sp(2;\mathbb{Z})\cap P_1(\mathbb{Q})$ such that $\xi^{-1}h\xi=\delta'(s_{12},s_2)$ for certain s_{12} ($s_{12}\neq0$), s_2.*

Lemma 4.18. *If $U\left(\begin{smallmatrix}0&s_{12}\\ s_{12}&s_2\end{smallmatrix}\right){}^tU=\left(\begin{smallmatrix}0&s'_{12}\\ s'_{12}&s'_2\end{smallmatrix}\right)$ ($s_{12}\neq0$) for $U\in GL(2;\mathbb{R})$, then $s'_{12}=\pm\det(U)s_{12}$.*

For an element $s_{12}\in L_{1,1}$ ($s_{12}\neq0$), let $\Delta_{n,s_{12}}$ denote the set of all elements γ in $g_n^{-1}\Gamma g_n\cap\Delta'$ such that there exist $\xi\in g_m^{-1}\Gamma g_n$ and $s_2\in L_{2,1}(s_{12})$, which satisfy $\xi\gamma\xi^{-1}=\delta'(s_{12},s_2)\in g_m^{-1}\Gamma g_m$. For $a\neq0$, $b\neq0$, we set

$$\phi_1(a)=\begin{pmatrix}a&0&0&0\\0&a^{-1}&0&0\\0&0&a^{-1}&0\\0&0&0&a\end{pmatrix},\qquad\phi_2(a)=\begin{pmatrix}0&a&0&0\\a^{-1}&0&0&0\\0&0&0&a^{-1}\\0&0&a&0\end{pmatrix},$$

$$\phi_3(b)=\begin{pmatrix}b&0&0&0\\0&b&0&0\\0&0&b^{-1}&0\\0&0&0&b^{-1}\end{pmatrix}.$$

Lemma 4.19. *For any $\xi \in g_m^{-1}\Gamma g_n$ and $\gamma \in \Delta_{n,s_{12}}$ such that $\xi\gamma\xi^{-1} = \delta'(s_{12}, s_2) \in g_m^{-1}\Gamma g_m$, we can express ξ as $\eta(-2^{-1}s_{12}^{-1}s_2)\xi = \epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5$, where $\epsilon_1 = \phi_1(a)$ or $\phi_2(a)$, $\epsilon_2 = \phi_3(b)$, $\epsilon_3 = \delta(S)$, $\epsilon_4 = \eta(t')$, and $\epsilon_5 \in Sp(2;\mathbb{Z}) \cap P_1(\mathbb{Q})$. If we fix s_{12}, then there exists a finite subset $\mathcal{J}$ in $\mathbb{R}^2$ such that the pair (a, b) belongs to $\mathcal{J}$ for any such ϵ_1 and ϵ_2.*

Proof. We have $\eta(-2^{-1}s_{12}^{-1}s_2)\xi\gamma\xi^{-1}\eta(2^{-1}s_{12}^{-1}s_2) = \delta'(s_{12}, 0)$. By $\gamma \in \Delta_{n,s_{12}} \subset \Delta'$ and Lemma 4.17, there exists $\epsilon_5 \in Sp(2;\mathbb{Z}) \cap P_1(\mathbb{Q})$ such that $\epsilon_5\gamma\epsilon_5^{-1} = \delta'(s'_{12}, s'_2)$. Since $\gamma = \epsilon_5^{-1}\delta'(s'_{12}, s'_2)\epsilon_5 = \xi^{-1}\delta'(s_{12}, s_2)\xi$, we have $\epsilon_5\xi^{-1}\delta'(s_{12}, s_2)\xi\epsilon_5^{-1} = \delta'(s'_{12}, s'_2)$. Hence, we can set $\epsilon_5\xi^{-1} = \left(\begin{smallmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} I_2 & T \\ 0 & I_2 \end{smallmatrix}\right)$. It follows from Lemma 4.18 that $s'_{12} = \pm\det(U)s_{12}$. We set $\epsilon_4 = \eta(-s'_2(2|\det(U)|s_{12})^{-1})$ and $\epsilon_2 = \phi_3(|\det(U)|^{-1/2})$. If $s'_{12} = |\det(U)|s_{12}$, then we get $\delta'(s_{12}, 0) = \epsilon_2\epsilon_4\epsilon_5\gamma\epsilon_5^{-1}\epsilon_4^{-1}\epsilon_2^{-1}$. If $s'_{12} = -|\det(U)|s_{12}$, then we get $\delta'(s_{12}, 0) = \epsilon_2\epsilon_4\epsilon'_5\gamma\epsilon_5'^{-1}\epsilon_4^{-1}\epsilon_2^{-1}$ for $\epsilon'_5 = \mathrm{diag}(1, -1, 1, -1)\epsilon_5$. Since $\mathrm{diag}(1, -1, 1, -1) \in Sp(2;\mathbb{Z}) \cap P_1(\mathbb{Q})$, we may replace ϵ'_5 with ϵ_5. Since

$$\begin{aligned}\epsilon_2\epsilon_4\epsilon_5\gamma\epsilon_5^{-1}\epsilon_4^{-1}\epsilon_2^{-1} &= \delta'(s_{12}, 0)\\ &= \eta\bigl(-2^{-1}s_{12}^{-1}s_2\bigr)\xi\gamma\xi^{-1}\eta\bigl(2^{-1}s_{12}^{-1}s_2\bigr),\end{aligned}$$

there exists an element $h \in C(\delta'(s_{12}, 0); G(\mathbb{R}))$ such that $\eta(-2^{-1}s_{12}^{-1}s_2)\xi = h \times \epsilon_2\epsilon_4\epsilon_5$. Thus, we get the first assertion of this lemma. Since $h \times \epsilon_2$ belongs to a certain lattice in $M(4;\mathbb{Q})$, the second assertion follows. □

Let $Z = X + iY \in \Sigma$ and $y_2 - y_1^{-1}y_{12}^2 \geqslant c(s_{12})$, where $c(s_{12})$ is a constant. We set $Z' = X' + iY' = \epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5 \cdot Z$ and $Y' = \left(\begin{smallmatrix} y'_1 & y'_{12} \\ y'_{12} & y'_2 \end{smallmatrix}\right)$ for $\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5$ in Lemma 4.19. If $\epsilon_1 = \phi_1(a)$, then we have $y'_1 \leqslant a^2b^2\alpha^{-1}\beta$ and $y'_2 - y_1'^{-1}y_{12}'^2 \geqslant a^{-2}b^2c(s_{12})$. The constants α and β have been used for the defining Σ (cf. Section 4.2). If $\epsilon_1 = \phi_2(a)$, then we have $y'_2 \leqslant a^{-2}b^2\alpha^{-1}\beta$ and $y'_1 - y_2'^{-1}y_{12}'^2 \geqslant a^2b^2c(s_{12})$. Let $c(s_{12}) = \mathrm{Max}_{\gamma\in\bigcup_{n=1}^{\nu}\Delta_{n,s_{12}}}\{a^4\alpha^{-1}\beta,\ a^{-4}\alpha^{-1}\beta\}$. Then, the domain $\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5 \cdot \{Z \in \Sigma;\ y_2 - y_1^{-1}y_{12}^2 \geqslant c(s_{12})\}$ is contained in $\{y_2 \geqslant y_1\}$ (resp. $\{y_1 \geqslant y_2\}$) if $\epsilon_1 = \phi_1(a)$ (resp. $\epsilon_1 = \phi_2(a)$). Therefore, we can use the argument in [25, Proof of Theorem 9] for the calculation below. We note that there exists only a finite number of $g_n^{-1}\Gamma g_n$-conjugacy classes in $g_n^{-1}\Gamma g_n$ which have intersections with $\Delta_{n,s_{12}}$.

Fix s_{12} and s_2. Let $c \in \mathbb{R}$ and $t' = 1$ or 2. Let $\mathfrak{T}_{n,s_{12},s_2,c,t'}$ denote the subset of $C_0(\delta'(s_{12}, s_2); g_m^{-1}\Gamma g_m)\backslash g_m^{-1}\Gamma g_n$, which consists of all elements ξ such that $\xi^{-1}\delta'(s_{12}, s_2)\xi \in \Delta_{n,s_{12}}$ and $y_1^{-1}\det(Y) \mapsto c \cdot y_{t'}^{-1}\det(Y)$ via the action of ξ. Note that we can apply Lemma 4.16 to the set $\{\xi^{-1}\delta'(s_{12}, s_2)\xi;\ \xi \in \mathfrak{T}_{n,s_{12},s_2,c,t'}\}$. Therefore, from Lemmas 4.16 and 4.19, we deduce

$$\begin{aligned}&\lim_{s\to+0}\sum_{n=1}^{\nu}\int_{g_n^{-1}\cdot F_n}\sum_{\omega\in\Delta_{n,s_{12}}\cap g_n^{-1}\{g_m\delta'(s_{12},s_2)g_m^{-1}\}_\Gamma g_n} H_\omega^{k,j}(Z)\bigl(y_1^{-1}\det(Y)\bigr)^{-s}\,dZ\\ &\quad= \bigl[\overline{C}\bigl(g_m\delta'(s_{12}, s_2)g_m^{-1};\Gamma\bigr) : \overline{C}_0\bigl(g_m\delta'(s_{12}, s_2)g_m^{-1};\Gamma\bigr)\bigr]^{-1}\\ &\qquad\times\lim_{s\to+0}\sum_{t'=1,2}\sum_{c}\sum_{\xi\in\mathfrak{T}_{n,s_{12},s_2,c,t'}}\int_{\xi\cdot g_n^{-1}\cdot F_n} H_{\delta'(s_{12},s_2)}^{k,j}(Z)\bigl(y_{t'}^{-1}\det(Y)\bigr)^{-s}\,dZ,\end{aligned}$$

where c runs over a finite set in $\mathbb{R}$. It follows from the above mentioned arguments, Proposition 4.14, Lemmas 4.15 and 4.16, and the argument in [25, Proof of Theorem 9] that

$$\begin{aligned}
&I\Bigl(\bigcup_{S\in L_1}\{g_m\delta(S)g_m^{-1}\}_\Gamma\Bigr)\\
&=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\sum_{n=1}^{v}\sum_{s_{12}\in L_{1,1}}\sum_{s_2\in L'_{2,1}(s_{12})}\Biggl\{\sum_{\omega\in g_n^{-1}\{g_m\delta'(s_{12},s_2)g_m^{-1}\}_\Gamma g_n-\Delta_{n,s_{12}}}\int_{g_n^{-1}\cdot F_n}H_\omega^{k,j}(Z)\,dZ\\
&\quad+\sum_{\omega\in\Delta_{n,s_{12}}\cap g_n^{-1}\{g_m\delta'(s_{12},s_2)g_m^{-1}\}_\Gamma g_n}\int_{(g_n^{-1}\cdot F_n)\cap\{y_1^{-1}\det(Y)<c(s_{12})\}}H_\omega^{k,j}(Z)\,dZ\\
&\quad+\lim_{s\to+0}\sum_{\omega\in\Delta_{n,s_{12}}\cap g_n^{-1}\{g_m\delta'(s_{12},s_2)g_m^{-1}\}_\Gamma g_n}\int_{(g_n^{-1}\cdot F_n)\cap\{y_1^{-1}\det(Y)\geqslant c(s_{12})\}}\frac{H_\omega^{k,j}(Z)}{(y_1^{-1}\det(Y))^s}\,dZ\Biggr\}\\
&=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\mathrm{vol}\bigl(C_0(\gamma;\Gamma)\backslash C_0\bigl(\gamma;G(\mathbb{R})\bigr)\bigr)\sum_{s_{12}\in L_{1,1}}\frac{1}{s_{12}^3}\\
&\quad\times\sum_{s_2\in L'_{2,1}(s_{12})}\bigl[\bar{C}\bigl(g_m\delta'(s_{12},s_2)g_m^{-1};\Gamma\bigr):\bar{C}_0\bigl(g_m\delta'(s_{12},s_2)g_m^{-1};\Gamma\bigr)\bigr]^{-1}\times\lim_{s\to+0}J_0\bigl(\delta'(1,0);s\bigr).
\end{aligned}$$

From this we have

$$I\Bigl(\bigcup_{\gamma'\in[\gamma]_\Gamma}\{\gamma'\}_\Gamma\Bigr)=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\mathrm{vol}\bigl(C_0(\gamma;\Gamma)\backslash C_0\bigl(\gamma;G(\mathbb{R})\bigr)\bigr)\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma/\sim}\frac{J_0(\gamma';s)}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]}.$$

Hence, we have only to calculate the integral $J_0(\delta'(1,0);s)$. First, we have

$$\begin{aligned}
&J_0\bigl(\delta'(1,0);s\bigr)\\
&=2\int_{0<y_1\leqslant y_2}\mathrm{tr}\Biggl\{\rho_{k,j}\Biggl(I_2+(2i)^{-1}Y^{-1}\begin{pmatrix}0&1\\1&0\end{pmatrix}\Biggr)^{-1}\Biggr\}\det(Y)^{-3}\bigl(y_1^{-1}\det(Y)\bigr)^{-s}\,dY\\
&=2\sum_{j_1+2j_2=j,\ j_1,j_2\geqslant0}a_{j_1,j_2}\int_{0<y_1\leqslant y_2}\frac{(2y_1y_2-2y_{12}^2+iy_{12})^{j_1}\det(Y)^{j_2+k-3-s}y_1^s}{(y_1y_2-y_{12}^2+iy_{12}+1/4)^{j_1+j_2+k}}\,dY\\
&=2\sum_{j_1+2j_2=j,\ j_1,j_2\geqslant0}a_{j_1,j_2}\sum_{p+q+r=j_1}\frac{j_1!(-4)^{-r}}{p!q!r!}\int_{0<y_1\leqslant y_2}\frac{\det(Y)^{j_2+k-3-s+q}y_1^s}{(y_1y_2-y_{12}^2+iy_{12}+1/4)^{j_1+j_2+k-p}}\,dY.
\end{aligned}$$

By an argument similar to that in [25, Proof of Theorem 9], we have

$$\begin{aligned}
J_0\bigl(\delta'(1,0);s\bigr)=-2&\sum_{j_1+2j_2=j,\ j_1,j_2\geqslant0}a_{j_1,j_2}\sum_{p+q+r=j_1}\frac{j_1!}{p!q!r!}(-4)^{-r}\times2^{2k'-2k''-4}\pi^{1/2}\\
&\times\frac{\Gamma(k''+1)\Gamma(k'-k''-3/2)\Gamma(k''+3/2)\Gamma(k'-k''-2)}{\Gamma(k')\Gamma(k'-1/2)}+o(s),
\end{aligned}$$

where $k' = j_1 + j_2 + k - p$ and $k'' = j_2 + k - 3 + q$. Therefore, it follows from the calculation of $J_0(\delta(I_2); 0)$ in Section 4.12 that

$$J_0\big(\delta'(1,0); s\big) = -J_0\big(\delta(I_2); 0\big) \times i + o(s) = -c_{k,j}^{-1} 2^{-3} \pi^{-2} (j+1) + o(s).$$

Thus, we obtain the result for (e-3) given in Section 3.

4.14. Quasi-unipotent contribution of (f-1)

If $G(\mathbb{Q})$ is not split, then the elements of type (f-1) do not appear in Γ.

Lemma 4.20. *Let $k \geqslant 5$. We have $\int_{g_n^{-1} \cdot F_n} \sum_{\gamma \in g_n^{-1} A(f1) g_n \cap \mathfrak{A}_{n,0}} |H_\gamma^{k,j}(Z)|\, dZ < +\infty$.*

Proof. By Lemma 4.2, we may assume $j = 0$. Let $\gamma = \left(\begin{smallmatrix} I_2 & S \\ 0 & I_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{smallmatrix}\right) \in \mathfrak{A}_{n,0}$, where the eigenvalues of U are 1, -1, $\mathrm{rank}(S) = 1$. Let $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ $(b \neq 0)$.

First, we assume $a \neq \pm 1$. If we set $V = \left(\begin{smallmatrix} 1+a & b \\ 1-a & -b \end{smallmatrix}\right)$, then we have $VUV^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = I'$. We note that $a + d = 0$ and $ad - bc = -1$. Hence, we have

$$\begin{pmatrix} V & 0 \\ 0 & {}^tV^{-1} \end{pmatrix} \gamma \begin{pmatrix} V^{-1} & 0 \\ 0 & {}^tV \end{pmatrix} = \begin{pmatrix} I_2 & VS\,{}^tV \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} I' & 0 \\ 0 & I' \end{pmatrix}, \qquad VS\,{}^tV = \begin{pmatrix} s_1' & s_{12}' \\ s_{12}' & s_2' \end{pmatrix},$$

$$\begin{aligned} s_1' &= (1+a)^2 s_1 + 2b(1+a) s_{12} + b^2 s_2, \\ s_{12}' &= \big(1 - a^2\big) s_1 - 2ab s_{12} - b^2 s_2, \\ s_2' &= (1-a)^2 s_1 - 2b(1-a) s_{12} + b^2 s_2. \end{aligned}$$

From this we have $s_1' = 0$ or $s_2' = 0$. Therefore, s_2 is determined by s_1, s_{12}, a, and b. Hence, we can reduce this proof to the proof of [25, Theorem 4].

Next, we assume $a = \pm 1$. We easily find $c = 0$ because $ad = -1$ and $ad - bc = -1$. If we replace $V = \left(\begin{smallmatrix} 1 & -2^{-1}b \\ 0 & 1 \end{smallmatrix}\right)$, then we get $VS\,{}^tV = \left(\begin{smallmatrix} s_1 - bs_{12} + 4^{-1}b^2 s_2 & s_{12} - 2^{-1}bs_2 \\ s_{12} - 2^{-1}bs_2 & s_2 \end{smallmatrix}\right)$. Hence, we can reduce this case also to that of [25]. $\square$

Lemma 4.21. *Let $k \geqslant 5$. For*

$$\mathfrak{E}_{n,4} = \left\{ \gamma = \pm \begin{pmatrix} a & 0 & b & * \\ * & 1 & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathfrak{A}_{n,1};\ \gamma \in g_n^{-1} A(f1) g_n,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq -I_2 \right\},$$

we have $\int_{g_n^{-1} \cdot F_n} \sum_{\gamma \in \mathfrak{E}_{n,4}} |H_\gamma^{k,j}(Z)|\, dZ < +\infty$.

Proof. Let

$$\gamma = \pm \begin{pmatrix} 1 & 0 & 0 & s_{12} \\ 0 & 1 & s_{12} & s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathfrak{E}_{n,4}.$$

Then, we easily find that s_2 is determined by s_{12}, a_3, a, b, c, and d, since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is unipotent. Hence, we can reduce this proof to the proof of [25, Theorem 4]. $\square$

Let γ be an element of $A(f1)$, and $\mathfrak{E}_{n,5} = g_n^{-1} A(f1) g_n \cap \mathfrak{A}_{n,1} - \mathfrak{E}_{n,4}$. It follows from Proposition 4.14 and Lemmas 4.20 and 4.21 that

$$I\Big(\bigcup_{\gamma' \in [\gamma]_\Gamma} \{\gamma'\}_\Gamma\Big) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \Bigg\{ \sum_{n=1}^{v} \sum_{\delta \in \bigcup_{\gamma' \in [\gamma]_\Gamma} \{\gamma'\}_\Gamma,\, \delta \notin \mathfrak{E}_{n,5}} \int_{g_n^{-1} \cdot F_n} H_\delta^{k,j}(Z)\, dZ$$

$$+ \lim_{s \to +0} \sum_{n=1}^{v} \sum_{\delta \in \bigcup_{\gamma' \in [\gamma]_\Gamma} \{\gamma'\}_\Gamma,\, \delta \in \mathfrak{E}_{n,5}} \int_{g_n^{-1} \cdot F_n} H_\delta^{k,j}(Z) \big(y_1^{-1} \det(Y)\big)^{-s}\, dZ \Bigg\}.$$

We may assume that γ belongs to $\Gamma \cap (g_m \mathfrak{E}_{m,5} g_m^{-1})$ for a certain m. For any element δ of $\mathfrak{E}_{m,5}$, we find that $\epsilon^{-1}\delta\epsilon \in \mathfrak{E}_{n,5}$ ($\epsilon \in g_m^{-1} \Gamma g_n$) if and only if $\epsilon \in g_m^{-1} \Gamma g_n \cap P_1(\mathbb{Q})$. Hence, by using (4.7), for $g_m^{-1} \gamma g_m \in \mathfrak{E}_{m,5}$, we get

$$I\Big(\bigcup_{\gamma' \in [\gamma]_\Gamma} \{\gamma'\}_\Gamma\Big)$$

$$= \frac{c_{k,j}}{\sharp(Z(\Gamma))} \Bigg\{ \sum_{\gamma' \in [\gamma]_\Gamma} \sum_{n=1}^{v} \sum_{\delta \in g_m^{-1} C(\gamma';\Gamma) g_m \backslash \mathfrak{Q}_{n,4,\gamma'}} \int_{\delta g_n^{-1} \cdot F_n} H^{k,j}_{g_m^{-1}\gamma' g_m}(Z)\, dZ$$

$$+ \lim_{s \to +0} \sum_{\gamma' \in [\gamma]_\Gamma} \sum_{n=1}^{v} \sum_{\delta \in g_m^{-1} C(\gamma';\Gamma) g_m \backslash \mathfrak{Q}_{n,5,\gamma'}} \int_{\delta g_n^{-1} \cdot F_n} H^{k,j}_{g_m^{-1}\gamma' g_m}(Z) \big(y_1^{-1} \det(Y)\big)^{-s}\, dZ \Bigg\},$$

where $\mathfrak{Q}_{n,5,\gamma'} = \{\delta \in g_m^{-1} \Gamma g_n;\ \delta^{-1} g_m^{-1} \gamma' g_m \delta \in \mathfrak{E}_{n,5}\}$ and $\mathfrak{Q}_{n,4,\gamma'} = g_m^{-1} \Gamma g_n - \mathfrak{Q}_{n,5,\gamma'}$. We get the following by using the argument in [25, p. 240].

$$I\Big(\bigcup_{\gamma' \in [\gamma]_\Gamma} \{\gamma'\}_\Gamma\Big) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \sum_{[\gamma]_\Gamma} \frac{\mathrm{vol}(C_0(\gamma;\Gamma) \backslash C_0(\gamma; G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma) : \bar{C}_0(\gamma;\Gamma)]} \lim_{s \to +0} \sum_{\gamma' \in [\gamma]_\Gamma} J_0\big(\gamma'; s\big).$$

Hence, we have only to calculate the integral $J_0(\gamma; s)$. From Hashimoto's calculation [12, p. 447], we deduce

$$J_0(\gamma; s) = \sum_{j_1 + 2j_2 = j,\ j_1, j_2 \geqslant 0} a_{j_1,j_2} \int_0^\infty \int_0^\infty \frac{(-1)^{j_1+j_2+k}((2i)^{-1}\lambda)^{j_1} v^{2j_2+2k-3-2s} t}{(v^2 + v^2 t^2 - (2i)^{-1}\lambda)^{j_1+j_2+k}}\, dt\, dv,$$

where γ is the same form as that in (f-1) of Section 3. Therefore, we can evaluate this integral by using the argument in Section 4.12, Lemma 4.12, and Hashimoto's calculations [12, p. 447].

4.15. Quasi-unipotent contribution of (f-2)

If $G(\mathbb{Q})$ is not split, then the elements of type (f-2) do not appear in Γ. Let γ be an element of $A(f2)$, which satisfies

$$\gamma = g \begin{pmatrix} 1 & 0 & \lambda_1 & 0 \\ 0 & -1 & 0 & \lambda_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} g^{-1}$$

for certain $g \in G(\mathbb{R})$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 \neq 0$. As a coordinate of $C_0(g^{-1}\gamma g; G(\mathbb{R}))\backslash\mathfrak{H}_2$, we take $\{\left(\begin{smallmatrix} y_1 & x_{12}+iy_{12} \\ x_{12}+iy_{12} & y_2 \end{smallmatrix}\right) \in \mathfrak{H}_2;\ Y>0,\ x_{12} \in \mathbb{R}\}$. Hence, we have $\hat{Z} = \left(\begin{smallmatrix} 0 & x_{12} \\ x_{12} & 0 \end{smallmatrix}\right) + iY$ and $d\hat{Z} = \det(Y)^{-3}\, dx_{12}\, dY$. On the coordinate, we have

$$\begin{aligned} H^{k,j}_{g^{-1}\gamma g}(\hat{Z}) &= \mathrm{tr}\left[\rho_{k,j}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} y_1 + (2i)^{-1}\lambda_1 & ix_{12} \\ ix_{12} & y_2 - (2i)^{-1}\lambda_2 \end{pmatrix}^{-1} Y\right\}\right] \\ &= \sum_{j_1+2j_2=j,\ j_1,j_2 \geqslant 0} a_{j_1,j_2} \frac{(-1)^{j_1+j_2+k}((2i)^{-1}(y_1\lambda_2 + y_2\lambda_1))^{j_1} \det(Y)^{j_2+k}}{(x_{12}^2 + (y_1 + (2i)^{-1}\lambda_1)(y_2 - (2i)^{-1}\lambda_2))^{j_1+j_2+k}}. \end{aligned}$$

Fix positive constants c_1 and c_2. We set

$$\gamma' = g \begin{pmatrix} 1 & 0 & \lambda_1' & 0 \\ 0 & -1 & 0 & \lambda_2' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} g^{-1}.$$

Using Lemma 4.2 and

$$\left|H^{k,0}_{g^{-1}\gamma g}(\hat{Z})\right| < \text{constant} \times \det(Y)^k \left(y_1^2 + \lambda_1^2\right)^{-k/2} \left(y_2^2 + \left(\lambda_2 + \left(x_{12}^2\lambda_1\right)\left(y_1^2 + \lambda_1^2\right)^{-1}\right)^2\right)^{-k/2},$$

for $-1/2 < \mu < k - 3/2$, we have

$$\int\limits_{Y>0} \int\limits_{\mathbb{R}} \left|H^{k,j}_{g^{-1}\gamma g}(\hat{Z})\right| \det(Y)^{-3-\mu}\, dx_{12}\, dY < \text{constant} \times |\lambda_1\lambda_2|^{-1-\mu}.$$

Therefore, for a small $\mu \geqslant 0$, we have

$$\sum_{\gamma' \in [\gamma]_\Gamma} \int\limits_{Y>0,\, y_1<c_1,\, y_2<c_2} \int\limits_{\mathbb{R}} \left|H^{k,j}_{g^{-1}\gamma' g}(\hat{Z})\right| \det(Y)^{-\mu}\, d\hat{Z} < +\infty.$$

For $\lambda_1' = \lambda_3' + \lambda_4'$ and a small $\mu \geqslant 0$, by the Poisson summation formula for λ_3', we have

$$\sum_{\lambda_4' \in b_3\mathbb{Z},\, |\lambda_4'|<b_4} \sum_{\lambda_2' \in b_2\mathbb{Z}-\{0\}} \int\limits_{Y>0,\, y_1>c_1,\, y_2<c_2} \int\limits_{\mathbb{R}} \left|\sum_{\lambda_3' \in b_1\mathbb{Z}} H^{k,j}_{g^{-1}\gamma' g}(\hat{Z})\right| \det(Y)^{-\mu} d\hat{Z} < +\infty.$$

In case of $\lambda_1' = 0$, we have $|H_{g^{-1}\gamma' g}(\hat{Z})| < \text{constant} \times |x_{12}^4 + y_1^2 y_2^2 + 4^{-1} y_1^2 \lambda_2'^2|^{-k/2}$. Hence, for $\lambda_1' = 0$, we also have

$$\sum_{\lambda_2' \in b_2\mathbb{Z}-\{0\}} \int\limits_{Y>0,\, y_1>c_1,\, y_2<c_2} \int\limits_{\mathbb{R}} \left|H^{k,j}_{g^{-1}\gamma' g}(\hat{Z})\right| \det(Y)^{-\mu}\, d\hat{Z} < +\infty.$$

For $\lambda_1' = \lambda_2' = 0$, we have $\int_{Y>0,\, y_1>c_1,\, y_2>c_2} |H^{k,j}_{g^{-1}\gamma' g}(\hat{Z})| \det(Y)^{-\mu}\, d\hat{Z} < +\infty$ ($\mu > -1$). For $\lambda_2' = 0$, by the Poisson summation formula for λ_1', we have $\int_{Y>0,\, y_1>c_1,\, y_2>c_2} \int_{\mathbb{R}} |\sum_{\lambda_1'} H^{k,j}_{g^{-1}\gamma' g}(\hat{Z})|\, d\hat{Z} < +\infty$. It follows from these facts and the Poisson summation formula in the proof of Lemma 4.4 that

$$\int\limits_{Y>0,\, y_1>c_1,\, y_2>c_2} \int\limits_{SM(2;\mathbb{R})/\mathfrak{L}} \left| \sum_{\gamma'\in[\gamma]_\Gamma} \sum_{\delta\in C_0(\gamma;\Gamma)\backslash gN_0(\mathbb{R})g^{-1}\cap\Gamma} H^{k,j}_{g^{-1}\gamma' g}\bigl(g^{-1}\delta g\cdot Z\bigr)\right| dZ<\infty,$$

where $\mathfrak{L}$ is the lattice such that $N_0(\mathbb{R})\cap g^{-1}\Gamma g=\{\delta(T);\ T\in\mathfrak{L}\}$. We also have

$$\sum_{\gamma'\in[\gamma]_\Gamma}\int\limits_{\mathbb{R}} \bigl|H^{k,j}_{g^{-1}\gamma' g}(\hat{Z})\bigr|\,dx_{12}<+\infty$$

by simple calculation. Therefore, by these inequalities and Proposition 4.14, we obtain

$$\begin{aligned} &I\Bigl(\bigcup_{\gamma'\in[\gamma]_\Gamma}\{\gamma'\}_\Gamma\Bigr)\\ &\quad=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\,\frac{\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]}\int\limits_{Y>0}\sum_{\gamma'\in[\gamma]_\Gamma}\int\limits_{\mathbb{R}} H^{k,j}_{g^{-1}\gamma' g}(\hat{Z})\,dx_{12}\det(Y)^{-3}\,dY. \end{aligned}$$

Lemma 4.22. *Let $k\geqslant 5$. For a family $[\gamma]_\Gamma$ of type* (f-2) *and a small $s\geqslant 0$, we have*

$$\int\limits_{Y>0}\left|\sum_{\gamma'\in[\gamma]_\Gamma}\int\limits_{\mathbb{R}} H^{k,j}_{g^{-1}\gamma' g}(Z)\,dx_{12}\right|\det(Y)^{-3-s}\,dY<+\infty.$$

Proof. By direct calculation, we have

$$\begin{aligned} &\int\limits_{\mathbb{R}} H^{k,j}_{g^{-1}\gamma' g}(Z)\det(Y)^{-3-s}\,dx_{12}\\ &\quad=\sum_{j_1+2j_2=j,\, j_1,j_2\geqslant 0} a_{j_1,j_2}\int_{-\infty}^{\infty}\frac{(-1)^{j_1+j_2+k}((2i)^{-1}(y_1\lambda_2+y_2\lambda_1))^{j_1}\det(Y)^{j_2+k-3-s}}{(x_{12}^2+(y_1+(2i)^{-1}\lambda_1)(y_2-(2i)^{-1}\lambda_2))^{j_1+j_2+k}}\,dx_{12}\\ &\quad=\sum_{j_1+2j_2=j,\, j_1,j_2\geqslant 0} a_{j_1,j_2}\frac{\Gamma(1/2)\Gamma(j_1+j_2+k-1/2)}{\Gamma(j_1+j_2+k)}\\ &\qquad\times\frac{(-1)^{j_1+j_2+k}((2i)^{-1}(y_1\lambda_2+y_2\lambda_1))^{j_1}\det(Y)^{j_2+k-3-s}}{(y_1+(2i)^{-1}\lambda_1)^{j_1+j_2+k-1/2}(y_2-(2i)^{-1}\lambda_2)^{j_1+j_2+k-1/2}}. \end{aligned}$$

By substituting $(2i)^{-1}(y_1\lambda_2+y_2\lambda_1)=y_2(y_1+(2i)^{-1}\lambda_1)-y_1(y_2-(2i)^{-1}\lambda_2)$, we have

$$\begin{aligned} &=\sum_{j_1+2j_2=j,\, j_1,j_2\geqslant 0} a_{j_1,j_2}\frac{\Gamma(1/2)\Gamma(j_1+j_2+k-1/2)}{\Gamma(j_1+j_2+k)}\\ &\quad\times\sum_{p+q=j_1}\frac{j_1!}{p!q!}\,\frac{(-1)^{j_1+j_2+k}y_2^p(-y_1)^q\det(Y)^{j_2+k-3-s}}{(y_1+(2i)^{-1}\lambda_1)^{j_1+j_2+k-1/2-p}(y_2-(2i)^{-1}\lambda_2)^{j_1+j_2+k-1/2-q}}. \end{aligned}$$

Thus, we have proved this lemma. $\square$

From Lemma 4.22, we deduce

$$I\Bigl(\bigcup_{\gamma'\in[\gamma]_\Gamma}\{\gamma'\}_\Gamma\Bigr)=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\sum_{[\gamma]_\Gamma}\frac{\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]}\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma}J_0\bigl(\gamma';s\bigr).$$

Hence, we have only to calculate the integral $J_0(\gamma;s)$. We can evaluate the integral $J_0(\gamma;s)$ by using the argument in Section 4.12, the proof of Lemma 4.22, Lemma 4.12, and Hashimoto's calculations [12, p. 445].

4.16. Quasi-unipotent contribution of (f-3)

Let γ be an element of $A(f3)$, which satisfies

$$\gamma=g\begin{pmatrix}\cos\theta & \sin\theta & \lambda\cos\theta & \lambda\sin\theta\\ -\sin\theta & \cos\theta & -\lambda\sin\theta & \lambda\cos\theta\\ 0&0&\cos\theta&\sin\theta\\ 0&0&-\sin\theta&\cos\theta\end{pmatrix}g^{-1}$$

for certain $g\in G(\mathbb{R})$, $\lambda\in\mathbb{R}$, $\lambda\neq 0$, $\sin\theta\neq 0$.

Lemma 4.23. *For the Γ-conjugacy class $\{\gamma\}_\Gamma$, there exists a constant $C_{k,j,\Gamma,\gamma,f3}$ depending only on k, j, Γ, and γ such that*

$$\sum_{\gamma'\in\{\gamma\}_\Gamma\cap gP_0(\mathbb{R})g^{-1}}\bigl|H^{k,j}_{g^{-1}\gamma' g}(Z)\bigr|<C_{k,j,\Gamma,\gamma,f3}\times y_1^{3/2}y_2^{1/2}.$$

Proof. For $\gamma'\in\{\gamma\}_\Gamma\cap gP_0(\mathbb{R})g^{-1}$, we have

$$g^{-1}\gamma' g=\begin{pmatrix}h^{-1}&\\&{}^th\end{pmatrix}\begin{pmatrix}I_2&T\\&I_2\end{pmatrix}\begin{pmatrix}I_2&\lambda I_2\\&I_2\end{pmatrix}\begin{pmatrix}k(\theta)&\\&k(\theta)\end{pmatrix}\begin{pmatrix}I_2&-T\\&I_2\end{pmatrix}\begin{pmatrix}h&\\&{}^th^{-1}\end{pmatrix},$$

where $h\in GL(2;\mathbb{R})$ and $h^{-1}\cdot k(\theta)\cdot h\in(g^{-1}\Gamma g)_{M_0}$. Hence, if we set $\gamma'=\left(\begin{smallmatrix}I_2&S\\&I_2\end{smallmatrix}\right)\left(\begin{smallmatrix}A&\\&{}^tA^{-1}\end{smallmatrix}\right)$, then S belongs to a subset of

$$\bigcup_{h^{-1}\cdot k(\theta)\cdot h\in(g^{-1}\Gamma g)_{M_0}}\left\{h^{-1}\left(\lambda I_2+\begin{pmatrix}t_1&t_{12}\\t_{12}&-t_1\end{pmatrix}\right){}^th^{-1}\in\mathcal{L}';\ t_1,t_{12}\in\mathbb{R}\right\},$$

for a certain lattice $\mathcal{L}'$ in $SM(2;\mathbb{R})$. From this, for $S=\left(\begin{smallmatrix}s_1&s_{12}\\s_{12}&s_2\end{smallmatrix}\right)$, we find that s_2 is determined by s_1, s_2, and A. Therefore, we reduce the proof of this lemma to that in [25, Theorem 4] by using Lemma 4.2. $\square$

From this lemma, for $s\in\mathbb{R}_{\geqslant 0}$, we get

$$\int_{C_0(g^{-1}\gamma g;G(\mathbb{R}))\backslash\mathfrak{H}_2}\bigl|H^{k,j}_{g^{-1}\gamma g}(\hat{Z})\bigr|\det(Y)^{-s}\,d\hat{Z}<\text{constant}\times|\lambda|^{-1-2s}.$$

Hence, it follows from Lemma 4.23, Proposition 4.14, and (4.7) that

$$I\left(\bigcup_{\gamma'\in[\gamma]_\Gamma}\{\gamma'\}_\Gamma\right)=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\frac{\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]}\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma}J_0(\gamma';s).$$

Therefore, we have only to calculate the integral $J_0(\gamma;s)$. As a coordinate of $C(g^{-1}\gamma g;G(\mathbb{R}))\backslash\mathfrak{H}_2$, we take $\{\left(\begin{smallmatrix}x_1+iy_1 & x_{12}\\ x_{12} & -x_1+iy_2\end{smallmatrix}\right)\in\mathfrak{H}_2 \mid 0<y_1<y_2\}$. We take the measure $(2\pi)^{-1}\,d\theta$ on $SO(2;\mathbb{R})=\{k(\theta);\ 0\leqslant\theta<2\pi\}\cong C_0(g^{-1}\gamma g;G(\mathbb{R}))\backslash C(g^{-1}\gamma g;G(\mathbb{R}))$. The measure on the coordinate is given by $(y_2-y_1)(y_1y_2)^{-3}\,dx_1\,dx_{12}\,dy_1\,dy_2$. For the above mentioned coordinate, we have

$$\begin{aligned}
&H^{k,j}_{g^{-1}\gamma g}(Z)\det(Y)^{-3-s}\\
&=\sum_{j_1+2j_2=j,\ j_1,j_2\geqslant 0}a_{j_1,j_2}\times(2i)^{-j_1}\times(y_1y_2)^{k+j_1+j_2-s-3}\\
&\quad\times\left\{2x_{12}\sin\theta\left(y_1^{-1}-y_2^{-1}\right)+\lambda\cos\theta\left(y_1^{-1}+y_2^{-1}\right)+4i\cos\theta\right\}^{j_1}\\
&\quad\times\left\{x_1^2\sin^2\theta+\left(x_{12}\sin\theta+2^{-1}i\cos\theta(y_1-y_2)\right)^2+4^{-1}(y_1+y_2-i\lambda)^2\right\}^{-k-j_1-j_2}.
\end{aligned}$$

By using [12, Lemma 3-5], we have

$$\begin{aligned}
J_0(\gamma;s)=&\sum_{j_1+2j_2=j,\ j_1,j_2\geqslant 0}a_{j_1,j_2}\times(2i)^{-j_1}\times\frac{\Gamma(k+j_1+j_2-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k+j_1+j_2)}\times(\sin\theta)^{-2k-2j_1-2j_2}\\
&\times\sum_{p=0,\ j_1-p\in 2\mathbb{Z}}^{j_1}\ \sum_{q=0}^{(j_1-p)/2}\frac{\Gamma(j_1+1)\Gamma((j_1-p)/2+1)}{\Gamma(p+1)\Gamma(j_1-p+1)\Gamma(q+1)\Gamma((j_1-p)/2-q+1)}\\
&\times\frac{\Gamma(k+j_1+j_2-q-1)\Gamma(1/2)}{\Gamma(k+j_1+j_2-q-1/2)}\times(2\sin\theta)^{2j_1-2p}(\cos\theta)^p\\
&\times\int_{0<y_1<y_2}(y_1y_2)^{k+j_2-s-3}(y_1+y_2)^p(y_2-y_1)^{j_1-p+1}(y_1+y_2-i\lambda)^{-2k-j+2}\,dy_1\,dy_2.
\end{aligned}$$

Lemma 4.24. *Let $a=\lambda/|\lambda|$ and $k_1\in\mathbb{R}_{>0}$, $k_2,k_3,k_4\in\mathbb{Z}$, $k_4-2k_1-2k_2-k_3-3>0$. Then, we have*

$$\begin{aligned}
&\int_{0<y_1<y_2}(y_1y_2)^{k_1}(y_2-y_1)^{2k_2+1}(y_1+y_2)^{k_3}(y_1+y_2-ia)^{-k_4}\,dy_1\,dy_2\\
&=(ia)^{k_4-2k_1-2k_2-k_3-3}\times 2^{-2k_1-2}\times\pi\\
&\quad\times\frac{\Gamma(k_1+1)\Gamma(k_2+1)}{\Gamma(k_1+k_2+2)}\times\frac{\Gamma(k_4-2k_1-2k_2-k_3-3)\Gamma(2k_1+2k_2+k_3+3)}{\Gamma(k_4)}.
\end{aligned}$$

Therefore, we can evaluate the integral $J_0(\gamma;s)$ by using the argument in Section 4.12 and Lemma 4.12.

4.17. Quasi-unipotent contribution of (f-4)

If $G(\mathbb{Q})$ is not split, then the elements of type (e-3) do not appear in Γ. Let γ be an element of $A(f4)$, which satisfies

$$\gamma = g \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & \lambda \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g^{-1}$$

for certain $g \in G(\mathbb{R})$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $\sin\theta \neq 0$. We can deduce the following lemma from [25, Theorem 4], Lemma 4.2, and an argument similar to Lemma 4.23.

Lemma 4.25. *For the Γ-conjugacy class $\{\gamma\}_\Gamma$, there exists a constant $C_{k,j,\Gamma,\gamma,f4}$ depending only on k, j, Γ, and γ such that*

$$\sum_{\gamma' \in \{\gamma\}_\Gamma \cap (gP_1(\mathbb{R})g^{-1})} \left| H^{k,j}_{g^{-1}\gamma' g}(Z) \right| < C_{k,j,\Gamma,\gamma,f4} \times y_2.$$

From this lemma, for $s \in \mathbb{R}_{\geqslant 0}$, we get

$$\int_{C_0(g^{-1}\gamma g; G(\mathbb{R}))\backslash \mathfrak{H}_2} \left| H^{k,j}_{g^{-1}\gamma g}(\hat{Z}) \right| \left(y_1^{-1} \det(Y) \right)^{-s} d\hat{Z} < \text{constant} \times |\lambda|^{-1-s}.$$

Hence, it follows from Lemma 4.25, Proposition 4.14, and (4.7) that

$$I\left(\bigcup_{\gamma' \in [\gamma]_\Gamma} \{\gamma'\}_\Gamma \right) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \frac{\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma; G(\mathbb{R})))}{[\bar{C}(\gamma;\Gamma) : \bar{C}_0(\gamma;\Gamma)]} \lim_{s\to +0} \sum_{\gamma' \in [\gamma]_\Gamma} J_0(\gamma'; s).$$

Hence, we have only to calculate the integral $J_0(\gamma; s)$. As a coordinate of $C(g^{-1}\gamma g; G(\mathbb{R}))\backslash \mathfrak{H}_2$, we take $\{ \left(\begin{smallmatrix} x_1+iy_1 & x_{12} \\ x_{12} & iy_2 \end{smallmatrix}\right) \in \mathfrak{H}_2 \mid x_1 \in \mathbb{R},\ x_{12} \geqslant 0,\ y_1, y_2 > 0\}$. We take the measure $(2\pi)^{-1}\, d\theta$ on $SO(2;\mathbb{R}) = \{k(\theta);\ 0 \leqslant \theta < 2\pi\} \cong C_0(g^{-1}\gamma g; G(\mathbb{R}))\backslash C(g^{-1}\gamma g; G(\mathbb{R}))$. The measure on the coordinate is given by $x_{12} y_1 (y_1 y_2)^{-3}\, dx_1\, dx_{12}\, dy_1\, dy_2$. For the above mentioned coordinate, we have

$$\begin{aligned} & H^{k,j}_{g^{-1}\gamma g}(Z) \det(Y)^{-3} \left(y_1^{-1} \det(Y) \right)^{-s} \\ &= \sum_{j_1+2j_2=j,\ j_1,j_2 \geqslant 0} a_{j_1,j_2} (2i)^{-j_1} (-4)^{k+j_1+j_2} y_1^{k+j_2-3} y_2^{k+j_2-3-s} \\ &\quad \times \left\{ 2x_{12}^2 (1-\cos\theta + iy_1\sin\theta) + (2iy_2+\lambda)\left(x_1^2 \sin\theta + y_1^2 \sin\theta + 2iy_1\cos\theta + \sin\theta \right) \right\}^{-k-j_1-j_2} \\ &\quad \times \left\{ x_{12}^2 y_1 \sin\theta + y_1(2iy_2+\lambda) + y_2\left(x_1^2\sin\theta + y_1^2\sin\theta + 2iy_1\cos\theta + \sin\theta \right) \right\}^{j_1}. \end{aligned}$$

By simple calculation, we get

$$\int_0^\infty \frac{(\delta x_{12}^2 + \omega)^{k_2}}{(\alpha x_{12}^2 + \beta)^{k_1}} x_{12}\, dx_{12} = \sum_{p=0}^{k_2} \frac{\Gamma(k_2+1)\Gamma(p+1)\Gamma(k_1-p-1)}{2\Gamma(p+1)\Gamma(k_2-p+1)\Gamma(k_1)} \alpha^{-p-1} \beta^{-k_1+p+1} \delta^p \omega^{k_2-p}.$$

Therefore, we have

$$J_0(\gamma;s)=\int\limits_{\mathbb{R}_{>0}}\int\limits_{\mathbb{R}_{>0}}\int\limits_{\mathbb{R}} dx_1\,dy_1\,dy_2$$
$$\times\sum_{p=0}^{j_1}\sum_{q=0}^{j_1-p}\frac{\Gamma(j_1+1)}{2\Gamma(j_1-p+1)}\frac{\Gamma(k+j_1+j_2-p-1)}{\Gamma(k+j_1+j_2)}\frac{\Gamma(j_1-p+1)}{\Gamma(q+1)\Gamma(j_1-p-q+1)}$$
$$\times y_1^{k+j_2-2+p+q}y_2^{k+j_1+j_2-p-q-3-s}\times 2^{-p-1}\times(1-\cos\theta+iy_1\sin\theta)^{-p-1}$$
$$\times(2iy_2+\lambda)^{-k-j_1-j_2+p+q+1}\big(x_1^2\sin\theta+y_1^2\sin\theta+2iy_1\cos\theta+\sin\theta\big)^{-k-j_2-q+1}.$$

Thus, we can evaluate the integral $J_0(\gamma;s)$ by using the integral for y_2, the argument in Section 4.12, and Lemma 4.12.

5. Prehomogeneous vector spaces

In this section, we give a formula (Theorem 5.7) for the contributions of (e-2), (e-3), and (e-4), which is a generalization of Shintani's result [28, Section 3] to the vector-valued case. By using this formula, we obtain a different proof for the contributions of (e-2), (e-3), and (e-4). The space $SM(n;\mathbb{C})$ is a prehomogeneous vector space, i.e., $SM(n;\mathbb{C})$ has a Zariski dense open $GL(n;\mathbb{C})$-orbit by the action $x\mapsto gx{}^tg$ ($x\in SM(n;\mathbb{C})$, $g\in GL(n;\mathbb{C})$). The contributions of (e-2), (e-3), and (e-4) coincide with zeta integrals of prehomogeneous vector spaces of symmetric matrices of degree one or two for certain test functions.

5.1. Poisson summation formula

We assume that r is equal to 1 or 2. We set $V_r=SM(r;\mathbb{R})$ and $\Omega_r=\{x\in V_r;\ x>0\}$. For $x\in V_r$, we set

$$f_r^*(x)=\operatorname{tr}\left[\rho_{k,j}\begin{pmatrix}1-ix&0\\0&1\end{pmatrix}^{-1}\right]\quad(r=1),\qquad \operatorname{tr}\big[\rho_{k,j}(I_2-ix)^{-1}\big]\quad(r=2).$$

For $x\in\Omega_1$, we set $f_1(x)=\sum_{l=0}^{j}(2\pi)^{k+l}\Gamma(k+l)^{-1}x^{k+l-1}\exp(-2\pi x)$. For $x\notin\Omega_1$, we set $f_1(x)=0$. The spherical polynomial $\Phi_m(x)$ for $m=(m_1,m_2)\in\mathbb{Z}_{\geqslant 0}$ ($m_1\geqslant m_2$) is defined by $\Phi_m(x)=\int_{SO(2;\mathbb{R})}\Delta_m({}^tgxg)\,dg$, where $\Delta_m(x)=x_1^{m_1-m_2}\det(x)^{m_2}$ and dg is the Haar measure on $SO(2;\mathbb{R})$ normalized by $\int_{SO(2;\mathbb{R})}dg=1$. Since $\operatorname{tr}(\rho_{k,j}(x))$ is invariant under the action $x\mapsto {}^tgxg$ ($g\in SO(2;\mathbb{R})$), we can express $\operatorname{tr}(\rho_{k,j}(x))$ as the linear combination $\operatorname{tr}(\rho_{k,j}(x))=\sum_{m_1+m_2=2k+j,\,m_2\geqslant k}a_m\Phi_m(x)$ ($a_m\in\mathbb{R}$) (cf. [7, Proposition XI.3.1]). For $x\in\Omega_2$, we set

$$f_2(x)=\sum_{m_1+m_2=2k+j,\,m_2\geqslant k}\frac{(2\pi)^{-(1/2)+m_1+m_2}a_m}{\Gamma(m_1)\Gamma(m_2-2^{-1})}\Phi_m(x)\det(x)^{-3/2}\exp\big(-2\pi\operatorname{tr}(x)\big).$$

For $x\notin\Omega_2$, we set $f_2(x)=0$. Let dx denote the Lebesgue measure on V_r. For the scalar-valued case ($j=0$), the following lemma is obtained from the works of Shintani [28] and Siegel [29].

Lemma 5.1.

(i) *If $-1<\operatorname{Re}(s)<k-r$, then the integral $\int_{V_r}f_r^*(x)|\det(x)|^s\,dx$ is absolutely convergent.*
(ii) *If $k>(r-1)/2$, then we get $\int_{V_r}f_r(x)\exp(2\pi i\operatorname{tr}(xy))\,dx=f_r^*(y)$. This integral is absolutely convergent.*

Proof. For $r = 1$, the proofs of (i) and (ii) are trivial. Hence, we consider only the case $r = 2$. By the proof of Lemma 4.2, we may assume $j = 0$ for the absolute convergence of $\int_{V_2} f_2^*(x)|\det(x)|^s\,dx$. Hence, (i) is proved by [28, Lemma 19]. The integral $\int_{\Omega_2} \Phi_m(x)\det(x)^{-3/2}\exp(-2\pi\,\mathrm{tr}(x))\,dx$ is absolutely convergent if $m_2 - (3/2) > -1$. Hence, if $k > 1/2$, then the integral of (ii) is absolutely convergent. From [7, Lemma XI.2.3], we know

$$\int_{\Omega_2} \Phi_m(x)\exp\bigl(-\mathrm{tr}(xy)\bigr)\det(x)^{-3/2}\,dx = \pi^{1/2}\Gamma(m_1)\Gamma\bigl(m_2 - 2^{-1}\bigr)\Phi_m\bigl(y^{-1}\bigr).$$

Thus, we have proved the equality in (ii). □

We use the following lemma to prove Theorem 5.7.

Lemma 5.2. *Suppose $k > 2$. Then, we have*

$$f_2(x) = \begin{cases} 2^{-5+2k+j}c_{k,j}^{-1}\,\mathrm{tr}(\rho_{k,j}(x)H_{k,j}^{-1})\det(x)^{-3/2}\exp(-2\pi\,\mathrm{tr}(x)) & (x \in \Omega_2), \\ 0 & (x \notin \Omega_2), \end{cases}$$

where

$$H_{k,j} = \int_{\Omega_2} \rho_{k,j}(x)\exp\bigl(-\pi\,\mathrm{tr}(x)\bigr)\det(x)^{-3}\,dx.$$

Proof. By [8, Expose 6, Théorème 6], for $y \in \Omega_2$, $Z \in \mathfrak{H}_2$, and $k > 2$, we get

$$c_{k,j}\cdot\rho_{k,j}(Z/2i)^{-1} = 2\int_{\Omega_2} H_{k,j}(4y)^{-1}\exp\bigl(2\pi i\,\mathrm{tr}(yZ)\bigr)\,dy,$$

where $H_{k,j}(y) = \int_{\Omega_2} \rho_{k,j}(x)\exp(-\pi\,\mathrm{tr}(yx))\det(x)^{-3}\,dx$. Since $\mathrm{tr}(\rho_{k,j}(x)H_{k,j}^{-1})$ is a $SO(2;\mathbb{R})$-invariant polynomial, we get the above mentioned lemma by substituting $Z = 2i(I_2 - ix)$. □

We identify V_2 with its dual vector space via the symmetric bilinear form $\langle x, x^*\rangle = \mathrm{tr}(xJ_1x^*J_1)$, $J_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ $(x, x^* \in V_2)$. We also identify V_1 with its dual space via the symmetric bilinear form $\langle x, x^*\rangle = xx^*$ $(x, x^* \in V_1)$. Let L_r be a lattice for a $\mathbb{Q}$-structure of V_r and L_r^* be the dual lattice to L_r, i.e., $L_r^* = \{x^* \in V_r;\ \langle x, x^*\rangle \in \mathbb{Z}\ (\forall x \in L_r)\}$. We denote the volume of the fundamental parallelogram of L_r by $\mathrm{vol}(L_r)$.

We get the following Poisson summation formula by the proof of Lemma 5.2 and the trace of the formula [8, Appendix of Expose 10]. For the scalar-valued case $(j = 0)$, the following lemma is obtained from the works of Siegel [29] and Braun [4].

Proposition 5.3. *(See [8, Appendix of Expose 10].) Suppose $k > 2$. For any $Z \in \mathfrak{H}_2$, we have*

$$\sum_{T \in L_2 \cap \Omega_2} F_2(T)\exp\bigl(2\pi i\,\mathrm{tr}(TZ)\bigr) = \mathrm{vol}(L_2)^{-1}\sum_{S \in L_2^*} \mathrm{tr}\bigl(\rho_{k,j}\bigl((Z+S)/i\bigr)^{-1}\bigr),$$

where $F_2(T) = f_2(T)\exp(2\pi\,\mathrm{tr}(T))$.

In the proof of Lemma 4.4, we have already used a Poisson summation formula, which is an analogue of this formula. By Lemma 5.1 and Proposition 5.3, we get the following.

Proposition 5.4. *If $k > r$, we have*

$$\sum_{x\in L_r} f_r(x) = \mathrm{vol}(L_r)^{-1}\sum_{x\in L_r^*} f_r^*(x) \quad (\textit{both sides are absolutely convergent}).$$

5.2. Zeta integrals

Let $\mathfrak{O}^1$ be the unit group with norm 1 of $\mathfrak{O}$, where $\mathfrak{O}$ is a maximal order of an indefinite division quaternion algebra $\mathbf{B}$ over $\mathbb{Q}$. Let D be an arithmetic subgroup of a $\mathbb{Q}$-form of $SL(r;\mathbb{R})$, i.e., D is commensurable with $SL(r;\mathbb{Z})$ or $\mathfrak{O}^1$ $(r=2)$. We assume that L_r is invariant for D. We define the zeta integral $Z(P_r, L_r, s)$ as

$$Z(P_r, L_r, s) = \int\limits_{G_+/D} \det(g)^{2s}\sum_{x\in L_r'} P_r\big(gx{}^tg\big)\,dg,$$

where $L_r' = L_r - \{x\in V_r\colon\ \det(x)=0\}$, P_r is a function on V_r, $G_+ = \{g\in GL(r;\mathbb{R});\ \det(g)>0\}$, and dg is the Haar measure on G_+ defined by $\det(g)^{-r}\prod_{1\leqslant i,j\leqslant r} dg_{ij}$. By Lemma 5.1 and Proposition 5.4, we can discuss the convergence, functional equation, and meromorphic continuity of the zeta integral using arguments similar to those of the scalar-valued case in [28] and [1].

Proposition 5.5.

(i) *The integral $Z(f_r, L_r, s)$ is absolutely convergent if* $\mathrm{Re}(s) > (r+1)/2$ *and* $\mathrm{Re}(k+s) > r$. *The integral $Z(f_r, L_r, s)$ is a meromorphic function of s on $\mathbb{C}$.*

(ii) *Suppose that D is commensurable with $SL(r;\mathbb{Z})$. If*

$$\begin{cases} k>1, & \mathrm{Re}(s)<k \quad (r=1),\\ k>4, & 2\mathrm{Re}(s)<k \quad (r=2)\end{cases}$$

and $\mathrm{Re}(s) > (r-1)/2$, *then the integral $Z(f_r^*, L_r^*, s)$ is absolutely convergent. The integral $Z(f_r^*, L_r^*, s)$ is a meromorphic function of s on $\mathbb{C}$.*

(ii)′ *Suppose that D is commensurable with $\mathfrak{O}^1$. If* $0<\mathrm{Re}(s)<k-1/2$, *then the integral $Z(f_2^*, L_2^*, s)$ is absolutely convergent. The integral $Z(f_2^*, L_2^*, s)$ is a meromorphic function of s on $\mathbb{C}$.*

(iii) *We have the functional equation* $Z(f_r, L_r, s) = \mathrm{vol}(L_r)^{-1} Z(f_r^*, L_r^*, (r+1)/2 - s)$.

Proof. We easily get (i), (ii) $r=1$, (ii)′, and (iii) by Lemma 5.1, Proposition 5.4, and the arguments of Shintani [28] and Arakawa [1]. Hence, it is sufficient to prove (ii) $r=2$. Let D be an arithmetic subgroup of $SL(2;\mathbb{Q})$. We set

$$\mathcal{R} = \left\{k\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0\\ u & 1\end{pmatrix}\in SL(2;\mathbb{R});\ k\in SO(2;\mathbb{R}),\ a\geqslant\alpha'',\ u\in\mathcal{W}'''\right\}$$

for a constant α'' and a compact subset $\mathcal{W}'''$ of $\mathbb{R}$. We may assume that $\mathcal{R}$ is a Siegel set of D. Then, there exist elements $h_1', h_2', \dots, h_{v'}'$ in $SL(2;\mathbb{Q})$ such that a fundamental domain of D on $SL(2;\mathbb{R})$ is contained in $\bigcup_{w=1}^{v'}\mathcal{R}h_w$. From the arguments in [28], we have only to show that the integral

$$\int\limits_{\mathcal{R}\times\mathbb{R}_{>0},\,\det(g)\geqslant 1}\left|\det(g)^{2s}\sum_{x\in(h_w' L_2{}^t h_w')'} f_2^*\big(gx{}^tg\big)\right|dg$$

is convergent for $k > 4$, where $(h'_w L_2 {}^t h'_w)' = h'_w L_2 {}^t h'_w - \{x \in V_2\colon \det(x) = 0\}$. We use the following lemma, which is proved in [28, Lemma 20], because we may assume $j = 0$ by the proof of Lemma 4.2. We slightly modified the result of [28, Lemma 20].

Lemma 5.6. *Let m_1, m_2, and m_{12} be positive real numbers and $\mathcal{D}$ be a relatively compact subset of $GL(2,\mathbb{R})$. Then, there exists a positive constant c'', which depends only on m_1, m_2, m_{12}, j, k, and $\mathcal{D}$, such that*

$$\left|f_2^*\left(gx\,{}^t g\right)\right| \leqslant c'' \times \left(1+|x_1|\right)^{-m_1}\left(1+|x_2|\right)^{-m_2}\left(1+|x_{12}|\right)^{-m_{12}} \quad (\textit{for any } g \in \mathcal{D})$$

if $k \geqslant m_1 + m_2 + m_{12}$ ($x = \left(\begin{smallmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{smallmatrix}\right)$).

By using this lemma and the argument in [28, Proof of Lemma 21], we prove the convergence of the above mentioned integral. We set $h'_w L_2 {}^t h'_w - \{\det(x) = 0\} = M_w \cup N_w$, $M_w = \{x_1 \neq 0\}$, $N_w = \{x_1 = 0\}$. We consider the absolute convergence for each the summation part of M_w or N_w. As for the summation part of M_w, by using Lemma 5.6 and the argument in [28], we have

$$\int\limits_{\mathcal{R}\times\mathbb{R}_{>0}} \det(g)^{2s}\left|\sum_{x\in M_w} f_2^*\left(gx\,{}^t g\right)\right| dg < +\infty$$

if there exist positive real numbers m_1, m_2, and m_{12} satisfying $m_1, m_2, m_{12} > 1$, $2s < m_1 + m_2 + m_{12} \leqslant k$, and $0 < m_1 - m_2 + 1$. Next, we evaluate the summation part of N_w. We write $f_2^*(x) = f_2^*(x_1, x_{12}, x_2)$. It follows from the argument in [28] that

$$\begin{aligned}
&\int\limits_{\mathcal{R}\times\mathbb{R}_{>0}} \det(g)^{2s}\left|\sum_{x\in N_w} f_2^*\left(gx\,{}^t g\right)\right| dg < \text{constant} \times \int_0^1 du \\
&\quad \times \int\limits_{(t_1,t_2)\in\mathcal{R}'} (t_1t_2)^s \sum_{l\in b_1\mathbb{Z}-\{0\}}\ \sum_{0\leqslant b_3 m<2b_1|l|}\left|\sum_{n\in b_4\mathbb{Z}} f_2^*\left(0, t_1t_2 l, t_2^2\left\{m + b_2 l + 2l(u+n)\right\}\right)\right| t_1^{-2}\,dt_1\,dt_2,
\end{aligned}$$

where $\mathcal{R}' = \{(t_1,t_2) \in \mathbb{R}^2_{>0};\ t_1t_2 \geqslant 1,\ t_1/t_2 \geqslant b\}$, b, b_1, b_2, b_3, and b_4 are positive constants. We also have

$$\begin{aligned}
f_2^*(0,p,q) &= \sum_{j_1+2j_2=j,\ j_1,j_2\geqslant 0} a_{j_1j_2}\cdot\left(2-\sqrt{-1}q\right)^{j_1}\left(1+p^2-\sqrt{-1}q\right)^{-k-j_1-j_2} \\
&= \sum_{j_1+2j_2=j,\ j_1,j_2\geqslant 0}\ \sum_{j'=0}^{j_1} a'_{j_1j_2j'}\cdot\left(1-p^2\right)^{j'}\left(1+p^2-\sqrt{-1}q\right)^{-k-j'-j_2},
\end{aligned}$$

where $a'_{j_1j_2j'}$ is a constant depending only on (j_1, j_2, j'). By using the Poisson summation formula for n, we find that the absolute convergence is reduced to that of

$$\int_0^1 du \int\limits_{(t_1,t_2)\in\mathcal{R}'} (t_1t_2)^s \sum_{l\in\mathbb{Z}-\{0\}}\sum_{n=1}\left|t_1^2 l\right|^{j'}\left|t_2^2 l\right|^{-k}|n|^{k+j'-1}\exp\left(-\pi n\left(l^{-1}t_2^{-2} + lt_1^2\right)\right)t_1^{-2}\,dt_1\,dt_2$$

($0 \leqslant j' \leqslant j$). Thus, we have proved the absolute convergence. $\square$

5.3. Unipotent contribution of ((e-2) and (e-3)) or (e-4)

We consider the contribution of ((e-2) and (e-3)) or (e-4). Let γ_1 be an element of Γ, which is $G(\mathbb{Q})$-conjugate to $\delta(S_1)$ ($\mathrm{rank}(S_1)=1$). Hence, γ_1 is of type (e-4). We set $A_1=\bigcup_{\gamma'\in[\gamma_1]_\Gamma}\{\gamma'\}_\Gamma$. Let γ_2 be an element of Γ, which is $G(\mathbb{Q})$-conjugate to $\delta(S_2)$ ($\det(S_2)\neq 0$). For $\gamma_2=g\delta(S_2)g^{-1}$ ($g\in G(\mathbb{Q})$), we set $\mathfrak{U}_{\gamma_2,\Gamma}=\{g\delta(T)g^{-1}\in\Gamma;\ \det(T)\neq 0\}$ and $A_2=\bigcup_{\gamma'\in\mathfrak{U}_{\gamma_2,\Gamma}}\{\gamma'\}_\Gamma$. The set $\mathfrak{U}_{\gamma_2,\Gamma}$ contains the two families for (e-2) $g\delta(T_2)g^{-1}$ ($\det(T_2)>0$) and (e-3) $g\delta(T_3)g^{-1}$ ($-\det(T_3)\in(\mathbb{Q}^\times)^2$). We set $D=\tilde{\Gamma}_+$ and $L_2=L$ if $r=2$, $D=\{1\}$ and $L_1=\mathbb{Z}$ if $r=1$ (cf. (e) Unipotent in Section 3). Using [28, Section 3] and Proposition 4.14, we have

$$I(A_r)=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\times c(r)\times Z\big(f_r^*,L_r,2-2^{-1}(r-1)\big),$$

where

$$c(1)=2^2\pi^{-1}\times\mathrm{vol}\big(C_0(\gamma;\Gamma)\backslash C_0\big(\gamma;G(\mathbb{R})\big)\big),$$
$$c(2)=2^4\pi^{-1}\times\sharp\big(Z(\Gamma)\big)\times[\tilde{\Gamma}:\tilde{\Gamma}_+]^{-1}\times\mathrm{vol}\big(C_0(\gamma;\Gamma)\backslash C_0\big(\gamma;G(\mathbb{R})\big)\big).$$

It follows from Proposition 5.5 that

$$I(A_r)=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\times c(r)\times\mathrm{vol}\big(L_r^*\big)\times Z\big(f_r,L_r^*,r-2\big).$$

For $x\in L_r^*\cap\Omega_r$, we set $G_x=\{g\in G_+\mid gx{}^tg=x\}$ and $D_x=D\cap G_x$. For any bounded domain U_x such that $U_x\subset\overline{U_x}\subset\Omega_r$, let $W_x=\{g\in G_+\mid gx{}^tg\in U_x\}$. Put

$$\mu(x)=\int_{W_x/D_x}dg\Big/\int_{U_x}\det(y)^{-(r+1)/2}\,dy.$$

The number $\mu(x)$ is finite and independent of the choice of U_x. Let $L_r^*\cap\Omega_r/\sim''$ denote the set of D-orbits in $L_r^*\cap\Omega_r$. We define the zeta functions $\xi(L_r^*,s)$ as follows:

$$\xi\big(L_r^*,s\big)=\sum_{x\in L_r^*\cap\Omega_r/\sim''}\frac{\mu(x)}{\det(X)^s}\quad\left(\mathrm{Re}(s)>\frac{1+r}{2}\right).$$

These zeta functions are called zeta functions associated to symmetric matrices. Since $SM(n;\mathbb{C})$ is a prehomogeneous vector space, they are examples of prehomogeneous zeta functions. $\xi(L_r^*,s)$ is a meromorphic function of s on $\mathbb{C}$ and has a pole at $s=(r+1)/2$ (cf. [28,1,27]). If D is commensurable with $SL(2;\mathbb{Z})$, $\xi(L_2^*,s)$ also has a pole at $s=1$.

Theorem 5.7. *The contribution of A_r for ((e-2)(e-3), $r=2$) or ((e-4), $r=1$) is given by*

$$I(A_r)=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\times c(r)\times\mathrm{vol}\big(L_r^*\big)\times Z\big(f_r,L_r^*,r-2\big)$$
$$=\frac{c_{k,j}}{\sharp(Z(\Gamma))}\times c(r)\times\mathrm{vol}\big(L_r^*\big)\times\xi\big(L_r^*,r-2\big)\times P(r),$$

where $P(1)=(2\pi)^2(j+1)(k-2)^{-1}(j+k-1)^{-1}$ and $P(2)=2^{-2}c_{k,j}^{-1}(j+1)$.

Proof. It follows from the relations between zeta integrals and zeta functions (cf. [28, Proof of Theorem 5] and [1, Proof of Proposition 1]) that

$$Z\big(f_r, L_r^*, r-2\big) = \xi\big(L_r^*, r-2\big) \times P(r),$$

where the integrals $P(1)$ and $P(2)$ are given by

$$P(1) = \int\limits_{\Omega_1} f_1(x) x^{-2}\,dx, \qquad P(2) = \int\limits_{\Omega_2} f_2(x)\det(x)^{-3/2}\,dx.$$

By direct calculation, we get $P(1) = (2\pi)^2 (j+1)(k-2)^{-1}(j+k-1)^{-1}$. By using Lemma 5.2, we get

$$\begin{aligned}
P(2) &= 2^{-5+2k+j} c_{k,j}^{-1} \int\limits_{\Omega_2} \operatorname{tr}\big(\rho_{k,j}(x) H_{k,j}^{-1}\big)\exp\big(-2\pi \operatorname{tr}(x)\big)\det(x)^{-3}\,dx \\
&= 2^{-2} c_{k,j}^{-1} \int\limits_{\Omega_2} \operatorname{tr}\big(\rho_{k,j}(x) H_{k,j}^{-1}\big)\exp\big(-\pi \operatorname{tr}(x)\big)\det(x)^{-3}\,dx \\
&= 2^{-2} c_{k,j}^{-1} \operatorname{tr}\left\{\left(\int\limits_{\Omega_2} \rho_{k,j}(x)\exp\big(-\pi \operatorname{tr}(x)\big)\det(x)^{-3}\,dx\right) H_{k,j}^{-1}\right\} \\
&= 2^{-2} c_{k,j}^{-1} \operatorname{tr}\big(H_{k,j} H_{k,j}^{-1}\big) = 2^{-2} c_{k,j}^{-1}(j+1). \qquad \square
\end{aligned}$$

If $r=1$, then we have $L_1^* = \mathbb{Z}$ and $\xi(L_1^*, -1) = -1/24$. If D is commensurable with $SL(2;\mathbb{Z})$, from [28, Theorem 2] and [27, Theorem 1], we know

$$\xi\big(L_2^*, 0\big) \times \frac{\sharp(Z(\Gamma))}{2} = \frac{\operatorname{vol}(D\backslash\mathfrak{H}_1)}{2^4} - \frac{\pi\cdot\operatorname{vol}(L_2)}{2^6\cdot 3} \times \sum_{u=1}^{t}\frac{c_u}{d_u^3}.$$

If D is commensurable with $\mathfrak{O}^1$, from [1, Proposition 1], we know

$$\xi\big(L_2^*, 0\big) \times \frac{\sharp(Z(\Gamma))}{2} = \frac{\operatorname{vol}(D\backslash\mathfrak{H}_1)}{2^4}.$$

From Theorem 5.7 and these results for special values, we obtain an alternative proof for the unipotent contributions, mentioned at (e) Unipotent of Section 3.

6. $\mathbb{Q}$-rank one case

Let $\mathbf{B}$ be an indefinite division quaternion algebra over $\mathbb{Q}$, $\mathfrak{O}$ be a maximal order of $\mathbf{B}$, and $D(\mathbf{B})$ be the discriminant of $\mathbf{B}$. Then, $G(\mathbb{Q})$ is a non-slit $\mathbb{Q}$-form of $Sp(2;\mathbb{R})$. We set $\Gamma^*(1) = G(\mathfrak{O})$ and $\Gamma^*(N) = \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma^*(1);\ a-1, b, c, d-1 \in N\mathfrak{O}\}$.

As for the scalar-valued case ($j=0$), the dimension formula for $S_{k,0}(\Gamma^*(1))$ has been derived by Hashimoto [13], and that for $S_{k,0}(\Gamma^*(N))$ ($N \geqslant 3$) has been derived by Arakawa [1] and Yamaguchi [37] (Yamaguchi used the Riemann–Roch theorem). In this section, we generalize their results to the vector-valued case. From [14, Section 5-1], the characteristic polynomials of the torsion elements of $G(\mathbb{Q})$ are as follows:

$f_1(x)=(x-1)^4,\ f_1(-x),$	$f_7(x)=(x^2+x+1)^2,\ f_7(-x),$
$f_2(x)=(x-1)^2(x+1)^2,$	$f_8(x)=(x^2+1)(x^2+x+1),\ f_8(-x),$
$f_3(x)=(x-1)^2(x^2+1),\ f_3(-x),$	$f_9(x)=(x^2+x+1)(x^2-x+1),$
$f_4(x)=(x-1)^2(x^2+x+1),\ f_4(-x),$	$f_{10}(x)=(x^4+x^3+x^2+x+1),\ f_{10}(-x),$
$f_5(x)=(x-1)^2(x^2-x+1),\ f_5(-x),$	$f_{11}(x)=x^4+1,$
$f_6(x)=(x^2+1)^2,$	$f_{12}(x)=x^4-x^2+1.$

In the notation $\prod_{p|D(\mathbf{B})}$, p runs over only prime numbers. The notation $t=[t_0,t_1,\dots,t_{l-1};l]_m$ implies that $t=t_n$ if $m\equiv n \pmod l$. We denote the Legendre symbol by $(\frac{*}{p})$. We note that $\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(1))=0$ if j is odd. The following is a generalization of the result obtained by Hashimoto [13] ($j=0$).

Theorem 6.1. *If $k\geqslant 5$ and j is even, then we have $\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(1))=\sum_{l=1}^{12} H_l$, where H_l, being the total contribution of elements of $\Gamma^*(1)$ with the characteristic polynomial $f_l(\pm x)$, are given as follows*:

$$H_1=2^{-7}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3)\times\prod_{p|D(\mathbf{B})}(p-1)\big(p^2+1\big)$$
$$+2^{-3}3^{-1}(j+1)\prod_{p|D(\mathbf{B})}(p-1).$$
$$H_2=2^{-7}3^{-2}(-1)^k(j+k-1)(k-2)\prod_{p|D(\mathbf{B})}(p-1)^2\times\begin{cases}7 & \text{if } 2\nmid D(\mathbf{B}),\\ 13 & \text{if } 2\mid D(\mathbf{B}).\end{cases}$$
$$H_3=2^{-5}3^{-1}\big[(-1)^{j/2}(k-2),-(j+k-1),(-1)^{j/2+1}(k-2),(j+k-1);4\big]_k$$
$$\times\prod_{p|D(\mathbf{B})}(p-1)\left(1-\left(\frac{-1}{p}\right)\right).$$
$$H_4=2^{-3}3^{-3}\big\{\big[(j+k-1),-(j+k-1),0;3\big]_k+\big[(k-2),0,-(k-2);3\big]_{j+k}\big\}$$
$$\times\prod_{p|D(\mathbf{B})}(p-1)\left(1-\left(\frac{-3}{p}\right)\right).$$
$$H_5=2^{-3}3^{-2}\big\{\big[-(j+k-1),-(j+k-1),0,(j+k-1),(j+k-1),0;6\big]_k$$
$$+\big[(k-2),0,-(k-2),-(k-2),0,(k-2);6\big]_{j+k}\big\}$$
$$\times\prod_{p|D(\mathbf{B})}(p-1)\left(1-\left(\frac{-3}{p}\right)\right).$$
$$H_6=-2^{-3}(-1)^{j/2}\prod_{p|D(\mathbf{B})}\left(1-\left(\frac{-1}{p}\right)\right)$$
$$+2^{-7}3^{-1}(-1)^{j/2+k}(j+1)\sum_{D_0|2D(\mathbf{B})}\prod_{q|D_0}(q-1)\times\prod_{p|2D(\mathbf{B})/D_0}\left(1-\left(\frac{-1}{p}\right)\right)\times A$$
$$+2^{-7}3^{-1}(-1)^{j/2}(j+2k-3)\sum_{D_e|2D(\mathbf{B})}\prod_{q|D_e}(q-1)\times\prod_{p|2D(\mathbf{B})/D_e}\left(1-\left(\frac{-1}{p}\right)\right)\times B,$$

where

$$A\ (\textit{resp. } B) = \begin{cases} 3 & \textit{if } 2 \nmid D(\mathbf{B}),\ 2 \mid D^*, \\ 5 & \textit{if } 2 \mid D(\mathbf{B}),\ 2 \mid D^*;\ \textit{or } 2 \nmid D(\mathbf{B}),\ 2 \nmid D^*, \\ 11 & \textit{if } 2 \mid D(\mathbf{B}),\ 2 \nmid D^* \end{cases}$$

and $D^* = D_0$ (*resp.* D_e) *runs through the set of divisors of* $2D(\mathbf{B})$, *which are the product of odd* (*resp. even*) *number of distinct primes.*

$$\begin{aligned} H_7 &= -2^{-1}3^{-1}[1,-1,0;3]_j \prod_{p \mid D(\mathbf{B})} \left(1 - \left(\frac{-3}{p}\right)\right) \\ &\quad + 2^{-3}3^{-3}(j+1)[0,1,-1;3]_{j+2k} \times \sum_{D_0 \mid 3D(\mathbf{B})} \prod_{q \mid D_0} (q-1) \times \prod_{p \mid 3D(\mathbf{B})/D_0} \left(1 - \left(\frac{-3}{p}\right)\right) \times A \\ &\quad + 2^{-3}3^{-3}(j+2k-3)[1,-1,0;3]_j \times \sum_{D_e \mid 3D(\mathbf{B})} \prod_{q \mid D_e} (q-1) \times \prod_{p \mid 3D(\mathbf{B})/D_e} \left(1 - \left(\frac{-3}{p}\right)\right) \times B, \end{aligned}$$

where

$$A\ (\textit{resp. } B) = \begin{cases} 1 & \textit{if } 3 \mid D^*, \\ 4 & \textit{if } 3 \nmid D(\mathbf{B}),\ 3 \nmid D^*, \\ 16 & \textit{if } 3 \mid D(\mathbf{B}),\ 3 \nmid D^* \end{cases}$$

and $D^* = D_0$ (*resp.* D_e) *runs through the set of divisors of* $3D(\mathbf{B})$, *which are the product of odd* (*resp. even*) *number of distinct primes.*

$$H_8 = 2^{-2}3^{-1}C_8(k,j) \times \prod_{p \mid D(\mathbf{B})} \left(1 - \left(\frac{-1}{p}\right)\right)\left(1 - \left(\frac{-3}{p}\right)\right),$$

$$C_8(k,j) = \begin{cases} [1,0,0,-1,-1,-1,-1,0,0,1,1,1;12]_k & \textit{if } j \equiv 0 \bmod 12, \\ [-1,1,0,1,1,0,1,-1,0,-1,-1,0;12]_k & \textit{if } j \equiv 2 \bmod 12, \\ [1,-1,0,0,-1,1,-1,1,0,0,1,-1;12]_k & \textit{if } j \equiv 4 \bmod 12, \\ [-1,0,0,-1,1,-1,1,0,0,1,-1,1;12]_k & \textit{if } j \equiv 6 \bmod 12, \\ [1,1,0,1,-1,0,-1,-1,0,-1,1,0;12]_k & \textit{if } j \equiv 8 \bmod 12, \\ [-1,-1,0,0,1,1,1,1,0,0,-1,-1;12]_k & \textit{if } j \equiv 10 \bmod 12. \end{cases}$$

$$H_9 = 2^{-1}3^{-2}C_9(k,j) \times \prod_{p \mid D(\mathbf{B}), p \neq 2} \left(1 - \left(\frac{-3}{p}\right)\right)^2 \times \begin{cases} 2 & \textit{if } 2 \nmid D(\mathbf{B}), \\ 5 & \textit{if } 2 \mid D(\mathbf{B}), \end{cases}$$

$$C_9(k,j) = \begin{cases} [1,0,0,-1,0,0;6]_k & \textit{if } j \equiv 0 \bmod 6, \\ [-1,1,0,1,-1,0;6]_k & \textit{if } j \equiv 2 \bmod 6, \\ [0,-1,0,0,1,0;6]_k & \textit{if } j \equiv 4 \bmod 6. \end{cases}$$

$$H_{10} = 2^{-1}5^{-1}C_{10}(k,j) \times \prod_{p \mid D(\mathbf{B})} 2 \times \prod_{p \in D(-1;5)} 2 \times \begin{cases} 0 & \textit{if } \bigcup_{i=1}^{3} D(i;5) \neq \emptyset, \\ 1 & \textit{if } \bigcup_{i=1}^{3} D(i;5) = \emptyset,\ 5 \mid D(\mathbf{B}), \\ 2 & \textit{if } \bigcup_{i=1}^{3} D(i;5) = \emptyset,\ 5 \nmid D(\mathbf{B}), \end{cases}$$

where we set $D(i;j) = \{p \mid D(\mathbf{B});\ p \equiv i \bmod j\}$,

$$C_{10}(k,j)=\begin{cases}[1,0,0,-1,0;5]_k & \text{if } j\equiv 0 \bmod 10,\\ [-1,1,0,0,0;5]_k & \text{if } j\equiv 2 \bmod 10,\\ 0 & \text{if } j\equiv 4 \bmod 10,\\ [0,0,0,1,-1;5]_k & \text{if } j\equiv 6 \bmod 10,\\ [0,-1,0,0,1;5]_k & \text{if } j\equiv 8 \bmod 10.\end{cases}$$

$$H_{11}=2^{-3}C_{11}(k,j)\times\prod_{p|D(\mathbf{B}),p\neq 2}2\times\prod_{p\in D(-1;8)}2\times\begin{cases}0 & \text{if } D(1;8)\neq\emptyset,\\ 1 & \text{if } D(1;8)=\emptyset,\end{cases}$$

$$C_{11}(k,j)=\begin{cases}[1,0,0,-1;4]_k & \text{if } j\equiv 0 \bmod 8,\\ [-1,1,0,0;4]_k & \text{if } j\equiv 2 \bmod 8,\\ [-1,0,0,1;4]_k & \text{if } j\equiv 4 \bmod 8,\\ [1,-1,0,0;4]_k & \text{if } j\equiv 6 \bmod 8.\end{cases}$$

$$H_{12}=0 \quad \text{if } D(1;12)\neq\emptyset,$$

otherwise

$$H_{12}=2^{-2}3^{-1}\prod_{p|D(\mathbf{B})}2\times\prod_{p\in D(-1;12)}2\times(-1)^{j/2+k}[1,-1,0;3]_j\times A$$
$$+2^{-2}3^{-1}\prod_{p|D(\mathbf{B})}2\times\prod_{p\in D(-1;12)}2\times(-1)^{j/2}[0,-1,1;3]_{j+2k}\times B,$$

where

(i) *if* $2\nmid D(\mathbf{B})$, $3\nmid D(\mathbf{B})$,

$$A\ (\text{resp. } B)=\begin{cases}1/2 & \text{if } D(-1;12)\neq\emptyset,\\ 0 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is even (resp. odd)},\\ 1 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is odd (resp. even)},\end{cases}$$

(ii) *if* $2\nmid D(\mathbf{B})$, $3\mid D(\mathbf{B})$,

$$A\ (\text{resp. } B)=\begin{cases}3/4 & \text{if } D(-1;12)\neq\emptyset,\\ 1/2 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is even (resp. odd)},\\ 1 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is odd (resp. even)},\end{cases}$$

(iii) *if* $2\mid D(\mathbf{B})$, $3\nmid D(\mathbf{B})$,

$$A\ (\text{resp. } B)=\begin{cases}3/4 & \text{if } D(-1;12)\neq\emptyset,\\ 1 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is even (resp. odd)},\\ 1/2 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is odd (resp. even)},\end{cases}$$

(iv) *if* $6\mid D(\mathbf{B})$,

$$A\ (\text{resp. } B)=\begin{cases}9/8 & \text{if } D(-1;12)\neq\emptyset,\\ 5/4 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is even (resp. odd)},\\ 1 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is odd (resp. even)}.\end{cases}$$

$$\left(\frac{-1}{p}\right)=\begin{cases}0 & \text{if } p=2,\\ 1 & \text{if } p\equiv 1 \pmod 4,\\ -1 & \text{if } p\equiv 3 \pmod 4,\end{cases}\qquad \left(\frac{-3}{p}\right)=\begin{cases}0 & \text{if } p=3,\\ 1 & \text{if } p\equiv 1 \pmod 3,\\ -1 & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Proof. We can generalize the proof of [13, Theorem 4-1] by using Theorem 3.1. Note that Theorem B.1 is used in the proof of this theorem (cf. Appendix B). □

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(1))$.

(i) $D(\mathbf{B}) = 2 \times 3$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15	16	17
0	2	0	4	2	8	5	15	10	25	15	34	26	53	38
2	2	2	5	7	15	17	33	34	53	58	91	96	138	140
4	4	6	14	19	35	42	67	77	114	126	179	200	264	287
6	9	17	30	40	65	82	118	145	195	224	299	341	432	484
8	19	27	49	67	106	131	188	223	298	346	448	514	642	717

(ii) $D(\mathbf{B}) = 2 \times 5$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15	16
0	4	2	13	5	26	19	56	41	98	70	149	123	232
2	9	12	28	39	82	99	170	185	285	316	470	513	714
4	23	33	76	99	180	227	346	408	587	675	926	1051	1364
6	46	83	150	203	330	423	607	742	1004	1173	1534	1771	2228
8	88	141	246	347	532	684	955	1157	1522	1805	2302	2669	3298

(iii) $D(\mathbf{B}) = 3 \times 5$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15
0	9	8	34	29	86	85	183	178	331	318	536	531
2	30	52	117	170	311	405	640	775	1120	1324	1821	2100
4	84	149	298	431	703	934	1357	1694	2316	2789	3644	4283
6	174	323	574	834	1281	1702	2373	2985	3936	4757	6044	7136
8	330	575	979	1416	2091	2756	3752	4681	6044	7305	9117	10746

$*$ Our theorem is not valid for $k = 4$. We formally substitute $k = 4$ in the formula of our theorem. When $D(\mathbf{B}) = 6$, we know $\dim_{\mathbb{C}} S_{4,0}(\Gamma^*(1)) = 2$ from [16, Theorem 4.4]. We conjecture that the dimension of $S_{k,j}(\Gamma^*(1))$ is given by substituting $k = 4$ in the formula (cf. [12,13]). We also conjecture the same for other arithmetic subgroups.

Theorem 6.2. *$k \geqslant 5$. j is even. If $2 \nmid D(\mathbf{B})$, then we have*

$$\dim S_{k,j}\bigl(\Gamma^*(2)\bigr) = 2^{-3}3^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) \prod_{p \mid D(\mathbf{B})} (p-1)\bigl(p^2+1\bigr)$$
$$+ 2^{-2} \cdot 3 \cdot 5(j+1) \prod_{p \mid D(\mathbf{B})} (p-1)$$
$$+ 2^{-3} \cdot 5(-1)^k (j+k-1)(k-2) \prod_{p \mid D(\mathbf{B})} (p-1)^2.$$

If $2 \mid D(\mathbf{B})$, then we have

$$\dim S_{k,j}\bigl(\Gamma^*(2)\bigr) = 3^{-1}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) \prod_{p \mid D(\mathbf{B})} (p-1)\bigl(p^2+1\bigr)$$
$$+ 2 \cdot 3 \cdot (j+1) \prod_{p \mid D(\mathbf{B})} (p-1) + (-1)^k (j+k-1)(k-2) \prod_{p \mid D(\mathbf{B})} (p-1)^2.$$

Proof. Let r be even, and $p_1, \dots, p_r$, and q be primes. The prime q satisfies $q \equiv 5 \pmod 8$ and $(q/p_m) = -1$ for all $p_m \neq 2$. Let $\alpha = p_1 \cdots p_r$ and $\beta = q$. We define the quaternion algebra $\mathbf{B}$ by $\mathbf{B} = \mathbb{Q} + \mathbb{Q}a + \mathbb{Q}b + \mathbb{Q}ab$, $a^2 = \alpha$, $b^2 = \beta$, $ab = -ba$. Then, $D(\mathbf{B}) = p_1 p_2 \cdots p_r$. We set

$$\mathfrak{O} = \mathbb{Z} + \mathbb{Z}\frac{1+b}{2} + \mathbb{Z}\frac{a(1+b)}{2} + \mathbb{Z}\frac{(a+\gamma)b}{q}$$

where $\gamma^2 \equiv \alpha \pmod{q}$. Ibukiyama constructed this integer ring $\mathfrak{O}$ and proved that this integer ring $\mathfrak{O}$ is maximal (cf. [17]). We set

$$g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin \Gamma^*(2) \quad \text{and} \quad g_2 = \begin{pmatrix} b & -2(2l+1) \\ 2 & -b \end{pmatrix} \in \Gamma^*(2),$$

where $b^2 = \beta = q = 4(2l+1)+1$. It follows from $H_2 = I(\{g_1\}_{\Gamma^*(1)}) + I(\{g_2\}_{\Gamma^*(1)})$ that the conjugacy classes $\{g_1\}_{\Gamma^*(1)}$ and $\{g_2\}_{\Gamma^*(1)}$ are all the $\Gamma^*(1)$-conjugacy classes whose eigenvalues are 1 and -1. Thus, we have obtained the dimension formula for $\Gamma^*(2)$ by using Theorem 3.2. $\square$

Using Theorem 3.2, we get the following dimension formula, which is a generalization of the result obtained by Arakawa [1] and Yamaguchi [37] ($j = 0$).

Theorem 6.3. *If $k \geqslant 5$ and $N \geqslant 3$, then we have*

$$\begin{aligned} \dim S_{k,j}\big(\Gamma^*(N)\big) = {} & \big[\Gamma^*(1) : \Gamma^*(N)\big] \\ & \times \Bigg\{ 2^{-8}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) \prod_{p \mid D(\mathbf{B})} (p-1)\big(p^2+1\big) \\ & + 2^{-4}3^{-1}(j+1)N^{-3} \prod_{p \mid D(\mathbf{B})} (p-1) \Bigg\}, \end{aligned}$$

where $[\Gamma^*(1) : \Gamma^*(N)] = N^{10} \prod_{p \mid N,\, p \nmid D(\mathbf{B})} (1-p^{-2})(1-p^{-4}) \prod_{p \mid N,\, p \mid D(\mathbf{B})} (1-p^{-2})(1+p^{-1})$.

7. $\mathbb{Q}$-rank two case

We can obtain the dimension formulas for some congruence subgroups of $Sp(2;\mathbb{Z})$ by using Theorems 3.1 and 3.2. We also use Theorem B.1 (cf. Appendix B) or the classifications of the Γ-conjugacy classes (cf. Gottschling [9], Ueno [34], and Hashimoto [12, Sections 6 and 7]). In this paper, we do not describe the classifications and local factors. Our proofs are different from Tsushima's proofs [32,33].

Let $\Gamma(1) = Sp(2;\mathbb{Z})$ and $\Gamma(N) = \{\gamma \in \Gamma(1);\ \gamma \equiv I_4 \pmod{N}\}$. As for the scalar-valued case ($j = 0$), the dimension formulas for $S_{k,0}(\Gamma(N))$ ($N = 1, 2$) have been derived by Igusa [22], Hashimoto [12] and Tsushima [31], the dimension formula for $S_{k,0}(\Gamma(N))$ ($N \geqslant 3$) has been derived by Christian [6], Morita [25], and Yamazaki [38] (Gunji also derived a formula for $N = 3$ in [10]).

Theorem 7.1. *If $k \geqslant 5$ and j is even, then we have* $\dim_{\mathbb{C}} S_{k,j}(\Gamma(1)) = \sum_{l=1}^{12} H_l$, *where H_l, being the total contribution of elements of $\Gamma(1)$ with the characteristic polynomial $f_l(\pm x)$, are given as follows*:

$$\begin{aligned} H_1 = {} & 2^{-7}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) \\ & - 2^{-5}3^{-2}(j+1)(j+2k-3) + 2^{-4}3^{-1}(j+1). \\ H_2 = {} & 2^{-7}3^{-2}7(-1)^k(j+k-1)(k-2) - 2^{-4}3^{-1}(-1)^k(j+2k-3) + 2^{-5}3(-1)^k. \\ H_3 = {} & -2^{-3}\big[(-1)^{j/2}, -1, (-1)^{j/2+1}, 1; 4\big]_k \\ & + 2^{-5}3^{-1}\big[(-1)^{j/2}(k-2), -(j+k-1), (-1)^{j/2+1}(k-2), (j+k-1); 4\big]_k. \\ H_4 = {} & -2^{-2}3^{-2}\big\{[1,-1,0;3]_k + [1,0,-1;3]_{j+k}\big\} - 3^{-2}\big\{[1,0,1;3]_k + [0,-1,-1;3]_{j+k}\big\} \\ & + 2^{-3}3^{-3}\big\{\big[(j+k-1), -(j+k-1), 0; 3\big]_k + \big[(k-2), 0, -(k-2); 3\big]_{j+k}\big\}. \end{aligned}$$

$$H_5 = -2^{-2}3^{-1}\{[-1,-1,0,1,1,0;6]_k + [1,0,-1,-1,0,1;6]_{j+k}\}$$
$$+ 2^{-3}3^{-2}\{\big[-(j+k-1),-(j+k-1),0,(j+k-1),(j+k-1),0;6\big]_k$$
$$+ \big[(k-2),0,-(k-2),-(k-2),0,(k-2);6\big]_{j+k}\}.$$
$$H_6 = -2^{-3}(-1)^{j/2} + 2^{-7}3^{-1}5(-1)^{j/2}(j+2k-3) + 2^{-7}(-1)^{j/2+k}(j+1).$$
$$H_7 = -2^{-1}3^{-1}[1,-1,0;3;j]$$
$$+ 2^{-1}3^{-3}(j+2k-3)[1,-1,0;3]_j + 2^{-2}3^{-3}(j+1)[0,1,-1;3]_{j+2k}.$$
$$H_8 = 2^{-2}3^{-1}C_8(k,j), \qquad H_9 = 3^{-2}C_9(k,j), \qquad H_{10} = 5^{-1}C_{10}(k,j),$$
$$H_{11} = 2^{-3}C_{11}(k,j), \qquad H_{12} = 2^{-2}3^{-1}(-1)^{j/2}[0,-1,1;3]_{j+2k},$$

where $C_8(k,j)$, $C_9(k,j)$, $C_{10}(k,j)$, and $C_{11}(k,j)$ are given in Theorem 6.1.

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(\Gamma(1))$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2	0	3	0	4	0
2	0	0	0	0	0	0	0	0	0	0	1	0	2	0	2	0	3	1	5	1
4	0	0	0	0	0	0	1	0	1	0	2	1	3	1	4	2	6	3	8	4
6	0	0	0	0	1	0	1	1	2	1	3	2	5	3	7	4	9	6	12	9
8	0	0	0	0	1	1	2	1	3	2	5	4	7	5	9	7	13	10	17	13

$*$ Our theorem is not valid for $k=4$. Igusa has calculated the dimensions for $(j,k)=(0,4)$ in [22]. For $(j,k)=(2,4)$, $(4,4)$, the values are trivial, because we can prove them by using $\dim_{\mathbb{C}} S_{8,2}(\Gamma(1)) = \dim_{\mathbb{C}} S_{8,4}(\Gamma(1)) = 0$ and the multiple of the Eisenstein series of weight 4. For $(j,k)=(6,4)$, $(8,4)$, Ibukiyama has calculated the dimensions by using the Witt operator.

Theorem 7.2. *If $k \geqslant 5$ and j is even, then we have*

$$\begin{aligned}\dim_{\mathbb{C}} S_{k,j}\big(\Gamma(2)\big) &= 2^{-3}3^{-1}(j+1)(k-2)(j+k-1)(j+2k-3)\\ &\quad - 2^{-3}\cdot 5(j+1)(j+2k-3) + 2^{-3}\cdot 3\cdot 5(j+1)\\ &\quad + 2^{-3}\cdot 5(-1)^k(k-2)(j+k-1)\\ &\quad - 2^{-3}\cdot 3\cdot 5(-1)^k(j+2k-3) + 2^{-3}\cdot 3^2\cdot 5(-1)^k.\end{aligned}$$

Theorem 7.3. *If $k \geqslant 5$ and $N \geqslant 3$, then we have*

$$\begin{aligned}\dim_{\mathbb{C}} S_{k,j}\big(\Gamma(N)\big) = \big[\Gamma(1):\Gamma(N)\big] \times \{&2^{-8}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3)\\ &- 2^{-6}3^{-2}(j+1)(j+2k-3)N^{-2} + 2^{-5}3^{-1}(j+1)N^{-3}\},\end{aligned}$$

where $[\Gamma(1):\Gamma(N)] = N^{10}\prod_{p:\text{prime},\, p|N}(1-p^{-2})(1-p^{-4})$.

For a prime p, we set $\Gamma_0(p) = \{\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma(1);\ C \equiv 0 \pmod p\}$. Let χ be a Dirichlet character modulo p. Let $\chi(\gamma) = \chi(\det(D))$ for $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma_0(p)$. We can prove $\chi(\gamma)=1$ for any unipotent element $\gamma \in \Gamma_0(p)$ by simple calculation. Hence, we can apply Theorem 3.2 to the calculation of $\dim_{\mathbb{C}} S_{k,j}(\Gamma_0(p),\chi)$. Hashimoto has not classified $\Gamma_0(p)$-conjugacy classes for $p=2$ in [12]. Since $\Gamma_0(2)$-conjugacy classes can be classified by using the same argument as that in [12], we omit the proof for $p=2$. For a polynomial $f(x)$ with $\mathbb{Z}$-coefficients, in the notation $\sum_{f(a)\equiv 0}$, a runs over all solutions of $f(a) \equiv 0 \bmod p$ on $\mathbb{Z}/p\mathbb{Z}$.

Theorem 7.4. *If $k \geqslant 5$, j is even, and p is prime, then we have $\dim_{\mathbb{C}} S_{k,j}(\Gamma_0(p),\chi) = \sum_{l=1}^{12} H_l$, where H_l, being the total contribution of elements of $\Gamma_0(p)$ with the characteristic polynomial $f_l(\pm x)$, are given as follows*:

$$\begin{aligned}
H_1 &= 2^{-7}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3)(p+1)\bigl(p^2+1\bigr) \\
&\quad - 2^{-4}3^{-2}(j+1)(j+2k-3)(p+1) + 2^{-2}3^{-1}(j+1). \\
H_2 &= 2^{-7}3^{-2}(-1)^k(j+k-1)(k-2)\chi(-1) \times \begin{cases} 7(p+1)^2 & \text{if } p \neq 2, \\ 57 & \text{if } p = 2 \end{cases} \\
&\quad - 2^{-3}3^{-1}(-1)^k(j+2k-3)(p+1)\chi(-1) + 2^{-4}(-1)^k\chi(-1)\left(7 - \left(\frac{-1}{p}\right)\right). \\
H_3 &= -2^{-2}\bigl[(-1)^{j/2}, -1, (-1)^{j/2+1}, 1; 4\bigr]_k \left(\sum_{a^2+1\equiv 0} \chi(a)\right) \\
&\quad + 2^{-5}3^{-1}\bigl[(-1)^{j/2}(k-2), -(j+k-1), (-1)^{j/2+1}(k-2), (j+k-1); 4\bigr]_k \\
&\quad \times (p+1)\left(\sum_{a^2+1\equiv 0} \chi(a)\right). \\
H_4 &= -2^{-1}3^{-2}\bigl\{[1,-1,0;3]_k + [1,0,-1;3]_{j+k}\bigr\}\left(\sum_{a^2+a+1\equiv 0} \chi(a)\right) \\
&\quad - 2\cdot 3^{-2}\bigl\{[1,0,1;3]_k + [0,-1,-1;3]_{j+k}\bigr\}\left(\sum_{a^2+a+1\equiv 0} \chi(a)\right) \\
&\quad + 2^{-3}3^{-3}\bigl\{\bigl[(j+k-1), -(j+k-1), 0; 3\bigr]_k + \bigl[(k-2), 0, -(k-2); 3\bigr]_{j+k}\bigr\} \\
&\quad \times (p+1)\left(\sum_{a^2+a+1\equiv 0} \chi(a)\right). \\
H_5 &= -2^{-1}3^{-1}\bigl\{[-1,-1,0,1,1,0;6]_k + [1,0,-1,-1,0,1;6]_{j+k}\bigr\} \\
&\quad \times \left(\sum_{a^2-a+1\equiv 0} \chi(a)\right) \\
&\quad + 2^{-3}3^{-2}\bigl\{\bigl[-(j+k-1), -(j+k-1), 0, (j+k-1), (j+k-1), 0; 6\bigr]_k \\
&\quad + \bigl[(k-2), 0, -(k-2), -(k-2), 0, (k-2); 6\bigr]_{j+k}\bigr\} \\
&\quad \times (p+1)\left(\sum_{a^2-a+1\equiv 0} \chi(a)\right). \\
H_6 &= -2^{-3}(-1)^{j/2}\left\{2 + \chi(-1)\left(1 + \left(\frac{-1}{p}\right)\right)\right\} \\
&\quad + 2^{-7}3^{-1}(-1)^{j/2}(j+2k-3) \times \begin{cases} 5\{p+1+\chi(-1)(1+(\frac{-1}{p}))\} & \text{if } p \neq 2, \\ 23 & \text{if } p = 2 \end{cases} \\
&\quad + 2^{-7}(-1)^{j/2+k}(j+1) \times \begin{cases} p+1+\chi(-1)(1+(\frac{-1}{p})) & \text{if } p \neq 2, \\ 3 & \text{if } p = 2. \end{cases}
\end{aligned}$$

$$H_7 = -2^{-1}3^{-1}[1,-1,0;3]_j\Bigl(2+\sum_{a^2+a+1\equiv 0}\chi(a)\Bigr)$$
$$+2^{-1}3^{-3}(j+2k-3)[1,-1,0;3]_j\times\begin{cases} p+1+\sum_{a^2+a+1\equiv 0}\chi(a) & \textit{if } p\neq 3,\\ 7 & \textit{if } p=3\end{cases}$$
$$+2^{-2}3^{-3}(j+1)[0,1,-1;3]_{j+2k}$$
$$\times\begin{cases} p-1+(\sum_{a^2+a+1\equiv 0}\chi(a))^2 & \textit{if } p\equiv 1 \bmod 3,\\ p+1 & \textit{if } p\equiv 2 \bmod 3,\\ 1 & \textit{if } p=3.\end{cases}$$

$$H_8 = 2^{-2}3^{-1}C_8(k,j)\Bigl(\sum_{a^2+a+1\equiv 0}\chi(a)\Bigr)\Bigl(\sum_{c^2+1\equiv 0}\chi(c)\Bigr).$$

$$H_9 = 3^{-2}C_9(k,j)\times\begin{cases} (\sum_{a^2-a+1\equiv 0}\chi(a))(\sum_{c^2+c+1\equiv 0}\chi(c)) & \textit{if } p\neq 2,\\ 3/2 & \textit{if } p=2.\end{cases}$$

$$H_{10} = 5^{-1}C_{10}(k,j)\Bigl(\sum_{a^4+a^3+a^2+a+1\equiv 0}\chi(a)\Bigr).$$

$$H_{11} = 2^{-3}C_{11}(k,j)\times\begin{cases} 2\chi(-1)+\sum_{a^2+1\equiv 0}\chi(a) & \textit{if } p\equiv 1 \bmod 8,\\ 2\chi(-1) & \textit{if } p\equiv 3 \bmod 8,\\ \sum_{a^2+1\equiv 0}\chi(a) & \textit{if } p\equiv 5 \bmod 8,\\ 0 & \textit{if } p\equiv 7 \bmod 8,\\ 1 & \textit{if } p=2.\end{cases}$$

$$H_{12} = 2^{-2}3^{-1}(-1)^{j/2}[0,-1,1;3]_{j+2k}\Bigl\{\chi(-1)\Bigl(1+\Bigl(\frac{-1}{p}\Bigr)\Bigr)+\sum_{a^2+a+1\equiv 0}\chi(a)\Bigr\}.$$

$C_8(k,j)$, $C_9(k,j)$, $C_{10}(k,j)$, and $C_{11}(k,j)$ are given in Theorem 6.1.

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(\Gamma_0(3))$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	1	0	2	0	5	0	10	0	16	0	23	1	35	3	47	4
2	0	0	2	0	7	3	16	6	26	12	44	24	67	37	92	54
4	1	0	5	3	14	10	29	20	49	36	79	61	116	90	163	130
6	3	4	11	11	27	25	51	46	84	74	128	116	187	168	258	232
8	5	7	18	19	42	43	77	74	123	118	187	181	269	256	365	349

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(\Gamma_0(3),(\frac{\det(D)}{3}))$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	1	0	4	0	7	0	12	0	20	1	29	1	39	4	55
2	0	1	0	5	1	10	3	21	10	36	17	53	28	79	47	112
4	0	2	2	9	6	20	14	38	29	63	47	95	74	139	111	191
6	1	7	7	19	17	38	33	66	59	106	94	156	138	220	199	301
8	3	10	14	29	30	56	56	98	97	154	148	223	214	314	304	426

∗ Our theorem is not valid for $k=4$. Tsushima has calculated the dimensions for $(j,k)=(0,4)$ in [33]. For $k=4$, $j>0$, the values are conjectural.

Appendix A. Non-cusp forms

In this appendix, we explain some properties of non-cusp forms for $\Gamma(1)$ and $\Gamma^*(1)$. A $\mathbb{C}^{j+1}$-valued holomorphic function f on $\mathfrak{H}_2$ is called a Siegel modular form of weight $\rho_{k,j}$ for Γ if f

satisfies $f(\gamma \cdot Z) = \rho_{k,j}(CZ + D) f(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ and $Z \in \mathfrak{H}_2$. Let $M_{k,j}(\Gamma)$ be the space of Siegel modular forms of weight $\rho_{k,j}$ for Γ. Let $N_{k,j}(\Gamma)$ be the orthogonal complement of $S_{k,j}(\Gamma)$ in $M_{k,j}(\Gamma)$ by the Petersson inner product. We have $M_{k,j}(\Gamma) = S_{k,j}(\Gamma) \oplus N_{k,j}(\Gamma)$. From [23,2], we know the following results for $\Gamma(1)$. We also obtain the following results for $\Gamma^*(1)$ similarly.

Theorem A.1. *Let $k \geqslant 5$. j is even. If k is odd, then $N_{k,j}(\Gamma(1)) = N_{k,j}(\Gamma^*(1)) = \{0\}$. If k is even, we have $\dim_{\mathbb{C}} N_{k,0}(\Gamma(1)) = \dim_{\mathbb{C}} M_k(SL(2;\mathbb{Z}))$, $\dim_{\mathbb{C}} N_{k,j}(\Gamma(1)) = \dim_{\mathbb{C}} S_{k+j}(SL(2;\mathbb{Z}))$ $(j > 0)$, $\dim_{\mathbb{C}} N_{k,0}(\Gamma^*(1)) = 1$, and $\dim_{\mathbb{C}} N_{k,j}(\Gamma^*(1)) = 0$ $(j > 0)$, where $M_k(SL(2;\mathbb{Z}))$ (resp. $S_k(SL(2;\mathbb{Z}))$) is the space of modular forms (resp. cusp forms) of weight k with respect to $SL(2;\mathbb{Z})$.*

Thus, we obtain the dimension formulas for $M_{k,j}(\Gamma(1))$ and $M_{k,j}(\Gamma^*(1))$. The Eisenstein series span the spaces $N_{k,j}(\Gamma(1))$ and $N_{k,j}(\Gamma^*(1))$. For details of the Eisenstein series, we refer to [23,2] (split $\mathbb{Q}$-form case) and [16] (non-split $\mathbb{Q}$-form case). For details of the L-functions of vector-valued Siegel modular forms, we refer to [2] (split $\mathbb{Q}$-form case) and [30] (non-split $\mathbb{Q}$-form case).

Appendix B. Elliptic contributions

Here, we describe the formula for elliptic contributions used in the proofs of Theorems 6.1 and 7.1. The formula was obtained by Hashimoto (cf. [11–13]). If H is an algebraic group defined over $\mathbb{Q}$, we denote the p-adic completion (resp. the adelization) of H by H_p (resp. $H_{\mathbb{A}}$). We set

$$\tilde{G}(\mathbb{Q}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2;\mathbf{B});\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^{\iota} & c^{\iota} \\ b^{\iota} & d^{\iota} \end{pmatrix} = n(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ n(g) \in \mathbb{Q}_{>0} \right\}$$

and $Z(g) = \{z \in M(2;\mathbf{B}) \mid zg = gz\}$ for $g \in \tilde{G}(\mathbb{Q})$. We assume that Γ satisfies the following conditions: (i) there exists a $\mathbb{Z}$-order R of $M(2;\mathbf{B})$ such that $\Gamma = R^\times \cap \tilde{G}(\mathbb{Q})$ and (ii) $n(R_p^\times \cap \tilde{G}_p) = \mathbb{Z}_p^\times$ for all p.

Theorem B.1. *(See [12, Theorem 2-4].) The elliptic contribution in Theorem 3.1 is equal to*

$$c_{k,j} \sum_{\{g\}_{\tilde{G}(\mathbb{Q})}} J_0'(g) \sum_{L_{\tilde{G}}(\Lambda)} M_{\tilde{G}}(\Lambda) \prod_p c_p(g, R_p, \Lambda_p).$$

The notations are defined below.

(1) The first sum is extended over the conjugacy classes in $\tilde{G}(\mathbb{Q})$ of the elements with finite orders, which are locally integral (cf. [12, Theorem 1-3]). (2) $L_{\tilde{G}}(\Lambda)$ runs over the $\tilde{G}$-genera of $\mathbb{Z}$-orders in $Z(g)$. The $\tilde{G}$-genus $L_{\tilde{G}}(\Lambda)$ containing Λ consists of all $\mathbb{Z}$-orders in $Z(g)$, which are conjugate in $Z(g)_p^\times \cap \tilde{G}_p$ with Λ_p for all p. (3) We decompose the group $(Z(g)^\times \cap \tilde{G})_{\mathbb{A}}$ into the disjoint union $(Z(g)^\times \cap \tilde{G})_{\mathbb{A}} = \bigcup_{k=1}^h (Z(g)^\times \cap \tilde{G}(\mathbb{Q})) y_k (\Lambda_{\mathbb{A}}^\times \cap \tilde{G}_{\mathbb{A}})$, $\Lambda_{\mathbb{A}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathbb{A}}$. Let $\Lambda_k = y_k \Lambda y_k^{-1} = \bigcap_p ((y_k)_p \Lambda_p (y_k)_p^{-1} \cap Z(g))$. Then, we define $M_{\tilde{G}}(\Lambda) = \mathrm{vol}(\Lambda_0^\times \cap C_0(g; G(\mathbb{R})) \backslash C_0(g; G(\mathbb{R})))$ $\sum_{k=1}^h [\Lambda_k^\times \cap \tilde{G}(\mathbb{Q}) : \Lambda_0^\times \cap C_0(g; G(\mathbb{Q}))]^{-1}$, where Λ_0 is a fixed $\mathbb{Z}$-order of $Z(g)$ ($M_{\tilde{G}}(\Lambda)$ is the $\tilde{G}$-Mass of Λ). (4) We set $c_p(g, R_p, \Lambda_p) = \sharp((Z(g)^\times \cap \tilde{G})_p \backslash M_p(g, R_p, \Lambda_p)/(R_p^\times \cap \tilde{G}_p))$, where $M_p(g, R_p, \Lambda_p) = \{x \in \tilde{G}_p;\ x^{-1} g x \in R_p,$ there exists an $a \in (Z(g)^\times \cap \tilde{G})_p$ such that $Z(g)_p \cap (x R_p x^{-1}) = a \Lambda_p a^{-1}\}$. (5) $J_0'(g) = J_0(g)$ if $-I_4 \notin C_0(g; G(\mathbb{R}))$, and $J_0'(g) = 2^{-1} J_0(g)$ if $-I_4 \in C_0(g; G(\mathbb{R}))$.

The explicit calculations of the local factors c_p for $R_p = M_2(\mathfrak{O}_p)$ have been carried out by Hashimoto and Ibukiyama [14].

References

[1] T. Arakawa, The dimension of the space of cusp forms on the Siegel upper half plane of degree two related to a quaternion unitary group, J. Math. Soc. Japan 33 (1981) 125–145.

[2] T. Arakawa, Vector valued Siegel's modular forms of degree two, Manuscripta Math. 44 (1983) 155–185.

[3] M. Božičević, A limit formula for elliptic orbital integrals, Duke Math. J. 113 (2002) 331–353.
[4] H. Braun, Konvergenz verallgemeinerter Eisensteinscher Reihen, Math. Z. 44 (1939) 387–397.
[5] U. Christian, Untersuchung einer Poincaréschen Reihe I, J. Reine Angew. Math. 233 (1968) 37–88; II, J. Reine Angew. Math. 237 (1969) 12–25.
[6] U. Christian, Berechnung des Ranges der Schar der Spitzenformen zur Modulgruppe zweiten Grades und Stufe $q > 2$, J. Reine Angew. Math. 277 (1975) 130–154;
U. Christian, Zur Berechnung des Ranges der Schar der Spitzenformen zur Modulgruppe zweiten Grades und Stufe $q > 2$, J. Reine Angew. Math. 296 (1977) 108–118.
[7] J. Faraut, A. Korányi, Analysis on Symmetric Cones, Oxford Science Publications, Clarendon Press, Oxford, 1994.
[8] R. Godement, Functions automorphes, in: Sem. Cartan, Paris, 1957/1958, Expose 5, 6, 10.
[9] E. Gottschling, Über die Fixpunkte der Siegelschen Modulgruppe, Math. Ann. 143 (1961) 111–149;
E. Gottschling, Über die Fixpunktuntergruppen der Siegelschen Modulgruppe, Math. Ann. 143 (1961) 399–430.
[10] K. Gunji, On the graded ring of Siegel modular forms of degree 2, level 3, J. Math. Soc. Japan 56 (2) (2004) 375–403.
[11] K. Hashimoto, On Brandt matrices associated with the positive definite quaternion Hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980) 227–245.
[12] K. Hashimoto, The dimension of the spaces of cusp forms on Siegel upper half-plane of degree two I, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 30 (1983) 403–488.
[13] K. Hashimoto, The dimension of the spaces of cusp forms on Siegel upper half-plane of degree two II. The $\mathbb{Q}$-rank one case, Math. Ann. 266 (1984) 539–559.
[14] K. Hashimoto, T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms (I), J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980) 549–601.
[15] K. Hashimoto, T. Ibukiyama, On relations of dimensions of automorphic forms of $Sp(2, R)$ and its compact twist $Sp(2)$ (II). Automorphic forms and number theory, Adv. Stud. Pure Math. 7 (1985) 31–102.
[16] Y. Hirai, On Eisenstein series on quaternion unitary groups of degree 2, J. Math. Soc. Japan 51 (1999) 93–128.
[17] T. Ibukiyama, On maximal orders of division quaternion algebras over the rational number field with certain optimal embeddings, Nagoya Math. J. 88 (1982) 181–195.
[18] T. Ibukiyama, On relations of dimensions of automorphic forms of $Sp(2, R)$ and its compact twist $Sp(2)$ (I). Automorphic forms and number theory, Adv. Stud. Pure Math. 7 (1985) 7–30.
[19] T. Ibukiyama, Siegel modular forms of weight three and conjectural correspondence of Shimura type and Langlands type, in: Conference on L-Functions, World Sci. Publ., Hackensack, NJ, 2007, pp. 55–69.
[20] T. Ibukiyama, H. Saito, On zeta functions associated to symmetric matrices I. An explicit form of zeta functions, Amer. J. Math. 117 (1995) 1097–1155.
[21] T. Ibukiyama, S. Wakatsuki, Siegel modular forms of small weight and the Witt operator, Contemp. Math. 493 (2009) 189–209.
[22] J. Igusa, On Siegel modular forms of genus two II, Amer. J. Math. 86 (1964) 392–412.
[23] H. Klingen, Introductory Lectures on Siegel Modular Forms, Cambridge Stud. Adv. Math., vol. 20, Cambridge University Press, Cambridge, 1990.
[24] R.P. Langlands, The dimension of spaces of automorphic forms, Amer. J. Math. 85 (1963) 99–125.
[25] Y. Morita, An explicit formula for the dimension of spaces of Siegel modular forms of degree two, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 21 (1974) 167–248.
[26] W. Rossmann, Nilpotent orbital integrals in a real semisimple Lie algebra and representations of Weyl groups, in: Progr. Math., vol. 92, Birkhäuser, Boston, 1990, pp. 263–287.
[27] F. Sato, On zeta functions of ternary zero forms, J. Fac. Sci. Univ. Tokyo Sect. IA 28 (1982) 585–604.
[28] T. Shintani, On zeta functions associated with the vector space of quadratic forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975) 25–65.
[29] C.L. Siegel, Über die analytische Theorie der quadratischen Formen, Ann. of Math. 36 (1935) 527–606.
[30] T. Sugano, On holomorphic cusp forms on quaternion unitary groups of degree 2, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 31 (1985) 521–568.
[31] R. Tsushima, On the spaces of Siegel cusp forms of degree two, Amer. J. Math. 65 (1982) 843–885.
[32] R. Tsushima, An explicit dimension formula for the spaces of generalized automorphic forms with respect to $Sp(2, \mathbb{Z})$, Proc. Japan Acad. Ser. A 59 (1983) 139–142.
[33] R. Tsushima, Dimension formula for the spaces of Siegel cusp forms and a certain exponential sum, Mem. Inst. Sci. Tech. Meiji Univ. 36 (1997) 1–56, http://www.math.meiji.ac.jp/~tsushima.
[34] K. Ueno, On fibre spaces of normally polarized abelian varieties of dimension 2. I, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 18 (1971) 37–95; II, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 19 (1972) 163–199.
[35] N. Wallach, On the constant term of a square integrable automorphic form, in: Operator Algebras and Group Representations, vol. II, Neptun, 1980, in: Monogr. Stud. Math., vol. 18, Pitman, Boston, MA, 1984, pp. 227–237.
[36] G. Warner, Harmonic Analysis on Semisimple Lie Groups I, II, Springer-Verlag, Berlin, Heidelberg, New York, 1972.
[37] H. Yamaguchi, The parabolic contributions to the dimension formula of the space of cusp forms on Siegel space of degree two, unpublished, 1976.
[38] T. Yamazaki, On Siegel modular forms of degree two, Amer. J. Math. 98 (1976) 39–53.