Journal of Number Theory 132 (2012) 54–78

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

L-functions of $S_3(\Gamma_2(2,4,8))$

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ARTICLE INFO

Article history:
Received 27 April 2010
Revised 29 January 2011
Accepted 25 June 2011
Available online 27 August 2011
Communicated by D. Zagier

MSC:
11F27
11F55
11F70
14K25

Keywords:
Theta function
Theta correspondence
Siegel threefold variety and modular form
Hermitian modular form

ABSTRACT

van Geemen and van Straten [B. van Geemen, D. van Straten, The cuspform of weight 3 on $\Gamma_2(2,4,8)$, Math. Comp. 61 (1993) 849–872] showed that the space of Siegel modular cusp forms of degree 2 of weight 3 with respect to the so-called Igusa group $\Gamma_2(2,4,8)$ is generated by 6-tuple products of Igusa theta constants, and each of them are Hecke eigenforms. They conjectured that some of these products generate Saito–Kurokawa representations, weak endoscopic lifts, or D-critical representations. In this paper, we prove these conjectures. Additionally, we obtain holomorphic Hermitian modular eigenforms of GU(2, 2) of weight 4 from these representations.

1. Introduction

Let $\mathfrak{H}_2 = \{Z = {}^tZ \in M_2(\mathbb{C}) \mid \Im(Z) > 0\}$ be the Siegel upper half space of degree 2. Let

$$\theta_m(Z) = \sum_{x\in\mathbb{Z}^2} \exp\left(2\pi i\left(\frac{1}{2}\left(x+\frac{m'}{2}\right)Z^t\left(x+\frac{m'}{2}\right)+\left(x+\frac{m'}{2}\right)^t\left(\frac{m''}{2}\right)\right)\right)$$

be the Igusa theta constant with $m = (m', m'') \in \mathbb{Q}^2 \times \mathbb{Q}^2$. For a congruence subgroup Γ of $\mathrm{Sp}_4(\mathbb{Z})$ ($\subset \mathrm{SL}_4(\mathbb{Z})$), let S_Γ denote the Siegel modular 3-fold and $S_3(\Gamma)$ denote the space of Siegel modular cusp forms of weight 3 with respect to Γ. van Geemen and van Straten showed that $S_3(\Gamma_2(2,4,8))$

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[1] This work was supported partially by Grant-in-Aid for JSPS Fellows.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2011.06.015

is spanned by certain 6-tuple products $\prod_{j=1}^{6}\theta_{m_j}(n_j Z)$ with $m_j \in \{0,1\}^4, n_j \in \{1,2\}$ using the theta embedding of $S_{\Gamma(2,4,8)}$ into $\mathbb{P}^{13}$ (cf. [6]), where $\Gamma_2(2,4,8)=\Gamma(2,4,8)$ is defined by

$$\left\{ I_4 + 4\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) \;\middle|\; A,B,C,D \in \mathrm{M}_2(\mathbb{Z}),\ \mathrm{diag}(B)\equiv \mathrm{diag}(C)\equiv 0,\ \mathrm{tr}(A)\equiv 0 \pmod 2 \right\}. \tag{1.1}$$

Through Igusa's transformation formula, $\mathrm{Sp}_4(\mathbb{Z})$ acts on these 6-tuple products. They showed that $S_3(\Gamma(2,4,8))$ is decomposed into eleven irreducible $\mathrm{Sp}_4(\mathbb{Z})$-modules, and each module is generated by acting $\mathrm{Sp}_4(\mathbb{Z})$ a 6-tuple product of Igusa theta constants. Further, they showed that these 6-tuple products is associated to irreducible cuspidal automorphic representations of $\mathrm{PGSp}_4(\mathbb{A})$ (cf. Proposition 2.2). Computing some eigenvalues of Evdokimov's Hecke operators on

$$g_1(Z) := \theta_{(0,0,0,0)}(2Z)\theta_{(1,0,0,0)}(Z)\theta_{(0,1,0,0)}(Z)\theta_{(0,0,1,0)}(Z)\theta_{(0,0,1,0)}(Z)\theta_{(0,0,0,1)}(Z),$$
$$g_4(Z) := \theta_{(0,0,0,0)}(2Z)\theta_{(1,0,0,0)}(2Z)\theta_{(0,1,0,0)}(2Z)\theta_{(0,0,1,0)}(Z)\theta_{(0,0,0,1)}(Z)\theta_{(0,0,1,1)}(Z),$$

they gave:

Conjecture. *(See van Geemen and van Straten [7].) Let Π_{g_i} be the irreducible cuspidal automorphic representation of* $\mathrm{PGSp}_4(\mathbb{A})$ *associated to g_i. Then the spinor L-functions (of degree 4) are*

$$L(s,\Pi_{g_1};\mathrm{spin}) = L(s,\lambda), \qquad L(s,\Pi_{g_4};\mathrm{spin}) = L\left(s-\frac{1}{2}, \left(\frac{-2}{*}\right)\right) L\left(s+\frac{1}{2}, \left(\frac{-2}{*}\right)\right) L(s,\rho_1),$$

up to the Euler factors at 2. Here λ is a größencharacter of the bi-quadratic CM-field $\mathbb{Q}(i,\sqrt{2})$ of conductor 2, ρ_1 is an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ *of lowest weight 4 of level 8, and $(\frac{*}{*})$ is the Legendre symbol.*

In this paper, we prove

Theorem A. *The conjecture is true.*

More precisely, their conjecture referred to Andrianov–Evdokimov's L-functions $L(s,g_i;\mathrm{AE})$. However, $L(s,g_i;\mathrm{AE})$ is essentially equal to the (partial) spinor L-functions of Π_{g_i} (cf. Proposition 2.1). Anyway, Theorem A means that Π_{g_1} is a D-critical representation in the sense of Weissauer [31], and Π_{g_4} is the $(\frac{-2}{*})$-twist of a Saito–Kurokawa representation associated to $\rho_1\otimes(\frac{-2}{*})$. Let $\mathrm{Gr}_3^W H^3(S_{\Gamma_{g_i}},\mathbb{C})$ be the graded quotient of degree 3 of a mixed Hodge structure on $H^3(S_{\Gamma_{g_i}},\mathbb{C})$. Theorem A also means that g_i corresponds to a generator of the 1-dimensional space $H^{3,0}(\mathrm{Gr}_3^W H^3(S_{\Gamma_{g_i}},\mathbb{C}))$ associated to a quotient $S_{\Gamma_{g_i}}$ of $S_{\Gamma(2,4,8)}$ (cf. Proposition 2.3). We are interested in the quotients $S_{\Gamma_{f_i}}, S_{\Gamma_{g_i}}$ of $S_{\Gamma(2,4,8)}$, for various reasons. Let $S'_{\Gamma_{f_5}}$ be a resolution of the Satake compactification of $S_{\Gamma_{f_5}}$. van Geemen and Nygaard [6] calculated the Hodge numbers $h^{3,0}$ and $h^{2,1}$ of $S'_{\Gamma_{f_5}}$ are both equal to one and showed that the L-function of the third etale cohomology of $S'_{\Gamma_{f_5}}$ is equal to $L(s-\frac{3}{2},\mu)L(s-\frac{3}{2},\mu^3)$, up to the Euler factors at 2, where μ is the unitary größencharacter related to the CM-elliptic curve $E_{/\mathbb{Q}}: y^2=x^3-x$. Because f_5 corresponds to the generator of $H^{3,0}(\mathrm{Gr}_3^W H^3(S_{\Gamma_{f_5}},\mathbb{C}))$, it was conjectured in [7,6] and verified in [17] that $L(s,\Pi_{f_5};\mathrm{spin}) = L(s,\mu)L(s,\mu^3)$, up to the Euler factors at 2. Thus, Π_{f_5} is a weak endoscopic lift of $(\pi(\mu),\pi(\mu^3))$ in the sense of [31] and we have

$$L\big(s, H^3_{\mathrm{et}}\big(S'_{\Gamma_{f_5}},\mathbb{Q}_2\big)\big) = L\left(s-\frac{3}{2},\Pi_{f_5};\mathrm{spin}\right),$$

up to the Euler factors at 2, where $\pi(\mu)$ indicates the irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ associated to μ. From the above Hodge numbers and these L-functions, it is natural to guess that a weak endoscopic lift of $(\pi(\mu), \pi(\mu^3))$ contributes to $H^{2,1}(\mathrm{Gr}_3^W H^3(S_{\Gamma_{f_5}}, \mathbb{C}))$. In Section 3.3, we will give the desired weak endoscopic lift.

We have verified in [17] their conjectures on $L(s, \Pi_{f_i}; \mathrm{spin})$ for $1 \leqslant i \leqslant 6$, and we will verify in another work in preparation their conjectures for Π_{f_7} and Π_{g_3}. Here f_i, g_j with $1 \leqslant i \leqslant 7$, $1 \leqslant j \leqslant 4$ are certain 6-tuple products of Igusa theta constants. Combining all these works, we will complete the proof for the conjectures given in [7].

By the way, our result means that there are irreducible automorphic representations of GSO(6) related to these representations of GSp(4) with the θ-correspondence. We find holomorphic Hermitian modular forms of GU(2,2) of weight 4 from the Siegel modular forms of weight 3 by the following theorem.

Theorem B. *Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field. Let $\mathrm{B}_{/\mathbb{Q}}$ be a definite quaternion algebra such that $\mathrm{B} \otimes K \simeq \mathrm{M}_2(K)$. Put $V = K + \mathrm{B}_{/\mathbb{Q}}$. Suppose that a Siegel modular eigen-cusp form F of degree 2 of weight 3 is given by a θ-lift from PGSO_V. Then, there is a holomorphic Hermitian modular form $\tilde{F}$ of $\mathrm{PGU}_{2,2}(K)$ of weight 4 with*

$$L\big(s, \tilde{F}; \wedge_t^2\big) = \zeta(s) L\left(s, F, \left(\frac{-d}{*}\right); r_5\right), \tag{1.2}$$

outside of finitely many bad places. If F satisfies the generalized Ramanujan conjecture at almost all good places, then $\tilde{F}$ is a cusp form. Here $L(s, F, (\frac{-d}{*}); r_5)$ is the $(\frac{-d}{*})$-twist of the L-function of degree five, and $L(s, \tilde{F}; \wedge_t^2)$ is the L-function of $\tilde{F}$ with respect to the twisted exterior square map from the L-group ${}^L\mathrm{GU}_{2,2}(\mathbb{C})$ to $\mathrm{GL}_6(\mathbb{C})$ introduced by Kim and Krishnamurthy [11].

Notice that a holomorphic Hermitian cusp form of GU(2,2) of weight 4 is canonically identified with a holomorphic differential 4-form on a modular 4-fold. A globally generic weak endoscopic lift of $\mathrm{PGSp}_4(\mathbb{A})$ is sent to a noncuspidal representation of $\mathrm{PGL}_4(\mathbb{A})$ through the generic transfer lift to $\mathrm{GL}_4(\mathbb{A})$ (cf. [2]). However, a holomorphic weak endoscopic lift as in Theorem B is sent to a cuspidal automorphic holomorphic representation.

The paper is organized as follows. After reviewing a result of van Geemen and van Straten [7], and summarizing our main tools θ-lifts, and Whittaker functions in Section 1, we prove Theorem A in Section 2. We prove Theorem B in Section 3.

Notation. For a reductive algebraic group G defined over a number field F, let $\mathcal{A}(G(\mathbb{A}))$ denote the space of automorphic forms on $G(\mathbb{A})$. At a place v of F, let $\mathrm{Irr}(G(F_v))$ denote the set of equivalence classes of irreducible admissible representations of $G(F_v)$. If σ is an element of $\mathrm{Irr}(G(F_v))$ or irreducible automorphic representation, then ω_σ denotes the central character of σ. For an irreducible automorphic representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A})$, let S_π denote the finite set of places for which π_v is ramified, and let $L_S(s, \pi; r) = \prod_{v \notin S} L(s, \pi_v; r)$ the partial Langlands L-function outside of $S (\supset S_\pi)$ with respect to a finite dimensional representation r of the L-group of $G(k_v)$. For a commutative ring R, we denote

$$\mathrm{GSp}_{2n}(R) = \left\{ g \in \mathrm{GL}_{2n}(R) \;\middle|\; {}^t g \eta_n g = \nu(g) \eta_n \right\}$$

where $\eta_n = \left[\begin{smallmatrix} & -I_n \\ I_n & \end{smallmatrix}\right]$ and $\nu(g) \in R^\times$ is the similitude norm of g. We will denote by $Z(R) (\simeq R^\times)$ the center of $\mathrm{GSp}_{2n}(R)$. For a quasi-character χ and a representation τ of $\mathrm{GSp}_{2n}(R)$, let $\chi\tau$ denote the representation sending g to $\chi(\nu(g))\tau(g)$.

2. Preliminaries

2.1. Review of van Geemen and van Straten's result

van Geemen and van Straten computed some local factors of Evdokimov's L-functions of the 6-tuple products f_i, g_j of Igusa theta constants. To begin with, we will compare Evdokimov's L-function of a Siegel modular cusp form of degree 2 with the spinor L-function of a unitary irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. We will relate Siegel modular forms to automorphic forms, in order to regard Evdokimov's Hecke operator for Siegel modular forms as an operator for automorphic forms. For $Z \in \mathfrak{H}_2$ and $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}_4(\mathbb{R})$, let $j(g, Z) = \det(CZ + D)$ and $g \cdot Z = (AZ + B)(CZ + D)^{-1}$. For a function f on $\mathfrak{H}_2$, an element $g \in \mathrm{Sp}_4(\mathbb{R})$, and a positive integer κ, we define

$$f|_\kappa g(Z) = j(g, Z)^{-\kappa} f(g \cdot Z).$$

Let $\mathbb{K}_\infty = \{g \in \mathrm{Sp}_4(\mathbb{R}) \mid g \cdot \mathrm{i}_2 = \mathrm{i}_2\} \simeq \mathrm{U}_2(\mathbb{C})$ where $\mathrm{i}_2 = \mathrm{i}I_2$. For a congruence subgroup $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$, let

$$\Gamma_{\mathbb{A}} = \mathbb{K}_\infty \otimes_{p<\infty} \Gamma_p, \qquad \Gamma_{\mathbb{A},0} = \bigotimes_{p<\infty} \Gamma_p,$$

where Γ_p is the p-adic completion of Γ. For a Siegel modular form f of degree 2 of weight κ with respect to a congruence subgroup $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$, we put $f^\sharp(g) = f(g \cdot \mathrm{i}_2) j(g, \mathrm{i}_2)^{-\kappa}$ with $g \in \mathrm{Sp}_4(\mathbb{R})$. Through the isomorphism: $\Gamma \backslash \mathfrak{H}_2 \simeq \Gamma \backslash \mathrm{Sp}_4(\mathbb{R})/\mathbb{K}_\infty \simeq \mathrm{Sp}_4(\mathbb{Q}) \backslash \mathrm{Sp}_4(\mathbb{A})/\Gamma_{\mathbb{A}}$, we extend $f^\sharp$ to an automorphic form on $\mathrm{Sp}_4(\mathbb{A})$, which is also denoted by $f^\sharp$. Let $\tilde{\Gamma}_p$ be the compact subgroup of $\mathrm{GSp}_4(\mathbb{Z}_p)$ generated by elements of Γ_p and $\left[\begin{smallmatrix} I_2 & \\ & zI_2 \end{smallmatrix}\right]$ with $z \in \mathbb{Z}_p^\times$. Let $\tilde{\Gamma}_{\mathbb{A}} = (Z(\mathbb{R})\mathbb{K}_\infty) \otimes_{p<\infty} \tilde{\Gamma}_p$. Because $\mathrm{Sp}_4(\mathbb{Q}) \backslash \mathrm{Sp}_4(\mathbb{A})/\Gamma_{\mathbb{A}} \simeq \mathrm{GSp}_4(\mathbb{Q}) \backslash \mathrm{GSp}_4(\mathbb{A})/\tilde{\Gamma}_{\mathbb{A}}$, we can write an element $g \in \mathrm{GSp}_4(\mathbb{A})$ as $\gamma t g_1 \left[\begin{smallmatrix} I_2 & \\ & zI_2 \end{smallmatrix}\right]$ with $g_1 \in \mathrm{Sp}_4(\mathbb{A})$, $\gamma \in \mathrm{GSp}_4(\mathbb{Q})$, $t \in Z(\mathbb{R})$, $z \in \bigotimes_p \mathbb{Z}_p^\times$. We put

$$\tilde{f}(g) = f^\sharp(g_1). \tag{2.1}$$

Then, $\tilde{f}$ is an automorphic form on $\mathrm{GSp}_4(\mathbb{A})$. Let χ_Γ be a congruence character of $\Gamma/\Gamma(N)$. Let

$$S_\kappa(\chi_\Gamma) = \bigl\{ f \in S_\kappa\bigl(\Gamma(N)\bigr) \bigm| f|_\kappa \gamma = \chi_\Gamma(\gamma) f \ (\gamma \in \Gamma) \bigr\}.$$

We identify χ_Γ with a character $\chi_\Gamma = \mathbf{1}_\infty \otimes_p \chi_{\Gamma_p}$ on $\Gamma_{\mathbb{A}}$. For an integer N, let $\Gamma^\sharp(N)_p$ be the subgroup generated by elements of $\Gamma(N)_p$ and $\left[\begin{smallmatrix} zI_2 & \\ & z^{-1}I_2 \end{smallmatrix}\right]$ with $z \in \mathbb{Z}_p^\times$. Define $\Gamma^\sharp(N) = \mathrm{Sp}_4(\mathbb{Q}) \cap \bigotimes_p \Gamma^\sharp(N)_p$. For a character $\chi_{\Gamma^\sharp(N)}$ on $\Gamma^\sharp(N)/\Gamma(N)$, we define $\tilde{\chi}_{\Gamma(N)_p}(u) = \chi_{\Gamma(N)_p}(u\left[\begin{smallmatrix} 1 & \\ & \nu(u)^{-1}I_2 \end{smallmatrix}\right])$ and $\chi_{\tilde{\Gamma}(N)} = \mathbf{1}_\infty \otimes_p \tilde{\chi}_{\Gamma(N)_p}$. Let

$$\mathcal{A}_\kappa(\chi_{\tilde{\Gamma}(N)}) = \bigl\{ f \in \mathcal{A}\bigl(\mathrm{GSp}_4(\mathbb{A})\bigr) \bigm| \varrho(u) f = j(u_\infty, \mathrm{i}_2)^{-\kappa} \otimes_p \tilde{\chi}_{\Gamma_p}(u_p) f \text{ for } u \in \tilde{\Gamma}(N)_{\mathbb{A}} \bigr\}. \tag{2.2}$$

Note that the central character of each $f \in \mathcal{A}_\kappa(\chi_{\tilde{\Gamma}(N)})$ is unitary. If $f \in S_\kappa(\chi_{\Gamma^\sharp(N)})$, then $\tilde{f} \in \mathcal{A}_\kappa(\chi_{\tilde{\Gamma}(N)})$. Now, we can regard Evdokimov's Hecke operators (cf. (2.13) of [5]) for Siegel modular forms as the following operator T'_{p^n} for $\mathcal{A}_\kappa(\chi_{\tilde{\Gamma}(N)})$ with $p \nmid N$:

$$T'_{p^n} \tilde{f}(g) = p^{n(\kappa-3)} \sum_j \tilde{f}\bigl(i_\infty(h_j) g\bigr) = p^{n(\kappa-3)} \sum_j \tilde{f}\bigl(g i_\infty(h_j) h_j^{-1}\bigr)$$

where $g \in \mathrm{Sp}_4(\mathbb{R})$, i_v denotes the embedding $\mathrm{GSp}_4(\mathbb{Q})$ to $\mathrm{GSp}_4(\mathbb{Q}_v)$, and $h_j \in \mathrm{GSp}_4(\mathbb{Q}) \cap \mathrm{M}_4(\mathbb{Z})$ is taken so that

$$h_j \equiv \begin{bmatrix} I_2 & \\ & p^n I_2 \end{bmatrix} \pmod N, \qquad \Gamma(N)\begin{bmatrix} I_2 & \\ & p^n I_2 \end{bmatrix}\Gamma(N) = \bigsqcup_j \Gamma(N) h_j. \tag{2.3}$$

Suppose that $f \in S_\kappa(\chi_{\Gamma^\sharp(N)})$ is a common eigenform and that $\tilde{f}$ lies in a (unitary) irreducible cuspidal automorphic representation π. Let λ'_{p^n} denote the eigenvalue of T'_{p^n} on f. The p-factor of Evdokimov's L-function of f is

$$\begin{aligned}&\bigl(1 - \lambda'_p p^{-s} + \bigl(\lambda'^2_p - \lambda'_{p^2} - \omega_{\pi_p}(p)^{-1} p^{2\kappa-4}\bigr) p^{-2s} - \omega_{\pi_p}(p)^{-1}\lambda'_p p^{2\kappa-3-3s}\\ &\quad + \omega_{\pi_p}(p)^{-2} p^{4\kappa-6-4s}\bigr)^{-1}.\end{aligned} \tag{2.4}$$

Let λ_{p^n} be the eigenvalue of the Hecke operator

$$T_{p^n}\tilde{f}(g) = \sum_j \tilde{f}\bigl(g i_p(h_j)\bigr) = \sum_j \omega_{\pi_p}\bigl(p^n\bigr)\tilde{f}\bigl(g i_p(h_j)^{-1}\bigr). \tag{2.5}$$

The spinor L-function of unramified π_p is

$$\bigl(1 - p^{-3/2}\lambda_p p^{-s} + p^{-3}\bigl(\lambda_p^2 - \lambda_{p^2} - p^2\omega_{\pi_p}(p)\bigr)p^{-2s} - p^{-3/2}\omega_{\pi_p}(p)\lambda_p p^{-3s} + \omega_{\pi_p}(p)^2 p^{-4s}\bigr)^{-1}.$$

In order to compare these L-functions, we recall generalized Whittaker function. Let F be a Siegel modular cusp form, and $\tilde{F}$ be the automorphic form on $\mathrm{GSp}_4(\mathbb{A})$ related to F as above. Let $\mathfrak{S}_2(\mathbb{Q}) = \{T = {}^tT \in \mathrm{M}_2(\mathbb{Q})\}$. For a $T \in \mathfrak{S}_2(\mathbb{Q})$, the Fourier coefficient $\tilde{F}_T$ with respect to ψ of $\tilde{F}$ is

$$\tilde{F}_T(g) = \int_{\mathfrak{S}_2(\mathbb{Q})\backslash\mathfrak{S}_2(\mathbb{A})} \psi\bigl(\mathrm{Trace}(Ts)\bigr)^{-1}\tilde{F}\left(\begin{bmatrix} I_2 & s \\ & I_2 \end{bmatrix} g\right) ds,$$

and that of F is $\tilde{F}_T(1)$. Because F is a cusp form, some $\tilde{F}_T(1)$ is not zero for some T with $\det T \neq 0$. For a character μ of $\mathrm{SO}_T(\mathbb{Q})\backslash\mathrm{SO}_T(\mathbb{A})$, the generalized Whittaker function $\tilde{F}_T^\mu$ is defined by

$$\tilde{F}_T^\mu(g) = \int_{\mathrm{SO}_T(\mathbb{Q})\backslash\mathrm{SO}_T(\mathbb{A})} \mu(z)^{-1}\tilde{F}_T\left(\begin{bmatrix} z & \\ & {}^tz^{-1} \end{bmatrix} g\right) dz$$

and factors as $\bigotimes_v \tilde{F}^\mu_{T,v}$ (cf. [19]). Because $\tilde{F}_T = \sum_\mu \tilde{F}_T^\mu$, some $\tilde{F}_T^\mu(1)$ is not zero.

Proposition 2.1. *Suppose that a Siegel modular form $f \in S_\kappa(\Gamma(N))$ of degree 2 is a common eigenfunction with respect to Evdokimov's Hecke operators. Suppose that $\tilde{f}$ lies in a (unitary) irreducible cuspidal automorphic representation π. Then, for $p \nmid N$,*

$$L(s, f; \mathrm{AE})_p = L\left(s - \kappa + \frac{3}{2}, \omega_{\pi,p}^{-1}\pi_p; \mathrm{spin}\right).$$

Proof. It suffices to show that

$$\lambda'_{p^n} = p^{n(\kappa-3)}\omega_{\pi,p}(p)^{-n}\lambda_{p^n} \tag{2.6}$$

for $n = 1, 2$. To do it, we will observe the actions of the operators on $\tilde{f}_T^\mu = \bigotimes_v \tilde{f}_{T,v}^\mu$ with $T \in \mathfrak{S}_2(\mathbb{Z})$ such that $\tilde{f}_T^\mu(1) \neq 0$. Then $\tilde{f}_{T,p}^\mu(1) \neq 0$. Abbreviate $\tilde{f}_{T,p}^\mu$ as B_p. In the case $n = 1$, as a complete system $\{h_j\}$ in (2.3), we can take the following types:

$$\begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & p & \\ & & & p \end{bmatrix}, \quad \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ & p & * & * \\ & & p & \\ & & * & 1 \end{bmatrix}, \quad \begin{bmatrix} p & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & p \end{bmatrix}$$

where $*$ indicate elements of $\mathbb{Z}$. But one can show $B_p(h_j^{-1}) = 0$, if h_j is not of the first type. Indeed, for example, using the property

$$\begin{aligned} B_p\left(\begin{bmatrix} pI_2 & \\ & I_2 \end{bmatrix}^{-1}\right) &= B_p\left(\begin{bmatrix} pI_2 & \\ & I_2 \end{bmatrix}^{-1} n(s)\right) \\ &= B_p\left(n\left(p^{-1}s\right)\begin{bmatrix} pI_2 & \\ & I_2 \end{bmatrix}^{-1}\right) \\ &= \psi_p\left(\frac{\mathrm{Trace}(Ts)}{p}\right) B_p\left(\begin{bmatrix} pI_2 & \\ & I_2 \end{bmatrix}^{-1}\right) \end{aligned}$$

for $s \in S_2(\mathbb{Z}_p)$, one can show that $B_p(\left[\begin{smallmatrix} pI_2 & \\ & I_2 \end{smallmatrix}\right]^{-1}) = 0$. Here $n(s) = \left[\begin{smallmatrix} I_2 & s \\ & I_2 \end{smallmatrix}\right]$, and note that B_p is right $\mathrm{GSp}_4(\mathbb{Z}_p)$-invariant. Then, (2.6) is derived from (2.1). The argument for the case $n = 2$ is similar to that for the case $n = 1$ and omitted. $\square$

Next, we recall the result of Sections 6, 7 of van Geemen and van Straten [7]. Let

$$\Gamma'(2) = \left\{ \begin{bmatrix} A & B \\ 2C' & D \end{bmatrix} \in \Gamma(2)\bigl(\subset \mathrm{Sp}_4(\mathbb{Z})\bigr) \;\middle|\; \mathrm{diag}\bigl(C'\bigr) \equiv 0 \ (\mathrm{mod}\ 2) \right\},$$

$$\Gamma(4, 8) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma(4)\bigl(\subset \mathrm{Sp}_4(\mathbb{Z})\bigr) \;\middle|\; \mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv 0 \ (\mathrm{mod}\ 8) \right\}.$$

Let f_i, g_j with $1 \leqslant i \leqslant 7$, $1 \leqslant j \leqslant 4$ be the 6-tuple products of Igusa theta constants in the table on p. 864 of [7]. We will abbreviate $f_i|_3\gamma, g_j|_3\gamma'$ for some $\gamma, \gamma' \in \mathrm{Sp}_4(\mathbb{Z})$ as f_i', g_j'. Through Igusa's transformation formula, from $F = f_i'$ (resp. g_j'), we obtain a congruence character χ_F of $\Gamma(2)$ (resp. $\Gamma'(2)$). In Theorem 6.4 of [7], they showed that $S_3(\chi_F)$ is 1-dimensional and

$$S_3\bigl(\Gamma(4)\bigr) = \sum_{f_1'} S_3(\chi_{f_1'}),$$

$$S_3\bigl(\Gamma(4,8)\bigr) = S_3\bigl(\Gamma(4)\bigr) + \sum_{i=2}^{7} \sum_{f_i'} S_3(\chi_{f_i'}),$$

$$S_3\bigl(\Gamma(2,4,8)\bigr) = S_3\bigl(\Gamma(4,8)\bigr) + \sum_{j=1}^{4} \sum_{g_j'} S_3(\chi_{g_j'}).$$

Proposition 2.2. *(See van Geemen and van Straten [7].) Let $\tilde{f}_i, \tilde{g}_j$ be the automorphic forms related to f_i, g_j as above. Then each $\tilde{f}_i$ (resp. $\tilde{g}_j$) lies in an irreducible cuspidal automorphic representation of* $\mathrm{PGSp}_4(\mathbb{A})$.

Proof. Let $f = f_i$. Write $\tilde{f} = \sum_l h_l \in \sum_l \pi_l$ where π_l's are irreducible cuspidal automorphic representations. From (2.1), it follows that $\varrho([\begin{smallmatrix} I_2 & \\ & zI_2 \end{smallmatrix}])\tilde{f} = \tilde{f}$ for any $z \in \mathbb{Z}_{\mathbb{A},0}^\times$. Thus,

$$\mathrm{vol}(\mathbb{Z}_{\mathbb{A},0}^\times)^{-1} \int_{\mathbb{Z}_{\mathbb{A},0}^\times} \sum_l \varrho\left(\begin{bmatrix} I_2 & \\ & zI_2 \end{bmatrix}\right) h_l \, \mathrm{d}z = \sum_l h_l.$$

Hence, we can assume that

$$\varrho\left(\begin{bmatrix} I_2 & \\ & zI_2 \end{bmatrix}\right) h_l = h_l, \quad z \in \mathbb{Z}_{\mathbb{A},0}^\times. \tag{2.7}$$

With the similar argument, we can assume that

$$\varrho(u_0) h_l = \chi_f(u_0) h_l, \quad u_0 \in \Gamma(2)_{\mathbb{A},0}, \tag{2.8}$$

$$\varrho(u_\infty) h_l = \det(-B\mathrm{i}_2 + A)^{-3} h_l, \quad u_\infty = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in \mathbb{K}_\infty. \tag{2.9}$$

Using Proposition 6.2 of [7], we find that $\chi_{f,p}([\begin{smallmatrix} zI_2 & \\ & z^{-1}I_2 \end{smallmatrix}]) = 1$ for any $z \in \mathbb{Z}_p^\times$. It follows that the central character of $\tilde{f}$ is trivial. Hence ω_{π_l} is also trivial. Consulting Eq. (2) of p. 505 of Oda and Schwermer [16], we find that $\pi_{l,\infty}|_{\mathrm{Sp}_4}$ is the holomorphic discrete series representation with Blattner parameter $(3,3)$. Define the function $h_l^\flat$ on $Z \in \mathfrak{H}_2$ by $h_l^\flat(Z) = h_l(g_\infty) j(g_\infty, \mathrm{i}_2)^3$, where $g_\infty \in \mathrm{Sp}_4(\mathbb{R})$ is taken so that $g_\infty \cdot \mathrm{i}_2 = Z$. Then, $h_l^\flat \in S_3(\chi_f)$. Because $S_3(\chi_f)$ is 1-dimensional, $h_l^\flat \in \mathbb{C}f$. One can show that $h_l \in \mathbb{C}\tilde{f}$, noting (2.7), (2.8), (2.9) and $\omega_{\pi_l} = 1$. This completes the proof for f_i. The proof for $\tilde{g}_j$ is similar. □

We will denote by Π_{f_i} (resp. Π_{g_j}) the irreducible cuspidal automorphic representation of $\mathrm{PGSp}_4(\mathbb{A})$ containing $\tilde{f}_i$ (resp. $\tilde{g}_j$).

Noting that $\Gamma'(2), \Gamma(2,4,8)$ are normal subgroups of $\Gamma(2)$ and $\Gamma(2)/\Gamma'(2) \simeq (\mathbb{Z}/2\mathbb{Z})^2$, one can extend χ_{g_j} in 4 ways, $\chi_{g_j,l}$ with $1 \leqslant l \leqslant 4$. Then $S_3(\chi_{g_j}) = \sum_l S_3(\chi_{g_j,l})$. However, because $\dim_{\mathbb{C}} S_3(\chi_{g_j}) = 1$, $\dim_{\mathbb{C}} S_3(\chi_{g_j,l}) = 1$ for an l and $\dim_{\mathbb{C}} S_3(\chi_{g_j,l}) = 0$ for other l. We define the character $\tilde{\chi}_{g_j}$ on $\Gamma(2)$ by $\dim_{\mathbb{C}} S_3(\tilde{\chi}_{g_j}) = 1$.

Proposition 2.3. *For $\Gamma = \ker(\chi_{f_i}), \ker(\tilde{\chi}_{g_j})$, $H^{3,0}(\mathrm{Gr}_3^W H^3(S_\Gamma, \mathbb{C}))$ is 1-dimensional.*

Proof. We give the proof for $\Gamma = \ker(\chi_{g_j})$. The proof for $\Gamma = \ker(\chi_{f_i})$ is similar and omitted. To prove $H^{3,0}(\mathrm{Gr}_3^W H^3(S_{\ker(\chi_g)}, \mathbb{C}))$ $(\simeq S_3(\ker(\chi_g)))$ is 1-dimensional, it suffices to show that $\ker(\chi_g) \not\subset \ker(\chi_{f_i'})$ for any f_i' and $\ker(\chi_g) \not\subset \ker(\chi_{g_l'})$ for any $g_l' \neq g$. Using the tables in Proposition 6.2 of [7], we find that χ_{f_1} is $\{\pm 1\}$-valued, and that χ_{f_i} for $i \neq 1$ and $\chi_{g_l'}$ are $\{\pm 1, \pm \mathrm{i}\}$-valued. Thus

$$\Gamma(2)/\ker(\chi_{f_1'}) \simeq \mathbb{Z}/2\mathbb{Z}, \qquad \Gamma'(2)/\ker(\chi_{g_l'}) \simeq \Gamma'(2)/\ker(\chi_{f_i'}|_{\Gamma'(2)}) \simeq \mathbb{Z}/4\mathbb{Z} \quad (i \neq 1).$$

Because the commutator subgroup of $\Gamma(2)$ is $\Gamma(4,8)$, and $g \notin S_3(\Gamma(4,8))$, it is impossible to extend χ_g to a character on $\Gamma(2)$. Hence, $\chi_{f_i}|_{\Gamma'(2)} \neq \chi_g, \overline{\chi}_g$ and $\ker(\chi_g) \not\subset \ker(\chi_{f_i'}|_{\Gamma'(2)})$ for $i \neq 1$. As described in the proof of Proposition 7.5 in [7], and $\chi_{g_l'} \neq \chi_g, \overline{\chi}_g$. Hence $\ker(\chi_g) \not\subset \ker(\chi_{g_l'})$ for $g_l' \neq g$. Finally, assume that $\ker(\chi_g) \subset \ker(\chi_{f_1'})$ for some f_1'. Then, $\chi_g^2 = \chi_{f_1}$, and hence $\ker(\chi_g^2) \supset \Gamma(4)$. But, this conflicts to the table of Proposition 6.2(b) in [7]. Hence $\ker(\chi_g) \not\subset \ker(\chi_{f_1'})$. This completes the proof. □

2.2. θ-lifts

In this section, we summarize the θ-correspondence for GSO(4) and GSp(4). Let $X_{/\mathbb{Q}}$ be a $2m$-dimensional space defined over $\mathbb{Q}$ with a nondegenerate quadratic form $(\,,)$. For $x = (x_i)$, $y = (y_i) \in X^n$, we denote $((x_i, y_j))$ also by (x, y). Let d_X be the discriminant of X. Let $\chi_X(*) = \{*, (-1)^m d_X\}_v$ where $\{*, *\}_v$ denotes the Hilbert symbol. We fix the standard additive character ψ on $\mathbb{Q}\backslash\mathbb{A}$. Let $\mathcal{S}(X(\mathbb{Q}_v)^n)$ be the space of Schwartz–Bruhat functions of $X(\mathbb{Q}_v)^n$. The Weil representation r_v^n of $\mathrm{Sp}_{2n}(\mathbb{Q}_v) \times \mathrm{O}_X(\mathbb{Q}_v)$ with respect to ψ_v is the unitary representation on $\mathcal{S}(X(\mathbb{Q}_v)^n)$ given by

$$r_v^n(1, h)\varphi_v(x) = \varphi_v\bigl(h^{-1}x\bigr), \tag{2.10}$$

$$r_v^n\left(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}, 1\right)\varphi_v(x) = \chi_X(\det a)|\det a|^m \varphi_v(xa), \tag{2.11}$$

$$r_v^n\left(\begin{bmatrix} I_n & b \\ 0 & I_n \end{bmatrix}, 1\right)\varphi_v(x) = \psi_v\left(\frac{\mathrm{Trace}(b(x,x))}{2}\right)\varphi_v(x), \tag{2.12}$$

$$r_v^n\left(\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, 1\right)\varphi_v(x) = \gamma\varphi_v^\vee(x). \tag{2.13}$$

The Weil constant γ is a fourth root of unity depending on the anisotropic kernel of X, n and ψ. The Fourier transformation $\varphi^\vee$ of φ is defined by

$$\varphi^\vee(x) = \int_{X(\mathbb{Q}_v)^n} \psi_v\bigl(\mathrm{Trace}(x, y)\bigr)\varphi(y)\,\mathrm{d}y$$

where $\mathrm{d}y$ is the self-dual Haar measure. As in [21], we extend r_v^n to the group $\{(g, h) \in \mathrm{GSp}_n(\mathbb{Q}_v) \times \mathrm{GO}_X(\mathbb{Q}_v) \mid \nu(g) = \nu(h)\}$, where $\nu(h)$ denotes the similitude norm of h. Let $r^n = \bigotimes_v r_v^n$. For $\varphi = \bigotimes_v \varphi_v \in \mathcal{S}(X(\mathbb{A})^n)$, we put

$$\theta_n(\varphi)(g, h) = \sum_{x \in X(\mathbb{Q})^n} r(g, h)\varphi(x).$$

This series converges absolutely. Let $\mathrm{d}h$ be a right Haar measure on $\mathrm{SO}_X(\mathbb{Q})\backslash\mathrm{SO}_X(\mathbb{A})$. Let σ be an irreducible cuspidal automorphic representation of $\mathrm{GSO}_X(\mathbb{A})$. Take an $f \in \sigma$. We define the θ-lift of f to $\mathrm{GSp}_n(\mathbb{A})$ with respect to φ by

$$\theta_n(\varphi, f)(g) = \int_{\mathrm{SO}_X(\mathbb{Q})\backslash\mathrm{SO}_X(\mathbb{A})} \theta_n(\varphi)(g, h) f(hh_0)\,\mathrm{d}h, \tag{2.14}$$

where h_0 is chosen so that $\nu(g) = \nu(h_0)$, and the value of $\theta_n(\varphi, f)(g)$ is independent of the choice of h_0. This integral converges absolutely and is an automorphic forms on $\mathrm{GSp}_{2n}(\mathbb{A})$. We will denote by $\Theta_n(\sigma)$ the subspace of $\mathcal{A}(\mathrm{GSp}_4(\mathbb{A}))$ spanned by $\theta_n(\varphi, f)$ with $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$ and $f \in \sigma$. We call $\Theta_n(\sigma)$ the global θ-lift of σ to GSp($2n$). In the case of $m = 2$, these θ-lifts are weak endoscopic lifts or D-critical representations under some situations as follows. For our later use and the sake of simplicity, we assume the central character of σ is trivial.

1) In the case that d_X is a square of a rational number, $X_{/\mathbb{Q}}$ is isometric to a quaternion algebra $\mathrm{B}_{/\mathbb{Q}}$ defined over $\mathbb{Q}$. Define $\rho(h_1, h_2)x = h_1^{-1}xh_2$ for $x \in \mathrm{B}(R), h_i \in \mathrm{B}(R)^\times$, where R denote $\mathbb{Q}$, $\mathbb{Q}_v$ or $\mathbb{A}$. Then ρ gives isomorphisms

$$i_\rho\colon\quad \begin{cases} \mathrm{B}(R)^\times \times \mathrm{B}(R)^\times/\Delta\big(R^\times\big) \simeq \mathrm{GSO}_X(R),\\ \big\{(h_1,h_2)\in \mathrm{B}(R)^\times\times\mathrm{B}(R)^\times \bigm| N_{\mathrm{B}/R}(h_1)=N_{\mathrm{B}/R}(h_2)\big\}/\Delta\big(R^\times\big)\simeq \mathrm{SO}_X(R), \end{cases} \tag{2.15}$$

where $\Delta(R^\times)$ denotes the diagonal embedding into $\mathrm{B}(R)^\times\times\mathrm{B}(R)^\times$. We identify a $\sigma_v\in\mathrm{Irr}(\mathrm{PGSO}_X(\mathbb{Q}_v))$ with a pair $(\sigma_{1,v},\sigma_{2,v})$ of $\mathrm{Irr}(\mathrm{PB}(\mathbb{Q}_v)^\times)$ through i_ρ. Then, σ is identified with $\sigma_1\boxtimes\sigma_2$ for a pair (σ_1,σ_2) of irreducible automorphic representations of $\mathrm{PGSO_B}(\mathbb{A})$. Then, $\Pi=\Theta_2(\sigma_1\boxtimes\sigma_2)$ is irreducible and factors as $\bigotimes_v\theta_2(\sigma_{1v}\boxtimes\sigma_{2v})$. For an irreducible cuspidal automorphic representation τ of $\mathrm{B}(\mathbb{A})^\times$, we will let τ^{JL} denote the Jacquet–Langlands transfer to $\mathrm{GL}_2(\mathbb{A})$. Let S_σ be the set of places v for which $\sigma_{1,v}^{\mathrm{JL}}\boxtimes\sigma_{2,v}^{\mathrm{JL}}$ is ramified. Then, $S_\Pi=S_\sigma$, and

$$L_{S_\sigma}(s,\Pi;\mathrm{spin})=L_{S_\sigma}(s,\sigma_1)L_{S(\sigma)}(s,\sigma_2),\qquad L_{S_\sigma}(s,\Pi;r_5)=\zeta_{S_\sigma}(s)L_{S_\sigma}(s,\sigma_1\times\sigma_2),$$

where r_5 indicates the 5-dimensional representation of $\mathrm{GSp}_4(\mathbb{C})$ as in Section 2 of [26]. If both of σ_1 and σ_2 are cuspidal and $\sigma_1\neq\sigma_2$, then Π is cuspidal, and thus Π is a weak endoscopic lift of $(\sigma_1^{\mathrm{JL}},\sigma_2^{\mathrm{JL}})$. If $\mathrm{B}_{/\mathbb{Q}}$ is a definite quaternion algebra, then Π is the so-called Yoshida lift of $\sigma=(\sigma_1,\sigma_2)$, and Π_∞ is holomorphic. Otherwise, Π is not holomorphic. In particular, if $\mathrm{B}_{/\mathbb{Q}}\simeq\mathrm{M}_2(\mathbb{Q})$, then Π is globally generic, i.e., every $F\in\Pi$ has a nontrivial global Whittaker function. Let $c_1,c_2\in\mathbb{Q}^\times$. A global Whittaker function of an automorphic form F on $\mathrm{GSp}_4(\mathbb{A})$ with respect to ψ_{c_1,c_2} is defined by

$$W_{F,\psi_{c_1,c_2}}(g)=\int\limits_{(\mathbb{Q}\backslash\mathbb{A})^4}\psi(-c_1t+c_2s_4)F\left(\begin{bmatrix}1&t&&\\&1&&\\&&1&\\&&-t&1\end{bmatrix}\begin{bmatrix}1&&s_1&s_2\\&1&s_2&s_4\\&&1&\\&&&1\end{bmatrix}g\right)\mathrm{d}t\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_4, \tag{2.16}$$

and factors as $\bigotimes_v W_{F,\psi_{c_1,c_2,v}}$. We call $W_{F,\psi_{1,1}}$ the standard Whittaker function and abbreviate as $W_{F,\psi}$. Let $\mathrm{B}=\mathrm{M}_2(\mathbb{Q})$. Let

$$e=\begin{bmatrix}&1\\&\end{bmatrix},\qquad \alpha=\begin{bmatrix}1&\\&-1\end{bmatrix}\in\mathrm{M}_2(\mathbb{Q}).$$

The pointwise stabilizer subgroup $Z_{(e,\alpha)}(R)\subset\mathrm{SO_B}(R)$ of e,α is isomorphic to

$$\left\{\left(\begin{bmatrix}1&s\\&1\end{bmatrix},\begin{bmatrix}1&s\\&1\end{bmatrix}\right)\Bigm| s\in R\right\}$$

via i_ρ. Let $\beta_{1,\psi}=\bigotimes_v\beta_{1,v}$, $\beta_{2,\psi}=\bigotimes_v\beta_{2,v}$ be the Whittaker functions of f_1,f_2 with respect to ψ. Then, the v-component of the global standard Whittaker function of $F=\theta_2(\varphi,f_1\boxtimes f_2)$ on $\mathrm{Sp}_4(\mathbb{Q}_v)$ is

$$W_{F,\psi_v}(g)=\int\limits_{Z_{(e_1,\alpha)}(\mathbb{Q}_v)\backslash\mathrm{SO}_X(\mathbb{Q}_v)} r_v^2\big(g,i_\rho(h_1,h_2)\big)\varphi_v(e_1,\alpha)\overline{\beta}_{1,v}(h_1)\beta_{2,v}(h_2)\,\mathrm{d}h_1\,\mathrm{d}h_2. \tag{2.17}$$

2) In the case that d_X is not a square of a rational number, $X_{\mathbb{Q}}$ is isometric to

$$X_{\mathrm{B},d_X}=X_{\mathrm{B}}:=\big\{b\in\mathrm{B}_{/\mathbb{Q}}\otimes\mathbb{Q}(\sqrt{d_X})\bigm| b^{\iota c}=-b\big\} \tag{2.18}$$

for a quaternion algebra $\mathrm{B}_{/\mathbb{Q}}$, where ι denotes the main involution of B, and c is the generator of $\mathrm{Gal}(\mathbb{Q}(\sqrt{d_X})/\mathbb{Q})$. Put $L=\mathbb{Q}(\sqrt{d_X})$. Let R be $\mathbb{Q},\mathbb{Q}_{\mathbb{A}}$ or $\mathbb{Q}_v$. But assume that $L_v\not\simeq\mathbb{Q}_v^2$. For $x\in X$, $t\in R^\times$, $h\in\mathrm{B}(RL)^\times$, define $\rho'(t,h)x=t^{-1}h^{\iota c}xh$. Then, ρ' gives isomorphisms

$$i_{\rho'}\colon\quad \begin{cases} \{(t,b)\in R^\times\times \mathrm{B}(LR)^\times\}/\{(N_{LR/R}(s),s)\mid s\in LR^\times\}\simeq \mathrm{GSO}_X(R),\\ \{(t,b)\mid t^2=N_{LR/R}\circ N_{\mathrm{B}(LR)/L}(b)\}/\{(N_{LR/R}(s),s)\mid s\in LR^\times\}\simeq \mathrm{SO}_X(R).\end{cases} \tag{2.19}$$

We identify a $\sigma_v\in \mathrm{Irr}(\mathrm{PGSO}_X(\mathbb{Q}_v))$ with one of $\mathrm{Irr}(\mathrm{PB}(L_v)^\times)$ through i'_ρ. If $L_v\simeq \mathbb{Q}_v^2$, then $\mathrm{GL}_2(L_{w_1})\times \mathrm{GL}_2(L_{w_2})\simeq \mathrm{GL}_2(\mathbb{Q}_v)^2$, and σ_v is identified with a pair of elements of $\mathrm{Irr}(\mathrm{PB}(\mathbb{Q}_v))$. Let σ be an irreducible cuspidal automorphic representation of $\mathrm{PB}(L_{\mathbb{A}})$, which is identified with an irreducible representation of $\mathrm{PGSO}_X(\mathbb{A})$. Contrary to the previous case, $\Theta_2(\sigma)$ is not irreducible in some cases. Anyway, every irreducible constituent τ of $\Theta_2(\sigma)$ factors as $\bigotimes_v \tau_v$, and

$$L_{S_\tau}(s,\tau;\mathrm{spin})=L_{S_\tau}(s,\sigma),\qquad L_{S_\tau}(s,\tau;r_5)=L_{S_\tau}(s,\chi_L)L_{S_\tau}(s,\tau,\chi_L;\mathrm{Asai}),$$

where χ_L is the quadratic character associated to the extension $L/\mathbb{Q}$, and the last L-function is the χ_L-twist of Asai's L-function (see [1] for the definition). Suppose that $d_X>0$ and each $\sigma^{\mathrm{JL}}_{\infty_i}$ is a holomorphic discrete series representation with lowest weight 2 or more. Employing the main result of Blasius [3], we find that σ^{JL} is tempered. Thus, in this case, every constituent of $\Theta_2(\sigma)$ is a D-critical representation in the sense of [31]. If $\mathrm{B}_{/\mathbb{Q}}$ is a definite quaternion algebra, then each irreducible constituent of $\Theta_2(\sigma)$ is holomorphic. If $\mathrm{B}_{/\mathbb{Q}}\simeq \mathrm{M}_2(\mathbb{Q})$, then an irreducible constituent of $\Theta_2(\sigma)$ is globally generic. Let $\mathrm{B}_{/\mathbb{Q}}=\mathrm{M}_2(\mathbb{Q})$. Define $\psi_L(z)=\bigotimes_v \psi_v(\mathrm{Trace}_{L_w/\mathbb{Q}_v}(z))$, where w denotes a place of L lying over v. Let $e,\alpha\in X_{\mathrm{M}_2,d_L}(\mathbb{Q})$ be the same as above. Then the pointwise stabilizer subgroup $Z_{(e,\alpha)}(\mathbb{A})\subset \mathrm{SO}_{X_{\mathrm{B}}}(\mathbb{A})$ is isomorphic to

$$\left\{\left(1,\begin{bmatrix}1 & s\\ & 1\end{bmatrix}\right)\,\middle|\, s\in\sqrt{d_X}\mathbb{A}\right\} \tag{2.20}$$

via $i_{\rho'}$. Let $f\in\sigma$, $\varphi=\bigotimes_v\varphi_v\in\mathcal{S}(X(\mathbb{A})^2)$, and $F=\theta_2(\varphi,f)$. Let $\beta_\psi=\bigotimes_w\beta_w$ be the global Whittaker function of f associated to ψ_L. If $L_v=L_{w_1}\times L_{w_2}\simeq \mathbb{Q}_v^2$, then W_{F,ψ_v} is similar to (2.17). If $L_v/\mathbb{Q}_v$ does not split, then

$$W_{F,\psi_v}(g)=\int_{Z_{(e,\alpha)}(\mathbb{Q}_v)\backslash \mathrm{SO}_{X_{\mathrm{M}_2(\mathbb{Q})}}(\mathbb{Q}_v)} r_v^2\bigl(g,i_{\rho'}(t,b)\bigr)\varphi_v(e,\alpha)\beta_w(b)\,\mathrm{d}t\,\mathrm{d}b. \tag{2.21}$$

The next lemma is needed to prove Theorem A.

Lemma 2.4. *Let L be a quadratic field. Let σ be an irreducible cuspidal automorphic representation of* $\mathrm{PGL}_2(L_{\mathbb{A}})$. *If σ is not a base change lift of an irreducible cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A})$, *then every irreducible constituent of $\Theta_2(\sigma)$ is not a weak endoscopic lift.*

Proof. Let τ be a constituent of $\Theta_2(\sigma)$. On the authority of Shahidi [25], Asai's L-function of σ does not vanish at $s=1$. Hence $L_{S_\tau}(s,\tau,\chi_L;r_5)$, the χ_L-twist of $L_{S_\tau}(s,\tau;r_5)$, has at least a simple pole at $s=1$. Assume that τ is a weak endoscopic lift. Then, $L_{S_\tau}(s,\tau,\chi_L;r_5)$ is equal to $L_{S_\tau}(s,\chi_L)L_{S_\tau}(s,\sigma_1\times\chi_L\sigma_2)$ for a cuspidal pair (σ_1,σ_2), and hence,

$$\mathrm{ord}_{s=1}L_{S_\tau}(s,\tau,\chi_L;r_5)=\begin{cases}-1 & \text{if } \sigma_1=\chi_L\sigma_2,\\ 0 & \text{otherwise.}\end{cases}$$

Hence the assertion. □

2.3. Degenerate Whittaker functions

Let R be a commutative ring. For $1 \leqslant r \leqslant 2$, let $P_r(R) = N_r(R)M_r(R) \subset \mathrm{GSp}_4(R)$ with

$$N_{P_r}(R) = \left\{ \begin{bmatrix} 1_r & & v & {}^t w \\ & 1_{2-r} & w & \\ & & 1_r & \\ & & & 1_{2-r} \end{bmatrix} \begin{bmatrix} 1_r & u & & \\ & 1_{2-r} & & \\ & & 1_r & \\ & & -{}^t u & 1_{2-r} \end{bmatrix} \,\middle|\, v = {}^t v \in \mathrm{M}_r(R),\ u, w \in \mathrm{M}_{r,2-r}(R) \right\},$$

$$M_{P_r}(R) = \left\{ \begin{bmatrix} z & & & \\ & a & & b \\ & & \det(g)^t z^{-1} & \\ & c & & d \end{bmatrix} \,\middle|\, g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GSp}_{4-2r}(R),\ z \in \mathrm{GL}_r(R) \right\} \simeq \mathrm{GL}_r(R) \times \mathrm{GSp}_{4-2r}(R),$$

where we understand $\mathrm{GSp}_0 = \mathrm{GL}_1$, $\mathrm{GSp}_2 = \mathrm{GL}_2$. We write $P_1 = Q$ (resp. $P_2 = P$) and call it Klingen (resp. Siegel) parabolic subgroup. Let e_Q, e_P denote the natural embedding of $\mathrm{GL}_2 \times \mathrm{GL}_1$ into M_{P_r}. If E is a noncuspidal automorphic form on $\mathrm{GSp}_4(\mathbb{A})$, then, for $\bullet = P$ or Q,

$$\Phi_\bullet(E)(g,z) := \mathrm{vol}\big(N_\bullet(\mathbb{Q})\backslash N_\bullet(\mathbb{A})\big)^{-1} \int\limits_{N_\bullet(\mathbb{Q})\backslash N_\bullet(\mathbb{A})} E\big(ne_\bullet(g,z)\big)\,\mathrm{d}n \tag{2.22}$$

is a nontrivial automorphic form on $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_1(\mathbb{A})$. Let $a \in \mathbb{Q}^\times$. We define $\psi_{(a)}(*) = \psi(a*)$. If a function $W^\bullet_{\psi_{(a)}}$ on $\mathrm{GSp}_4(\mathbb{A})$ (resp. $\mathrm{GSp}_4(\mathbb{Q}_v)$) satisfies

$$W^\bullet_{\psi_{(a)}}\left(\begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & \\ & & -u & 1 \end{bmatrix} \begin{bmatrix} 1 & & * & * \\ & 1 & * & z \\ & & 1 & \\ & & & 1 \end{bmatrix} g \right) = W^\bullet_{\psi_{(a)}}(g) \times \begin{cases} \psi(au) & (\bullet = P), \\ \psi(az) & (\bullet = Q), \end{cases} \tag{2.23}$$

then we say $W^\bullet_{\psi_{(a)}}$ is a $\bullet$-degenerate global (resp. local) Whittaker function.

3. Automorphic forms on $\mathrm{GSp}_4(\mathbb{A})$

Let Π_{f_i}, Π_{g_j} be the irreducible cuspidal automorphic representations associated to f_i, g_j (cf. Proposition 2.2). The idea of our proof of Theorem A is as follows. We will show that a D-critical representation associated to the Hilbert modular form $\pi(\lambda)$ of $\mathbb{Q}(\sqrt{2})$, and the $(\frac{-2}{*})$-twist of a Saito–Kurokawa representation associated to ρ_1 has a $\Gamma(2,4,8)_2$-fixed vector. Because the 2-component of this D-critical representation, and that of this $(\frac{-2}{*})$-twist of the Saito–Kurokawa representation are given by local θ-lifts from GSO(4), we will do it by constructing local Whittaker functions, or local degenerate Whittaker functions defined in 2.3 of these local θ-lifts. If it is done, then each of these representation has an automorphic form related to a Siegel modular form belonging to $S_3(\Gamma(2,4,8))$. From the eigenvalues of Π_{f_i}, Π_{g_j} computed in [7], one concludes Π_{g_1} is this D-critical representation and Π_{g_4} is this $(\frac{-2}{*})$-twist of the Saito–Kurokawa representation. In this way, the conjecture is verified.

3.1. D-critical representation, proof for $L(s, \Pi_{g_1}; \mathrm{spin})$

Let L be a quadratic field with the ring of integers $\mathfrak{o}$. Let δ_L be the discriminant of L. For an integral ideal $\mathfrak{m}$ of a Dedekind ring R, let

$$\tilde{\Gamma}_0^{(n)}(\mathfrak{m}) = \left\{ g = \begin{bmatrix} A_g & B_g \\ C_g & D_g \end{bmatrix} \in \mathrm{GSp}_{2n}(R) \;\middle|\; C_g \in \mathrm{M}_n(\mathfrak{m}) \right\},$$

$$\Gamma_0^{(n)}(\mathfrak{m}) = \tilde{\Gamma}_0^{(n)}(\mathfrak{m}) \cap \mathrm{Sp}_{2n}(R).$$

First, we show the following proposition.

Proposition 3.1. *Let p be a prime which does not split in $L/\mathbb{Q}$, and $\mathfrak{p}$ denote the unique prime ideal of L lying over p. Let π be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(L_{\mathbb{A}})$ of level $\mathfrak{n}$. Then, there is an automorphic form $F \in \Theta_2(\pi)$ such that*

$$\varrho(g)F = \chi_{L,p}\big(\det(A_g)\big)F, \quad g \in \Gamma_0^{(2)}\big(p^N \mathbb{Z}_p\big), \tag{3.1}$$

where χ_L is the quadratic character of $\mathbb{A}^\times$ associated to the extension $L/\mathbb{Q}$, and

$$N = \begin{cases} \frac{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}{2} + \mathrm{ord}_{\mathfrak{p}}(\delta_L) & \text{if } p \text{ is ramified and } \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) \text{ is even}, \\ \frac{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})+1}{2} + \mathrm{ord}_{\mathfrak{p}}(\delta_L) & \text{if } p \text{ is ramified and } \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) \text{ is odd}, \\ \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) & \text{otherwise}. \end{cases}$$

Proof. For a $\varphi = \bigotimes_v \varphi_v \in \mathcal{S}(X_{\mathrm{M}_2}(\mathbb{A})^2)$ and an $f \in \pi$, each component W_{F,ψ_v} of the global standard Whittaker function of $F = \theta_2(\varphi, f)$ is given by (2.17) or (2.21). Therefore, it suffices to construct a nontrivial W_{F,ψ_p} which is right $\Gamma_0(p^N\mathbb{Z}_p)$-semi invariant as in (3.1). We will give a proof for the first case with $L = \mathbb{Q}(\sqrt{2})$ and $p = 2$. The other cases are easier and omitted. For an ideal $\mathfrak{m} \subset \delta_L \mathfrak{o}_{\mathfrak{p}}$ of $\mathfrak{o}_{\mathfrak{p}}$, let

$$\tilde{\Gamma}_0'(\mathfrak{m}) = \begin{bmatrix} \mathfrak{o}_{\mathfrak{p}} & \delta_L^{-1}\mathfrak{o}_{\mathfrak{p}} \\ \mathfrak{m} & \mathfrak{o}_{\mathfrak{p}} \end{bmatrix} \cap \mathrm{GL}_2(L_{\mathfrak{p}}).$$

In the case $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) = 0$, the proof is easy and omitted. Suppose that $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})$ is a positive (even) integer. Then, $\pi_{\mathfrak{p}}$ is a ramified principal series representation or a supercuspidal representation. The conductor of the additive character $\psi_{L_{\mathfrak{p}}} = \psi_p \circ \mathrm{Trace}_{L_{\mathfrak{p}}/\mathbb{Q}_p}$ is $\mathfrak{p}^{-3}$. Using the local newform theory for GL(2), we find that $\pi_{\mathfrak{p}}$ has a right $\tilde{\Gamma}_0'(\delta_L\mathfrak{n})$-invariant local Whittaker function $\beta_{\mathfrak{p}}$ associated to $\psi_{L_{\mathfrak{p}}}$ such that

$$\beta_{\mathfrak{p}}\left(\begin{bmatrix} 1 & z \\ & 1 \end{bmatrix}\begin{bmatrix} t & \\ & 1 \end{bmatrix}\right) = \begin{cases} \psi_{L_{\mathfrak{p}}}(z) & \text{if } t \in \mathfrak{o}_{\mathfrak{p}}^\times, \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

$$\varrho\left(\begin{bmatrix} & -1 \\ p^N & \end{bmatrix}\right)\beta_{\mathfrak{p}} = \pm\beta_{\mathfrak{p}}. \tag{3.3}$$

For an integral ideal $\mathfrak{m}$ of a Dedekind ring R, let $R_0(\mathfrak{m}) = \{\left[\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right] \in \mathrm{M}_2(R) \mid c \in \mathfrak{m}\}$ be the so-called Eichler order of $\mathrm{M}_2(R)$ of level $\mathfrak{m}$. We set

$$\phi(x_1, x_2) = \mathrm{ch}\big(x_1; R_0\big(p^N\big) \cap X_{\mathrm{M}_2(\mathbb{Q}_p)}\big)\mathrm{ch}\big(x_2; R_0\big(p^N\big) \cap X_{\mathrm{M}_2(\mathbb{Q}_p)}\big)$$

where ch indicates the characteristic function. Put $\mathbb{K}_p = i_{\rho'}(\mathbb{Q}_p^\times \times \tilde{\Gamma}_0^{(1)}(p^N)) \cap \mathrm{SO}_X(\mathbb{Q}_p)$. If $g \in \Gamma_0^{(2)}(p^N)$ and $h \in \mathbb{K}_p$, then

$$r_p^2(g,h)\phi = \chi_{L,p}(\det A_g) r_p^2(1,h)\phi. \tag{3.4}$$

From (2.21),

$$W_{F,\psi_p}(g) = \mathrm{vol}(\mathbb{K}_p) \int_{Z_{(e,\alpha)}(\mathbb{Q}_p)\backslash \mathrm{SO}_{X_{\mathrm{M}_2}}(\mathbb{Q}_p)/\mathbb{K}_p} r_p^2(g,h)\phi(e,\alpha)\beta_{\mathfrak{p}}(\bar{h})\,\mathrm{d}\bar{h}, \tag{3.5}$$

where $\bar{h}$ indicates the projection of $h \in \mathrm{GL}_2(L_{\mathfrak{p}})$ to $\mathrm{SO}_X(\mathbb{Q}_p)$ (see (2.20) for the definition of $Z_{(e,\alpha)}$). Then, we are going to see $W_{F,\psi_p}(1) \neq 0$. Using the Iwasawa decomposition of $\mathrm{GL}_2(L_{\mathfrak{p}})$, we can take the following complete system of representatives for $Z_{(e,\alpha)}(\mathbb{Q}_p)\backslash \mathrm{SO}_X(\mathbb{Q}_p)/\mathbb{K}_p$:

$$\left(2^m, \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix} \begin{bmatrix} 2^m & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ l & 1 \end{bmatrix}\right), \qquad \left(2^{m+\frac{N}{2}}, \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix} \begin{bmatrix} 2^m & \\ & 1 \end{bmatrix} \begin{bmatrix} & -1 \\ 2^N & \end{bmatrix}\right)$$

where $s \in \mathbb{Q}_2$, $m \in \mathbb{Z}$ and $l \in \mathfrak{o}_{\mathfrak{p}}$ modulo 2^N. We will observe the contributions of these types to the integral (3.5). We will denote $\rho'(t,h)(e,\alpha) = (\left[\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right])$. For the former types, we calculate

$$\rho'(t,h)(e,\alpha) = \left(\begin{bmatrix} 2^{-m}l & 2^{-m} \\ -2^{-m}ll^c & -2^{-m}l^c \end{bmatrix}, \begin{bmatrix} 1+2^{-m+1}ls & 2^{-m+1}s \\ -(l+l^c)-2^{-m+1}ll^c s & -1-2^{-m+1}l^c s \end{bmatrix}\right)$$

where c is the generator of $\mathrm{Gal}(L/\mathbb{Q})$. Suppose $\rho'(t,h)(e,\alpha) \in \mathrm{supp}(\phi)$. Observing b_1, we find $m \leqslant 0$. If $m < 0$, then

$$\rho'\left(t, \begin{bmatrix} 1 & \frac{1}{4} \\ & 1 \end{bmatrix} h\right)(e,\alpha) \in \mathrm{supp}(\phi).$$

Because $\beta_2(\left[\begin{smallmatrix} 1 & \frac{1}{4} \\ & 1 \end{smallmatrix}\right]h) = -\beta_2(h)$, we can ignore the contribution if $m < 0$. Therefore, we can assume $m = 0$. Then, observing c_1, we find $l \in \mathfrak{p}^N$. Observing b_2, we find $s \in 2^{-1}\mathbb{Z}_2$. We see that, if $m = 0$, $l \in \mathfrak{p}^N$ and $s \in 2^{-1}\mathbb{Z}_2$, then $\rho'(t,h)(e,\alpha) \in \mathrm{supp}(\phi)$. Now, recall that $\beta_{\mathfrak{p}}$ is a local new vector, which is right $\Gamma_0'(\delta_L\mathfrak{n})$-invariant. Hence, if $c \in \mathfrak{p}^{-1}\delta_L\mathfrak{n} \setminus \delta_L\mathfrak{n}$, then

$$\varrho\left(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}\right)\beta_2 = -\beta_2.$$

Using this property, we conclude that the sum of the contributions of the former types are none. For the latter types, we calculate

$$\rho'(t,h)(e,\alpha) = \left(\begin{bmatrix} 0 & 0 \\ 2^{N-m} & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -2^{N+1-m}s & 1 \end{bmatrix}\right).$$

Suppose $\rho'(t,h)(e,\alpha) \in \mathrm{supp}(\phi)$. Using (3.2) and (3.3), we can assume $m = 0$. Observing c_2, we find that $s \in 2^{-1}\mathbb{Z}_2$. Then, using (3.2) again, we see that the total contribution of the latter types is non-trivial. This completes the proof. □

Let $\zeta_8 = \frac{(1+\mathrm{i})}{\sqrt{2}}$. Let $L = \mathbb{Q}(\sqrt{2})$ (resp. $K = \mathbb{Q}(\zeta_8)$) with the ring of integers $\mathfrak{o}$ (resp. $\mathfrak{O}$). Let $\mathfrak{p}$ (resp. $\mathfrak{P}$) be the unique (ramified) prime ideal of $\mathfrak{o}$ (resp. $\mathfrak{O}$) lying over the prime ideal 2 of $\mathbb{Q}$. Next, we observe the irreducible cuspidal automorphic representation $\pi(\lambda)$ of $\mathrm{GL}_2(L_{\mathbb{A}})$ obtained from the größencharacter λ of $K_{\mathbb{A}}^{\times}$ on p. 870 of [7]. The definition of λ is as follows. For the two archimedean places ∞_1, ∞_2 of K, $\lambda_{\infty_1}(z) = |z|^3/z^3$, $\lambda_{\infty_2}(z) = |z|/z$, $z \in \mathbb{C}^{\times}$. Thus, the lowest weights of the archimedean components of $\pi(\lambda)$ are 4, 2, respectively. The conductor of λ is $\mathfrak{P}^4 = (2)$, and

$$\left(\mathfrak{O}/\mathfrak{P}^4\right)^{\times} = \left\langle \zeta_8 \pmod 2\right\rangle \oplus \left\langle (1+\sqrt{2}) \pmod 2\right\rangle \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Then, $\lambda_{\mathfrak{P}}$ is defined by

$$\lambda_{\mathfrak{P}}\big(\zeta_8 \pmod 2\big) = 1, \qquad \lambda_{\mathfrak{P}}\big((1+\sqrt{2}) \pmod 2\big) = -1.$$

We define the quasi-character μ on $L_{\mathfrak{p}}^{\times}$ with conductor $\mathfrak{p}^3$ by

$$\mu\big((1+\sqrt{2})\ \big(\mathrm{mod}\ \mathfrak{p}^3\big)\big) = \mathrm{i},$$

where $(\mathfrak{o}/\mathfrak{p}^3)^{\times} = \langle (1+\sqrt{2}) \pmod{\mathfrak{p}^3}\rangle \simeq \mathbb{Z}/4\mathbb{Z}$. Then, it holds $\lambda_{\mathfrak{P}} = \mu \circ N_{K/L}$. Let $\chi_{K/L}$ be the quadratic character of $L_{\mathbb{A}}^{\times}$ associated to the extension K/L. The central character of $\pi(\lambda)$ is $\lambda|_{L_{\mathbb{A}}^{\times}} \chi_{K/L}$. Because both of $\lambda_{\infty_i}\chi_{K/L,\infty_i}$ and $\lambda_{\mathfrak{P}}\chi_{K/L,\mathfrak{p}} = \mu \circ N_{K/L,\mathfrak{p}}\chi_{K/L,\mathfrak{p}}$ are trivial, so is the central character of $\pi(\lambda)$. Employing Theorem 4.6(iii) of [9], we find that $\pi(\lambda)_{\mathfrak{p}}$ is the principal series representation

$$\pi(\mu, \mu\chi_{K/L,\mathfrak{p}}) = \pi(\mu, \overline{\mu}) \tag{3.6}$$

of level $\mathfrak{p}^6$.

Finally, we prove the conjecture. One can construct an automorphic form $F \in \Theta_2(\pi(\lambda))$ satisfying $\varrho(u)F = F$ for $u \in \mathrm{Sp}_4(\mathbb{Z}_p)$ at $p \neq 2$, and (3.1) at 2. The local standard Whittaker function $W_{F,\psi,2}$ is right $\Gamma_0^{(2)}(2^6)_2$-semi invariant and $W_{F,\psi,2}(1) \neq 0$. Let $g_0 = \mathrm{diag}(2^5, 2^3, 2^{-2}, 1) \in \mathrm{GSp}_4(\mathbb{Q})$, and $F'(g) = F(g_0 g g_0^{-1}) = F(g g_0^{-1})$. Let

$$\Gamma' := g_0^{-1}\Gamma_0^{(2)}\big(2^6\mathbb{Z}_2\big)g_0 = \begin{bmatrix} \mathbb{Z}_2 & 2^2\mathbb{Z}_2 & 2^6\mathbb{Z}_2 & 2^5\mathbb{Z}_2 \\ 2^{-2}\mathbb{Z}_2 & \mathbb{Z}_2 & 2^5\mathbb{Z}_2 & 2^3\mathbb{Z}_2 \\ \mathbb{Z}_2 & 2\mathbb{Z}_2 & \mathbb{Z}_2 & 2^{-2}\mathbb{Z}_2 \\ 2\mathbb{Z}_2 & 2^3\mathbb{Z}_2 & 2^2\mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix} \cap \mathrm{Sp}_4(\mathbb{Q}_2).$$

Then, F' is right Γ'-semi invariant, and so is $W_{F',\psi_{4,8}}$. Note that $\Gamma(2,4,8)_2 \cap \Gamma_0^{(2)}(8\mathbb{Z}_2) \subset \Gamma'$. Because

$$\varrho\left(\begin{bmatrix} 1 & & s_1 & s_2 \\ & 1 & s_2 & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) W_{F',\psi_{4,8},2}(1) = W_{F',\psi_{4,8},2}(1) \neq 0$$

for $s_1, s_2 \in \mathbb{Q}_2$,

$$\int_{\Gamma(2,4,8)_2} \varrho(u) W_{F',\psi_{4,8}}(1)\, \mathrm{d}u \neq 0. \tag{3.7}$$

Hence, there is an irreducible globally generic constituent of $\Theta_2(\pi(\lambda))$, which has a right $\Gamma(2,4,8)_2 \times \prod_{p\neq 2} \mathrm{Sp}_4(\mathbb{Z}_p)$-invariant vector. We denote this representation by Π^{gen}.

Theorem 3.2. *The irreducible cuspidal automorphic representation Π_{g_1} is a* D*-critical representation associated to $\pi(\lambda)$. The conjecture is true.*

Proof. First, employing the result of local θ-correspondence for $\mathrm{Sp}_4(\mathbb{R})$ and $\mathrm{O}_{2,2}(\mathbb{R})$ due to Przebinda [20], we find that $\Pi_\infty^{\mathrm{gen}}|_{\mathrm{Sp}_4}$ is the large discrete series repesentation with Blattner parameter $(3,-1)$, a cohomological weight. Next, we claim that Π^{gen} is not a weak endoscopic lift, nor a CAP representation. Recall that the lowest weights of the archimedean components of $\pi(\lambda)$ are $(4,2)$. Hence, $\pi(\lambda)$ is not a base change lift. From Lemma 2.4, Π^{gen} is not a weak endoscopic lift. On the authority of Piatetski-Shapiro [18], and Soudry [26], every partial spinor L-function of a CAP representation is, up to finitely many Euler factors, in the form of $L(s-\frac{1}{2},\chi)L(s+\frac{1}{2},\chi)L(s-\frac{1}{2},\chi')L(s+\frac{1}{2},\chi')$, $L(s-\frac{1}{2},\mu)L(s+\frac{1}{2},\mu)$, or $L(s-\frac{1}{2},\chi)L(s+\frac{1}{2},\chi)L(s,\sigma_1)$. Here χ,χ' are some quadratic character of $\mathbb{A}^\times$, μ is a quasi-character of $L_{\mathbb{A}}^\times$ for a quadratic field L, and σ_1 is an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. But, $L(s,\pi(\lambda))=L(s,\lambda)$ satisfies the Ramanujan conjecture. Hence the claim. Finally, according to Theorem III and Proposition 1.5 of Weissauer [31], there is an irreducible cuspidal automorphic representation Π^{hol} such that

- $\Pi_\infty^{\mathrm{hol}}|_{\mathrm{Sp}_4}$ is the holomorphic discrete series representation with Blattner parameter $(3,3)$.
- $\Pi_v^{\mathrm{hol}}\simeq\Pi_v^{\mathrm{gen}}$ at $v\neq\infty$.

Thus, Π^{hol} contributes to $H^{3,0}(\mathrm{Gr}_3^W(S_{\Gamma(2,4,8)},\mathbb{C}))\simeq S_3(\Gamma(2,4,8))$, i.e., Π^{hol} is one of the 11 irreducible representations $\Pi_{f_1},\dots,\Pi_{g_4}$. Observing some L-factors of them calculated in [7], one can conclude that $\Pi^{\mathrm{hol}}=\Pi_{g_1}$. This completes the proof. □

Remark 1. Using the definition of μ, one can show that $\pi(\lambda)$ is invariant but not distinguished in the sense of Roberts [22]. Employing Theorem 8.5 of [22], we find that the set of D-critical representations associated to $\pi(\lambda)$ consists of four irreducible representations $\Pi^{\mathrm{gen}}=\Pi_1,\Pi_2,\Pi_3,\Pi_4$. They are all given by a θ-lift from GSO(4). Further, $\Pi_{2,\infty}\simeq\Pi_{3,\infty}$ (resp. $\Pi_{1,\infty}\simeq\Pi_{4,\infty}$) is the holomorphic (resp. large) discrete series representation with Blattner parameter $(3,3)$ (resp. $(3,-1)$), and $\Pi_{1,p}\simeq\Pi_{2,p},\Pi_{3,p}\simeq\Pi_{4,p}$ at every nonarchimedean place. Noting this fact, one can show the above theorem.

3.2. Saito–Kurokawa representation, proof for $L(s,\Pi_{g_4};\mathrm{spin})$

First, we will recall some known results on Saito–Kurokawa representation. For a square free integer a, let $\chi^{(a)}$ denote the quadratic character of $\mathbb{A}^\times$ associated to the extension $\mathbb{Q}(\sqrt{a})/\mathbb{Q}$. For an irreducible cuspidal automorphic representation τ of $\mathrm{GSp}_{2n}(\mathbb{A})$, we will abbreviate $\chi^{(a)}\tau$ as $\tau^{(a)}$. Let $\mathrm{B}_{/\mathbb{Q}}$ be a quaternion algebra. Let σ be an irreducible cuspidal automorphic representation of $\mathrm{PB}(\mathbb{A})^\times$. Suppose that $\sigma_\infty^{\mathrm{JL}}$ is the holomorphic discrete series representation of lowest weight 4. Let $\mathbf{1}_{\mathrm{B}(\mathbb{A})^\times}=\mathbf{1}$ denote the trivial representation of $\mathrm{B}(\mathbb{A})$. For a $\{\pm1\}$-valued character χ of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$, we denote by $\chi\sigma$ the representation of $\mathrm{PB}(\mathbb{A})^\times$ sending $h\in\mathrm{B}(\mathbb{A})^\times$ to $\chi(N_{B/\mathbb{Q}}(h))\sigma(h)$. We will abbreviate $\chi\mathbf{1}_{\mathrm{B}(\mathbb{A})^\times}$ as χ. If $\mathrm{B}_{/\mathbb{Q}}$ is not split, then $\Theta_2(\chi\boxtimes\sigma)$ is cuspidal. It is easy to show that $\Theta_2(\chi\boxtimes\sigma)$ is not vanishing, if and only if $L(\frac{1}{2},\chi\sigma)\neq0$, by using a result of Waldspurger [29]. On the other hand, if $\mathrm{B}_{/\mathbb{Q}}$ is split, then $\Theta_2(\chi\boxtimes\sigma)$ is non-vanishing and noncuspidal. Indeed, one can construct an $f\in\Theta_2(\chi\boxtimes\sigma)$ so that the P-degenerate Whittaker function $W_{f,\psi}^P$ is nontrivial as is explained below (hence, $\Phi_P(f)$ defined in (2.22) is nontrivial). We will recall the result of Cogdell and Piatetski-Shapiro [4] and Schmidt [24]. Let π be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$. The global cuspidal Saito–Kurokawa packet $\mathrm{SK}_0(\pi)$ is defined as the set of irreducible cuspidal automorphic representations of $\mathrm{PGSp}_4(\mathbb{A})$ whose spinor L-functions are equal to $\zeta(s-\frac{1}{2})\zeta(s+\frac{1}{2})L(s,\pi)$, up to finitely many Euler factors. Let D_v be the unique division quaternion algebra over $\mathbb{Q}_v$. When π_v is square-integrable, let π_v' denote the Jacquet–Langlands transfer to $\mathrm{D}_v^\times$. The local Saito–Kurokawa packet is the following set:

$$\mathrm{SK}(\pi_v) = \begin{cases} \{\theta_2(\mathbf{1}_v \boxtimes \pi_v), \theta_2(\mathbf{1}_v \boxtimes \pi'_v)\}, & \text{if } \pi_v \text{ is square-integrable}, \\ \{\theta_2(\mathbf{1}_v \boxtimes \pi_v)\}, & \text{otherwise}. \end{cases}$$

At a nonarchimedean place $v = p$, as is explained on pp. 230–233 of [24], $\theta_2(\mathbf{1}_p \boxtimes \pi_p)$ is the local Saito–Kurokawa representation that is the unique irreducible quotient of the Siegel parabolically induced representation $|*|^{1/2}\pi_p \rtimes |*|^{-1/2}$ (cf. [24,23]). For a $\{\pm 1\}$-valued character χ_p, $\theta_2(\chi_p \boxtimes \pi_p)$ is the χ_p-twist of the local Saito–Kurokawa representation $\theta_2(\mathbf{1}_p \boxtimes \chi_p\pi_p)$.

Next, we will observe the global cuspidal Saito–Kurokawa packet of ρ_1, and that of $\rho_1^{(-2)}$. For a moment, let

$$\mathrm{B}_{/\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}I + \mathbb{Q}J + \mathbb{Q}IJ, \qquad I^2 = J^2 = -1, \qquad IJ = -JI. \tag{3.8}$$

This quaternion algebra splits outside of $\{\infty, 2\}$. As is seen in Section 4 of [17], ρ_1 has the Jacquet–Langlands transfer to $\mathrm{PB}(\mathbb{A})^\times$. Denote it by ρ'_1. In [17], the Siegel modular form F_1 is constructed by the Yoshida lift of $(\mathbf{1}, \rho'_1)$. This implies

$$L\left(\frac{1}{2}, \rho_1\right) \neq 0, \qquad \varepsilon\left(\frac{1}{2}, \rho_1\right) = \varepsilon\left(\frac{1}{2}, \rho_{1,2}, \psi_2\right) = 1.$$

The 2-component $\rho'_{1,2}$ is the finite dimensional representation of $\mathrm{B}_2^\times \simeq \mathrm{D}_2^\times$ described as follows. We fix the maximal order $\mathcal{R} = \mathbb{Z}_2 + \mathbb{Z}_2 I + \mathbb{Z}_2 J + \mathbb{Z}(\frac{1+I+J+IJ}{2}) \subset \mathrm{B}_2$. Let $\varpi \in \mathrm{B}_2$ be an uniformizer. Let $\mathcal{R}(2) = \mathbb{Z}_2 + \varpi^2\mathcal{R}$. As a complete system of representatives U of $\mathcal{R}^\times/\mathcal{R}(2)^\times$, we can take $\{1, I, J, \frac{1\pm I\pm J\pm IJ}{2}\}$. Let $W = \mathbb{C}I + \mathbb{C}J + \mathbb{C}IJ$. Then, we obtain a finite dimensional representation τ_2 of $\mathrm{B}_2^\times$ from the automorphism of W defined by $u^{-1}wu$. Because $\mathrm{B}_\mathbb{A}^\times = \mathrm{B}_\mathbb{Q}^\times \mathcal{R}(2)_\mathbb{A}^\times$, from this representation, one can obtain an automorphic representation τ of $\mathrm{PB}_\mathbb{A}^\times$. One can construct a right $\Gamma_0^{(1)}(8)$-invariant vector in $\Theta_1(\tau \boxtimes \tau)$ (see also Proposition 3.8). This means $\rho'_1 = \tau$, because the space of elliptic cusp form of weight 4 of level 8 is 1-dimensional. Hence τ_2 is irreducible and equivalent to $\rho'_{1,2}$.

Lemma 3.3. *The root number of $\rho_1^{(-2)}$ is -1.*

Proof. Because $\rho_{1,p}^{(-2)}$ is unramified for $p \neq 2$ and $\rho_{1,\infty}$ is the holomorphic discrete series representation of lowest weight 4, it suffices to show that $\varepsilon(\frac{1}{2}, \rho_{1,2}^{(-2)}, \psi_2) = -1$. We will see the ε-factor of the base change lift $\rho_{1,\mathfrak{p}}^{\mathrm{BC}}$ to $\mathrm{GL}_2(\mathbb{Q}(\sqrt{-2})_\mathfrak{p})$ with $\mathfrak{p} = \sqrt{-2}$. Let $L = \mathbb{Q}(\sqrt{-2})$. Let $\psi_L = \psi \circ \mathrm{Trace}_{L/\mathbb{Q}}$. We identify $L \simeq \mathbb{Q}(I + J) \subset \mathrm{B}_{/\mathbb{Q}}$ for the above $\mathrm{B}_{/\mathbb{Q}}$. Then $\mathcal{R}(2) \cap L_\mathfrak{p}$ is the maximal order of $L_\mathfrak{p}$. Thus, every character (constituent) of the restriction $\rho'_{1,2}|_{L_\mathfrak{p}^\times}$ is unramified. Because $(I + J)^{-1}(I + J)(I + J) = I + J \in W$, the trivial character of $L_\mathfrak{p}^\times$ appears in this restriction. Applying Lemma 14 of [10], we have

$$\begin{aligned} -1 &= -\omega_{\rho_1,2}(-1) \\ &= \varepsilon\left(\frac{1}{2}, \rho_{1,\mathfrak{p}}^{\mathrm{BC}}, \psi_{L,\mathfrak{p}}\right) \\ &= \varepsilon\left(\frac{1}{2}, \rho_{1,2}, \psi_2\right)\varepsilon\left(\frac{1}{2}, \rho_{1,2}^{(-2)}, \psi_2\right) \\ &= \varepsilon\left(\frac{1}{2}, \rho_{1,2}^{(-2)}, \psi_2\right). \qquad \square \end{aligned}$$

From Lemma 3.3, it follows that $L(s,\rho_1^{(-2)})=-L(1-s,\rho_1^{(-2)})$, and hence

$$L\left(\frac{1}{2},\rho_1^{(-2)}\right)=0,\qquad \varepsilon\left(\frac{1}{2},\rho_1^{(-2)}\right)=\varepsilon\left(\frac{1}{2},\rho_{1,2}^{(-2)},\psi_2\right)=-1.$$

Employing the main lifting theorem of [24], and Theorem 3.1 of [4], we conclude

$$\mathrm{SK}_0(\rho_1)=\left\{\left(\bigotimes_{v=\infty,2}\theta_2\big(\mathbf{1}\boxtimes\rho'_{1,2}\big)\right)\otimes\left(\bigotimes_{v\neq\infty,2}\theta_2(\mathbf{1}\boxtimes\rho_{1,v})\right)\right\},$$

$$\mathrm{SK}_0\big(\rho_1^{(-2)}\big)=\left\{\theta_2\big(\mathbf{1}\boxtimes\rho_{1,2}'^{(-2)}\big)\otimes\left(\bigotimes_{v\neq2}\theta_2\big(\mathbf{1}\boxtimes\rho_{1,v}^{(-2)}\big)\right),\ \theta_2\big(\mathbf{1}\boxtimes\rho_{1,\infty}'^{(-2)}\big)\otimes\left(\bigotimes_{v\neq\infty}\theta_2\big(\mathbf{1}\boxtimes\rho_{1,v}^{(-2)}\big)\right)\right\}.$$

Note that $\theta_2(\mathbf{1}\boxtimes\rho_{1,\infty}^{(-2)})|_{\mathrm{Sp}(4)}$ is the holomorphic discrete series representation with Blattner parameter $(3,3)$. Therefore, we guess that the latter constituent of $\mathrm{SK}_0(\rho_1^{(-2)})$ is $\chi^{(-2)}\Pi_{g_4}$. We want to show that $\theta_2(\chi_p^{(-2)}\boxtimes\rho_{1,p})$ has a right $\Gamma(2,4,8)_p$-invariant vector for every p. The local θ-lift $\theta_2(\chi_v^{(-2)}\boxtimes\rho_{1,v})=\chi_v^{(-2)}\theta_2(\mathbf{1}\boxtimes\rho_{1,v}^{(-2)})$ does not have a local Whittaker function. But it has a local P-degenerate Whittaker function $W_{\psi_v}^P$ as follows. Let $e'=\left[\begin{smallmatrix}1&\\&\end{smallmatrix}\right]$. Let $Z_{(e,e')}\subset\mathrm{SO}_X$ be the pointwise stabilizer subgroup of e,e', which is isomorphic to

$$\left\{\left(\begin{bmatrix}1&s\\&1\end{bmatrix},\begin{bmatrix}1&\\&1\end{bmatrix}\right)\;\middle|\;s\in\mathbb{Q}_v\right\}$$

via i_ρ. Then, $W_{\psi_v}^P(g)$ of $\theta_2(\chi_v^{(-2)}\boxtimes\rho_{1,v})$ is

$$\int_{Z_{(e,e')}(\mathbb{Q}_v)\backslash\mathrm{SO}_{\mathrm{M}_2}(\mathbb{Q}_v)} r_v^2\big(g,i_\rho(h_1,h_2)\big)\varphi_v\big(e,e'\big)\chi_v^{(-2)}\big(\det(h_1)\big)\beta_v(h_2)\,\mathrm{d}h_1\,\mathrm{d}h_2 \tag{3.9}$$

where β_v is a Whittaker function of $\rho_{1,v}$ with respect to ψ_v. It is easy to construct a right $\mathrm{Sp}_4(\mathbb{Z}_p)$-invariant $W_{\psi_p}^P$ for a nonarchimedean place $p\neq2$. We will construct a right $\Gamma(2,4,8)_2$-invariant P-degenerate Whittaker function of $\theta(\chi_2^{(-2)}\boxtimes\rho_{1,2})$. From $\rho_{1,2}$, we take the right $\Gamma_0^{(1)}(8\mathbb{Z}_p)$-invariant local Whittaker function β_2 with respect to ψ_2 such that $\beta_2(1)=1$. We define

$$\phi'(x_1,x_2)=\chi_2^{(-2)}(b_1)\mathrm{ch}\big(x_2;\mathrm{M}_2(\mathbb{Z}_2)\big)\times\begin{cases}1 & \text{if } \mathrm{ord}_2(a_1)\geqslant0,\ \mathrm{ord}_2(b_1)=0,\ \mathrm{ord}_2(c_1),\mathrm{ord}_2(d_1)\geqslant3,\\ 0 & \text{otherwise,}\end{cases}$$

where we write $x_1=\left[\begin{smallmatrix}a_1&b_1\\c_1&d_1\end{smallmatrix}\right]\in\mathrm{M}_2(\mathbb{Q}_2)$. Let

$$\Gamma''=\begin{bmatrix}1+2^3\mathbb{Z}_2&\mathbb{Z}_2&\mathbb{Z}_2&\mathbb{Z}_2\\2^3\mathbb{Z}_2&\mathbb{Z}_2&\mathbb{Z}_2&\mathbb{Z}_2\\2^6\mathbb{Z}_2&2^3\mathbb{Z}_2&1+2^3\mathbb{Z}_2&2^3\mathbb{Z}_2\\2^3\mathbb{Z}_2&\mathbb{Z}_2&\mathbb{Z}_2&\mathbb{Z}_2\end{bmatrix}\cap\mathrm{Sp}_4(\mathbb{Z}_2).$$

Then, ϕ' is right Γ''-invariant. One can calculate (3.9) is not zero at $g=1$, directly. Let $g_0'=\mathrm{diag}(2^4,2^3,2^{-1},1)$. Then

$$g_0'^{-1}\Gamma'' g_0' = \begin{bmatrix} 1+2^3\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2^5\mathbb{Z}_2 & 2^4\mathbb{Z}_2 \\ 2^2\mathbb{Z}_2 & \mathbb{Z}_2 & 2^4\mathbb{Z}_2 & 2^3\mathbb{Z}_2 \\ 2\mathbb{Z}_2 & 2^{-1}\mathbb{Z}_2 & 1+2^3\mathbb{Z}_2 & 2^2\mathbb{Z}_2 \\ 2^{-1}\mathbb{Z}_2 & 2^{-3}\mathbb{Z}_2 & 2\mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix} \cap \mathrm{Sp}_4(\mathbb{Q}_2).$$

There is a right $g_0'^{-1}\Gamma'' g_0'$-invariant $W^P_{\psi_{(1/2)},2} \in \theta_2(\chi_2^{(-2)} \boxtimes \rho_{1,2})$ such that $W^P_{\psi_{(1/2)},2}(1) \neq 0$. Then, an integral similar to (3.7) gives a nontrivial right $\Gamma(2,4,8)_2$-invariant local P-degenerate Whittaker function of $\theta_2(\chi_2^{(-2)} \boxtimes \rho_{1,2})$. Consequently,

Theorem 3.4. *The irreducible cuspidal automorphic representation Π_{g_4} is the $\chi^{(-2)}$-twist of the irreducible (holomorphic) constituent of* $\mathrm{SK}_0(\rho_1)$. *The conjecture is true.*

Finally, we give a remark. Observing the eigenvalues of g_4 in the table of Section 8 of [7], we find that Π_{g_4} does not satisfy the generalized Ramanujan conjecture. Indeed

$$|\alpha_{p1}| = |\alpha_{p2}| = p^{\frac{3}{2}}, \qquad |\alpha_{p3}| = p, \qquad |\alpha_{p4}| = p^2$$

for $p = 3, 5, 7, 11, 13, 17, 19$, if we write the Hecke polynomial of $\Pi_{g_4,p}$ as $\prod_{i=1}^4 (X - \alpha_{pi})$. Then, one can see that Π_{g_4} is a twist of a Saito–Kurokawa representation with the following proposition.

Proposition 3.5. *For a Siegel modular* 3*-fold S_Γ, if an irreducible cuspidal automorphic representation Π contributes to $H^{3,0}(\mathrm{Gr}_3^W(S_\Gamma, \mathbb{C}))$ and does not satisfy the Ramanujan conjecture, then Π is a twist of a Saito–Kurokawa representation.*

Proof. As stated by Theorem I of Weissauer [31], there is a $\mathrm{GL}_4(\overline{\mathbb{Q}}_2)$-valued Galois representation ρ_Π of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that

$$L_{S_\Pi}\left(s - \frac{3}{2}, \Pi; \mathrm{spin}\right) = L_{S_\Pi}(s, \rho_\Pi).$$

Assume that Π is not a CAP representation. Then ρ_Π is pure of weight 3, the eigenvalues of $\rho_\Pi(\mathrm{Frob}_p)$ has absolute value $p^{3/2}$, and hence Π does not satisfy the Ramanujan conjecture. This is a contradiction. Hence Π is a CAP representation, i.e., an irreducible cuspidal automorphic representation associated to a parabolically induced representation. As stated by Theorem A of Soudry [26], every CAP representation associated to a Borel or Klingen parabolically induced representation is a constituent of a global θ-lift of an irreducible automorphic representation σ_T of $\mathrm{GO}_T(\mathbb{A})$ for a quadratic field T. It is not hard to see the local θ-lift to $\mathrm{Sp}_4(\mathbb{R})$ of $\sigma_{T,\infty}$ is not a holomorphic discrete series representation with Blattner parameter $(3,3)$. Hence Π is a CAP representation associated to a Siegel parabolically induced representation. On the authority of Piatetski-Shapiro [18], such a representation is a twist of a Saito–Kurokawa representation. □

3.3. Weak endoscopic lift

Let f_5 be the 6-tuple product of Igusa theta constants defined in [7], and χ_{f_5} be the character of $\Gamma(2)$ obtained from f_5 through the Igusa transformation formula (cf. Lemmas 5.2, 5.3 in [7]). Let Π_{f_5} be the irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ associated to f_5 in Proposition 2.2. Our aim is to prove

Theorem 3.6. *An irreducible cuspidal automorphic representation which is weakly equivalent to Π_{f_5} contributes to $H^{2,1}(\mathrm{Gr}_3^W(S_{\ker(\chi_{f_5})}, \mathbb{C}))$.*

First, we recall that Π_{f_5} is a weak endoscopic lift of the pair $(\pi(\mu), \pi(\mu^3))$ of the following CM-elliptic cusp forms. Let $E_{/\mathbb{Q}}$ be the CM-elliptic curve defined by the equation $y^2 = x^3 - x$. Let μ be the größencharacter of $\mathbb{Q}(\mathrm{i})_{\mathbb{A}}^\times$ such that $L(s - \frac{1}{2}, \mu) = L(s, E_{/\mathbb{Q}})$. At $v = \infty$, $\mu_\infty(z) = |z|/z$, $z \in \mathbb{C}^\times$. Thus, the lowest weights of the holomorphic discrete series representations $\pi(\mu)_\infty, \pi(\mu^3)_\infty$ are 2, 4, respectively. Let $\mathfrak{o} = \mathbb{Z}[\mathrm{i}]$. Let $\mathfrak{p} \subset \mathfrak{o}$ be the prime ideal lying over 2. The conductor of μ is $\mathfrak{p}^3$, and thus $\pi(\mu)_p, \pi(\mu^3)_p$ are unramified at $p \neq 2$. The group $(\mathfrak{o}/\mathfrak{p}^3)^\times$ is the cyclic group of order 4 generated by i (mod $\mathfrak{p}^3$), and $\mu_\mathfrak{p}$ is defined by $\mu_\mathfrak{p}(\mathrm{i} \pmod{\mathfrak{p}^3}) = \mathrm{i}$.

Lemma 3.7. *The* 2*-components* $\pi(\mu)_2, \pi(\mu^3)_2$ *are equivalent and supercuspidal.*

Proof. From the definition, $\mu_\mathfrak{p}$ is $\{\pm 1, \pm \mathrm{i}\}$-valued on $\mathfrak{o}_\mathfrak{p}^\times$. Thus $\mu_\mathfrak{p} = \overline{\mu}_\mathfrak{p}^3$ on $\mathfrak{o}_\mathfrak{p}^\times$. Noting that the central character of $\pi(\mu)$ is trivial, we have

$$\pi(\mu)_2 = \pi\left(\overline{\mu}^3\right)_2 = \overline{\pi\left(\mu^3\right)_2} = \pi\left(\mu^3\right)_2.$$

There is no quasi-character ξ of $\mathbb{Q}_2^\times$ such that $\xi \circ N_{\mathbb{Q}(\mathrm{i})_\mathfrak{p}/\mathbb{Q}_2} = \mu$. Employing Lemma 4.6 of [9], we find that $\pi(\mu)_2$ is supercuspidal. This completes the proof. $\square$

Employing this lemma and the Jacquet–Langlands theory, we find that both of $\pi(\mu), \pi(\mu^3)$ have the Jacquet–Langlands transfers $\pi(\mu)', \pi(\mu^3)'$ to $\mathrm{PB}(\mathbb{A})^\times$ for the definite quaternion algebra $\mathrm{B}_{/\mathbb{Q}}$ defined in (3.8). In [17], we really construct a Siegel modular form lying in Π_{f_5} by the Yoshida lift $\Theta_2(\pi(\mu)' \boxtimes \pi(\mu^3)')$. Thus, $\Pi_{f_5} = \Theta_2(\pi(\mu)' \boxtimes \pi(\mu^3)')$. Further, employing Theorem 8.5 of [22], we find that the set of all weak endoscopic lifts of $(\pi(\mu), \pi(\mu^3))$ is

$$\left\{\Theta_2\left(\pi(\mu) \boxtimes \pi\left(\mu^3\right)\right), \Theta_2\left(\pi(\mu)' \boxtimes \pi\left(\mu^3\right)'\right)\right\}.$$

Therefore, we guess that the irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ as in Theorem 3.6 is $\Theta_2(\pi(\mu) \boxtimes \pi(\mu^3))$, which is globally generic.

Next, in order to show the theorem, we will observe the local θ-lift $\theta_2(\pi(\mu)_2 \boxtimes \pi(\mu^3)_2) = \theta_2(\pi(\mu)_2 \boxtimes \pi(\mu)_2)$, which is the 2-component of $\Theta_2(\pi(\mu) \boxtimes \pi(\mu))$. For the sake of generality, let $\mathrm{B}_{/\mathbb{Q}}$ be a general quaternion algebra and consider $\Theta_2(\sigma \boxtimes \sigma)$ for an irreducible cuspidal automorphic representation σ of $\mathrm{PB}(\mathbb{A})^\times$.

Proposition 3.8. *Let* σ *be an irreducible cuspidal automorphic representation of* $\mathrm{PB}(\mathbb{A})$. *Let* Φ_Q *be the operator defined in Section* 2.3. *Then,* $\Phi_\mathrm{Q}(\Theta_2(\sigma \boxtimes \sigma))|_{\mathrm{GL}(2)} = \sigma^{\mathrm{JL}}$.

Proof. For a $\varphi \in \mathcal{S}(\mathrm{M}_2(\mathbb{A})^2)$, put $\varphi_0(x) = \varphi(0, x) \in \mathcal{S}(\mathrm{M}_2(\mathbb{A}))$. Take an $f \in \sigma$, and put $F = \theta_2(\varphi, f \boxtimes f)$. We calculate $\Phi_\mathrm{Q}(F)|_{\mathrm{GL}(2)} = \theta_1(\varphi_0, f \boxtimes f)$. We abbreviate $W_{F,\psi}^{\mathrm{Q}}(e_\mathrm{Q}(g, 1))$ as $W^1(g)$ for $g \in \mathrm{SL}_2(\mathbb{A})$. Then

$$W^1(1) = \int\limits_{Z_1(\mathbb{A})\backslash \mathrm{SO}_\mathrm{B}(\mathbb{A})} r^1\left(g, i_\rho(h_1, h_2)\right)\varphi_0(1)\left(\int\limits_{Z_1(\mathbb{Q})\backslash Z_1(\mathbb{A})} \overline{f}(bh_1) f(bh_2)\, \mathrm{d}b\right) \mathrm{d}h_1\, \mathrm{d}h_2, \tag{3.10}$$

where Z_1 denotes the stabilizer subgroup of $1 \in \mathrm{B}(\mathbb{Q})$, which is isomorphic to $\{(b, b) \mid b \in \mathrm{B}(\mathbb{A})^\times\}$ via i_ρ. Obviously, the integral in the parenthesis is nontrivial, and so is $W^1(1)$. Thus $\theta_1(\varphi_0, f \boxtimes f)$ is nontrivial. Because $\theta_1(\varphi_0, f \boxtimes f)$ is right $\mathrm{GL}_2(\mathbb{Z}_p)$-invariant for almost all p, it is easy to see that $\Phi_\mathrm{Q}(F)|_{\mathrm{GL}_2(\mathbb{Q}_p)} \in \sigma_p^{\mathrm{JL}}$. Noting the strong multiplicity theorem for GL(2), we find $\Phi_\mathrm{Q}(F)|_{\mathrm{GL}(2)} \in \sigma^{\mathrm{JL}}$. Hence the assertion. $\square$

Remark 2. This proof implies that $\Phi_\mathrm{Q}(\Theta_2(\sigma_1 \boxtimes \sigma_2)) = 0$ if $\sigma_1 \neq \sigma_2$.

Remark 3. If π_p is a supercuspidal representation, then $\theta_2(\pi_p \boxtimes \pi_p)$ (resp. $\theta_2(\pi'_p \boxtimes \pi'_p)$) is the constituent $\tau(S, \pi_p)$ (resp. $\tau(T, \pi_p)$) of the parabolically induced representation $1 \rtimes \pi_p$ (see [23] for the meanings of these symbols).

From this proof, there are a pair of $\phi_1 \in \mathcal{S}(\mathrm{B}(\mathbb{A}))$ and $f_0 \in \sigma$ such that $\theta_1(\phi_1, f_0 \boxtimes f_0)$ is a newform of σ^{JL}. In particular, if we set a $\varphi \in \mathcal{S}(\mathrm{B}(\mathbb{A})^2)$ so that $\varphi_0 = \phi_1$, then $\theta_2(\varphi, f \boxtimes f)$ is nontrivial. For example, set $\varphi(x_1, x_2) = \phi_1(x_2)\varphi'_\infty(x_1) \otimes_p \mathrm{ch}(x_1; \mathcal{R}_p)$, where $\mathcal{R}$ is a maximal order of $\mathrm{B}(\mathbb{Q})$ and φ'_∞ is an arbitrary Schwartz–Bruhat function on B_∞ such that $\varphi'_\infty(0) \neq 0$. Then, $\theta_2(\varphi, f_0 \boxtimes f_0)$ is right $\mathrm{Kl}_p(\mathrm{ord}_p(N))$-invariant if B_p is split, and $\mathrm{Kl}'_p(\mathrm{ord}_p(N))$-invariant otherwise, where N is the level of σ^{JL}, and

$$\mathrm{Kl}_p(n) := \begin{bmatrix} \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \cap \mathrm{GSp}_4(\mathbb{Z}_p),$$

$$\mathrm{Kl}'_p(n) := \begin{bmatrix} \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \cap \mathrm{GSp}_4(\mathbb{Z}_p)$$

for an integer n. van Geemen and van Straten [7] conjectured that, up to the Euler factors at 2,

$$L(s, \Pi_{f_i}; \mathrm{spin}) = L\bigl(s, \chi_i \pi(\mu)\bigr) L\bigl(s, \chi_i \pi\bigl(\mu^3\bigr)\bigr)$$

for $4 \leqslant i \leqslant 6$, where $\chi_4 = \chi^{(-2)}$, $\chi_5 = \mathbf{1}$, $\chi_6 = \chi^{(2)}$.

Corollary 3.9. *The above conjecture is true.*

Proof. It is possible to show the level of $\pi(\mu)$ (resp. $\chi^{(\pm 2)}\pi(\mu)$) is 2^5 (resp. 2^6) (cf. Proposition 4.8 of [17]). From the above argument, the local θ-lift $\theta_2(\pi(\mu)'_2 \boxtimes \pi(\mu)'_2)$ (resp. $\theta_2(\chi^{(\pm 2)}\pi(\mu)'_2 \boxtimes \chi^{(\pm 2)}\pi(\mu)'_2)$) has a local right $\mathrm{Kl}'_2(5)$ (resp. $\mathrm{Kl}'_2(6)$)-invariant Q-degenerate Whittaker function. Now, noting that

$$\mathrm{Kl}'_2(6) \simeq \begin{bmatrix} \mathbb{Z}_2 & 2^7\mathbb{Z}_2 & 2^5\mathbb{Z}_2 & 2^4\mathbb{Z}_2 \\ 2^{-1}\mathbb{Z}_2 & \mathbb{Z}_2 & 2^4\mathbb{Z}_2 & 2^3\mathbb{Z}_2 \\ 2^{-4}\mathbb{Z}_2 & 2^2\mathbb{Z}_2 & \mathbb{Z}_2 & 2^{-1}\mathbb{Z}_2 \\ 2^2\mathbb{Z}_2 & 2^3\mathbb{Z}_2 & 2^7\mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix} \cap \mathrm{GSp}_4(\mathbb{Q}_2),$$

one can show that the local θ-lift has a right $\Gamma(4,8)_2$-invariant vector and verify the conjecture in the same manner as in 3.1. $\square$

Finally, we will prove the theorem. Put

$$f'_5(Z) := \frac{\theta_{(1,0,0,0)}(Z)\theta_{(1,1,0,0)}(Z)}{\theta_{(1,0,0,1)}(Z)\theta_{(0,0,0,0)}(Z)}.$$

From f'_5, a character of $\Gamma(2)$ is obtained through the Igusa transformation formula. Using Proposition 6.2 of [7], we check that this character coincide with χ_{f_5}. For our computation, we put

$$f_5''(Z) = f_5'|_0\eta_2(Z) = c\frac{\theta_{(0,0,1,0)}(Z)\theta_{(0,0,1,1)}(Z)}{\theta_{(0,1,1,0)}(Z)\theta_{(0,0,0,0)}(Z)}$$

with $c \neq 0$. Let $\chi_{f_5''}$ be the character of $\Gamma(2)$ obtained from f_5''. Then $\ker(\chi_{f_5}) \simeq \ker(\chi_{f_5''})$. We can regard f_5'' as the θ-kernel $\theta_2(\phi'')(g,1)$ with $\phi'' = \bigotimes_v \phi_v'' \in \mathcal{S}(\mathrm{M}_2(\mathbb{A})^2)$. In particular, $\phi_2''(x_1, x_2)$ is in the form $\phi_1''(x_1) \times \phi_0''(x_2)$ such that

- $\phi_1''(0) \neq 0$.
- $\phi_0''(\varrho(h_1,h_2)x_2) = \phi_0''(x_2)$ if $h_1, h_2 \in \tilde{\Gamma}_0^{(1)}(32)_{\mathbb{A}}$.

For a positive integer κ and a congruence subgroup $\Gamma_1 \subset \mathrm{GL}_2(\mathbb{Q})$, let $S_\kappa^{(1)}(\Gamma_1)$ denote the space of elliptic cusp forms of weight κ with respect to Γ_1. Identifying this space with a subspace of automorphic forms on $\mathrm{GL}_2(\mathbb{A})$, we define the subspace

$$S_\kappa^{(1)}(\Gamma_1)^{\otimes 2,\mathrm{dis}} = \left\{ (f_1, f_2) \in S_\kappa^{(1)}(\Gamma_1)^{\otimes 2} \;\middle|\; \int_{Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})} \overline{f}_1(g) f_2(g)\,dg \neq 0 \right\}$$

of automorphic forms on $\mathrm{GL}_2(\mathbb{A})^{\otimes 2}$. Composing Remark 2 and the proof of Theorem 2 of Oda [15], we can obtain the following lemma.

Lemma 3.10. *Let κ be a positive integer. Let Γ_1 be a congruence subgroup of $\mathrm{GL}_2(\mathbb{Q})$. Suppose that a $\varphi \in \bigotimes_{p<\infty} \mathcal{S}(\mathrm{M}_2(\mathbb{Q}_p))$ satisfies that $\varphi(\varrho(h_1,h_2)x) = \varphi(x)$ for any $h_1, h_2 \in \Gamma_{1,\mathbb{A}}$. Then, there is a $\varphi_\infty \in \mathcal{S}(\mathrm{M}_2(\mathbb{R}))$ such that $\theta_1(\varphi_\infty \times \varphi, f) \not\equiv 0$ for a certain $f \in S_\kappa^{(1)}(\Gamma_1)^{\otimes 2,\mathrm{dis}}$.*

Applying this lemma to the above $\bigotimes_{p<\infty} \phi_{0,p}''$, we find that there is ϕ''' such that $\phi_p''' = \phi_p''$ for all $p < \infty$ and $\theta_1(\phi''', f)$ is not trivial for a certain $f \in S_2^{(1)}(\Gamma_0^{(1)}(32))^{\otimes 2,\mathrm{dis}}$. However, $S_2^1(\Gamma_0^{(1)}(32))$ is 1-dimensional, generated by a newform f^{new} of $\pi(\mu)$. Thus

$$\theta_1\left(\phi''', f^{\mathrm{new}} \boxtimes f^{\mathrm{new}}\right) \not\equiv 0.$$

From the above argument, $\Theta_2(\pi(\mu) \boxtimes \pi(\mu))$ has a right $\ker(\chi_{f_5''})_{\mathbb{A}}$-invariant vector. Thus $\Theta_2(\pi(\mu) \boxtimes \pi(\mu^3))$ also has a right $\ker(\chi_{f_5''})_{\mathbb{A}}$-invariant vector, and Theorem 3.6 follows immediately.

4. Hermitian modular forms

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field. For a Hermitian space W over K, let $\mathrm{U}_W(K)$ denote the unitary group acting on W and $\mathrm{GU}_W(K)$ the similitude one. In particular, we write

$$\mathrm{GU}_{n,n}(K) = \left\{ g \in \mathrm{GL}_{2n}(K) \;\middle|\; g\eta_n{}^t\overline{g} = \nu(g)\eta_n,\ \nu(g) \in \mathbb{Q}^\times \right\}$$

and the $2n$-dimensional split Hermitian space as $W_{n,n}$. Let $\mathrm{B}_{/\mathbb{Q}}$ be a definite quaternion algebra such that $\mathrm{B}_\mathbb{Q} \otimes K \simeq \mathrm{M}_2(K)$. We set the 6-dimensional positive quadratic space $V = K + \mathrm{B}_\mathbb{Q}$. Then, $\mathrm{PGSO}_V(\mathbb{Q}) \simeq \mathrm{PGU}_{W_\mathrm{B}}(K)$ for a certain 4-dimensional Hermitian space W_B (cf. Section 11 of [12]). Let $r_{U_{W_{n,n}\otimes W_\mathrm{B}}}$ be the global Weil representation of $\mathrm{U}_{W_{n,n}\otimes W_\mathrm{B}}(K_\mathbb{A})$ associated to the trivial character of $\mathbb{A}^\times$ and the additive character $\psi_K = \psi \circ \mathrm{Trace}_{K/\mathbb{Q}}$ (cf. [8,30]). We get the Weil representation $r_{U,n}$ of $\{(g,h) \in \mathrm{GU}_{n,n} \times \mathrm{GU}_{W_\mathrm{B}} \mid \nu(g) = \nu(h)\}$ by restricting $r_{U_{W_{n,n}\otimes W_\mathrm{B}}}$. For a $\varphi \in \mathcal{S}(W_\mathrm{B}(K_\mathbb{A})^n)$, we define

$$\theta_{U,n}(\varphi)(g,h) = \sum_{y \in W_\mathrm{B}(K)^n} r_{U,n}(g,h)\varphi(y).$$

For an automorphic form f on $\mathrm{GU}_4(K_{\mathbb{A}})$, define

$$\theta_{U,n}(\varphi, f)(g) = \int\limits_{\mathrm{U}_{W_{\mathrm{B}}}(K)\backslash \mathrm{U}_{W_{\mathrm{B}}}(K_{\mathbb{A}})} \theta_{U,n}(\varphi)(g, hh_1) f\bigl(hh'\bigr)\,\mathrm{d}h,$$

where h' is chosen so that $\nu(g) = \nu(h')$ and $\mathrm{d}h$ is a right Haar measure on $\mathrm{U}_{W_{\mathrm{B}}}(K)\backslash \mathrm{U}_{W_{\mathrm{B}}}(K_{\mathbb{A}})$. Because W_{B} is positive definite, this integral converges absolutely, and $\theta_{n,n}(\varphi, f)$ is an automorphic form on $\mathrm{GU}_{n,n}(K_{\mathbb{A}})$. For an irreducible cuspidal automorphic representation σ of $\mathrm{GU}_4(K_{\mathbb{A}})$, let $\Theta_{U,n}(\sigma)$ denote the space spanned by $\theta_{U,n}(\varphi, f)$ with $f \in \sigma$ and $\varphi \in \mathcal{S}(W_{\mathrm{B}}(K_{\mathbb{A}})^n)$. In the case $n = 2$, imitating the method in Section 4 of [27], it is possible to show that

$$\Theta_{U,2}(\sigma)_w \simeq \sigma_w,$$

if σ_w, $K_w/\mathbb{Q}_v$ and B_v are all unramified, where w is a place of K lying over a place v of $\mathbb{Q}$. We will identify irreducible cuspidal automorphic representations of $\mathrm{PGSO}_V(\mathbb{A})$ and those of $\mathrm{PGU}_{W_B}(K_{\mathbb{A}})$ via the isomorphism. Then, consider global θ-lifts of σ to $\mathrm{GSp}_4(\mathbb{A})$. Let σ' be an irreducible constituent of $\sigma|_{\mathrm{SO}_V}$. Assume $\Theta_2(\sigma) \neq 0$. Let Π' be an irreducible constituent of $\Theta_2(\sigma)$. Using [14], we calculate

$$L_{S_{\sigma'}}\bigl(s, \sigma'\bigr) = \zeta_{S_{\sigma'}}(s) L_{S_{\sigma'}}\left(s, \Pi', \left(\frac{-d}{*}\right); r_5\right), \tag{4.1}$$

where $L_{S_{\sigma'}}(s, \sigma')$ is the standard Langlands L-function of σ' (of degree 6) and $L_{S_{\sigma'}}(s, \Pi', \chi_K; r_5)$ is the $(\frac{-d}{*})$-twist of $L_{S_{\sigma'}}(s, \Pi'; r_5)$ (note $S_{\sigma'} = S_{\Pi'}$). Assume $\Theta_{U,2}(\sigma) \neq 0$. Let τ' be an irreducible constituent of $\Theta_{U,2}(\sigma)$. Using the description of L-functions of unramified $\tau'_w \in \mathrm{Irr}(\mathrm{GU}_2(K_w))$ in Section 3 of [11], we calculate

$$L_{S_{\sigma'}}\bigl(s, \tau'; \wedge_t^2\bigr) = L_{S_{\sigma'}}\bigl(s, \sigma'\bigr).$$

Now (1.2) is shown. We will show the existence of $\tilde{F}$ of Theorem B.

Proposition 4.1. *Let K, $\mathrm{B}_{/\mathbb{Q}}$, V and W_{B} be as above. Let σ be an irreducible automorphic representation of $\mathrm{PGSO}_V(\mathbb{A}) \simeq \mathrm{PGU}_{W_{\mathrm{B}}}(\mathbb{A})$. If $\Theta_2(\sigma)$ is cuspidal and nontrivial, then $\Theta_{U,2}(\sigma) \neq 0$.*

Proof. Since $\Theta_2(\sigma) \neq 0$, there is an automorphic form $f \in \mathrm{Ind}_{\mathrm{GSO}_V}^{\mathrm{GO}_V} \sigma$ and $\phi \in \mathcal{S}(V(\mathbb{A})^2)$ such that

$$F(g) := \int\limits_{\mathrm{O}_V(\mathbb{Q})\backslash \mathrm{O}_V(\mathbb{A})} \theta_2(\phi)(g, hh_0) f(hh_0)\,\mathrm{d}h$$

is nontrivial, where $h_0 \in \mathrm{GO}_V(\mathbb{A})$ is chosen so that $\nu(g) = \nu(h_0)$. Since V is positive definite, F is a cusp form on $\mathrm{GSp}_4(\mathbb{A})$ is related to a (holomorphic) Siegel modular form. Since F is a cusp form, $F_T(1) \neq 0$ for a positive $T = {}^tT$. Take $x_1, x_2 \in V$ so that $(x_1, x_2) = T$. Let $Z_{(x_1,x_2)}(\mathbb{Q}) \subset \mathrm{O}_V(\mathbb{Q})$ be the pointwise stabilizer subgroup of (x_1, x_2). Then,

$$F_T(1) = \int\limits_{Z_{(x_1,x_2)}(\mathbb{Q})\backslash \mathrm{O}_V(\mathbb{A})} r^2(1, h)\phi(x_1, x_2) f(h)\,\mathrm{d}h.$$

Hence,

$$\int_{Z_{(x_1,x_2)}(\mathbb{Q})\backslash Z_{(x_1,x_2)}(\mathbb{A})} f(zh)\,\mathrm{d}z \not\equiv 0.$$

Because $Z_{(x_1,x_2)}(\mathbb{Q}) \simeq \mathrm{O}_4(\mathbb{Q})$, there is a subgroup $\mathrm{U}_x(K)$ $(\simeq \mathrm{U}_2(K))$ of $Z_{(x_1,x_2)}(\mathbb{Q})$ such that

$$\int_{\mathrm{U}_x(K)\backslash \mathrm{U}_x(K_{\mathbb{A}})} f(zh)\,\mathrm{d}z \not\equiv 0.$$

Now then, we will consider $\Theta_{U,2}(\sigma)$. Let $\langle *, * \rangle$ denote the Hermite form of W_{B}. Notice that U_x stabilizes a pair $(y_1, y_2) \in W_{\mathrm{B}}(K)^2$. Put $Y = \left[\begin{smallmatrix} \langle y_1,y_1\rangle & \langle y_1,y_2\rangle \\ \langle y_2,y_1\rangle & \langle y_2,y_2\rangle \end{smallmatrix}\right]$, which is positive definite. Then, for a $\varphi \in \mathcal{S}(W_{\mathrm{B}}(K_{\mathbb{A}})^2)$, the Fourier coefficient of $\theta_{U,2}(\varphi, f)(g)$ at Y is

$$\int_{\mathrm{U}_x(K)\backslash \mathrm{U}_{W_{\mathrm{B}}}(K_{\mathbb{A}})} r_{U,2}(g,h)\varphi(y_1,y_2) f(h)\,\mathrm{d}h$$

$$= \mathrm{vol}\big(\mathrm{U}_x(K)\backslash \mathrm{U}_x(K_{\mathbb{A}})\big)^{-1} \int_{\mathrm{U}_x(K_{\mathbb{A}})\backslash \mathrm{U}_{W_{\mathrm{B}}}(K_{\mathbb{A}})} r_{U,2}(g,h)\varphi(y_1,y_2)\left(\int_{\mathrm{U}_x(K)\backslash \mathrm{U}_x(K_{\mathbb{A}})} f(zh)\,\mathrm{d}z\right)\mathrm{d}\dot{h},$$

where $\mathrm{d}\dot{h}$ indicates the Haar measure of $\mathrm{U}_x(K)\backslash \mathrm{U}_x(K_{\mathbb{A}})$ associated to $\mathrm{d}h$. Since the integral in the parenthesis is nontrivial, it is possible to choose φ so that this value does not vanish at $g = 1$ (cf. concluding remarks in [28]). Hence the assertion. □

Finally, we will show the last assertion of the theorem, observing the L-function $L_{S_\tau}(s, \tau; \wedge_t^2)$ for an irreducible, noncuspidal, automorphic representation τ of $\mathrm{GU}_{2,2}(K_{\mathbb{A}})$. Let $K^1 = \{z \in K^\times \mid N_{K/\mathbb{Q}}(z) = 1\}$. Let $P_1(K) = N_1(K)M_1(K)$ with

$$N_1(K) = \left\{ \begin{bmatrix} 1 & & v & w \\ & 1 & \overline{w} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & \\ & & -u & 1 \end{bmatrix} \,\middle|\, v \in \mathbb{Q},\ u, w \in K \right\},$$

$$M_1(K) = \left\{ \begin{bmatrix} tz & & & \\ & z^c\alpha & & z^c\beta \\ & & t^{-1}z\nu(g_1) & \\ & z^c\gamma & & z^c\delta \end{bmatrix} \,\middle|\, g_1 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GU}_{1,1}(K),\ z \in K^1,\ t \in \mathbb{Q}^\times \right\}.$$

The modular character δ_{P_1} of $P_1(K_{\mathbb{A}})$ is given by $\delta_{P_1}(nm) = |\nu(g)|^{-4}|t|^6$. We embed $\mathrm{GU}_{1,1}(K) \times K^1 \times \mathbb{Q}^\times$ into $M_1(K)$, naturally. For a triple of irreducible automorphic representations π, μ, ξ of $\mathrm{GU}_{1,1}(K_{\mathbb{A}}) \times K^1_{\mathbb{A}} \times \mathbb{A}^\times$, let $\pi \otimes \mu \otimes \xi$ denote the representation of $P_1(K_{\mathbb{A}})$ sending $nm = n(g_1, z, t)$ to $\pi(g_1)\mu(z)\xi(t)$. Hermitian modular forms of $\mathrm{SU}_{2,2}(K)$ are related to automorphic forms on $\mathrm{GU}_{2,2}(K_{\mathbb{A}})$ with a manner similar to that in Section 2.1. We will identify them. A Hermitian modular form is noncuspidal, if and only if

$$\Phi_U(F)(g,t,z;h) := \mathrm{vol}\big(N_1(k)\backslash N_1(\mathbb{A})\big)^{-1} \int_{N_1(K)\backslash N_1(K_{\mathbb{A}})} F\big(n(g_1,t,z)h\big)\,\mathrm{d}n$$

is not a zero function of (g_1, t, z) at some $h \in \mathrm{GU}_{2,2}(K_{\mathbb{A}})$, where Φ_U is equal to the Siegel operator in [13], essentially. Hence, if a noncuspidal τ is generated by a Hermitian modular form, then τ is a constituent of an induced representation from $\pi \otimes \mu \otimes \xi$. In this case, there is an automorphic form $f \in \tau$, such that

$$\Phi_U(f)(nmh) = \left|\nu(g_1)\right|^{-2}|t|^3\pi(g_1)\mu(z)\xi(t)\Phi_U(f)(h).$$

Further, if the central character of π_1 is trivial, with regarding π_1 as an irreducible automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ $(\simeq \mathrm{SO}_{2,1}(\mathbb{A}) \simeq \mathrm{PGU}_{1,1}(K_{\mathbb{A}}))$, we write

$$L_{S_\tau}\left(s, \tau; \wedge_t^2\right) = L_{S_\tau}\left(s - \frac{1}{2}, \sigma_1\right) L_{S_\tau}\left(s - \frac{1}{2}, \sigma_1, \xi\right) L_{S_\tau}(s, \mu). \tag{4.2}$$

Now, apply the above argument to our case. Since every automorphic form of $\Theta_{U,2}(\sigma)$ is related to a Hermitian modular form of weight 4, the weight of ξ is $4 - 3 = 1$, if $\Theta_{U,2}(\sigma)$ is noncuspidal. Since the central character of σ is trivial, so is that of $\Theta_{U,1}(\sigma)$. Then, obviously, (4.2) does not satisfy the Ramanujan conjecture. The last assertion of the theorem follows, immediately. This completes the proof.

Acknowledgments

I express my thanks to Professor R. Salvati Manni for suggesting to solve the conjectures, and to Professor T. Ibukiyama and Professor T. Yamauchi for their kind advice and encouragement.

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