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Dimensions of Cusp Forms for $\Gamma_0(p)$ in Degree Two and Small Weights

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Abstract. We investigate degree two Siegel cusp forms of small weight for $\Gamma_0(p)$. Using the Restriction Technique we compute some dimensions and verify the conjectures of HASHIMOTO in some examples of weights three and four. For weight two we determine the dimension for primes $p \leq 41$ and find only lifts. We explain in general how to compute spaces of Siegel cusp forms for subgroups of finite index in Γ_n.

1 Introduction

See the end of this section for a list of basic notations used in this article. For weights $k \geq 5$, the dimensions of the spaces of cusp forms in degree two for $\Gamma_0(p)$ were computed by K. HASHIMOTO [7]. He also gave conjectural dimension formulas in the cases of weights 3 and 4, leaving only weights 1 and 2 untouched. The intervening years have not seen many examples to test his conjectures. It is the proof of the upper bound that makes the computation of $\dim S_2^k(\Gamma_0(p))$ difficult. Recent techniques make the computation of this upper bound feasible for Siegel modular forms. We use Vanishing Theorems [16] and the Restriction Technique [17, 21] to compute $\dim S_2^k(\Gamma_0(p))$ for $k = 2, 3, 4$ and for small primes p. For $k = 1$, all examples were trivial and we refer to [12] by T. IBUKIYAMA and N. SKORUPPA, where it is shown that $S_2^1(\Gamma_0(N)) = \{0\}$ for all positive integers N. Lower bounds are given by constructing Siegel modular cusp forms. This paper both explains how to use the Vanishing Theorems and the Restriction Technique for subgroups of finite index and performs the following computations. For primes $p = 2$ and $p = 3$ the results can be found in [26, 9].

Theorem 1.1. *For weight $k = 4$, we have the following dimensions:*

p	2	3	5	7	11	13
$\dim S_2^4(\Gamma_0(p))$	0	1	1	3	7	11

Conjecture 7-1 in [7], pg. 485–486 *of* K. HASHIMOTO *is true in all these cases.*

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Theorem 1.2. *For weight $k = 3$, we have the following dimensions:*

p	2	3	5	7	11	13	17	19	23
$\dim S_2^3(\Gamma_0(p))$	0	0	0	0	0	0	1	1	2

Conjecture 7-2 in [7], pg. 486 *of* K. HASHIMOTO *is true in all these cases.*

Theorem 1.3. *For weight $k = 2$, we have the following dimensions:*

p	2	3	5	7	11	13	17	19	23	29	31	37	41
$\dim S_2^2(\Gamma_0(p))$	0	0	0	0	1	0	1	1	3	3	3	2	6

For primes $p \leq 41$, the Hecke eigenforms in $S_2^2(\Gamma_0(p))$ are all lifts of elliptic eigenforms; they are either Yoshida lifts, Saito-Kurokawa lifts, or both.

These computations are feasible because of two theoretical innovations. First, although SIEGEL had estimated the number of Fourier coefficients needed to determine a Siegel modular form, these estimates were rough and superior ones were discovered in [16]. Section 2 surveys these estimates, relaxes some restrictions found in [16], and provides an improved list of constants for estimations with Siegel modular forms on subgroups of finite index. Section 3 works out specific details for the subgroup $\Gamma_0(p)$. Second, the Restriction Technique, introduced in [17], efficiently produces linear relations among the Fourier coefficients of Siegel modular forms. The restriction of a Siegel modular form to a modular curve gives an elliptic modular form; known linear relations among the Fourier coefficients of elliptic modular forms may then be pulled back to produce linear relations among the Fourier coefficients of Siegel forms. Along with a determining set of Fourier coefficients, these linear relations provide upper bounds for $\dim S_2^k(\Gamma_0(p))$. Whether or not this method always generates a complete set of linear relations is unknown. An exposition of the Restriction Technique for level one and some partial converses to the generation question can be found in [21]. Section 5 here explains the Restriction Technique for subgroups of finite index but refers to [17, 21] for full details.

The weight two case is interesting because the L-functions of the rational non-lift Hecke eigenforms may also be those of rational abelian varieties. It would be most interesting to find a weight two rational Hecke eigenform that is not a lift of elliptic eigenforms but we evidently need to extend our search to higher levels to reach this goal. Also, the dimension of $\dim S_2^2(\Gamma_0(p))$ may grow more slowly than $O(p^3)$, which is the growth rate of HASHIMOTO's dimension formulas for $k > 2$. The weight three case is also interesting as it corresponds to holomorphic differential forms on the modular threefold $\Gamma_0(p)\backslash\mathcal{H}_2$. We thank R. SCHARLAU for discussions at AIM in Palo Alto in 2003 about the paper [24]. Many of the experimental results in [24] become theorems by using Theorem 2.5 (or 3.3) here. We thank S. BÖCHERER for communicating the general result in Section 6. We thank A. BRUMER and T. IBUKIYAMA for discussions about this work and for their encouragement. We thank the referee for improving the Introduction and for shortening a number of proofs.

Notations. Let $n, k \in \mathbb{Z}^+$.

- $\mathcal{P}_n(\mathbb{R})$ = real positive definite $n \times n$ matrices and $\mathcal{P}_n^{\text{semi}}(\mathbb{R})$ = real positive semidefinite $n \times n$ matrices.
- $\mathcal{H}_n = \{\Omega \in \mathbb{C}^{n\times n} \mid \Omega \text{ symmetric and } \mathrm{Im}(\Omega) > 0\}$ = Siegel upper half space.
- $\mathrm{Sp}_n(F)$ = symplectic $2n \times 2n$ matrices over a ring F.
- Define $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ = the Siegel modular group.
- $\Gamma_0(N) = \left\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z}) \mid C \equiv 0 \mod N\right\}$, and $\Delta_n(\mathbb{Z}) = \left\{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})\right\}$.
- $V_n(\mathbb{Z})$ = symmetric $n \times n$ matrices over $\mathbb{Z}$. For $S \in V_n(\mathbb{Z})$, define $t(S) = \left(\begin{smallmatrix} I & S \\ 0 & I \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$. For $U \in \mathrm{GL}_n(\mathbb{Z})$, define $u(U) = \left(\begin{smallmatrix} U & 0 \\ 0 & U^* \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$.
- For $\Omega \in \mathcal{H}_n$ and $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$, define $\sigma\langle\Omega\rangle = (A\Omega + B)(C\Omega + D)^{-1}$, and for $f : \mathcal{H}_n \to \mathbb{C}$, define $(f|_k\sigma)(\Omega) = \det(C\Omega + D)^{-k} f(\sigma\langle\Omega\rangle)$.
- For $\Gamma \subseteq \Gamma_n$ of finite index, define the Siegel modular forms of weight k with respect to the subgroup Γ to be $M_n^k(\Gamma)$ and the Siegel cusp forms to be $S_n^k(\Gamma)$, see [6].
- When $t\,(V_n(\mathbb{Z})) \subseteq \Gamma$, we have the Fourier expansion
$$f(\Omega) = \sum_{t \in \mathcal{X}_n} a(t; f) e(\langle t, \Omega\rangle)$$
for $f \in S_n^k(\Gamma)$. Here, $\langle t, \Omega\rangle = \mathrm{tr}(t\Omega)$, $e(z) = e^{2\pi i z}$ and $\mathcal{X}_n$ = integral-valued half-integral positive definite $n \times n$ matrices.
- For $f \in S_n^k(\Gamma)$ as above, define the support of f to be $\mathrm{supp}(f) = \{t \in \mathcal{X}_n \mid a(t; f) \neq 0\}$, and the semihull of f to be
$$\nu(f) = \mathrm{Closure}(\mathrm{ConvexHull}(\mathbb{R}_{\geq 1}\,\mathrm{supp}(f))) \text{ inside } \mathcal{P}_n^{\text{semi}}(\mathbb{R}).$$
- For $T, u \in \mathrm{GL}_n(\mathbb{R})$, define $T[u] = u^{\mathrm{t}} T u$.
- For $s \in \mathcal{P}_n^{\text{semi}}(\mathbb{R})$, define
  (1) $m(s) = \inf_{u \in \mathbb{Z}_n \setminus \{0\}} u^{\mathrm{t}} s u$, the Minimum function.
  (2) $\widetilde{\mathrm{tr}}(s) = \inf_{u \in \mathrm{GL}_n(\mathbb{Z})} \mathrm{tr}(u^{\mathrm{t}} s u)$, the reduced trace function.
  (3) $\delta(s) = \det(s)^{1/n}$, the reduced determinant function.
  (4) $w(s) = \inf_{u \in \mathcal{P}_n(\mathbb{R})} \frac{\langle u, s\rangle}{m(u)}$, the dyadic trace function.
- $\mu_n = \sup_{u \in \mathcal{P}_n(\mathbb{R})} \frac{m(s)}{\delta(s)}$, the Hermite constant.

2 Vanishing Theorems

For computational purposes it is convenient to choose a function ϕ to linearly order the support of a Siegel modular form.

Definition 2.1. A function $\phi : \mathcal{P}_n^{\text{semi}}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ is called *type one* if

(1) For all $s \in \mathcal{P}_n(\mathbb{R})$, $\phi(s) > 0$,
(2) for all $\lambda \in \mathbb{R}_{\geq 0}$ and $s \in \mathcal{P}_n^{\text{semi}}(\mathbb{R})$, $\phi(\lambda s) = \lambda\phi(s)$,
(3) for all $s_1, s_2 \in \mathcal{P}_n^{\text{semi}}(\mathbb{R})$, $\phi(s_1 + s_2) \geq \phi(s_1) + \phi(s_2)$.

A type one function is continuous on $\mathcal{P}_n(\mathbb{R})$ and respects the partial order on $\mathcal{P}_n^{\text{semi}}(\mathbb{R})$. The following vanishing theorem is essentially from [16], pg. 215.

Theorem 2.2. *Let ϕ be type one. For all $n \in \mathbb{Z}^+$ there exists a $c_n(\phi) \in \mathbb{R}_{>0}$ such that*

$$\forall k \in \mathbb{Z}^+, \forall f \in S_n^k, \quad \inf \phi(\mathrm{supp}(f)) > c_n(\phi)k \Rightarrow f \equiv 0. \qquad (2.1)$$

The constant $c_n(\phi)$ may be taken to be $\frac{1}{4\pi} \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left((\mathrm{Im}\,\sigma\langle\Omega\rangle)^{-1}\right)$.

Proof. There is an $\Omega_0 \in \mathcal{H}_n$ where $\phi_f(\Omega) = \det(\mathrm{Im}\,\Omega)^{k/2}|f(\Omega)|$ attains its maximum. The Semihull Theorem from [16], pg. 211 says that $\frac{k}{4\pi}(\mathrm{Im}\,\Omega_0)^{-1} \in \nu(f)$ if f is nontrivial. Therefore, for some $\alpha_s \geq 0$ with $\sum \alpha_s \geq 1$, we have $x = \sum_{s \in \mathrm{supp}(f)} \alpha_s s$ arbitrarily close to $\frac{k}{4\pi}(\mathrm{Im}\,\Omega_0)^{-1}$. By the continuity of ϕ we have

$$\phi(x) = \phi\left(\sum \alpha_s s\right) \geq \sum \alpha_s \phi(s) \geq \sum \alpha_s \inf \phi(\mathrm{supp}(f)) \geq \inf \phi(\mathrm{supp}(f))$$

arbitrarily close to $\frac{k}{4\pi}\phi\left((\mathrm{Im}\,\Omega_0)^{-1}\right)$ so that $\frac{k}{4\pi}\phi\left((\mathrm{Im}\,\Omega_0)^{-1}\right) \geq \inf \phi(\mathrm{supp}(f))$. Any $\Omega \in \Gamma_n\langle\Omega_0\rangle$ also has this property so that $\inf_{\sigma \in \Gamma_n} \frac{k}{4\pi}\phi\left((\mathrm{Im}\,\sigma\langle\Omega_0\rangle)^{-1}\right) \geq \inf \phi(\mathrm{supp}(f))$ and $\frac{k}{4\pi} \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left((\mathrm{Im}\,\sigma\langle\Omega\rangle)^{-1}\right) \geq \inf \phi(\mathrm{supp}(f))$. □

Theorem 2.3. *Equation* (2.1) *holds if we select for $c_n(\phi)$ the following:*

(1) *For the Minimum function m, $c_n(m) = \frac{1}{4\pi}\frac{2}{\sqrt{3}}\mu_n^2$.*
(2) *For the reduced trace $\widetilde{\mathrm{tr}}$, $c_n(\widetilde{\mathrm{tr}}) = \frac{1}{4\pi}\frac{2}{\sqrt{3}} n\mu_n^n$.*
(3) *For the reduced determinant δ, $c_n(\delta) = \frac{1}{4\pi}\frac{2}{\sqrt{3}}\mu_n$.*
(4) *For the dyadic trace w, $c_n(w) = \frac{1}{4\pi}\frac{2}{\sqrt{3}} n$.*
(5) *For $n = 1$, $c_1(\phi) = \frac{1}{12}\phi(1)$ and this is optimal.*
(6) *For $n = 2$, $c_2(m) = \frac{1}{10}$ and this is optimal.*
(7) *For $n = 2$, $c_2(\widetilde{\mathrm{tr}}) = \frac{1}{5}$ and this is optimal.*
(8) *For $n = 2$, $c_2(w) = \frac{1}{6}$.*
(9) *For $n = 3$, $c_3(m) = \frac{1}{9}$ and this is optimal.*
(10) *For $n = 3$, $c_3(\widetilde{\mathrm{tr}}) = \frac{3}{4\pi}\frac{2}{\sqrt{3}}\frac{1}{2}$.*
(11) *For $n = 4$, $c_4(m) = \frac{1}{8}$ and this is optimal.*
(12) *For $n = 4$, $c_4(\widetilde{\mathrm{tr}}) = \frac{4}{4\pi}\frac{2}{\sqrt{3}}\frac{1}{2}$.*
(13) *For $n = 5$, $c_5(\widetilde{\mathrm{tr}}) = \frac{5}{4\pi}\frac{2}{\sqrt{3}}\frac{21}{50}$.*
(14) *For $n = 5$, $c_5(m) = \frac{1}{4\pi}\frac{2}{\sqrt{3}}2$.*
(15) *For $n = 6$, $c_6(m) = \frac{1}{4\pi}\frac{2}{\sqrt{3}}\frac{8}{3}$.*
(16) *For $n = 7$, $c_7(m) = \frac{1}{4\pi}\frac{2}{\sqrt{3}}3$.*

Proof. Estimates (1)–(4) were proven in [16], pp. 216–218. The formula (5) $c_1(\phi) = \frac{1}{12}\phi(1)$ follows from the Valence Inequality. Estimates (6)–(8) were proven in [19], pg. 71. A reference for (9) $c_3(m) = \frac{1}{9}$ and (11) $c_4(m) = \frac{1}{8}$ is [23]. Estimates (10), (12) and (13) were proven in [19], pg. 63. Estimates (14)–(16) were published in [20]. □

The constants in Theorem 2.3 are the best currently known to the authors. Item (8) is the estimate used in the computations of this paper. The following

Theorem for cusp forms on subgroups of finite index is also essentially from [16], pp. 215–216.

Theorem 2.4. *Let ϕ be type one. For all $n \in \mathbb{Z}^+$ there exists a $c_n(\phi) \in \mathbb{R}_{>0}$ such that: For any subgroup $\Gamma \subseteq \Gamma_n$ with finite index I and coset decomposition $\Gamma_n = \bigcup_{i=1}^{I} \Gamma M_i$, we have*

$$\forall k \in \mathbb{Z}^+, \forall f \in S_n^k(\Gamma), \quad \frac{1}{I}\sum_{i=1}^{I} \inf\phi\,(\operatorname{supp}(f|M_i)) > c_n(\phi)k \Rightarrow f \equiv 0. \quad (2.2)$$

Any constant $c_n(\phi)$ for which equation (2.1) *holds makes equation* (2.2) *hold, and conversely.*

Proof. If f is level one then

$$\frac{1}{I}\sum_{i=1}^{I} \inf\phi\,(\operatorname{supp}(f|M_i)) = \frac{1}{I}\sum_{i=1}^{I} \inf\phi\,(\operatorname{supp}(f)) = \inf\phi\,(\operatorname{supp}(f))$$

and so any constant valid for equation (2.2) is also valid for equation (2.1). On the other hand, assume that $c_n(\phi)$ is valid for equation (2.1). If $f \in S_n^k(\Gamma)$ then $\operatorname{Norm}(f) = \prod_{i=1}^{I} f|M_i \in S_n^{Ik}$ is level one. We have $\operatorname{supp}(\operatorname{Norm}(f)) \subseteq \sum_1^I \operatorname{supp}(f|M_i)$ so that $\inf\phi\,(\operatorname{supp}(\operatorname{Norm}(f))) \geq \inf\phi\big(\sum_1^I \operatorname{supp}(f|M_i)\big) \geq \sum_1^I \inf\phi\,(\operatorname{supp}(f|M_i))$. Thus $\frac{1}{I}\sum_1^I \inf\phi\,(\operatorname{supp}(f|M_i)) > c_n(\phi)k$ implies $\inf\phi\,(\operatorname{supp}(\operatorname{Norm}(f))) \geq \sum_{i=1}^{I} \inf\phi\,(\operatorname{supp}(f|M_i)) > c_n(\phi)Ik$ so that $\operatorname{Norm}(f) = 0$ by Theorem 2.2 and hence $f = 0$. □

3 $\Gamma_0(p)$

We explain how to use the Vanishing Theorems for a subgroup of finite index in Γ_2. We work out the details of Section 2 and find determining sets of Fourier coefficients for $S_2^k(\Gamma_0(p))$, p a prime. The dimension of $S_2^k(\Gamma_0(p))$ was determined in [7] for $k \geq 5$, and in [12] for $k = 1$, so we focus on weights $k = 2, 3, 4$ here. Recall the definitions $\Gamma_0(N) = \left\{\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z}) \mid C \equiv 0 \mod N\right\}$, and $\Delta_n(\mathbb{Z}) = \left\{\left(\begin{smallmatrix} A & B \\ 0 & D\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})\right\}$. The index is $[\Gamma_2 : \Gamma_0(p)] = (1+p)(1+p^2)$ and the double coset decomposition has three double cosets:

$$\Gamma_2 = \Gamma_0(p)E_0 \cup \Gamma_0(p)E_1\Delta_2(\mathbb{Z}) \cup \Gamma_0(p)E_2\Delta_2(\mathbb{Z}),$$

where $E_r = I_{4-2r} \oplus J_r$ for $r = 0, 1, 2$; namely,

$$E_0 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}; \quad E_1 = \begin{pmatrix} 1&0&0&0\\0&0&0&1\\0&0&1&0\\0&-1&0&0\end{pmatrix}; \quad E_2 = \begin{pmatrix} 0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}.$$

The width of a double coset is the number of distinct single cosets it contains and the widths of the above double cosets are 1, $p+p^2$ and p^3, respectively. Our interest is in the double coset decomposition because for $f \in S_n^k(\Gamma_0(\ell))$ and $\delta = \left(\begin{smallmatrix} u & \zeta u^* \\ 0 & u^*\end{smallmatrix}\right) \in \Delta_n(\mathbb{Z})$ we have $\operatorname{supp}(f|\delta) = u^{\mathrm{t}}\operatorname{supp}(f)u$. If ϕ is a class function then $\phi\,(\operatorname{supp}(f|M))$ depends only on the double coset $\Gamma_0(\ell)M\Delta_n(\mathbb{Z})$. Let

$f \in S_2^k(\Gamma_0(p))$ and consider the three Fourier expansions:

$$\begin{aligned}
(f|E_0)(\Omega) &= \sum_{t \in \mathcal{X}_2} a_0(t) e(\langle t, \Omega\rangle),\\
(f|E_1)(\Omega) &= \sum_{t \in \mathcal{X}_2^{(1)}} a_1(t) e(\langle t, \Omega\rangle),\\
(f|E_2)(\Omega) &= \sum_{t \in \frac{1}{p}\mathcal{X}_2} a_2(t) e(\langle t, \Omega\rangle).
\end{aligned} \tag{3.1}$$

In the first Fourier series the translation subgroup of $\Gamma_0(p)$ is $\left\{\zeta \in V_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} I & \zeta \\ 0 & I\end{smallmatrix}\right) \in \Gamma_0(p)\right\} = V_2(\mathbb{Z})$ so that t runs over $\mathcal{X}_2$. The similarity subgroup of $\Gamma_0(p)$ is $\left\{u \in \mathrm{GL}_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} u & 0 \\ 0 & u^*\end{smallmatrix}\right) \in \Gamma_0(p)\right\} = \mathrm{GL}_2(\mathbb{Z})$ so that $a_0(u^{\mathrm{t}}tu) = \det(u)^k a_0(t)$ for all $u \in \mathrm{GL}_2(\mathbb{Z})$. For the third Fourier series, the translation subgroup is $\left\{\zeta \in V_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} I & \zeta \\ 0 & I\end{smallmatrix}\right) \in E_2\Gamma_0(p)E_2\right\} = pV_2(\mathbb{Z})$ so that t runs over $\frac{1}{p}\mathcal{X}_2$. The similarity subgroup is $\left\{u \in \mathrm{GL}_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} u & 0 \\ 0 & u^*\end{smallmatrix}\right) \in E_2\Gamma_0(p)E_2\right\} = \mathrm{GL}_2(\mathbb{Z})$ so that $a_2(u^{\mathrm{t}}tu) = \det(u)^k a_2(t)$ for all $u \in \mathrm{GL}_2(\mathbb{Z})$. For the middle cusp $\Gamma_0(p)E_1\Delta_2(\mathbb{Z})$, the translation subgroup is $\left\{\zeta \in V_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} I & \zeta \\ 0 & I\end{smallmatrix}\right) \in E_1\Gamma_0(p)E_1\right\} = \left\{\left(\begin{smallmatrix} a & b \\ b & cp\end{smallmatrix}\right) \mid \left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right) \in V_2(\mathbb{Z})\right\}$ so that t runs over $\left(\begin{smallmatrix} a & b \\ b & c/p\end{smallmatrix}\right) \in \mathcal{P}_2(\mathbb{Q})$ with $\left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right) \in \mathcal{X}_2$, we call this set $\mathcal{X}_2^{(1)}$ for convenience. The similarity subgroup is $\left\{u \in \mathrm{GL}_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} u & 0 \\ 0 & u^*\end{smallmatrix}\right) \in E_1\Gamma_0(p)E_1\right\} = \langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1\end{smallmatrix}\right), \Gamma_0(p)\rangle$ so that $a_1(u^{\mathrm{t}}tu) = \det(u)^k a_1(t)$ for all $u \in \left\langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1\end{smallmatrix}\right), \Gamma_0(p)\right\rangle$.

Our goal is to choose sets $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2 \in \mathcal{P}_2(\mathbb{Q})$ so that the following map is injective:

$$\begin{aligned}
S_2^k(\Gamma_0(p)) &\to \prod_{\mathcal{C}_0}\mathbb{C} \times \prod_{\mathcal{C}_1}\mathbb{C} \times \prod_{\mathcal{C}_2}\mathbb{C}\\
f &\mapsto (a_0(t))_{t\in\mathcal{C}_0} \times (a_1(t))_{t\in\mathcal{C}_1} \times (a_2(t))_{t\in\mathcal{C}_2}.
\end{aligned} \tag{3.2}$$

Theorem 3.1. *For $p, k \in \mathbb{Z}^+$, define*

$$\begin{aligned}
\mathcal{C}_0 &= \left\{t \in \mathcal{X}_2 \mid w(t) < \tfrac{1}{6}(1+p)k\right\},\\
\mathcal{C}_1 &= \emptyset,\\
\mathcal{C}_2 &= \left\{t \in \tfrac{1}{p}\mathcal{X}_2 \mid w(t) < \tfrac{1}{6}\left(\tfrac{1+p}{p}\right)k\right\}.
\end{aligned}$$

For these $\mathcal{C}_r$ the map (3.2) *is injective. That is, $\mathcal{C}_0$ and $\mathcal{C}_2$ are a determining set of Fourier coefficients for elements of $S_2^k(\Gamma_0(p))$.*

Proof. Let $\gamma = \frac{1}{6}(1+p)k$. Let $I = [\Gamma_2 : \Gamma_0(p)] = (1+p)(1+p^2)$. Take an $f \in S_2^k(\Gamma_0(p))$ and suppose that $a_r(t) = 0$ for all $t \in \mathcal{C}_r$ and $0 \le r \le 2$. Since the dyadic trace is a class function we have

$$\frac{1}{I}\sum_{i=1}^{I}\inf w\,(\mathrm{supp}(f|M_i)) = \frac{1}{(1+p)(1+p^2)} \times$$
$$\big(1\inf w\,(\mathrm{supp}(f)) + (p+p^2)\inf w\,(\mathrm{supp}(f|E_1)) + p^3 \inf w\,(\mathrm{supp}(f|E_2))\big).$$

Since $\inf w\,(\mathrm{supp}(f|E_1)) > 0$ we have

$$\frac{1}{I}\sum_{i=1}^{I}\inf w\,(\mathrm{supp}(f|M_i)) > \frac{1}{(1+p)(1+p^2)}\Big(1\gamma + 0 + p^3\frac{1}{p}\gamma\Big) = \frac{\gamma}{1+p} = \frac{k}{6}.$$

Hence we have $f = 0$ by Theorem 2.4 and item (8) of Theorem 2.3. □

Note that $\mathcal{C}_0$ and $p\mathcal{C}_2$ in Theorem 3.1 represent the same classes so that we immediately have the upper bound

$$\dim S_2^k\,(\Gamma_0(p)) \le 2\,\#\big\{\text{classes } [t] \;\big|\; t \in \mathcal{X}_2 \text{ and } w(t) < \tfrac{1}{6}(p+1)k\big\}.$$

We will write $a^b c$ for $\frac{1}{2}\left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right) \in V_2(\mathbb{Q})$. From [16], pg. 224 we know that for reduced $a^b c$ with $2|b| \le a \le c$ we have $w(a^b c) = \frac{1}{2}(a + c - |b|)$. This already proves that $\dim S_2^2\,(\Gamma_0(p)) = 0$ for $p = 2, 3$. For odd weights k, we automatically have $a_r(t) = 0$ for $r = 0, 2$ if t has an improper automorphism; an improper automorphism of t is a $u \in \mathrm{GL}_n(\mathbb{Z})$ with $\det(u) = -1$ and $u^t t u = t$. In the theory of quadratic forms, forms possessing an improper automorphism are ususally called ambiguous. We define t to be *nonamibiguous* if it has no improper automorphisms. Thus for *odd* k,

$$\dim S_2^k\,(\Gamma_0(p)) \le$$
$$2\,\#\big\{\text{classes } [t] \;\big|\; t \in \mathcal{X}_2,\, w(t) < \tfrac{1}{6}(p+1)k \text{ and } t \text{ nonamibiguous}\big\}.$$

This already proves that $\dim S_2^1\,(\Gamma_0(p)) = 0$ for $p = 2, \dots, 23$ and that $\dim S_2^3\,(\Gamma_0(p)) = 0$ for $p = 2, 3, 5, 7$. To illustrate, we give the following sets of determining Fourier coefficients:

$$\text{For } S_2^2\,(\Gamma_0(11)),\quad \mathcal{C}_0 = \big\{t \in \mathcal{X}_2 \;\big|\; w(t) < \tfrac{12}{3}\big\}$$
$$= [2^1 2] \cup [2^0 2] \cup [2^1 4] \cup [2^0 4] \cup [4^2 4] \cup [2^1 6] \cup [4^1 4];$$
$$\mathcal{C}_2 = \tfrac{1}{11}\mathcal{C}_0.$$
$$\text{For } S_2^3\,(\Gamma_0(11)),\quad \mathcal{C}_0 = \big\{t \in \mathcal{X}_2 \;\big|\; w(t) < \tfrac{36}{6} \text{ and } t \text{ nonamibiguous}\big\}$$
$$= [4^1 6] \cup [4^1 8];$$
$$\mathcal{C}_2 = \tfrac{1}{11}\mathcal{C}_0.$$

4 Restriction Technique for $S_2^k(\Gamma_0(p))$

Consider $f \in S_2^k\,(\Gamma_0(p))$ with Fourier expansions at the cusps E_0 and E_2 given by equation (3.1). Let $\mathcal{C}_0$ and $\mathcal{C}_2$ be as in Theorem 3.1. Our goal is to generate linear relations among the Fourier coefficients of f at $\mathcal{C}_0$ and $\mathcal{C}_2$. Toward this end we will generate linear relations on the Fourier coefficients from possibly larger sets $\mathcal{B}_0$ and $\mathcal{B}_2$ with $\mathcal{C}_0 \subseteq \mathcal{B}_0$ and $\mathcal{C}_2 \subseteq \mathcal{B}_2$. After fixing $\mathcal{B}_0$ and $\mathcal{B}_2$, we then choose a set $\mathcal{A} \subseteq \mathcal{P}_n(\mathbb{Z})$ and for each $s \in \mathcal{A}$ we apply the technique of restriction to modular curves. We summarize this technique and refer to [21] for more details. Denote $\phi_s : \mathcal{H}_1 \to \mathcal{H}_n$ by $\phi_s(\tau) = s\tau$. Then we have (see [18], pg. 375)

$$\phi_s^* : S_2^k\left(\Gamma_0(p)\right) \to S_1^{2k}\left(\Gamma_0(p\ell)\right)$$

where $\ell \in \mathbb{Z}^+$ satisfies $\ell s^{-1} \in \mathcal{P}_2(\mathbb{Z})$. Let $g_1, \dots, g_N$ be a basis of $M_1^{2k}\left(\Gamma_0(p\ell)\right)$. For any $f \in S_2^k\left(\Gamma_0(p)\right)$ there must be parameters $c_1, \dots, c_N \in \mathbb{C}$ such that

$$\phi_s^* f = \sum_{m=1}^{N} c_m g_m. \tag{4.1}$$

For each $\sigma \in \Gamma_1$ we have

$$\left(\phi_s^* f\right)|_{2k}\sigma = \sum_{m=1}^{N} c_m g_m|_{2k}\sigma. \tag{4.2}$$

We obtain a countable set of linear equations by expanding both sides of equation (4.2) into Fourier series. The equations on the c_m that one obtains in this manner depend only upon the coset $\Gamma_0(p\ell)\sigma$. The point is that much is known about elements of $M_1^{2k}\left(\Gamma_0(p\ell)\right)$ and that each $g_m|_{2k}\sigma$ can be expanded in a Fourier series by known methods. For example, we may generate $M_1^{2k}\left(\Gamma_0(p\ell)\right)$ via theta series and transform these theta series using the shadow theory of modular forms [22].

On the other hand, we can compute the Fourier series of $\left(\phi_s^* f\right)|_{2k}\sigma$ in terms of the Fourier coefficients of f. First, we compute $\phi_s^* f$ as follows [17]

$$\begin{aligned}\left(\phi_s^* f\right)(\tau) &= \sum_{j\in\mathbb{Z}^+}\Big(\sum_{t\in\mathcal{X}_2:\langle t,s\rangle=j} a_0(t)\Big)q^j \\ &= \sum_{j\in\mathbb{Z}^+}\Big(\sum_{t\in\mathcal{X}_2^{\mathrm{red}}} \nu(j,s,t)a_0(t)\Big)q^j\end{aligned} \tag{4.3}$$

where $q = e(\tau)$. Here $\mathcal{X}_2^{\mathrm{red}} = \left\{\left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right) \in \mathcal{X}_2 \;\middle|\; 0 \le 2b \le a \le c\right\}$ and $\nu(j,s,t) = \mathrm{card}\{v \in [t] \mid \langle v, s\rangle = j\}$ for even k, while for odd k we define

$$\nu(j,s,t) = \begin{cases} 0, & \text{if } t \text{ is not nonamibiguous,} \\ \sum_{v\in[t]:\langle v,s\rangle=j}\delta(t,v), & \text{if } t \text{ is nonamibiguous,}\end{cases}$$

where for nonamibiguous t we define

$$\delta(t,v) = \begin{cases} +1, & \text{if } t \text{ and } v \text{ are properly equivalent,} \\ -1, & \text{if } t \text{ and } v \text{ are improperly equivalent.}\end{cases}$$

The $\nu(j,s,t)$ are readily computable number theoretic functions. Also, note that for odd k, $\phi_s^* f$ is identically zero if s is not nonamibiguous. The Fourier expansion of $\left(\phi_s^* f\right)|\sigma$ is computed in the next three Lemmas and Propositions. The following Lemma may be proven from the general recipe given in [10], Proposition 3.4.

Lemma 4.1. *Let* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$. *Then we have*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = ME_r \begin{pmatrix} W & Z \\ 0 & W^* \end{pmatrix}$$

for some $M \in \Gamma_0(p)$ *and some* $\left(\begin{smallmatrix} W & Z \\ 0 & W^* \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$, *and where* $r = \mathrm{rank}_{\mathbb{F}_p}(C) \in \{0, \dots, n\}$.

Although the following Proposition is stated for $n = 2$, it is true for general n, and is an easy consequence of Lemma 4.1.

Proposition 4.2. *Let* $s \in \mathcal{P}_2(\mathbb{Z})$ *and* $\ell \in \mathbb{Z}^+$ *such that* $\ell s^{-1} \in \mathcal{P}_2(\mathbb{Z})$. *Let* $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. *Then we have*

$$\begin{pmatrix} aI_2 & bs \\ cs^{-1} & dI_2 \end{pmatrix} = ME_r \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{pmatrix}$$

for some $M \in \Gamma_0(p)$, *some* $r \in \{0, 1, 2\}$ *and some* $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{smallmatrix}\right) \in \mathrm{Sp}_2(\mathbb{Q})$. *Furthermore, if* $p \nmid \ell$ *then* $r = 0$ *if* $p \mid c$ *and* $r = 2$ *if* $p \nmid c$.

Proposition 4.3. *Let* $f \in S_2^k(\Gamma_0(p))$ *along with the hypothesis of Proposition* 4.2, *we have*

$$\left((\phi_s^* f)|_{2k}\sigma\right)(\tau) = \det(\mathcal{A})^k \,(f|_k E_r)\,(\mathcal{A} s \mathcal{A}^{\mathrm{t}} \tau + \mathcal{B}\mathcal{A}^{\mathrm{t}}).$$

Proof. This follows as in the proof of Proposition 2.3 in [17]. □

Now we are able to expand both sides of equation (4.2) into Fourier series; however, for computational purposes we truncate the series (4.3) once Fourier coefficients from outside the set $\mathcal{B}_0$ appear. For $J \in \mathbb{Q}^+$, define the truncation operator Trunc_J as truncation at order q^J. Define:

$$J(s, m, \mathcal{T}, \mathcal{B}) = \sup\left\{ j \in \tfrac{1}{m}\mathbb{Z} \;\middle|\; \{t \in \mathcal{T} \mid \langle s, t\rangle \le j\} \subseteq \mathcal{B} \right\}.$$

Applying the operator $\mathrm{Trunc}_{J(s,1;\mathcal{X}_2,\mathcal{B}_0)}$ to the Fourier expansion of (4.1) gives us $1 + J(s, 1; \mathcal{X}_2, \mathcal{B}_0)$ number of equations involving the Fourier coefficients of f with indices from $\mathcal{B}_0$ and the parameters $c_1, \dots, c_N$. We get other sets of such equations by considering $(\phi_s^* f)|\sigma$ for other cusps σ. We truncate the Fourier series of equation (4.2) at

$$J(\mathcal{A} s \mathcal{A}^{\mathrm{t}}, \mathrm{width}_{p\ell}(\sigma); \mathcal{X}_2, \mathcal{B}_0), \text{ if } r = 0 \text{ and}$$
$$J(\mathcal{A} s \mathcal{A}^{\mathrm{t}}, \mathrm{width}_{p\ell}(\sigma); \tfrac{1}{p}\mathcal{X}_2, \mathcal{B}_2), \text{ if } r = 2.$$

In this manner we get $1 + \mathrm{width}_{p\ell}(\sigma) J$ more linear equations involving the Fourier coefficients of f with indices from $\mathcal{B}_0$, $\mathcal{B}_2$ along with the parameters $c_1, \dots, c_N$.

The hope is that by using the collection of equations over all cusps σ, we can eliminate the parameters $c_1, \dots, c_N$ and thus obtain relations among the Fourier coefficients of f with indices from $\mathcal{B}_0$, $\mathcal{B}_2$. Finally, by using s from a large enough set $\mathcal{A}$, we hope we can generate enough linear relations among the Fourier coefficients of f with indices from $\mathcal{B}_0$, $\mathcal{B}_2$ to deduce enough linear relations among the Fourier coefficients of f with indices from $\mathcal{C}_0$, $\mathcal{C}_2$ to yield an optimal upper bound on $\dim S_2^k(\Gamma_0(p))$.

Although Proposition 4.3 completes the theoretical desription of the Restriction Technique, for computational purposes it is beneficial to be even more specific. The following is a straightforward generalization of [17], Prop 2.4.

Proposition 4.4. *With the hypotheses of Proposition* 4.3 *and with the additional hypothesis that* $\gcd(c, p\ell/c) = 1$, *let* $\hat{c} \in \mathbb{Z}$ *such that* $\hat{c}c \equiv 1 \mod p\ell/c$. *Then we have*

$$\begin{aligned}\left((\phi_s^* f)|_{2k}\sigma\right)(\tau) &= \det(\mathcal{A})^k \,(f|_k E_r)\,(\mathcal{A}s\mathcal{A}^{\mathrm{t}}(\tau + d\hat{c}))\\ &= \det(\mathcal{A})^k \phi^*_{\mathcal{A}s\mathcal{A}^{\mathrm{t}}}\,(f|_k E_r)\,(\tau + d\hat{c})\end{aligned}$$

Furthermore, if $r = 0$ *or* $r = 2$ *this is*

$$\det(\mathcal{A})^k \sum_{j\in\mathbb{Q}^+}\Big(\sum_{t\in\mathcal{X}_2^{(r)}} \nu(j, \mathcal{A}s\mathcal{A}^{\mathrm{t}}, t)a_r(t)\Big)e(\tau + d\hat{c})^j$$

where $\mathcal{X}_2^{(r)} = \begin{cases}\mathcal{X}_2^{red}, & \text{if } r = 0\\ \frac{1}{p}\mathcal{X}_2^{red}, & \text{if } r = 2\end{cases}.$

These calculations can be further simplified by choosing s so that $p\ell$ is squarefree. By choosing $p \nmid \ell$, Proposition 4.2 insures that only the cases $r = 0$ and $r = 2$ will occur. And choosing ℓ squarefree allows us to use Proposition 4.4 to further advantage in the following way. Since $p\ell$ is squarefree, each cusp σ has a representative of the form

$$\sigma = \left(\begin{smallmatrix}1 & 0\\ c & 1\end{smallmatrix}\right)$$

where c ranges over the divisors of $p\ell$. Let $\hat{c}$ be as in Proposition 4.4 for this σ. Then

$$\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix} = \begin{pmatrix}p\ell/c & -\hat{c}\\ p\ell & 1-\hat{c}c\end{pmatrix}\begin{pmatrix}1 & \hat{c}\\ 0 & p\ell/c\end{pmatrix}\frac{c}{p\ell}.$$

Note that $\begin{pmatrix}p\ell/c & -\hat{c}\\ p\ell & 1-\hat{c}c\end{pmatrix}$ is an Atkin-Lehner involution, denoted by $W_{\tilde{c}}$, where we write $\tilde{c} = \frac{p\ell}{c}$. Note that $W_{p\ell}$ is the Fricke involution. For $g \in M_1^{2k}(\Gamma_0(p\ell))$,

$$g|_{2k}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}(\tau) = (g|_{2k}W_{\tilde{c}})\,|\begin{pmatrix}1 & \hat{c}\\ 0 & \tilde{c}\end{pmatrix}(\tau) = (\tilde{c})^{-k}\,(g|_{2k}W_{\tilde{c}})\left(\frac{\tau + \hat{c}}{\tilde{c}}\right).$$

Thus we can avoid the whole $\tau + \hat{c}$ business by replacing $(\tau + \hat{c})$ by τ to get the first part of the following Corollary.

Corollary 4.5. *Let* $f \in S_2^k(\Gamma_0(p))$ *have Fourier expansions as in equation* (3.1). *Let* $s \in \mathcal{P}_2(\mathbb{Z})$ *and* $\ell \in \mathbb{Z}^+$ *such that* $\ell s^{-1} \in \mathcal{P}_2(\mathbb{Z})$. *Let* $\sigma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. *Assume that* $c \mid p\ell$. *Denote* $\tilde{c} = \frac{p\ell}{c}$. *Assume* $\gcd(c, \tilde{c}) = 1$, *which is automatically true if* $p\ell$ *is square free. We have*

$$\begin{pmatrix}aI_2 & bs\\ cs^{-1} & dI_2\end{pmatrix} = ME_r\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ 0 & \mathcal{A}^*\end{pmatrix}$$

for some $M \in \Gamma_0(p)$, *some* $r \in \{0, 2\}$ *and some* $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{smallmatrix}\right) \in \mathrm{Sp}_2(\mathbb{Q})$. *We have* $r = 0$ *if* $p \mid c$ *and* $r = 2$ *otherwise. For any such choice of* $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{smallmatrix}\right)$ *we have*

$$(\tilde{c})^{-k}\left((\phi_s^* f)|W_{\tilde{c}}\right)\left(\frac{1}{\tilde{c}}\tau\right) = \det(\mathcal{A})^k \sum_{j \in \mathbb{Q}^+} \Big(\sum_{t \in \mathcal{X}_2^{(r)}} \nu(j, \mathcal{A}s\mathcal{A}^{\mathrm{t}}, t) a_r(t) \Big) q^j,$$

and equivalently,

$$\begin{aligned}\left((\phi_s^* f)|W_{\tilde{c}}\right)(\tau) &= (\tilde{c})^k \det(\mathcal{A})^k \sum_{j \in \mathbb{Z}^+} \Big(\sum_{t \in \mathcal{X}_2^{(r)}} \nu(j, \tilde{c}\mathcal{A}s\mathcal{A}^{\mathrm{t}}, t) a_r(t) \Big) q^j \\ &= (\tilde{c})^k \det(\mathcal{A})^k \left(\phi^*_{\tilde{c}\mathcal{A}s\mathcal{A}^{\mathrm{t}}}(f|E_r)\right)(\tau).\end{aligned}$$

Proof. The second equation is gotten by replacing τ by $\tilde{c}\tau$ on both sides of the first equation. □

Using the second form above that avoids fractional exponents speeds up calculations.

Proposition 4.6. *Let* $f \in S_2^k(\Gamma_0(p))$. *Let* $s \in \mathcal{P}_2(\mathbb{Z})$ *and* $\ell = \det(s)$. *Assume that* $p \nmid \ell$. *Then*

$$(\phi_s^* f)|W_\ell = \phi_s^* f.$$

Proof. We will apply the second part of Corollary 4.5 with $c = p$ and $\tilde{c} = \ell$ and $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ to get that

$$(\phi_s^* f)|W_\ell = \left(\frac{p\ell}{p}\right)^k \det(\mathcal{A})^k \phi^*_{\frac{p\ell}{p}\mathcal{A}s\mathcal{A}^{\mathrm{t}}}(f|E_r).$$

Note $r = 0$ because $p \mid c$, and we have to compute $\mathcal{A}$ according to Proposition 4.2. Let $\hat{c}$ be as in Proposition 4.4 so that $\ell \mid (\hat{c}c - 1)$. Observe

$$\begin{pmatrix} I & 0 \\ cs^{-1} & I \end{pmatrix} = \begin{pmatrix} s & -\hat{c}I \\ cI & (1 - \hat{c}c)s^{-1} \end{pmatrix} \begin{pmatrix} s^{-1} & \hat{c}I \\ 0 & s \end{pmatrix}$$

has $\left(\begin{smallmatrix} s & -\hat{c}I \\ cI & (1-\hat{c}c)s^{-1} \end{smallmatrix}\right) \in \Gamma_0(p)$ because $c = p$ and because $(1 - \hat{c}c)s^{-1}$ is integral. Thus we may take $\mathcal{A} = s^{-1}$. Then observing that $\left(\frac{p\ell}{p}\right)^k \det(\mathcal{A})^k = 1$ and that $\frac{p\ell}{p}\mathcal{A}s\mathcal{A}^{\mathrm{t}} = \ell s^{-1}$ is properly equivalent to s (because s is 2×2) completes the proof. □

Proposition 4.7. *Let* $f \in S_2^k(\Gamma_0(p))$. *Let* $s \in \mathcal{P}_2(\mathbb{Z})$ *and* $\ell \in \mathbb{Z}^+$ *such that* $\ell s^{-1} \in \mathcal{P}_2(\mathbb{Z})$. *Assume that* $p \nmid \ell$. *Then*

$$(\phi_s^* f)|W_p = p^k \phi^*_{ps}(f|E_2).$$

Proof. We will apply the second part of Corollary 4.5 with $c = \ell$ and $\tilde{c} = p$ and $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ \ell & 1 \end{smallmatrix}\right)$ to get that

$$(\phi_s^* f)|W_p = \left(\frac{p\ell}{\ell}\right)^k \det(\mathcal{A})^k \phi^*_{\frac{p\ell}{\ell}\mathcal{A}s\mathcal{A}^{\mathrm{t}}}(f|E_r).$$

Note $r = 2$ because $p \nmid c$, and we have to compute $\mathcal{A}$ according to Proposition 4.2. Let $\hat{c}$ be as in Proposition 4.4 so that $p \mid (\hat{c}c - 1)$. Observe

$$\begin{pmatrix} I & 0 \\ cs^{-1} & I \end{pmatrix} = \begin{pmatrix} -\hat{c}s & -I \\ (1-\hat{c}c)I & -cs^{-1} \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & \hat{c}s \\ 0 & I \end{pmatrix}$$

has $\begin{pmatrix} -\hat{c}s & -I \\ (1-\hat{c}c)I & -cs^{-1} \end{pmatrix} \in \Gamma_0(p)$ because $p \mid (1 - \hat{c}c)$ and because cs^{-1} is integral. Thus we may take $\mathcal{A} = I$. The Proposition follows immediately. $\square$

In the case where ℓ is prime, the above two propositions tell us how to get the other expansions $(\phi_s^* f)|W_{\hat{c}}$ from just the $\phi_s^* f$ expansion.

5 Upper Bounds

For $m \in \mathbb{Z}^+$, $\alpha \in \mathbb{R}$, denote $\lfloor \alpha \rfloor_m = \max\{\beta \in \frac{1}{m}\mathbb{Z} \mid \beta < \alpha\}$. Note that we have $\lfloor \alpha \rfloor_m < \alpha$. Recall in $n = 2$ that the dyadic trace of a half-integral form takes values in $\frac{1}{2}\mathbb{Z}_{\geq 0}$. From a weight k and a prime p we construct our set of determining Fourier coefficients $\mathcal{C}_0 \amalg \mathcal{C}_2$ and, using an auxillary parameter β, our net $\mathcal{B}_0$ and $\mathcal{B}_2$ as follows:

$$\mathcal{C}_0 = \left\{t \in \mathcal{X}_2 \mid w(t) \leq \lfloor \tfrac{k}{6}(p+1) \rfloor_2 \text{ and } t \text{ nonamibiguous if } k \text{ odd}\right\} \quad \text{and } \mathcal{C}_2 = \tfrac{1}{p}\mathcal{C}_0,$$

$$\mathcal{B}_0 = \{t \in \mathcal{X}_2 \mid w(t) \leq \beta \text{ and } t \text{ nonamibiguous if } k \text{ odd}\} \text{ and } \mathcal{B}_2 = \tfrac{1}{p}\mathcal{B}_0.$$

Our choices are given in Tables 1 through 3. Also, we choose a set $\mathcal{A} \subseteq \mathcal{P}_2(\mathbb{Z})$. Note that for k odd, we only need nonamibiguous forms in $\mathcal{C}_0$, $\mathcal{C}_2$, $\mathcal{B}_0$, $\mathcal{B}_2$ and $\mathcal{A}$. We ran the Restriction Technique with the choices in Tables 1 through 3 and obtained upper bounds for $\dim S_2^k(\Gamma_0(p))$ as reported in these tables. By $|[\mathcal{C}_0]|$ and $|[\mathcal{B}_0]|$ we denote the number of classes in $\mathcal{C}_0$ and $\mathcal{B}_0$, respectively. For the first Table we use the sets: $F_0 = \{2^1 2, 2^1 3, 3^1 4, 4^1 4\}$, $F_1 = F_0 \cup \{2^1 4, 3^1 5, 5^2 5\}$, $F_2 = \{3^0 5, 4^1 5, 5^0 7, 5^2 6, 6^1 6\}$ and $F_3 = \{2^1 6, 2^1 7, 2^1 9, 3^0 3, 3^0 9, 3^1 6, 4^1 6, 4^1 9, 5^1 6, 5^1 7, 5^1 8, 5^2 7, 6^2 7, 6^3 7, 6^3 8, 7^1 8, 8^3 8\}$.

Instead of going through each Table, we give an example for weight two. To enjoy any brevity of exposition the reader must cede us the ability to compute with elliptic forms. Our programs used theta series to span spaces of elliptic forms. This allowed us to compute the expansion of an elliptic form in a Fourier series at any cusp. MAGMA will also give these cusp expansions when the cusp is given by an Atkin-Lehner involution, as in the following example.

Example. We consider $S_2^2(\Gamma_0(11))$ and use $\phi_s^* : S_2^2(\Gamma_0(11)) \to S_1^4(\Gamma_0(11\ell))$ for $s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ and $\ell = 1, 2, 3, 5$, respectively. The determining set is given by $\mathcal{C}_0$ and $\mathcal{C}_2$ where $\mathcal{C}_0 = [2^1 2] \cup [2^0 2] \cup [2^1 4] \cup [2^0 4] \cup [4^2 4] \cup [2^1 6] \cup [4^1 4]$ and $\mathcal{C}_2 = \frac{1}{11}\mathcal{C}_0$ and the net is given by $\mathcal{B}_0 = \mathcal{C}_0 \cup [4^0 4] \cup [4^2 6] \cup [2^0 6]$ and $\mathcal{B}_2 = \frac{1}{11}\mathcal{B}_0$.

p	$\lfloor \frac{1}{3}(p+1) \rfloor_2$	$\|[\mathcal{C}_0]\|$	β	$\|[\mathcal{B}_0]\|$	$\mathcal{A}$	$\dim S_2^2(\Gamma_0(p)) \le$
2	0.5	0				0
3	1.0	0				0
5	1.5	1	2	2	$2^1 2$	0
7	2.5	3	2.5	3	$2^1 2$	0
11	3.5	7	4	10	$2^1 2, 2^1 3, 1^0 1, 1^0 2$	1
13	4.5	13	5	17	same as above	0
17	5.5	21	7	39	$2^1 2, 3^1 4, 4^1 4$	1
19	6.5	32	7.5	46	F_0	1
23	7.5	46	9.5	84	$F_0 \cup 3^0 3, 2^1 4, 3^1 6$	3
29	9.5	84	12	156	$F_1 \cup 4^1 5$	3
31	10.5	109	12.5	172	$F_1 \cup 3^0 5, 5^2 6$	3
37	12.5	172	15	281	$F_1 \cup F_2$	2
41	13.5	211	16.5	361	$F_1 \cup F_2 \cup F_3$	6

Table 1. Upper Bounds for $k = 2$

p	$\lfloor \frac{1}{2}(p+1) \rfloor_2$	$\|[\mathcal{C}_0]\|$	β	$\|[\mathcal{B}_0]\|$	$\mathcal{A}$	$\dim S_2^3(\Gamma_0(p)) \le$
2	1.0	0				0
3	1.5	0				0
5	2.5	0				0
7	3.5	0				0
11	5.5	2	7	6	$3^1 5$	0
13	6.5	6	8.5	15	$3^1 4, 3^1 5$	0
17	8.5	15	13	72	$3^1 5, 5^2 6$	1
19	9.5	23	13.5	84	$3^1 5, 5^2 6$	1
23	11.5	47	17	185	$3^1 4, 3^1 5, 3^1 6, 4^1 5,$ $4^1 8, 5^1 6, 5^2 6, 5^2 7$	2

Table 2. Upper Bounds for $k = 3$

p	$\lfloor \frac{2}{3}(p+1) \rfloor_2$	$\|[\mathcal{C}_0]\|$	β	$\|[\mathcal{B}_0]\|$	$\mathcal{A}$	$\dim S_2^4(\Gamma_0(p)) \le$
2	1.5	1	1.5	1	$2^1 2$	0
3	2.5	3	3.5	7	$2^1 3$	1
5	3.5	7	6	27	$2^1 2, 3^0 3, 2^1 4, 3^1 4$	1
7	5.0	17	8	55	$2^1 2, 3^0 3,$ $2^1 3, 3^1 4, 4^1 4$	3
11	7.5	46	11.5	138	$2^1 2, 2^1 3, 2^1 4,$ $3^0 3, 3^1 5, 3^1 6, 4^1 4$	7
13	9.0	74	13	192	$2^1 2, 3^1 4, 3^0 3, 3^1 5, 3^1 6,$ $4^1 4, 4^1 5, 4^2 8, 5^2 5, 6^3 9$	11

Table 3. Upper Bounds for $k = 4$

First, consider $s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For $f \in S_2^2(\Gamma_0(11))$ we have $\phi^*_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} f \in S_1^4(\Gamma_0(11))$ and

$$\begin{aligned}
(\phi^*_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} f)(\tau) &= \big(2a_0(2^1 2) + a_0(2^0 2)\big)q^2 \\
&\quad + \big(4a_0(2^0 2) + 2a_0(2^0 4) + 4a_0(2^1 4)\big)q^3 \\
&\quad + \big(4a_0(2^0 4) + 2a_0(2^0 6) + 4a_0(2^1 2) + 2a_0(2^1 4) \\
&\quad + 4a_0(2^1 6) + a_0(4^0 4) + 2a_0(4^1 4) + 2a_0(4^2 4)\big)q^4 + \dots, \\
\big((\phi^*_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} f)|W_{11}\big)(\tau) &= 121\phi^*_{11\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}(f|E_2)(\tau) \\
&= 121\big(2a_2(\tfrac{2^1 2}{11}) + a_2(\tfrac{2^0 2}{11})\big)q^2 + \dots.
\end{aligned}$$

There is only one cusp form of weight 4 whose q-expansion begins with q^2 and it is in the Fricke plus space:

$$(\eta(\tau)\eta(11\tau))^4 = q^2 - 4q^3 + 2q^4 + 8q^5 + \cdots .$$

If we set $\phi^*_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} f = c\,(\eta(\tau)\eta(11\tau))^4$ and eliminate the parameter c in $121\phi^*_{11\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}(f|E_2) = c\,(\eta(\tau)\eta(11\tau))^4$ as well, we obtain the equations

$$\begin{aligned}
0 &= 4a_0(2^02) + a_0(2^04) + 4a_0(2^12) + 2a_0(2^14),\\
0 &= 4a_2\big(\tfrac{2^02}{11}\big) + a_2\big(\tfrac{2^04}{11}\big) + 4a_2\big(\tfrac{2^12}{11}\big) + 2a_2\big(\tfrac{2^14}{11}\big),\\
0 &= -2a_2\big(\tfrac{2^02}{11}\big) + 4a_2\big(\tfrac{2^04}{11}\big) + 2a_2\big(\tfrac{2^06}{11}\big) + 2a_2\big(\tfrac{2^14}{11}\big) + 4a_2\big(\tfrac{2^16}{11}\big) + a_2\big(\tfrac{4^04}{11}\big)\\
&\quad + 2a_2\big(\tfrac{4^14}{11}\big) + 2a_2\big(\tfrac{4^24}{11}\big),\\
0 &= -2a_0(2^02) + 4a_0(2^04) + 2a_0(2^06) + 2a_0(2^14) + 4a_0(2^16) + a_0(4^04)\\
&\quad + 2a_0(4^14) + 2a_0(4^24),\\
0 &= a_0(2^02) + 2a_0(2^12) - 121a_2\big(\tfrac{2^02}{11}\big) - 242a_2\big(\tfrac{2^12}{11}\big).
\end{aligned}$$

A similar analysis for $s = \left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)$ gives the equations

$$\begin{aligned}
0 &= 4a_0(2^02) + a_0(2^04) + 4a_0(2^12) + 2a_0(2^14),\\
0 &= 4a_2\big(\tfrac{2^02}{11}\big) + a_2\big(\tfrac{2^04}{11}\big) + 4a_2\big(\tfrac{2^12}{11}\big) + 2a_2\big(\tfrac{2^14}{11}\big),\\
0 &= a_0(2^02) + 2a_0(2^04) + a_0(2^06) + 4a_0(2^12) + 2a_0(2^16),\\
0 &= a_2\big(\tfrac{2^02}{11}\big) + 2a_2\big(\tfrac{2^04}{11}\big) + a_2\big(\tfrac{2^06}{11}\big) + 4a_2\big(\tfrac{2^12}{11}\big) + 2a_2\big(\tfrac{2^16}{11}\big),\\
0 &= a_0(2^02) + 2a_0(2^12) - 121a_2\big(\tfrac{2^02}{11}\big) - 242a_2\big(\tfrac{2^12}{11}\big).
\end{aligned}$$

For $s = \left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$ we have $\phi^*_{\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)} : S_2^2(\Gamma_0(11)) \to S_1^4(\Gamma_0(33))$ and the expansions under the Atkin-Lehner involutions $W_{11} = \left(\begin{smallmatrix}3&1\\-33&12\end{smallmatrix}\right)$, $W_3 = \left(\begin{smallmatrix}11&-4\\33&11\end{smallmatrix}\right)$, $W_{33} = \left(\begin{smallmatrix}0&1\\-33&0\end{smallmatrix}\right)$ are

$$\begin{aligned}
\big(\phi^*_{\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)} f\big)(\tau) &= a_0(2^12)q^3 + 3a_0(2^02)q^4 + \big(3a_0(2^12) + 3a_0(2^14)\big)q^5\\
&\quad + \big(6a_0(2^04) + a_0(4^24)\big)q^6\\
&\quad + \big(6a_0(2^14) + 3a_0(2^16) + 3a_0(4^14)\big)q^7\\
&\quad + \big(6a_0(2^02) + 6a_0(2^06) + 3a_0(4^04) + 3a_0(4^26)\big)q^8 + \cdots,
\end{aligned}$$

$$\big((\phi^*_{\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)} f)|W_{11}\big)(\tau) = 121\big(a_2\big(\tfrac{2^12}{11}\big)q^3 + 3a_2\big(\tfrac{2^02}{11}\big)q^4 + \cdots\big),$$

along with $\big(\phi^*_{\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)} f\big)|W_3 = \phi^*_{\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)} f$, and $\big(\phi^*_{\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)} f\big)|W_{33} = \big(\phi^*_{\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)} f\big)|W_{11}$.

The subspace $\mathrm{SV} \subseteq S_1^4(\Gamma_0(33))$ of cusp forms g for which the vanishing of g, $g|W_3$, $g|W_{11}$, $g|W_{33}$ is each of at least order q^3 and for which $g|W_3 = g$ is

2-dimensional. A basis for SV is

$$q^3 - 3q^5 - 2q^6 + 6q^8 + \cdots,$$
$$q^4 - 2q^5 - q^6 + 5q^8 + \cdots.$$

This space SV is stable under the Atkin-Lehner involutions and in fact is fixed under each Atkin-Lehner involution. Using these facts and setting $\left(\phi^*_{\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)} f\right)(\tau) = \alpha(q^3 - 3q^5 + \cdots) + \beta(q^4 - 2q^5 + \cdots)$ gives the equations

$$\begin{aligned}
a_0(2^12) &= \alpha,\\
3a_0(2^02) &= \beta,\\
3a_0(2^12) + 3a_0(2^14) &= -3\alpha - 2\beta,\\
6a_0(2^04) + a_0(4^24) &= -2\alpha - \beta,\\
6a_0(2^14) + 3a_0(2^16) + 3a_0(4^14) &= 0,\\
6a_0(2^02) + 6a_0(2^06) + 3a_0(4^04) + 3a_0(4^26) &= 6\alpha + 5\beta
\end{aligned}$$

and corresponding equations in $a_2(\ldots)$s. Eliminating the parameters gives the linear relations

$$\begin{aligned}
0 &= 2a_0(2^02) + 2a_0(2^12) + a_0(2^14)\\
0 &= 2a_0(2^14) + a_0(2^16) + a_0(4^14)\\
0 &= 2a_0(2^12) + 3a_0(2^02) + a_0(4^24) + 6a_0(2^04)\\
0 &= 2a_0(2^12) + 3a_0(2^02) - 2a_0(2^06) - a_0(4^04) - a_0(4^26)\\
0 &= a_0(2^12) - 121a_2\left(\tfrac{2^12}{11}\right)\\
0 &= a_0(2^02) - 121a_2\left(\tfrac{2^02}{11}\right)\\
0 &= 2a_2\left(\tfrac{2^02}{11}\right) + 2a_2\left(\tfrac{2^12}{11}\right) + a_2\left(\tfrac{2^14}{11}\right)\\
0 &= 2a_2\left(\tfrac{2^14}{11}\right) + a_2\left(\tfrac{2^16}{11}\right) + a_2\left(\tfrac{4^14}{11}\right)\\
0 &= 2a_2\left(\tfrac{2^12}{11}\right) + 3a_2\left(\tfrac{2^02}{11}\right) + a_2\left(\tfrac{4^24}{11}\right) + 6a_2\left(\tfrac{2^04}{11}\right)\\
0 &= 2a_2\left(\tfrac{2^12}{11}\right) + 3a_2\left(\tfrac{2^02}{11}\right) - 2a_2\left(\tfrac{2^06}{11}\right) - a_2\left(\tfrac{4^04}{11}\right) - a_2\left(\tfrac{4^26}{11}\right)
\end{aligned}$$

A similar analysis for $s = \left(\begin{smallmatrix}2&1\\1&3\end{smallmatrix}\right)$ gives the linear relations

$$\begin{aligned}
0 &= 4a_0(2^02) + a_0(2^04) + 4a_0(2^12) + 2a_0(2^14),\\
0 &= a_0(2^04) - 2a_0(2^14) + 4a_0(2^16) + 4a_0(4^24),\\
0 &= 3a_0(2^04) + 2a_0(2^06) + 2a_0(2^14) + 2a_0(4^14)\\
0 &= a_0(2^12) - 121a_2\left(\tfrac{2^12}{11}\right)\\
0 &= a_0(2^02) - 121a_2\left(\tfrac{2^02}{11}\right)\\
0 &= a_0(2^14) - 121a_2\left(\tfrac{2^14}{11}\right)
\end{aligned}$$

$$0 = 4a_2\left(\tfrac{2^02}{11}\right) + a_2\left(\tfrac{2^04}{11}\right) + 4a_2\left(\tfrac{2^12}{11}\right) + 2a_2\left(\tfrac{2^14}{11}\right),$$
$$0 = a_2\left(\tfrac{2^04}{11}\right) - 2a_2\left(\tfrac{2^14}{11}\right) + 4a_2\left(\tfrac{2^16}{11}\right) + 4a_2\left(\tfrac{4^24}{11}\right),$$
$$0 = 3a_2\left(\tfrac{2^04}{11}\right) + 2a_2\left(\tfrac{2^06}{11}\right) + 2a_2\left(\tfrac{2^14}{11}\right) + 2a_2\left(\tfrac{4^14}{11}\right)$$

The solution space of these 29 equations has a one-dimensional projection onto the Fourier coefficients from $\mathcal{C}_0$ and $\mathcal{C}_2$ and is spanned by the following solution.

$a_0(2^12)$	$a_0(2^02)$	$a_0(2^14)$	$a_0(2^04)$	$a_0(4^24)$	$a_0(4^14)$	$a_0(2^16)$
1	-1	0	0	1	1	-1

with all $a_2(\frac{a^bc}{11}) = \frac{1}{121}a_0(a^bc)$. Thus $\dim S_2^2\,(\Gamma_0(11)) \le 1$.

The technique illustrated in the Example almost tells the whole story. For odd weights, one must additionally keep track of the proper equivalence classes of the indices of the Fourier coefficients. Finally, in the case of $S_2^2\,(\Gamma_0(41))$ the linear relations provided by the forms in $\mathcal{A}$ have an 11-dimensional nullspace on the Fourier coefficients from $\mathcal{C}_0 \amalg \mathcal{C}_2$. We used the fact that if $a_0(T;\, f)$ are the Fourier coefficients of a Siegel modular cusp form then

$$a_0(T;\, T_q f) = q^{2k-3}a_0\left(\tfrac{1}{q}T;\, f\right) + q^{k-2}a_0\left(\tfrac{1}{q}T\left[\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)\right];\, f\right)$$
$$+\, q^{k-2}\sum_{\alpha=0}^{q-1} a_0\left(\tfrac{1}{q}T\left[\left(\begin{smallmatrix} 1 & 0 \\ \alpha & q \end{smallmatrix}\right)\right];\, f\right) + a_0(qT;\, f)$$

are the Fourier coefficients of the Siegel modular cusp form $T_q f$. Here T_q is the standard Hecke operator on $S_2^k\,(\Gamma_0(p))$, see [8] or [25]. The intersection of the 11-dimensional nullspace and its image under T_2 was 6-dimensional, hence $\dim S_2^2\,(\Gamma_0(41)) \le 6$.

6 Lower Bounds

Until this point we have discussed only upper bounds for $\dim S_2^k\,(\Gamma_0(p))$. We address the question of lower bounds by actually constructing cusp forms. The charm of the subject has always been the diversity of ways in which modular forms arise. Although our topic remains the same, this section has a decidedly different flavor as we cast about for constructions of cusp forms.

For $k = 2$, the work of S. BÖCHERER and R. SCHULZE-PILLOT [2] on the injectivity of the Yoshida lift provides the dimension Y of the subspace of cusp forms that is spanned by Yoshida lifts. We quote the results from the thesis of M. KLEIN [13] from part of his Tabelle 2.3:

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43
g	0	0	0	0	1	0	1	1	2	2	2	2	3	3
d_+	0	0	0	0	0	0	0	0	0	0	0	1	0	1
d_-	0	0	0	0	1	0	1	1	2	2	2	1	3	2
Y	0	0	0	0	1	0	1	1	3	3	3	1	6	3

Table 4. Dimension of Yoshida lifts

In the Table above: $g = \dim S_1^2(\Gamma_0(p))$; $d_\pm = \dim\{f \in S_1^2(\Gamma_0(p)) \mid f|W_p = \pm f\}$; We have $Y = d_- + \binom{d_-}{2} + \binom{d_+}{2}$ for primes $p < 389$. In general, $Y = d_* + \binom{d_-}{2} + \binom{d_+}{2}$ where $d_* \le d_-$ is the dimension of the space spanned inside of $\{f \in S_1^2(\Gamma_0(p)) \mid f|W_p = -f\}$ by Hecke eigenforms whose L-function does not vanish at $s = 1$.

The case of $p = 37$ requires further comment, the space of Yoshida lifts is one-dimensional but we need a lower bound of 2 on $\dim S_2^2(\Gamma_0(37))$. We will see that there is also a (generalized) Saito-Kurokawa lift in $S_2^2(\Gamma_0(37))$, see [15]. From Table 4 we see that $S_1^2(\Gamma_0(37))$ has two eigenforms, one each in the Fricke plus and minus spaces. By the Shimura correspondence these correspond to distinct, and hence linearly independent, eigenforms of half integral weight. As sharpened by KOHNEN [14], pg. 64 we have the noncanonical isomorphism, $S_1^2(\Gamma_0(37)) \cong S_1^{\frac{3}{2}}(\Gamma_0(4{\cdot}37))^+$. The generalized Saito-Kurokawa lift

$$SK : S_1^{\frac{3}{2}}(\Gamma_0(4{\cdot}37))^+ \to S_2^2(\Gamma_0(37))$$

is linear so that $\dim S_2^2(\Gamma_0(37)) \ge 2$.

One minor difficulty with the above discussion is that in [15] the generalized Saito-Kurokawa lift was demonstrated only for even $k > 2$, whereas we need the case $k = 2$. In [15], the map SK was factored

$$S_1^{k-\frac{1}{2}}(\Gamma_0(4N))^+ \to J_{k,1}^{\text{cusp}}(\Gamma_0(N)) \to S_2^k(\Gamma_0(N))$$

for odd squarefree N. The second map holds for general k but the proof of the first map used Poincare series and so required $k > 2$. The following ad hoc Lemma amends Theorem 2 from [15] to include the case $k = 2$ but we should mention that we have received from T. IBUKIYAMA [11] a development of the theory of the Saito-Kurokawa lift to $S_2^k(\Gamma_0(N))$ that treats all even weights in a uniform manner for any $N \in \mathbb{Z}^+$. See [5] for the definition of $J_{k,1}^{\text{cusp}}(\Gamma_0(N))$.

Lemma 6.1. *The linear map $\mathcal{S}$ defined by*

$$\sum_{D<0, r\in\mathbb{Z}, D\equiv r^2 \bmod 4} c(D)e\Big(\frac{r^2-D}{4}\tau + rz\Big) \mapsto \sum_{D<0, D\equiv 0,1 \bmod 4} c(D)e(|D|\tau)$$

induces an isomorphism between $J_{k,1}^{cusp}(\Gamma_0(N))$ and $S_1^{k-\frac{1}{2}}(\Gamma_0(4N))^+$ in the case $k = 2$.

Proof. The space of Jacobi forms is an $M_1(\Gamma_0(N))$-module. For $g \in M_1(\Gamma_0(N))$ and $F \in J_{k,1}^{\text{cusp}}(\Gamma_0(N))$ we have $\mathcal{S}(g(\tau)F(\tau,z)) = g(4\tau)\mathcal{S}(F(\tau,z))$; indeed, this is true even as a map on formal series. We may use the statement of the Lemma for even $k > 2$ by Theorem 2 of [15]. Take $f \in S_1^{\frac{3}{2}}(\Gamma_0(4N))^+$. If $f(\tau) = \sum_{D<0, D\equiv 0,1 \bmod 4} c(D)e(|D|\tau)$, define F by the convergent power series $F(\tau,z) = \sum_{D<0, r\in\mathbb{Z}, D\equiv r^2 \bmod 4} c(D)e\big(\frac{r^2-D}{4}\tau + rz\big)$, so that $F : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C}$ is holomorphic. Let E_4 be the weight 4 Eisenstein series of level one. There exists F_6

such that $\mathcal{S}(F_6) = E_4(4\tau)f(\tau)$. We have $F_6(\tau, z) = E_4(\tau)F(\tau, z)$ because they have the same series expansion. Thus

$$F(\tau, z) = \frac{F_6(\tau, z)}{E_4(\tau)} \tag{6.1}$$

and we conclude $F \in J_{k,1}(\Gamma_0(N))$. Since the q-expansions of Eisenstein series begin with 1 at the cusps, equation (6.1) shows that F is a Jacobi cusp form. Therefore $\mathcal{S}$ is surjective in the case $k = 2$. It is clearly injective from the definition. □

More generally, S. BÖCHERER has explained to us [1] that the Saito-Kurokawa lifts of elliptic eigenforms in $S_2^2(\Gamma_0(p))$ whose L-function vanishes at $s = 1$ are always linearly independent from the space of Yoshida lifts. The reason is essentially that in this case the standard L-function of the Saito-Kurokawa lift does not have a pole at $s = 1$ as it would were it in the span of the theta series, see Theorem 4.1 in [3]. Arguing from this result one may increase the dimension of the space of known lifts in $S_2^2(\Gamma_0(p))$ to $d_+(d_+ + 1)/2 + d_-(d_- + 1)/2$.

For $k = 3$, the work of S. BÖCHERER and R. SCHULZE-PILLOT [3] on Yoshida lifts, while not giving a general injectivity result, does allow us to construct lifts in specific cases. For example, in $p = 17$ we have the nontrivial Yoshida lift computed in [3]. These are theta series with pluriharmonic coefficients. Let $\Lambda \subseteq \mathbb{R}^m$ be an even lattice of rank m and square determinant $\det \Lambda = N^2$. Let $P : M_{n\times m}(\mathbb{C}) \to \mathbb{C}$ be a pluri-harmonic polynomial [6], p. 161 of degree ν and define $\vartheta_{\Lambda,P} : \mathcal{H}_n \to \mathbb{C}$ by

$$\vartheta_{\Lambda,P}(\Omega) = \sum_{L\in\Lambda^n} P(L)e\big(\tfrac{1}{2}\langle LL^{\mathrm{t}}, \Omega\rangle\big).$$

The function $\vartheta_{\Lambda,P}$ is then a Siegel modular cusp form of weight $\frac{m}{2} + \nu$ and level $\Gamma_0(N)$ and degree n, see [6]. Furthermore for $B, X \in M_{n\times m}(\mathbb{C})$, the polynomial $P(X) = \det(BX^{\mathrm{t}})^{\nu}$ is pluri-harmonic when $\nu = 1$ or whenever B satisfies $BB^{\mathrm{t}} = 0$. To list some Fourier coefficients of $\vartheta_{\Lambda,P}$ we let $\Lambda = \mathbb{Z}^m M$ for $M \in \mathrm{GL}_m(\mathbb{R})$, give the Gram matrix MM^{t}, and write the Fourier series $\vartheta_{\Lambda,P}(\Omega) = (\text{cont.}) \sum_T a(T)e(\langle\Omega, T\rangle)$.

level	MM^{t}	B	cont.	$4^1 6$	$4^1 8$	$4^1 10$	$6^1 12$	$6^2 8$	$6^2 10$	$6^2 12$	$8^1 12$	$8^2 12$
$\Gamma_0(17)$	$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & -1 & 1 \\ 1 & -1 & 6 & 2 \\ 0 & 1 & 2 & 10 \end{pmatrix}$	0010 0100	$4\sqrt{17}$	1	−1	−1	−1	1	1	−1	1	−3
$\Gamma_0(19)$	$\begin{pmatrix} 4 & 0 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 2 & 1 & 6 & 1 \\ 1 & 2 & 1 & 6 \end{pmatrix}$	0010 0100	$-4\sqrt{19}$	1	0	−1	0	−1	0	1	0	−1
$\Gamma_0(23)$	$\begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 6 \end{pmatrix}$	1000 0100	$2\sqrt{23}$	1	0	2	−2	0	0	0	2	0
$\Gamma_0(23)$	$\begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 12 \end{pmatrix}$	1000 0100	$2\sqrt{23}$	1	2	0	−2	0	0	0	0	−2

This table of the Fourier coefficients of $\vartheta_{\Lambda,P}$ for $\Lambda = \mathbb{Z}^m M$ and $P(X) = \det(BX^{\mathrm{t}})$ shows that $\dim S_2^3(\Gamma_0(17)) \geq 1$, $\dim S_2^3(\Gamma_0(19)) \geq 1$ and $\dim S_2^3(\Gamma_0(23)) \geq 2$.

For $k = 4$ we construct modular forms from the theta series of 8-by-8 integral positive definite even quadratic forms with square determinant. We obtain cusp forms by taking linear combinations. For $Q \in \mathcal{P}_m(\mathbb{Q})$, let $\vartheta^Q(\Omega) = \sum_{N \in \mathbb{Z}^{m\times n}} e\big(\frac{1}{2}\langle N^t QN, \Omega\rangle\big)$. If $Q = MM^t$ is the Gram matrix of the lattice $\Lambda = \mathbb{Z}^m M$ then $\vartheta^Q = \vartheta_{\Lambda,1}$. For the construction of cusp forms from theta series, the following Lemma is useful.

Lemma 6.2. *Let $f \in M_2^k(\Gamma_0(p))$, and let W_p denote the Fricke involution. If $\Phi_0(f) = 0$ and $\Phi_0(f|W_p) = 0$ then $f \in S_2^k(\Gamma_0(p))$.*

Proof. From [6], pg. 127 it suffices to check $\Phi_0(f|M) = 0$ for a complete set of representatives $\Gamma_2 = \bigcup_M \Gamma_0(p)M$. For the M we may take the 1 representative E_0; the p^3 representatives $E_2 t(S)$, where $S \in M_{2\times 2}^{\rm sym}(\mathbb{Z})$ represents each class in $M_{2\times 2}^{\rm sym}(\mathbb{F}_p)$; and the $p(p+1)$ representatives $E_1 t\left(\left(\begin{smallmatrix}0&0\\0&x\end{smallmatrix}\right)\right) u(U)$, where $x \in \mathbb{Z}$ represents each class in $\mathbb{F}_p$ and $U = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ or $U = \left(\begin{smallmatrix}0&-1\\1&j\end{smallmatrix}\right)$ where $j \in \mathbb{Z}$ represents each class in $\mathbb{F}_p$. These choices of U have bottom rows which represent each one-dimensional subspace of $\mathbb{F}_p^2$. Our assumptions are equivalent to assuming that $\Phi_0(f|E_0) = 0$ and $\Phi_0(f|E_2) = 0$ and we will show that all other $\Phi_0(f|M) = 0$ follow from these.

Let f be any Siegel form for a group of finite index. One elementary relation is $\Phi_0(f) = 0$ if and only if $\Phi_0(f|t(S)) = 0$. From this we see that $\Phi_0(f|E_2) = 0$ implies $\Phi_0(f|E_2 t(S)) = 0$. Another elementary relation is $\Phi_0(f|E_1) = \Phi_0(f)|J_1$. This takes care of the representatives in the double coset $\Gamma_0(p)E_1\Delta_2(\mathbb{Z})$ that have $U = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. For those with $U = \left(\begin{smallmatrix}0&-1\\1&j\end{smallmatrix}\right)$ we note that $E_1 t\big(\left(\begin{smallmatrix}0&0\\0&x\end{smallmatrix}\right)\big) u\big(\left(\begin{smallmatrix}0&-1\\1&j\end{smallmatrix}\right)\big) = u(J_1)E_2 t\big(\left(\begin{smallmatrix}x&j\\j&0\end{smallmatrix}\right)\big)E_1$, so that

$$\begin{aligned}\Phi_0\Big(f|E_1 t\big(\left(\begin{smallmatrix}0&0\\0&x\end{smallmatrix}\right)\big)u\big(\left(\begin{smallmatrix}0&-1\\1&j\end{smallmatrix}\right)\big)\Big) &= \Phi_0\Big(f|u(J_1)E_2 t\big(\left(\begin{smallmatrix}x&j\\j&0\end{smallmatrix}\right)\big)E_1\Big)\\ &= \Phi_0\Big(f|u(J_1)E_2 t\big(\left(\begin{smallmatrix}x&j\\j&0\end{smallmatrix}\right)\big)\Big)|J_1.\end{aligned}$$

Now we make use of $f \in M_2^k(\Gamma_0(p))$ so that $f|u(J_1) = f$; then the vanishing of $\Phi_0(f|E_2)$ is equivalent to that of $\Phi_0\Big(f|u(J_1)E_2 t\big(\left(\begin{smallmatrix}x&j\\j&0\end{smallmatrix}\right)\big)\Big)|J_1$. □

This Lemma, along with the standard action of the Fricke operator on theta series,

$$\vartheta^Q|W_\ell = i^{nk}\det(Q)^{-n/2}\ell^{nk/2}\vartheta^{\ell Q^*},$$

allows us to check if linear combinations of theta series are indeed cusp forms. For standard lattices like E_6, A_2, etc., we refer to [4]. For more obscure lattices we give the Gram matrices here. For convenience in typesetting, if Q and $\hat{Q}$ are even forms and $\hat{Q}$ is the Gram matrix of the lattice Λ, then we define $\vartheta(Q^{(r)} \oplus \ell\cdot\Lambda)$ to be $\vartheta^{(\bigoplus^r Q)\oplus\ell\hat{Q}} = (\vartheta^Q)^r\vartheta_{\sqrt{\ell}\Lambda}$.

A nontrivial cusp form in $S_2^4(\Gamma_0(3))$ is

$$10\vartheta\,(A_2 \oplus E_6) - 90\vartheta(A_2 \oplus E_6^*) - \vartheta\,(E_8) + 81\vartheta\,(3\cdot E_8)$$

so that $\dim S_2^4(\Gamma_0(3)) \geq 1$. A nontrivial cusp form in $S_2^4(\Gamma_0(5))$ is

$$625\vartheta(Q_5 \oplus Q_5) - 36\vartheta(A_4 \oplus A_4) - 900\vartheta(A_4^* \oplus A_4^*) - 315\vartheta(A_4 \oplus A_4^*) \\ + \vartheta(E_8) + 625\vartheta(5 \cdot E_8)$$

so that $\dim S_2^4(\Gamma_0(5)) \geq 1$. Here we set

$$Q_5 = \begin{pmatrix} 2 & 1 & -1 & -1 \\ 1 & 2 & 0 & -1 \\ -1 & 0 & 4 & 2 \\ -1 & -1 & 2 & 4 \end{pmatrix} \text{ and note } \det(Q_5) = 5^2.$$

We have $S_2^4(\Gamma_0(7)) \supseteq \mathrm{Span}(f, T_2 f, T_2^2 f)$ where $f = 3626\vartheta(B^{(4)}) - 1232\vartheta(B \oplus A_6^{\mathrm{sup}}) - 192\vartheta(B \oplus A_6) - 9408\vartheta(B \oplus A_6^*) + 3\vartheta(E_8) + 7203\vartheta(7 \cdot E_8)$. Here we have

$$A_6^{\mathrm{sup}} = \begin{pmatrix} 4 & 1 & 1 & 1 & 0 & 2 \\ 1 & 4 & -1 & 0 & 2 & 2 \\ 1 & -1 & 4 & -2 & 1 & 1 \\ 1 & 0 & -2 & 4 & -2 & -1 \\ 0 & 2 & 1 & -2 & 4 & 2 \\ 2 & 2 & 1 & -1 & 2 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}.$$

These forms are linearly independent so that $\dim S_2^4(\Gamma_0(7)) \geq 3$. On this space we have $T_2^3 = 35T_2^2 - 324T_2 + 516I_2$.

We have $S_2^4(\Gamma_0(11)) \supseteq \mathrm{Span}(f_1, f_2, f_3, T_2 f_1, T_2 f_2, T_2 f_3, T_2^2 f_1)$ where $f = -3\vartheta(Q_1) + 2\vartheta(Q_2) + \vartheta(Q_3) \in S_2^2(\Gamma_0(11))$ and $f_i = f\vartheta(Q_i)$ for $i = 1, 2, 3$, and where

$$Q_1 = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 6 & 0 \\ 0 & -1 & 0 & 6 \end{pmatrix}; \quad Q_2 = \begin{pmatrix} 2 & 1 & -1 & -1 \\ 1 & 2 & 0 & -1 \\ -1 & 0 & 8 & 4 \\ -1 & -1 & 4 & 8 \end{pmatrix}; \quad Q_3 = \begin{pmatrix} 4 & 0 & -2 & -1 \\ 0 & 4 & 1 & 2 \\ -2 & 1 & 4 & 0 \\ -1 & 2 & 0 & 4 \end{pmatrix}.$$

These forms are linearly independent so that $\dim S_2^4(\Gamma_0(11)) \geq 7$.

We have $S_2^4(\Gamma_0(13)) \supseteq \mathrm{Span}(f, T_2 f, T_2^2 f, \ldots, T_2^7 f, g, T_2 g, T_2^2 g)$ where $f = 15\vartheta(Q_a^{(2)}) + 3\vartheta(Q_b \oplus Q_c) - 2\vartheta(Q_\alpha) - 16\vartheta(Q_\beta)$ and $g = -\vartheta(Q_a^{(2)}) - 13\vartheta(Q_b \oplus Q_c) + 14\vartheta(Q_\alpha) + 16\vartheta(Q_\gamma) - 16\vartheta(Q_\delta)$. Here we have

$$Q_a = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 4 & 1 & 2 \\ -1 & 1 & 4 & 0 \\ -1 & 2 & 0 & 8 \end{pmatrix}; \quad Q_b = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}; \quad Q_c = \begin{pmatrix} 4 & -1 & 1 & 1 \\ -1 & 10 & 3 & 3 \\ 1 & 3 & 10 & -3 \\ 1 & 3 & -3 & 10 \end{pmatrix}.$$

$$Q_\alpha = \begin{pmatrix} 6 & -3 & 3 & -3 & -3 & 2 & -2 & -1 \\ -3 & 6 & -3 & 0 & 2 & -3 & 1 & 0 \\ 3 & -3 & 6 & -3 & -1 & 2 & -2 & -2 \\ -3 & 0 & -3 & 6 & 1 & -1 & 3 & 0 \\ -3 & 2 & -1 & 1 & 6 & 1 & 3 & -2 \\ 2 & -3 & 2 & -1 & 1 & 6 & -1 & 0 \\ -2 & 1 & -2 & 3 & 3 & -1 & 6 & 0 \\ -1 & 0 & -2 & 0 & -2 & 0 & 0 & 6 \end{pmatrix}; \quad Q_\beta = \begin{pmatrix} 2 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & 1 & 0 & -1 & 1 & -2 \\ 1 & -2 & 4 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 4 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 6 & -2 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 & 6 & 3 & 1 \\ 0 & 1 & 0 & 2 & 1 & 3 & 12 & 5 \\ 0 & -2 & 0 & -1 & 3 & 1 & 5 & 12 \end{pmatrix};$$

$$Q_\gamma = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 6 & 3 & 3 & 3 \\ -1 & 0 & 1 & 0 & 3 & 12 & 5 & 5 \\ 0 & 0 & 1 & 1 & 3 & 5 & 12 & -1 \\ 0 & 0 & 1 & 1 & 3 & 5 & -1 & 12 \end{pmatrix}; \quad Q_\delta = \begin{pmatrix} 4 & 1 & -1 & 0 & 0 & 1 & 1 & 2 \\ 1 & 4 & 1 & 1 & -1 & 0 & 2 & -1 \\ -1 & 1 & 4 & -1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & 4 & 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 1 & 4 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 4 & 0 & 0 \\ 1 & 2 & 2 & -1 & 0 & 0 & 6 & -2 \\ 2 & -1 & -1 & -1 & -1 & 0 & -2 & 10 \end{pmatrix}.$$

We mention that although we have an 11-dimensional space, the minimal polynomial of T_2 has degree 8. These linearly independent theta series show that $\dim S_2^4(\Gamma_0(13)) \geq 11$.

7 Conclusion

The lower bounds of Section 6 all coincide with the upper bounds in the Tables of Section 5. This proves the dimensions in the Theorems of the Introduction. The results used modest computing power, mainly a desktop personal computer. We plan a more computationally intensive search for paramodular cusp forms of weight two.

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