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# On the Graded Ring of Modular Forms of the Siegel Paramodular Group of Level 2

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In this paper, we shall describe the concrete ring structure of the graded rings of modular forms belonging to the Siegel paramodular group $\Gamma^{para}(2)$ of degree two with polarization diag(1, 2). We also show that the Satake compactification of the quotient variety by this group is rational. Here, for each prime $p$, we define the group $\Gamma^{\mathrm{para}}(p)$ by

$$\Gamma^{\mathrm{para}}(p) := \{g \in M_4(\mathbb{Z}) \mid {}^t g J_2(p) g = J_2(p)\},$$

where for any number $d$, we put

$$J_2(d) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \\ -1 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{pmatrix}.$$

The main results will be given in Section 1.

Historically, FREITAG [1] has obtained the ring structure for a certain group which contains our group $\Gamma^{\mathrm{para}}(2)$ with index 2. He used some geometrical method. Since the dimension formula for $\Gamma^{\mathrm{para}}(p)$ has been known by IBUKIYAMA [9], we can use more direct method here, and his result is also obtained as a corollary of our result. Various generators of the ring have been considered by various approach (cf. GRITSENKO [2], [3], GRITSENKO and NIKULIN [4], [5], or RUNGE [15]). But the ring structure was not known as far as the authors know.

Actually we treat the discrete subgroup $K(p)$ of $\mathrm{Sp}(2, \mathbb{R})$ which is $\mathrm{GL}_4(\mathbb{Q})$-conjugate to $\Gamma^{\mathrm{para}}(p)$ and defined by

$$K(p) = \mathrm{Sp}(2, \mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

(The fact that $K(p)$ is conjugate to $\Gamma^{\mathrm{para}}(p)$ is well known and was remarked also in the introduction of IBUKIYAMA [9] without proof. As for the written

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proof, see e.g. HULEK, KAHN and WEINTRAUB [7] or GRITSENKO [2].) This group $K(p)$ has been treated in IBUKIYAMA [8], [9] as one of standard parahoric subgroups in some different context and several results there are applicable here. For example, the dimension formula for Siegel cusp forms of weight $k \geq 5$ belonging to $K(p)$ was given in [9] for each prime $p$, and some forms of small weights belonging to $K(2)$ have been given explicitly with their L functions in [8]. If we take the Iwahori subgroup $B(2)$ of level 2 defined e.g. in [8], then $K(2)$ contains $B(2)$ (cf. [6]) and the ring structure of modular forms belonging to $B(2)$ has been known in IBUKIYAMA [10]. We shall use these facts. By the way, the structure of $A(\Gamma(2))$ for the principal congruence subgroup of degree 2 is well known by IGUSA [12] and our $K(2)$ contains $\Gamma(2)$. But $\Gamma(2)$ is not a normal subgroup of $K(2)$, and we need some work to get $A(K(2))$. In Section 1, we shall state the main result, and the proof will be given in Section 2.

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## 1 Main results

**1.1 Preliminary definitions.** For any ring $S$, we denote by $\mathrm{Sp}(n, S)$ the usual symplectic group of size $2n$ defined by

$$\mathrm{Sp}(n, S) := \{g \in M_{2n}(S) \mid {}^t gJg = J\},$$

where

$$J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

We denote by $H_n$ the Siegel upper half space of degree $n$ defined by

$$H_n := \{Z \in M_n(\mathbb{C}) \mid {}^t Z = Z,\ \mathrm{Im}(Z) > 0\}.$$

Let $\Gamma$ be a discrete subgroup of $Sp(n, \mathbb{R})$ with covolume finite. We denote by $A_k(\Gamma)$ or $S_k(\Gamma)$ the space of modular forms, or cusp forms, of weight $k$ belonging to $\Gamma$, respectively. We define two graded rings as follows.

$$A(\Gamma) = \bigoplus_{k=0}^{\infty} A_k(\Gamma) \quad \text{and} \quad A_{\mathrm{even}}(\Gamma) = \bigoplus_{k=0}^{\infty} A_{2k}(\Gamma).$$

For any $F(Z) \in A_k(\Gamma)$, we write

$$F \mid [g] = F \mid_k [g] = F(gZ) \det(CZ + D)^{-k}, \quad \text{for } g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbb{R}).$$

Next, we define several discrete subgroups of $\mathrm{Sp}(2, \mathbb{R})$. For each prime $p$, we define "Iwahori subgroup" $B(p)$ by

$$B(p) = \mathrm{Sp}(2, \mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

We need several groups which contain $B(2)$. Put

$$s_0 = \begin{pmatrix} 0 & 0 & -p^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We also put $\Gamma_0'(p) = B(p) \cup B(p)s_2B(p)$, and $\Gamma_0''(p) = B(p) \cup B(p)s_0B(p)$. These are groups. As for more explicit description, cf.[8] p.601. By the general theory of Bruhat-Tits, we get $K(p) = B(p) \cup B(p)s_0B(p) \cup B(p)s_2B(p) \cup B(p)s_0s_2B(p)$ and the group $K(p)$ is generated by $\Gamma_0'(p)$ and $\Gamma_0''(p)$. Now, put

$$\rho = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

Then we get $\Gamma_0''(p) = \rho\Gamma_0'(p)\rho^{-1}$ and $\rho K(p)\rho^{-1} = K(p)$. We denote by $K^*(p)$ the group generated by $K(p)$ and $\rho$. We have $[K^*(p) : K(p)] = 2$.

**1.2 Generators of Modular forms.** For any $m = (m', m'') \in \mathbb{Z}^{2n}$ $(m', m'' \in \mathbb{Z}^n)$, we define a theta constant $\theta_{m',m''} = \theta_{m',m''}(Z)$ by the following function of $Z \in H_n$.

$$\theta_{m',m''}(Z) = \sum_{p \in Z^n} \exp\big(2\pi i \left({}^t(p + m'/2)Z(p + m'/2)/2 + {}^t(p + m'/2)m''/2\right)\big).$$

We also put

$$\begin{aligned}
X &= (\theta_{0000}^4 + \theta_{0001}^4 + \theta_{0010}^4 + \theta_{0011}^4)/4, \\
Y &= (\theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011})^2, \\
Z &= (\theta_{0100}^4 - \theta_{0110}^4)^2/16384, \\
K &= (\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111})^2/4096, \\
T &= (\theta_{0100}\theta_{0110})^4/256,
\end{aligned}$$

where the theta constants are for $n = 2$. Each of the above is a modular form which belongs to $B(2)$ of weight 2, 4, 4, 6, or 4, respectively. We define the following functions.

$$\begin{aligned}
F_4 &= X^2 + 3Y + 3072Z + 960T, \\
F_6 &= X^3 - 9XY - 9216XZ + 27648K + 4032TX, \\
F_8 &= 16YZ - 16XK + 64T^2 - TX^2 + 1024TZ + TY,
\end{aligned}$$

$$F_{12} = 32X^3K + 64X^2YZ - 96XYK - 98304XZK + 5X^4T - 14X^2YT - 14336X^2ZT - 6144XKT + 9Y^2T + 18432YZT + 9437184Z^2T - 896X^2T^2 + 1152YT^2 + 1179648ZT^2 + 36864T^3,$$

$$G_{10} = 4X^2K - 16XYZ + 12YK + 12288ZK + X^3T - XYT - 1024XZT + 768KT - 64XT^2,$$

$$G_{12} = 3014656TX^2Z - 2944TX^2Y + 12582912KXZ - 12288KXY + 184320T^2Y - 188743680T^2Z - 1152TY^2 + 1207959552TZ^2 - 1024X^4Z + 2097152X^2Z^2 + 3145728YZ^2 - 1073741824Z^3 + X^4Y - 2X^2Y^2 - 3072Y^2Z + Y^3,$$

$$G_{11} = \theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111} \times (\theta_{1000}^{12} - \theta_{1001}^{12} - \theta_{1100}^{12} + \theta_{1111}^{12})/1536 \quad (= \chi_{11} \text{ in [10]}).$$

**Proposition 1.** *The function $F_4$, $F_6$, $F_8$, $F_{12}$, $G_{10}$, $G_{12}$, or $G_{11}$ defined above is a modular form which belongs to $K(2)$ and of weight 4, 6, 8, 12, 10, 12 or 11, respectively. The first 5 forms belong also to $K^*(2)$, and we get $G_{11} \mid [\rho] = -G_{11}$ and $G_{12} \mid [\rho] = -G_{12}$. Besides, $F_8$, $F_{12}$, $G_{10}$ and $G_{11}$ are cusp forms.*

The proof of this Proposition will be given in Section 2.

**1.3 Main results.** We denote by $B$ the following subring of $A(K(2))$

$$B = \mathbb{C}[F_4, F_6, F_8, F_{12}].$$

**Theorem 1.** *The modular forms $F_4$, $F_6$, $F_8$, $F_{12}$ are algebraically independent and $B$ is a weighted polynomial ring. The graded ring $A_{\text{even}}(K(2))$ is given by*

$$A_{\text{even}}(K(2)) = B \oplus (G_{12})B \oplus (G_{10})B \oplus (G_{10}G_{12})B,$$

*and we get*

$$A(K(2)) = A_{\text{even}}(K(2)) \oplus (G_{11})A_{\text{even}}(K(2)),$$

*where $\oplus$ means the direct sum as modules. The ideal of cusp forms of $A(K(2))$ is spanned by $F_8$, $F_{12}$, $G_{10}$, and $G_{11}$.*

*The fundamental relations of the generators of the above graded ring are given as follows:*

$$G_{10}^2 = F_8F_{12}/4,$$

$$729G_{12}^2 = 26873856F_{12}^2 - 10368F_4^3F_{12} - 71663616F_4F_8F_{12} - 10368F_6^2F_{12} + F_4^6 - 6912F_4^4F_8 - 2F_4^3F_6^2 + 15925248F_4^2F_8^2 - 13824F_4F_6^2F_8 + F_6^4 - 12230590464F_8^3 + G_{10}(82944F_4^2F_6 + 63700992F_6F_8),$$

$$G_{11}^2 = 3^{-3} \cdot 2^6(-F_6F_8^2 + 3F_4F_8G_{10} - F_{12}G_{10}).$$

Next Corollary was first proved by Freitag [1] for even weights.

**Corollary 1.** *We get*

$$A(K^*(2)) = B \oplus (G_{10})B \oplus (G_{11}G_{12})B \oplus (G_{11}G_{12}G_{10})B.$$

Now, we will give a result of the structure of the variety. For the sake of simplicity, we put

$$\begin{aligned} \alpha &= G_{10}/F_4F_6, \\ \beta &= F_6^2/F_4^3, \\ \gamma &= F_8/F_4^2. \end{aligned}$$

Further, we define the automorphic functions $A$, $B$, $C$ belonging to $K(2)$ by

$$\begin{aligned} A &= 27\gamma\left(20736\alpha^2 - \gamma\right)\left(G_{12}/F_4^3\right), \\ B &= \beta\left(20736\alpha^2 - \gamma\right)^2 - 20736\alpha^2\gamma - 143327232\alpha^2\gamma^2 - \gamma^2 \\ &\quad - 6912\gamma^3 + 41472\alpha\gamma^2 + 31850496\alpha\gamma^3, \\ C &= \gamma\left(768\gamma^2 + \gamma - 13824\alpha\gamma + 15925248\alpha^2\gamma - 2\alpha + 20736\alpha^2\right). \end{aligned}$$

As an application of the above theorem, we get

**Corollary 2.** *The Satake compactification* $\mathcal{S}(K(2)\backslash H_2) = \mathrm{Proj}(A(K(2)))$ *is a rational variety. The function field is given by*

$$\mathbb{C}\left(\tfrac{G_{10}}{F_4F_6}, \tfrac{A}{C}, \tfrac{B}{C}\right).$$

## 2 Proofs

**2.1** We first review some dimension formulas.

**Proposition 2.** (cf. [9]) *We get*

$$\begin{aligned} \sum_{k=0}^{\infty} \dim A_k(K(2))\, t^k &= \frac{(1+t^{10})(1+t^{12})(1+t^{11})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})}, \\ \sum_{k=0}^{\infty} \dim S_k(K(2))\, t^k &= \frac{(t^8+t^{10}+t^{12}-t^{20})(1+t^{12})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})}, \\ \sum_{k=0}^{\infty} \dim A_k(\Gamma_0''(2))\, t^k &= \frac{(1+t^{11})(1+t^6+t^8+t^{10}+t^{12}+t^{18})}{(1-t^4)^2(1-t^6)(1-t^{12})}. \end{aligned}$$

*Proof.* As for the formula for $\Gamma_0'(2)$, this is an easy corollary to Igusa [12]. When the weight $k \leq 4$, the results for $K(2)$ is obtained from explicit structure of $A_k(B(2))$ very easily. For general $k$ with $k \geq 5$, as for $\dim S_k(K(2))$, the above formula is the special case of [9] Theorem 4. As for $\dim A_k(K(2))$, it is easily obtained by the surjectivity of $\Phi$ operator by Satake [16] and the explicit description of cusps of $K(p)$ given e.g. in [11] for each prime $p$. We omit the details. □

*Proof of Proposition* 1. It has been written in [8] how to obtain forms in $A_k(K(2))$. We review this shortly for readers convenience. For automorphic form $F \in A_k(B(2))$, write the Fourier expansion as

$$F(Z) = \sum_T a(T) \exp(2\pi i \operatorname{tr}(TZ)),$$

where $T$ runs over positive semi-definite half integral matrices. The subspace $A_k(\Gamma_0''(2))$ is characterized by those forms $F \in A_k(B(2))$ such that $a(T) = 0$ for all $T$ with odd (1,1) component (As for the proof, see [8]). Since we know $\dim A_k(\Gamma_0''(2))$ and the generators of $A(B(2))$ (cf. [10]), we can determine $A_k(\Gamma_0''(2))$ if $k$ is given explicitly and enough Fourier coefficients are known. Now, we have $A_k(K(2)) = A_k(\Gamma_0''(2)) \cap A_k(\Gamma_0'(2))$ and $A_k(\Gamma_0'(2)) = A_k(\Gamma_0''(2)) \mid_k [\rho]$. So we can get $A_k(K(2))$ for given small $k$. We know that $X$, $K$, $T$ are invariant by the action of $\rho$ and $Y \mid_4 [\rho] = 1024 Z$, $Z \mid_4 [\rho] = Y/1024$. Hence, we can also show that $F_4$, $F_6$, $F_8$, $F_{12}$, $G_{10} \in A_k(K^*(2))$. We can show $G_{12} \mid [\rho] = -G_{12}$ easily from the above. If we put $G = G_{11} \mid [\rho_1]$ for

$$\rho_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

then obviously $(G_{11} \mid [\rho])(Z) = 2^{11} G(2Z)$. By the theta transformation formula (cf. [13]), we get

$$G(Z) = -(\theta_{0001}^{12} - \theta_{1001}^{12} - \theta_{0011}^{12} + \theta_{1111}^{12}) \prod_m \theta_m / 1536,$$

where $m$ runs over ten even characteristics mod 2. Since $\dim A_{11}(K(2)) = 1$, obviously $G_{11} \mid [\rho]$ is $G_{11}$ or $-G_{11}$, hence comparing one non vanishing Fourier coefficient, we can show it is $-G_{11}$. If you prefer more theoretical proofs, you can prove this by using the following relations (cf. [13] p. 232)

$$\theta_{0000}(2Z)\theta_{0100}(2Z) = (\theta_{0100}(Z)^2 + \theta_{0110}(Z)^2)/4,$$
$$\theta_{1000}(2Z)\theta_{1100}(2Z) = (\theta_{0100}(Z)^2 - \theta_{0110}(Z)^2)/4,$$
$$\theta_{0011}(2Z)\theta_{1111}(2Z) = (\theta_{1100}(Z)\theta_{1111}(Z))/2,$$
$$\theta_{0010}(2Z)\theta_{0110}(2Z) = (\theta_{0100}(Z)\theta_{0110}(Z))/2,$$
$$\theta_{0001}(2Z)\theta_{1001}(2Z) = (\theta_{1000}(Z)\theta_{1001}(Z))/2,$$
$$\theta_{0001}(2Z)^2 = (\theta_{0000}(Z)\theta_{0001}(Z) + \theta_{0010}(Z)\theta_{0011}(Z))/2,$$
$$\theta_{1001}(2Z)^2 = (\theta_{0000}(Z)\theta_{0001}(Z) - \theta_{0010}(Z)\theta_{0011}(Z))/2,$$
$$\theta_{0011}(2Z)^2 = (\theta_{0000}(Z)\theta_{0011}(Z) + \theta_{0010}(Z)\theta_{0001}(Z))/2,$$
$$\theta_{1111}(2Z)^2 = (\theta_{0000}(Z)\theta_{0011}(Z) - \theta_{0010}(Z)\theta_{0001}(Z))/2,$$

and Riemann's theta formula (cf. [12].) Since $K$, $YZ$, and $T(X^2 - Y - 1024 Z - 64 T)$ are cusp forms (cf. [10]), it is easy to see $F_8$, $F_{12}$ and $G_{10}$ are cusp forms. Hence, Proposition 1 is proved. □

Now we introduce the Witt operator. For any function $F(Z)$ on $H_2$, we put

$$(WF)(z_1, z_2) = F\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right),$$

$z_1, z_2 \in H_1$. We denote by $E_k(z)$, $z \in H_1$ the Eisenstein series of weight $k$ belonging to $\mathrm{SL}_2(\mathbb{Z})$ having 1 as the constant term. For short we write $E_k = E_k(z_2)$ and $E_k' = E_k(2z_1)$ for mutually independent variables $z_1, z_2 \in H_1$. The image of theta constants by $W$ is again easily expressed by theta constants. Also, it is easy to see the following relations:

$$\begin{aligned}
E_4 &= \theta_{01}^8(z_2) + \theta_{01}^4(z_2)\theta_{10}^4(z_2) + \theta_{10}^8(z_2),\\
E_4' &= \left(16\theta_{01}^8(z_1) + 16\theta_{01}^4(z_1)\theta_{10}^4(z_1) + \theta_{10}^8(z_1)\right)/16,\\
E_6 &= \left(2\theta_{01}^4(z_2) + \theta_{10}^4(z_2)\right)\left(\theta_{01}^8(z_2) + \theta_{01}^4(z_2)\theta_{10}^4(z_2) - 2\theta_{10}^8(z_2)\right)/2,\\
E_6' &= \left(2\theta_{01}^4(z_1) + \theta_{10}^4(z_1)\right)\left(32\theta_{01}^8(z_1) + 32\theta_{01}^4(z_1)\theta_{10}^4(z_1) - \theta_{10}^8(z_1)\right)/64.
\end{aligned}$$

Hence, it is easy to show that $W(F_8) = W(G_{10}) = 0$ and

$$\begin{aligned}
W(F_4) &= 4E_4'E_4,\\
W(F_6) &= -8E_6'E_6,\\
W(F_{12}) &= (E_4'^3 - E_6'^2)\,(E_4^3 - E_6^2)/81,\\
W(G_{12}) &= 3^{-3}\cdot 2^6\,(E_4'^3E_6^2 - E_4^3E_6'^2).
\end{aligned}$$

**Lemma 1.** *Let $P$ and $Q$ be polynomials of three variables which satisfy the following relation*

$$P\left(W(F_4), W(F_6), W(F_{12})\right) + W(G_{12})Q\left(W(F_4), W(F_6), W(F_{12})\right) = 0.$$

*Then we get $P = Q = 0$.*

*Proof.* It is clear that $E_4$, $E_4'$, $E_6$, $E_6'$ are algebraically independent. The forms $W(F_4)$, $W(F_6)$ and $W(F_{12})$ are invariant under the exchange between $E_k$ and $E_k'$ ($k = 4, 6$) and $G_{12}$ becomes $-G_{12}$ by this exchange. So, we get $P(W(F_4), W(F_6), W(F_{12})) = 0$. But obviously, $W(F_4)$, $W(F_6)$ and $W(F_{12})$ are algebraically independent. Hence $P = Q = 0$ as polynomials. □

*Proof of Theorem* 1. First, it is easy to show the formula for $G_k^2$ for $k = 10$, 11, 12 in Theorem 1, since we know the relation

$$64K^2 = -16XTK - T\left(-16YZ + X^2T - YT - 1024ZT - 64T^2\right),$$

and the formula to express $G_{11}^2$ by $X$, $Y$, $Z$, $K$ and $T$ in Appendix of [10]. All we must do is to express both sides of the formulas as polynomials of $X$, $Y$, $Z$, $K$, $T$ of degree one with respect to $K$ to find out they are the same. In particular we get

$$G_{10}^2 = F_8F_{12}/4.$$

**Proposition 3.** *Let $P_i(x_1, x_2, x_3, x_4)$, $1 \le i \le 4$ be polynomials of four variables which satisfy the following relation*

$$P_1(F_4, F_6, F_8, F_{12}) + G_{12}P_2(F_4, F_6, F_8, F_{12}) + G_{10}P_3(F_4, F_6, F_8, F_{12}) + G_{10}G_{12}P_4(F_4, F_6, F_8, F_{12}) = 0.$$

*Then $P_i = 0$, $i = 1, \dots, 4$ as polynomials. In particular, $F_4$, $F_6$, $F_8$, $F_{12}$ are algebraically independent.*

*Proof.* We take the image under the Witt operator of the both sides of the above relation. Since $W(G_{10}) = 0$, by the above lemma we get

$$P_1(x_1, x_2, 0, x_4) = P_2(x_1, x_2, 0, x_4) = 0.$$

That is, for $i = 1$, 2, we have $P_i = x_3 Q_i$ for some polynomials $Q_i$. Now, multiplying $G_{10}$ to both sides, we get

$$F_8 G_{10} Q_1(F_4, F_6, F_8, F_{12}) + F_8 G_{10} G_{12} Q_2(F_4, F_6, F_8, F_{12}) + 4^{-1} F_8 F_{12} P_3(F_4, F_6, F_8, F_{12}) + 4^{-1} F_8 F_{12} G_{12} P_4(F_4, F_6, F_8, F_{12}) = 0.$$

Now dividing both sides by $F_8$, then applying $W$ on both sides, and dividing by $W(F_{12}) \neq 0$, we can see as before that $P_i = x_3 Q_i$ for $i = 3$, 4, for some polynomials $Q_i$. Repeating this process, we can conclude that $P_i = 0$ for all $i = 1, \dots, 4$. □

By the above proposition, we can calculate the dimensions of $(B + G_{10}B + G_{12}B + G_{10}G_{12}B) \cap A_k(K(2))$ for each $k$ to find that it is equal to $\dim A_k(K(2))$. Hence our main theorem is proved. □

Corollary 1 is clear from Theorem.

*Proof of Corollary* 2. We denote by $\mathcal{K}$ the function field of $\mathrm{Proj}(A(K(2)))$. We define elements $A$, $B$, $C$, $\alpha$, $\beta$, $\gamma \in \mathcal{K}$ as in section 1. By the formula for $G_{10}^2$, we get $F_{12}/F_4^3 = 4\alpha^2\beta/\gamma$. By Theorem 1, it is easy to see that $\mathcal{K}$ is generated by $\alpha$, $\beta$, $\gamma$, and $G_{12}/F_4^3$. Now, we define the field $\mathcal{K}'$ by $\mathcal{K}' = \mathbf{C}(\alpha, A/C, B/C)$. By modifying the formula for $G_{12}^2$, we can show that $A^2 = B^2 - \gamma C^2$. Hence $\gamma = (B/C)^2 - (A/C)^2 \in \mathcal{K}'$. Since $C$ is a polynomial of $\alpha$ and $\gamma$, we get $C \in \mathcal{K}'$. Hence $A$, $B \in \mathcal{K}'$ and also $G_{12}/F_4^3$, $\beta \in \mathcal{K}'$. Thus we get $\mathcal{K}' = \mathcal{K}$ and Corollary 2. □

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