ON SIEGEL MODULAR VARIETIES OF LEVEL 3

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In this paper, first, we shall give the explicit structure of the graded ring $A(\Gamma_0(3))$ or $A(\Gamma_0^*(3))$, of Siegel modular forms of genus two (of even weights) belonging to the discrete subgroup $\Gamma_0(3)$ or $\Gamma_0^*(3)$ of $Sp(2, \mathbb{R})$, respectively. (Sec. 1. Theorem 1, 3) We shall also give the ideals of cusp forms of these rings explicitly there. Secondly, we shall show that the Satake compactifications $S(\Gamma_0(3)\backslash H_2)$ and $S(\Gamma_0^*(3)\backslash H_2)$ of the quotients of the Siegel upper half space H_2 of genus two by the above groups are rational varieties (Sec. 1. Theorem 2), and besides, we shall show that these are weighted projective spaces (Sec. 1. Theorem 4), although the graded rings above are *not* weighted polynomial rings. Here, for any natural number p, we denote by $\Gamma_0(p)$ the group defind by:

$$\Gamma_0(p) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{Z}); C \equiv 0 \bmod p \right\},$$

and by $\Gamma_0^*(p)$ the subgroup of $Sp(2, \mathbb{R})$ generated by $\Gamma_0(p)$ and ρ, where

$$\rho = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \end{pmatrix}.$$

In the appendix, we also give graded rings of Siegel modular forms belonging to some level two subgroups.

Historically, first, Igusa [6] [7] [8] gave explicitly the graded rings of Siegel modular forms belonging to the several discrete subgroups of $Sp(2, \mathbb{R})$ with 2 power level. It was also shown by Coble [1] and van der Geer [2] that the Siegel modular variety with respect to the level three principal congruence subgroup $\Gamma(3)$ of $Sp(2, \mathbb{Z})$ is a rational variety. So, it is clear that $S(\Gamma_0(3)\backslash H_2)$ is *unirational*, because it is covered by the rational variety $S(\Gamma(3)\backslash H_2)$. But it was not known whether it is *rational*, because our variety is of dimension three, and Castelnuovo's lemma is false in general in such cases. Besides, explicit structures of the graded rings were not known in any cases of level three.

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Now, we outline our proof and the content of each section of this paper. In Sec. 1, after reviewing the definitions of several known Siegel modular forms, we shall state our main results as Theorem 1, 2, 3, 4. In Sec. 2, we shall write down the dimension formula of Siegel modular forms belonging to $\Gamma_0(3)$. This is obtained easily by using the dimension formula of cusp forms belonging to $\Gamma_0(3)$ of weight $k \geq 5$ by Hashimoto [3], of $k \leq 4$ by Yoshida [11], and the surjectivity of ϕ-operator given by Satake [9] for $k \geq 6$. In Sec. 3, we shall define a certain submodule A' of $A(\Gamma_0(3))$, and we shall show that the image of A' under the Witt operator W (that is, the restriction of Siegel modular forms to the diagonals of H_2) is equal to the space of *symmetric* modular forms on the diagonals of H_2. (The precise definition will be given there.) The graded ring of the symmetric modular forms is also given there. In Sec. 4, we shall show that A' coincides with $A(\Gamma_0(3))$ by using the results in Secs. 1, 2, and prove Theorem 1, 2. In Sec. 5, we shall prove Theorem 3, 4. In the appendix, we shall give graded rings of Siegel modular forms belonging to some level two subgroups. The results in this appendix was a part of [4], but this part has not been published before. We shall omit the proof there, because the proof is almost the same as in the case of level three and it can also be proved, at least in principle, by using Igusa's result on the level two principal congruence subgroup. (cf. [7])

Notations. We shall use the standard notation $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{Z}$ for the complex numbers, the real numbers, or the rational integers. For any ring R, we shall denote by $M_n(R)$ the set of all n by n matrices and by 1_n the n by n unit matrix. When R is an algebra over $\mathbb{C}$ and $r_1, \ldots, r_m \in R$, we denote by $C[r_1, \ldots, r_m]$ the ring generated by $r_1, \ldots, r_m$ over $\mathbb{C}$.

1. Main Results

1.1. *Preliminaries*

We denote by H_n the Siegel upper half space of genus n defined by:

$$H_n = \{X + iY \in M_n(\mathbb{C});\, {}^tX = X, {}^tY = Y \in M_n(\mathbb{R}),\, Y > 0\ (Y\text{: positive definite})\}.$$

We denote by $Sp(n, \mathbb{R})$ the usual real symplectic group of size $2n$:

$$Sp(n, \mathbb{R}) = \{g \in M_{2n}(\mathbb{R});\, gJ_n{}^tg = J_n\},$$

where

$$J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{R}).$$

Any element g of $Sp(n, \mathbb{R})$ acts on H_n by:

$$g(Z) = (AZ + B)(CZ + D)^{-1} \text{ for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

For any element $g \in Sp(n, \mathbb{R})$, any non-negative integer k, and any function F on H_n, we write

$$F|[g]_k = F(g(Z)) \det(CZ + D)^{-k}.$$

For each non-negative integer k and each discrete subgroup Γ of $Sp(n, \mathbb{R})$ with covolume finite, we say that any holomorphic function F on H_n is a modular form of weight k belonging to Γ, if it satisfies the following conditions (1) and (2):

(1) $F|[\gamma]_k = F$ for any $\gamma \in \Gamma$

(2) F is holomorphic at each cusp.

(It is well known that the condition (2) is automatically satisfied when $n \geq 2$.) If a modular form F vanishes on every cusp of the Satake compactification $S(\Gamma\backslash H_n)$ of $\Gamma\backslash H_n$, then we say that F is a cusp form. We denote by $A_k(\Gamma)$, or $S_k(\Gamma)$, the space of Siegel modular forms, or cusp forms, of weight k belonging to Γ, respectively. It is well known that, for any natural number d, we have

$$S(\Gamma\backslash H_n) = \mathrm{Proj}\left(\sum_{k=0}^{\infty} A_{kd}(\Gamma)\right).$$

We denote by $A(\Gamma)$ the graded ring of Siegel modular forms of *even* weights belonging to Γ:

$$A(\Gamma) = \sum_{k=0}^{\infty} A_{2k}(\Gamma),$$

and by $K(\Gamma)$ the function field of $S(\Gamma\backslash H_2)$.

1.2. *Modular forms*

Now, we review the definition of several modular forms on H_2 given explicitly. For any natural number $k \geq 4$, we denote by $E_k(Z)$ the Eisenstein series on H_2 of weight k belonging to $Sp(2, \mathbb{Z})$, normalized so that the constant term of the Fourier expansion of E_k is 1. We put $E'_k = E_k(3Z)$. It is trivial that E'_k is a modular form belonging to $\Gamma_0(3)$ and that $E_k + 3^k E'_k$ to $\Gamma_0^*(3)$. Yoshida [11] defined severed modular forms belonging to $\Gamma_0(3)$. We quote some of them below. We denote by S the 4×4 matrix defined by:

$$S = \begin{pmatrix} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 3/2 \\ 3/2 & 0 & 3 & 0 \\ 0 & 3/2 & 0 & 3 \end{pmatrix}.$$

For column vectors $x = {}^t(x_1, x_2, x_3, x_4)$, $y = {}^t(y_1, y_2, y_3, y_4) \in R^4$, set

$$Q(x,y) = \begin{pmatrix} {}^t xSx & {}^t xSy \\ {}^t xSy & {}^t ySy \end{pmatrix}$$

and

$$c = (x_1y_3 - y_1x_3) + (x_2y_4 - y_2x_4)$$

$$d = (x_1y_4 - y_1x_4) + (x_3y_2 - y_3x_2) + (x_1y_2 - y_1x_2).$$

According to Yoshida, we define three theta functions θ_2, θ_4, and θ_6 as follows:

$$\theta_2(Z) = \sum_{(x,y)} \exp(2\pi i(\mathrm{tr}(Q(x,y)Z))),$$

$$\theta_4(Z) = \sum_{(x,y)} (c^2 - d^2)\exp(2\pi i(\mathrm{tr}(Q(x,y)Z))),$$

$$\theta_6(Z) = \sum_{(x,y)} (c^4 - 6c^2d^2 + d^4)\exp(2\pi i(\mathrm{tr}(Q(x,y)Z))),$$

where $Z \in H_2$ *and* (x, y) extends over $\mathbb{Z}^4 \oplus \mathbb{Z}^4$. The above functions $\theta_2, \theta_4, \theta_6$ are Siegel modular form of weight 2, 4, 6, respectively, which belong not only to $\Gamma_0(3)$ but also to $\Gamma_0^*(3)$, and θ_4 and θ_6 are cusp forms (Yoshida [11] pp. 358–359).

1.3. *Main results*

We denote by f_6 the element of $A_6(\Gamma_0^*(3))$ defined by:

$$f_6 = \frac{1}{22464}(-1098\theta_2^3 + 3888\theta_6 + 49\theta_2(E_4 + 81E_4') - 4(E_6 + 729E_6')) + \frac{2271}{3328}\theta_2\theta_4.$$

We denote by B the subalgebra of $A(\Gamma_0^*(3))$ generated by $\theta_2, \theta_4, \theta_6$ and f_6 over $\mathbb{C}$.

Theorem 1. *Notations being as above, the ring B is a weighted polynomial ring. Each graded ring $A(\Gamma_0(3))$, or $A(\Gamma_0^*(3))$ is a free B-module of rank 4, or of rank 2, respectively. More explicitly, we have*

$$A(\Gamma_0^*(3)) = B \oplus B(E_4 + 81E_4'),$$

$$A(\Gamma_0(3)) = B \oplus B(E_4 + 81E_4') \oplus B(E_4 - 81E_4') \oplus B(E_6 - 729E_6'),$$

where we mean by $\oplus$ the direct sum as B-modules. Each ideal of cusp forms of each graded ring is spanned by θ_4 and θ_6.

Theorem 2. *The 3 dimensional varieties $S(\Gamma_0^*(3)\backslash H_2)$ and $S(\Gamma_0(3)\backslash H_2)$ are rational. The function field of each variety is given by*

$$K(\Gamma_0^*(3)) = C\left(\frac{\theta_4}{\theta_2^2}, \frac{(E_4 + 81E_4')}{\theta_2^2}, \frac{\theta_6}{\theta_2^3}\right), \quad or$$

$$K(\Gamma_0(3)) = C\left(\frac{\theta_4}{\theta_2^2}, \frac{E_4}{\theta_2^2}, \frac{E_4'}{\theta_2^2}\right).$$

To write down relations between the generators of the graded rings above as simply as possible, we introduce some more notations. We define

$$t_2 = \theta_2,$$

$$u_4 = \theta_4,$$

$$v_6 = -18\theta_2^3 - 1728f_6 + \frac{3321}{4}\theta_2\theta_4 + \theta_2(E_4 + 81E_4'),$$

$$w_6 = (82\theta_2^3 - 1728f_6 + \frac{3321}{4}\theta_2\theta_4 - \theta_2(E_4 + 81E_4') + 1944\theta_6)/4,$$

$$x_4 = ((E_4 + 81E_4') - 82\theta_2^2 + 324\theta_4)/16,$$

$$y_4 = (E_4 - 81E_4')/10,$$

$$z_6 = \frac{1}{7}(E_6 - 729E_6') - \frac{13}{10}(E_4 - 81E_4').$$

We denote by T_2, U_4, V_6, W_6, X_4, Y_4, Z_6 seven algebraically independent *weighted* variables, where each variable of $T_2, U_4, V_6, W_6, X_4, Y_4$, or Z_6, is of weight 2, 4, 6, 6, 4, 4, or 6, respectively. Denote by I^*, or I the ideal of the weighted polynomial ring $C[T_2, U_4, V_6, W_6, X_4]$, or $C[T_2, U_4, V_6, W_6, X_4, Y_4, Z_6]$, generated respectively by the following elements in each ring:

$I^* = (X_4^2 - T_2W_6)$, or

$$I = (X_4^2 - T_2W_6, X_4Y_4 - T_2Z_6, Y_4^2 - T_2V_6, X_4Z_6 - W_6Y_4, Y_4Z_6 - V_6X_4, Z_6^2 - V_6W_6).$$

Theorem 3. *We have the following isomorphisms between graded rings:*

$$A(\Gamma_0^*(3)) \cong C[T_2, U_4, V_6, W_6, X_4]/I^*,$$

$$A(\Gamma_0(3)) \cong C[T_2, U_4, V_6, W_6, X_4, Y_4, Z_6]/I.$$

The isomorphisms are obtained by mapping each T_2, U_4, V_6, W_6, X_4, Y_4, or Z_6 to each t_2, u_4, v_6, w_6, x_4, y_4, or z_6, respectively.

Now, we shall state some more details on these varieties. For each quadruple of natural numbers i_1, i_2, i_3, i_4, we denote by $\alpha_{i_1}, \beta_{i_2}, \gamma_{i_3}, \delta_{i_4}$, algebraically independent weighted variables of weight i_1, i_2, i_3, or i_4, respectively. We denote by $P(i_1, i_2, i_3, i_4)$ the three dimensional weighted projective space of weight i_1, i_2, i_3, i_4. By definition, we have

$$P(i_1, i_2, i_3, i_4) = \mathrm{Proj}(C[\alpha_{i_1}, \beta_{i_2}, \gamma_{i_3}, \delta_{i_4}]).$$

We denote by $C[\alpha_{i_1}, \beta_{i_2}, \gamma_{i_3}, \delta_{i_4}]_{(2)}$ the graded subring consisting of elements of even weights of $C[\alpha_{i_1}, \beta_{i_2}, \gamma_{i_3}, \delta_{i_4}]$

Theorem 4. *We have the following isomorphisms between graded rings*:

$$A(\Gamma_0^*(3)) \cong C[\alpha_1, \beta_3, \gamma_4, \delta_6]_{(2)}$$

$$A(\Gamma_0(3)) \cong C[\alpha_1, \beta_3, \gamma_4, \delta_3]_{(2)}.$$

These isomorphisms induce the following (*biregular*) *isomorphisms between algebraic varieties*:

$$S(\Gamma_0^*(3)\backslash H_2) \cong P(1,3,4,6), \quad \text{and}$$

$$S(\Gamma_0(3)\backslash H_2) \cong P(1,3,3,4).$$

Each isomorphism between graded rings will be given explicitly below.

Isomorphisms between graded rings:
(1) in case of $\Gamma_0(3)$
An isomorphism is given by mapping the generators as follows:

$$\alpha_1^2 \to \theta_2, \qquad \beta_3^2 \to w_6,$$

$$\delta_3^2 \to v_6, \qquad \gamma_4 \to \theta_4,$$

$$\alpha_1\beta_3 \to x_4, \qquad \alpha_1\delta_3 \to y_4,$$

$$\beta_3\delta_3 \to z_6,$$

(2) in case of $\Gamma_0(3)^*$
In this case, an isomorphism is given by restricting the above isomorphism to the subring $C[\alpha_1, \beta_3, \gamma_4, \delta_6]_{(2)}$ of $C[\alpha_1, \beta_3, \gamma_4, \delta_3]_{(2)}$, where we identify δ_6 with δ_3^2.

Remark 1. For odd $k \le 13$, there is *no* Siegel modular form of weight k belonging to $\Gamma_0(3)$. So, the graded ring $\sum_{k=1}^{\infty} A_k(\Gamma_0(3))$ is *not* isomorphic to $C[\alpha_1, \beta_3, \gamma_4, \delta_3]$.

Remark 2. By virtue of Abel's fundamental theorem on $\mathbb{Z}$-modules, it is trivial that any weighted projective space is rational. So, Theorem 2 is a direct consequence of our Theorem 4. But, in this paper, we shall prove Theorem 2 more directly without using Theorem 4.

2. Dimension Formula

First, we review the following Proposition.

Proposition 2.1. (*Hashimoto* [3] *for* $k \ge 5$, *Yoshida* [11] *for* $k \le 4$) *We have*

$$\sum_{k=1}^{\infty} (\dim S_k(\Gamma_0(3)))t^k = \frac{(1 + 2t^4 + t^6)(t^4 + t^6 - t^{10}) + t^{15}(1 + 2t^2 + t^6)}{(1 - t^2)(1 - t^4)(1 - t^6)^2}.$$

Proof. This is obtained just by gathering the individual data for each dim $S_k(\Gamma_0(3))$ in [3] [11] together. The details will be omitted here. □

Next, we show

Proposition 2.2. *We have*

$$\sum_{k=1}^{\infty} (\dim A_k(\Gamma_0(3)))t^k = \frac{1 + 2t^4 + t^6 + t^{15}(1 + 2t^2 + t^6)}{(1 - t^2)(1 - t^4)(1 - t^6)^2}.$$

Proof. The boundary of $S(\Gamma_0(3)\backslash H_2)$ have two 1-dimensional components, both of which are isomorphic to $S(\Gamma_0^1(3))$, where we denote by $\Gamma_0^1(3)$ the subgroup of $SL_2(\mathbb{Z})$ defined by:

$$\Gamma_0^1(3) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); c \equiv 0 \bmod 3\right\}.$$

And these 1-dimensional components intersect with each other at a point p which corresponds to a cusp of $S(\Gamma_0^1(3))$ (cf. [4], [5]). In this case, for each k, the generalized Siegel Φ-operator in the sense of Satake [9] is the mapping Φ_k of $A_k(\Gamma_0(3))$ into the following C-linear space ∂A_k:

$$\partial A_k = \{(f, g) \in A_k(\Gamma_0^1(3)) \times A_k(\Gamma_0^1(3)); f = g \text{ on } p\},$$

and defined by:

$$\Phi_k(F(Z)) = \left(\lim_{\tau' \to i\infty} F\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}, \lim_{\tau' \to i\infty} (F|[J_2]_k)\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}\right).$$

If k is odd, then $\partial A_k = 0$ and $A_k(\Gamma_0(3)) = S_k(\Gamma_0(3))$. Hereafter, we assume that k is even. The above Φ_k is surjective for any even k. In fact, when $k \geq 6$, this is contained in the more general theorem of Satake (loc.cit.,). When $k = 2$, or 4, we can show the surjectivity of Φ_k directly by mapping θ_2, θ_2^2, E_4, and E_4' by Φ_k. Hence, for even k, we have

$$\dim A_k(\Gamma_0(3)) = \dim S_k(\Gamma_0(3)) + \dim \partial A_k.$$

As we have

$$\dim \partial A_k = 2 \dim S_k(\Gamma_0^1(3)) + \begin{cases} 3 \ldots \text{for } k \geq 4 \\ 1 \ldots \text{for } k = 2 \end{cases},$$

and

$$\sum_{k=1}^{\infty} (\dim S_k(\Gamma_0^1(3)))t^k = \frac{t^6 + t^{10}}{(1 - t^2)(1 - t^6)},$$

we get our Proposition 2.2. □

3. Witt Operator

We denote by B the subalgebra of $A(\Gamma_0(3))$ defined as in Sec. 1 and by A' the B-submodule of $A(\Gamma_0(3))$ generated by $1, E_4 + 81E'_4, E_4 - 81E'_4$, and $E_6 - 729E'_6$ over B (as B-module). In the next section, we shall see that A' is actually a ring and equal to $A(\Gamma_0(3))$. In this section, we shall see the explicit structure of the image of A' by the Witt operator W. The Witt operator W is defined to be the mapping of the set of functions F on H_2 to the functions on $H_1 \times H_1$ as follows:

$$W(F)(\tau, \tau') = F\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}.$$

It is trivial that W is a ring homomorphism.

Now, we shall call a holomorphic function $F(\tau, \tau')$ on $H_1 \times H_1$ a symmetric modular form of weigth k belonging to $\Gamma_0^1(3)$, if it satisfies the following two conditions:

(1) $F(\tau, \tau') = F(\tau', \tau)$,

(2) For any fixed $\tau' \in H_1$, the function $F(\tau, \tau')$ on $\tau \in H_1$ belongs to $A_k(\Gamma_0^1(3))$.

By a similar argument as in Witt [10], we can show that these conditions (1), (2) are equivalent to the following single condition (3):

(3) For some integer $r \geq 0$ and some forms $f_i, g_i \in A_k(\Gamma_0^1(3))$ $(i = 1, \ldots, r)$, we have

$$F(\tau,\tau') = \sum_{i=1}^{\tau} (f_i(\tau)g_i(\tau') + g_i(\tau)f_i(\tau')).$$

We denote by $\mathrm{Sym}_k(\Gamma_0^1(3))$ the space of all symmetric modular forms of weight k belonging to $\Gamma_0^1(3)$. We also define a graded ring $\mathrm{Sym}(\Gamma_0^1(3))$ by:

$$\mathrm{Sym}(\Gamma_0^1(3)) = \bigoplus_{k=1}^{\infty} \mathrm{Sym}_k(\Gamma_0^1(3)).$$

It is easy to see that

$$W(A(\Gamma_0(3))) \subset \mathrm{Sym}(\Gamma_0^1(3)).$$

We shall describe the explicit structure of $\mathrm{Sym}(\Gamma_0^1(3))$. First, we clarify the structure of $A(\Gamma_0^1(3))$. We denote by g_2(resp. χ_6) the unique normalized modular (resp. cusp) form of weight 2 (resp. 6) belonging to $\Gamma_0^1(3)$. We also denote by g_4 the normalized Eisentein series of weight 4 belonging to $SL_2(\mathbb{Z})$. The forms g_2 and χ_6 are algebraically independent. In fact, g_2 is not a cusp form, while χ_6 is, and if $P(x, y)$ is any polynomial such that $P(g_2, \chi_6) = 0$, then $P(x, y)$ is divisible by y, so, if we replace $P(x, y)$ several times by $P(x, y)/y$, we can assume that $P(x, y)$ is a polynomial only on x, which is a contradiction. Now, we see that g_2, g_4, and χ_6 span $A(\Gamma_0^1(3))$. More precisely, we get

$$A(\Gamma_0^1(3)) = C[g_2, \chi_6] \oplus g_4 C[g_2, \chi_6],$$

$$g_4^2 = -9g_2^4 + 10g_2^2 g_4 - 1729 g_2 \chi_6,$$

where $\oplus$ means a direct sum as modules. In fact, as $\dim A_8(\Gamma_0^1(3)) = 3$, there exists a linear relation between g_4^2, g_2^4, $g_2^2 g_4$, and $g_2 \chi_6$. We get the above explicit relation by using the following Fourier coefficients:

$$g_2(\tau) = 1 + 12q + 36q^2 + 12q^3 + \cdots,$$

$$g_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \cdots,$$

$$\chi_6(\tau) = q - 6q^2 + 9q^3 + \cdots,$$

where $\tau \in H_1$ and $q = \exp(2\pi i \tau)$. Now, take a polynomial $P(x, y, z)$ such that $P(g_2, \chi_6, g_4) = 0$. We can assume that the degree of P with respect to z is one and that the degree with respect to y is minimal among those which satisfy the above relation. Then, we have $P(x, 0, z) = ax^n + bzx^{n-2}$ for some a, $b \in \mathbb{C}$ and a natural number $n \geq 2$. As χ_6 is a cusp form and g_2 does not vanish at each cusp, the form $ag_2^2 + bg_4$ must be a cusp form. This implies that $a = b = 0$, because g_2^2 and g_4 are linearly independent and $\dim S_4(\Gamma_0^1(3)) = 0$. Hence, $P(x, y, z)$ is divisible by y and we can replace $P(x, y, z)$ by $P(x, y, z)/y$, which is a contradiction.

Next, we shall determine the ring structure of $\mathrm{Sym}(\Gamma_0^1(3))$. We denote by τ (resp. τ') the first (resp. second) argument of an element of $H_1 \times H_1$. For a pair $(f, g) \in A(\Gamma_0^1(3))^2$, we shall often consider a function $f(\tau)g(\tau')$ on $H_1 \times H_1$. To distinguish the variables τ and τ' in such cases, we shall denote $f(\tau)$ by f and $f(\tau')$ by f'.

Proposition 3.1. *The symmetric modular forms $g_2 g_2'$, $g_2^3 \chi_6' + g_2'^3 \chi_6$, and $\chi_6 \chi_6'$ are algebraically independent. Besides, we get*

$$\mathrm{Sym}(\Gamma_0^1(3)) = C[g_2 g_2', g_2^3 \chi_6' + g_2'^3 \chi_6, \chi_6 \chi_6', g_4 g_4', g_2^2 g_4' + g_2'^2 g_4, g_2 g_4 \chi_6' + g_2' g_4' \chi_6]$$

$$= S \oplus (g_4 g_4') S \oplus (g_2^2 g_4' + g_2'^2 g_4) \oplus (g_2 g_4 \chi_6' + g_2' g_4' \chi_6) S,$$

where

$$S = C[g_2 g_2', g_2^3 \chi_6' + g_2'^3 \chi_6, \chi_6 \chi_6'],$$

and $\oplus$ means the direct sum as S-modules.

Proof. We denote by Sym' the right hand side of the above equality, that is, the subring of $\mathrm{Sym}(\Gamma_0^1(3))$ spanned by the six symmetric modular forms mentioned above. First, we shall show that $\mathrm{Sym}(\Gamma_0^1(3)) = \mathrm{Sym}'$. By definition, every symmetric modular form of weight k is linear combination of several forms $fg' + f'g$, where $(f, g) \in A_k(\Gamma_0^1(3))^2$. But, by the explicit structure of $A(\Gamma_0^1(3))$, this is also a linear combination of the following three kinds of symmetric modular forms:

(1) $g_2^a \chi_6^b g_2'^c \chi_6'^d + g_2'^a \chi_6'^b g_2^c \chi_6^d \qquad (2a + 6b = 2c + 6d = k)$,

(2) $g_2^a \chi_6^b g_2'^c \chi_6'^d g_4' + g_2'^a \chi_6'^b g_2^c \chi_6^d g_4 \qquad (2a + 6b = 2c + 6d + 4 = k)$,

(3) $(g_4 g_4')(g_2^a \chi_6^b g_2'^c \chi_6'^d + g_2'^a \chi_6'^b g_2^c \chi_6^d) \qquad (2a + 6b + 4 = 2c + 6d + 4 = k)$,

where a, b, c, d are non-negative integers. Now, we must show that these three kinds of forms belong to Sym$'$. Dividing the above forms by $(g_2g_2')^{\min(a,c)}$ and $(\chi_6\chi_6')^{\min(b,c)}$, we can assume that $\min(a,c) = \min(b,d) = 0$. Hence, the proof reduces to show that the forms $g_2^{3e}\chi_6'^e + g_2'^{3e}\chi_6^e$, $g_2^{3e-2}\chi_6'^e g_4 + g_2'^{3e-2}\chi_6^e g_4'$, and $g_2^{3e+2}\chi_6'^e g_4 + g_2'^{3e+2}\chi_6^e g_4'$ belong to Sym$'$ for each $e \geq 1$. This can be proved by induction, using the following relations:

$$(g_2^{3e}\chi_6'^e + g_2'^{3e}\chi_6^e)(g_2^3\chi_6' + g_2'^3\chi_6) = (g_2^{3(e+1)}\chi_6'^{e+1} + g_2'^{3(e+1)}\chi_6^{e+1}) + (g_2g_2')(\chi_6\chi_6')(g_2^{3(e-1)}\chi_6'^{e-1} + g_2'^{3(e-1)}\chi_6^{e-1}),$$

$$g_2^{3e+2}\chi_6'^e g_4 + g_2'^{3e+2}\chi_6^e g_4' = (g_2^2g_4' + g_2'^2g_4)(g_2^{3e}\chi_6'^e + g_2'^{3e}\chi_6^e) - (g_2^{3e-2}\chi_6'^e g_4 + g_2'^{3e-2}\chi_6^e g_4')$$

$$g_2^{3(e+1)-2}\chi_6'^e g_4 + g_2'^{3(e+1)-2}\chi_6^e g_4' = (g_2g_4\chi_6' + g_2'g_4'\chi_6)(g_2^{3e}\chi_6'^e + g_2'^{3e}\chi_6^e) - (g_2g_2')(\chi_6\chi_6')(g_2^{3(e-1)+2}\chi_6'^e g_4 + g_2'^{3(e-1)+2}\chi_6^e g_4').$$

Hence, we get $\mathrm{Sym}(\Gamma_0^1(3)) = \mathrm{Sym}'$. Next, we shall prove the second equality in Proposition 3.1. By the explicit structure of $A(\Gamma_0^1(3))$ which was shown before, it is clear that $\mathrm{Sym}(\Gamma_0^1(3))$ is spanned by 1, g_4g_4', $g_2^2g_4' + g_2'^2g_4$, and $g_2g_4\chi_6' + g_2'g_4'\chi_6$ as S-module. As we have shown before, g_2, g_2', χ_6, and χ_6' are algebraically independent. This implies also the algebraic independence of the three forms g_2g_2', $g_2^3\chi_6' + g_2'^3\chi_6$, and $\chi_6\chi_6'$. Now, assume that

$$P_1 + g_4g_4'P_2 + (g_2^2g_4' + g_2'^2g_4)P_3 + (g_2g_4\chi_6' + g_2'g_4'\chi_6)P_4 = 0$$

for some polynomials P_1, P_2, P_3, and P_4 of $g_2g_2', g_2^3\chi_6' + g_2'^3\chi_6$, and $\chi_6\chi_6'$. Then, we get

$$P_1 + g_4'(g_2^2P_3 + g_2'\chi_6P_4) = 0$$

$$(g_2'^2P_3 + g_2\chi_6'P_4) + g_4'P_2 = 0,$$

and

$$P_1 = g_2^2P_3 + g_2'\chi_6P_4 = 0$$

$$P_2 = g_2'^2P_3 + g_2\chi_6'P_4 = 0.$$

So, we get $P_4(g_2'^3\chi_6 - g_2^3\chi_6') = 0$. But, by calculating the Fourier coefficients, we get

$$g_2'^3\chi_6 - g_2^3\chi_6' = 756qq'^2 + \cdots \neq 0.$$

Hence, we get $P_i = 0$ for all $i = 1, \ldots, 4$, and that the sum is direct. Thus, we have proved Proposition 3.1. □

Corollary 3.2. *The notations being as above, we have*

$$\sum_{k=1}^{\infty} \dim(\mathrm{Sym}(\Gamma_0^1(3)))t^k = \frac{1 + 2t^4 + t^6}{(1 - t^2)(1 - t^6)^2}.$$

Proof. Obvious. □

Proposition 3.3. *We have*

$$W(A') = \mathrm{Sym}(\Gamma_0^1(3)).$$

Proof. It is easy to see that the image of each generator of B under the Witt operator W is given as follows:

$$W(\theta_2) = g_2 g_2',$$

$$W(\theta_4) = 0,$$

$$W(\theta_6) = 96\chi_6\chi_6',$$

$$W(f_6) = g_2^3\chi_6' + g_2'^3\chi_6,$$

We also get

$$W(E_4) = g_4 g_4',$$

$$W(E_4') = (10g_2^2 - g_4)(10g_2'^2 - g_4')/81,$$

$$W(E_4 + 81E_4') = 2g_4g_4' + 100(g_2g_2')^2 - 10(g_2^2g_4' + g_2'^2g_4),$$

$$W(E_4 - 81E_4') = 100(g_2g_2')^2 - 10(g_2^2g_4' + g_2'^2g_4),$$

$$W(E_6 - 729E_6') = -910(g_2g_2')^3 + 7560(g_2^3\chi_6' + g_2'^3\chi_6) + 91(g_2g_2')(g_2^2g_4' + g_2'^2g_4) - 1512(g_2g_4\chi_6' + g_2'g_4'\chi_6).$$

Hence, by Proposition 3.1, we get the required results. □

Remark. We also get

$$W(B + B(E_4 + 81E_4')) = S \oplus (g_4g_4' + (10g_2^2 - g_4)(10g_2'^2 - g_4))S.$$

This means that the Witt operator is 'surjective' in our cases in a certain sense. It seems interesting to ask whether this is always true.

4. Proof of Theorems 1, 2

4.1. First, we shall prove Theorem 1. The four modular forms θ_2, θ_4, θ_6, and f_6 are algebracially independent. In fact, assume that

$$P(\theta_2, \theta_4, \theta_6, f_6) = P_1(\theta_2, \theta_6, f_6) + \theta_4 P_2(\theta_2, \theta_4, f_6, \theta_6) = 0$$

for some polynomials $P(x, y, z, w) = P_1(x, z, w) + yP_2(x, y, z, w)$. Taking the images of both sides of the above equation under the Witt operator W, we get

$$P_1(W(\theta_2), W(\theta_6), W(f_6)) = 0.$$

As we have mentioned before in Sec. 3, $W(\theta_2)$, $W(\theta_6)$ and $W(f_6)$ are algebraically independent, so we get $P_1 = 0$. Hence, we can replace P by P/y, and by induction, we get $P = 0$. Next, define the ring B as in Theorem 1 and denote by A' the B-module defined as in Sec. 3. We shall prove now that $1, E_4 + 81E_4', E_4 - 81E_4'$, and $E_6 - 729E_6'$ are the B-free basis of A' as a B-module. In fact, assume that

$$P_1 + (E_4 + 81E_4')P_2 + (E_4 - 81E_4')P_3 + (E_6 - 729E_6')P_4 = 0$$

for some elements P_i $(i = 1, \ldots, 4)$ in B. Taking the images under the Witt operator, we get

$$\begin{aligned}&\{W(P_1) + 100(g_2g_2')^2W(P_2 - P_3) - 910(g_2g_2')^3W(P_4) + 7560(g_2^3\chi_6' + g_2'^3\chi_6)W(P_4)\}\\ &\quad + 2(g_4g_4')W(P_2) + (g_2^2g_4' + g_2'^2g_4)\{-10W(P_2 - P_3) + 91(g_2g_2')W(P_4)\}\\ &\quad - 1512((g_2g_4\chi_6' + g_2'g_4'\chi_6)W(P_4)\\ &= 0.\end{aligned}$$

Hence, by Proposition 3.1, we get $W(P_i) = 0$ for all $i = 1, \ldots, 4$. This fact also implies that P_i is divisible by θ_4 for each $i = 1, \ldots, 4$, and by induction, we get $P_i = 0$ $(i = 1, \ldots, 4)$. Hence, we proved that the sum is direct. We proved here also that, if $W(F) = 0$ for $F \in A'$, then $F \in \theta_4 A'$. Now, denote by A_k' the linear subspace of A' consisting of all the modular forms of A' of weight k. By the above results on the B-module structure of A', it is easy to see that

$$\sum_{k=1}^{\infty} (\dim A_{2k}')t^{2k} = \frac{1 + 2t^4 + t^6}{(1 - t^2)(1 - t^4)(1 - t^6)^2}.$$

Hence, by Proposition 2.2, we get $A_{2k}(\Gamma_0(3)) = A_{2k}'$ for each $k \geq 1$ and $A(\Gamma_0(3)) = A'$. Incidentally, we can give an alternative proof of this fact by using Corollary 3.2 and the following relation:

$$\dim A'_{2k} = \dim A'_{2k-4} + \dim \mathrm{Sym}_{2k}(\Gamma_0^1(3)).$$

As $E_4 + 81E'_4$ and all the elements of B belong to $A(\Gamma_0^*(3))$, while $(E_k - 3^k E'_k)|[\rho]_k = -(E_k - 3^k E'_k)$ for any $k \geq 4$, we get $A(\Gamma_0^*(3)) = B + (E_4 + 81E'_4)B$.

Now, we shall determine the ideal of cusp forms. We denote by S'_k the linear space consisting of all elements of degree k in the ideal of $A(\Gamma_0(3))$ spanned by θ_4 and θ_6. If $\theta_4 F + \theta_6 G = 0$ for some $F \in A_{k-4}(\Gamma_0(3))$ and $G \in A_{k-6}(\Gamma_0(3))$, then $W(G) = 0$. Hence, G is divisible by θ_4, and we get

$$\dim S'_{2k} = \dim A_{2k-4}(\Gamma_0(3)) + \dim A_{2k-6}(\Gamma_0(3)) - \dim A_{2k-10}(\Gamma_0(3)).$$

Hence, comparing the dimensions by Proposition 2.1 and 2.2, we get $S'_{2k} = S_{2k}(\Gamma_0(3))$. The ideal of cusp forms in $A(\Gamma_0^*(3))$ is obtained almost in the same way and the proof will be omitted here. Thus, we proved Theorem 1.

Secondly, we shall prove Theorem 2. For any discrete group Γ, the function field $K(\Gamma)$ of Proj $A(\Gamma)$ is generated by the elements f/g of degree 0 in the quotient field of $A(\Gamma)$ such that $f, g \in A_k(\Gamma)$ for some k. Hence, it is obvious that $K(\Gamma_0^*(3))$ is spanned by θ_4/θ_2^2, θ_6/θ_2^3, f_6/θ_2^3, and $(E_4 + 81E_4)/\theta_2^2$. By routine calculation, we can show that the image of the modular form

$$(E_4 + 81E'_4)^2 - 100\theta_2^2(E_4 + 81E'_4) - 1476\theta_2^4 + 432\theta_2 f_6 + 124416\theta_2\theta_6$$

under the Witt operator is 0. Hence, this modular form belongs to $\theta_4 A_4(\Gamma_0^*(3))$. In other words, we get

$$\frac{f_6}{\theta_2^3} \in C\left(\frac{\theta_4}{\theta_2^2}, \frac{\theta_6}{\theta_2^3}, \frac{E_4 + 81E_4}{\theta_2^4}\right).$$

By the similar argument as in the proof of Theorem 1, we can show that θ_2, θ_4, θ_6 and $E_4 + 81E'_4$ are algebraically independent. Hence, the three-dimensional variety $S(\Gamma_0^*(3)\backslash H_2)$ is rational. Incidentally, we can show also by the same argument that $K(\Gamma_0^*(3))$ is spanned by θ_4/θ_2^2, $(E_4 + 81E'_4)/\theta_2^2$, and f_6/θ_2^3. Next, as $(E_4 - 81E'_4) \times (E_6 - 729E'_6) \in A(\Gamma_0^*(3))$, the function field $K(\Gamma_0(3))$ is spanned by $K(\Gamma_0^*(3))$ and $(E_4 - 81E'_4)/\theta_2^2$. Calculating the image under the Witt operator, we see that

$$(E_4 - 81E'_4)^2 - 100\theta_2(E_4 + E'_4) + 1800\theta_2^4 + \frac{675}{256}\theta_2 f_6 + 83025\theta_2^2\theta_4 \in \theta_4 A_4(\Gamma_0^*(3)).$$

Hence, we get

$$\frac{f_6}{\theta_2^3} \in C\left(\frac{\theta_4}{\theta_2^2}, \frac{E_4 + 81E'_4}{\theta_2^2}, \frac{E_4 - 81E'_4}{\theta_2^2}\right).$$

This implies that

$$K(\Gamma_0(3)) = C\left(\frac{\theta_4}{\theta_2^2}, \frac{E_4}{\theta_2^2}, \frac{E_4'}{\theta_2^2}\right).$$

Hence, the variety $S(\Gamma_0(3)\backslash H_2)$ is rational. Thus, we proved Theorem 2.

Remark. By Theorem 1, we get the following dimension formulae:

$$\sum_{k=1}^{\infty} (\dim A_{2k}(\Gamma_0^*(3)))t^{2k} = \frac{1+t^4}{(1-t^2)(1-t^4)(1-t^6)^2}, \quad \text{and}$$

$$\sum_{k=1}^{\infty} (\dim S_{2k}(\Gamma_0^*(3)))t^{2k} = \frac{t^4+t^6+t^{10}}{(1-t^2)(1-t^4)(1-t^6)^2},$$

which have not been known before.

5. Proof of Theorems 3, 4

First, we shall prove Theorem 3. By Theorem 1, every modular form $F \in A(\Gamma_0(3))$ is equal to $a + (E_4 + 81E_4')b + (E_4 - 81E_4')c + (E_6 - 729E_6')d$ for some a, b, c, and $d \in B$. To get relations between generators, it is sufficient to give the above coefficients a, b, c, and $d \in B$ for each of the following six modular forms:

$$(E_4 + 81E_4')^2, \qquad (E_4 - 81E_4)^2, \qquad (E_4 + 81E_4')(E_4 - 81E_4'),$$

$$(E_4 + 81E_4')(E_4 - 729E_6'), \quad (E_4 - 81E_4')(E_6 - 729E_4'), \quad (E_6 - 729E_6')^2.$$

We already know each image of each generator of $A(\Gamma_0(3))$ under the Witt operator W (cf. Sec. 3), and we have shown in Sec. 3 that $\ker(W|_{A(\Gamma_0(3))}) = \theta_4 A(\Gamma_0(3))$. Hence, by Proposition 3.1, we can calculate a, b, c, d at least up to the elements of $\theta_4 A(\Gamma_0(3))$. For example, we get

$$\begin{aligned}(E_4 - 81E_4')(E_6 - 729E_6') = 70\{&13\theta_2^3(E_4 + 81E_4') + 7776\theta_2^2\theta_6 - 234\theta_2^5 \\ &- 180(E_4 + 81E_4')f_6 - 34200\theta_2^2 f_6\} + \theta_4 G \qquad (1)\end{aligned}$$

for some $G \in A_6(\Gamma_0(3))$. Here, by Theorem 1, we have

$$G = c_1\theta_2^3 + c_2\theta_2\theta_4 + c_3\theta_2(E_4 + 81E_4') + c_4\theta_6 + c_5 f_6$$

for some constants $c_i \in \mathbb{C}$ ($i = 1, \ldots, 5$). As is well known, any modular form $F \in A(\Gamma_0(3))$ has the Fourier expansion of the following form:

$$F(Z) = \sum a(T)e^{2\pi i \operatorname{tr}(TZ)},$$

where $a(T) \in \mathbb{C}$ and T runs over all positive semi-definite half-integral symmetric matrices. We can determine the above coefficients c_i $(i = 1, \ldots, 5)$, if we have sufficiently many explicit Fourier coefficients $a(T)$ for each modular form which appears in the above relation. For the sake of simplicity, we denote the matrix

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

by (a, c, b). By computer calculation, we get the following table of the Fourier coefficients:

(a,c,b)	$\theta_4 G$	$\theta_2^3\theta_4$	$\theta_2\theta_4^2$	$\theta_2\theta_4(E_4+81E_4')$	$\theta_4\theta_6$	$\theta_4 f_6$
(1, 1, 0)	43069320	−48	0	−3936	0	0
(1, 2, 0)	1249214400	−1728	0	−58752	0	−48
(1, 3, 0)	13799124360	−25968	0	−387552	0	288
(2, 2, 0)	28946665440	−48480	3456	−1279680	−1152	−3849/2
(2, 2, 1)	−13193296200	26640	−2304	401184	0	843

By using this table, we get

$$G = -\frac{1}{32}(154620900\theta_2^3 + 523567800\theta_2\theta_4 + 1615950\theta_2(E_4 + 81E_4') + 78382080f_6). \quad (2)$$

By the similar argument, we get the following relations:

$$(E_4 + 81E_4')^2 = 100\theta_2^2(E_4 + 81E_4') - 1476\theta_2^4 - 110592\theta_2 f_6 + 124416\theta_2\theta_6$$
$$- 104976\theta_4^2 - 648\theta_4(E_4 + 81E_4'), \quad (3)$$

$$(E_4 - 81E_4')^2 = 100\theta_2^2(E_4 + 81E_4') - 1800\theta_2^4 - 172800\theta_2 f_6 - 83025\theta_2^2\theta_4, \quad (4)$$

$$(E_4 + 81E_4')(E_4 - 81E_4') = \frac{160}{7}\theta_2(E_6 - 729E_6') - 126\theta_2^2(E_4 - 81E_4')$$
$$- 324\theta_4(E_4 - 81E_4'), \quad (5)$$

$$(E_4 + 81E_4')(E_6 - 729E_6') = \frac{1}{5}\{27216\theta_6(E_4 - 81E_4') - 6552\theta_2^3(E_4 - 81E_4')$$
$$-24192(E_4 - 81E_4')f_6\} + 226\theta_2^2(E_6 - 729E_6')$$
$$- 324\theta_4(E_6 - 729E_6') - \frac{2835}{2}\theta_2\theta_4(E_4 - 81E_4'), \quad (6)$$

$$
\begin{aligned}
64(E_6 - 729E'_6)^2 = {} & 529984\theta_2^4(E_4 + 81E'_4) - 88058888\theta_2 f_6(E_4 + 81E'_4) \\
& + 15240960\theta_2\theta_6(E_4 + 81E'_4) - 95397120\theta_2^6 \\
& - 7573063680\theta_2^3 f_6 + 5090480640\theta_2^3\theta_6 \\
& + 2341011456 f_6^2 - 2633637888 f_6\theta_6 - 1283062599\theta_2^2\theta_4^2 \\
& - 53740260\theta_2^2\theta_4(E_4 + 81E'_4) - 1644594840\theta_2^4\theta_4 \\
& - 6035420160\theta_2\theta_4 f_6 - 1265380704\theta_2\theta_4\theta_6. \qquad (7)
\end{aligned}
$$

Some of these relations can be also deduced by the other relations without knowing the Fourier coefficients. For example, we can get the above expression (7) for $(E_6 - 729E'_6)^2$ by using the above relations (1), (2), (3) and (4). In fact, we can obtain the coefficients $a, b \in B$ such that

$$(E_4 - 81E'_4)^2(E_6 - 729E'_6)^2 = a + b(E_4 + 81E'_4) \qquad (8)$$

by using only (1), (2), (3), (4) and not (7). Dividing each side of (8) by each side of (4) we get (7). These calculations are fairly complicated but essentially a routine work, and the details will be omitted here. By substituting the variables as in Sec. 1, we get the following relations:

$$x_4^2 = t_2 w_6 \qquad x_4 y_4 = t_2 z_6 \qquad y_4^2 = t_2 v_6$$

$$x_4 z_6 = w_6 y_4 \qquad y_4 z_6 = v_6 x_4 \qquad z_6^2 = v_6 w_6.$$

Hence, we proved Theorem 3. Next, take variables α_1, β_3, γ_4, and δ_3 as in Theorem 4. Then, the fundamental relations between α_1^2, β_3^2, δ_3^2, γ_4, $\alpha_1\beta_3$, $\alpha_1\delta_3$, and $\beta_3\delta_3$ are just the same as those between t_2, u_6, v_6, w_6, x_4, y_4, and z_6. So, we get Theorem 4. Thus, we proved all the theorems in Sec. 1.

6. Appendix

In this appendix, we shall give the explicit structure of the graded rings $A(\Gamma)$ for the subgroups $\Gamma = \Gamma_0(2)$ and $B(2)$ of $Sp(2, \mathbb{R})$, where $\Gamma_0(2)$ is defined as in the introduction, and $B(2)$ is the Iwahori subgroup of $Sp(2, \mathbb{Q})$ defined by:

$$B(2) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(2); A, {}^tD \text{ mod } 2 \text{ are lower triangular.} \right\}.$$

The content of this appendix was a part of [4], and is used to prove the main results there, but has been omitted in the publication. We shall omit any proof here, because the proof is more or less similar to the one given in the preceding sections. (See also Igusa [7].)

For any $m = {}^t(m', m'') \in \mathbb{Z}^4 (m', m'' \in \mathbb{Z}^2)$ (row vectors), we define a theta constant with characteristic m, as usual, by the following function on $Z \in H_2$:

$$\theta_m = \theta_m(Z) = \sum_{p \in \mathbb{Z}^2} \exp 2\pi i \left[\frac{1}{2} {}^t\left(p + \frac{m'}{2}\right) Z \left(p + \frac{m'}{2}\right) + {}^t\left(p + \frac{m'}{2}\right) \frac{m''}{2} \right].$$

We also put

$$x = (\theta_{0000}^4 + \theta_{0001}^4 + \theta_{0010}^4 + \theta_{0011}^4)/4,$$

$$y = (\theta_{0000}\theta_{0010}\theta_{0001}\theta_{0011})^2,$$

$$z = (\theta_{0100}^4 - \theta_{0110}^4)^2/16384,$$

$$k = (\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111})^2/4096,$$

$$t = (\theta_{0100}\theta_{0110})^4/256.$$

The functions x, y, z, k belong to $A(\Gamma_0(3))$, and t to $A(B(2))$. The weight of each of these forms is 2, 4, 4, 6, or 4, respectively. For convenience, we put

$$r = (x^2 - y - 1024z - 64t)/64.$$

Incidentally, it can be proved that all the Fourier coefficients of these five forms are rational integers. (The proof is omitted here.)

Theorem A. *The modular forms x, y, z, k are algebraically independent. We get*

$$A(\Gamma_0(2)) = C[x, y, z, k],$$

$$\mathrm{Proj}(A(\Gamma_0(2))) \cong P(2, 4, 4, 6), \quad \textit{and}$$

$$K(\Gamma_0(2)) = C\left(\frac{y}{x^2}, \frac{z}{x^2}, \frac{k}{x^3}\right).$$

In particular, the variety $S(\Gamma_0(2)\backslash H_2)$ is rational. The ideal of $A(\Gamma_0(2))$ consisting of cusp forms is spanned by two cusp forms k and yz.

Next, denote by X, Y, Z, K, T, five algebraically independent variables and by J the ideal of the polynomical ring $C[X, Y, Z, K, T]$ spanned by

$$64K^2 + 16XTK + T(-16YZ + X^2T - YT - 1024ZT - 64T^2).$$

Theorem B. *We get*

$$A(B(2)) = C[x, y, z, k, t]$$
$$\cong C[X, Y, Z, K, T]/J, \quad \textit{and}$$
$$K(B(2)) = C\left(\frac{y}{x^2}, \frac{t}{x^2}, \frac{k}{x^2}\right).$$

In particular, $S(B(2)\backslash H_2)$ is rational. The ideal of $A(B(2))$ consisting of cusp forms is spanned by three forms k, yz, and tr.

So far, we considered only modular forms of even weights. Finally, we give results on modular forms of odd weights. We get $A_k(B(2)) = 0$ for odd $k \leq 9$, and $A_k(\Gamma_0(3)) = 0$ for odd $k \leq 17$. Define functions θ, θ', χ_{11}, and χ_{19} on H_2 by:

$$\theta = \theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111},$$
$$\theta' = (\theta_{1000}^{12} - \theta_{1001}^{12} - \theta_{1100}^{12} + \theta_{1111}^{12})/1536,$$
$$\chi_{11} = \theta\theta',$$
$$\chi_{19} = \chi_{11}(8yz - x^2t + yt + 1024zt + 96t^2 - 8xk).$$

Then, we can show that $\chi_{11} \in S_{11}(B(2))$ and $\chi_{19} \in S_{19}(\Gamma_0(2))$.

Theorem C. *For odd $k \geq 11$, we get $A_k(B(2)) = S_k(B(2)) = \chi_{11}A_{k-11}(B(2))$, and for odd $k \geq 19$, we get $A_k(\Gamma_0(2)) = S_k(\Gamma_0(2)) = \chi_{19}A_{k-19}(\Gamma_0(2))$. Besides, we have*

$$\theta^2 = 4096yk, \quad \textit{and}$$
$$\theta'^2 = z(x^4 - 2048x^2z + 1048576z^2 - 64x^2t + 65536tz$$
$$- 2x^2y + y^2 - 2048yz + 12288tr + 64yt + 4096xk).$$

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