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On Certain Vector Valued Siegel Modular Forms of Degree Two

Takakazu Satoh

Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguroku, Tokyo, 152 Japan

Introduction

In this paper we explicitly construct vector valued Siegel modular forms of degree two and the automorphic factor $\det^k \otimes \mathrm{Sym}^2 \mathrm{St}$ for even k where St denotes the standard representation of $\mathrm{GL}(2, \mathbf{C})$. As an application, we prove some congruences between eigenvalues of Hecke operators.

For a positive integer n, let Γ_n be the full Siegel modular group of degree n and H_n the Siegel upper half plane of degree n. For $M \in \Gamma_n$ and $Z \in H_n$, we put

$$M\langle Z \rangle = (AZ+B)(CZ+D)^{-1} \quad \text{where} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let $V(k,r)$ be a representation space of the holomorphic representation $\det^k \otimes \mathrm{Sym}^r \mathrm{St}$ of $\mathrm{GL}(2, \mathbf{C})$. A C^∞-Siegel modular form f of type (k,r) and degree two is a $V(k,r)$ valued C^∞-function on H_2 satisfying the equation

$$f(M\langle Z \rangle) = (\det^k \otimes \mathrm{Sym}^r \mathrm{St})(CZ+D) f(Z)$$

and the usual growth rate condition (see Borel [4, Sect. 7]), which is satisfied for f treated in this paper. We denote by $M^\infty_{k,r}(\Gamma_2)$ the $\mathbf{C}$-vector space of all such functions. We put

$$M_{k,r}(\Gamma_2) = \{ f \in M^\infty_{k,r}(\Gamma_2) | f \text{ is holomorphic on } H_2 \}.$$

If $r=0$, the subscript k, r is abbreviated as k and type (k,r) is mentioned as weight k for simplicity. Let S_2 be the $\mathbf{C}$-vector space of complex symmetric matrices of size two. The action of $G \in \mathrm{GL}(2, \mathbf{C})$ defined by

$$A \to \det(G)^k G A {}^tG \qquad (A \in S_2)$$

is equivalent to $\det^k \otimes \mathrm{Sym}^2 \mathrm{St}$ where ${}^tG$ is the transpose of G. Henceforth, we set $V(k,2) = S_2$. For the variable $Z = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ on H_2 and $f \in M_k(\Gamma_2) = M_{k,0}(\Gamma_2)$, we define the differential operator $\nabla=\nabla_k$ by

$$\nabla f=\frac{k}{2\pi i}(2iY)^{-1}f+\frac{1}{2\pi i}\frac{d}{dZ}f, \tag{0.1}$$

where

$$\frac{d}{dZ}=\begin{pmatrix}\partial_1 & (1/2)\partial_3\\ (1/2)\partial_3 & \partial_2\end{pmatrix} \quad \text{with} \quad \partial_j=\frac{\partial}{\partial z_j}$$

and $Y=\frac{1}{2i}(Z-\bar{Z})$. By Shimura [24, (4.5)], we see that $\nabla f\in M_{k,2}^{\infty}(\Gamma_2)$. For $f\in M_k(\Gamma_2)$ and $g\in M_j(\Gamma_2)$, we put

$$[f,g]=\frac{1}{2\pi i}\left(\frac{1}{j}f\frac{d}{dZ}g-\frac{1}{k}g\frac{d}{dZ}f\right). \tag{0.2}$$

By (0.1), we have

$$[f,g]=\frac{1}{j}f\nabla g-\frac{1}{k}g\nabla f. \tag{0.3}$$

Hence $[f,g]$ is a holomorphic function [by (0.2)] belonging $M_{k+j,2}^{\infty}(\Gamma_2)$ by (0.3), so we obtain $[f,g]\in M_{k+j,2}(\Gamma_2)$. $\left(\text{This also follows from } [f,g]=\frac{1}{jk}(f^{j+1}/g^{k-1})\frac{d}{dZ}(g^k/f^j).\right)$ Our result (Theorem 2.2) is that $\bigoplus_{k:\text{even}} M_{k,2}(\Gamma_2)$ is spanned by $f[g,h]$ where f,g and h are (usual) scalar valued modular forms. There we obtain a minimal generator set over **C**. Unfortunately, $M_{k,2}(\Gamma_2)$ for an odd k is not spanned by such forms. We have $\dim M_{21,2}(\Gamma_2)=1$ by Tsushima [25, Theorem 4] whereas the least odd integer k such that nonzero $f[g,h]$ belongs to $M_{k,2}(\Gamma_2)$ is 39. Using our structure theorem, we prove some congruence formulas between eigenvalues of Hecke operators in Sect. 4. In principle, this is done by comparison of Fourier coefficients. However on congruences between eigen functions of different type, say type $(k,2)$ and weight $k+2$ ($=$type $(k+2,0)$), we cannot compare them immediately. For this purpose, we construct a map from $M_{k,2}(\Gamma_2)$ to $M_{k+2}^{\infty}(\Gamma_2)$ in Sect. 3. This is essentially the particular case considered abstractly in Harris and Jakobsen [9, Sect. 1]. But our result is so explicit that each Fourier coefficient can be computed effectively (and we can prove congruences).

Notation

If R is a ring with 1, we denote by $M(n,R)$ the ring of matrices of size n whose entries belong to R and by $\mathrm{GL}(n,R)$ the unit group of $M(n,R)$. For a holomorphic function f on H_n satisfying $f(Z+S)=f(Z)$ for all $Z\in H_n$ and all symmetric integral matrices S of size n, we denote the Fourier expansion of f by

$$f(Z)=\sum_N a(N,f)\exp(2\pi i\,\mathrm{Tr}(NZ)), \tag{0.4}$$

where N runs over all semi-integral matrices and $a(N,f)$ stands for the Fourier coefficient of f at N.

1. Supplement to the Dimension Formula

We denote by $S_{k,r}(\Gamma_2)$ the space of cusp forms of type (k,r):

$$S_{k,r}(\Gamma_2)=\{f\in M_{k,r}(\Gamma_2)|\Phi f=0\},$$

where Φ is the Siegel-Φ operator. The dimension formula of $S_{k,r}(\Gamma_2)$ for $r=0$ and $k\geqq 4$ or $r\geqq 1$ and $k\geqq 5$ is obtained by Tsushima [25, 26]. Here we show $\dim S_{k,2}(\Gamma_2)=0$ for $k\leqq 6$. The proof is technical and a reader may skip this section except for the statement of Corollary 1.3.

For a complex matrix A of size n, we define its norm $\|A\|$ by $\sqrt{\operatorname{Tr}(A^t\bar{A})}$. The following properties are easily verified.

$\|U^{-1}AU\|=\|A\|$ for a unitary matrix U of size n.

$$\|A\|<\operatorname{Tr}(A) \text{ if } A \text{ is real positive definite.} \tag{1.1}$$

If $S\in M(n,\mathbf{R})$ is a symmetric matrix satisfying $\|S^2-E\|<\varepsilon$ with $0<\varepsilon<1$, then

$$(1-\varepsilon)\,\|A\|\leqq\|SAS\|\leqq(1+\varepsilon)\,\|A\|. \tag{1.2}$$

We denote the identity matrix by E. We put $A^{\otimes 0}=E$ and $A^{\otimes r}=A^{\otimes(r-1)}\otimes A$ for an integer $r\geqq 1$ where $\otimes$ is the Kronecker product.

Proposition 1.1 (Maximum principle). *Let D be a simply connected domain in $\mathbf{C}$ and C a simple closed curve of finite length L contained in D. Let $f: D\to M(n,\mathbf{C})$ be a holomorphic function. Put*

$$M=\sup_{z\in C}\|f(z)\|.$$

If z lies in the inside of C, then

$$\|f(z)\|\leqq M. \tag{1.3}$$

Proof. Let $d>0$ be the distance between C and z. For any positive integer r, we have

$$f(z)^{\otimes r}=\frac{1}{2\pi i}\int_C\frac{1}{w-z}f(w)^{\otimes r}dw$$

by the holomorphy of f, hence,

$$\|f(z)^{\otimes r}\|\leqq\frac{L}{2\pi d}\operatorname*{Max}_{w\in C}\|f(w)^{\otimes r}\|.$$

Since $\|A\otimes B\|=\|A\|\cdot\|B\|$ for square matrices A and B, we obtain

$$\|f(z)\|\leqq\left(\frac{L}{2\pi d}\right)^{1/r}M.$$

Since r is arbitrary, we obtain (1.3). Q.E.D.

Let $\Omega_n=\Gamma_n\backslash H_n$ be the fundamental domain specified in Maass [17, p. 169]. We put

$$\sigma_n=\sup_{Z\in\Omega_n}\operatorname{Tr}(\operatorname{Im}(Z)^{-1}).$$

It is known that σ_n is finite for all $n \geqq 1$ (see Maass [17, p. 178]). For a real positive definite matrix A, we denote by $\sqrt{A}$ a unique positive definite matrix satisfying $(\sqrt{A})^2 = A$. Further, we denote by S_n the set of all complex symmetric matrices of size n.

Theorem 1.2. *Let $n \geqq 2$ be an integer. Let $f: H_n \to S_n$ be a holomorphic function satisfying*

$$f(M\langle Z\rangle) = |CZ+D|^k (CZ+D) f(Z) {}^t(CZ+D) \tag{1.4}$$

for all $Z \in H_n$ and all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. (In other words, f is a holomorphic modular form of type $\det^k \otimes \mathrm{Sym}^2 \mathrm{St}$ and degree n.) If $a(T, f) = 0$ for all $T \geqq 0$ satisfying $\mathrm{Tr}(T) \leqq \frac{k+2}{4\pi} \sigma_n$, then $f = 0$.

Proof. We closely follow Maass [17, pp. 189–194]. As in [17, p. 191], it is sufficient to prove this theorem when $a(T, f) = 0$ unless $T > 0$ (i.e. f is a cusp form). In this case, we obtain

$$\lim_{|Y| \to \infty} \phi(Z) = 0 \quad \text{for} \quad Z = X + iY \in \Omega_n,$$

where

$$\phi(Z) = |Y|^{k/2} \|\sqrt{Y} f(Z) \sqrt{Y}\|,$$

as in [17, p. 192]. We see that $\phi(Z)$ is invariant under Γ_n by (1.4). Therefore $\phi(Z)$ attains its maximum value M at some point $W = U + iV \in \Omega_n$ where U and V are real. Let $z = x + iy \in \mathbf{C}$. Set $Z = W + zE$, $t = \exp(2\pi i z)$ and

$$g(t) = \exp(-i\lambda \mathrm{Tr}(Z)) \sqrt{V} f(Z) \sqrt{V},$$

where λ is defined by

$$\frac{\lambda n}{2\pi} = 1 + \left[\frac{k+2}{4\pi} \sigma_n\right]. \tag{1.5}$$

We have the expansion

$$g(t) = \sqrt{V} \left(\sum_T a(T, f) \exp(2\pi i \mathrm{Tr}(TW) - i\lambda \mathrm{Tr}(W)) t^{\mathrm{Tr}(T) - \frac{\lambda n}{2\pi}} \right) \sqrt{V},$$

where T runs over all semi-integral matrices $T > 0$ satisfying $\mathrm{Tr}(T) \geqq \frac{\lambda n}{2\pi}$. Therefore exponents of t are positive integers. There exists $\varepsilon > 0$ such that $Z \in H_n$ for $y \geqq -\varepsilon$. The function $g(t)$ is holomorphic in a disk $|t| \leqq \varrho$ for some ϱ satisfying $e^\varepsilon > \varrho > 1$. By Proposition 1.1, there exists t with $|t| = \varrho$ such that $\|g(t)\| \geqq \|g(1)\|$, i.e.,

$$|V|^{k/2} \|\sqrt{V} f(Z) \sqrt{V}\| \exp(\lambda \mathrm{Tr}(V + yE)) \geqq M \exp(\lambda \mathrm{Tr}(V)). \tag{1.6}$$

Since $Y = V + yE$, matrices $\sqrt{Y}$ and $\sqrt{V}^{-1}$ commute. Hence $\sqrt{Y}\sqrt{V}^{-1}$ is symmetric and $\|YV^{-1} - E\| = |y| \, \|V^{-1}\|$. Therefore, using (1.2) and maximality of

M, we have

$$|V|^{k/2}\|\sqrt{V}f(Z)\sqrt{V}\| \leqq \frac{1}{1-|y|\,\|V^{-1}\|}|V|^{k/2}\|\sqrt{Y}f(Z)\sqrt{Y}\|$$
$$\leqq \frac{1}{1-|y|\,\|V^{-1}\|}|V|^{k/2}|Y|^{-k/2}M\,. \tag{1.7}$$

Combining (1.6) and (1.7), we obtain

$$M \leqq M\exp(\psi(y)),$$

where $y=-\frac{1}{2\pi}\log\varrho<0$ and

$$\psi(y)=n\lambda y-\log(1-|y|\,\|V^{-1}\|)-\frac{k}{2}\log|E+yV^{-1}|\,.$$

Noting $\psi(0)=0$ and

$$\psi'(0)\geqq n\lambda-\|V^{-1}\|-\frac{k}{2}\operatorname{Tr}(V^{-1})$$
$$>n\lambda-\left(1+\frac{k}{2}\right)\operatorname{Tr}(V^{-1})>0$$

by (1.1) and (1.5), we have $\psi(y)<0$ if $\varrho>1$ is sufficiently close to 1. This proves $M=0$ and consequently $f=0$. Q.E.D.

Corollary 1.3. *Let* $k\leqq 6$ *be an integer. Then* $\dim S_{k,2}(\Gamma_2)=0$.

Proof. Note that $\sigma_2\leqq\frac{16}{3\sqrt{3}}$ from Maass [17, pp. 195–196] and that $\operatorname{Tr}(T)\geqq 2$ for semi-integral $T>0$. Hence this corollary follows from the preceding theorem. Q.E.D.

2. Construction of Vector Valued Modular Forms of Type $(k, 2)$

In this section, we determine the structure of $M_{k,2}(\Gamma_2)$ in Theorem 2.2 and prove the integrality of eigenvalues in Corollary 2.3. Recall that the graded $\mathbf{C}$-algebra $\bigoplus_k M_k(\Gamma_2)$ where k runs over even integers is generated over $\mathbf{C}$ by four algebraically independent elements. [We understand that $M_k(\Gamma_2)=\{0\}$ for a negative k.] They are $\varphi_4\in M_4(\Gamma_2)$, $\varphi_6\in M_6(\Gamma_2)$, $\chi_{10}\in S_{10}(\Gamma_2)$ and $\chi_{12}\in S_{12}(\Gamma_2)$. For an odd k, we have $M_k(\Gamma_2)=\chi_{35}M_{k-35}(\Gamma_2)$ where χ_{35} is a cusp form of weight 35 (see Igusa [10] and Maass [18]). We prepare one lemma concerning linear independency of derivatives of modular forms.

Lemma 2.1. *Let k be an integer. For* $j=4$, 6, 10, *and* 12, *let* $f_j\in M_{k-j}(\Gamma_2)$. *If*

$$f_4\frac{d}{dZ}\varphi_4+f_6\frac{d}{dZ}\varphi_6+f_{10}\frac{d}{dZ}\chi_{10}+f_{12}\frac{d}{dZ}\chi_{12}=0\,, \tag{2.1}$$

then we have

$$f_4 = f_6 = f_{10} = f_{12} = 0\,. \tag{2.2}$$

Proof. In the case of an odd k, all the f_j are divisible by χ_{35}. Therefore it is enough to prove the case of an even k. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2$. For $f \in M_k(\Gamma_2)$, we have

$$\left(\frac{d}{dZ} f\right)(M\langle Z\rangle) = |CZ+D|^k (CZ+D)\left(k(CZ+D)^{-1} Cf(Z) + \frac{d}{dZ} f(Z)\right){}^t(CZ+D)$$

from Shimura [24, (4.2) and (4.3)]. Hence considering (2.1) at $M\langle Z\rangle$, we see that

$$4f_4\varphi_4 + 6f_6\varphi_6 + 10f_{10}\chi_{10} + 12f_{12}\chi_{12} = 0\,. \tag{2.3}$$

From (2.1) and (2.3) we have $P^t(f_4\ f_6\ f_{10}\ f_{12}) = 0$ where

$$P = \begin{pmatrix} 4\varphi_4 & 6\varphi_6 & 10\chi_{10} & 12\chi_{12} \\ \partial_1\varphi_4 & \partial_1\varphi_6 & \partial_1\chi_{10} & \partial_1\chi_{12} \\ \partial_2\varphi_4 & \partial_2\varphi_6 & \partial_2\chi_{10} & \partial_2\chi_{12} \\ \partial_3\varphi_4 & \partial_3\varphi_6 & \partial_3\chi_{10} & \partial_3\chi_{12} \end{pmatrix}$$

with $\partial_i = \dfrac{\partial}{\partial z_i}$. Using values of Fourier coefficients in Resnikoff and Saldaña [20, Tables III, IV, V], we see that the Fourier coefficient of $|P|$ at $\begin{pmatrix} 3 & 1/2 \\ 1/2 & 2 \end{pmatrix}$ is $6912\pi^3 i$. We note that $|P|$ is holomorphic on H_2. Therefore there exists an open domain $\Omega \subset H_2$ such that $|P|(Z) \neq 0$ for $Z \in \Omega$. Then f_j for $j = 4$, 6, 10, and 12 identically vanish on Ω, hence on H_2 by the holomorphy of f_j. Q.E.D.

Theorem 2.2. *For each even integer k, we have (as a* **C***-vector space)*

$$\begin{aligned} M_{k,2}(\Gamma_2) = {} & M_{k-10}(\Gamma_2)[\varphi_4, \varphi_6] \oplus M_{k-14}(\Gamma_2)[\varphi_4, \chi_{10}] \\ & \oplus M_{k-16}(\Gamma_2)[\varphi_4, \chi_{12}] \oplus V_{k-16}(\Gamma_2)[\varphi_6, \chi_{10}] \\ & \oplus V_{k-18}(\Gamma_2)[\varphi_6, \chi_{12}] \oplus W_{k-22}(\Gamma_2)[\chi_{10}, \chi_{12}]\,, \end{aligned} \tag{2.4}$$

where

$$V_k(\Gamma_2) = M_k(\Gamma_2) \cap \mathbf{C}[\varphi_6, \chi_{10}, \chi_{12}]$$

and

$$W_k(\Gamma_2) = M_k(\Gamma_2) \cap \mathbf{C}[\chi_{10}, \chi_{12}]\,.$$

Proof. The inclusion $\supset$ is clear. We show that subspaces appearing in the right hand side of (2.4) are mutually linearly independent. Suppose

$$\begin{aligned} & F_{4,6}[\varphi_4, \varphi_6] + F_{4,10}[\varphi_4, \chi_{10}] + F_{4,12}[\varphi_4, \chi_{12}] \\ & \qquad + F_{6,10}[\varphi_6, \chi_{10}] + F_{6,12}[\varphi_6, \chi_{12}] + F_{10,12}[\chi_{10}, \chi_{12}] = 0 \end{aligned}$$

with

$$F_{4,j} \in M_{k-4-j}(\Gamma_2),\ F_{6,j} \in V_{k-6-j}(\Gamma_2) \quad \text{and} \quad F_{10,12} \in W_{k-22}(\Gamma_2)\,. \tag{2.5}$$

Then we have

$$\tfrac{1}{4}(-F_{4,6}\varphi_6-F_{4,10}\chi_{10}-F_{4,12}\chi_{12})\frac{d}{dZ}\varphi_4$$

$$+\tfrac{1}{6}(F_{4,6}\varphi_4-F_{6,10}\chi_{10}-F_{6,12}\chi_{12})\frac{d}{dZ}\varphi_6 \tag{2.6}$$

$$+\tfrac{1}{10}(F_{4,10}\varphi_4+F_{6,10}\varphi_6-F_{10,12}\chi_{12})\frac{d}{dZ}\chi_{10} \tag{2.7}$$

$$+\tfrac{1}{12}(F_{4,12}\varphi_4+F_{6,12}\varphi_6+F_{10,12}\chi_{10})\frac{d}{dZ}\chi_{12}=0. \tag{2.8}$$

By (2.5), (2.8) and Lemma 2.1, we have $F_{4,12}\varphi_4=-F_{6,12}\varphi_6-F_{10,12}\chi_{12}$, i.e. $F_{4,12}\varphi_4\in\mathbf{C}[\varphi_6,\chi_{10},\chi_{12}]$. This means $F_{4,12}=0$ since φ_4, φ_6, χ_{10}, and χ_{12} are algebraically independent. Then we have $F_{6,12}\varphi_6=-F_{10,12}\chi_{12}$ and $F_{6,12}=F_{10,12}=0$ by the same reason. Similarly, from (2.7) we obtain $F_{4,10}\varphi_4=-F_{6,10}\varphi_6$ and $F_{4,10}=F_{6,10}=0$. Finally, from (2.6) we have $F_{4,6}=0$. Thus linear independency is proved.

Now we show that the equality holds in (2.4). Let d_k be the dimension of the the right hand side of (2.4). By the linear independency we have

$$\sum_{k:\text{even}} d_kT^k=\frac{T^{10}+T^{14}+T^{16}}{(1-T^4)(1-T^6)(1-T^{10})(1-T^{12})}+\frac{T^{16}+T^{18}}{(1-T^6)(1-T^{10})(1-T^{12})}$$
$$+\frac{T^{22}}{(1-T^{10})(1-T^{12})}$$
$$=\frac{T^{10}+T^{14}+2T^{16}+T^{18}-T^{20}-T^{26}-T^{28}+T^{32}}{(1-T^4)(1-T^6)(1-T^{10})(1-T^{12})}, \tag{2.9}$$

where T is an indeterminate. On the other hand, by Arakawa [3, Proposition 1.3] we have

$$M_{k,2}(\Gamma_2)=E_{k,2}(\Gamma_2)\oplus S_{k,2}(\Gamma_2),$$

where $E_{k,2}$ is the space of Eisenstein series of type $(k,2)$ and

$$\sum_{k=0}^{\infty}\dim E_{k,2}(\Gamma_2)T^k=\frac{T^{10}}{(1-T^4)(1-T^6)}. \tag{2.10}$$

By Tsushima [25, Theorem 4] (cf. Tsushima [26, Table 1]) and Corollary 1.3 we obtain

$$\sum_{k:\text{even}}\dim S_{k,2}(\Gamma_2)T^k=\frac{T^{14}+2T^{16}+T^{18}+T^{22}-T^{26}-T^{28}}{(1-T^4)(1-T^6)(1-T^{10})(1-T^{12})}. \tag{2.11}$$

Comparing (2.9), (2.10), and (2.11) we see that $d_k=\dim M_{k,2}(\Gamma_2)$ for each even k, so the right hand side of (2.4) spans the left hand side. Q.E.D.

We recall some properties on a Hecke operator (in a classical language). For each positive integer m, we put

$$G_m=\{M\in M(4,\mathbf{Z})|{}^tMJM=mJ\},$$

where $J=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ with $E=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We define the m-th Hecke operator $T_{k,r}(m)$ acting on $M_{k,r}(\Gamma_2)$ by

$$(T_{k,r}(m)f)(Z)=m^{2k+r-3}\sum_{M\in\Gamma_2\backslash G_m}((\det^k\otimes\mathrm{Sym}^r\mathrm{St})(CZ+D))(f(M\langle Z\rangle)), \tag{2.12}$$

where $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs over (finitely many) complete representatives of the coset $\Gamma_2\backslash G_m$. We shall omit subscripts k, r if there arises no confusion. It is known that we can take representatives satisfying $C=0$ (cf. Andrianov [1, (1.3.12)]). In what follows we always take such representatives. Let $N=\begin{pmatrix} n_1 & n_3/2 \\ n_3/2 & n_2 \end{pmatrix}$ be a symmetric matrix of size two. For integers p and q, we put

$$\tilde{a}(p,q,N,f)=\begin{cases} a(N,f) & \text{if } N \text{ is semi-integral}, \\ & n_1\equiv n_3\equiv 0 \bmod p,\ n_2\equiv 0 \bmod pq, \\ 0 & \text{otherwise}. \end{cases}$$

By the similar method to Andrianov [1, (2.1.11)], we obtain

$$a(N,T_{k,r}(p^\delta)f)=\sum_{\substack{\alpha+\beta+\gamma=\delta\\ \alpha,\beta,\gamma\geqq 0}} p^{(k-2)\beta+(2k+r-3)\gamma} \times\sum_{U\in R(p^\beta)}(\mathrm{Sym}^r\mathrm{St})\left(\begin{pmatrix} p^\beta & 0 \\ 0 & 1 \end{pmatrix}\right)\tilde{a}(p^\alpha,p^\beta,p^{-\delta}D_{\alpha\beta}UN{}^tUD_{\alpha\beta},f), \tag{2.13}$$

where

$$D_{\alpha\beta}=\begin{pmatrix} p^\alpha & 0 \\ 0 & p^{\alpha+\beta} \end{pmatrix},$$

$$R(p^\beta)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \mathrm{SL}(2,\mathbf{Z})\,\middle|\,(c,d) \bmod p^\beta\right\}.$$

A modular form $f\in M_{k,r}^\infty(\Gamma_2)$ is said to be an eigenform if f is a non zero common eigen function of all Hecke operators. Let f be an eigenform. We denote the eigenvalue of $T(m)$ by $\lambda(m,f)$ and put $\mathbf{Q}(f)=\mathbf{Q}(\lambda(m,f)|m\geqq 1)$. For a subring R of $\mathbf{C}$, we put

$$M_k(\Gamma_2)_R=\{f\in M_k(\Gamma_2)|a(N,f)\in R \quad \text{for all} \quad N\geqq 0\},$$
$$M_{k,2}(\Gamma_2)_R=\{f\in M_{k,2}(\Gamma_2)|a(N,f)\in M(2,R) \quad \text{for all} \quad N\geqq 0\}$$

and

$$S_{k,r}(\Gamma_2)_R=S_{k,r}(\Gamma_2)\cap M_{k,r}(\Gamma_2)_R$$

for $r=0$ and 2.

Corollary 2.3. *Let $f\in M_{k,2}(\Gamma_2)$ be an eigenform for an even integer k. Then, $\mathbf{Q}(f)/\mathbf{Q}$ is a totally real finite extension, and the eigenvalues $\lambda(m,f)$ are algebraic integers for all $m\geqq 1$. For a subring R of $\mathbf{C}$, the R module $M_{k,2}(\Gamma_2)_R$ is stable under $T(m)$ for all $m\geqq 1$.*

Proof. Since Hecke operators are Hermitian with respect to the Petersson inner product by Arakawa [3, (2.3)], eigenvalues are real. By Igusa [11] we have $\varphi_4 \in M_4(\Gamma_2)_{\mathbf{Z}}$, $\varphi_6 \in M_6(\Gamma_2)_{\mathbf{Z}}$, $4\chi_{10} \in S_{10}(\Gamma_2)_{\mathbf{Z}}$ and $12\chi_{12} \in S_{12}(\Gamma_2)_{\mathbf{Z}}$. Hence it holds that

$$\begin{aligned} M_{k,2}(\Gamma_2)_{\mathbf{Z}} \supset M_{k-10}(\Gamma_2)_{\mathbf{Z}} 12[\varphi_4, \varphi_6] + M_{k-14}(\Gamma_2)_{\mathbf{Z}} 20[\varphi_4, 4\chi_{10}] \\ + M_{k-16}(\Gamma_2)_{\mathbf{Z}} 12[\varphi_4, 12\chi_{12}] + M_{k-16}(\Gamma_2)_{\mathbf{Z}} 30[\varphi_6, 4\chi_{10}] \\ + M_{k-18}(\Gamma_2)_{\mathbf{Z}} 12[\varphi_6, 12\chi_{12}] + M_{k-22}(\Gamma_2)_{\mathbf{Z}} 60[4\chi_{10}, 12\chi_{12}]. \end{aligned}$$

So, by using Theorem 2.2 we have,

$$M_{k,2}(\Gamma_2)_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C} = M_{k,2}(\Gamma_2). \tag{2.14}$$

Since $f \neq 0$ we obtain $k \geqq 10$ from Theorem 2.2. Using $k \geqq 10$ and (2.13) we see that

$$T(m) M_{k,2}(\Gamma_2)_{\mathbf{Z}} \subset M_{k,2}(\Gamma_2)_{\mathbf{Z}} \quad \text{for all} \quad m \geqq 1. \tag{2.15}$$

We obtain the last assertion from (2.15) and

$$M_{k,2}(\Gamma_2)_R = M_{k,2}(\Gamma_2)_{\mathbf{Z}} \otimes_{\mathbf{Z}} R.$$

The other assertions follow from (2.14) and (2.15) by the same argument as Kurokawa [15, p. 48]. Q.E.D.

Remark 2.4. Let $k \geqq 39$ be an odd integer, R a subring of $\mathbf{C}$. By the same reason as above, we see that $M_{k,2}(\Gamma_2)_R$ is a non-zero R-submodule of $M_{k,2}(\Gamma_2)$ and that $M_{k,2}(\Gamma_2)_R$ is stable under $T(m)$ for all $m \geqq 1$.

3. Nonholomorphic Scalar Valued Realization

In this section, we construct a map from $M_{k,2}(\Gamma_2)$ to $M^{\infty}_{k+2}(\Gamma_2)$ which commutes Hecke operators upto constants. Following Maass [16], we define a differential operator δ_k acting on a C^{∞}-function f on H_2 by

$$\begin{aligned} \delta_k f &= (2\pi i)^{-2} |Y|^{-k+(1/2)} \left| \frac{d}{dZ} \right| (|Y|^{k-(1/2)} f) \\ &= \frac{1}{4} |2\pi Y|^{-1} k\left(k-\frac{1}{2}\right) f - \frac{1}{2i}\left(k-\frac{1}{2}\right) |2\pi Y|^{-1} \operatorname{Tr}\left(Y \frac{d}{dZ} f\right) + \left|\frac{1}{2\pi i}\frac{d}{dZ}\right| f. \end{aligned} \tag{3.1}$$

By Harris [8, 1.5.3], δ_k maps $M^{\infty}_k(\Gamma_2)$ to $M^{\infty}_{k+2}(\Gamma_2)$. We define a subspace $PM^1_k(\Gamma_2)$ of $M^{\infty}_k(\Gamma_2)$ by

$$\begin{aligned} PM^1_k(\Gamma_2) = M_k(\Gamma_2) + \delta_{k-2} M_{k-2}(\Gamma_2) \\ + \{f \delta_j g \mid f \in M_{k-2-j}(\Gamma_2), g \in M_j(\Gamma_2)\}_{\mathbf{C}}, \end{aligned}$$

where $\{\ \}_{\mathbf{C}}$ stands for a $\mathbf{C}$-linear span.

Theorem 3.1. *Let $F \in M_{k,2}(\Gamma_2)$ for an even integer k. Then there exists the unique element $D(F)$ of $PM^1_{k+2}(\Gamma_2)$ satisfying the following conditions (a) and (b):*

(a) *With respect to the Petersson inner product, $D(F)$ lies in the orthogonal complement of $S_{k+2}(\Gamma_2)$ in $PM^1_{k+2}(\Gamma_2)$.*

(b) *The function $H(F)$ defined by*

$$H(F)=D(F)-\tfrac{1}{2}|2\pi Y|^{-1}\,\mathrm{Tr}(2\pi YF) \tag{3.2}$$

is a holomorphic function having Fourier expansion of the following form

$$H(F)(Z)=\sum_{N>0} a(N,H(F))\exp(2\pi i\,\mathrm{Tr}(NZ))\,, \tag{3.3}$$

where N runs over all positive definite semi integral matrices of size two.

Proof. We first prove the uniqueness. Let D_1, $D_2\in PM^1_{k+2}(\Gamma_2)$ be functions satisfying conditions (*a*) and (*b*). Then $D_1-D_2\in PM^1_{k+2}(\Gamma_2)$ is holomorphic on H_2 by (3.2) and moreover is a holomorphic cusp form of weight $k+2$ by (3.3). Using (*a*), we have that

$$D_1-D_2\in S_{k+2}(\Gamma_2)\cap S_{k+2}(\Gamma_2)^{\perp}=\{0\}\,.$$

Hence $D_1=D_2$. By Theorem 2.2, it is enough to show the existence of $D(F)$ when $F=f[g,h]$ for $f\in M_p(\Gamma_2)$, $g\in M_q(\Gamma_2)$, $h\in M_r(\Gamma_2)$ with $k=p+q+r$. Using (3.1), we observe

$$\frac{2}{q(2q-1)}\,h\delta_q g-\frac{2}{r(2r-1)}\,g\delta_r h=\tfrac{1}{2}|2\pi Y|^{-1}\,\mathrm{Tr}(2\pi Y[g,h])+G(h,g)\,, \tag{3.4}$$

where

$$G(h,g)=\frac{2}{q(2q-1)}\,h\left|\frac{1}{2\pi i}\frac{d}{dZ}\right|g-\frac{2}{r(2r-1)}\,g\left|\frac{1}{2\pi i}\frac{d}{dZ}\right|h\,. \tag{3.5}$$

We note that $a(N,G(h,g))=0$ unless $N>0$ [cf. (0.4)]. Put

$$A=\tfrac{1}{2}|2\pi Y|^{-1}\,\mathrm{Tr}(2\pi YF)+fG(h,g)\,.$$

By (3.4), we have $A\in PM^1_{k+2}(\Gamma_2)$. Therefore we can take

$$D(F)=\tfrac{1}{2}|2\pi Y|^{-1}\,\mathrm{Tr}(2\pi YF)+H(F)$$

with

$$H(F)=fG(h,g)-\sum_{i=1}^{d}\frac{\langle A,\psi_1\rangle}{\langle\psi_i,\psi_i\rangle}\,\psi_i\,, \tag{3.6}$$

where $\{\psi_1,\psi_2,\dots,\psi_d\}$ is an eigen basis of $S_{k+2}(\Gamma_2)$ and $\langle\ ,\ \rangle$ is the Petersson inner product. Q.E.D.

Corollary 3.2. *Let k be an even integer. If $F\in M_{k,2}(\Gamma_2)$ is an eigenform, then $D(F)\in PM^1_{k+2}(\Gamma_2)$ is an eigenform satisfying*

$$\lambda(m,D(F))=m\lambda(m,F)$$

for all $m\geqq 1$.

Proof. Since $F\neq 0$, clearly $D(F)\neq 0$. Fix representatives of the coset $\Gamma_2\backslash G_m$ satisfying $C=0$ in (2.12). By Shimura [24, (4.2)], we see that

$$T_{k+2}(m)\,(|2\pi Y|^{-1}\,\mathrm{Tr}(2\pi YF))=m|2\pi Y|^{-1}\,\mathrm{Tr}(2\pi YT_{k,2}(m)F)\,. \tag{3.7}$$

(Hence in fact the left hand side does not depend on the choice of representatives satisfying $C=0$.) Therefore $T(m)D(F)-m\lambda(m,F)D(F)$ is a holomorphic cusp form lying in $S_{k+2}(\Gamma_2)^{\perp}$, hence is zero. Q.E.D.

Remark 3.3. Let $F\in M_{k,2}(\Gamma_2)$ be an eigenform for an even k. Suppose there is a positive integer m satisfying

$$m\lambda(m,F)\neq\lambda(m,\psi) \quad \text{for each eigen form } \psi\in S_{k+2}(\Gamma_2). \tag{3.8}$$

Then the following method is useful in the actual computation of $D(f)$. Write F as the form

$$F=\sum_j f_j[g_j,h_j],$$

where f_j, g_j, and h_j are suitable modular forms (preferably having integral coefficients). Let $\{\psi_1,\psi_2,\ldots,\psi_d\}$ be a basis of $S_{k+2}(\Gamma_2)$. Then $D(F)$ is given by

$$\tfrac{1}{2}|2\pi Y|^{-1}\operatorname{Tr}(2\pi YF)+\sum_j f_jG(h_j,g_j)+\sum_{i=1}^{d}c_i\psi_i, \tag{3.9}$$

where $c_i\in\mathbf{C}$ are uniquely determined so that (3.9) is the eigen function of $T(m)$ with eigenvalue $m\lambda(m,F)$. Since $\sum_{i=1}^{d}c_i\psi_i$ belongs to $S_k(\Gamma_2)$, without loss of generality we may assume that $\{\psi_1,\psi_2,\ldots,\psi_d\}$ is an eigen basis of $S_k(\Gamma_2)$. In this case, the existence follows from (3.6) and uniqueness is shown by

$$m\lambda(m,F)\sum_j f_j[g_j,h_j]-T(m)\sum_j f_j[g_j,h_j]=\sum_{i=1}^{d}c_i(m\lambda(m,F)-\lambda(m,\psi_i))\psi_i.$$

Theorem 3.4. *Let k be even. Let $\mathbf{T}$ be the Hecke ring for Γ_2 over $\mathbf{C}$. As a $\mathbf{T}$ module, we have*

$$PM_k^1(\Gamma_2)=M_k(\Gamma_2)\oplus\delta_{k-2}M_{k-2}(\Gamma_2)\oplus D(M_{k-2,2}(\Gamma_2)). \tag{3.10}$$

Especially, $PM_k^1(\Gamma_2)$ is stable under $\mathbf{T}$.

Proof. Put $\theta_k=(k-\frac{1}{2})^{-1}\delta_k$. Let $f\in M_{k-j-2}(\Gamma_2)$ and $g\in M_j(\Gamma_2)$. By (3.1), we have

$$f\theta g+g\theta f-\theta(fg)\in S_k(\Gamma_2).$$

On the other hand, using (3.4) we have

$$\frac{1}{j}f\theta g-\frac{1}{k-2-j}g\theta f-D([f,g])\in S_k(\Gamma_2).$$

Therefore $f\delta_j g=(j-\frac{1}{2})f\theta g$ belongs to the right hand side of (3.10), so we have (3.10) as a $\mathbf{C}$-vector space using the uniqueness of the Fourier coefficient. We see that $D(M_{k-2,2}(\Gamma_2))$ is stable under $\mathbf{T}$ by Corollary 3.2. Let $\mathbf{A}$ be the adele ring of $\mathbf{Q}$. Then, commutation of Hecke operators acting on functions on $Sp(2,\mathbf{A})$ and differential operators induced from the universal enveloping algebra of $Sp(2,\mathbf{R})$ is well known. Since δ_k corresponds to such a differential operator by Harris [8, Theorem 6.8], we see that for $f\in M_{k-2}(\Gamma_2)$

$$T_k(m)\delta_{k-2}f=m^2\delta_{k-2}T_{k-2}(m)f \tag{3.11}$$

using the correspondence between automorphic functions on $Sp(2,\mathbf{A})$ and modular forms on H_2 (see e.g. Yoshida [27, Sect. 6]). Any way, $\delta_{k-2}M_{k-2}(\Gamma_2)$ is stable under $\mathbf{T}$ by (3.11). Therefore all the $\mathbf{C}$-subspaces appearing in the right hand side of (3.10) is stable under $\mathbf{T}$, so we obtain (3.10) as a $\mathbf{T}$ module. Q.E.D.

4. Congruence Formulas

We prove some congruence formulas between eigenvalues of Hecke operators. Unfortunately, the method is not so systematic as that of Serre [22]. We denote by E_k the Eisenstein series of degree one and weight k normalized as $a(1,E_k)=1$. Let $S_k(\Gamma_1)$ be the space of cusp forms of degree one and weight k. If $\dim S_k(\Gamma_1)=1$, we denote by Δ_k the eigen cusp form of weight k normalized as $a(1,\Delta_k)=1$. For simplicity, we put $E=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $P=\begin{pmatrix}1&1/2\\1/2&1\end{pmatrix}$, $Q=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $\eta_{14}=[\chi_{10},\varphi_4]$.

For a cusp form $f\in S_{k+2}(\Gamma_1)$, we denote by $[f]_2\in M_{k,2}(\Gamma_2)$ the Klingen type Eisenstein series attached to f defined by $[f]_2(Z)=E_{k,2}(Z,f,Q)$ in the notation of Arakawa [3, (1.4)]. If $f\in S_{k+2}(\Gamma_1)$ is an eigenform, we see that $[f]_2$ is determined as the unique eigenform satisfying $(\Phi[f]_2)(z)=f(z)Q$ by the same method as Kurokawa [14, Theorem 2]. We also have

$$\lambda(p,[f]_2)=(1+p^{k-2})\lambda(p,f) \tag{4.1}$$

for a rational prime p by Arakawa [3, Proposition 2.1]. Using Theorem 2.2 we see that an eigen basis of $M_{14,2}(\Gamma_2)$ is $\{[\Delta_{16}]_2,\eta_{14}\}$, while an eigen basis of $S_{16}(\Gamma_2)$ is $\{\chi_{16}^{(+)},\chi_{16}^{(-)}\}$ where

$$\chi_{16}^{(\pm)}=185\cdot 4\chi_{10}\varphi_6+(-128\pm\sqrt{51349})\,12\chi_{12}\varphi_4,$$

respectively by Kurokawa [12, Sect. 3].

Theorem 4.1. *The following congruences hold for all $m\geqq 1$:*

$$\lambda(m,\eta_{14})\equiv\lambda(m,[\Delta_{16}]_2)\bmod 373, \tag{4.2}$$

and

$$\mathrm{N}_{K/\mathbf{Q}}(m\lambda(m,\eta_{14})-\lambda(m,\chi_{16}^{(\pm)}))\equiv 0\bmod 13, \tag{4.3}$$

where $K=\mathbf{Q}(\sqrt{51349})$ and $\mathrm{N}_{K/\mathbf{Q}}$ is the norm map.

Proof. As to (4.2), we follow the proof of Kurokawa [13, Theorem 1]. Since

$$\begin{aligned}\Phi[\varphi_6,\varphi_8]&=\left(\frac{1}{8}E_6\frac{d}{dz}E_8-\frac{1}{6}E_8\frac{d}{dz}E_6\right)Q\\&=144\Delta_{16}Q,\end{aligned}$$

there is a constant c satisfying

$$[\Delta_{16}]_2=\tfrac{1}{144}[\varphi_6,\varphi_8]+c\eta_{14}.$$

Calculating Fourier coefficients of $[\varphi_6,\varphi_8]$ and η_{14} we have

$$a(P,[\Delta_{16}]_2)=(-28+\tfrac{1}{40}c)P$$

and

$$a(2P,[\Delta_{16}]_2)=(-344736-480c)P.$$

Using (4.1) and (2.13) we obtain

$$\begin{aligned}a(P,T(2)[\Delta_{16}]_2)&=(1+2^{12})\lambda(2,\Delta_{16})a(P,[\Delta_{16}]_2)\\&=884952a(P,[\Delta_{16}]_2)\end{aligned}$$

and

$$a(P,T(2)[\Delta_{16}]_2)=a(2P,[\Delta_{16}]_2),$$

respectively. These yield

$$(-344736-480c)P=884952(-28+\tfrac{1}{40}c)P.$$

Consequently, we have

$$\tfrac{1}{144}[\varphi_6,\varphi_8]=[\Delta_{16}]_2-\tfrac{403200}{373}\eta_{14}. \tag{4.4}$$

Considering $\lambda(m,[\Delta_{16}]_2)(4.4)-T(m)(4.4)$, we obtain

$$\begin{aligned}&\lambda(m,[\Delta_{16}]_2)\,a(P,\tfrac{1}{144}[\varphi_6,\varphi_8])-a(P,\tfrac{1}{144}T(m)[\varphi_6,\varphi_8])\\&=-(\lambda(m,[\Delta_{16}]_2)-\lambda(m,\eta_{14}))\,a(P,\tfrac{403200}{373}\eta_{14}).\end{aligned} \tag{4.5}$$

Let $R=\mathbf{Z}_{(373)}$ be the localization of $\mathbf{Z}$ at 373. Since $\varphi_6\in M_6(\Gamma_2)_{\mathbf{Z}}$ and $\varphi_8\in M_8(\Gamma_2)_{\mathbf{Z}}$, we have $144^{-1}[\varphi_6,\varphi_8]\in M_{14,2}(\Gamma_2)_R$. By Corollary 2.3, the left hand side of (4.5) belongs to $M(2,R)$. Since $a(P,403200\eta_{14})=10080P$ belongs to $\mathrm{GL}(2,R)$, we have (4.2).

Let us prove (4.3). We first determine $H(\eta_{14})$ and $D(\eta_{14})$. Noting Remark 3.3, we put

$$4H(\eta_{14})=G(\varphi_4,4\chi_{10})+c_1 4\chi_{10}\varphi_6+c_2 12\chi_{12}\varphi_4$$

[cf. (3.5)]. To determine c_1 and c_2 we use the following table.

N	$a(N,G(\varphi_4,4\chi_{10}))$	$a(N,4\chi_{10}\varphi_6)$	$a(N,12\chi_{12}\varphi_4)$	$a(N,\eta_{14})$
E	2/95	2	10	$(-1/20)E$
$2E$	$-267968/95$	280192	283520	$(6848/5)E$
P	$-3/380$	-1	1	$(1/40)P$
$2P$	11808/19	-47616	23424	$-480P$

Since $a(2P,\eta_{14})=\lambda(2,\eta_{14})a(P,\eta_{14})$, we have $2\lambda(2,\eta_{14})=-38400$. Therefore we put $m=2$ in (3.8). [Note that $\lambda(2,\chi_{16}^{(\pm)})$ are not rational integers.] Using relations

$$a(2P,H(\eta_{14}))=a(P,T(2)H(\eta_{14}))$$

and

$$a(2E,H(\eta_{14}))=a(E,T(2)H(\eta_{14}))-2^{14}a(E,H(\eta_{14}))$$

we obtain

$$-\tfrac{267968}{95}+280192c_1+283520c_2=(-38400-2^{14})(\tfrac{2}{95}+2c_1+10c_2),$$
$$\tfrac{11808}{19}-47616c_1+23424c_2=-38400(-\tfrac{3}{380}-c_1+c_2).$$

Solving these equations, we obtain

$$4H(\eta_{14})=\frac{1}{5\cdot 19}\varphi_4\begin{vmatrix}\frac{1}{2\pi i}\frac{d}{dZ}\end{vmatrix}4\chi_{10}-\frac{1}{14}4\chi_{10}\begin{vmatrix}\frac{1}{2\pi i}\frac{d}{dZ}\end{vmatrix}\varphi_4$$
$$+\frac{1}{4\cdot 5\cdot 13}4\chi_{10}\varphi_6+\frac{1}{4\cdot 5\cdot 13\cdot 19}12\chi_{12}\varphi_4$$

and

$$2^4\cdot 5\cdot 7\cdot 13\cdot 19D(\eta_{14})=2^2\cdot 7\cdot 13\varphi_4\delta_{10}\chi_{10}-2\cdot 5\cdot 13\cdot 19\cdot 4\chi_{10}\delta_4\varphi_4$$
$$-2^2\cdot 13\cdot 4\chi_{10}\varphi_6+(135\mp\sqrt{51349})12\chi_{12}\varphi_6+\chi_{16}^{(\pm)}$$

Since $N_{K/\mathbf{Q}}(135+\sqrt{51349})\equiv 0 \bmod 13$, we have

$$N_{K/\mathbf{Q}}(a(E, 138320H(\eta_{14}))-a(E, \chi_{16}^{(\pm)}))\equiv 0 \bmod 13 \tag{4.6}$$

using the uniqueness of Fourier coefficients. On the other hand, we obtain

$$a(E, T(m)H(\eta_{14}))=m\lambda(m, \eta_{14})a(E, H(\eta_{14})) \tag{4.7}$$

from (3.2) and Corollary 3.2. Hence noting that eigenvalues are algebraic integers, we have

$$N_{K/\mathbf{Q}}((m\lambda(m, \eta_{14})-\lambda(m, \chi_{16}^{(\pm)}))a(E, 138320H(\eta_{14})))\equiv 0 \bmod 13$$

by (4.6) and (4.7). Since $a(E, 138320H(\eta_{14}))=1064\equiv 11 \bmod 13$, we have (4.3). Q.E.D.

With respect to congruences of eigenvalues between eigen cusp forms of type $(k, 2)$ and weight k, we have the following general result. We denote by $\mathbf{Z}(f)$ the integer ring of $\mathbf{Q}(f)$.

Lemma 4.2. *Let $f\in S_k(\Gamma_2)$ be an eigenform. Then there exists an eigenform $F\in S_k(\Gamma_2)_{\mathbf{Z}(f)}$ satisfying*

$$\lambda(m, f)=\lambda(m, F) \quad \textit{for all} \quad m\geqq 1 \tag{4.8}$$

and

$$F\notin S_k(\Gamma_2)_{\ell\mathbf{Z}(f)} \quad \textit{for any prime ideal } \ell \textit{ of } \mathbf{Z}(f). \tag{4.9}$$

Proof. Note that $S_k(\Gamma_2)_{\mathbf{Z}}\otimes_{\mathbf{Z}}\mathbf{C}=S_k(\Gamma_2)$ for all non-negative integers k. By the same argument as the proof of Kurokawa [15, Theorem 2], there exists an eigenform $g\in S_k(\Gamma_2)_{\mathbf{Z}(f)}$ satisfying $\lambda(m, f)=\lambda(m, g)$ for all $m\geqq 1$. Take a semi-integral matrix N of size two such that $a(N, g)\neq 0$. Since $a(N, g)$ is contained in finitely many prime ideals, we can take $c^{-1}g$ as F with a suitable constant $c\in\mathbf{Z}(f)$. Q.E.D.

Theorem 4.3. *Let $F \in S_k(\Gamma_2)$ be an eigenform. Let ℓ_0 be a prime number dividing k satisfying*

$$\ell_0 \neq 2, 3, 5 \quad \text{if } k \text{ is even},$$
$$\ell_0 \neq 5, 7 \quad \text{if } k \text{ is odd}.$$

Let ℓ be a prime ideal of $\mathbf{Z}(F)$ lying above ℓ_0. Then, there exists an eigenform $G \in S_{k,2}(\Gamma_2)$ such that

$$\mathrm{N}_{K(G)/K}(\lambda(m, G) - m\lambda(m, F)) \equiv 0 \bmod \ell \quad \text{for all} \quad m \geqq 1,$$

where $K = \mathbf{Q}(F)$ and $K(G) = K(\lambda(m, G) | m \geqq 1)$.

Proof. Noting Lemma 4.2, we may assume that F satisfies $F \in S_k(\Gamma_2)_{\mathbf{Z}(F)}$ and condition (4.9). Let R be the localization of $\mathbf{Z}(F)$ at ℓ. Suppose $a(N, F) \in \ell R$ for all semi integral positive definite N satisfying $N \not\equiv 0 \bmod \ell_0$ in R. Note that $\lambda(\ell_0^\delta, f)$ is an algebraic integer for each positive integer δ by Corollary 2.3. Using (2.13) with $r = 0$, we obtain

$$a(\ell_0^\delta N, f) = \lambda(\ell_0^\delta, f) a(N, f) - \sum_Q c(Q) a(Q, f),$$

where Q runs over finitely many semi integral positive definite matrices and $c(Q) \in \mathbf{Z}$ is non zero only for $Q \not\equiv 0 \bmod \ell_0^\delta$. Therefore we have $a(\ell_0^\delta N, f) \in \ell R$ for all δ and for all $N \not\equiv 0 \bmod \ell_0$ by induction on δ. This contradicts to (4.9). Hence there exists N satisfying

$$a(N, F) N \not\equiv 0 \bmod \ell. \tag{4.10}$$

In the case of the even k, using $F \in S_k(\Gamma_2)_R$, we can put

$$F = \chi_{10} f_1 + \chi_{12} f_2,$$

where $f_1 \in M_{k-10}(\Gamma_2)_R$ and $f_2 \in M_{k-12}(\Gamma_2)_R$. (See, Igusa [11, Theorem 1].) Then we have

$$\frac{1}{k} \nabla F - \frac{1}{10} f_1 \nabla \chi_{10} - \frac{1}{12} f_2 \nabla \chi_{12} = \frac{k-10}{k} [\chi_{10}, f_1] + \frac{k-12}{k} [\chi_{12}, f_2],$$

namely

$$\nabla F = H + k(\tfrac{1}{10} f_1 \nabla \chi_{10} + \tfrac{1}{12} f_2 \nabla \chi_{12}) \in M_{k,2}^\infty(\Gamma_2), \tag{4.11}$$

where

$$H = (k-10)[\chi_{10}, f_1] + (k-12)[\chi_{12}, f_2] \in M_{k,2}(\Gamma_2)_R.$$

In the case of the odd k, we have

$$\nabla F = H + \frac{k}{35} f \nabla 4i\chi_{35} \in M_{k,2}^\infty(\Gamma_2) \tag{4.12}$$

with

$$H = (k-35)[4i\chi_{35}, f] \in M_{k,2}(\Gamma_2)_R,$$

where $f \in M_{k-35}(\Gamma_2)_R$ is determined by $F = 4i\chi_{35} f$. In both cases, by the conditions on ℓ_0 and (4.10)–(4.12) we have

$$a(N, H) \equiv a(N, F) N \not\equiv 0 \bmod \ell. \tag{4.13}$$

So the coefficient wise reduction mod ℓ of H is non-zero. Using the same method to the proof of (3.7), we see that ∇F is an eigenform with $\lambda(m, \nabla F) = m\lambda(m, F)$ for all $m \geqq 1$. Hence by (4.13) we see that H is an common eigen function of $T(m)$ mod ℓ for all $m \geqq 1$ in the sense of Deligne and Serre [7, 6.6]. Since H is a cusp form, the existence of the desired cusp form G follows from Deligne and Serre [7, Lemme 6.11] and Corollary 2.3 and Remark 2.4. Q.E.D.

As an example, let $F = \chi_{14} \in S_{14}(\Gamma_2)$, $K = \mathbf{Q}$, $\ell = 7$ and $R = \mathbf{Z}_{(7)}$. Here $\chi_{14} = \varphi_4 \chi_{10}$ is the eigen cusp form of weight 14. Then $G = \eta_{14}$ since $\dim S_{14,2}(\Gamma_2) = 1$ and we have

$$\lambda(m, \eta_{14}) \equiv m\lambda(m, \chi_{14}) \bmod 7.$$

In this case we have moreover

$$\lambda(m, \eta_{14}) \equiv m\lambda(m, \chi_{14}) \bmod 35 \tag{4.14}$$

using

$$\nabla 4\chi_{14} - \tfrac{7}{2} 4\chi_{10} \nabla \varphi_4 = -10 \cdot 4\eta_{14}$$

and $a(N, \varphi_4) \equiv 0 \bmod 240$ for all non-zero semi integral N.

We notice some interpretation concerning the above congruences. Let $f \in S_k(\Gamma_1)$ be an eigenform. We denote by $\mathbf{Z}(f)$ the integer ring of $\mathbf{Q}(f)$ and by $\mathbf{Z}(f)_\ell$ its ℓ adic completion where ℓ is a prime ideal of $\mathbf{Z}(f)$. Let ℓ_0 be the rational prime satisfying $\ell \cap \mathbf{Z} = (\ell_0)$. Then there exists an ℓ-adic representation attached to f

$$\varrho_\ell(f) : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}(2, \mathbf{Z}(f)_\ell)$$

in the sense of Deligne [6]. We may conjecture that for each eigenform $f \in M_{k,r}(\Gamma_2)$, there exists an ℓ-adic representation attached to f

$$\varrho_\ell(f) : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}(4, \mathbf{Z}(f)_\ell).$$

We note that we can take

$$\varrho_\ell([f]_2) = \varrho_\ell(f) \oplus \chi_\ell^{k-2} \otimes \varrho_\ell(f) \quad \text{for each eigenform } f \in S_{k+2}(\Gamma_1)$$

and

$$\varrho_\ell(\sigma_k(f)) = \chi_\ell^{k-2} \oplus \chi_\ell^{k-1} \oplus \varrho_\ell(f) \quad \text{for each eigenform } f \in S_{2k-2}(\Gamma_1),$$

where $\sigma_k : M_{2k-2}(\Gamma_1) \to M_k(\Gamma_2)$ is the Saito-Kurokawa lifting (cf. Kurokawa [12] and Andrianov [2]) and χ_ℓ is the cyclotomic ℓ_0-adic character. Assume the existence of $\varrho_\ell(\eta_{14})$. [In this case $\mathbf{Q}(\eta_{14}) = \mathbf{Q}$ and ℓ is a rational prime.] Then, for example, congruences (4.2) and (4.14) are ascribed to

$$\tilde{\varrho}_{373}(\eta_{14}) \cong \tilde{\varrho}_{373}(\Delta_{16}) \oplus \tilde{\chi}_{373}^{14} \otimes \tilde{\varrho}_{373}(\Delta_{16})$$

and

$$\tilde{\varrho}_\ell(\eta_{14}) \cong \tilde{\chi}_\ell^{15} \oplus \tilde{\chi}_\ell^{16} \oplus \tilde{\chi}_\ell \otimes \tilde{\varrho}_\ell(\Delta_{26}) \quad \text{with} \quad \ell = 5 \text{ or } 7,$$

where $\tilde{\varrho}_\ell(f)$ and $\tilde{\chi}_\ell$ are reduction modulo ℓ of $\varrho_\ell(f)$ and χ_ℓ, respectively. In particular, the image of $\tilde{\varrho}_\ell(\eta_{14})$ would not be so large for $\ell = 5, 7$, and 373. See Serre

[21, 22] for the elliptic modular case. Moreover the congruence (4.2) would be related to a special value of the second L-function of Δ_{16}. Let $f \in S_k(\Gamma_1)$ be an eigen form, $L_2(s,f)$ the second L-function attached to f and $\langle f, f\rangle$ its Petersson inner product normalized as in Shimura [23, (2.1)]. Put

$$L_2^*(s,f)=L_2(s,f)\,(2\pi)^{-(2s-k+2)}\Gamma(s)/\langle f,f\rangle .$$

Then, $L_2^*(s,f)$ belongs to $\mathbf{Q}(f)$ for an even integer s with $k \leqq s \leqq 2k-2$ by Zagier [28, Theorem 2]. Using this theorem we have

$$L_2^*(28,\Delta_{16})=\frac{2^9\cdot 373}{3^2 5^2 7^2 11}.$$

In the computation, we use the following value of the function $H(r,N)$ defined by Cohen [5, Sect. 2]: $H(13,0)=-657931/12$, $H(13,3)=111202/3$ and $H(13,4)=2702765/2$. Here we note $28=2(k+r)-2-r$ with $k=14$ and $r=2$. More generally we expect that $L_2^*(2(k+r)-2-r,f)$ appears in the denominator of Fourier coefficients of $E_{k,r}(Z,f,v_0)$ with suitable choice of v_0 in Arakawa [3, (1.4)]. We notice that the case $r=0$ is proved in Mizumoto [19] (cf. Kurokawa [13]).

Appendix: Eigenvalues of η_{14}

Using a computer calculation we obtain six eigenvalues of η_{14} in the following table. From this table we see that Euler factors of the Andrianov's L-function $L(s,\eta_{14})$ (see Arakawa [3, p. 173]) at 2, 3, and 5 satisfy the Ramanujan conjecture. We first compute Fourier coefficients of φ_4 and χ_{10} by the method of Maass [18], then those of $\eta_{14}=[\chi_{10},\varphi_4]$. In the computation of eigenvalues, we employ the device of Kurokawa [12, Sect. 7] to reduce amount of a computation. (The program is about 1000 line long using the C-language.)

m	$\lambda(m,\eta_{14})$
2	−19200
3	2251800
4	35454976
5	−311252100
9	−4797957991599
25	1336090571170425625

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Note added in proof. Prof. T. Oda pointed out to the author that the proof of Theorem 2.2 also gives the structure of $S_{k,2}(\Gamma_2)$ for an even k. The result is:

$$S_{k,2}(\Gamma_2) = S_{k-10}(\Gamma_2)\,[\varphi_4, \varphi_6] \oplus M_{k-14}(\Gamma_2)\,[\varphi_4, \chi_{10}] \oplus M_{k-16}(\Gamma_2)\,[\varphi_4, \chi_{12}]$$
$$\oplus V_{k-16}(\Gamma_2)\,[\varphi_6, \chi_{10}] \oplus V_{k-18}(\Gamma_2)\,[\varphi_6, \chi_{12}] \oplus W_{k-22}(\Gamma_2)\,[\chi_{10}, \chi_{12}]\,.$$