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On Relations of Dimensions of Automorphic Forms of $Sp(2, \boldsymbol{R})$ and Its Compact Twist $Sp(2)$ (I)

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Let p be a fixed prime. In the previous paper [9], we have given some examples and conjectures on correspondence between automorphic forms of $Sp(2, \boldsymbol{R})$ (size four) and $Sp(2)=\{g \in \boldsymbol{H}; g\,{}^t\bar{g}=1_2\}$ ($\boldsymbol{H}$: Hamilton quaternions) which preserves L-functions, where the p-adic closures of the discrete subgroups (to which automorphic forms belong) are minimal parahoric. This was an attempt to a generalization of Eichler's correspondence between $SL_2(\boldsymbol{R})$ and $SU(2)$. Ihara raised such a problem for symplectic groups and Langlands [15] has given a quite general philosophy on correspondence of automorphic forms of any reductive groups (functoriality with respect to L-groups). In this paper, we give good global dimensional relations of automorphic forms of $Sp(2, \boldsymbol{R})$ and $Sp(2)$, when the p-adic closures of discrete subgroups in question are maximal compact. (As for similar results for other groups, see [8].) More precisely, put

$$K(p)=Sp(2, \boldsymbol{Q}) \cap \gamma M_4(\boldsymbol{Z})\gamma^{-1}$$
$$=Sp(2, \boldsymbol{Q}) \cap \begin{pmatrix} * & * & */p & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & * & * & * \end{pmatrix}, \quad \text{where } \gamma=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $*$'s run through all integers. For any $\Gamma \subset Sp(2, \boldsymbol{R})$, denote by $A_k(\Gamma)$ (resp. $S_k(\Gamma)$) the space of automorphic (resp. cusp) forms belonging to Γ. We shall calculate the dimension of $S_k(K(p))$ for all primes p (Theorem 4 in § 4). By comparing these with those of certain automorphic forms (i.e., certain spherical functions) of $Sp(2)$, we shall show certain interesting relations of dimensions (Theorem 1 below). Some philosophical aspects of relations of orbital integrals have been explained in Langlands [16]. But except for the case of GL_n, or the product of its copies, as far as I know, this is the first global result concerning on the comparison of

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dimensions of spaces of automorphic forms belonging to different $\boldsymbol{R}$-forms of a complex Lie group. We propose a precise conjecture on the correspondence of these spaces which is suggested by these relations (Conjecture 1.4). (Some examples of pairs of automorphic forms whose Euler 3-factors fit this conjecture have been given in [9].) In a sense, the situation is fairly different from the case of GL_2. For example, it is noteworthy that, nevertheless the discrete subgroups in question are 'maximal', some 'old forms' come in these spaces. This is *not* because there exist some forms obtained by the Saito-Kurokawa lifting. To state the relation more explicitly, we need some more notations. Let B be the definite quaternion algebra with the prime discriminant p, O a maximal order of B. Put $B_p = B\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$ and $O_p = O\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$. Put

$$G=\{g \in M_2(B);\ g\,{}^t\bar{g}=n(g)1_2,\ n(g) \in \boldsymbol{Q}_+^\times\}.$$

Let G_A be the adelization of G, and G_∞ (resp. G_q) be the infinite (resp. q-adic) component of G_A. For any open subgroup U of G_A, denote by $\mathfrak{M}_\nu(U)$ the space of automorphic forms on G_A belonging to U with 'weight ρ_ν', where ρ_ν is the irreducible representation of $Sp(2)$ which corresponds to the Young diagram $\begin{array}{|c|c|c|}\hline 1 & \cdots & \nu \\ \hline 1 & \cdots & \nu \\ \hline \end{array}$ (cf. Ihara [11], Hashimoto [5]). We take an open subgroup $U_2 = G_\infty U_p^2 \prod_{q\neq p} U_q^1$ of G_A, where $U_q^1 = GL_2(O_q)\cap G_q$, and U_q^2 is the unit group of the right order of a maximal left O_p-lattice in the non principal genus in the quaternion hermitian space B_p^2 with the metric $n(x)+n(y)$ for $(x, y) \in B_p^2$, where $n(*)$ is the reduced norm of B. (cf. § 1). Put

$$\Gamma_0(p)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\boldsymbol{Z});\ c\equiv 0 \bmod p\right\}.$$

Theorem 1. *For each integer $k\geqq 5$ and each prime integer p, we have the following relation of the dimensions*:

$$\begin{aligned}&\dim S_k(K(p))-2\dim S_k(Sp(2, \boldsymbol{Z}))\\ &\qquad =\dim \mathfrak{M}_{k-3}(U_2)-\dim A_2(\Gamma_0(p))\times \dim S_{2k-2}(SL_2(\boldsymbol{Z})).\end{aligned}$$

The conjectural meaning of this Theorem will be explained in Section 1. The dimension of $S_k(Sp(2, \boldsymbol{Z}))$ has been known by Igusa [12], and the dimension of $\mathfrak{M}_{k-3}(U_2)$ has been given in [7] (II). So, only $\dim S_k(K(p))$ is to be calculated. Recently, Hashimoto [6] obtained a general (but not explicit) formula of dimensions of cusp forms belonging to any discrete subgroups Γ of $Sp(2, \boldsymbol{R})$. Roughly spoken, his assertion is as follows: apparently, we have to calculate the contribution of each Γ-conjugacy class to the dimension, but at least for the semi-simple conjugacy classes, we can calculate everything from some data on integral property of their local conjugacy classes in $Sp(2, \boldsymbol{Q}_p)$ and $Sp(2, \boldsymbol{R})$ (so, in these cases, we can avoid the classification of Γ-conjugacy classes), and besides, for all conjugacy classes, 'local data' at the infinite place can be explicitly written down. (As for the further details such as 'family', see his paper.) But in order to obtain the dimensions explicitly by using his formula, we must calculate such local data (the number of 'optimal embeddings' and some local masses) of semi-simple conjugacy classes, and classify $K(p)$-conjugacy classes of parabolic type or some mixed type. (Since $K(p)$ is not contained in $Sp(2, \boldsymbol{Z})$, there were no known results on such classification.) These calculations are rather elaborate and have been done in somewhat lengthy case by case process similar to [7], and here, we shall often omit the proofs, or content ourselves with some sketchy proofs. (As for an expository review on results in [5], [6], [7] how to calculate dimensions in general, confer [8], § 4.) In Section 2, we give local data of semi-simple conjugacy classes. In Section 3, we classify $K(p)$-conjugacy classes of parabolic or mixed type. In Section 4, we sum up them and prove Theorem 1.

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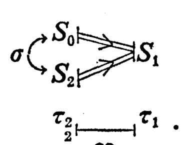
§ 1. Conjectural meaning of Theorem 1

To explain the situation more clearly, we recall some local theory of p-adic algebraic groups (cf. Tits [18]). The extended Dynkin diagram for G_p can be obtained from the one for $Sp(2, \boldsymbol{Q}_p)$ by dividing by the non trivial graph automorphism σ, and each vertex can be regarded as a double coset of a minimal parahoric subgroups. The diagrams are given as follows: (See C_2 and ${}^2C_2$ in the table of [16], p. 64.)



These double cosets are explicitly given as follows: put

$$B(p)=\left\{g\in Sp(2,\boldsymbol{Z}): g\equiv\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix} \bmod p\right\} \quad (*\text{: integers})$$

and let $B(p)_p$ be the p-adic closure of $B(p)$. Then, $B(p)_p$ is an Iwahori subgroup of $Sp(2, \boldsymbol{Q}_p)$. We can take

$$S_0=B(p)_p w_0 B(p)_p, \quad S_1=B(p)_p w_1 B(p)_p, \quad \text{and} \quad S_2=B(p)_p w_2 B(p)_p,$$

where

$$w_0=\begin{pmatrix} 0 & 0 & -p^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad w_1=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{and}$$

$$w_2=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

On the other hand, put

$$G_p^*=\left\{g\in M_2(B_p): g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}{}^t\bar{g}=n(g)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, n(g)\in \boldsymbol{Q}_p^\times\right\}.$$

Then, $G_p^*\cong G_p$. We fix such an isomorphism and regard subgroups of G_p as those of G_p^* if necessary. Put

$$U_p^0=\begin{pmatrix} O_p & O_p \\ \pi O_p & O_p \end{pmatrix}^\times \cap G_p^*,$$

where π is a prime element of O_p such that $\pi^2=p$. Then, U_p^0 is a minimal parahoric subgroup of G_p, and we can take

$$\tau_2=U_p^0=\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}U_p^0, \qquad \tau_1=U_p^0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}U_p^0.$$

There are three maximal compact subgroups (up to conjugation) in $Sp(2, \boldsymbol{Q}_p)$, that is,

$$K(p)_p=B(p)_p\cup S_0\cup S_2\cup S_0S_2, \quad Sp(2,\boldsymbol{Z}_p), \quad \text{and} \quad \rho Sp(2,\boldsymbol{Z}_p)\rho^{-1},$$

where

$$\rho=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

Among these, only $K(p)_p$ is invariant by σ, and the group which 'corresponds' with $K(p)_p$ by 'folding' is $U_p^2 = U_p^0 \cup \tau_2$. So, it is natural to consider that there exists some good correspondence between $S_k(K(p))$ and $\mathfrak{M}_{k-3}(U_2)$. But, in spite of the fact that these are 'maximal' groups, we must subtract the 'old forms' from each space. Now, we shall explain this. We intend to regard the cusp forms in $S_k(K(p))$ obtained 'from' $S_k(Sp(2, Z)) + S_k(\rho Sp(2, Z)\rho^{-1})$ as old forms. But $K(p)$ is not conjugate to $Sp(2, Z)$ or $\rho Sp(2, Z)\rho^{-1}$, and is not contained in, or does not contain any of these groups. So, we must define some mapping between these spaces. Define $\mathrm{Tr}_{K(p)/B(p)}$: $S_k(B(p)) \to S_k(K(p))$ by:

$$\mathrm{Tr}_{K(p)/B(p)}(f) = \Big(\sum_{\gamma \in B(p)\backslash K(p)} f\,|\,[\gamma]_k\Big)/[K(p)\colon B(p)]$$

for any $f \in S_k(B(p))$, where $f\,|\,[\gamma]_k = f(\gamma z) \det(Cz+D)^{-k}$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, Q)$: Denote by Tr the restriction of $\mathrm{Tr}_{K(p)/B(p)}$ on $S_k(Sp(2, Z)) + S_k(\rho Sp(2, Z)\rho^{-1})$. We define new forms of $S_k(K(p))$ to be the orthogonal complement of $\mathrm{Tr}(S_k(Sp(2, Z)) + S_k(\rho Sp(2, Z)\rho^{-1})$ in $S_k(K(p))$, and denote it by $S_k^0(K(p))$. The map Tr does not vanish in general. For example, we have

Lemma 1.2. *Let* $f \in S_k(Sp(2, Z))$ *be an eigen form of the Hecke operators* $T(p)$ *and* $T(p^2)$ *with eigenvalues* $\lambda(p)$ *and* $\lambda(p^2)$*, respectively. Assume that* $\lambda(p) \neq 0$ *or* $\lambda(p^2) \neq p^{2k-2}$. *(For example, this is satisfied for all eigen forms of the Maass space* M_k.*) Then* $\mathrm{Tr}(f) \neq 0$.

The proof consists of an easy argument on Fourier coefficients, and will be omitted here. In view of the Ramanujan Conjecture, it is very plausible that the assumption of Lemma 1.2 is always satisfied. On the other hand, the map Tr is not injective in general:

Lemma 1.3. *Let* k *be an even integer. Then, for* $f \in M_k$, *we have* $\mathrm{Tr}(f) = \mathrm{Tr}(f\,|\,[\rho]_k)$.

The proof is easy and omitted here. It seems that, if k is odd, then Tr is injective, and if k is even, then $\ker \mathrm{Tr} = \{f - f\,|\,[\rho]_k ; f \in M_k\}$. If this is true, we have $\dim S_k^0(K(p)) = \dim S_k(K(p)) - 2 \dim S_k(Sp(2, Z))$ for odd k, and $\dim S_k^0(K(p)) = \dim S_k(K(p)) - 2 \dim S_k(Sp(2, Z)) + \dim S_{2k-2}(SL_2(Z))$ for even k. (Numerical examples in [9] support this.) On the other hand, we can show that, if a common eigen form $f \in \mathfrak{M}_\nu(U_2)$ satisfies a certain condition, then $L(s, f) = L(s, g)L(s, h)$ for some $g \in A_2(\Gamma_0(p))$ and $h \in S_{2\nu+4}(SL_2(Z))$. (This is a slight modification of Ihara [13].) So, denote by $\mathfrak{M}_\nu^E(U_2)$ the space spanned by common eigen forms $f \in \mathfrak{M}_\nu(U_2)$ such that $L(s,f)=L(s,g)L(s,h)$ (up to Euler p-factors) for some $g \in A_2(\Gamma_0(p))$ and $h \in S_{2k-2}(SL_2(Z))$. We define the space of new forms of $\mathfrak{M}_\nu(U_2)$ to be the orthogonal complement of $\mathfrak{M}_\nu^E(U_2)$ in $\mathfrak{M}_\nu(U_2)$. Theorem 1 and some examples seem to suggest that

$$\dim \mathfrak{M}_\nu^0(U_2)=\dim \mathfrak{M}_\nu(U_2)-\dim A_2(\Gamma_0(p))\times \dim S_{2\nu+4}(SL_2(Z))$$

for even ν, and

$$\dim \mathfrak{M}_\nu^0(U_2)=\dim \mathfrak{M}_\nu(U_2)-\dim S_2(\Gamma_0(p))\times \dim S_{2\nu+4}(SL_2(Z))$$

for odd ν.

Conjecture 1.4. *For any integer $k\geqq 5$, there exists an isomorphism ϕ of $\mathfrak{M}_{k-3}^0(U_2)$ onto $S_k^0(K(p))$ such that $L(s,f)=L(s,\phi(f))$ (up to Euler p-factors) for any common eigen form $f \in \mathfrak{M}_{k-3}^0(U_2)$ of all the Hecke operators $T(n)$ $(n\neq p)$.*

Now, we point out one important fact. There exist some *new* forms of $S_k(K(p))$ which can be obtained by lifting cusp forms in $S_{2k-2}(\Gamma_0(p))$ (see examples in [9]). So, also in the case of $\mathfrak{M}_\nu(U_2)$, it seems more natural to define new forms in the same point of view as in the case of $S_k(K(p))$. Put $U_p^1=GL_2(O_p)\cap G_p^*$. Put

$$U_1=G_\infty \prod_q U_q^1, \quad \text{and} \quad U_0=G_\infty U_p^0 \prod_{q\neq p} U_p^1.$$

The 'trace map' Tr_{U_2/U_0} of $\mathfrak{M}_\nu(U_0)$ to $\mathfrak{M}_\nu(U_2)$ can be defined as before. Denote the orthogonal complement of $\mathrm{Tr}_{U_2/U_0}(\mathfrak{M}_\nu(U_1))$ in $\mathfrak{M}_\nu(U_2)$ by $\mathfrak{M}_\nu^1(U_2)$. (We note here that U_p^1 is not conjugate to U_p^2, which causes the difference from the case of SL_2.) Then, it seems natural to expect $\mathfrak{M}_\nu^0(U_2)=\mathfrak{M}_\nu^1(U_2)$. In representation theoretic language, our conjecture seems to be stated as follows: Let $\pi=\otimes\pi_q$, or $\pi'=\otimes\pi_q'$ be an irreducible (admissible) automorphic representation of $GSp(2, Q_A)$, or G_A, respectively. (Here, GSp means the group of symplectic similitudes.) Assume that π_∞ corresponds to $\det^{\nu+3}$, π_∞' to ρ_ν, and that π_q or π_q' $(q\neq p, \infty)$ has a $Sp(2, Z_q)$-fixed vector. Further, assume that π_p has a $K(p)_p$-fixed vector, but no $Sp(2, Z_p)$- or $\rho Sp(2, Z_p)\rho^{-1}$-fixed vector, and that π_p' has a U_p^2-fixed vector, but no U_p^1-fixed vector. Let A (resp. B) be the set of all such π (resp. π'). Then, there exists a bijection φ: $A\to B$ such that $L(s,\pi)=L(s,\varphi(\pi))$?

§ 2. Semi-simple conjugacy classes

In this section, we shall give 'local data' at p of semi simple conjugacy classes, then, give their contribution to dim $S_k(K(p))$ as Theorem 2. (The local data at $q \neq p$ have been given in [7].) The proofs are lengthy and elaborate but similar technique can be found in [7], and we will omit them here. We review some notations. Put

$$R = \gamma M_4(\mathbf{Z}_p)\gamma^{-1}, \quad \text{where } \gamma = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & 1 \end{pmatrix},$$

and put

$$GSp = \left\{ g \in M_4(\mathbf{Q}_p);\ g\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}{}^t g = n(g)\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \right\}.$$

Let $R^\times$ be the invertible elements of R. For $g \in GSp$, let $Z(g)$ be the commutor algebra of $\mathbf{Q}_p(g)$ in $M_4(\mathbf{Q}_p)$. For any $\mathbf{Z}_p$-order Λ_1, Λ_2 of $Z(g)$, write $\Lambda_1 \sim \Lambda_2$ when $a^{-1}\Lambda_1 a = \Lambda_2$ for some $a \in Z(g) \cap GSp$. For any torsion element $g \in GSp$ and $\mathbf{Z}_p$-order $\Lambda \subset Z(g)$, put $c_p(g, R, \Lambda)$ = the number of elements of $M(g, \Lambda)$, where $M(g, \Lambda) = (Z(g) \cap GSp)\backslash M(g, R, \Lambda)/R^\times$ and $M(g, R, \Lambda) = \{x \in GSp;\ x^{-1}gx \in R^\times,\ Z(g) \cap xRx^{-1} \sim \Lambda\}$. In the following sentences, we always denote by $f(x)$ the principal polynomial of the elements in conjugacy classes treated there.

Proposition 2.1. *The total contribution of* $\pm 1 \in K(p)$ *to* $\dim S_k(K(p))$ *is given by*:

$$(p^2+1)(2k-2)(2k-3)(2k-4)/2^9 3^3 5.$$

Proof. Obvious, because $[Sp(2, \mathbf{Z}): B(p)] = (p^2+1)(p+1)^2$ and $[K(p): B(p)] = (p+1)^2$. q.e.d.

Proposition 2.2. *The representatives of* $K(p)/\{\pm 1\}$*-conjugacy classes with* $f(x) = (x-1)^2(x+1)^2$ *are given by*:

$$\delta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \delta_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The contribution $t(\delta_1)$, $t(\delta_2)$ *of each conjugacy class to* $\dim S_k(K(p))$ *for* $k \geqq 5$ *is given by*:

$$t(\delta_1) = (-1)^k(2k-2)(2k-4)/2^8 3^2,$$

$$t(\delta_2) = \begin{cases} (-1)^k(2k-2)(2k-4)/2^7 3, & \text{if } p \neq 2, \\ (-1)^k(2k-2)(2k-4)/2^9, & \text{if } p = 2. \end{cases}$$

Next, we treat the case where $f(x)=(x-1)^2 g(x)$ and $g(x)$ is an irreducible quadratic polynomial. Put $F=\boldsymbol{Q}[x]/g(x)$. We identify the algebra $M_2(\boldsymbol{Q}_p)\times M_2(\boldsymbol{Q}_p)$ with the algebra

$$\left\{\begin{pmatrix} a & 0 & b & 0 \\ 0 & x & 0 & y \\ c & 0 & d & 0 \\ 0 & z & 0 & w \end{pmatrix};\ a, b, c, d, x, y, z, w \in \boldsymbol{Q}_p\right\}.$$

Put $g=\left(\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \omega\right)$, where $\omega=\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 1\end{pmatrix}$, or $\begin{pmatrix} 0 & -1 \\ 1 & -1\end{pmatrix}$ for $f(x)=(x-1)^2(x^2+1)$, $(x-1)^2(x^2+x+1)$, or $(x-1)^2(x^2-x+1)$, respectively.

Proposition 2.3. *Let notations be as above.*

(i) *If* $\left(\dfrac{F}{p}\right)=1$, *then*

$$c_p(g, R, \Lambda)=\begin{cases} 2\cdots \text{if } \Lambda\sim M_2(\boldsymbol{Z}_p)\oplus \boldsymbol{Z}_p^2, \\ 0\cdots \text{otherwise.}\end{cases}$$

(ii) *If* $\left(\dfrac{F}{p}\right)=-1$, *or* $p=3$ *and* $f(x)=(x-1)^2(x^2-x+1)$, *then*

$$c_p(g, R, \Lambda)=\begin{cases} 2\cdots \text{if } \Lambda\sim M_2(\boldsymbol{Z}_p)\oplus \boldsymbol{Z}_p=\Lambda_1, \\ 0\cdots \text{otherwise,}\end{cases}$$

$$\text{and } M(g, \Lambda_1)=\left\{1_4, \begin{pmatrix} 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0\end{pmatrix}\right\}.$$

(iii) *If* $p=3$ *and* $f(x)=(x-1)^2(x^2+x+1)$, *then*

$$c_p(g, R, \Lambda)=\begin{cases} 2\cdots \text{if } \Lambda\sim \Lambda_1=M_2(\boldsymbol{Z}_p)\oplus \boldsymbol{Z}_p^2, \\ 1\cdots \text{if } \Lambda\sim\Lambda_2, \\ 0\cdots \text{otherwise,}\end{cases}$$

where

$$\Lambda_2=\left\{\left(\begin{pmatrix} a & 3b \\ c & d\end{pmatrix}, x+y\omega\right);\ a, b, c, d, x, y \in \boldsymbol{Z}_3,\ x+y\equiv d \bmod 3\right\},$$

$[\Lambda_1\cap GSp : \Lambda_2\cap GSp]=6$, *and* $M(g, \Lambda_1)$ *is as in* (ii),

$$M(g, \Lambda_2)=\left\{\begin{pmatrix} -3 & 3 & -1 & 0 \\ 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1\end{pmatrix}\right\}.$$

(iv) *If* $\left(\dfrac{F}{p}\right)=0$ *and* $p=2$,

$$c_p(g, R, \Lambda)=\begin{cases}2\cdots \textit{if } \Lambda\sim\Lambda_1=M_2(\boldsymbol{Z}_p)\oplus \boldsymbol{Z}_p[\omega],\\ 1\cdots \textit{if } \Lambda\sim\Lambda_2,\\ 0\cdots \textit{otherwise},\end{cases}$$

where

$$\Lambda_2=\left\{\left(\begin{pmatrix} a & 2b\\ c & d\end{pmatrix}, x+y\omega\right);\ a, b, c, d, x, y\in \boldsymbol{Z}_2,\ x-y\equiv d \bmod 2\right\},$$

$[\Lambda_1\cap GSp\colon \Lambda_2\cap GSp]=3$, *and* $M(g, \Lambda_1)$ *is as in* (ii),

$$M(g, \Lambda_2)=\left\{\begin{pmatrix} -2 & 2 & 1 & 0\\ 2 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 1\end{pmatrix}\right\}.$$

Next, we treat the case where $f(x)=g(x)^2$ and $g(x)$ is an irreducible quadratic polynomial. First, we treat the case where $Z_0(g)$ is split. (As for the notation $Z_0(g)$, see [7] (I), § 2.) Put $F=\boldsymbol{Q}[x]/g(x)$.

Proposition 2.4. *Let assumptions be as above,*

(i) *If* $\left(\frac{F}{p}\right)=-1$, *then* $c_p(g, R, \Lambda)=0$ *for any* Λ.

(ii) *If* $\left(\frac{F}{p}\right)=1$, *take* $g=\begin{pmatrix} a1_2 & 0\\ 0 & b1_2\end{pmatrix}$, *where* $a, b\in \boldsymbol{Q}_p$ *and* $g(x)=(x-a)(x-b)$, *then*

$$c_p(g, R, \Lambda)=\begin{cases}1\cdots \textit{if } \Lambda\sim\Lambda_1=M_2(\boldsymbol{Z}_p)\oplus\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}^{-1}M_2(\boldsymbol{Z}_p)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix},\\ 0\cdots \textit{otherwise},\end{cases}$$

$[GSp\cap GL_2(\boldsymbol{Z}_p)^2\colon \Lambda_1\cap GSp]=p+1$, *where we embed* $M_2(\boldsymbol{Q}_p)^2$ *in* $M_4(\boldsymbol{Q}_p)$ *diagonally.*

(iii) *If* $\left(\frac{F}{p}\right)=0$ *and* $p=2$, *take* $g=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 1 & 0 & 0 & -1\end{pmatrix}$, *then*

$$c_p(g, R, \Lambda)=\begin{cases}1\cdots \textit{if } \Lambda\sim\Lambda_1=xRx^{-1}\cap Z(g),\\ 1\cdots \textit{if } \Lambda\sim\Lambda_2=yRy^{-1}\cap Z(g),\\ 0\cdots \textit{otherwise},\end{cases}$$

where

$$x=\begin{pmatrix} 1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ -1 & 1 & -1 & -1\\ 1 & 0 & 0 & -1\end{pmatrix}\quad \textit{and}\quad y=\begin{pmatrix} 4 & 0 & 1 & 2\\ -4 & 4 & 1 & 0\\ -4 & 4 & -3 & -4\\ 4 & 0 & 1 & -2\end{pmatrix},$$

$d_2(\Lambda_1)=3$, $e_2(\Lambda_1)=2$, $d_2(\Lambda_2)=6$, $e_2(\Lambda_2)=2$.

(iv) *If* $\left(\frac{F}{p}\right)=0$ *and* $p=3$, *take* $g=\begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$, *then*

$$c_p(g, R, \Lambda)=\begin{cases} 1 \cdots \textit{if } \Lambda \sim \Lambda_1 = xRx^{-1} \cap Z(g), \\ 1 \cdots \textit{if } \Lambda \sim \Lambda_2 = yRy^{-1} \cap Z(g), \\ 0 \cdots \textit{otherwise}, \end{cases}$$

where

$$x=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad y=\begin{pmatrix} 3 & 0 & 0 & 1 \\ 3 & 3 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$d_3(\Lambda_1)=1$, $e_3(\Lambda_1)=1$, $d_3(\Lambda_2)=8$, $e_3(\Lambda_2)=2$, *where* $d_p(\Lambda)$ *and* $e_p(\Lambda)$ *are as in* [7] (I), *Proposition* 12.

Next, we treat the case where $Z_0(g)$ is division. Then, $\left(\frac{F}{p}\right)\neq 1$ by definition of $Z_0(g)$.

Proposition 2.5.

(i) *If* $\left(\frac{F}{p}\right)=-1$, *take* $g=\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\omega\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1}, \omega\right)$,

where ω *are as in Proposition* 2.3, *and* g *are regarded as elements of GSp as in Proposition* 2.3. *Then,*

$$c_p(g, R, \Lambda)=\begin{cases} 1 \cdots \textit{if } \Lambda \sim \Lambda_1, \\ 0 \cdots \textit{otherwise}, \end{cases}$$

and $d_p(\Lambda_1)=e_p(\Lambda_1)=1$, *where*

$$\Lambda_1=\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} ; A, B, C, D \in Z_p[\omega] \subset M_2(Z_p)\right\}.$$

(ii) *If* $\left(\frac{F}{p}\right)=0$ *and* $p=2$, *take* $g=\begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}$, *then*

$$c_p(g, R, \Lambda)=\begin{cases} 1 \cdots \textit{if } \Lambda \sim \Lambda_1 = xRx^{-1} \cap Z(g), \\ 1 \cdots \textit{if } \Lambda \sim \Lambda_2 = yRy^{-1} \cap Z(g), \\ 0 \cdots \textit{otherwise}, \end{cases}$$

where

$$x=\begin{pmatrix} p1_2 & 1_2 \\ 0 & 1_2 \end{pmatrix}, \quad y=\begin{pmatrix} 2 & 0 & 0 & 1 \\ -2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$d_2(\Lambda_1)=6$, $e_2(\Lambda_1)=2$, $d_2(\Lambda_2)=1$, *and* $e_2(\Lambda_2)=2$.

(iii) *If* $\left(\frac{F}{p}\right)=0$ *and* $p=3$, *take* $g=\pm\begin{pmatrix} 0 & 1_2 \\ -1_2 & 1_2 \end{pmatrix}$, *then,*

$$c_p(g, R, \Lambda)=\begin{cases} 1\cdots \textit{if } \Lambda\sim\Lambda_1=xRx^{-1}\cap Z(g), \\ 0\cdots \textit{otherwise}, \end{cases}$$

where

$$x=\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \textit{and} \quad d_3(\Lambda_1)=4,\ e_3(\Lambda_1)=2.$$

Next, we treat the regular elements $g \in K(p)$. When $Z[g]$ is the maximal order of $Q[g]$, it is fairly easy to classify global conjugacy classes. We sketch it here. Let $\zeta \in Sp(2, Z)$ be an element whose principal polynomial is $f(x)=(x^2+1)(x^2\pm x+1)$, $x^4\pm x^3+x^2\pm x+1$, x^4+1, or x^4-x^2+1. (It exists and we fix it.) When $f(x)=(x^2+1)(x^2\pm x+1)$, more explicitly, put

$$\zeta=\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Put $J=\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$. Assume that $g^{-1}\zeta g \in K(p)$ for some $g \in GL_4(Q)$. Then, $\zeta(gJ^tgJ^{-1})=(gJ^tgJ^{-1})\zeta$, and $gJ^tgJ^{-1} \in Q(\zeta)$. The map $Q(\zeta) \ni h \mapsto J^thJ^{-1} \in Q(\zeta)$ is the complex conjugation on $Q(\zeta)$, and gJ^tgJ^{-1} is invariant by this map. So, $gJ^tgJ^{-1} \in Q(\zeta+\zeta^{-1})$. Put

$$\gamma=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, $\gamma^{-1}g^{-1}\zeta g\gamma \in M_4(Z)$. Now, the class number of $Q(\zeta)$ is one. So, by virtue of Chevalley [2], $ag\gamma \in GL_4(Z)$ for some $a \in Q(\zeta)$.

Lemma 2.6. *Let $f(x)$ be one of the above polynomials. Then, the set of $K(p)$-conjugacy classes with principal polynomial $f(x)$ corresponds bijectively to the set*

$$\{\alpha/p;\ \alpha\in \boldsymbol{Z}[\zeta+\zeta^{-1}],\ N(\alpha)=\pm p\}/N_{\boldsymbol{Q}(\zeta)/\boldsymbol{Q}(\zeta+\zeta^{-1})}(\boldsymbol{Z}[\zeta]^{\times})$$

The map is given by:

$$\{g^{-1}\zeta g;\ g\gamma\in GL_4(\boldsymbol{Z})\}\longrightarrow gJ^t gJ^{-1}.$$

Proof. The injectivity is obvious. The surjectivity is proved by case by case process. q.e.d.

Proposition 2.7. *The numbers of $K(p)$-conjugacy classes of above types are given as follows*:

$$(x^2+1)(x^2\pm x+1)\ \cdots\cdot 8$$

$$x^4+1 \qquad \cdots\begin{cases} 0\cdots \textit{if } \left(\dfrac{F}{p}\right)=-1,\\ 4\cdots \textit{if } \left(\dfrac{F}{p}\right)=0,\\ 8\cdots \textit{if } \left(\dfrac{F}{p}\right)=1,\end{cases}$$

$x^4+x^3+x^2+x+1$, *and*
$x^4-x^3+x^2-x+1$ $\quad\cdots$ *same as in* x^4+1,

$$x^4-x^2+1 \qquad \cdots\begin{cases} 0\cdots \textit{if } \left(\dfrac{F}{p}\right)=-1,\\ 2\cdots \textit{if } \left(\dfrac{F}{p}\right)=0,\\ 4\cdots \textit{if } \left(\dfrac{F}{p}\right)=1,\end{cases}$$

where $F=\boldsymbol{Q}(\zeta+\zeta^{-1})$.

Next, we treat the case where $f(x)=(x^2+x+1)(x^2-x+1)$. In this case, $\boldsymbol{Z}[x]/f(x)$ is not the maximal order, and we give the local data instead of giving global conjugacy classes. Put $F=\boldsymbol{Q}[x]/(x^2+x+1)$. Put

$$g=\begin{pmatrix} a & 0 & 0 & 0\\ 0 & -a & 0 & 0\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & -b\end{pmatrix}, \quad \text{where } \left(\frac{F}{p}\right)=1,$$

where $f(x)=(x^2-a^2)(x^2-b^2)$, $a,\ b\in \boldsymbol{Q}_p$,

$$g_1=\begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&-1&0\\0&-1&0&1\end{pmatrix},\quad g_2=\begin{pmatrix}0&0&-1/p&0\\0&0&0&1\\p&0&-1&0\\0&-1&0&1\end{pmatrix},\quad \text{when } \left(\frac{F}{p}\right)=-1,$$

and

$$g_1=\begin{pmatrix}0&0&-1/p&0\\0&0&0&1\\p&0&-1&0\\0&-1&0&1\end{pmatrix},\quad g_2=\begin{pmatrix}0&0&1/p&0\\0&0&0&1\\-p&0&-1&0\\0&-1&0&1\end{pmatrix},\quad \text{when } p=3.$$

Proposition 2.8.

(i) *If* $\left(\frac{F}{p}\right)=1$, *then*

$$c_p(g, R, \Lambda)=\begin{cases}2\cdots \textit{if } \Lambda\sim \boldsymbol{Z}_p^4\\ 0\cdots \textit{otherwise},\end{cases}$$

where $\boldsymbol{Z}_p^4$ *is embedded diagonally in* $M_4(\boldsymbol{Z}_p)$, *and*

$$M(g, \boldsymbol{Z}_p^4)=\left\{1_4, \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right\}.$$

(ii) *If* $\left(\frac{F}{p}\right)=-1$ *then,*

$$c_p(g_1, R, \Lambda)=0 \textit{ for any } \Lambda, \textit{ and}$$

$$c_p(g_2, R, \Lambda)=\begin{cases}2\cdots \textit{if } \Lambda\sim o_p,\\ 0\cdots \textit{otherwise},\end{cases}$$

where o_p *is the maximal order of* $F_p=\boldsymbol{Q}_p(g_2)$, *and*

$$M(g_2, o_p)=\left\{1_4, \begin{pmatrix}0&1&0&0\\p&0&0&0\\0&0&0&p\\0&0&1&0\end{pmatrix}\right\}.$$

(iii) *If* $\left(\frac{F}{p}\right)=0$ $(p=3)$, *then*

$$c_p(g_i, R, \Lambda)=\begin{cases}2\cdots \textit{if } \Lambda\sim o_p,\\ 0\cdots \textit{otherwise}, \quad \textit{for } i=1, 2,\end{cases}$$

where o_p *and* $M(g_i, o_p)$ *are the same as in* (ii).

Now, denote by H_i the total contribution to dim $S_k(K(p))$, of those semi-simple conjugacy classes whose principal polynomials are of the form $f_i(\pm x)$, where the polynomials $f_i(x)$ are defined as in [7] (I), p. 590. We can give H_i explicitly as a corollary to the above results by using [7] and Hashimoto [6].

Theorem 2. *Assume that* $k\geqq 5$, *then* H_1 *and* H_2 *have been given in Proposition* 2.1, 2.2, *and* H_i $(i\geqq 3)$ *are given as follows*:

$$H_3=\begin{cases}[k-2,\ -k+1,\ -k+2,\ k-1;\ 4]/2^4 3, & \cdots \textit{if } p\neq 2,\\ 5[k-2,\ -k+1,\ -k+2,\ k-1;\ 4]/2^5 3, & \cdots \textit{if } p=2,\end{cases}$$

$$H_4=\begin{cases}[2k-3,\ -k+1,\ -k+2;\ 3]/2^2 3^3, & \cdots \textit{if } p\neq 3,\\ 5[2k-3,\ -k+1,\ -k+2;\ 3]/2^2 3^3, & \cdots \textit{if } p=3,\end{cases}$$

$$H_5=[-1,\ -k+1,\ -k+2,\ 1,\ k-1,\ k-2;\ 6]/2^2 3^2,$$

$$H_6=\begin{cases}\dfrac{5(2k-3)(p+1)}{2^7 3}+\dfrac{(-1)^k(p+1)}{2^7} & \cdots \textit{if } p\equiv 1 \bmod 4,\\[2ex] \dfrac{(2k-3)(p-1)}{2^7}+\dfrac{5(-1)^k(p-1)}{2^7 3} & \cdots \textit{if } p\equiv 3 \bmod 4,\\[2ex] \dfrac{3(2k-3)}{2^7}+\dfrac{7(-1)^k}{2^7 3} & \cdots \textit{if } p=2,\end{cases}$$

$$H_7=\begin{cases}\dfrac{(2k-3)(p+1)}{2\cdot 3^3}+\dfrac{(p+1)}{2^2 3^3}[0,\ -1,\ 1;\ 3] & \cdots \textit{if } p\equiv 1 \bmod 3,\\[2ex] \dfrac{(2k-3)(p-1)}{2^2\cdot 3^3}+\dfrac{(p-1)}{2\cdot 3^3}[0,\ -1,\ 1;\ 3] & \cdots \textit{if } p\equiv 2 \bmod 3,\\[2ex] \dfrac{5(2k-3)}{2^2 3^3}+\dfrac{1}{3^3}[0,\ -1,\ 1;\ 3] & \cdots \textit{if } p=3,\end{cases}$$

$$H_8=[1,0,0,-1,-1,-1,-1,0,0,1,1,1;12]/2\cdot 3,$$

$$H_9=\begin{cases}2[1,0,0,-1,0,0;6]/3^2 & \cdots \textit{if } p\neq 2,\\ [1,0,0,-1,0,0;6]/2\cdot 3^2 & \cdots \textit{if } p=2,\end{cases}$$

$$H_{10}=\left(1+\left(\frac{5}{p}\right)\right)[1,0,0,-1,0;5]/5,$$

$$H_{11}=\left(1+\left(\frac{2}{p}\right)\right)[1,0,0,-1;4]/2^3, \quad \textit{and}$$

$$H_{12}=\begin{cases}[0,\ 1,\ -1;\ 3]/2\cdot 3 & \cdots \textit{if}\ p\equiv 1 \bmod 12,\\ (-1)^k/2\cdot 3 & \cdots \textit{if}\ p\equiv 11 \bmod 12,\\ (-1)^k/2^2\cdot 3 & \cdots \textit{if}\ p=2,\ 3,\\ 0 & \cdots \textit{if}\ p\equiv 5,\ 7 \bmod 12,\end{cases}$$

where $\left(\frac{*}{p}\right)$ *is the Legendre symbol, and* $t=[t_0, t_1, \cdots, t_{q-1}; q]$ *means that* $t=t_j$ *if* $k\equiv j \bmod q$.

§ 3. Conjugacy classes of non-semi-simple types

In this section, we shall give the representatives of non semi-simple $K(p)$-conjugacy classes which have non-zero contribution to $\dim S_k(K(p))$, and give their contribution to $\dim S_k(K(p))$ $(k\geqq 5)$. Put

$$P_0=\left\{\begin{pmatrix} A & B\\ 0 & D\end{pmatrix}\in Sp(2, Q)\right\} \quad \text{and}$$

$$P_1=\left\{\begin{pmatrix} * & 0 & * & *\\ * & * & * & *\\ * & 0 & * & *\\ 0 & 0 & 0 & *\end{pmatrix}\in Sp(2, Q)\right\}.$$

Lemma 3.1. *Assume that* $g\in Sp(2, Q)$ *is not semi-simple. Then, some* $Sp(2, Q)$*-conjugate of* g *is contained in* P_0 *or* P_1.

As for the proof, see Borel-Tits [1]. Next two lemmata are easy and the proof will be omitted.

Lemma 3.2. *The Satake compactification of* $K(p)\backslash Sp(2, \boldsymbol{R})$ *has a unique zero-dimensional cusp and two one-dimensional cusps, that is*

$$\begin{aligned} Sp(2, Q)&=K(p)P_0\\ &=K(p)P_1\cup K(p)\begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}P_1.\end{aligned}$$

Lemma 3.3. *Assume that* $g\in K(p)$ *is not semi-simple. Then, some* $K(p)$*-conjugate of* g *is contained in* P_0, P_1, *or* P_1', *where*

$$P_1'=\left\{\begin{pmatrix} * & * & * & *\\ 0 & * & * & *\\ 0 & 0 & * & 0\\ 0 & * & * & *\end{pmatrix}\in Sp(2, Q)\right\}.$$

By this Lemma, we can assume that $g \in P_0$, P_1, or P_1'. Then, by case by case direct calculations, we can give a complete list of $K(p)$-conjugacy classes which are not semi-simple and which have contribution to $\dim S_k(K(p))$. The proofs are lengthy but routine, and will be omitted here.

Theorem 3. *The representatives of $K(p)$-conjugacy classes which are of elliptic/parabolic, δ-parabolic, parabolic, or paraelliptic (in the sense of Hashimoto [6]), are given in the following list, together with their contribution to* $\dim S_k(K(p))$ $(k \geqq 5)$. *The contribution to* $\dim S_k(K(p))$, *of each set of conjugacy classes below, is denoted by I_i.*

(I) *Elliptic/parabolic*

(1) $f(x)=(x-1)^2(x^2-x+1)$ *and* $(x+1)^2(x^2+x+1)$,

$$\pm\begin{pmatrix} 0 & 0 & 1/p & 0 \\ 0 & 1 & 0 & n \\ -p & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \pm\begin{pmatrix} 1 & 0 & -1/p & 0 \\ 0 & 1 & 0 & n \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pm\begin{pmatrix} 1 & 0 & n/p & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad \pm\begin{pmatrix} 1 & 0 & n/p & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (n \in Z,\ n \neq 0)$$

The total contribution of the above conjugacy classes to $\dim S_k(K(p))$ *is given by*:

$I_1=[0, 1, 1, 0, -1, -1; 6]/6$,

(2) $f(x)=(x-1)^2(x^2+x+1)$ *and* $(x+1)^2(x^2-x+1)$

(i) $$\pm\begin{pmatrix} 0 & 0 & -1/p & 0 \\ 0 & 1 & 0 & n \\ p & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \pm\begin{pmatrix} -1 & 0 & 1/p & 0 \\ 0 & 1 & 0 & n \\ -p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pm\begin{pmatrix} 1 & 0 & n/p & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad \pm\begin{pmatrix} 1 & 0 & n/p & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (n \in Z,\ n \neq 0):$$

$I_2=[-2, 1, 1; 3]/2\cdot 3^2$,

(ii) $$\pm\begin{pmatrix} 1 & 0 & n/p & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix} \quad \pm\begin{pmatrix} 1 & 0 & -n/p & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

$$\pm\begin{pmatrix}0&0&-1/p&0\\0&1&1&n\\p&0&-1&p\\0&0&0&1\end{pmatrix}\qquad\pm\begin{pmatrix}-1&0&1/p&-1\\0&1&1&-n\\-p&0&0&-p\\0&0&0&1\end{pmatrix}$$

$(n\in Z,\ n\neq 0 \textit{ if } p\neq 3 \textit{ and } n\neq -1 \textit{ if } p=3)$:

$$I_3=\begin{cases}[-2,1,1;3]/3^2 \cdots \textit{if } p=3,\\ 2[-1,1,0;3]/3^2\cdots \textit{if } p\equiv 1 \bmod 3,\\ 2[-1,0,1;3]/3^2\cdots \textit{if } p\equiv 2 \bmod 3,\end{cases}$$

(3) $f(x)=(x-1)^2(x^2+1)$ *and* $(x+1)^2(x^2+1)$,

$$\pm\begin{pmatrix}0&0&1/p&0\\0&1&0&n\\-p&0&0&0\\0&0&0&1\end{pmatrix}\qquad\pm\begin{pmatrix}0&0&-1/p&0\\0&1&0&n\\p&0&0&0\\0&0&0&1\end{pmatrix}$$

$$\pm\begin{pmatrix}1&0&n/p&0\\0&0&0&1\\0&0&1&0\\0&-1&0&0\end{pmatrix}\qquad\pm\begin{pmatrix}1&0&n/p&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix}\qquad (n\in Z,\ n\neq 0),$$

$$\pm\begin{pmatrix}0&0&1/p&0\\0&1&1&n\\-p&0&0&-p\\0&0&0&1\end{pmatrix}\qquad\pm\begin{pmatrix}0&0&-1/p&0\\0&1&1&-n\\p&0&0&p\\0&0&0&1\end{pmatrix}$$

$$\pm\begin{pmatrix}1&0&n/p&1\\0&0&0&1\\0&0&1&0\\0&-1&-1&0\end{pmatrix}\qquad\pm\begin{pmatrix}1&0&-n/p&1\\0&0&0&-1\\0&0&1&0\\0&1&1&0\end{pmatrix}$$

$(n\in Z,\ n\neq 0 \textit{ if } p\neq 2 \textit{ and } n\neq 1 \textit{ if } p=2)$:

$I_4=[-1,1,1,-1;4]/2^2$,

(II) *δ-parabolic*: $f(x)=(x-1)^2(x+1)^2$

(i) $$\pm\begin{pmatrix}1&0&n/p&0\\0&-1&0&m\\0&0&1&0\\0&0&0&-1\end{pmatrix}\qquad\pm\begin{pmatrix}1&0&n/p&-1\\0&-1&1&m\\0&0&1&0\\0&0&0&-1\end{pmatrix}$$

$(n, m\in Z,\ n\neq 0,\ m\neq 0)$:

$I_5=(-1)^k/2^3$,

(ii) $$\begin{pmatrix}1&1&n/p&m\\0&-1&m&-2m\\0&0&1&0\\0&0&1&-1\end{pmatrix}\qquad\begin{pmatrix}1&1&n/p&m-1\\0&-1&m&-2m+1\\0&0&1&0\\0&0&1&-1\end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 2m/p & m \\ -p & 1 & m-1 & n \\ 0 & 0 & -1 & -p \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & (2m-1)/p & m \\ -p & 1 & m-1 & n \\ 0 & 0 & -1 & -p \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

($m, n \in Z$, *and* $(2n+pm, -2m)$, $(4n+p(2m-1), -2m+1)$, $(2m, 2n-pm)$, or $(2m-1, 4n-p(2m-1))$, *is not equal to* $(0, 0)$, *respectively.*):

$$I_6=(-1)^k\left(2-\left(\frac{-1}{p}\right)\right)\Big/2^4,$$

(iii) $$\pm\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & S \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where $S=\begin{pmatrix} n/p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}, \begin{pmatrix} n/p & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix}$ $(n \in Z, n\neq 0)$

$I_7=-(-1)^k(2k-3)/2^3 3.$

(III) *Parabolic*: $f(x)=(x-1)^4$ *and* $(x+1)^4$

(1) $\pm\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}$; $S=\begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}, \begin{pmatrix} n/p & 0 \\ 0 & 0 \end{pmatrix}$ $(n \in Z, n\neq 0)$,

$I_8=-p(2k-3)/2^4\cdot 3^2.$

Next, put $L=\left\{\begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{pmatrix}; s_1 \in p^{-1}Z, s_{12}, s_2 \in Z\right\}$ *and for* $S_1, S_2 \in L$, *write* $S_1 \sim S_2$ *when* $S_1=US_2{}^tU$ *for some* $U \in \Gamma_0(p) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Gamma_0(p)$.

(2) $\pm\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}$; $S \in \{S \in L, \det S \in (Q^\times)^2\}/\sim$,

$I_9=-1/2^3 3$

(3) $\pm\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}$; $S \in \{S \in L, S$ *definite*$\}/\sim$,

$I_{10}=(p+1)/2^3 3,$

(4) $\pm\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}$; $S \in \{S \in L, S$ *indefinite*, $\det S \in (Q^\times)^2\}/\sim$,

(*the contribution to the dimension is zero*),

(IV) *Paraelliptic*:

Put

$$g(d)=\begin{pmatrix}1&0&0&0\\ d&1&0&0\\ 0&0&1&-d\\ 0&0&0&1\end{pmatrix},$$

where d is some integer.

(1) $f(x)=(x^2+1)^2$:

(i) *If* $\left(\frac{-1}{p}\right)=-1$, *there exists none in* $K(p)$,

(ii) *if* $\left(\frac{-1}{p}\right)=1$, *then*

$$g(d)^{-1}\begin{pmatrix}\begin{matrix}0&-1\\1&0\end{matrix}&S\\ \begin{matrix}0&0\\0&0\end{matrix}&\begin{matrix}0&-1\\1&0\end{matrix}\end{pmatrix}g(d),$$

$$S=\begin{pmatrix}0&-n\\n&0\end{pmatrix},\ \begin{pmatrix}1&n\\-n&1\end{pmatrix}\quad (n\in \boldsymbol{Z},\ n\neq 0),$$

$$\begin{pmatrix}0&-n\\n+1&0\end{pmatrix},\ \begin{pmatrix}1&n+1\\-n&1\end{pmatrix}\quad (n\in \boldsymbol{Z}),$$

where d runs through a set of the representatives in $\boldsymbol{Z}$ *of the solutions of* $d^2+1\equiv 0 \bmod p$, *and*

(iii) *if* $p=2$, *then*

$$g(1)^{-1}\begin{pmatrix}\begin{matrix}0&-1\\1&0\end{matrix}&S\\ \begin{matrix}0&0\\0&0\end{matrix}&\begin{matrix}0&-1\\1&0\end{matrix}\end{pmatrix}g(1),$$

$$S=\begin{pmatrix}0&-n\\n&0\end{pmatrix},\ \begin{pmatrix}2^{-1}&2^{-1}-n\\2^{-1}+n&-2^{-1}\end{pmatrix}\quad (n\in \boldsymbol{Z},\ n\neq 0),$$

$$\begin{pmatrix}0&-n\\n+1&0\end{pmatrix},\ \begin{pmatrix}2^{-1}&-2^{-1}-n\\2^{-1}+n&-2^{-1}\end{pmatrix}\quad (n\in \boldsymbol{Z})$$

$$I_{11}=-\left(1+\left(\frac{-1}{p}\right)\right)\Big/8.$$

(2) $f(x)=(x^2+x+1)^2$ *and* $(x^2-x+1)^2$:

(i) *If* $\left(\frac{-3}{p}\right)=-1$, *then, there exists none in* $K(p)$,

(ii) *if* $\left(\frac{-3}{p}\right)=1$, *then,*

$$\pm g(d)^{-1}\begin{pmatrix} 0 & -1 & \multicolumn{2}{c}{S} \\ 1 & -1 & & \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} g(d),$$

$$S=\begin{pmatrix} -n & -2n \\ n & -n \end{pmatrix} \quad (n \in Z,\ n\neq 0),$$

$$\begin{pmatrix} -n & -2n \\ n+1 & -n \end{pmatrix}, \begin{pmatrix} -n & -2n \\ n+2 & -n \end{pmatrix} \quad (n \in Z),$$

where d runs through a set of the representatives of the solutions of $x^2+x+1\equiv 0 \bmod p$, *and*

(iii) *if $p=3$, then, besides the above conjugacy classes in* (ii) (*here, we put* $d=1$), *there exist following conjugacy classes*:

$$\begin{pmatrix} 1 & -1 & \multicolumn{2}{c}{B} \\ 3 & -2 & & \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \text{where } B=\begin{pmatrix} -m-2e-h & -3m-6e-h \\ 0 & -3m-6e+h \end{pmatrix},$$

$e=\pm 1/3$, $h=0$, ± 1, *and m is any integer such that* $3m+6e+h\neq 0$:

$$I_{12}=-\left(1+\left(\frac{-3}{p}\right)\right)\Big/6.$$

§ 4. Proof of Theorem 1

In this section, we prove Theorem 1. First, we get

Theorem 4. *For any integer $k\geqq 5$ and any prime p, we have*

$$\dim S_k(K(p))=\sum_{i=1}^{12} H_i+\sum_{i=1}^{12} I_i,$$

where H_i or I_i is given in Theorem 2 *or Theorem* 3, *respectively.*

By virtue of [7] and Igusa [12], our Theorem 1 is a corollary to Theorem 4. But it is interesting to see the details of contribution of each conjugacy classes. We denote by J_i the contribution to

$$\dim S_k(K(p))-2\dim S_k(Sp(2, Z))-\dim \mathfrak{M}_{k-3},$$

of those *semi-simple* conjugacy classes whose principal polynomials are of the form $f_i(\pm x)$ $(i=1, \cdots, 12)$. (As for the notations $f_i(x)$, see [7], p. 590, e.g., $f_6(x)=(x^2+1)^2$, $f_7(x)=(x^2+x+1)^2$, and $f_{12}(x)=x^4-x^2+1$.) We get the following result.

Numerical examples of dim $S_k(K(p))$

p \ k	5	6	7	8	9	10	11	12	13	14	15	16
2	0	0	0	1	0	1	1	2	0	2	1	4
3	0	1	0	1	1	2	1	4	1	4	3	6
5	1	1	1	2	2	4	4	6	5	9	8	13
7	1	2	2	4	4	7	7	11	11	16	16	24
11	2	3	3	6	7	12	14	20	22	32	36	48
13	3	5	7	10	13	19	23	31	37	48	56	72

Proposition 4.1. *The numbers* J_i $(i=1,\cdots,12)$ *are given as follows*: $J_i=0$ if $i\neq 6, 7, 12$, *and*

$$J_6=\frac{1}{2^4}\left(1-\left(\frac{-1}{p}\right)\right)+\frac{(p-1)}{2^4 3}(-1)^k-\frac{k}{2^3 3}\left(1-\left(\frac{-1}{p}\right)\right),$$

$$J_7=\frac{1}{2^2 3}\left(1-\left(\frac{-3}{p}\right)\right)+\frac{(p-1)}{2^2 3^2}[0,-1,1;3]-\frac{k}{2\cdot 3^2}\left(1-\left(\frac{-3}{p}\right)\right),$$

$$J_{12}=\frac{1}{2^2 3}\left(1-\left(\frac{-3}{p}\right)\right)(-1)^k+\frac{1}{2^2 3}\left(1-\left(\frac{-1}{p}\right)\right)[0,-1,1;3].$$

Proof. The contribution to dim $\mathfrak{M}_{k-3}(U_2)$ has been given in [7], dim $S_k(Sp(2, \boldsymbol{Z}))$ in Hashimoto [6], and dim $S_k(K(p))$ in Theorem 2 of this paper. q.e.d.

Remark. This result is rather mysterious. Those elements with the principal polynomials $f_i(x)$ $(i=8,\cdots,12)$ are regular elements. Among those, as stated above, only J_{12} is exceptionally non-zero. I do not know the intrinsic reason of this.

Next, we shall give the contribution to

$$\dim S_k(K(p))-2\dim S_k(Sp(2,\boldsymbol{Z})),$$

of *non-semi-simple* conjugacy classes. (Note that there is no such contribution to $\mathfrak{M}_{k-3}(U_2)$.) More precisely, take a set $\{\gamma\}\subset Sp(2,\boldsymbol{R})$ of non semi-simple elements, and denote by $K(\{\gamma\})$ the contribution to

$$\dim S_k(K(p))-2\dim S_k(Sp(2,\boldsymbol{Z})),$$

of those $K(p)$-conjugacy classes whose elements are $Sp(2, \mathbf{R})$-conjugates of one of $\{\gamma\}$. Put

$$\hat{\delta}(\pm 1, \pm 1)=\begin{pmatrix}1 & 0 & \pm 1 & 0\\ 0 & 1 & 0 & \pm 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad a=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

$$b=\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad \hat{\beta}(\theta, \lambda)=\begin{pmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & \lambda\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

$$\hat{\gamma}(\theta, \lambda)=\begin{pmatrix}\cos\theta & \sin\theta & \lambda\cos\theta & \lambda\sin\theta\\ -\sin\theta & \cos\theta & -\lambda\sin\theta & \lambda\cos\theta\\ 0 & 0 & \cos\theta & \sin\theta\\ 0 & 0 & -\sin\theta & \cos\theta\end{pmatrix}.$$

Proposition 4.2. *For $k \geqq 5$, we have*

$$K\left(\pm\hat{\beta}\left(\frac{2\pi}{3}, +1\right)\right)=\left(1-\left(\frac{-3}{p}\right)\right)[0, -1, 1; 3]/3^2,$$

$$K(\hat{\delta}(\pm 1, \pm 1))=\frac{(-1)^k}{2^4}\left(1-\left(\frac{-1}{p}\right)\right),$$

$$K(\pm a)=-\frac{p-1}{2^4 3^2}(2k-3),$$

$$K(\pm b)=\frac{p-1}{2^3 3},$$

$$K\left(\hat{\gamma}\left(\frac{\pi}{2}, \pm 1\right)\right)=\frac{1}{2^3}\left(1-\left(\frac{-1}{p}\right)\right),$$

$$K\left(\pm\hat{\gamma}\left(\frac{2\pi}{3}, \pm 1\right)\right)=\frac{1}{2\cdot 3}\left(1-\left(\frac{-3}{p}\right)\right),$$

and $K(\gamma)=0$ for any other $\gamma \in Sp(2, \mathbf{R})$ which is not $Sp(2, \mathbf{R})$-conjugate to one of the above.

Proof is obvious by virtue of Theorem 3 and Hashimoto [6], Theorem 6.2. Now, denote six non zero values in Proposition 4.2 by K_i ($i=1, \cdots, 6$), that is, $K_1=K(\pm\hat{\beta}(2\pi/3, \pm 1))$, and so on. Then, for $k \geqq 5$, we have,

$$\dim S_k(K(p))-2\dim S_k(Sp(2, \mathbf{Z}))-\dim \mathfrak{M}_{k-3}(U_2)$$
$$=J_6+J_7+J_{12}+\sum_{i=1}^{6} K_i$$

$$=-\left\{\frac{p-1}{12}+\frac{1}{4}\left(1-\left(\frac{-1}{p}\right)\right)+\frac{1}{3}\left(1-\left(\frac{-3}{p}\right)\right)\right\}$$
$$\times\left\{\frac{k}{6}-\frac{1}{3}[0,\ -1,\ 1;\ 3]-\frac{1}{4}(3+(-1)^k)\right\}$$
$$=-\dim A_2(\Gamma_0(p))\times\dim S_{2k-2}(SL_2(\mathbf{Z})).$$

So, we obtain Theorem 1.

Remark. We get also the following interesting result. Put

$$\Gamma_0(p)=B(p)\cup B(p)w_1B(p),\quad \Gamma_0'(p)=B(p)\cup B(p)w_2B(p),$$

and

$$\Gamma_0''(p)=B(p)\cup B(p)w_0B(p).$$

When $p=2$, the dimensions of cusp forms belonging to these groups are easily calculated by using Igusa [14] (II) (cf. [11]). We get the following equality for $k\geqq 3$:

$$\begin{aligned}&\dim S_k(B(2))-\dim S_k(\Gamma_0(2))-\dim S_k(\Gamma_0'(2))-\dim S_k(\Gamma_0''(2))\\ &\quad+\dim S_k(K(2))+2\dim S_k(Sp(2,\mathbf{Z}))\\ &\qquad=\dim \mathfrak{M}_{k-3}(U_0)-\dim \mathfrak{M}_{k-3}(U_1)-\dim \mathfrak{M}_{k-3}(U_2)\end{aligned}$$

where the discriminant of B is two. This supports the conjecture in [9]. This relation is extended in [8] for all p.

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