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VECTOR VALUED SIEGEL'S MODULAR FORMS OF DEGREE TWO AND THE ASSOCIATED ANDRIANOV L-FUNCTIONS

TSUNEO ARAKAWA

In [1], [2], Andrianov constructed a remarkable Hecke theory for Siegel's modular forms of degree two. In this article we extend some of his results to the case of vector valued Siegel's modular forms of degree two.

0. Introduction

0.1. We summarize our results. For non-negative integers k, ν, let $\rho_{k,\nu}$ be the holomorphic irreducible representation of $GL_2(\mathbb{C})$ defined by

$$(0.1) \qquad \rho_{k,\nu}(g) = \det(g)^k \sigma_\nu(g) \qquad (g \in GL_2(\mathbb{C})),$$

where σ_ν denotes a symmetric tensor representation of $GL_2(\mathbb{C})$ of degree ν. We simply write ρ for $\rho_{k,\nu}$ and denote by V the representation space of ρ. There exists a non zero vector $v_0 \in V$ which satisfies the condition:

$$(0.2) \qquad \rho(b)v_0 = \det(b)^k b_1^\nu v_0 \qquad \text{for all } b=\begin{pmatrix} b_1 & x \\ 0 & b_2 \end{pmatrix} \in GL_2(\mathbb{C}).$$

Such a vector v_0 is uniquely determined up to a constant multiple. Here we note that any holomorphic representation of $GL_2(\mathbb{C})$ with finite dimension is equivalent to a direct sum of representations $\rho_{k,\nu}$ of the form (0.1). We set

$$GSp(2,\mathbb{R}) = \left\{ M \in M_4(\mathbb{R}) \;\middle|\; MJ\,{}^tM=\nu(M)J,\ \nu(M)>0 \right\}, \qquad J=\begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}.$$

The real symplectic group $GSp(2,\mathbb{R})$ with similitudes acts on the Siegel upper half plane H_2 of degree two in a usual manner:

$$(0.3) \qquad Z \longrightarrow M\langle Z\rangle=(AZ+B)(CZ+D)^{-1} \qquad (Z \in H_2,\ M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp(2,\mathbb{R})).$$

The canonical automorphic factor $J(M,Z)$ is defined by

$$J(M,Z) = CZ+D \qquad (Z\in H_2,\ M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp(2,\mathbb{R})).$$

Denote by Γ the Siegel modular group $Sp(2,\mathbb{Z})$ of degree two. A V-valued holomorphic function f on H_2 is called a modular form of weight ρ with respect to Γ, if f satisfies the equalities:

$$(0.4) \qquad f(M\langle Z\rangle) = \rho(J(M,Z))f(Z) \qquad \text{for all } M \in \Gamma.$$

Let $M_{k,\nu}$ be the space of modular forms of weight ρ with respect to Γ. If ν is odd, then, $M_{k,\nu}$ is trivially zero. Denote by Φ the Φ-operator of Siegel on $M_{k,\nu}$ and by $S_{k,\nu}$ the space of cusp forms of $M_{k,\nu}$: $S_{k,\nu}=\left\{ f \in M_{k,\nu} \mid \Phi f=0 \right\}$. Let $N_{k,\nu}$ be the orthogonal complement of the subspace $S_{k,\nu}$ in $M_{k,\nu}$ with respect to the Petersson inner product on $M_{k,\nu}$. For ν even and k odd, we have $N_{k,\nu}=\{0\}$. The space $M_{k,\nu}$ decomposes into the direct sum of subspaces $S_{k,\nu}$ and $N_{k,\nu}$: $M_{k,\nu} = S_{k,\nu} \oplus N_{k,\nu}$. Let $T(m)$ $(m=1,2,\ldots)$ be the Hecke operators acting on the space $M_{k,\nu}$ (for the definition, see § 2). The properties of Hecke operators make it possible to show that the subspaces $S_{k,\nu}$, $N_{k,\nu}$ are invariant under the action of $T(m)$ $(m=1,2,\ldots)$ and that each subspace has a basis consisting of common eigen forms. If $k>4$ and $\nu>0$, then one can show that the space $N_{k,\nu}$ is isomorphic to the space $S^1_{k+\nu}$ of elliptic cusp forms of weight $k+\nu$ with respect to $SL_2(\mathbb{Z})$. The isomorphism of $S^1_{k+\nu}$ to $N_{k,\nu}$ is given by Klingen's

Eisenstein series (see § 1).

For $f \in M_{k,\nu}$, $f(Z)$ has a Fourier expansion of the form

$$(0.5) \qquad f(Z) = \sum a_f(T)\exp(2\pi\sqrt{-1}\,\mathrm{tr}(TZ)) \qquad (a_f(T) \in V),$$

where T is over all semi-positive definite half-integral symmetric matrices of size two. Let $f \in M_{k,\nu}$ be a common eigen form, i.e., $T(m)f=\lambda(m)f$ $(m=1,2,\ldots)$. The Andrianov L-function associated to f is defined by

$$(0.6) \qquad L_f(s) = \zeta(2s-2k-\nu+4)\sum_{m=1}^{\infty} \lambda(m)m^{-s}.$$

The Dirichlet series $L_f(s)$ is absolutely convergent, if $\mathrm{Re}(s)$ is sufficiently large. We attach the Γ-factor to $L_f(s)$:

$$(0.7) \qquad \Psi_f(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)L_f(s).$$

When $\nu=0$ (the original case), it has been proved by Andrianov [1], [2] that $L_f(s)$ can be continued analytically to a meromorphic function in the whole complex plane which satisfies the functional equation:

$$\Psi_f(2k-2-s) = (-1)^k\Psi_f(s).$$

Suppose $\nu > 0$. For $f \in M_{k,\nu}$, $\Phi f(z)$ $(\mathrm{Im} z > 0)$ has the form:

$$\Phi f(z) = \psi(z)v_0 \quad \text{with some } \psi \in S^1_{k+\nu}.$$

It will be shown that $f \in N_{k,\nu}$ is a common eigen form, if and only if the corresponding $\psi \in S^1_{k+\nu}$ is a common eigen form. Moreover, for a common eigen form $f \in N_{k,\nu}$, the L-function $L_f(s)$ can be written in the form:

(0-8) $$L_f(s) = L_\psi(s)L_\psi(s-k+2),$$

where $L_\psi(s)$ is the associated L-function to ψ.

For cusp forms, we obtain the following theorem, which will be regarded as generalization of Andrianov's results to the case $\nu > 0$.

Theorem. *Suppose $\nu > 0$. Let $f \in S_{k,\nu}$ be a common eigen form and assume that $a_f(E_2) \neq 0$. Then, the L-function $L_f(s)$ is continued analytically to an entire function of s which satisfies the functional equation:*

(0-9) $$\Psi_f(2k+\nu-2-s) = (-1)^k \Psi_f(s).$$

If $f \in N_{k,\nu}$ is a common eigen form, then the analytic continuation of $L_f(s)$ and the functional equation (0-9) can be derived directly from the identity (0-8).

Example. By virtue of the explicit calculation of the dimension of the space $S_{k,\nu}$ due to Tsushima [11], [12], it is known that $\dim_{\mathbb{C}} S_{17,4} = 1$. It will be shown that the unique cusp form χ of $S_{17,4}$ gives an example of a common eigen form with the condition $a_\chi(E_2) \neq 0$.

0.2. The method for the proof is similar to that of Andrianov [1]. However, in the vector valued case, there is one difficulty which did not appear in the original case of Andrianov. The difficulty arises from the following property of Fourier coefficients $a_f(T)$ of $f \in M_{k,\nu}$:

(0-10) $$a_f(UT\,{}^tU) = \rho(U)a_f(T) \qquad \text{for all } U \in GL_2(\mathbb{Z}).$$

Because of this property of Fourier coefficients, we are not able to prove the theorem without the assumption $a_f(E_2)\neq 0$ for the moment.

0.3. The present article consists of four sections. In § 1, basic properties of vector valued modular forms of degree two are explained. In § 2, we study the relation between Fourier coefficients of $f\in M_{k,\nu}$ and those of $T(m)f$. The aim of § 3 is to translate the results of § 2 into terms of the Andrianov L-function and to prove the theorem with the use of a certain integral representation of $L_f(s)$. In § 4, we give a method of constructing cusp forms f with the non-zero Fourier coefficients $a_f(E_2)$.

Notation

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For any natural number n and for any commutative ring S with an identity element, $M_n(S)$, $GL_n(S)$, and $SL_n(S)$ denote the ring of all matrices of size n with entries in S, the group of all invertible elements in $M_n(S)$, and the group of elements in $M_n(S)$ whose determinants are one, respectively. For any element A of $M_n(S)$, we denote by ${}^tA$, tr(A), and det(A) the transposed matrix of A, the trace of A, and the determinant of A, respectively. Moreover, we denote by E_n the unit matrix of $M_n(S)$.

For any element Z of $M_n(\mathbb{C})$, Re(Z), Im(Z), and $\bar{Z}$ denote the real part of Z, the imaginary part of Z, and the complex conjugate, respectively. We denote by $\Gamma(s)$ and $\zeta(s)$ the gamma function and the Riemann zeta function, respectively. The symbol $e[w]$ $(w\in\mathbb{C})$ is used as an abbreviation for $\exp(2\pi\sqrt{-1}w)$.

1. Vector valued modular forms of degree two

1.1. Throughout the present paper, we keep the notation used in the introduction.

The general theory of vector valued Siegel's modular forms has been fruitfully investigated by Godement in [9, Exposé 5 ~ 10]. Here we only consider the degree two case.

Let $(\rho_{k,\nu},V)$ be the representation of $GL_2(\mathbb{C})$ defined by (0.1). Note that $\rho_{k,\nu}$ is holomorphic and irreducible. We fix k and ν, and write ρ for $\rho_{k,\nu}$, if there is no fear of confusion. We choose a positive definite hermitian scalar product (v,w) $(v,w\in V)$ on V satisfying

$$(1.1) \qquad (\rho(g)v,w) = (v,\rho({}^t\bar{g})w) \qquad (v,w\in V,\ g\in GL_2(\mathbb{C})).$$

Put $\|v\| = (v,v)^{1/2}$ $(v\in V)$. Throughout the first three sections, we assume $\nu > 0$.

The Siegel upper half plane of degree two is denoted by H_2: $H_2=\{Z\in M_2(\mathbb{C}) \mid {}^tZ=Z,\ \mathrm{Im}Z \text{ is positive definite}\}$. The action on H_2 of the real symplectic group $GSp(2,\mathbb{R})$ of degree two with similitudes is given by the map (0.3). Denote by $M_{k,\nu}$ the space consisting of V-valued holomorphic functions on H_2 which satisfy the condition (0.4). Any function of $M_{k,\nu}$ is called a modular form of weight ρ with respect to $\Gamma=Sp(2,\mathbb{Z})$. Note that, for an odd integer ν, $M_{k,\nu}=\{0\}$. The Fourier expansion of $f\in M_{k,\nu}$ is given by the equality (0.5). Then the property (0.4) implies the relation (0.10).

Let H_1 be the upper half plane. The Φ-operator of Siegel on the space $M_{k,\nu}$ is defined by

$$(1.2) \qquad \Phi f(z) = \lim_{\lambda \to +\infty} f\begin{pmatrix} z & 0 \\ 0 & \sqrt{-1}\lambda \end{pmatrix} \qquad (f \in M_{k,\nu},\ z \in H_1).$$

Denote by S_m^1 the space of elliptic cusp forms of weight m with respect to $SL_2(\mathbb{Z})$. Then the image of Φ is characterized by the following lemma.

Lemma 1.1. Assume $\nu > 0$. Then, for $f \in M_{k,\nu}$, there exists a ψ of $S_{k+\nu}^1$ satisfying $\Phi f(z) = \psi(z) v_0$ (for the vector v_0, see (0.2)).

Proof. It is not difficult to see from the property (0.10) that each Fourier coefficient $a_f\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$ $(m \in \mathbb{Z},\ m > 0)$ is a scalar multiple of the vector v_0. Moreover, from (0.10) and the assumption $\nu > 0$, we easily get $a_f(0) = 0$. Thus one can set

$$\Phi f(z) = \psi(z) v_0 \quad (z \in H_1) \qquad \text{with } \psi(z) = \sum_{m=1}^{\infty} a(m) e[mz] \quad (a(m) \in \mathbb{C}).$$

With the use of the property (0.4), we easiy deduce that $\psi \in S_{k+\nu}^1$.

1.2. The subspace $S_{k,\nu}$ of cusp forms in $M_{k,\nu}$ is given by $S_{k,\nu} = \{f \in M_{k,\nu} \mid \Phi f = 0\}$. Let F be a fundamental domain of Γ in H_2. Following Godement [9, Expose 7], we define the Petersson inner product on $M_{k,\nu}$: Let f_j $(j=1,2) \in M_{k,\nu}$, and let one of f_j be a cusp form. Set

$$(1.3) \qquad \langle f_1, f_2 \rangle = \int_F (\rho(\mathrm{Im}(Z)) f_1(Z), f_2(Z))\, dZ,$$

dZ being an invariant measure on H_2. Note that the integrand is invariant under Γ, and that the integral is absolutely convergent (see [9, Exposé 7, Théorème 1]). Let $N_{k,\nu}$ be the orthogonal complement of $S_{k,\nu}$ in $M_{k,\nu}$ with respect to the inner product (1.3). Then we easily get $M_{k,\nu}= S_{k,\nu}\oplus N_{k,\nu}$ (a direct sum). If k is odd, then it easily follows that $N_{k,\nu}=\{0\}$. Now we define Klingen's Eisenstein series following [5]. For $Z=\begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \in H_2$, we write $Z^*=z_1$. Denote by Γ_∞ the subgroup of Γ consisting of matrices of Γ with the last rows $(0,0,0,\pm 1)$. For an elliptic cusp form ψ of $S^1_{k+\nu}$, we set

$$(1.4) \qquad E_{k,\nu}(Z,\psi,v_0) = \sum_{M\in\Gamma_\infty\backslash\Gamma} \psi(M\langle Z\rangle^*)\rho(J(M,Z)^{-1})v_0.$$

Provided that the infinite series in (1.4) is absolutely convergent, then $E_{k,\nu}(Z,\psi,v_0)$ is well-defined. The next proposition is a modification of Klingen's results [5] to our situation.

<u>Proposition</u> 1.2. <u>Let</u> k, ν <u>be</u> <u>even</u> <u>integers</u> <u>with</u> $k>4$, $\nu>0$. <u>Then</u>, $E_{k,\nu}(Z,\psi,v_0)$ <u>is</u> <u>absolutely</u> <u>convergent</u> <u>for</u> <u>any</u> $Z\in H_2$. <u>Moreover</u>, <u>we</u> <u>have</u>

$$E_{k,\nu}(Z,\psi,v_0)\in N_{k,\nu}, \quad \text{and} \quad \Phi E_{k,\nu}(*,\psi,v_0)=\psi(z)v_0.$$

Proof. The proof is essentially due to Godement [9, Exposé 9, Théorème 1] and [5]. Put, for $Z\in H_2$ and $M\in\Gamma$,

$$I_\rho(Z)=\rho((\mathrm{Im}(Z))^{1/2}) \quad \text{and} \quad J_\rho(M,Z)=I_\rho(M\langle Z\rangle)\rho(J(M,Z))I_\rho(Z)^{-1}.$$

From (1.1), we get $\|J_\rho(M,Z)v\| = \|v\|$ for any $v\in V$. Further, we have

$$\|I_\rho(Z)v_0\| = \det(\mathrm{Im}(Z))^{k/2}\mathrm{Im}(Z^*)^{\nu/2}\|v_0\|.$$

Since $\psi\in S^1_{k+\nu}$, there exists a positive constant C such that

$$(\mathrm{Im}(z))^{(k+\nu)/2}|\psi(z)| < C \qquad \text{for any } z\in H_1.$$

Therefore, we easily get

$$\|I_\rho(Z)E_{k,\nu}(Z,\psi,v_0)\| \leqq C \sum_{M\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(M\langle Z\rangle^*)^{-(k+\nu)/2}\|I_\rho(M\langle Z\rangle)v_0\|$$

$$= C\|v_0\|\det(\operatorname{Im}(Z))^{k/2} \sum_{M\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(M\langle Z\rangle^*)^{-k/2}|\det(J(M,Z))|^{-k},$$

where the last infinite series is convergent if $k>4$, due to [5]. Thus the absolutely convergence of $E_{k,\nu}(Z,\psi,v_0)$ is verified. Other assertions of the proposition are similarly proved as in [5], so we omit the proof. Q.E.D.

Lemma 1.1 and Proposition 1.2 imply the proposition.

Proposition 1.3. *Let* k, ν *be even integers with* $k>4$, $\nu>0$. *Then, the space* $N_{k,\nu}$ *is isomorphic to* $S^1_{k+\nu}$, *and the isomorphism of* $S^1_{k+\nu}$ *to* $N_{k,\nu}$ *is given via the map*: $\psi \longrightarrow E_{k,\nu}(Z,\psi,v_0)$.

1.3. We denote by $P_{\mathbb{Z}}$ the set of positive definite half-integral symmetric matrices of size two:

$$(1.5) \qquad P_{\mathbb{Z}} = \left\{ T=\begin{pmatrix} t_1 & t_{12} \\ t_{12} & t_2 \end{pmatrix} \;\middle|\; t_1,\ t_2,\ 2t_{12}\in\mathbb{Z},\ t_1>0,\ \det(T)>0 \right\}.$$

For the later use, we need some estimates for the Fourier coefficients $a_f(T)$ ($T\in P_{\mathbb{Z}}$) of a cusp form f. By virtue of a result of Godement [9, Expose 7, Corollaire 3 of Theoreme 1], we have, for $f\in S_{k,\nu}$,

$$(1.6) \qquad \|\rho(T^{-1/2})a_f(T)\| < c(f) \qquad \text{for all } T\in P_{\mathbb{Z}},$$

where $c(f)$ is some positive constant independent of T.

2. Hecke operators and Fourier coefficients.

2.1. Set, for each natural number m,

$$S_m = \left\{ M \in M_4(\mathbb{Z}) \;\middle|\; MJ\,{}^tM = mJ \right\}, \qquad J = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}.$$

Put $S = \bigcup_{m=1}^{\infty} S_m$. For $f \in M_{k,\nu}$ and $M \in S$, we define a function $f|M$ on H_2 by

$$(f|M)(Z) = \rho(J(M,Z))^{-1} f(M\langle Z\rangle) \qquad (Z \in H_2).$$

Then we get $(f|M)|M' = f|MM'$ for all $M, M' \in S$. The Hecke operators $T(m)$ $(m \in Z,\ m > 0)$ on $M_{k,\nu}$ are defined by

$$T(m)f = m^{2k+\nu-3} \sum_{M \in \Gamma\backslash S_m} f|M \qquad (f \in M_{k,\nu}),$$

where M runs over a complete set of representatives of all left cosets of S_m modulo Γ. Since $f|M=f$ for all $M \in \Gamma$, the operation of $T(m)$ is well-defined and $T(m)f \in M_{k,\nu}$. From the properties of the abstract Hecke ring due to Shimura [10], it follows that, for natural numbers m, m',

$$(2.1) \qquad \begin{cases} T(m)T(m') = T(m')T(m) \\ T(m)T(m') = T(mm') \quad \text{if } (m,m')=1, \end{cases}$$

and that, for a prime p, the formal power series $\sum_{\delta=0}^{\infty} T(p^\delta)t^\delta$ is given by

$$(2.2) \qquad \sum_{\delta=0}^{\infty} T(p^\delta)t^\delta = (1-p^{\mu-1}t^2)\times$$

$$(1-T(p)t+\{T(p)^2-T(p^2)-p^{\mu-1}\}t^2-p^{\mu}T(p)t^3+p^{2\mu}t^4)^{-1},$$

where we put $\mu=2k+\nu-3$ for simplicity. In the same manner as in Maass [8, Satz 4], it is not difficult to verify that the Hecke operator $T(m)$ is self-dual on $M_{k,\nu}$ with respect to the inner product (1.3): Namely, for $f_1, f_2 \in M_{k,\nu}$,

$$(2.3) \qquad \langle T(m)f_1, f_2\rangle = \langle f_1, T(m)f_2\rangle \qquad (m=1,2,\ldots),$$

where one of $f_i \in S_{k,\nu}$. Therefore, the subspaces $S_{k,\nu}$, $N_{k,\nu}$ are invariant under the action of the Hecke operators $T(m)$. Further, in view of (2.1) and (2.3), the subspace $S_{k,\nu}$ has a basis consisting of common eigen forms of all $T(m)$.

2.2. Fix a prime p. For a non-negative integer β, let $R(p^\beta)$ be the same set introduced as in [2, §2.1]. Namely, $R(p^\beta)$ is the set of matrices $\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ of $SL_2(\mathbb{Z})$ whose first rows (u_1, u_2) run over a complete set of representatives modulo the equivalence relation:

$$(u_1, u_2) \sim (u_1', u_2') \qquad (\mathrm{mod}\, p^\beta)$$

which means that there exists $v \in \mathbb{Z}$, $(v,p)=1$ such that $vu_1 \equiv u_1'$ $(\mathrm{mod}\, p^\beta)$ and $vu_2 \equiv u_2'$ $(\mathrm{mod}\, p^\beta)$ (the second rows (u_3, u_4) are chosen so that $u_1u_4 - u_2u_3 = 1$).

Denote by N_2 the set of half-integral symmetric matrices of size two and by A_2 the set of all V-valued functions b on N_2 which satisfy the relations $b(UT{}^tU) = \rho(U)b(T)$ for all $U \in SL_2(\mathbb{Z})$ and all $T \in N_2$. The Fourier coefficients $a_f(T)$ $(f \in M_{k,\nu})$ define a function of A_2, if we put $a_f(T)=0$ when T is not semi-positive definite. Following [1, §2], [2, §2], we define some operators on the set N_2: For non-negative integers α, β, γ, and $b \in A_2$, put

$$(2.4)\quad \begin{cases} (\Delta^{+}(p^{\alpha})b)(T) = \rho(p^{-\alpha}E_2)b(p^{\alpha}T), \\ (\Delta^{-}(p^{\gamma})b)(T) = b(p^{-\gamma}T), \\ (\Pi(p^{\beta})b)(T) = \sum_{U\in R(p^{\beta})} \rho(U)^{-1}\rho\begin{pmatrix}1 & 0\\ 0 & p^{-\beta}\end{pmatrix} b\left(\begin{pmatrix}p^{-\beta} & 0\\ 0 & 1\end{pmatrix} UT\,{}^tU \begin{pmatrix}1 & 0\\ 0 & p^{\beta}\end{pmatrix}\right), \end{cases}$$

where we understand $b(S)=0$, if a rational symmetric matrix S does not belong to N_2.

For $f\in M_{k,\nu}$, we set, as a Fourier expansion of $T(p^{\delta})f$,

$$T(p^{\delta})f(Z) = \sum a_f(p^{\delta};T)e[\mathrm{tr}(TZ)].$$

In the same manner employed as in [1, §2], [2, §2.1], we easily obtain

$$(2.5)\qquad a_f(p^{\delta};T) = \sum_{\alpha+\beta+\gamma=\delta} p^{\delta\mu+3\alpha+\beta}(\Delta^{-}(p^{\gamma})\Pi(p^{\beta})\Delta^{+}(p^{\alpha})a_f)(T),$$

where $\mu=2k+\nu-3$ and α, β, γ run over all non-negative integers satisfying $\alpha+\beta+\gamma=\delta$.

2.3. Let $T_{k+\nu}(m)$ $(m=1,2,\ldots)$ be the usual Hecke operators acting on the space $S^1_{k+\nu}$ (see [4, §2] or [1, (2.3)]). The following commutation formulae for the Hecke operators $T(p^{\delta})$ $(\delta=1, 2)$ and the operator Φ are essentially due to Maass [8, Satz 20].

Proposition 2.1. *Suppose* $\nu>0$. *Let* $f\in M_{k,\nu}$ *and set* $\Phi f(z)=\psi(z)v_0$ *with some* $\psi\in S^1_{k+\nu}$. *Then, for any prime* p,

(i) $\Phi(T(p)f)(z) = (1+p^{k-2})T_{k+\nu}(p)\psi(z)v_0$,

(ii) $\Phi(T(p^2)f)(z) = \{(1+p^{k-2}+p^{2k-4})T_{k+\nu}(p^2)\psi(z)+p^{\mu-1}(p-1)\psi(z)\}v_0$.

Proof. We give only the proof of (i) (the assertion (ii) is similarly verified). Let the Fourier expansion of ψ be

$$\psi(z) = \sum_{m=1}^{\infty} a(m)e[mz] \qquad (z \in H_1),$$

and note that $a_f\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} = a(m)v_0$ $(m=1,2,\ldots)$. We may choose the following set as $R(p)$:

$$R(p) = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} (0 \leqq m \leqq p-1),\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Put $T=\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$ $(t \in \mathbb{Z},\ t > 0)$. Then, in view of the relations (2.5), (0.10), and (0.2), we easily get

$$a_f(p;T) = a_f\begin{pmatrix} pt & 0 \\ 0 & 0 \end{pmatrix} + p^{\mu} a_f\begin{pmatrix} t/p & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ p^{\mu+1}\{\sum_{m=0}^{p-1} \rho\begin{pmatrix} 1 & -m/p \\ 0 & 1/p \end{pmatrix} a_f\begin{pmatrix} t/p & 0 \\ 0 & 0 \end{pmatrix} + \rho\begin{pmatrix} 1/p & 0 \\ 0 & 1 \end{pmatrix} a_f\begin{pmatrix} pt & 0 \\ 0 & 0 \end{pmatrix}\}$$

$$= (1+p^{k-2})\{a(pt)+p^{k+\nu-1}a(t/p)\}v_0,$$

which completes the proof of the assertion (i). Q.E.D.

2.4. Now we study the Fourier coefficients $a_f(p^{\delta};T)$ $(f \in M_{k,\nu})$ under certain conditions on $T \in P_{\mathbb{Z}}$ (for the set $P_{\mathbb{Z}}$, see (1.5)).

We set $T=\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in P_{\mathbb{Z}}$ and require the following conditions for T:

(2.6)
- (i) T is primitive, i.e, $(a,b,c)=1$,
- (ii) $d=b^2-4ac$ is the discriminant of the imaginary quadratic field $K=\mathbb{Q}(\sqrt{d})$,

(iii) the class number h_K of K is one.

Put $z=\frac{b-\sqrt{d}}{2a}$. For any θ of K, a 2×2 rational matrix $L(\theta)$ is defined by

$$\theta\begin{pmatrix}1\\z\end{pmatrix} = L(\theta)\begin{pmatrix}1\\z\end{pmatrix}.$$

For any ideal $\mathfrak{a}$ (resp. any number θ) of K, denote by $N(\mathfrak{a})$ (resp. $N(\theta)$) the norm of $\mathfrak{a}$ (resp. θ). We see easily that

$$L(\theta)T\,{}^tL(\theta) = N(\theta)T.$$

Now recall that (σ_ν, V) is a symmetric tensor representation of $GL_2(\mathbf{C})$ of degree ν (see (0.1) in the introduction). Define a subspace $V(T)$ of V by

(2.7) $$V(T) = \{v \in V \mid \sigma_\nu(L(\varepsilon))v=v \quad \text{for all units } \varepsilon \text{ of } K\}.$$

Any ideal $\mathfrak{a}$ of K is written in the form $\mathfrak{a}=(\theta)$ with some $\theta \in K$, due to the property (iii) of (2.6). Then, a linear transformation $\sigma_T(\mathfrak{a})$ of the space $V(T)$ is defined by

$$\sigma_T(\mathfrak{a}) = \sigma_\nu(L(\theta)) \qquad \text{for } \mathfrak{a}=(\theta),\ \theta \in K.$$

Obviously, σ_T is well-defined. It is easy to see that σ_T is a direct sum of Hecke's Grössencharacters of K. Moreover, a L-function $L(s,\sigma_\nu,T)$ is defined by

(2.8) $$L(s,\sigma_\nu,T) = \sum_{\mathfrak{a}} \sigma_T(\mathfrak{a})N(\mathfrak{a})^{-s},$$

where $\mathfrak{a}$ is over all integral ideals of K. The L-function $L(s,\sigma_\nu,T)$ is absolutely convergent, if Re(s) is sufficiently large, and has the expression as Euler products:

$$L(s,\sigma_\nu,T) = \prod_{\mathfrak{p}} (1-\sigma_T(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1},$$

where $\mathfrak{p}$ is over all prime ideals of K.

Proposition 2.2. Let $f \in M_{k,\nu}$ and let p be a prime. Suppose that $T=\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in P_{\mathbb{Z}}$ satisfies the conditions of (2.6). Let m be a positive integer coprime to p. Then the following relations hold:

(i) If p splits in K/Q ($p=\mathfrak{p}\bar{\mathfrak{p}}$, $\mathfrak{p}\neq\bar{\mathfrak{p}}$), then,

$$(\Pi(p^\beta)a_f)(mT) = p^{-k\beta}\{\sigma_T(\mathfrak{p})^{-\beta}a_f(mT)+\sigma_T(\bar{\mathfrak{p}})^{-\beta}a_f(mT)\} \qquad (\beta\geq 1).$$

(ii) If p ramifies in K/Q ($p=\mathfrak{p}^2$), then,

$$(\Pi(p^\beta)a_f)(mT) = \begin{cases} p^{-k}\sigma_T(\mathfrak{p})^{-1}a_f(mT) & \text{if } \beta=1, \\ 0 & \text{if } \beta>1 \end{cases}$$

(iii) If p remains prime in K/Q, then, $(\Pi(p^\beta)a_f)(mT)=0$ $(\beta\geq 1)$.

Proof. Suppose that p splits in K/Q and put $p=\mathfrak{p}\bar{\mathfrak{p}}$, $\mathfrak{p}\neq\bar{\mathfrak{p}}$. By the condition (iii) of (2.6), we have $\mathfrak{p}=(\pi)$ for some integer π of K.

First assume that a is coprime to p. If we put $\pi^\beta=u_1+u_2z$ $(\beta\geq 1$, $u_1, u_2\in \mathbb{Q})$, then, $u_1, u_2 \in \mathbb{Z}$ and a divides u_2. We easily get

$$L(\pi^\beta) = \begin{pmatrix} u_1 & u_2 \\ -cu_2/a & u_1+bu_2/a \end{pmatrix} \quad (\in M_2(\mathbb{Z})).$$

Since $\det(L(\pi^\beta))=N(\pi^\beta)=p^\beta$ and $(u_1,u_2)=1$, there exists a rational integer k_1 such that

$$k_1u_1-\frac{c}{a}u_2\equiv 0 \pmod{p^\beta}, \quad \text{and} \quad k_1u_2+u_1+\frac{b}{a}u_2\equiv 0 \pmod{p^\beta}.$$

Determine integers u_3, u_4 by the relation

$$\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{-\beta} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k_1 & 1 \end{pmatrix} L(\pi^{\beta}).$$

If we put $U_1=\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$, then, $U_1 \in SL_2(\mathbb{Z})$. Similarly if we set $\bar{\pi}^{\beta}= v_1+v_2 z$ with v_1, $v_2 \in \mathbb{Z}$, then there exist $k_2 \in \mathbb{Z}$ and $U_2=\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \in SL_2(\mathbb{Z})$ satisfying the relation $U_2=\begin{pmatrix} 1 & 0 \\ 0 & p^{-\beta} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k_2 & 1 \end{pmatrix} L(\bar{\pi}^{\beta})$. As is shown in the proof of Theorem 2.3.1 of [2], the matrices U_1, U_2 give a complete set of representatives of $R(p^{\beta})$ so that

$$m\begin{pmatrix} p^{-\beta} & 0 \\ 0 & 1 \end{pmatrix} UT\,{}^tU \begin{pmatrix} 1 & 0 \\ 0 & p^{\beta} \end{pmatrix} \qquad (U \in R(p^{\beta}))$$

is a half-integral symmetric matrix. Then we have

$$m\begin{pmatrix} p^{-\beta} & 0 \\ 0 & 1 \end{pmatrix} U_i T\,{}^tU_i \begin{pmatrix} 1 & 0 \\ 0 & p^{\beta} \end{pmatrix} = m\begin{pmatrix} 1 & 0 \\ k_i & 1 \end{pmatrix} T \begin{pmatrix} 1 & k_i \\ 0 & 1 \end{pmatrix} \qquad (i=1,2).$$

Therefore, from the definition of $\Pi(p^{\beta})$ and (0.10), we get

$$(\Pi(p^{\beta})a_f)(mT) = \rho(L(\pi^{\beta}))^{-1}a_f(mT)+\rho(L(\bar{\pi}^{\beta}))^{-1}a_f(mT)$$

$$= p^{-k\beta}\{\sigma_T(\mathfrak{p})^{-\beta}a_f(mT)+\sigma_T(\bar{\mathfrak{p}})^{-\beta}a_f(mT)\}.$$

Next assume that p divides a. There exists some $U \in SL_2(\mathbb{Z})$ such that, if we put $UT\,{}^tU=\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}$, then a' is coprime to p. Set $z'=\frac{b'-\sqrt{d}}{2a'}$. For any $\theta \in K$, define a 2×2 rational matrix $L'(\theta)$ by

$$\theta\begin{pmatrix} 1 \\ z' \end{pmatrix} = L'(\theta)\begin{pmatrix} 1 \\ z' \end{pmatrix}.$$

Further, determine a matrix W of $M_2(\mathbb{Q})$ by $\begin{pmatrix} 1 \\ z' \end{pmatrix}=W\begin{pmatrix} 1 \\ z \end{pmatrix}$. Then we have

$L'(\theta)=WL(\theta)W^{-1}$. Put $\theta=\xi+\eta z=\xi'+\eta' z'$ with ξ, η, ξ', $\eta'\in Q$. Immediately, $(\xi,\eta)=(\xi',\eta')W$. On the other hand, we have

$$N(\theta) = \frac{1}{a}(\xi,\eta)T\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \frac{1}{a'}(\xi',\eta')T'\begin{pmatrix}\xi'\\ \eta'\end{pmatrix},$$

where we put $T'=UT{}^tU$. From these equalities, we easily get $T'=\frac{a'}{a}WT{}^tW$. Therefore, we have

$$\frac{a'}{a}W = UP \quad \text{with some } P\in SO_0(T),$$

where $SO_0(T)=\{A\in SL_2(\mathbb{R}) \mid AT{}^tA=T\}$. Note that the orthogonal group $SO_0(T)$ is commutative, and hence we see that $L(\theta)$ commutes with P. Therefore, we get

$$(2.9) \qquad L'(\theta) = UL(\theta)U^{-1} \qquad (\theta\in K).$$

Since $(a',p)=1$, we have, as is shown above,

$$(2.10) \qquad (\Pi(p^\beta)a_f)(mT') = \rho(L'(\pi^\beta))^{-1}a_f(mT')+\rho(L'(\bar{\pi}^\beta))^{-1}a_f(mT').$$

Taking the relations (0.10), (2.9), and (2.10) into account, we obtain

$$\begin{aligned}(\Pi(p^\beta)a_f)(mT) &= \rho(U)^{-1}(\Pi(p^\beta)a_f)(mT') \\ &= \rho(L(\pi^\beta))^{-1}a_f(mT)+\rho(L(\bar{\pi}^\beta))^{-1}a_f(mT),\end{aligned}$$

which completes the proof of the assertion (i). The assertions (ii), (iii) are easily to be seen. Q.E.D.

As an easy corollary of Proposition 2.2, we obtain the following.

<u>Proposition</u> 2.3. <u>Let the assumption be the same as in</u> <u>Proposition</u> 2.2.

(i) *If* p *splits in* K/Q $(p=\mathfrak{p}\bar{\mathfrak{p}},\ \mathfrak{p}\neq\bar{\mathfrak{p}})$, *then*

$$a_f(p^{\delta};mT) = a_f(p^{\delta}mT) + \sum_{\beta=1}^{\delta} p^{(k+\nu-2)\beta}\{\sigma_T(\mathfrak{p})^{-\beta}a_f(p^{\delta-\beta}mT)+\sigma_T(\bar{\mathfrak{p}})^{-\beta}a_f(p^{\delta-\beta}mT)\}.$$

(ii) *If* p *ramifies in* K/Q $(p=\mathfrak{p}^2)$, *then*,

$$a_f(p^{\delta};mT) = a_f(p^{\delta}mT)+p^{k+\nu-2}\sigma_T(\mathfrak{p})^{-1}a_f(p^{\delta-1}mT).$$

(iii) *If* p *remains prime in* K/Q, *then*, $a_f(p^{\delta};mT)=a_f(p^{\delta}mT)$.

§ 3. Andrianov's L-functions.

3.1. To define an L-function associated to a common eigen form of $M_{k,\nu}$, we have to get some information on eigen values.

Let $f\in S_{k,\nu}$. By virtue of the inequality (1.6) and the relation (2.5), we easily get, for any prime p, any $\delta\geq 1$, and for all $T\in P_{\mathbb{Z}}$,

$$\|\rho(T^{-1/2})a_f(p^{\delta};T)\| < c_1(f)p^{\delta c}, \tag{3.1}$$

where $c_1(f)$ and c are some positive constants independent of T, p, and δ. Then the estimate (3.1) and the property (2.1) immediately imply the proposition:

Proposition 3.1. *Let* $f\in S_{k,\nu}$ *be a common eigen form, and put* $T(m)f=\lambda(m)f$ $(m=1,2,\ldots)$. *Then we have*

$$|\lambda(m)| < C_1m^c$$

for some positive constants C_1 *and* c.

In case of modular forms of $N_{k,\nu}$, the following proposition holds.

Proposition 3.2. Let $f \in N_{k,\nu}$ and put $\Phi f(z)=\psi(z)v_0$ with some $\psi \in S^1_{k+\nu}$. Then, f is a common eigen form, if and only if ψ is a common eigen form. In this situation, set $T(m)f=\lambda(m)f$ and $T_{k+\nu}(m)\psi=\lambda_0(m)\psi$ $(m=1,2,\ldots)$. Then, for any prime p, we have

$$(3.2) \qquad \begin{cases} \lambda(p) = (1+p^{k-2})\lambda_0(p), \\ \lambda(p^2) = (1+p^{k-2}+p^{2k-4})\lambda_0(p^2)+(p-1)p^{2k+\nu-4}. \end{cases}$$

The assertion of Proposition 3.2 is easily derived from Proposition 2.1 and the relation (2.3), so we omit the proof.

Now let $f \in M_{k,\nu}$ be a common eigen form and put $T(m)f=\lambda(m)f$ $(m=1,2,\ldots)$. The Andrianov L-function $L_f(s)$ associated to f is defined by (0.6) in the introduction. Then, Proposition 3.1 immediately implies that the Dirichlet series $L_f(s)$ for $f \in S_{k,\nu}$ is absolutely convergent, if Re(s) is sufficiently large. Moreover, it is not difficult to see from Proposition 3.2 and some properties of Hecke operators T(m) that $L_f(s)$ for $f \in N_{k,\nu}$ also converges absolutely for a sufficiently large number of Re(s). By virtue of the relations (2.1), (2.2), we get the expression for $L_f(s)$ as Euler products:

$$L_f(s) = \prod_p \left(1-\lambda(p)p^{-s}+\{\lambda(p)^2-\lambda(p^2)-p^{\mu-1}\}p^{-2s}-\lambda(p)p^{\mu-3s}+p^{2\mu-4s} \right)^{-1},$$

where we put $\mu=2k+\nu-3$.

3.2. In case of $f \in N_{k,\nu}$, the L-function $L_f(s)$ is reduced to some products of L-functions associated to an elliptic eigen cusp form.

Proposition 3.3. Let $f \in N_{k,\nu}$ be a common eigen form and put $\Phi f(z)=\psi(z)v_0$ with a common eigen form $\psi \in S^1_{k+\nu}$ ($T_{k+\nu}(m)\psi=\lambda_0(m)\psi$, $m=1,2,\ldots$). Then we have

$$L_f(s) = L_\psi(s)L_\psi(s-k+2),$$

where we set $L_\psi(s)= \sum_{m=1}^{\infty} \lambda_0(m)m^{-s}$.

Proposition 3.3 is easily derived from (3.2)

3.3. Due to the estimate (1.6), note that the Dirichlet series

$$\sum_{m=1}^{\infty} a_f(mT)m^{-s} \qquad (f \in S_{k,\nu},\ T \in P_{\mathbb{Z}})$$

is absolutely convergent, if Re(s) is sufficiently large. Following the method used in [1, Theorem 2], we easily obtain the proposition which plays a key role for us to prove the theorem in the introduction:

Proposition 3.4. Let $f \in S_{k,\nu}$ be a common eigen form, and let $T \in P_{\mathbb{Z}}$ satisfy the conditions of (2.6). Then we have

$$L(s-k+2,\sigma_\nu,T) \sum_{m=1}^{\infty} a_f(mT)m^{-s} = L_f(s)a_f(T),$$

if Re(s) is sufficiently large.

3.4. To represent $L_f(s)$ in a certain integral of Rankin's type, we shall review some preliminary tools from [1, §3] (cf. [2, Chapter 3]).

We define a 3-dimensional symmetric space H^* by

$$H^* = \left\{ (y,w) \mid y > 0,\ w \in \mathbb{C} \right\}.$$

Then the group $SL_2(\mathbb{C})$ acts on H^* in the manner:

$$u=(y,w) \longrightarrow \sigma\langle u\rangle = \left(\frac{y}{\Delta_\sigma(u)},\ \frac{\alpha\bar{\gamma}y^2+(\alpha w+\beta)(\bar{\gamma}\bar{w}+\bar{\delta})}{\Delta_\sigma(u)} \right),$$

where we put $\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL_2(\mathbb{C})$ and $\Delta_\sigma(u)=|\gamma|^2y^2+|\gamma w+\delta|^2$. We fix an embedding ϕ of $\mathbb{C}$ into $M_2(\mathbb{R})$ by setting

$$\phi(\alpha) = \begin{pmatrix} a & -b\\ b & a\end{pmatrix} \qquad (\alpha=a+\sqrt{-1}b,\ a,\ b\in \mathbb{R}).$$

Let $Sp(2,\mathbb{R})$ be the real symplectic group of degree two: $Sp(2,\mathbb{R})=\{M\in GSp(2,\mathbb{R}) \mid \nu(M)=1\}$. For $\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL_2(\mathbb{C})$, we set

$$\psi(\sigma) = \begin{pmatrix} I_0 & 0\\ 0 & E_2\end{pmatrix}\begin{pmatrix} \phi(\alpha) & \phi(\beta)\\ \phi(\gamma) & \phi(\delta)\end{pmatrix}\begin{pmatrix} I_0 & 0\\ 0 & E_2\end{pmatrix}, \qquad I_0=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

Then the map ψ gives an injective homomorphism of $SL_2(\mathbb{C})$ into $Sp(2,\mathbb{R})$. Denote by G the image of ψ in $Sp(2,\mathbb{R})$. Furthermore, define the subset H of H_2 by

$$H = \{Z=X+\sqrt{-1}yE_2 \mid y>0,\ X\in M_2(\mathbb{R}),\ {}^tX=X,\ tr(X)=0\}.$$

For each $M\in G$, the automorphism $Z\longrightarrow M\langle Z\rangle$ of H_2 induces an automorphism of H. For $Z=\begin{pmatrix} x+\sqrt{-1}y & -t\\ -t & -x+\sqrt{-1}y\end{pmatrix} \in H$, put

$$u(Z) = (y,w) \qquad \text{with } w=x+\sqrt{-1}t.$$

Then it is easy to see that the action of the group G on H is compatible with the action of $SL_2(\mathbb{C})$ on H^*: Namely,

$$u(\psi(\sigma)\langle Z\rangle) = \sigma\langle u(Z)\rangle \qquad \text{for all } Z\in H,\ \sigma\in SL_2(\mathbb{C}).$$

Set

$$\Gamma_0 = SL_2(O)/\{\pm E_2\}, \text{ and } \Gamma_{0,\infty} = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \,\middle|\, \beta \in O \right\},$$

where we put $O=\mathbb{Z}[\sqrt{-1}]$, the ring of integers in $\mathbb{Q}(\sqrt{-1})$. Then, $\Gamma_{0,\infty}$ is naturally regarded as a subgroup of Γ_0. For $\sigma=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{C})$ and $u=(y,w) \in H^*$, we define an automorphic factor $j(\sigma,u)$ by

$$j(\sigma,u) = \phi(\gamma w+\delta)+\sqrt{-1}y\phi(\gamma)I_0 \qquad (I_0=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}).$$

We easily get

$$j(\sigma\sigma',u) = j(\sigma,\sigma'\langle u\rangle)j(\sigma',u),$$

$${}^t\overline{j(\sigma,u)}j(\sigma,u) = \Delta_\sigma(u)E_2 \qquad (\sigma,\ \sigma' \in SL_2(\mathbb{C}),\ u \in H^*).$$

3.5. Now we put $T=E_2$ and $K=\mathbb{Q}(\sqrt{-1})$. Let $f \in S_{k,\nu}$ and assume that $a_f(E_2) \neq 0$. Set

$$R_f(s) = \sum_{m=1}^{\infty} a_f(mE_2)m^{-s}.$$

Define a function F_f on H^* induced from f by

$$F_f(u) = f(Z) \qquad \text{for } Z \in H \text{ with } u=u(Z).$$

It easily follows that, for any $\sigma \in \Gamma_0$,

$$F_f(\sigma\langle u\rangle) = \rho(j(\sigma,u))F_f(u).$$

If we put

$$S = \{u=(y,x+\sqrt{-1}t) \in H^* \mid x,\ t \in \mathbb{R},\ |x| \leq 1/2,\ |w| \leq 1/2\},$$

then in the same manner as in [1,(3.16)], the following identity holds:

$$(3.3) \qquad (4\pi)^{-s}\Gamma(s)R_f(s) = \int_S y^{s-1}F_f(u)du,$$

where we put $du=dxdtdy$ for $u=(y,x+\sqrt{-1}t)\in H^*$.

Now we introduce the $\mathrm{End}_{\mathbf{C}}(V)$-valued Eisenstein series on H^*:

$$E(u,s;\sigma_\nu) = \sum_{\sigma\in\Gamma_{0,\infty}\backslash\Gamma_0} \frac{y^s\sigma_\nu(j(\sigma,u))}{\Delta_\sigma(u)^s} \qquad (u=(y,w)\in H^*).$$

The Eisenstein series $E(u,s;\sigma_\nu)$ converges absolutely, if $\mathrm{Re}(s) > 2+\nu/2$. Further, we see easily that

$$E(\sigma\langle u\rangle,s;\sigma_\nu) = E(u,s;\sigma_\nu)\sigma_\nu(j(\sigma,u))^{-1} \qquad \text{for all } \sigma\in\Gamma_0.$$

In case of $\nu=0$, the Eisenstein series $E(u,s;\sigma_\nu)$ is precisely studied by Kubota [6]. Moreover, it is known by a general theory of Eisenstein series due to Langlands [7] that $E(u,s;\sigma_\nu)$ has an analytic continuation to a meromorphic function of s in the whole complex plane which satisfies a certain functional equation. Here we treat $E(u,s;\sigma_\nu)$ in an elementary manner.

For ξ, $\eta\in\mathbf{C}$, and $u=(y,w)\in H^*$, put

$$\Delta_{\{\xi,\eta\}}(u) = |\xi|^2y^2+|\xi w+\eta|^2.$$

We set, for $u=(y,w)\in H^*$,

$$\theta(u,s;\sigma_\nu) = \frac{1}{2}\left(\frac{y}{\pi}\right)^s\Gamma(s)\sum_{\{\xi,\eta\}}{}' \frac{\sigma_\nu(\phi(\xi w+\eta)+\sqrt{-1}y\phi(\xi)I_0)}{\Delta_{\{\xi,\eta\}}(u)^s} \quad ,$$

where ξ, η run over all elements of $O=\mathbf{Z}[\sqrt{-1}]$ with the condition $(\xi,\eta)\neq(0,0)$. The infinite series $\theta(u,s;\sigma_\nu)$ is absolutely convergent if $\mathrm{Re}(s) > 2+\frac{\nu}{2}$. Then we easily get

$$(3.4) \qquad \theta(u,s;\sigma_\nu) = \pi^{-s}\Gamma(s)L_K(s,\sigma_\nu)E(u,s;\sigma_\nu),$$

where we put

$$L_K(s,\sigma_\nu) = \frac{1}{4}\sum_{\alpha\in O,\ \alpha\neq 0} \sigma_\nu(\phi(\alpha))|\alpha|^{-2s}.$$

Taking notice of the fact $L(\alpha)=\phi(\alpha)$ $(\alpha\in K)$ for $T=E_2$, we have

$$(3.5) \qquad L_K(s,\sigma_\nu) = L(s,\sigma_\nu,E_2) \quad \text{on } V(E_2) \qquad (\text{see } (2.7),\ (2.8)).$$

Moreover we define a kind of theta series:

$$K(v;u,\sigma_\nu) = \sum_{\{\xi,\eta\}}{}' \sigma_\nu(\phi(\xi w+\eta)+\sqrt{-1}y\phi(\xi)I_0)\exp\Big(-\frac{\pi v}{y}\Delta_{\{\xi\ \eta\}}(u)\Big),$$

where $u=(y,w)\in H^*$ and $v>0$. Immediately, we have

$$(3.6) \qquad \int_0^\infty v^s K(v;u,\sigma_\nu)\frac{dv}{v} = 2\theta(u,s;\sigma_\nu) \qquad (\mathrm{Re}(s)>2+\frac{\nu}{2}).$$

Proposition 3.5. *Let ν be an even integer >0. Then,*

$$K(\frac{1}{v};u,\sigma_\nu) = v^{\nu+2}\sigma_\nu(I_0)K(v;u,\sigma_\nu).$$

Proof. Since (σ_ν,V) is a polynomial representation of $GL_2(\mathbb{C})$, we may naturally extend $\sigma_\nu(g)$ to $g\in M_2(\mathbb{C})$. Set, for $\xi,\ \eta\in\mathbb{C}$ and $u=(y,w)\in H^*$,

$$f(\xi,\eta) = \sigma_\nu(\phi(\xi w+\eta)+\sqrt{-1}y\phi(\xi)I_0)\exp\Big(-\frac{\pi v}{y}\Delta_{\{\xi\ \eta\}}(u)\Big).$$

An euclidean measure $d\xi$ on $\mathbb{C}$ $(\xi=\xi_1+\sqrt{-1}\xi_2,\ \xi_1,\ \xi_2\in\mathbb{R})$ is normalized by $d\xi=d\xi_1 d\xi_2$. The Fourier transform of $f(\xi,\eta)$ is defined by

$$f^*(\xi',\eta') = \int_{\mathbf{C}^2} f(\xi,\eta)e[\mathrm{Re}(\xi\xi'+\eta\eta')]d\xi d\eta \quad (\xi',\ \eta' \in \mathbf{C}).$$

By an easy calculation, we get

$$f^*(\xi',\eta')=v^{-\nu-2}\sigma_\nu(I_0)\sigma_\nu(\phi(\xi'-w\eta')-\sqrt{-1}y\phi(\eta')I_0)\exp\left(-\frac{\pi}{yv}\Delta_{\{-\eta',\xi'\}}(u)\right)$$

(cf. Eichler [3, Chapter I]).

Therefore, the Poisson summation formula easily implies Proposition 3.5. Q.E.D.

By the relation (3.6) and Proposition 3.5, we have

$$(3.7) \qquad 2\theta(u,s;\sigma_\nu) = \int_1^\infty (v^s+\sigma_\nu(I_0)v^{\nu+2-s})K(v;u,\sigma_\nu)\frac{dv}{v} \qquad (\mathrm{Re}(s) > 2+\frac{\nu}{2})$$

The identity (3.7) gives an analytic continuation of $\theta(u,s;\sigma_\nu)$ to an entire function of s, and also implies that

$$\theta(u,\nu+2-s;\sigma_\nu) = \sigma_\nu(I_0)\theta(u,s;\sigma_\nu).$$

3.6. Finally, we give a proof of the theorem in the introduction.

Following [1, §3], we easily get the integral representation of $R_f(s)$ from (3.3):

$$(3.8) \qquad (4\pi)^{-s}\Gamma(s)R_f(s) = \int_D E(u,s-k+2;\sigma_\nu)y^k F_f(u)\frac{du}{y^3} \qquad (\mathrm{Re}(s) \gg 0),$$

D being a fundamental domain of Γ_0 in H*.

Now suppose that f is a common eigen form. It easily follows from Proposition 3.4 and (3.5) that

$$(3.9) \qquad L_K(s-k+2,\sigma_\nu)R_f(s) = L_f(s)a_f(E_2).$$

If we put

$$\Psi_f(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)L_f(s),$$

then, by virtue of (3.4), (3.8), and (3.9), we obtain

$$\Psi_f(s)a_f(E_2) = \pi^{2-k} \int_D \theta(u,s-k+2;\sigma_\nu)y^k F_f(u) \frac{du}{y^3}, \tag{3.10}$$

if Re(s) is sufficiently large. As is known, we may take as D the following set (see [6]):

$$D = \left\{ u=(y,x+\sqrt{-1}t) \mid 0 \leqq x+t, \quad x \leqq 1/2, \quad t \leqq 1/2, \quad 1 \leqq x^2+t^2+y^2 \right\}.$$

From (3.7) and (3.10), we get

$$2\pi^{k-2}\Psi_f(s)a_f(E_2) =$$

$$\int_D y^k F_f(u) \left[\int_1^\infty (v^s + \sigma_\nu(I_0)v^{\nu+2-s})K(v;u,\sigma_\nu) \frac{dv}{v} \right] \frac{du}{y^3}$$

Therefore, following the argument of [1, §3, (3.29)], we easily deduce that, under the assumption $a_f(E_2) \neq 0$, $\Psi_f(s)$ can be continued analytically to an entire function of s and that $\Psi_f(s)$ satisfies the functional equation (0.9) in the introduction (note that $\sigma_\nu(I_0)a_f(E_2)=(-1)^k a_f(E_2)$).

Thus we have completed the proof of the theorem in the introduction.

§ 4. Construction of cusp forms with $a_f(E_2)\neq 0$.

4.1. Let (τ,W) be a finite dimensional holomorphic representation of $GL_2(\mathbb{C})$. We denote by M_τ the space of modular forms of weight τ with respect to $\Gamma=Sp(2,\mathbb{Z})$ consisting of all W-valued holomorphic functions f on H_2 which satisfy

$$f(M\langle Z\rangle) = \tau(J(M,Z))f(Z) \qquad \text{for all } M\in\Gamma.$$

The Φ-operator on M_τ is also given by (1.2) for $f\in M_\tau$. If τ is equivalent to a direct sum $\bigoplus_{j=1}^{m}\tau_j$ of irreducible holomorphic representations (τ_j,W_j) of $GL_2(\mathbb{C})$, then it is clear that M_τ is isomorphic to the direct sum $\bigoplus_{j=1}^{m} M_{\tau_j}$ (as $\mathbb{C}$-vector spaces). Let (ρ_{k_1,ν_1},V_1) and (ρ_{k_2,ν_2},V_2) be two representations of $GL_2(\mathbb{C})$ given as in (0.1) in the introduction. For simplicity, we put $\tau_1=\rho_{k_1,\nu_1}$ and $\tau_2=\rho_{k_2,\nu_2}$. We consider the product representation $\tau_1\otimes\tau_2$ of $GL_2(\mathbb{C})$ with the representation space $V_1\otimes V_2$. It is known that $\tau_1\otimes\tau_2$ decomposes into a direct sum of irreducible representations of type (0.1) in the following manner:

$$(4.1) \qquad \tau_1\otimes\tau_2 \sim \bigoplus_{j=0}^{\mu} \rho_{k_1+k_2+j,\,\nu_1+\nu_2-2j} \qquad (\mu=\mathrm{Min}(\nu_1,\nu_2)),$$

where $\sim$ denotes the equivalence of representations. Then there exist invariant subspaces W_j $(0\leq j\leq\mu)$ of $V_1\otimes V_2$ such that the restriction of $\tau_1\otimes\tau_2$ to W_j is equivalent to $\rho_{k_1+k_2+j,\nu_1+\nu_2-2j}$ and

$$V_1 \otimes V_2 = \bigoplus_{j=0}^{\mu} W_j .$$

We write $M_{k,\nu}$ for M_ρ, when $\rho=\rho_{k,\nu}$. Due to the relation (4.1), one can identify $M_{\tau_1 \otimes \tau_2}$ with the space $\bigoplus_{j=0}^{\mu} M_{k_1+k_2+j,\nu_1+\nu_2-2j}$. For $f_j \in M_{\tau_j}$ (j=1, 2), the function $f_1(Z) \otimes f_2(Z)$ is naturally regarded as a modular form of $M_{\tau_1 \otimes \tau_2}$. We may put

$$(4.2) \quad f_1(Z) \otimes f_2(Z) = \sum_{j=0}^{\mu} \Psi_j(Z) \quad \text{with } \Psi_j \in M_{k_1+k_2+j,\nu_1+\nu_2-2j}.$$

Assume that k_i, ν_i (i=1, 2) are all even integers. Let v_0 (resp. v_0') be a vector of V_1 (resp. V_2) satisfying the condition (0.2) with respect to k_1, ν_1 (resp. k_2, ν_2). By Lemma 1.1, we have

$$\Phi f_1(z)=\phi_1(z)v_0 \quad \text{and} \quad \Phi f_2(z)=\phi_2(z)v_0' \qquad (z \in H_1)$$

with some $\phi_i \in S^1_{k_i+\nu_i}$ (i=1, 2). Therefore, we easily get

$$\phi_1(z)\phi_2(z)(v_0 \otimes v_0') = \sum_{j=0}^{\mu} \Phi\Psi_j(z) \qquad (z \in H_1).$$

Since it is easily verified that $v_0 \otimes v_0' \in W_0$, for each j ($1 \leqq j \leqq \mu$), Ψ_j is a cusp form. Let the Fourier expansion of ϕ_i (i=1, 2) be

$$\phi_i(z)= \sum_{m=1}^{\infty} a_i(m)e[mz] \qquad (z \in H_1).$$

Thus, for the Fourier coefficient $a_{f_1 \otimes f_2}(E_2)$ of the modular form $f_1(Z) \otimes f_2(Z)$, we easily have

$$a_{f_1 \otimes f_2}(E_2) = a_1(1)a_2(1)w^*,$$

where we put

$$w^* = v_0 \otimes \tau_2(J_0)v_0' + \tau_1(J_0)v_0 \otimes v_0', \qquad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We choose ϕ_1, ϕ_2 so that $a_j(1) \neq 0$ $(j=1,2)$, and write $w^* = \sum_{j=1}^{\mu} w_j$ $(w_j \in W_j)$. If $\nu_1 \neq \nu_2$, then it is not difficult to verify that $\mathbb{C}$-linear spans of $(\tau_1(g) \otimes \tau_2(g))w^*$ for all $g \in GL_2(\mathbb{C})$ generate the space $V_1 \otimes V_2$. Therefore, we have $w_j \neq 0$ for all j. Thus we see that, if $\nu_1 \neq \nu_2$, then for each j $(1 \leq j \leq \mu)$, Ψ_j is a cusp form with the condition $a_{\Psi_j}(E_2) \neq 0$.

4.2. We give an example. The same notation as above will be kept. Set $k_1=6$, $\nu_1=6$, $k_2=8$, $\nu_2=4$, and $\tau_1=\rho_{6,6}$, $\tau_2=\rho_{8,4}$. Let $\Delta(z)$ be the Ramanujan Δ-function:

$$\Delta(z) = q \prod_{m=1}^{\infty} (1-q^m), \qquad q=e[z] \qquad (z \in H_1).$$

Define $f_1 \in M_{6,6}$ and $f_2 \in M_{8,4}$ as Klingen's Eisenstein series by the following:

$$f_1(Z)=E_{6,6}(Z,\Delta,v_0) \quad \text{and} \quad f_2(Z)=E_{8,4}(Z,\Delta,v_0') \qquad (\text{see } (1.4)).$$

We write $(f_1 \otimes f_2)(Z) = \sum_{j=0}^{4} \Psi_j(Z)$ with $\Psi_j \in M_{14+j,10-2j}$ $(0 \leq j \leq 4)$. As is shown in 4.1, we have, for each j $(1 \leq j \leq 4)$,

$$\Psi_j \in S_{14+j,10-2j} \quad \text{and} \quad a_{\Psi_j}(E_2) \neq 0.$$

By virtue of the explicit calculation of the dimension of the space $S_{k,\nu}$ due to Tsushima [11], [12], we know that

$$\dim_{\mathbb{C}} S_{15,8}=4, \quad \dim_{\mathbb{C}} S_{16,6}=5, \quad \dim_{\mathbb{C}} S_{17,4}=1, \quad \dim_{\mathbb{C}} S_{18,2}=2.$$

Hence one can deduce that the unique cusp form χ of $S_{17,4}$ satisfies the condition $a_\chi(E_2)\neq 0$ and that the theorem in the introduction holds for χ.

References

[1] A. N. Andrianov, Dirichlet series with Euler products in the theory of Siegel modular forms of genus 2, Trudy Mat. Inst. Steklov 112 (1971), 73-94.

[2] A. N. Andrianov, Euler products corresponding to Siegel modular forms of genus 2, Uspekhi Mat. Nauk 29 (1974), 43-110.

[3] M. Eichler, The basis problem for modular forms and the traces of the Hecke operators, Lect. Notes in Math. 320. Berlin-Heidelberg-New York: Springer 1973.

[4] E. Hecke, Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung I, Math. Ann. 114 (1937), 1-28.

[5] H. Klingen, Zum Darstellungssatz fur Siegelsche Modulformen, Math. Z. 102 (1967), 30-43.

[6] T. Kubota, Über diskontinuierliche Gruppen Picardschen Typus und zugehörige Eisensteinsche Reihen, Nagoya Math. J. 32 (1968), 259-271.

[7] R. P. Langlands, On the functional equations satisfied by Eisenstein series, Lect. Notes in Math. 544. Berlin-Heidelberg-New York: Springer 1976.

[8] H. Maass, Die Primzahlen in der Theorie der Siegelschen Modulfuktionen, Math. Ann. 124 (1951), 87-122.

[9] Séminaire H. Cartan, Fonctions automorphes, 1957-1958.

[10] G. Shimura, On modular correspondences for Sp(n,$\mathbb{Z}$) and their congruence relations, Proc. Nat. Acad. U.S.A. 49 (1963), 824-828.

[11] R. Tsushima, An explicit dimension formula for the spaces of generalized automorphic forms with respect to Sp(2,$\mathbb{Z}$), to appear in Proc. Japan Acad.

[12] R. Tsushima, An explicit dimension formula for the spaces of generalized automorphic forms with respect to Sp(2,$\mathbb{Z}$), to appear.

Department of Mathematics
Rikkyo University
Nishi-Ikebukuro
Tokyo 171, Japan

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