42. An Explicit Dimension Formula for the Spaces of Generalized Automorphic Forms with Respect to $Sp(2, Z)$

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Let $\mathfrak{S}_g$ be the Siegel upper half plane of degree g. The real symplectic group $Sp(g, \boldsymbol{R})$ acts on $\mathfrak{S}_g$ as

$$Z \longmapsto M \cdot Z := (AZ+B)(CZ+D)^{-1},$$

for

$$Z \in \mathfrak{S}_g \quad \text{and} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \boldsymbol{R}).$$

Let Z and M be as above, and put

$$J(M, Z) = CZ + D \qquad (\in GL(g, \boldsymbol{C})).$$

This satisfies the following relation for any M, $M' \in Sp(g, \boldsymbol{R})$ and $Z \in \mathfrak{S}_g$:

$$J(MM', Z) = J(M, M' \cdot Z) J(M', Z),$$

and this is called the *canonical automorphic factor*. Let μ be a holomorphic representation of $GL(g, \boldsymbol{C})$ into $GL(r, \boldsymbol{C})$. Then $\mu(J(M, Z)) = \mu(CZ+D)$ also satisfies the above relation.

Let μ be as above and let Γ be a subgroup of finite index of $Sp(g, \boldsymbol{Z})$. By an *automorphic form of type* μ with respect to Γ, we mean a holomorphic mapping f of $\mathfrak{S}_g$ to the r dimensional complex vector space $\boldsymbol{C}^r$ which satisfies the equalities:

$$f(M \cdot Z) = \mu(CZ+D) f(Z),$$

for any $M \in \Gamma$ and $Z \in \mathfrak{S}_g$ (we need to assume the holomorphy of f at "cusps" if $g=1$). An automorphic form of type μ with respect to Γ is called a *cusp form*, if it belongs to the kernel of Φ-operator ([1] Exposé 8). We denote by $A_\mu(\Gamma)$ and $S_\mu(\Gamma)$ the spaces of automorphic forms and cusp forms of type μ with respect to Γ, respectively. They are finite dimensional vector spaces. In case $\mu(CZ+D) = \det(CZ+D)^k$, an automorphic form of type μ is also called an automorphic form of *weight* k, and $A_\mu(\Gamma)$ is also denoted by $A_k(\Gamma)$. Similarly $S_\mu(\Gamma)$ is also denoted by $S_k(\Gamma)$.

Let Γ be as above. Then it is known that Γ contains the principal congruence subgroup $\Gamma_g(l)$ of $Sp(g, \boldsymbol{Z})$ for some l, if $g \geq 2$. We may assume that $l \geq 3$. Then $\Gamma_g(l)$ has no torsion elements and the quotient space $\mathfrak{S}_g^*(l) := \Gamma_g(l) \backslash \mathfrak{S}_g$ is non-singular. In the case of degree two the author calculated the dimension of $S_k(\Gamma)$ and represented it by the group theoretical conditions of $\Gamma/\Gamma_2(l)$ by applying the formula of Riemann-Roch-Hirzebruch with action of finite groups to the smooth compactification $\tilde{\mathfrak{S}}_2^*(l)$ of $\mathfrak{S}_2^*(l)$ ([7]). In [8] the author generalized this result to the case of the general holomorphic representation μ of $GL(2, \boldsymbol{C})$. In this note we announce the explicit formula of $\dim S_\mu(\Gamma_2(l))$ with $l \geq 1$ (as to the work of other authors see the Introduction of [7]).

We define the action of $Sp(g, \boldsymbol{R})$ on the product manifold $\mathcal{E}_\mu := \mathfrak{S}_g \times \boldsymbol{C}^r$ by

$$M(Z, \xi) = (M \cdot Z, \mu(J(M, Z))\xi),$$

for $M \in Sp(g, \boldsymbol{R})$, $Z \in \mathfrak{S}_g$ and $\xi \in \boldsymbol{C}^r$. We denote $\Gamma_g(l) \backslash \mathcal{E}_\mu$ by E_μ. E_μ has a structure of a holomorphic vector bundle on $\mathfrak{S}_g^*(l)$. $A_\mu(\Gamma_g(l))$ is naturally identified with $H^0(\mathfrak{S}_g^*(l), \mathcal{O}(E_\mu))$. Let $\tilde{\mathfrak{S}}_g^*(l)$ be Namikawa's compactification of $\mathfrak{S}_g^*(l)$ ([5]). Then E_μ has a natural extension to a holomorphic vector bundle $\tilde{E}_\mu$ on $\tilde{\mathfrak{S}}_g^*(l)$. An element f of $H^0(\mathfrak{S}_g^*(l), \mathcal{O}(E_\mu))$ has an extension to an element $\tilde{f}$ of $H^0(\tilde{\mathfrak{S}}_g^*(l), \mathcal{O}(\tilde{E}_\mu))$, since f has a Fourier expansion at each cusp ([1]). Hence $A_\mu(\Gamma_g(l))$ is also identified with $H^0(\tilde{\mathfrak{S}}_g^*(l), \mathcal{O}(\tilde{E}_\mu))$. It is known that $\tilde{\mathfrak{S}}_g^*(l)$ is non-singular and $\Delta(g) := \tilde{\mathfrak{S}}_g^*(l) - \mathfrak{S}_g^*(l)$ is a divisor with normal crossings, if $g \leq 4$. $S_\mu(\Gamma_g(l))$ is identified with $H^0(\tilde{\mathfrak{S}}_g^*(l), \mathcal{O}(\tilde{E}_\mu - \Delta(g)))$.

By using the Kawamata-Viehweg's generalization of Kodaira-Ramanujam's vanishing theorem ([3], [9]), we can prove the following

Theorem 1. *Let μ be an irreducible holomorphic representation of $GL(2, \boldsymbol{C})$ and $(j+k, k)$ its signature, where j and k are integers with $j \geq 0$. If $j=0$, $k \geq 4$ or $j \geq 1$, $k \geq 5$, then for $p>0$, we have*

$$H^p(\tilde{\mathfrak{S}}_2^*(l), \mathcal{O}(\tilde{E}_\mu - \Delta(2))) \simeq 0.$$

Remark 1. Let μ be an irreducible holomorphic representation of $GL(g, \boldsymbol{C})$ and $(f_1, f_2, \cdots, f_g)$ with $f_1 \geq f_2 \geq \cdots \geq f_g$ its signature. In case Γ is a discrete subgroup of $Sp(g, \boldsymbol{R})$ without torsion elements such that $\Gamma \backslash \mathfrak{S}_g$ is compact, we can prove that if $f_g \geq g+2$, then for $p>0$

$$(*) \qquad H^p(\Gamma \backslash \mathfrak{S}_g, \mathcal{O}(E_\mu)) \simeq 0,$$

by the vanishing theorem of Nakano ([4]). So it is expected that the above theorem holds always under the condition that $k \geq 4$. But we cannot prove this now. M. Ise calculated the dimension of the spaces of automorphic forms in the case of compact quotients ([2]). He proved the vanishing theorem under the assumption that $f_g \geq 2g+1$ by using the original vanishing theorem of Kodaira. So the result (*) is more strict than his and this is the best possible.

If $l \geq 3$, we get the following theorem by Theorem 1 and the formula of Riemann-Roch-Hirzebruch:

Theorem 2. *Under the assumption of Theorem 1, the dimension of $S_\mu(\Gamma_2(l))$ is equal to*

$$(2^{-8}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3)l^{10} - 2^{-6}3^{-2}(j+1)(j+2k-3)l^{8}$$

$+2^{-5}3^{-1}(j+1)l^{7})\Pi_{p|l,p:\text{ prime}}(1-p^{-2})(1-p^{-4})$.

Remark 2. If j is odd and $-1_4 \in \Gamma$, then since $\mu(-1_2)=-1_{i+1}$, we have

$$S_\mu(\Gamma)\simeq 0.$$

By using the results of [8] Theorem (3.2), we get the following theorems similarly as in [7] Section 5. We assume that the signature of μ is $(2j+k, k)$.

Theorem 3. *Under the assumption of Theorem* 1, *the dimension of* $S_\mu(\Gamma_2(2))$ *is equal to*

$$\begin{aligned}&2^{-8}3^{-1}(2j+1)(k-2)(2j+k-1)(2j+2k-3)-2^{-8}5^{1}(2j+1)(2j+2k-3)\\&\quad+2^{-8}3^{1}5^{1}(2j+1)+(-1)^{k}(2^{-8}5^{1}(k-2)(2j+k-1)-2^{-8}3^{1}5^{1}(2j+2k-3)\\&\quad+2^{-8}3^{2}5^{1}).\end{aligned}$$

Let i, ρ, ω and σ be $\sqrt{-1}$, $e^{2\pi i/3}$, $e^{2\pi i/5}$ and $e^{\pi i/6}$, respectively. We denote $\mathrm{tr}_{Q[\alpha]/Q}$ by tr_α for an algebraic number α.

Theorem 4. *Under the assumption of Theorem* 1, *the dimension of* $S_\mu(\Gamma_2(1))$ *is equal to*

$$\begin{aligned}&2^{-7}3^{-3}5^{-1}(2j+1)(k-2)(2j+k-1)(2j+2k-3)-2^{-5}3^{-2}(2j+1)(2j+2k-3)\\&\quad+2^{-4}3^{-1}(2j+1)+(-1)^{k}(2^{-7}3^{-2}7^{1}(k-2)(2j+k-1)-2^{-4}3^{-1}(2j+2k-3)\\&\quad+2^{-5}3^{1})+(-1)^{j}(2^{-7}3^{-1}5^{1}(2j+2k-3)-2^{-3})+(-1)^{k}(-1)^{j}2^{-7}(2j+1)\\&\quad+\mathrm{tr}_i\,(i)^{k}(2^{-6}3^{-1}(i)(2j+k-1)-2^{-4}(i))+\mathrm{tr}_i\,(-1)^{k}(i)^{j}2^{-5}(i+1)\\&\quad+\mathrm{tr}_i\,(i)^{k}(-1)^{j}(2^{-6}3^{-1}(k-2)-2^{-4})+\mathrm{tr}_i\,(-i)^{k}(i)^{j}2^{-5}(i+1)\\&\quad+\mathrm{tr}_\rho\,(-1)^{k}(\rho)^{j}3^{-3}(\rho+1)+\mathrm{tr}_\rho\,(\rho)^{k}(\rho)^{j}2^{-2}3^{-4}(2\rho+1)(2j+1)\\&\quad-\mathrm{tr}_\rho\,(\rho)^{k}(-\rho)^{j}2^{-2}3^{-2}(2\rho+1)+\mathrm{tr}_\rho\,(-\rho)^{k}(\rho)^{j}3^{-3}\\&\quad+\mathrm{tr}_\rho\,(\rho)^{j}(2^{-1}3^{-4}(1-\rho)(2j+2k-3)-2^{-1}3^{-2}(1-\rho))\\&\quad+\mathrm{tr}_\rho\,(\rho)^{k}(2^{-3}3^{-4}(\rho+2)(2j+k-1)-2^{-2}3^{-3}(5\rho+6))\\&\quad-\mathrm{tr}_\rho\,(-\rho)^{k}(2^{-3}3^{-3}(\rho+2)(2j+k-1)-2^{-2}3^{-2}(\rho+2))\\&\quad+\mathrm{tr}_\rho\,(\rho)^{k}(\rho^{2})^{j}(2^{-3}3^{-4}(1-\rho)(k-2)+2^{-2}3^{-3}(\rho-5))\\&\quad+\mathrm{tr}_\rho\,(-\rho)^{k}(\rho^{2})^{j}(2^{-3}3^{-3}(1-\rho)(k-2)-2^{-2}3^{-2}(1-\rho))\\&\quad+\mathrm{tr}_\omega\,(\omega)^{k}(\omega^{4})^{j}5^{-2}-\mathrm{tr}_\omega\,(\omega)^{k}(\omega^{3})^{j}5^{-2}\omega^{2}\\&\quad+\mathrm{tr}_\sigma\,(\sigma^{7})^{k}(-1)^{j}2^{-3}3^{-2}(\sigma^{2}+1)-\mathrm{tr}_\sigma\,(\sigma^{7})^{k}(\sigma^{8})^{j}2^{-3}3^{-2}(\sigma+\sigma^{3}).\end{aligned}$$

Remark 3. Let μ be as in Remark 1. If $f_g<0$, then

$$S_\mu(\Gamma)\simeq 0$$

for any subgroup of finite index Γ of $Sp(g, Z)$ ([1] Exposé 8).

The group $\Gamma_2(1)/\Gamma_2(l)$ acts on $\mathfrak{S}_2^*(l)$. The values in Theorem 4 are equal to the Euler-Poincaré characteristics

$$\chi_{kj}:=\sum_{i=0}^{3}(-1)^{i}\dim H^{i}(\mathfrak{S}_2^*(l), \mathcal{O}(\tilde{E}_\mu-\Delta(2)))^{\Gamma_2(1)/\Gamma_2(l)},$$

for any j (≥ 0) and k. The generating function of χ_{kj}:

$$\sum_{k,j=0}^{\infty}\chi_{kj}t^{k}s^{j}$$

is a rational function of t and s whose denominator is

$$(1-t^{4})(1-t^{5})(1-t^{6})(1-t^{12})(1-s^{3})(1-s^{4})(1-s^{5})(1-s^{6}).$$

Let $f(t, s)$ be the numerator. $f(t, s)$ is of degree 26 with respect to t

and of degree 17 with respect to s. We list the coefficients of $t^k s^j$ in $f(t, s)$ in Table 1 in [8], and list the values in Theorem 4 in Table 2 in [8] for $0 \leq j \leq 13$ and $0 \leq k \leq 30$.

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