ON SIEGEL MODULAR FORMS OF DEGREE TWO.

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Introduction. Let H_n be the Siegel upper half plane of degree n and $\Gamma_n(\ell)$ the principal congruence subgroup of $\mathrm{Sp}(n,\mathbf{Z})$ of level ℓ. Let $A(\Gamma_n(\ell))_k$ be the space of modular forms of weight k with respect to $\Gamma_n(\ell)$ and put

$$A(\Gamma_n(\ell)) = \oplus_{k\geqslant 0} A(\Gamma_n(\ell))_k.$$

Then $A(\Gamma_n(\ell))$ is a positively graded, integral domain and finitely generated over $\mathbf{C}$, and the projective variety $\mathfrak{S}(\Gamma_n(\ell))$ associated with this graded ring is the Satake compactification of the quotient $\Gamma_n(\ell)\backslash H_n$. In [9] Igusa showed that the blowing up $\mathfrak{M}(\Gamma_n(\ell))$ of $\mathfrak{S}(\Gamma_n(\ell))$ with respect to the sheaf of ideals defined by all cusp forms is non-singular for $n=2$ or 3 and $\ell\geqslant 3$.

We shall examine the condition under which multiple forms on $\Gamma_2(\ell)\backslash H_2$ can be extended to $\mathfrak{M}(\Gamma_i(\ell))$ (Sec. 2). It follows immediately from this study that the variety $\mathfrak{M}(\Gamma_2(\ell))$ is of general type for $\ell\geqslant 4$.

We can construct a line bundle L on $\mathfrak{M}(\Gamma_2(\ell))$ which corresponds to modular forms of weight one with respect to $\Gamma_2(\ell)$ for $\ell\geqslant 3$. It is a natural problem to establish the explicit Riemann-Roch theorem for this line bundle L. In Sec. 3 we shall calculate the related intersection numbers. The result is given as follows;

(i) $c(L)^3 = 2^{-6}3^{-2}5^{-1}\ell^{10} \prod_{p/\ell} (1-p^{-2})(1-p^{-4})$,

(ii) $c(L)^2 c(D) = 0$,

(iii) $c(L)c(D)^2 = -2^{-3}3^{-1}\ell^{8}\prod(1-p^{-2})(1-p^{-4})$,

(iv) $c(D)^3 = -11\cdot 2^{-2}3^{-1}\ell^{7}\prod(1-p^{-}2)(1-p^{-4})$,

(v) $c_2 c(D) = 2^{-3}\ell^{7}\prod(1-p^{-2})(1-p^{-4})$,

(vi) $c_2 c(L) = 4c(L)^3$,

where D is a divisor determined by the complement $\mathfrak{M}(\Gamma_2(\ell)) - \Gamma_2(\ell)\backslash H_2$.

It follows from the results in Sec. 2 that the canonical bundle K of

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$\mathfrak{M}(\Gamma_2(\ell))$ is given by $3L-[D]$. Therefore by the Riemann-Roch theorem and the vanishing theorem of Kodaira's type, we obtain the following dimension formula for the vector space $S(\Gamma_2(\ell))_k$ of cusp forms of weight $k \geqslant 4$: (Sec. 4)

$$\dim S(\Gamma_2(\ell))_k = \ell^{10}\cdot 2^{-10}3^{-3}5^{-1}(2k-2)(2k-3)(2k-4)\prod(1-p^{-2})(1-p^{-4})$$
$$-2^{-6}3^{-2}(2k-3)\ell^{8}\prod(1-p^{-2})(1-p^{-4})$$
$$+2^{-5}3^{-1}\ell^{7}\prod(1-p^{-2})(1-p^{-4}).$$

This formula was also obtained by Y. Morita (under a slightly stronger restriction on the weight k) by using the Selberg trace formula ([11]).

1. The principal congruence subgroup $\Gamma_n(\ell)$ of level ℓ is defined by

$$\Gamma_n(\ell) = \{M \in \mathrm{Sp}(n,\mathbf{Z});\ M \equiv I_{2n} \bmod \ell\},$$

and the index is given by

$$[\Gamma_n(1);\Gamma_n(\ell)] = \ell^{n(2n+1)} \prod_{p|\ell} \prod_{1\leqslant k\leqslant n} (1-p^{-2k}).$$

The boundary of the Satake compactification $\mathfrak{S}(\Gamma_n(\ell))$ of the quotient $\Gamma_n(\ell)\backslash H_n$ is a disjoint union of quasi-projective varieties, each of which is a conjugate of the image of $\Gamma_m(\ell)\backslash H_m$ under the dual Φ^* of the Siegel Φ-operator for some $m<n$.

Let $\mathfrak{M}(\Gamma_n(\ell)) \to \mathfrak{S}(\Gamma_n(\ell))$ be the monoidal transform of $\mathfrak{S}(\Gamma_n(\ell))$ along its boundary.

THEOREM. ([9]). *The monoidal transform $\mathfrak{M}(\Gamma_n(\ell))$ is non-singular for $n=2$ or 3 and $\ell \geqslant 3$.*

Now the local parameters for $n=2$ and $\ell \geqslant 3$ are given explicitly as follows. Let ω be a point of $\mathfrak{M}(\Gamma_2(\ell))$ such that its projection is the image point of a point t_0 of H_1. Then take a sequence of points in $\Gamma_2(\ell)\backslash H_2$ which converges to ω, and take representatives of these points in H_2 to obtain a sequence of points with $(t,z,w) = \begin{pmatrix} t & z \\ z & w \end{pmatrix}$, say, as a typical term. By taking a subsequence if necessary, we can assume that (t,z) converges to (t_0,z_0) and $\mathrm{Im}\ \omega \to \infty$. Let $\xi = e(\omega/\ell)$; then $\xi \to \xi_0 = 0$, where $e(x)$ stands for $e^{2\pi i x}$. If we denote the local parameters at t_0, z_0, and ξ_0 by $t-t_0$, $z-z_0$, and ξ respectively, then $(t-t_0, z-z_0, \xi)$ is a local coordinate system of $\mathfrak{M}(\Gamma_2(\ell))$ at ω. ([9])

2. Let m be a vector in $\mathbf{Z}^{2n}$ and m', m'' be vectors in $\mathbf{Z}^n$ determined by the first and the last n components of m. Now if τ is a point in H_n and z is a point in $\mathbf{C}^n$, the following series

$$\theta_m(\tau,z)=\sum_{p\in Z^n} e\Big(\frac{1}{2}{}^t(p+m'/2)\tau(p+m'/2)+{}^t(p+m'/2)(z+m''/2)\Big)$$

converges absolutely and uniformly in every compact subset of $H_n\times\mathbf{C}^n$. Therefore for a fixed m, it represents an analytic function of the two variables τ and z, which is called the theta-function of characteristic m. If we put $z=0$, we get an analytic function $\theta_m(\tau)=\theta_m(\tau,0)$ on H_n, which is called the theta-constant. There are ten theta-constants which are not identically zero for $n=2$. We denote by $\theta(\tau)$ the product of all such functions.

PROPOSITION. ([8]). *Let $\psi(\tau)=\theta(\tau)^2$, then it is a unique cusp form of weight ten with respect to* $\Gamma_2(1)$.

The modular form in the above proposition has the following Fourier-Jacobi expansion;

$$\psi(\tau)=\big[-6(\theta_{00}\theta_{10}\theta_{01})(t)^6\theta_{11}(t,z)^2+\cdots\big]e(w),$$

where the unwritten part is a convergent power series in t, z, and $e(w)$.

Let $\tau=(t,z,w)$ be the coordinate of H_2 and $d\tau=dt\wedge dz\wedge dw$. Using the above cusp form $\psi(\tau)$, we set

$$\varphi=\psi(\tau)^6(d\tau)^{10};$$

then it is $\Gamma_2(1)$-invariant 10-ple 3-form on H_2. Therefore it is, in particular, $\Gamma_2(\ell)$-invariant, so it can be regarded as a 10-ple 3-form on $\Gamma_2(\ell)\backslash H_2 \subset \mathfrak{M}(\Gamma_2(\ell))$. Now we examine the condition under which φ can be extended to the whole of $\mathfrak{M}(\Gamma_2(\ell))$.

The differential $d\tau$ is expressed as

$$d\tau=\frac{1}{2\pi i}\,\ell\, dt\wedge dz\wedge\xi^{-1}d\xi,$$

with respect to the local coordinate system $(t-t_0,z-z_0,\xi)$. Now φ has the following expansion;

$\varphi=\text{const.}\ [(\theta_{00}\theta_{01}\theta_{10})(t)^{18}\theta_{11}(t,z)^6+\cdots]\xi^{3\ell-10}(dt\wedge dz\wedge d\xi)^{10}$, where the unwritten part is a convergent power series in t, z, and ξ^{ℓ}. Therefore φ is holomorphic with respect to $(t-t_0,z-z_0,\xi)$ if and only if $3\ell-10\geqslant 0$. Therefore if $\ell\geqslant 4$, φ can be extended to $\mathfrak{M}(\Gamma_2(\ell))$ by the continuation theorem as a holomorphic 10-ple 3-form.

By a well-known asymptotic behaviour of the dimensions of the vector spaces of modular forms with respect to $\mathrm{Sp}(2,\mathbf{Z})$, we obtain:

THEOREM *. *The non-singular model* $\mathfrak{M}(\Gamma_2(\ell))$ *is of general type for* $\ell \geqslant 4$. *In particular, in this case, it is non-rational.*

3. From now on we fix a level $\ell \geqslant 3$. Throughout this section we shall denote by Y the Satake compactification of the quotient $\Gamma_2(\ell)\backslash H_2$, and by $\pi: X \to Y$ the Igusa's desingularization. We denote by D and B the complements of $\Gamma_2(\ell)\backslash H_2$ in X and Y respectively. Then D and B are decomposed into the same number of irreducible components,

$$D=\sum D_i, \qquad B=\sum B_i,$$

where the number $\mu(\ell)$ of irreducible components is given by ([2])

$$\mu(\ell)=\frac{1}{2}\ell^4\prod_{p\mid\ell}(1-p^{-4}).$$

Each B_i is isomorphic to the standard compactification of $\Gamma_1(\ell)\backslash H_1$, namely it is set-theoretically the union of $\Gamma_1(\ell)\backslash H_1$ and cusps P_j;

$$B_i=(\Gamma_1(\ell)\backslash H_u)\cup P_1\cup\cdots\cup P_{\nu(\ell)},$$

where the number $\nu(\ell)$ of cusps is given by

$$\nu(\ell)=\frac{1}{2}\ell^2\prod_{p\mid\ell}(1-p^{-2}).$$

The restriction of π to D_i, which we also denote by π, gives rise to a projection $D_i \to B_i$. By this projection, D_i is the elliptic modular surface of level ℓ in the sense of Shioda [13]. That is, its general fibers are elliptic curves with level ℓ structures and it has singular fibers over the cusps of B_i. The singular fibers consist of ℓ lines with multiplicity one and with self-intersection number -2, and ℓ lines intersect like edges of an ℓ-gon. [9] (For terminology see [10].)

The group $\Gamma_2(1)/\Gamma_2(\ell)$ operates on X as a group of automorphisms and D_i's are mapped isomorphically to each other by this group.

LEMMA 1. *The Euler number* $e(D_i)$ *of* D_i *is given by*

$$e(D_i)=\ell\nu(\ell)$$

*I was informed by Prof. Igusa that this result is already known among some of the specialists, but there is no statements with complete proofs in the literature.

Proof. By the theory of elliptic surface, $e(D_i)$ is equal to the sum of the Euler number of singular fibers of D_i, so that we have,

$$e(D_i)=\nu(\ell)(1-1+\ell)=l\nu(\ell).$$

Q.E.D.

LEMMA 2. ([13]). *Let $K(D_i)$ be the canonical bundle of D_i and let $\pi: D_i \to B_i$ be the natural projection. Then we have*

$$K(D_i)=\pi^* M_i,$$

where M_i is a line bundle on B_i which corresponds to cusp forms of weight three with respect to $\Gamma_1(\ell)$. Moreover the degree of M_i is given by

$$\deg(M_i)=2^{-3}\ell^2(\ell-4)\Pi(1-p^{-2}).$$

As in Section 2, we denote by $\theta(\tau)$ the product of all even theta constants of degree two. We know that it is a cusp form of weight five with respect to $\Gamma_2(2)$ and its square is a cusp form of weight ten with respect ot $\Gamma_2(1)$. We have the following.

THEOREM. ([5]). *Let Δ be the set of diagonal elements in H_2. Then the zero set of $\theta(\tau)$ is precisely the union of all $\Gamma_2(1)$-conjugates of Δ.*

Let E be the closure of $\Gamma_2(\ell)\backslash\Gamma_2(1)\Delta$ in X, and decompose E into irreducible components;

$$E=\sum E_\alpha.$$

LEMMA 3 *Under the decomposition $E=\Sigma E_\alpha$, the number $\lambda(\ell)$ of irreducible components is given by*

$$\lambda(\ell)=\frac{1}{2}\ell^4\Pi(1+p^{-2}).$$

Proof. Let

$$G=\{M\in\Gamma_2(1); M\Delta=\Delta\}$$

and

$$G'=\left\{\begin{pmatrix} a_1 & 0 & b_1 & 0\\ 0 & a_2 & 0 & b_2\\ c_1 & 0 & d_1 & 0\\ 0 & c_2 & 0 & d_2\end{pmatrix}\in\Gamma_2(1)\right\}.$$

It is easy to see that $G' \cong \Gamma_1(1)\times\Gamma_1(1)$, $G' \subset G$ with $[G;G']=2$, and $G=G'\cup G'V$, where

$$V=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since $G\cap\Gamma_2(\ell)=G'\cap\Gamma_2(\ell)\cong\Gamma_1(\ell)\times\Gamma_1(\ell)$, we have

$$\begin{aligned}\lambda(\ell)&=[\Gamma_2(1);\Gamma_2(\ell)G]\\ &=[\Gamma_2(1);\Gamma_2(\ell)][\Gamma_2(\ell)G;\Gamma_2(\ell)]^{-1}\\ &=\frac{1}{2}[\Gamma_2(1);\Gamma_2(\ell)][G';G'\cap\Gamma_2(\ell)]^{-1}\\ &=\frac{1}{2}\ell^{10}\Pi(1-p^{-2})(1-p^{-4})[\ell^3\Pi(1-p^{-2})]^{-2}\\ &=\frac{1}{2}\ell^4\Pi(1+p^{-2}).\end{aligned}$$

Q.E.D.

As we have remarked before, the group $\Gamma_2(1)/\Gamma_2(\ell)$ operates on X as a group of automorphisms and by this action the sets of components $\{D_i\}$ and $\{E_\alpha\}$ are homogeneous. Therefore, to study the intersection properties among them, it suffices to see at special places. Let D_1 be the component of D at the infinity in the sense that $\operatorname{Im} w\to\infty$, where $\tau=\begin{pmatrix} t & z \\ z & w\end{pmatrix}$ is the coordinates of H_2. With the same notations as in the proof of lemma 2, let E_1 be the closure of $\Gamma_2(\ell)\cap G\backslash\Delta$ in X. Obviously the quotient $\Gamma_2(\ell)\cap G\backslash\Delta$ is isomorphic to $(\Gamma_1(l)\backslash H_1)\times(\Gamma_1(\ell)\backslash H_1)$. We remark that, if w is the coordinate of H_1, we can take $e(w/\ell)$ as the local coordinate of the cusp at the infinity in the standard compactification $(\Gamma_1(\ell)\backslash H_1)^*$ of $\Gamma_1(\ell)\backslash H_1$. This is the same as that of X which determines the divisor D_1. Therefore it follows from the form of the local coordinate system of X at D_1 (Sec. 1), that D_1 and E_1 intersect transversally with multiplicity one. The intersection $D_1\cdot E_1$ is isomorphic to the standard compactification of $\Gamma_1(\ell)\backslash H_1$. More precisely, on D_1 it consists of origins of general fibers of π, and on E_1 it is isomorphic to the product $(\Gamma_1(\ell)\backslash H_1)^*\times P$, where P is the cusp at the infinity in the standard compactification of $\Gamma_1(\ell)\backslash H_1$. Therefore E_1, hence each E_α, is isomorphic to $(\Gamma_1(\ell)\backslash H)^*\times(\Gamma_1(\ell)\backslash H_1)^*$.

If D_i intersects with E_1, the intersection $D_i\cdot E_1$ takes form on E_1 of either $(\Gamma_1(\ell)\times H_1)^*\times\{\text{cusp}\}$ or $\{\text{cusp}\}\times(\Gamma_1(\ell)\backslash H_1)^*$. There are $2\nu(\ell)D_i$'s which intersect with E_1, and they are given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{bmatrix} D_1, \quad \text{or} \quad \begin{bmatrix} 0 & a & 0 & b \\ 1 & 0 & 0 & 0 \\ 0 & c & 0 & d \\ 0 & 0 & 1 & 0 \end{bmatrix} D_1,$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over a complete set of representatives of $\Gamma_1(1)/\Gamma_1(\ell)\Gamma_{1\infty}$ with

$$\Gamma_{1\infty} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL(2, \mathbf{Z}) \right\}.$$

On the other hand, if E_α intersects with D_1, the intersection $E_\alpha \cdot D_1$ is a image of a section of π which consists of points of order ℓ of general fibers of π. There are ℓ^2 such sections so that there are the same number E_α's which intersect with D_1, and they are given by

$$\begin{bmatrix} 1 & 0 & 0 & b \\ a & 1 & b & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix} E_1, \qquad 0 \leqslant a < \ell, \quad 0 \leqslant b < \ell.$$

Now the partial derivative $\partial\theta(\tau)/\partial z$ does not vanish on Δ, where $\tau = \begin{pmatrix} t & z \\ z & w \end{pmatrix}$. ([5]) So that if $\alpha \neq \beta$, E_α does not intersect with E_β in $\Gamma_2(\ell)\backslash H_2$. On the other hand it is easy to see that if $\alpha = \beta$, then $E_\alpha \cap E_\beta \cap D_i = \phi$ for every i. We summarize the results.

Lemma 4. *The divisor E is a disjoint union of non-singular surfaces each of which is isomorphic to the product* $R_1 \times R_2$, *where* R_i *is the standard compactification of* $\Gamma_1(\ell)\backslash H_1$.

Lemma 5. *Let* $E_\alpha \cong R_1 \times R_2$, *and let* p_i *be the i-th projection of* $R_1 \times R_2$. *Let* L_i *be a line bundle on* R_i *which corresponds to modular forms of weight one with respect to* $\Gamma_1(\ell)$. *Then the normal bundle* $N(E_\alpha)$ *of* E_α *in X is given by*

$$N(E_\alpha) = -(p_1{}^*L_1 + p_i{}^*L_2).$$

Proof. Since the E_α's are conjugate under the group $\Gamma_2(1)$, we may assume E_α is the closure E_1 of $\Gamma_2(\ell) \cap G\backslash\Delta$ in X, where G and Δ are the same as before.

Take an element

$$M=\begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}$$

in $\Gamma_2(\ell)\cap G$, and if we set $\tau=\begin{pmatrix} t & z \\ z & w \end{pmatrix}$ and $M\tau=\tau'=\begin{pmatrix} t' & z' \\ z' & w' \end{pmatrix}$, then

$$z'=z/\{(c_1t+d_1)(c_2w+d_2)-c_1c_2z^2\}$$

Therefore we have

$$\lim_{z\to 0} z'/z=(c_1t+d_1)^{-1}(c_2w+d_2)^{-1}.$$

Since the local coordinate of $(\Gamma_1(\ell)\backslash H_1)^*$ at a cusp is the same as that of E_1 induced from X, we obtain the lemma. Q.E.D.

On the Satake compactification Y, we have a natural ample line bundle M which corresponds to modular forms of weight one. We set

$$L=\pi^*M.$$

Since the graded ring $A(\Gamma_2(\ell))$ is normal, it follows from the definition of the Satake compactification that the 0-th cohomology group $H^0(Y,\mathcal{O}(kM))$ is canonically isomorphic to the vector space $A(\Gamma_2(\ell))_k$ of modular forms of weight k with respect to $\Gamma_2(\ell)$. Since Y is a normal variety,

$$H^0(X,\mathcal{O}(kL))=H^0(Y,\mathcal{O}(kM)),$$

hence we have

$$H^0(X,\mathcal{O}(kL))=A(\Gamma_2(\ell))_k.$$

LEMMA 6. *The restriction $L|E_\alpha$ of L to E_α is expressed as*

$$P_1{}^*L_1+P_2{}^*L_2,$$

where the notations are the same as in Lemma 5.

The proof is straight forward, so we omit the proof.

LEMMA 7. *Let $[E]$ and $[D]$ be line bundles which are determined by the* *divisors E and D. Then the line bundle* $10L$ *has the following expression*;

$$10L = 2[E] + \ell[D].$$

Proof. As we have observed, we have the cusp form θ^2 of weight ten with respect to $\Gamma_2(1)$, and it is naturally interpreted as a section of the line bundle $10L$ on X. Since the divisor of zeroes of θ^2 is $2E + \ell D$, we have $10L = 2[E] + \ell[D]$. Q.E.D.

We shall always identify a cohomology class in $H^6(X, \mathbf{Z})$ with its value at the fundamental cycle X.

THEOREM 1. *Let* $c(E)$ *be the Chern class of the line bundle* $[E]$. *Then we have*

$$c(E)^3 = 2^{-6}3^{-2}\ell^{10}\prod(1-p^{-2})(1-p^{-4}).$$

Proof. Since $E = \sum E_\alpha$ is a disjoint union,

$$c(E)^3 = \sum c(E_\alpha)^3.$$

As in Lemma 5, let $E_c \cong R_1 \times R_2$ and let L_i be a line bundle on R_i which corresponds modular forms of weight one with respect to $\Gamma_1(\ell)$. Then we have

$$\begin{aligned} c(E_\alpha)^3 &= c(N(E_\alpha))^2 \\ &= [-c(p_1{}^*L_1 + p_2{}^*L_2)]^2 \\ &= 2c(p_1{}^*L_1)c(p_2{}^*L_2) \\ &= 2\left[2^{-3}3^{-1}\ell^3\prod(1-p^{-2})\right]^2 \\ &= 2^{-5}3^{-2}\ell^6\prod(1-p^{-2})^2, \end{aligned}$$

hence

$$\begin{aligned} c(E)^3 &= \lambda(\ell)c(E_\alpha)^3 \\ &= 2^{-6}3^{-2}\ell^{10}\prod(1-p^{-2})(1-p^{-4}). \end{aligned}$$

Q.E.D.

THEOREM 2. *Let* $c(D)$ *be the Chern class of the line bundle* $[D]$. *Then we have*

$$c(E)^2c(D) = -2^{-4}3^{-1}\ell^9\prod(1-p^{-2})(1-p^{-4}).$$

Proof. Since the sum ΣE_α is disjoint,

$$c(E)^2c(D)=\sum c(E_\alpha)^2c(D).$$

On the other hand, by the intersection properties among E_α and D_i's, we have

$$\begin{aligned}c(E_\alpha)^2c(D)&=c(N(E_\alpha))c(D|E_\alpha)\\&=-2\nu(\ell)2^{-3}3^{-1}\ell^3\Pi(1-p^{-2}),\end{aligned}$$

hence

$$\begin{aligned}c(E)^2c(D)&=\lambda(\ell)c(E_\alpha)^2c(D)\\&=-2^{-4}3^{-1}\ell^9\Pi(1-p^{-2})(1-p^{-4}).\end{aligned}$$

Q.E.D.

THEOREM 3. *We have*

$$c(E)c(D)^2=2^{-2}\ell^8\Pi(1-p^{-2})(1-p^{-4}).$$

Proof. By the observation at the beginning of this section, we have

$$\begin{aligned}c(E_\alpha)c(D)^2&=\sum c(E_\alpha)c(D_i)c(D_j)\\&=2\nu(\ell)^2,\end{aligned}$$

hence

$$\begin{aligned}c(E)c(D)^2&=\lambda(\ell)2\nu(\ell)^2\\&=2^{-2}\ell^8\Pi(1-p^{-2})(1-p^{-4}).\end{aligned}$$

THEOREM 4. *Let $c(L)$ be the Chern class of the line bundle L. Then we have*

$$c(L)^2c(D)=0.$$

Proof. Let $\pi:D_i\to B_i$ be the natural projection. Then the restriction $L|D_i$ of L to D_i is isomorphic to π^*L_i', where L_i' is a line bundle on B_i which corresponds to modular forms of weight one with respect to $\Gamma_1(\ell)$. Therefore we have

$$\begin{aligned} c(L)^2c(D) &= \sum c(L)^2c(D_i) \\ &= \sum c(L|D_i)^2 \\ &= \mu(\ell)c(\pi^* L_i')^2 \\ &= 0. \end{aligned}$$

Q.E.D.

COROLLARY. *With the same notations as above, we have*

(i) $c(D)^3 = -11\cdot 2^{-2}3^{-1}\ell^7\Pi(1-p^{-2})(1-p^{-4})$,

(ii) $c(L)^3 = 2^{-6}3^{-2}5^{-1}\ell^{10}\Pi(1-p^{-2})(1-p^{-4})$,

(iii) $c(L)c(D)^2 = -2^{-3}3^{-1}\ell^8\Pi(1-p^{-2})(1-p^{-4})$.

These are direct numerical calculations based on Theorem 1, 2, 3 and 4 and Lemma 7, so we omit the proof.

THEOREM 5. *Let c_2 be the second Chern class of the tangent bundle $T(X)$ of X. Then we have*

$$c_2c(D) = 2^{-3}\ell^7(\ell-2)\Pi(1-p^{-2})(1-p^{-4}).$$

Proof. We have an exact sequence of vector bundles on D_i;

$$0 \longrightarrow T(D_i) \longrightarrow T(X)|D_i \longrightarrow N(D_i) \longrightarrow 0,$$

where $T(D_i)$ is the tangent bundle of D_i and $N(D_i)$ is the normal bundle of D_i in X. Therefore we have

$$c_2(T(X)|D_i) = c_2(T(D_i)) + c_1(T(D_i))c(N(D_i)).$$

Since the second Chern class of a surface is its Euler number,

$$c_2(T(D_i)) = \ell\nu(\ell).$$

Since

$$\begin{aligned} c(L)c(D)^2 &= \sum c(L)c(D_i)^2 \\ &= \mu(\ell)c(L|D_i)c(N(D_i)), \end{aligned}$$

it follows from the corollary to Theorem 4 that

$$c(L|D_i)c(N(D_i)) = -2^{-2}3^{-1}\ell^4\Pi(1-p^{-2}).$$

Now we remark that for any line bundle N on B_i, the intersection number of π^*N with a fixed line bundle on D_i is proportional with the degree of N.

As we have observed in the proof of theorem 4, the line bundle $L|D_i$ is given by

$$L|D_i = \pi^* L_i',$$

where L_i' is a line bundle on B_i which corresponds to modular forms of weight one.

On the othe hand, the canonical bundle $K(D_i)$ of D_i is given in Lemma 2, so that we have

$$\begin{aligned} c(K(D_i))c(N(D_i)) &= (\deg M_i/\deg L_i')(-2^{-2}3^{-1}\ell^4)\Pi(1-p^{-2}) \\ &= (-2^{-2}\ell^4+\ell^3)\Pi(1-p^{-2}). \end{aligned}$$

Hence we have

$$\begin{aligned} c_2c(D) &= \sum c_2(T(X)|D_i) \\ &= \mu(\ell)[c_2(T(D_i)) - c(K(D_i))c(N(D_i))] \\ &= \mu(\ell)[\ell\nu(\ell) + (2^{-2}\ell^4 - \ell^3)\Pi(1-p^{-2})] \\ &= \frac{1}{2}(\ell-2)\ell^7\Pi(1-p^{-2})(1-p^{-4}). \end{aligned}$$

Q.E.D.

Theorem 6. *We have*

$$c_2c(E) = 2^{-4}3^{-2}\ell^8(\ell-3)(\ell-6)\Pi(1-p^{-2})(1-p^{-4}).$$

Proof. As in the proof of Theorem 5, we have an exact sequence of vector bundles on E_α;

$$0 \longrightarrow T(E_\alpha) \longrightarrow T(X)|E_\alpha \longrightarrow N(E_\alpha) \longrightarrow 0,$$

therefore

$$c_2(T(X)|E_\alpha) = c_2(T(E_\alpha)) + c_1(T(E_\alpha))c(N(E_\alpha)).$$

The Euler number $c_2(T(E_\alpha))$ of E_α is given by

$$\begin{aligned} c_2(T(E_\alpha)) &= e(R_1)\times e(R_2) \\ &= [2^{-2}3^{-1}\ell^2(\ell-6)\Pi(1-p^{-2})]^2, \end{aligned}$$

where $E_\alpha \cong R_1 \times R_2$ and $e(R_i)$ is the Euler number of R_i. On the other hand, if K_i is the canonical bundle of R_i and if p_i is the i-th projection of $R_1 \times R_2$, then the canonical bundle $K(E_\alpha)$ of E_α is given by

$$K(E_\alpha) = p_1{}^*K_1 + p_2{}^*K_2.$$

Therefore we have

$$\begin{aligned} c(K(E_a))c(N(E_a)) &= -c(p_1{}^*K_1 + p_2{}^*K_2)c(p_1{}^*L_1 + p_2{}^*L_2) \\ &= -2c(p_1{}^*K_1)c(p_2{}^*L_2) \\ &= -2(2^{-2}3^{-1}\ell^3 - 2^{-1}\ell^2)2^{-3}3^{-1}\ell^3\Pi(1-p^{-2}) \\ &= -2^{-4}3^{-2}\ell^5(\ell-6)\Pi(1-p^{-2}). \end{aligned}$$

Hence we obtain

$$\begin{aligned} c_2c(E) &= \sum c_2(T(X)|E_\alpha) \\ &= \lambda(\ell)\left[2^{-4}3^{-2}\ell^4(\ell-6)^2 + 2^{-4}3^{-2}\ell^5(\ell-6)\right]\Pi(1-p^{-2})^2 \\ &= 2^{-4}3^{-2}\ell^8(\ell-6)(\ell-3)\Pi(1-p^{-2})(1-p^{-4}). \end{aligned}$$

Q.E.D.

COROLLARY. *We have*

$$c_2c(L) = 4c(L)^3.$$

Proof. Since $10L = 2[E] + \ell[D]$, we have

$$\begin{aligned} c_2c(L) &= 5^{-1}c_2c(E) + 10^{-1}c_2c(D) \\ &= 2^{-4}3^{-2}5^{-1}\ell^{10}\Pi(1-p^{-2})(1-p^{-4}). \end{aligned}$$

Q.E.D.

4. As an application of the results in Section 3, we shall calculate the dimension of the vector space $S(\Gamma_2(\ell))_k$ of cusp forms of weight k with respect to $\Gamma_2(\ell)$. Let L, M, X, Y be the same as in Section 3. In Section 3 we have observed the following isomorphism

$$H^0(X, \mathcal{O}(kL)) \cong A(\Gamma_2(\ell))_k.$$

As for the space $S(\Gamma_2(\ell))_k$ of cusp forms, it is easy to verify the isomorphism:

$$H^0(X, \mathcal{O}(kL - [D])) \cong S(\Gamma_2(\ell))_k.$$

Now from the consideration in Section 2, it follows that the canonical bundle K of the Igusa's non-singular model is given by

$$K=3L-[D],$$

so that the first Chern class c_1 is given by

$$\begin{aligned} c_1 &= -c(K) \\ &= -3c(L)+c(D). \end{aligned}$$

If we apply the Riemann-Roch-Hirzebruch theorem to the line bundle $L_k = kL-[D]$ on X, [6] we obtain

$$\begin{aligned} &\sum(-1)^p \dim H^p(X, \mathcal{O}(L_k)) \\ &= 6^{-1}c(L_k)^3 + 4^{-1}c(L_k)^2 c_1 + 12^{-1}c(L_k)(c_1^2+c_2) + 24^{-1}c_1c_2 \\ &= 2^{-2}3^{-1}(k-1)(k-2)(2k-3)c(L)^3 + (2^{-2}3^{-1}k-2^{-2})c(L)c(D)^2 \\ &\quad -2^{-3}3^{-1}c(D)c_2. \end{aligned}$$

To estimate the higher cohomology groups, we need the following vanishing theorem.

THEOREM.* ([4], [12]). *Let Z be a normal projective variety, let $\pi: Z' \to Z$ be a resolution and let K' be the canonical bundle of Z'. If B is an ample line bundle on Z, then*

$$H^p(Z', \mathcal{O}(\pi^*B+K'))=0,$$

for $p>0$.

In our case, the Satake compactification is normal, the line bundle M is ample and the canonical bundle K of the Igusa's non-singular model is given by

$$\begin{aligned} K &= 3\pi^*M-[D] \\ &= 3L-[D]; \end{aligned}$$

therefore it follows from the above theorem that

$$H^p(X, \mathcal{O}(kL-[D]))=0,$$

for $k \geqslant 4$ and $p>0$.

So we obtain the following.

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*I was informed this theorem by Prof. Freitag.

THEOREM. *Let $S(\Gamma_2(\ell))_k$ be the space of cusp forms of weight k with respect to $\Gamma_2(\ell)$. Then we have the following dimension formula for $\ell \geqslant 3$ and $k \geqslant 4$;*

$$\begin{aligned}\dim S(\Gamma_2(l))_k &= \dim H^0(X, \mathcal{O}(kL-[D])) \\ &= 2^{-10}3^{-3}5^{-1}(2k-2)(2k-3)(2k-4)\, l^{10} \prod(1-p^{-2})(1-p^{-4}) \\ &\quad -2^{-6}3^{-2}(2k-3)l^8\prod(1-p^{-2})(1-p^{-4}) \\ &\quad +2^{-5}3^{-1}l^7\prod(1-p^{-2})(1-p^{-4}).\end{aligned}$$

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