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Author(s): Jun-Ichi Igusa
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ON SIEGEL MODULAR FORMS OF GENUS TWO.*[1]

By Jun-ichi Igusa.

I. Introduction. Since the publication of Siegel's *Einführung in die Theorie der Modulfunktionen n-ten Grades*, the theory has been enriched by several mathematicians. However, even in the case of degree two, the whole subject is rather misty compared with the clarity we have in the elliptic case. We are not talking here about the superstructure but the very foundation of the theory. As Siegel's treatment already shows, one of the fundamental objects of study is the graded ring of (finite sums of) modular forms. However, we know very little about this ring beyond the facts that it is finitely generated and that an operator Φ introduced by Siegel is almost an epimorphism of the graded rings of degree n to degree $n-1$ [3, 6]. Now, among modular forms, Eisenstein series (in the original sense of Siegel) are singled out by their importance in the analytic theory of quadratic forms [9]. We shall show that, *in the degree two case, every modular form is a polynomial of Eisenstein series of weight four, six, ten and twelve.* These four Eisenstein series are, of course, algebraically independent. Thus, we have a *complete structure theorem* of the ring and it gives answers to some well-known problems in this field, e. g. the dimension of the complex vector space of modular forms of a given weight and the structure of "Satake's compactification" of Siegel's fundamental domain (in the degree two case). We shall also determine the birational correspondence between the projective varieties associated with the graded ring of "even" projective invariants of binary sextics and with the graded ring of modular forms. In other words, we shall obtain explicit rational expressions for the three fundamental absolute invariants in terms of the four Eisenstein series.

We shall give an outline of our method. The results in AVM imply that the projective variety associated with the graded ring of even projective invariants of binary sextics is a compactification of the variety of moduli of curves of genus two. However, this projective variety does not contain the Siegel fundamental domain. In fact, those points of the Siegel fundamental domain representing products of elliptic curves are all mapped to one simple

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point of the projective variety. The first thing we do is, therefore, to "blow up" this point to a two dimensional affine space to construct the Siegel fundamental domain explicitly. We note that the blowing up is not a monoidal transformation. At any rate, once we know the structure of the Siegel fundamental domain as an (incomplete) algebraic variety, we can very easily characterize "multi-canonical differentials" on the variety which correspond to modular forms whose weights are multiples of six. In this way, we are able to determine the structure of the graded ring of (finite sums of) such modular forms. Then, by taking its normalization in the field of fractions of modular forms, we get the graded ring of all modular forms and can prove that it is generated by four modular forms of weight four, six, ten and twelve. The relation between these modular forms and Eisenstein series will finally be obtained by comparing their Fourier expansions. In this paper, we shall use classical formulae on elliptic theta-functions due to Jacobi and well-known results on elliptic modular forms, i. e. those we can find in Hurwitz's papers, without specific references. We shall use, also, the following standard notations:

$\mathfrak{S}_n$ = Siegel upper-half plane of degree n, i. e. the variety of complex symmetric matrices of degree n with positive-definite imaginary parts

Γ_n = homogeneous modular group of degree n $(=Sp(n,\boldsymbol{Z}))$ operating in $\mathfrak{S}_n$ as

$$\tau \rightarrow (a\tau+b)(c\tau+d)^{-1}$$

F_n = Siegel fundamental domain of degree n $(=\Gamma_n\backslash\mathfrak{S}_n)$.

II. Blowing up of proj $\boldsymbol{C}[A,B,C,D]$.

1. Let A, B, C, D be the projective invariants of binary sextics of degree two, four, six, ten defined in AVM. If $\xi_1,\xi_2,\cdots,\xi_6$ are roots of a sextic

$$u_0X^6+u_1X^5+\cdots+u_6$$

and if we denote $\xi_j-\xi_k$ by (jk), the values of A, B, C, D at this sextic have the following irrational expressions

$$A(u)=u_0{}^2\sum_{\text{fifteen}}(12)^2(34)^2(56)^2$$

$$B(u)=u_0{}^4\sum_{\text{ten}}(12)^2(23)^2(31)^2(45)^2(56)^2(64)^2$$

$$C(u)=u_0{}^6\sum_{\text{sixty}}(12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2$$

$$D(u)=u_0{}^{10}\prod_{j<k}(jk)^2.$$

In general, if S is a finitely generated graded integral ring over the field $\boldsymbol{C}$ of complex numbers, we shall denote by proj S the (complex) projective variety associated with S.[2] We are interested in the following projective variety:

$$X = \operatorname{proj} \boldsymbol{C}[A, B, C, D].$$

We have shown in AVM that points of X which are not on $D = 0$ form the variety of moduli of curves of genus two, hence also of their jacobian varieties (with canonical polarizations) over $\boldsymbol{C}$. Now, the jacobian varieties can degenerate to products of ellipitic curves, and they form a two dimensional affine space. However, since the projective invariants B, C, D vanish simultaneously at sextics with triple roots, such abelian varieties are mapped to one point of X. We shall, therefore, try to blow up this point to get the two dimensional affine space of products of elliptic curves. We note that, since the points of X which are not on $A = 0$ form a three dimensional affine space, the point in question is, at any rate, simple on X. In fact, the following three absolute invariants

$$B/A^2 \qquad C/A^3 \qquad D/A^5$$

form a set of uniformizing parameters of X around this point. Therefore, we take a point

$$\tau = \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix}$$

of $\mathfrak{S}_2$ and try to expand the above three absolute invariants into power-series of $\epsilon = \tau_{12}$ (assuming that ϵ is small). The actual calculation was guided by the following observation.

Suppose that $\tau_{12} = 0$ in τ. Then, certainly τ corresponds to a product of elliptic curves whose Weierstrass invariants are $j(\tau_1)$ and $j(\tau_2)$. However, if we start from a product of elliptic curves, there exist infinitely many points in $\mathfrak{S}_2$, some not even satisfying $\tau_{12} = 0$, which correspond to the product and ϵ does not have any intrinsic meaning. Consider the subvariety of $\mathfrak{S}_2$ defined by the equation $\tau_{12} = 0$. Then, we can see by a simple matrix calcu-

[2] In Grothendieck's language, what we are defining is

$$\operatorname{Proj}(S)_{\boldsymbol{C}}(\boldsymbol{C}) = \operatorname{Hom}_{\operatorname{Spec}(\boldsymbol{C})}(\operatorname{Spec}(\boldsymbol{C}), \operatorname{Proj}(S)).$$

Since we are assuming that S is finitely generated, this point-set has the unique structure of a projective variety, i. e. a closed subvariety of a projective space.

lation that elements of Γ_2 which keep this subvariety stable are those we can expect, i.e. of the following two types

$$\begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}.$$

By these elements, the parameter $\epsilon = \tau_{12}$ of points of a tubular neighborhood of the subvariety $\tau_{12} = 0$ undergoes the following transformation:

$$\epsilon \to \epsilon/(c_1\tau_1 + d_1)(c_2\tau_2 + d_2) \bmod \epsilon^2.$$

Therefore, if we take powers of ϵ and multiply elliptic modular forms in τ_1 and τ_2 of the corresponding weight, we can make it invariant modulo higher powers of ϵ. If we require, in addition, that the modular forms to be multiplied have no zeros, which is certainly a reasonable condition to get a good parameter, the smallest weight of the modular forms will be twelve. In this way, we can foresee that the parameter we need is $\delta(\tau_1)\delta(\tau_2)\epsilon^{12}$ up to some normalizing constant with

$$\delta(\omega) = e(\omega) \prod_{n=1}^{\infty} (1 - e(n\omega))^{24}$$

in which $e(\ )$ stands for $\exp(2\pi i\ )$ and ω is a point of $\mathfrak{S}_1$. This was a very discouraging conclusion because it forced us to calculate expansions of absolute invariants up to order twelve in ϵ. However, very fortunately, the blowing up is not monoidal and this complication on the part of the transformation brought about an unexpected simplification of the other.

2. As in the previous section, let τ be a point of $\mathfrak{S}_2$. Then, four column vectors of a two-by-four matrix $(\tau 1_2)$ generate a discrete subgroup of two dimensional complex vector space and the corresponding quotient group T is a complex torus carrying a positive divisor Θ satisfying $\deg(\Theta, \Theta) = 2$. Moreover, by a suitable translation in T, we can make Θ symmetric in the sense that it is stable under the transformation $z \to -z$. There exist sixteen such divisors and they are zeros of the following theta-functions:

$$\theta_{{}^t\mathfrak{g}{}^t\mathfrak{h}}(\tau, z) = \sum_n e[\tfrac{1}{2} \cdot {}^t(n + \mathfrak{g}/2)\tau(n + \mathfrak{g}/2) + {}^t(n + \mathfrak{g}/2)(z + \mathfrak{h}/2)]$$

in which $\mathfrak{g}$ and $\mathfrak{h}$ are column vectors with 0, 1 as their coefficients. On the other hand, if the point τ is not equivalent (with respect to Γ_2) to a point on $\tau_{12}=0$, we know that the divisor Θ is actually a non-singular curve of genus two and T is its jacobian variety [11]. In this case, the three lambdas which appear in what we called Rosenhain normal form of Θ in AVM can be expressed by theta-functions of zero argument

$$\theta_{t\mathfrak{g}t\mathfrak{h}}(\tau,0)=\theta_{t\mathfrak{g}t\mathfrak{h}}(\tau)=\theta_{t\mathfrak{g}t\mathfrak{h}}.$$

We have a choice of seven-hundred twenty expressions, and we shall use the following one

$$\lambda_1=(\theta_{1100}\theta_{1000}/\theta_{0100}\theta_{0000})^2 \qquad \lambda_2=(\theta_{1001}\theta_{1100}/\theta_{0001}\theta_{0100})^2$$
$$\lambda_3=(\theta_{1001}\theta_{1000}/\theta_{0001}\theta_{0000})^2$$

which is given by Rosenhain [7]. Conversely, if λ_1, λ_2, λ_3 are three complex numbers different from each other and from $0, 1, \infty$, we can consider a period matrix belonging to the corresponding Rosenhain normal form. If we normalize the period matrix suitably in the form $(\tau 1_2)$, the three lambdas can be expressed as above with respect to this τ. We shall try to expand the lambdas into power-series of $\epsilon=\tau_{12}$. We observe that, in case neither (g_1,h_1) nor (g_2,h_2) is $(1,1)$, we have

$$\theta_{g_1g_2h_1h_2}\begin{pmatrix}\tau_1 & \epsilon\\ \epsilon & \tau_2\end{pmatrix}=\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}\,\frac{d^n}{d\tau_1{}^n}\theta_{g_1h_1}(\tau_1)\,\frac{d^n}{d\tau_2{}^n}\theta_{g_2h_2}(\tau_2)\cdot\epsilon^{2n}.$$

The verification is formal and straightforward. Therefore, we can express the coefficients of the expansions

$$\lambda_1=a_0+a_1\epsilon^2+a_2\epsilon^4+\cdots$$
$$\lambda_2=b_0+b_1\epsilon^2+b_2\epsilon^4+\cdots$$
$$\lambda_3=c_0+c_1\epsilon^2+c_2\epsilon^4+\cdots$$

in terms of elliptic theta-functions of zero argument and their derivatives. For instance, we have

$$a_0=b_0=c_0=\lambda(\tau_1)$$

in which

$$\lambda(\omega)=(\theta_{10}(\omega)/\theta_{00}(\omega))^4.$$

Now, we evaluate the projective invariants A, B, C, D at a sextic $X(X-1)(X-\lambda_1)(X-\lambda_2)(X-\lambda_3)$ and replace the three lambdas by their expansions in ϵ. Then, writing $\lambda=\lambda(\tau_1)$ for the sake of simplicity, we get the following expansions

$$A=6\lambda^2(1-\lambda)^2+4\lambda(1-\lambda)(1-2\lambda)\sum a_1\epsilon^2+\cdots$$

$$\begin{aligned}B=&2\lambda^2(1-\lambda)^2(1-\lambda+\lambda^2)\sum(b_1-c_1)^2\epsilon^4\\&+4\lambda(1-\lambda)(1-2\lambda)\sum a_1(b_1-c_1)^2\epsilon^6\\&-2\lambda^2(1-\lambda)^2(1-2\lambda)\sum a_1\sum(b_1-c_1)^2\epsilon^6\\&+4\lambda^2(1-\lambda)^2(1-\lambda+\lambda^2)\sum(b_1-c_1)(b_2-c_2)\epsilon^6+\cdots\end{aligned}$$

$$\begin{aligned}C=&4\lambda^4(1-\lambda)^4(1-\lambda+\lambda^2)\sum(b_1-c_1)^2\epsilon^4\\&+2\lambda^3(1-\lambda)^3(1-2\lambda)(2-\lambda+\lambda^2)\sum a_1(b_1-c_1)^2\epsilon^6\\&+2\lambda^3(1-\lambda)^3(1-2\lambda)(2-3\lambda+3\lambda^2)\sum a_1\sum(b_1-c_1)^2\epsilon^6\\&+8\lambda^4(1-\lambda)^4(1-\lambda+\lambda^2)\sum(b_1-c_1)(b_2-c_2)\epsilon^6+\cdots\end{aligned}$$

$$D=\lambda^6(1-\lambda)^6(b_1-c_1)^2(c_1-a_1)^2(a_1-b_1)^2\epsilon^{12}+\cdots$$

in which the summations are symmetrizations in a, b, c. On the other hand, we need the following classical identities:

$$\theta_{10}{}^4+\theta_{01}{}^4=\theta_{00}{}^4 \qquad (\theta_{00}\theta_{10}\theta_{01})^8=2^8\delta$$

$$\frac{d}{d\omega}\log(\theta_{10}/\theta_{00})=\frac{\pi i}{4}\theta_{01}{}^4 \qquad \frac{d}{d\omega}\log(\theta_{00}/\theta_{01})=\frac{\pi i}{4}\theta_{10}{}^4.$$

Using some of them, we get

$$\left.\begin{matrix}a_1\\b_1\\c_1\end{matrix}\right\}=\pi i\lambda(1-\lambda)\theta_{00}(\tau_1)^4\frac{d}{d\tau_2}\left\{\begin{matrix}\log\theta_{00}(\tau_2)\theta_{10}(\tau_2)\\\log\theta_{10}(\tau_2)\theta_{01}(\tau_2)\\\log\theta_{01}(\tau_2)\theta_{00}(\tau_2).\end{matrix}\right.$$

Therefore, if we put

$$t=\delta(\tau_1)\delta(\tau_2)(\pi\epsilon)^{12},$$

we have

$$D/A^5=2^{-1}3^{-5}t+\cdots.$$

Furthermore, since the Weierstrass invariant j and λ are related as

$$j=2^8(1-\lambda+\lambda^2)^3/\lambda^2(1-\lambda)^2,$$

we have

$$(B/A^2)^3=2^{-12}3^{-6}j(\tau_1)j(\tau_2)t+\cdots.$$

Finally, powers of C/A^3 themselves are of no use, because $3C/A^3$ and B/A^2 have the same leading term. However, if we examine their difference, terms involving a_2, b_2, c_2 cancel each other and we get

$$((3C-AB)/A^3)^2 = 2^{-12}3^{-6}(j(\tau_1)-2^63^3)(j(\tau_2)-2^63^3)t+\cdots.$$

3. We, now, introduce following set of uniformizing parameters

$$x_1 = 2^43^2B/A^2 \qquad x_2 = 2^63^3(3C-AB)/A^3 \qquad x_3 = 2\cdot 3^5D/A^5$$

of $X = \operatorname{proj} \boldsymbol{C}[A,B,C,D]$ at the point $B=C=D=0$. Also, in view of the calculation we made in the previous section, we put

$$y_1 = x_1{}^3/x_3 \qquad y_2 = x_2{}^2/x_3.$$

Then, the normalization of the integral ring $\boldsymbol{C}[x_1,x_2,x_3,y_1,y_2]$ (in its field of fractions) is generated by

$$y_3 = x_1{}^2x_2/x_3.$$

The proof is as follows. Clearly y_3 is an element of the field of fractions. Moreover, since we have $y_3{}^2 = x_1y_1y_2$, certainly y_3 is integral over the ring. We have only to show, therefore, that $\boldsymbol{C}[x_1,x_2,x_3,y_1,y_2,y_3]$ is normal. However, if we operate a cyclic group of order six to a ring of polynomials in three letters t_1, t_2, t_3 as

$$t_1, t_2, t_3 \to \zeta^2t_1, \zeta^3t_2, \zeta^5t_3 \quad (\zeta^6=1),$$

the ring of invariant elements of $\boldsymbol{C}[t_1,t_2,t_3]$ can be identified with $\boldsymbol{C}[x_1,x_2,x_3,y_1,y_2,y_3]$, in fact, in the following way

$$x_1 = t_1t_3{}^2 \qquad x_2 = t_2t_3{}^3 \qquad x_3 = t_3{}^6$$
$$y_1 = t_1{}^3 \qquad y_2 = t_2{}^2 \qquad y_3 = t_1{}^2t_2t_2.$$

Therefore, certainly $\boldsymbol{C}[x_1,x_2,x_3,y_1,y_2,y_3]$ is normal, and this proves the assertion. Also, we can write down very easily a base of the ideal of relations of x_1, x_2, x_3, y_1, y_2, y_3 consisting of six polynomials. We also note that, if V_0 and V are the affine varieties with $\boldsymbol{C}[x_1,x_2,x_3,y_1,y_2]$ and $\boldsymbol{C}[x_1,x_2,x_3,y_1,y_2,y_3]$ as their co-ordinate rings, the holomorphic map of V to V_0 associated with the inclusion of the rings is a bijection. In fact, over the point of V_0 not on

$x_3=0$, there exists only one point, and over the point of V_0 on $x_3=0$, there also exists only one point, because we necessarily have $y_3=0$.

We shall, now, show that the variety of moduli and the variety V form an open affine covering of $F_2=\Gamma_2\backslash\mathfrak{S}_2$. We recall that the variety of moduli is the complement of $D=0$ in X and it is isomorphic to a quotient variety of a three dimensional affine space modulo a cyclic group of order five operating in this space as follows

$$t_j\to\zeta^j t_j \quad j=1,2,3 \quad (\zeta^5=1).$$

Since the three lambdas considered as meromorphic functions in $\mathfrak{S}_2$ by means of the expressions in terms of theta-functions of zero argument undergo seven-hundred twenty transformations by operations of Γ_2, which are just automorphisms of the field of lambdas relative to the field of absolute invariants, the absolute invariants can be considered as meromorphic functions in $\mathfrak{S}_2$ invariant by operations of Γ_2. Therefore, the absolute invariants are meromorphic functions on the the corresponding quotient variety F_2. We shall show that, if we denote by $(x_3)=(x_3)_0-(x_3)_\infty$ the divisor on F_2 of the absolute invariant x_3, elements of the co-ordinate ring of the variety of moduli are holomorphic in F_2 minus the support of $(x_3)_0$, simply $F_2-\mathrm{supp.}(x_3)_0$, and elements of the co-ordinate ring of V are holomorphic in $F_2-\mathrm{supp.}(x_3)_\infty$. First of all, if a point τ of $\mathfrak{S}_2$ corresponds to a curve of genus two, values of the three lambdas at τ are different from each other and from 0, 1, ∞. Therefore, the A, B, C, D, written as polynomials in the lambdas, are all finite at τ and D is different from 0. On the other hand, if τ corresponds to a product of elliptic curves, i.e., if τ is (equivalent to) a point on $\epsilon=0$, we have $x_1=x_2=x_3=y_3=0$ and

$$y_1=j(\tau_1)j(\tau_2) \qquad y_2=(j(\tau_1)-2^6 3^3)(j(\tau_2)-2^6 3^3).$$

Therefore, $\mathrm{supp.}(x_3)_0$ consists of points of F_2 representing products of elliptic curves and it has no point in common with $\mathrm{supp.}(x_3)_\infty$. Also, elements of the co-ordinate ring of the variety of moduli are holomorphic in $\mathfrak{S}_2$ at points τ which correspond to curves of genus two, hence they are holomorphic in $F_2-\mathrm{supp.}(x_3)_0$. Furthermore, the corresponding holomorphic map is a bijection of $F_2-\mathrm{supp.}(x_3)_0$ to the variety of moduli. On the other hand, the six absolute invariants x_1, x_2, x_3, y_1, y_2, y_3 are all holomorphic in $F_2-\mathrm{supp.}(x_3)_\infty$. It is clear that they are holomorphic in $F_2-\mathrm{supp.}(x_3)$. However, their expansions in ϵ show immediately that they are also holomorphic at every point of $\mathrm{supp.}(x_3)_0$. Furthermore, the corresponding holomorphic map is a bijection of $F_2-\mathrm{supp.}(x_3)_\infty$ to V. In fact, it gives a bijection of $F_2-\mathrm{supp.}(x_3)$ to V

minus $x_3 = 0$ and of supp.$(x_3)_0$ to the set of points of V on $x_3 = 0$. Now, since the variety of moduli and V are both normal, the bijective holomorphic maps are necessarily isomorphisms. In the present case, since the varieties concerned are "V-manifolds" [cf. 8], we have only to use an elementary lemma on removable singularities [2]. This completes the proof.

The observations made so far permit us also to prove the following important lemma:

LEMMA. *The field of meromorphic functions on F_2 can be identified with the field of absolute invariants such that holomorphic functions correspond to constants.*

Since this is an immediate consequence of a general theorem proved by Baily [1] (and of some properties of absolute invariants), we shall give only an outline. Knowing the absolute invariants which generate the co-ordinate rings of the variety of moduli and V, we can easily find a set of projective invariants of the same degree which gives a projective embedding of F_2. In this way, we get a normal projective variety which is a compactification of F_2 such that the complement is one dimensional. This is all we need. Actually, the compactification is a "V-manifold" and the elementary lemma on removable singularities is again sufficient.

III. Ring of algebraic modular forms.

4. In general, if

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $Sp(n, \boldsymbol{R})$, the jacobian of the corresponding transformation in $\mathfrak{S}_n$

$$\tau \to M\tau = (a\tau + b)(c\tau + d)^{-1}$$

is known to be $\det(c\tau + d)^{-n-1}$. We need this fact only for elements of $Sp(n, \boldsymbol{Z})$ (in the case $n = 2$) and an easy proof is to verify it for some generators of $Sp(n, \boldsymbol{Z})$. This being remarked, a modular form ψ of degree n is defined by the following two conditions:

(1) *For every element M of $Sp(n, \boldsymbol{Z})$, ψ satisfies a functional equation of the form*

$$\psi(M\tau) = \det(c\tau + d)^w \psi(\tau)$$

with some even integer w;

(2) *it is holomorphic in* $\mathfrak{S}_n$.

The integer w in (1) is called the weight of the modular form. In the above definition, the elliptic case $n=1$ is exceptional and we have to assume that ψ is holomorphic also at $i\infty$. More precisely, since, in general, a modular form is invariant under a transformation of the form $\tau \to \tau + b$, in which b is a symmetric integer matrix of degree n, it admits a Fourier expansion of the form

$$\sum_T a(T) e(tr(T\tau)).$$

The summation extends over all symmetric half-integer matrices (i.e. symmetric matrices such that diagonal coefficients and twice all other coefficients are integers) of degree n. Since the modular form is also invariant under a transformation of the form $\tau \to u\tau{}^tu$, in which u is a unimodular integer matrix of degree n, the coefficient $a(T)$ depends only on the class of T in the sense $a({}^tuTu) = a(T)$. Now, a closer examination first made by Koecher shows that $a(T)$ is zero unless T is positive semi-definite [5]. The elliptic case is, however, exceptional and this is not a consequence but an assumption. At any rate, we shall not use this theorem of Koecher and, in the case $n=2$, it will simply come out from our later considerations.

If we consider the set of finite sums of modular forms, we get a subring of the ring of all holomorphic functions in $\mathfrak{S}_n$ and it is obviously graded. We are interested in the structure of this graded ring in the simplest unknown case $n=2$. We say that a modular form (in the case $n=2$) is algebraic if its weight is a non-negative integer divisible by six. Finite sums of algebraic modular forms form a graded subring of the graded ring of all modular forms and we shall determine its structure.

We take an algebraic modular form ψ of weight w and, using the absolute invariants x_1, x_2, x_3, we consider

$$\phi = \psi(\tau)\,(\partial(x_1, x_2, x_3)/\partial(\tau_1, \tau_2, \tau_{12}))^{-w/3}.$$

Then, it is a meromorphic function in $\mathfrak{S}_2$ and, because of the property of the jacobian, it is invariant under operations of Γ_2; hence it is meromorphic on F_2. The lemma in Section 3 implies, therefore, that ϕ is a rational function of x_1, x_2, x_3. Suppose, conversely, that ϕ is a rational function of x_1, x_2, x_3 and consider

$$\psi(\tau) = \phi\,(\partial(x_1, x_2, x_3)/\partial(\tau_1, \tau_2, \tau_{12}))^{w/3}.$$

Then, it satisfies the functional equation, but it may not have the property

(2). We shall, therefore, try to obtain necessary and sufficient conditions (in terms of the rational function ϕ) for ψ to be holomorphic in $\mathfrak{S}_2$.

We first recall that the three lambdas are holomorphic in $\mathfrak{S}_2$ minus those points which are equivalent to points on $\epsilon = 0$. Therefore, we get a holomorphic map of this open subvariety of $\mathfrak{S}_2$ to the space of lambdas, which is a three dimensional affine space minus nine planes defined by $D = 0$, and this map is surjective. The point is that we have an unramified covering, i. e. the holomorphic map is a local isomorphism. In fact, the inverse map is given locally by representing τ as a part of a period matrix belonging to a Rosenhain normal form, and it is certainly holomorphic. We can construct another proof using the fact that no operation of Γ_2 belonging to the so-called principal congruence group modulo 2 has a fixed point in the said open subvariety of $\mathfrak{S}_2$. On the other hand, the lambda space is a ramified covering of the variety of moduli and the ramification takes place along those points which represent curves of genus two having "many automorphisms," i. e. at the singular point $A = B = C = 0$ and along a surface which corresponds to lambda triples satisfying $\lambda_3 - \lambda_1 + \lambda_1\lambda_2 - \lambda_3\lambda_1 = 0$. For this, the reader is referred to the last section of AVM. This surface is related to the "skew-invariant" in the following way. Using the same notation as in Section 1, we consider the following expression:

$$u_0^{15} \prod_{\text{fifteen}} \det \begin{pmatrix} 1 & \xi_1 + \xi_2 & \xi_1\xi_2 \\ 1 & \xi_3 + \xi_4 & \xi_3\xi_4 \\ 1 & \xi_5 + \xi_6 & \xi_5\xi_6 \end{pmatrix}.$$

Then, since each factor has an invariant property, the symmetrized product defines a projective invariant E of degree fifteen. Since the degree is odd, it can not possibly be expressed by A, B, C, D, but its square E^2 can be expressed by these projective invariants. This being remarked, if we express E in terms of the lambdas, we get fifteen distinct irreducible factors one of which is $\lambda_3 - \lambda_1 + \lambda_1\lambda_2 - \lambda_3\lambda_1$. Therefore, the surface in question is defined by $E^2 = 0$ on the variety of moduli and the ramification index is two. We need also an expansion of E^2 into power-series of ϵ. The calculation is straightforward and we get

$$E^2/A^{15} = 2^{-23}3^{-15}(j(\tau_1) - j(\tau_2))^2 t^2 + \cdots.$$

5. Now, consider $\mathfrak{S}_2$ minus points equivalent to points on $\epsilon = 0$ and

points on $A=0$. This open subvariety is ramified over the variety of moduli minus points on $A=0$ along $E^2=0$. We shall translate the condition for ψ to be holomorphic in the said open subvariety of $\mathfrak{S}_2$. We observe that the co-ordinate ring of the variety of moduli minus points on $A=0$ is generated over $\boldsymbol{C}$ by absolute invariants whose denominators are power-products of A and D. Therefore, the co-ordinate ring in question is the ring of fractions of $\boldsymbol{C}[x_1, x_2, x_3]$ with respect to powers of x_3. Since the ramification index is two along $E^2=0$, the condition we are looking for is that ϕ multiplied by E^2/A^{15} to the power w over 6 is in the co-ordinate ring, i.e. that ϕ is a linear combination of

$$(A^{15}/E^2)^{w/6} x_1^{e_1} x_2^{e_2}/x_3^{e_3}$$

in which e_1, e_2, e_3 are integers and e_1, e_2 are non-negative. We shall next write down the condition for ψ to be holomorphic at points on $A=0$. Since the condition is (by the lemma on removable singularities) along $A=0$ and not at special points on $A=0$, we have only to examine those points of it which are not equivalent to points on $\epsilon=0$ and not on $BE^2=0$. Then, instead of the τ's, we can use

$$u_1 = x_1^2/x_3 = 2^7 3^{-1} AB^2/D \qquad u_2 = x_1^5/x_3^2 = 2^{18} B^5/D^2$$
$$u_3 = x_1 x_2/x_3 = 2^9 B(3C - AB)/D$$

as local co-ordinates. Since we have

$$x_1 = u_2/u_1^2 \qquad x_2 = u_2 u_3/u_1^3 \qquad x_3 = u_2^2/u_1^5,$$

we get

$$(A^{15}/E^2)^{w/6}(x_1^{e_1} x_2^{e_2}/x_3^{e_3})(\partial(x_1, x_2, x_3)/\partial(u_1, u_2, u_3))^{w/3}$$
$$= {*}u_1^{-2e_1-3e_2+5e_3-7w/6}\, u_2^{e_1+e_2-2e_3+w} u_3^{e_2}$$

in which * is a unit depending on w but not on e_1, e_2, e_3. Therefore, for a fixed w, the condition is that a linear combination of the form

$$\sum \text{const.}\, u_1^{-2e_1-3e_2+5e_3-7w/6}\, u_2^{e_1+e_2-2e_3+w} u_3^{e_2}$$

be finite along $u_1=0$. However, in case two distinct triples (e) and (e') give the same exponent of u_1 in the above linear combination, certainly they give different exponents either to u_2 or to u_3. Consequently, for each (e), the exponent of u_1 has to be non-negative. Therefore, the condition is simply

(C1) $$5e_3 - 7w/6 \geqq 2e_1 + 3e_2$$

for every $(e)=(e_1,e_2,e_3)$. In particular, the integer e_3 is also non-negative. Finally, we shall write down the condition for ψ to be holomorphic at those points which are equivalent to points on $\epsilon=0$, i.e. along $\epsilon=0$. In order to get this condition, we shall use the parameters t_1, t_2, t_3 introduced in Section 3. Since t_3 is of order two in ϵ, however, we have to use u defined by $t_3=u^2$. Then, we have

$$(A^{15}/E^2)^{w/6}(x_1{}^{e_1}x_2{}^{e_2}/x_3{}^{e_3})\,(\partial(x_1,x_2,x_3)/\partial(t_1,t_2,u))^{w/3} \\ = *\, t_1{}^{e_1}t_2{}^{e_2}u^{2(2e_1+3e_2-6e_3)+3w}$$

in which * is a unit depending on w but not on e_1, e_2, e_3. Therefore, in the same way as before, we get

$$\text{(C2)} \qquad 2e_1+3e_2\geqq 6e_3-\tfrac{3}{2}w$$

for every $(e)=(e_1,e_2,e_3)$. We have thus finished the translation of the condition completely. The result can, obviously, be stated by saying that there exists a monomorphism of the graded ring of algebraic modular forms to the graded ring of multi-canonical differentials on X and that the image ring consists of finite sums of

$$(A^{15}/E^2)^{w/6}(x_1{}^{e_1}x_2{}^{e_2}/x_3{}^{e_3})\,(dx_1dx_2dx_3)^{w/3}$$

in which e_1, e_2, e_3 are non-negative integers satisfying both (C1) and (C2). We note that these differentials are linearly independent and, for a given w, we can compute the number of such differentials quite easily. For instance, we get 1, 1, 3, 4, 8, 11, $\cdots$ for $w=0$, 6, 12, 18, 24, 30, $\cdots$.

6. We shall examine the structure of the graded ring of algebraic modular forms. For this purpose, we put $w=6e_4$ and consider an additive monoid of non-negative integer quadraples (e_1,e_2,e_3,e_4) satisfying

$$5e_3-7e_4\geqq 2e_1+3e_2\geqq 6e_3-9e_4.$$

The connection between the ring in question and this monoid is clear. We shall determine the structure of this monoid. If we replace the above two inequalities by equalities, we get a submonoid, and this submonoid is generated by $(0,1,2,1)$ and $(3,0,4,2)$. Now, suppose that (e_1,e_2,e_3,e_4) is an arbitrary element of the monoid. Then, the difference $(e_1,e_2,e_3,e_4)-e_2(0,1,2,1)$ $=(e_1,0,e_3-2e_2,e_4-e_2)$ is again an element of the monoid, and this reduces our consideration to its submonoid consisting of elements of that type. In fact, if four real numbers $\epsilon_1, \epsilon_2 = 0, \epsilon_3, \epsilon_4$ satisfy the above inequalities and if ϵ_1 is non-negative, the others are also non-negative, This simple remark will be used repeatedly. Suppose, now, that $(e_1, 0, e_3, e_4)$ is an arbitrary element of the submonoid. This time we subtract $(3, 0, 4, 2)$ from $(e_1, 0, e_3, e_4)$ so that the first coefficient of the difference becomes 0, 1 or 2. Then, as we know, the other coefficients are non-negative; hence the difference is still an element of the monoid. Suppose, therefore, that $(e_1, 0, e_3, e_4)$ is an element of the monoid and that $e_1 = 0, 1, 2$. If we have $e_1 = 0$, the element is in the submonoid consisting of elements of the form $(0, 0, e_3, e_4)$ in which e_3, e_4 are, of course, non-negative and satisfy

$$5e_3 - 7e_4 \geqq 0 \qquad 3e_4 - 2e_3 \geqq 0.$$

If we replace one of the inequalities by an equality, we get $(0, 0, 7, 5)$ and $(0, 0, 3, 2)$, and these elements generate the submonoid. In fact, the element $(0, 0, e_3, e_4)$ decomposes into $5e_3 - 7e_4$ times $(0, 0, 3, 2)$ and $3e_4 - 2e_3$ times $(0, 0, 7, 5)$. If we have $e_1 = 1$, we try to reduce the element $(1, 0, e_3, e_4)$ using $(0, 0, 3, 2)$ and $(0, 0, 7, 5)$. The reduction will fail to work if both $(1, 0, e_3, e_4) - (0, 0, 3, 2)$ and $(1, 0, e_3, e_4) - (0, 0, 7, 5)$ are outside the monoid. This happens if and only if we have both $5e_3 - 7e_4 = 2$ and $2e_3 - 3e_4 = 0$. Thus, we get $(1, 0, 6, 4)$. Finally, if we have $e_1 = 2$, we again try to reduce the element $(2, 0, e_3, e_4)$ using $(0, 0, 3, 2)$ and $(0, 0, 7, 5)$. The reduction fails to work if and only if we have $5e_3 - 7e_4 = 4$ and $2e_3 - 3e_4 = 1$. This gives $(2, 0, 5, 3)$. We have, thus, shown that our monoid is generated by the following six elements:

$$\begin{array}{ccc} (0,1,2,1) & (0,0,3,2) & (3,0,4,2) \\ (0,0,7,5) & (2,0,5,3) & (1,0,6,4). \end{array}$$

In other words, we have found generators of the graded ring of algebraic modular forms. The structure of this ring will be known, therefore, if we determine all possible relations between the above six generators. For this purpose, we observe that a column vector with six coefficients is annihilated by the matrix

$$\begin{pmatrix} 0 & 0 & 3 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 7 & 5 & 6 \\ 1 & 2 & 2 & 5 & 3 & 4 \end{pmatrix}$$

if and only if the coefficients are of the form 0, 0, x, y, $-2x+y$, $x-2y$. In particular, we have

$$2(2,0,5,3)=(3,0,4,2)+(1,0,6,4)$$
$$2(1,0,6,4)=(0,0,7,5)+(2,0,5,3)$$
$$(2,0,5,3)+(1,0,6,4)=(3,0,4,2)+(0,0,7,5).$$

Using these relations, we can lower the sum of multiplicities of $(2,0,5,3)$ and $(1,0,6,4)$ in any expression of an element of the monoid by the six generators as long as it is at least equal to two. In this way, every element of the monoid can be expressed in one of the three forms

$$p(0,1,2,1)+q(0,0,3,2)+r(3,0,4,2)+s(0,0,7,5)$$
$$+\,0,\ \ (2,0,5,3),\ \ (1,0,6,4)$$

in which p, q, r, s are non-negative integers. Moreover, the expression is unique. Thus, the structure of the ring of algebraic modular forms is completely determined. In particular, the dimension N_w of the complex vector space of modular forms of weight w is, in case w is (non-negative and) of the form $6m$, equal to the number of partitions of m into the form $p+2(q+r)+5s+0,3,4$. There are ten formulas for N_w according to the values of w modulo sixty and, for example, we have

$$N_{60k}=150\binom{k}{3}+190\binom{k}{2}+51\binom{k}{1}+1.$$

IV. Main theorems.

7. Since the structure of the ring of algebraic modular forms is determined, using this result, we shall start investigating the ring of all modular forms in connection with Eisenstein series. In general, if τ is a point of $\mathfrak{S}_n$, the Eisenstein series of degree n and of weight w is defined as follows

$$\psi_w(\tau)=\sum_{\{c,d\}}\det(c\tau+d)^{-w}.$$

The summation extends over all classes of coprime symmetric pairs, i. e. over all inequivalent bottom rows of elements of $Sp(n,\mathbf{Z})$ with respect to left multiplications by unimodular integer matrices of degree n. It is a classical theorem of H. Braun that the series is absolutely convergent for $w>n+1$ and represents a modular form of weight w. The modular forms defined by

Eisenstein series are connected with positive-definite quadratic forms by their Fourier expansions. In the Einführung, Siegel gives a formula for the coefficients of Fourier expansions of Eisenstein series in terms of what he called p-adic densities [9]. Therefore, in order to calculate the Fourier coefficients using this formula, we have to calculate certain p-adic densities for all p including $p=2$. In some cases, however, the so-called Siegel main theorem can be used to go around this tedious calculation. On the other hand, if $a(T)$ is a Fourier coefficient of Eisenstein series of degree n, the said Siegel formula shows that

$$a(T_1)=a\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$$

is a Fourier coefficient of Eisenstein series of degree $n-1$ and of the same weight. This allows us, in some cases, to reduce the calculation of the Fourier coefficient from degree n to degree $n-1$, thus finally to the elliptic case. In this case, if we denote the Eisenstein series of weight w by ϕ_w, we have

$$\phi_w(\omega)=1+((2\pi i)^w/\Gamma(w)\zeta(w))\sum_{t=1}^{\infty}(\sum_{d\mid t}d^{w-1})e(t\omega).$$

Also, in the case $n=2$, we have the following table:

$w=4$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=2^4 3\cdot 5 \qquad a\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}=2^4 3^3 5$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=2^5 3^3 5\cdot 7 \qquad a\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}=2^7 3\cdot 5\cdot 7$$

$w=6$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=-2^3 3^2 7 \qquad a\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}=-2^3 3^3 7\cdot 11$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=2^4 3^3 5\cdot 7\cdot 11 \qquad a\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}=2^6 3^2 7\cdot 11$$

$w=8$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=2^5 3\cdot 5 \qquad a\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}=2^5 3^2 5\cdot 43$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=2^6 3^2 5\cdot 61 \qquad a\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}=2^8 3\cdot 5\cdot 7$$

$$w = 10$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -2^3 3 \cdot 11 \qquad a\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = -2^3 3^4 11 \cdot 19$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2^4 3^4 5 \cdot 7 \cdot 11 \cdot 19 \cdot 277 \cdot 43867^{-1}$$

$$a\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = 2^6 3 \cdot 7 \cdot 11 \cdot 19 \cdot 809 \cdot 43867^{-1}$$

$$w = 12$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 2^4 3^2 5 \cdot 7 \cdot 13 \cdot 691^{-1} \qquad a\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2^4 3^3 5 \cdot 7 \cdot 13 \cdot 683 \cdot 691^{-1}$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2^5 3^3 5 \cdot 7 \cdot 13 \cdot 19 \cdot 23 \cdot 2659 \cdot 131^{-1} 593^{-1} 691^{-1}$$

$$a\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = 2^7 3^2 5 \cdot 7 \cdot 13 \cdot 23 \cdot 1847 \cdot 131^{-1} 593^{-1} 691^{-1}.$$

Some of the large prime numbers we find in this table come from Bernoullian numbers. For instance 43867 is the numerator of the ninth Bernoullian number. At any rate, if we observe that we always have $a(0) = 1$, we see that the above table gives Fourier coefficients $a(T)$ of Eisenstein series of weight at most twelve for all (symmetric half-integer positive semi-definite) matrices T satisfying $tr(T) \leqq 2$. Also, in order to avoid a possible misunderstanding by some reader, we note that the table is not really necessary until we start proving explicit rational expressions of absolute invariants by Eisenstein series.

Now, it is known (in the case $n = 2$) that the dimension N_w is zero for $w = 2$ and one for $w = 4, 6, 8$. This fact was proved by Maass [6] and partly by Witt [12]. One of their main ideas is to use a remarkable lemma by Siegel on a unique determination of modular forms (of a given weight) by their first few Fourier coefficients. Incidentally, this lemma shows also that the Fourier coefficients of the five Eisenstein series in the previous table determine these modular forms uniquely. When we started working on the subject, beside some general results, that was practically all we knew about the ring of modular forms. At any rate, we shall not make any use of the above results and they will come out from our subsequent considerations.

We shall denote by $\psi_{e_1e_2e_3e_4}$ the algebraic modular form defined by

$$\psi_{e_1e_2e_3e_4}(\tau) = (2^{-23}3^{-15}A^{15}/E^2)^{e_4}(x_1{}^{e_1}x_2{}^{e_2}/x_3{}^{e_3})\,(\partial(x_1, x_2, x_3)/\partial(\tau_1, \tau_2, \tau_{12}))^{2e_4}.$$

It will become necessary to expand this modular form into a Fourier series or, more precisely, into a power-series of $e(\tau_1)$, $e(\tau_2)$ and $\epsilon = \tau_{12}$. For this calculation, we need following expansions

$$j = 2^6 3^3 \phi_4^3/(\phi_4^3 - \phi_6^2) = e(-\omega) + 2^3 3\cdot 31 + 2^2 3^3 1823 \cdot e(\omega) + \cdots$$

$$\partial(x_1, x_2, x_3)/\partial(\tau_1, \tau_2, \tau_{12})$$

$$= 2^7 3^3 (j(\tau_1) - j(\tau_2)) j'(\tau_1) j'(\tau_2) (y_3^{11}/y_1^8 y_2^6)(1/\epsilon) + \cdots.$$

This being remarked, we note that ψ_{0032}, ψ_{0242}, ψ_{3042} form a base of the vector space of modular forms of weight twelve. Since ψ_4^3 is a modular form of the same weight, therefore, it is a linear combination of these modular forms. The coefficients of the linear combination can be calculated with the aid of Fourier expansions. If we observe that the three modular forms are linearly independent also on $\epsilon = 0$, we have only to calculate the Fourier expansion of ψ_4 on $\epsilon = 0$, but this is simply $\phi_4(\tau_1)\phi_4(\tau_2)$. In this way, without really using the Fourier coefficients of ψ_4 itself, we get

$$\psi_{3042} = 2^{36} 3^{12} \pi^{12} \psi_4^3.$$

We can also use the following argument to see that ψ_{3042} and ψ_4^3 differ only by a constant factor. Because of $N_{12} = 3$, we have $N_4 \leqq 1$, hence $N_4 = 1$. On the other hand, the divisor of ψ_{3042} in $\mathfrak{S}_2$ is three times another divisor. Therefore, its cubic root divided by ψ_4 defines a "multiplicative function," i.e. a function whose absolute value is single-valued, on F_2. However, using the structure of the compactification, we can conclude very easily that the function itself is single-valued. Consequently, the cubic root of ψ_{3042} is a modular form of weight four and, because of $N_4 = 1$, it differs from ψ_4 by a constant factor. In this way, we can minimize the use of numerical computation. Besides, the exact constant factor is not necessary in proving most of our theorems.

At any rate, if we adjoin ψ_4 to the ring of algebraic modular forms, we get a new graded ring. We shall show that the normalization, say S, of this graded ring (in its field of fractions) is precisely the graded ring of all modular forms. Suppose that ψ is an arbitrary modular form of weight w. Then, first of all, the weight w is non-negative. Otherwise, the product $\psi^{12}\psi_4^{-3w}$ will be a holomorphic function on F_2, hence it is a constant by a lemma in Section 3. In particular, both ψ and ψ_4 will be units in the ring of holomorphic functions in $\mathfrak{S}_2$. But, certainly ψ_4 is not a unit. Hence w is non-negative. This being remarked, let $2e$ be the least residue of w modulo six. Then, the product $\psi\psi_4^e$ is certainly an algebraic modular form.

Hence, every modular form is in the field of fractions. Furthermore, since the sixth power of any modular form is an algebraic modular form, it is integral over the ring. Therefore, every modular form is in S. On the other hand, we know in general that a normalization of a graded integral ring over a field is itself a graded ring [13]. In particular, our S is a graded ring. Moreover, a homogeneous element ϕ of S can be expressed as the quotient of two homogeneous elements of the ring, and they are modular forms. Therefore, it satisfies the functional equation. Also, since ϕ is integral over the ring of holomorphic functions in $\mathfrak{S}_2$, it is holomorphic there. This shows that ϕ is a modular form, hence S is the graded ring of modular forms. We shall, now, determine the structure of S explicitly. In doing this, we shall not use the second half of the above proof, i. e. the fact that every homogeneous element of S is a modular form. It will come out as a consequence.

We recall that the ring S is the normalization of a graded ring generated over $\boldsymbol{C}$ by

$$\psi_4, \quad \psi_{0121}, \quad \psi_{0032}, \quad \psi_{0075}, \quad \psi_{2053}, \quad \psi_{1064}.$$

These elements, except ψ_{0121} and ψ_{0032}, are related as

$$\psi_{2053}{}^3 = \text{const.}\, \psi_4{}^6\psi_{0075} \qquad \psi_{1064}{}^3 = \text{const.}\, \psi_4{}^3\psi_{0075}{}^2.$$

Therefore, the field of fractions of $\boldsymbol{C}[\psi_4, \psi_{0075}, \psi_{2053}, \psi_{1064}]$ is of degree of transcendency at most two over $\boldsymbol{C}$. Since the field of fractions of the ring of modular forms is of degree of transcendency four over $\boldsymbol{C}$, the degree of transcendency in question is precisely two and the two elements ψ_{0121} and ψ_{0032} are algebraically independent over that field. Therefore, by recalling that a ring of polynomials with coefficients in a normal integral ring is itself normal, we conclude that, if R is the normalization of $\boldsymbol{C}[\psi_4, \psi_{0075}, \psi_{2053}, \psi_{1064}]$ (in its field of fractions), we get S by just adjoining ψ_{0121} and ψ_{0032} to R. Now, if we put $\chi = \psi_{2053}/\psi_4{}^2$, we have

$$\chi^3 = \text{const.}\, \psi_{0075}.$$

Hence χ is in R, and R is also the normalization of $\boldsymbol{C}[\psi_4, \chi, \psi_{1064}]$. On the other hand, we have

$$\psi_{1064}{}^3 = \text{const.}\, (\psi_4\chi^2)^3,$$

hence $\psi_{1064} = \text{const.}\, \psi_4\chi^2$. Since $\boldsymbol{C}[\psi_4, \chi]$ is certainly normal (as a ring of polynomials in two letters with coefficients in $\boldsymbol{C}$), therefore, we get $R = \boldsymbol{C}[\psi_4, \chi]$, hence finally

$$S = \boldsymbol{C}[\psi_4, \chi, \psi_{0121}, \psi_{0032}].$$

We state our result in the following way:

THEOREM 1. *The graded ring of modular forms is generated over* $\boldsymbol{C}$ *by four (algebraically independent) modular forms*

$$\psi_4, \quad \psi_{0121}, \quad \chi = \psi_{2053}/\psi_4{}^2, \quad \psi_{0032}$$

of respective weight four, six, ten and twelve.

COROLLARY. *The dimension* N_w *of the complex vector space of modular forms of weight* w *is equal to the number of non-negative integer solutions of the linear Diophantine equation*

$$w = 4p + 6q + 10r + 12s.$$

8. We have shown that the graded ring of modular forms is generated by four modular forms of which one is the Eisenstein series ψ_4. We shall show that other three can also be expressed by Eisenstein series. First of all, since we have $N_6 = 1$, two modular forms ψ_6 and ψ_{0121} differ only by a constant factor. This constant factor can be determined immediately. In order to make the rest of the argument clear, we shall explicitly use Siegel's operator Φ which maps a modular form of degree n to a modular form of degree $n-1$ of the same weight for $n = 1, 2, 3, \cdots$. If ψ is a modular form of degree n and if $a(T)$ is its Fourier coefficient, the Fourier coefficient of $\Phi\psi$ is given by

$$a(T_1) = a\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, Eisenstein series are mapped by Φ to Eisenstein series. The operator Φ gives rise to a homomorphism of the graded rings of modular forms [10] and it is almost surjective [6]. In the case $n = 2$, the surjectivity is obvious because the graded ring of elliptic modular forms is generated by the Eisenstein series ϕ_4 and ϕ_6. A modular form is called a cusp form if it is in the kernel of Φ. In the case $n = 2$, therefore, there exists one cusp form and only one up to a constant factor of weight ten and of weight twelve. These cusp forms can be obtained in two ways. Since we have $\phi_4\phi_6 = \phi_{10}$, we see that $\psi_4\psi_6 - \psi_{10}$ vanishes along $\epsilon = 0$, hence, certainly, it is a cusp form. Also, since we have

$$3^2 7^2 \phi_4{}^3 + 2 \cdot 5^3 \phi_6{}^2 - 691\phi_{12} = 0,$$

we get a cusp form by replacing phi by psi. We have thus found cusp forms of weight ten and twelve, and they have Fourier expansions of the following form

$$\psi_4\psi_6 - \psi_{10} = -2^{12}3^5 5^2 7\cdot 53\cdot 43867^{-1} e(\tau_1)e(\tau_2)(\pi\epsilon)^2 + \cdots$$

$$3^2 7^2\psi_4{}^3 + 2\cdot 5^3\psi_6{}^2 - 691\psi_{12}$$
$$= 2^{13}3^7 5^3 7^2 337\cdot 131^{-1}593^{-1} e(\tau_1)e(\tau_2) + \cdots.$$

Therefore, we shall define normalized cusp forms by

$$\chi_{10} = -43867\cdot 2^{-12}3^{-5}5^{-2}7^{-1}53^{-1}(\psi_4\psi_6 - \psi_{10})$$
$$\chi_{12} = 131\cdot 593\cdot 2^{-13}3^{-7}5^{-3}7^{-2}337^{-1}(3^2 7^2\psi_4{}^3 + 2\cdot 5^3\psi_6{}^2 - 691\psi_{12}).$$

On the other hand, the modular forms χ and ψ_{0032} in Theorem 1 are also cusp forms of weight ten and twelve, and they have Fourier expansions of the form

$$\chi = 2^{54}3^{18}\pi^{18} e(\tau_1)e(\tau_2)(\pi\epsilon)^2 + \cdots$$
$$\psi_{0032} = 2^{36}3^{12}\pi^{12} e(\tau_1)e(\tau_2) + \cdots.$$

Therefore, we have the following theorem:

THEOREM 2. *The three modular forms ψ_{0121}, χ and ψ_{0032} can be expressed by Eisenstein series in the form*

$$\psi_{0121} = 2^{18}3^6\pi^6\psi_6 \qquad \chi = 2^{54}3^{18}\pi^{18}\chi_{10}$$
$$\psi_{0032} = 2^{36}3^{12}\pi^{12}\chi_{12}.$$

COROLLARY. *The graded ring of modular forms is generated over $\mathbf{C}$ by the Eisenstein series ψ_4, ψ_6, ψ_{10} and ψ_{12}.*

Now, we recall that elements of weight zero of the field of fractions of the graded ring of modular forms are called modular functions. By the lemma in Section 3, modular functions are absolute invariants. The converse is also true, i.e. absolute invariants are modular functions. They can be expressed, therefore, rationally in terms of Eisenstein series. Now, there is a problem once proposed by Siegel to get these rational expressions explicitly [9, p. 604]. In order to solve this problem, we have only to write down the three absolute invariants x_1, x_2, x_3 by ψ_4, ψ_6, χ_{10} and χ_{12}. However, using the relation betwen algebraic modular forms and multi-canonical differentials on X, we immediately get the following result:

THEOREM 3. *The three absolute invariants can be expressed by the four modular forms in the form*

$$x_1 = \psi_4\chi_{10}{}^2/\chi_{12}{}^2 \qquad x_2 = \psi_6\chi_{10}{}^3/\chi_{12}{}^3 \qquad x_3 = \chi_{10}{}^6/\chi_{12}{}^5.$$

Thus, the problem raised by Siegel is completely solved.

9. In this last section, we shall investigate the structure of Satake's compactification in the case $n=2$. Because of Theorem 1, the problem is reduced to determining the structure of

$$Y = \operatorname{proj} \boldsymbol{C}[T_2, T_3, T_5, T_6]$$

in which T_2, T_3, T_5, T_6 are indeterminates of degree four, six, ten, twelve and the ring is graded by the total degree. The projective variety Y admits a covering by four open affine subvarieties Y_2, Y_3, Y_5, Y_6 in which Y_j is the complement of $T_j = 0$ for $j = 2, 3, 5, 6$. We shall start by examining these affine varieties.

First, if we operate a cyclic group of order two to a ring of polynomials in three letters t_1, t_2, t_3 as

$$t_1, t_2, t_3 \to -t_1, -t_2, t_3,$$

the ring of invariant elements of $\boldsymbol{C}[t_1, t_2, t_3]$ can be identified with the co-ordinate ring of Y_2 in the following way:

$$T_3{}^{e_3} T_5{}^{e_5} T_6{}^{e_6} / T_2{}^{e_2} = t_1{}^{e_3} t_2{}^{e_5} t_3{}^{e_6}.$$

In fact, the condition on one side is $3e_3 + 5e_5 + 6e_6 = 2e_2$ and the condition on the other side is $e_3 + e_5 \equiv 0 \bmod 2$. These two conditions are clearly equivalent. Since the co-ordinate ring of Y_2 is the subring of $\boldsymbol{C}[t_1, t_2, t_3]$ generated by $t_1{}^2, t_1 t_2, t_2{}^2, t_3$, the variety Y_2 is isomorphic to a product of a representative cone of a non-degenerate conic and a straight line. The singular locus of Y_2 is, therefore, the locus of the vertex of the cone, which is a straight line defined by $T_3 = T_5 = 0$. As for Y_3, if we operate a cyclic group of order three to a ring of polynomials in t_1, t_2, t_3 as

$$t_1, t_2, t_3 \to \zeta t_1, \zeta t_2, t_3 \quad (\zeta^3 = 1),$$

the ring of invariant elements of $\boldsymbol{C}[t_1, t_2, t_3]$ can be identified with the co-ordinate ring of Y_3 in the following way:

$$T_2{}^{e_2} T_5{}^{e_5} T_6{}^{e_6} / T_3{}^{e_3} = t_1{}^{e_2} t_2{}^{e_5} t_3{}^{e_6}.$$

In fact, the two conditions $2e_2 + 5e_5 + 6e_6 = 3e_3$ and $e_2 + e_5 \equiv 0 \bmod 3$ are equivalent. Since the co-ordinate ring of Y_3 is the subring of $\boldsymbol{C}[t_1, t_2, t_3]$ generated by $t_1{}^3, t_1{}^2 t_2, t_1 t_2{}^2, t_2{}^3, t_3$, the variety Y_3 is isomorphic to a product of a representative cone of a cubic space curve and a straight line. The singular locus of Y_3 is, therefore, the locus of the vertex of the cone, which is a straight line defined by $T_2 = T_5 = 0$. We have to examine two more varieties. If we operate a cyclic group of order five to a ring of polynomials in t_1, t_2, t_3 as

$$t_j \to \zeta^j t_j \qquad j=1,2,3 \qquad (\zeta^5=1),$$

the ring of invariant elements of $\boldsymbol{C}[t_1, t_2, t_3]$ can be identified with the co-ordinate ring of Y_5 in the following way:

$$T_2{}^{e_2}T_3{}^{e_3}T_6{}^{e_6}/T_5{}^{e_5} = t_1{}^{e_6}t_2{}^{e_2}t_3{}^{e_3}.$$

Therefore, as we have shown in AVM, the co-ordinate ring of Y_5 is the subring of $\boldsymbol{C}[t_1, t_2, t_3]$ generated by $t_1{}^5$, $t_1{}^3t_2$, $t_1t_2{}^2$, $t_1{}^2t_3$, t_2t_3, $t_1t_3{}^3$, $t_2{}^5$, $t_3{}^5$ and the variety Y_5 is isomorphic to the variety of moduli. The singular locus of Y_5 is the point defined by $T_2 = T_3 = T_6 = 0$. Finally, if we operate a cyclic group of order six to a ring of polynomials in t_1, t_2, t_3 as

$$t_1, t_2, t_3 \to \zeta^2 t_1, \zeta^3 t_2, \zeta^5 t_3 \quad (\zeta^6 = 1),$$

the ring of invariant elements of $\boldsymbol{C}[t_1, t_2, t_3]$ can be identified with the co-ordinate ring of Y_6 in the following way:

$$T_2{}^{e_2}T_3{}^{e_3}T_5{}^{e_5}/T_6{}^{e_6} = t_1{}^{e_2}t_2{}^{e_3}t_3{}^{e_5}.$$

Therefore, the variety Y_6 is isomorphic to the affine variety V in Section 3 and its singular locus consists of two straight lines defined by $T_2 = T_5 = 0$ and by $T_3 = T_5 = 0$. Hence, the singular locus of Y itself consists of a point defined by $T_2 = T_3 = T_6 = 0$ and of two projective straight lines defined by $T_2 = T_5 = 0$ and by $T_3 = T_5 = 0$ intersecting at $T_2 = T_3 = T_5 = 0$.

It is, now, easy to determine Betti numbers b_j of the variety Y for all j. Since Y_2 admits a three dimensional affine space as a two-sheeted covering, if a singular chain of Y does not cover the whole Y_2, twice this chain is homotopic to a singular chain on $T_2 = 0$. Therefore, beside $b_0 = b_6 = 1$, we have $b_5 = 0$. Since the intersection of Y_3 and $T_2 = 0$ is a two dimensional affine space, we have $b_4 = 1$ and $b_3 = 0$. Finally, since the intersection of Y_5 and $T_2 = T_3 = 0$ is a straight line, we have $b_2 = 1$ and $b_1 = 0$. Therefore, the projective variety Y has the same Betti numbers as three dimensional complex projective space.

We have thus investigated the structure of Y which is defined abstractly. We shall, now, translate the results into a language of modular forms. For this purpose, we put

$$T_2 = a\psi_4 \qquad T_3 = b\psi_6 \qquad T_5 = c\psi_4\psi_6 + c'\psi_{10}$$
$$T_6 = d\psi_4{}^3 + d'\psi_6{}^2 + d''\psi_{12}$$

in which a, b, c, c', d, d', d'' are constants and $abc'd''$ is different from zero. Then, we get a holomorphic map of F_2 to Y so that the variety of moduli and the variety V are mapped isomorphically to Y_5 and Y_6. Since the verification is straightforward, we shall leave it to the reader. Moreover, the complement of the image is a projective straight line isomorphic to proj $\boldsymbol{C}[\phi_4, \phi_6]$. This can be identified with j-line compactified by a point at infinity, i. e. with the union of F_1 and F_0. Therefore Y is isomorphic to Satake's compactification of F_2 and it is a "V-manifold" whose structure we know completely. We note that the singular locus of Y consists of a point representing the jacobian variety of $y^2 = 1 - x^5$ and of two projective straight lines of points representing products of elliptic curves of which at least one factor is $y^2 = 1 - x^3$ or $y^2 = 1 - x^4$ (allowing another factor to degenerate to $y^2 = 1 - x^2$). Equations in terms of the Eisenstein series of these cases are $\psi_4 = \psi_6 = \psi_{12} = 0$, $\psi_4 = \psi_{10} = 0$ and $\psi_6 = \psi_{10} = 0$.

Appendix. The dimension N_w of the complex vector space of modular forms of weight w is equal to the number of non-negative integer solutions of $w = 4p + 6q + 10r + 12s$. This is derived from a structure theorem of the graded ring of modular forms. It is known, on the other hand, that A. Selberg has a general "trace formula" which contains a formula for N_w as a special case. However, since his formula is not given in a "finite form" even in the case of modular group of degree two (because of serious complications coming from non-compact boundaries), we did not try to use this formula. Nevertheless, we feel that we have to spend a few lines connected with this approach to the problem of determining the structure of the graded ring of modular forms, because it is the only general method we can think of at the present moment.

The result we have for N_w can be written in the form

$$N_{2k} = \mathrm{Res}_{x=0}\, dx/(1-x^2)(1-x^3)(1-x^5)(1-x^6)x^{k1+}$$

for $k = 0, \pm 1, \pm 2, \cdots$. We transform this formula using the theorem of residues and calculate all residues. In this way, we get the following formula

$$\begin{aligned} N_{2k} = {} & 2^{-4}3^{-3}5^{-1}(2k^3 + 48k^2 + 347k + 728) \\ & + (-1)^k 2^{-4}3^{-1}(k+8) \\ & + 2^{-2}3^{-4}(\rho^{2k+1} + \bar{\rho}^{2k+1})(6k+41) \\ & + 2^{-1}3^{-4}7(\rho^{2k} + \bar{\rho}^{2k}) \\ & + 2^{-2}3^{-2}(\rho^k + \bar{\rho}^k + \rho^{k+1} + \bar{\rho}^{k+1}) \\ & + 5^{-2}\sum_{\zeta^5=1} \zeta^k - \zeta^{k+1} \end{aligned}$$

for $k=0,1,2,\cdots$, where ρ stands for $e(\frac{1}{6})$ and $\bar{\rho}$ is its complex conjugate. Now, it may be possible to prove this formula by Selberg's method. Therefore, it will be of some importance if we shall show that this dimension formula implies the structure theorem. At any rate, we get the two cusp forms χ_{10} and χ_{12} as in Section 8. We shall show that ψ_4, ψ_6, χ_{10} and χ_{12} are algebraically independent over $\boldsymbol{C}$. If they are not algebraically independent, there exists a homogeneous element $P(T)$ not equal to zero in the graded ring $\boldsymbol{C}[T_2,T_3,T_5,T_6]$ (of Section 9) satisfying $P(\psi_4,\psi_6,\chi_{10},\chi_{12})=0$. We take as $P(T)$ one which has a smallest degree and write it in the form $P_0(T_2,T_3,T_5,T_6)T_5+P_1(T_2,T_3,T_6)$. Now, we introduce an operator Ψ to be the restriction of (finite sums of) modular forms to the subvariety of $\mathfrak{S}_2$ defined by $\epsilon=0$. Then Ψ is a homomorphism and χ_{10} is in the kernel. Therefore, applying Ψ to

$$P_0(\psi_4,\psi_6,\chi_{10},\chi_{12})\chi_{10}+P_1(\psi_4,\psi_6,\chi_{12}) = 0,$$

we get $P_1(\Psi\psi_4,\Psi\psi_6,\Psi\chi_{12})=0$. We shall show that $\Psi\psi_4$, $\Psi\psi_6$ and $\Psi\chi_{12}$ are algebraically independent (over $\boldsymbol{C}$). For this purpose, we observe that we have a relation of the form

$$\begin{aligned}\Psi\chi_{12}(\tau_1,\tau_2) &= \text{const.}\,\phi_4(\tau_1)^3\phi_4(\tau_2)^3+\text{const.}\,\phi_6(\tau_1)^2\phi_6(\tau_2)^2\\ &\quad+\text{const.}\,(\phi_4(\tau_1)^3\phi_6(\tau_2)^2+\phi_4(\tau_2)^3\phi_6(\tau_1)^2)\end{aligned}$$

in which the third constant coefficient is different from zero. On the other hand, if x, y, x', y' are algebraically independent, certainly xx', yy', $x^3y'^2+x'^3y^2$ are also algebraically independent. Therefore $\Psi\psi_4$, $\Psi\psi_6$, $\Psi\chi_{12}$ are algebraically independent. Hence, we have $P_1(T_2,T_3,T_6)=0$. This will imply that $P_0(T)$ is different from zero and $P_0(\psi_4,\psi_6,\chi_{10},\chi_{12})=0$. Since $P_0(T)$ is of a smaller degree than $P(T)$, we get a contradiction. Therefore ψ_4, ψ_6, χ_{10} and χ_{12} are algebraically independent. Then, by the dimension formula, the graded ring generated by these four modular forms will be the ring of all modular forms. This is the structure theorem.

THE JOHNS HOPKINS UNIVERSITY.

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