# THETA BLOCKS

MARTIN RAUM FOLLOWING A TALK BY D. ZAGIER

The theory of Theta Blocks is based on work by V. Gritsenko, N. Skoruppa, and Z.

## Jacobi Forms (Eichler-Z.)

$\tau \in \mathcal{H}$, $z \in \mathbb{C}$, $q = e^{2\pi i \tau} = \underline{e}(\tau)$, $\zeta = e^{2\pi i z} = \underline{e}(z)$.

$$\begin{aligned}\Theta(\tau, z) &= \sum_{n\in\mathbb{Z}} (\frac{-4}{n}) q^{n^2/8} \zeta^{n/2} \\ &= q^{\frac{1}{8}}\big(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\big) - q^{\frac{9}{8}}\big(\zeta^{\frac{3}{2}} - \zeta^{-\frac{3}{2}}\big) + \cdots \\ &= q^{\frac{1}{8}}\zeta^{\frac{1}{2}} \prod_{n=1}^{\infty} (1-q^n)(1-q^n\zeta)(1-q^{n-1}\zeta^{-1}) \quad \text{(Jacobi triple product identity).}\end{aligned}$$

where

$$(\frac{-4}{n}) = \begin{cases} \pm 1 & n \equiv \pm 1 \pmod 4 \\ 0 & 2 \,|\, n \end{cases}.$$

$$\begin{aligned}\Theta(\tau, -z) &= \Theta(\tau, z) \\ \Theta(\tau+1, z) &= \underline{e}(\frac{1}{8})\Theta(\tau, z) \\ \Theta(\tau, z+1) &= -\Theta(\tau, z) \\ \Theta(\tau, z+\tau) &= -q^{-\frac{1}{2}}\zeta^{-1}\Theta(\tau, z) \\ \Theta(-\frac{1}{\tau}, \frac{z}{\tau}) &= \sqrt{\frac{\tau}{i}}\underline{e}(\frac{z^2}{2\tau})\Theta(\tau, z).\end{aligned}$$

$$\eta(\tau) = \frac{1}{2}\ sum_{n\in\mathbb{Z}}(\frac{12}{n}) q^{\frac{n^2}{24}}.$$

**Definition 0.1.** *A Jacobi form is a (holomorphic) function $\phi(\tau, z)$ satisfying ($k$ = weight, $m$ = index)*

*(i)* $\phi\big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\big) = (c\tau+d)^k \underline{e}\big(\frac{mcz^2}{c\tau+d}\big)\phi(\tau, z)$.
*(ii)* $\phi(\tau, z + r\tau + s) = \underline{e}(-mr^2\tau - 2mrz)\phi(\tau, z)$.
*(iii) holomorphy at infinity:* $\phi(\tau, z) = \sum_{4nm - r^2 \geq 0} c(n, r) q^n \zeta^r$.

*Notation:* $J_{k,m}$.

$\Theta \in J_{\frac{1}{2},\frac{1}{2}}$.

weak Jacobi forms: $\phi(\tau, z) = \sum_{n\geq 0} c(n,r) q^n \zeta^r$; weakly holomorphic: $\phi(\tau, z) = \sum_{n \geq n_0} c(n, r) q^n \zeta^r$; Notation: $J_{k,m} \subset \widetilde{J}_{k,m} \subset J^!_{k,m}$.

**Examples 0.2.** $k > 2$

$$E_{k,m} = \sum_{\mathrm{SL}_2(\mathbb{Z}} \ltimes \mathbb{Z}^2 1\Big| \ldots \in J_{k,m}.$$

$$E_{4,1}(\tau, 0) = E_4(\tau) \qquad E_{6,1}(\tau, 0) = E_6(\tau)$$
$$\phi_{10,1}(\tau, z) = E_{4,1}E_6 - E_{6,1}E_4 \in J_{10,1}^{cusp}$$
$$\phi_{11,1} = E_{4,1}E'_{6,1} - E_{6,1}E'_{4,1}.$$

$$A = \frac{\phi_{10,1}}{\Delta} \in \widetilde{J}_{-2,1} \quad B = \frac{\phi_{12,1}}{\Delta} \in \widetilde{J}_{0,1}$$
$$C = \frac{\phi_{11,1}}{\Delta} \in \widetilde{J}_{-1,2}.$$

$$M_* = \bigoplus M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6].$$

$$\widetilde{J}_{even,*} = \mathbb{C}[E_4, E_6, A, B] = M_*[A, B]$$
$$\widetilde{J}_{*,*} = M_*[A, B, C] \,/ (432C^2 = A(B^3 - 3E_4BA^2 + 2E_6A^4)).$$

$$\frac{B}{A} = \frac{\phi_{12,1}}{\phi_{10,1}} = \frac{\neq 0}{O(z^2)} = c\surd$$
$$\frac{C}{A^2} = \surd'.$$

$$\surd'^2 = \surd^3 + E_4\surd + E_6.$$

0.1. **"Jacobi like".** $k$ even $\Rightarrow \phi(\tau, -z) = \phi(\tau, z)$.

$\phi(\tau, z) = \xi_0(\tau) + \big(\frac{\xi_1(\tau)}{2} - \frac{m\xi_0'(\tau)}{k}\big)(2\pi iz)^2 + \big(\frac{xi_2(\tau)}{24} + \frac{m\xi_1'(\tau)}{2(k+2)} + \frac{m^2\xi_0''(\tau)}{2k(k+1)}\big)(2\pi iz)^4 + \cdots,$
where $(f(\tau) = \sum a_nq^n,\ f'(\tau) := \sum na_nq^n)$

$$\xi_0 \in M_k \quad \xi_1 \in M_{k+2} \quad \xi_\nu \in M_{k+2\nu}.$$

Cohen-Kuznetsov

$$\xi_0 \to \widehat{xi}_0(\tau, z) = \xi_0 + \frac{m}{k}\xi_0'(2\pi iz)^2 + \cdots$$

is Jacobi-like (satisfies (1)).

$$\phi = \sum \widehat{\xi}_\nu z^{2\nu}.$$

$$J_{k,m}^{(\mathrm{like})} \cong \big(\prod_{\nu=0}^{\infty} M_{k+2\nu} \quad \text{(convergent part)}.$$

$k$ odd

$$phi(\tau, z) = \xi_0(\tau)(2\pi iz) + (\xi_0' + \frac{xi_1}{6})(2\pi iz)^3 + \cdots.$$

Odd weight forms have zeros at $\frac{\tau+1}{2}$, $\frac{\tau}{2}$, and $\frac{1}{2}$.

Any Jacobi function has at most $2m$ zeros in a fundamental domain. So we get

**Theorem 0.3.** *$k$ even*

$$\widetilde{J}_{k,m} \overset{\cong}{\hookrightarrow} \bigoplus_{\nu=0}^{m} M_{k+2\nu}.$$

*$k$ odd*

$$\widetilde{J}_{k,m} \overset{\cong}{\hookrightarrow} \bigoplus_{\nu=0}^{m-2} M_{k+2\nu+1}.$$

We call this the Taylor expansion of a weak Jacobi form.

0.2. **Elliptic forms.** Functions satisfying only (2).

$$\phi = \sum c(n,r) q^n \zeta^r.$$

(2) tells us

$$\phi(\tau,z) = q^{m\lambda^2}\zeta^{2m\lambda}\phi(\tau, z+\lambda\tau+\mu) = \sum c(n,r) q^{n+\lambda r+m\lambda^2}\zeta^{r+2m\lambda}.$$

So we have

$$c(n,r) = c(n+\lambda r + m\lambda^2, r+2m\lambda) \quad \forall \lambda \in \mathbb{Z}.$$

Set

$$\Delta := 4nm - r^2 = 4m(m\lambda^2 + r\lambda + n) - (2m\lambda + r)^2.$$

$$c(n,r) = C(\Delta, r \pmod{2m})$$

$$\begin{aligned}\phi &= \sum_{l \pmod{2m}} \Big( \sum_{\Delta \equiv -l^2 \pmod{4m}} C(\Delta, l) q^{\frac{Delta}{4m}} \Big) \Big( \sum_{r \equiv l \pmod{2m}} q^{\frac{r^2}{4m}}\zeta^r \Big) \\ &= \sum_{l \pmod{2m}} h_l(\tau) \underbrace{\Theta_{m,l}(\tau,z)}\text{ universal.}\end{aligned}$$

$$\phi \mapsto (h_0, h_1, \dots, h_{2m-1}) = \text{vector}.$$

$\phi \in J_{k,m} \Rightarrow h_l \in M_{k-\frac{1}{2}}$(some level).

**Summary:** (1) ("Jacobi-like") $\Leftrightarrow$ $\phi = \xi_0 + (\xi_1 + \dots)z^2 + \cdots$. (2) ("elliptic") $\Leftrightarrow \phi = \sum_{l=1}^{2m} h_l(\tau)\Theta_{m,l}(\tau,z)$.

$$\xi_\nu = \sum_{l \pmod{2m}} [h_l \Theta^0_{m,l}]_\nu \in M_{k+2\nu}.$$

$m = 1$ then $\widetilde{J}_{k,m} = M_{k+2}A \oplus M_k B$, $J_{k,1} \cong M_k \oplus S_{k+2}$, $J^{\text{cusp}}_{k,1} = S_k \oplus S_{k+2}$.

| | $\Delta$ | -1 | 0 | 3 | 4 | 7 | 8 | 11 | 12 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|
| Coefficients | $\phi_{10,1}$ | 0 | 0 | 1 | -2 | -16 | 36 | 99 | -272 | -240 |
| | $\phi_{12,1}$ | 0 | 0 | 1 | 10 | -88 | -132 | 1275 | | |
| | $A$ | 1 | -2 | 8 | -12 | 39 | -56 | 153 | | |
| | $B$ | 1 | 10 | -64 | 108 | | | | | |

$$A = \frac{\Theta(\tau,z)^2}{\eta(\tau)^6} \quad B = A\Big|_{\surd} = 4\sum_{j=2}^{4} \frac{\Theta_j(\tau,z)^2}{\Theta_j(\tau,0)^2} \quad C = \frac{\Theta(\tau,2z)}{\eta\tau^3}.$$

First weight 2 Jacobi form at $m = 25$, first cusp form at $m = 37$.

**Theorem 0.4** (Sk. - Z.)**.**

$$J_{k,m} \cong \underbrace{\mathcal{M}} \text{ “pure”}$$
$$M_k^{new}(\Gamma_0(N)) \subset \mathcal{M}_k(N) \subset M_k(\Gamma_0(N)).$$

Connection to elliptic curves $X_0^*(37) = E_{37}$ with odd function equation. $L'(E,1) \cong h(P)$, $P$ a Heegner point. Heegner points $P_\Delta$ are labeled by discriminants. To compute the corresponding Jacobi form $\phi$ use projection to $S_2$.

$$\phi = \frac{\Theta(z)^3\Theta(2z)^3\Theta(3z)^2\Theta(4z)\Theta(5z)}{\eta(\tau)^6}.$$

**Example 0.5.** $\widetilde{J}_{2,1} = M_4 A \oplus M_2 B = \langle E_4 A\rangle$

| $\Delta$ | -1 | 0 | 3 | 4 | 7 | 8 | 11 |
|---|---|---|---|---|---|---|---|
| $C(E_4A;\Delta$ | 1 | -2 | 248 | -492 | 4119 | -7256 | 33512 |

$$j(\tau) = \frac{E_4^3}{\Delta} = q^{-1} + 744 + 196884q + \cdots$$

*is a Hauptmodule* $\overline{\mathcal{H}/\Gamma} \to \mathbb{P}^1(\mathbb{C})$. *Set* $J = j - 744$.

$$\sqrt[3]{J(\tau)} = \frac{E_4(\tau)}{\eta(\tau)^8} = q^{-\frac{1}{3}} + 248q^{\frac{2}{3}} + \cdots.$$

*We find*

$$J\Big(\frac{1+i\sqrt{3}}{3}\Big) = \frac{0-744}{3} = -248 \quad \frac{J(i)}{2} = \frac{1728-744}{2} = 492 \quad J\Big(\frac{1+i\sqrt{7}}{2}\Big) = -3375 - 744 = -4119.$$

*This is the above coefficients.*

$\phi \in J^!_{k,m}$ *order at* $\infty$: $\mathrm{Ord}_\infty(\phi) = \min\{n - \frac{r^2}{4m} \,:\, c(n,r) \neq 0\} = \min_l \mathrm{Ord}_\infty(h_l)$.
*weak order*: $\widetilde{\mathrm{Ord}}_\infty(\phi) = \min\{n \,:\, c(n,r) \neq 0\}$.
$\widetilde{\mathrm{Ord}}_\infty(\phi) \geq 0 \Leftrightarrow \phi \in \widetilde{J}_{k,m}$ $\mathrm{Ord}_\infty(\phi) \geq 0 \Leftrightarrow \phi \in J_{k,m}$.
Right notion of order is a function on $\mathbb{R}/\mathbb{Z}$.

**Definition 0.6.**

$$\mathrm{Ord}(\phi)(x) = \min\{mx^2 + rx + n \,:\, c(n,r) \neq 0\}.$$

$\min_{x\in\mathbb{R}/\mathbb{Z}} \mathrm{Ord}(\phi)(x) = \mathrm{Ord}_\infty(\phi)$. We have $\mathrm{Ord}(\phi_1\phi_2)(x) = \mathrm{Ord}(\phi_1) + \mathrm{Ord}(\phi_2)$. The minimum in the definition are attained at $x = \frac{-r}{2m} \in \frac{1}{2m}\mathbb{Z}/\mathbb{Z}$.

$$\Theta(\tau,z) = \sum \Big(\frac{-4}{n}\Big) q^{\frac{n^2}{8}} \zeta^{\frac{n}{2}}$$
$$\mathrm{Ord}(\Theta)(x) = \min_{n \text{ odd}} \Big(\frac{1}{2}x^2 + \frac{n}{2}x + \frac{n^2}{8}\Big) = \frac{1}{2}\|x + \frac{1}{2}\|^2.$$

**Example 0.7.** *The above theta block corresponds to* $(a_1,\dots,a_{10}) = (1,1,1,2,2,2,3,3,4,5)$.

$$\min_x \Big( \sum_{i=1}^{10} \frac{1}{2}\|a_i x - \frac{1}{2}\|^2 \geq \frac{6}{24}.$$

*So we can divide by* $\eta^6$.

**Second lecture**

$$\mathrm{ord}_\phi(x) = \min_{c(n,r)\neq 0} (mx^2 + nx + r).$$

$\phi$ hol. $\Rightarrow 4mn \geq r^2 \Rightarrow mx^2 + rx + n = \frac{(2mx+r)^2+(4mn-r^2)}{4m} \geq 0.$

$\phi$ hol $\Leftrightarrow \mathrm{ord}_\phi(x) \geq 0 \forall x$, because

$$\min_{x\in\mathbb{R}}(\mathrm{ord}_\phi(x)) = \min_x \min_{c(n,r)\neq 0} (mx^2+rx+n) = \min_{n,r}\min_x(mx^2+rx+n) = \min_{n,r}\frac{4mn-r^2}{4m} = \frac{\Delta_{\min}}{4m} = \mathrm{Ord}$$

Since $\mathrm{ord}_\phi$ has discontinuities at non-minimal points only, it the minimum is attained at $x \in \frac{1}{2m}\mathbb{Z}/\mathbb{Z}$.

Properties of $g(x) = \mathrm{ord}_\phi(x)$:

- continuous.
- $g(x+1) = g(x)$.
- $g(-x) = g(x)$.
- $g''(x) = 2m$ a.e.
- $\mathrm{ord}_{\phi_1\phi_2} = \mathrm{ord}_{\phi_1} + \mathrm{ord}_{\phi_2}$.

The order is multiplicative, because for $\phi \in J_{k,m}$, then $\phi(\tau,0) \in M_k, \phi(\tau,\frac{2}{5}) \in M_k$, but $\phi(\tau,\frac{2\tau-3}{7}) \notin M_k$, $q^{\frac{4m}{49}}\phi(\tau,\frac{2\tau-3}{7}) \in M_k$. Generally, $q^{m\lambda^2}\phi(\tau,\lambda\tau+\nu)$. Thus

$$\mathrm{ord}_\phi(x) = \text{limiting value as } \lambda \to x \in \mathbb{R},\ \mu \to 0 \text{ of } \mathrm{Ord}_\infty\big(q^{m\lambda^2}\phi(\tau,\lambda\tau+\mu)\big).$$

## 1. Theta Blocks

**Definition 1.1** (temporary)**.** *A theta block is:*

$$\frac{\Theta_{l_i}\cdots\Theta_{l_N}}{\eta^*},$$

*where* $l_i \in \mathbb{Z}$, $* \in \mathbb{Q}$ *and* $\Theta_l(\tau,z) = \Theta(\tau,lz)$.

*More generally: allow* $\Theta_l^{\pm 1}$.

Example of the more general case: $\eta(\tau)\frac{\Theta(2z)}{\Theta(z)}$, which is holomorphic, since all zeros cancel.

$$\Theta_l^0 := \prod_{d\,|\,l}\Theta_{l/d}^{\mu(d)}$$

$$\Theta_l = \prod_{d\,|\,l}\Theta_d^0,$$

which are holomorphic.

Notation: $a = \underline{a} = (a_1,\dots,a_N)$. $\Theta_{\underline{a}} = \prod_{i=1}^N \Theta(\tau,a_i z)$. $M(\underline{a}) = 24\mathrm{Ord}_\infty(\Theta_{\underline{a}}) \in \mathbb{Q}$

$M(\underline{a})$ is the biggest power of $\eta$ by which $\Theta_{\underline{a}}$ can be divided and remain holomorphic.

**Example 1.2.**

$$\eta^{-6}\Theta_{(1,1,1,2,2,2,3,3,4,5)} \in J_{2,37}^{\mathrm{cusp}}$$

Question: How good does this get?

Open question: Does $N - M(\underline{a})$ have to go to $\infty$ as $N \to \infty$? Assuming it does, what is the biggest $N$ you can find where $\frac{\Theta_{l_1} \cdots \Theta_{l_N}}{\eta^{N-4}}$ has weight 2?

Record: $N = 37$, $\frac{\Theta_1(\Theta_1\Theta_2\Theta_3)(\Theta_1 \cdots \Theta_{33})}{\eta^{33.02}}$.

Fact: $M(\underline{a}) < N$. $\eta^{-M}\Theta_1 \cdots \Theta_N$ hol $\Rightarrow \frac{N-M}{2} =$ weight $> 0$.

$\underline{a} = (a_1, \ldots, a_N) \in \mathbb{Z}^N$. WLOG $\underline{a} \in \mathbb{N}^N$, since $\Theta_{-l} = -\Theta_l$, $\gcd(\underline{a}) = 1$, and $1 \geq a_1 \geq \ldots \geq a_N$.

**Example 1.3** ("Theta quarks")**.** *$a, b \in \mathbb{Z} \Rightarrow \Theta_a\Theta_b\Theta_{a+b}$ is divisible by $\eta$, i.e.* $\mathrm{Ord}_\infty(\Theta_a\Theta_b\Theta_{a+b}) \geq \frac{1}{24}$. *$m = \frac{a^2}{2} + \frac{b^2}{2} + \frac{(a+b)^2}{2} = a^2 + ab + b^2 \in \mathbb{Z}$. $\eta^{-M}\Theta_{\underline{a}} \in J_{\frac{N}{2},m}(\epsilon^{3N-M} v_{\mathrm{H}}^{2m})$.*

$$\mathrm{Ord}_\infty = \begin{cases} \frac{1}{24} & \text{if } 3 \mid m \\ \frac{1}{24}(1 + \frac{2}{m}) & \text{if } 3 \nmid m. \end{cases}$$

*Proof 1: Show* $\mathrm{ord}(\Theta_a\Theta_b\Theta_{a+b})(x) \geq \frac{1}{24}$.

$$\mathrm{ord}(\theta)(x) = \frac{1}{2}\|x - \frac{1}{2}\|^2,$$

$\|x\| = d(x, \mathbb{Z})$.

$$\mathrm{ord}_{\underline{a}} = \min_{x \in \mathbb{R}} \big(\frac{1}{2}\sum_{i=1}^N \|a_i x - \frac{1}{2}\|^2.$$

*Average of function is $\frac{N}{24}$, giving an upper bound for the minimum* $\mathrm{ord}_\infty$. *So*

$$M(\underline{a}) = 24\mathrm{Ord}_\infty(\Theta_{\underline{a}}) = 12 \min_x \sum_{i=1}^N \|a_i x - \frac{1}{2}\|^2$$

$$\frac{M}{12} = \min_x \min_{\underline{k} \in (\mathbb{Z}+\frac{1}{2})^N} \|\underline{a}x - \frac{k}{\|}^2 = \min_{\underline{k} \in (\mathbb{Z}+\frac{1}{2})^N} \|\pi_{\underline{a}}(\underline{k})\|^2 = d\big(\pi_{\underline{a}}(\frac{1}{2}, \ldots, \frac{1}{2}), \pi_{\underline{a}}(\mathbb{Z}^N)\big)^2,$$

*where* $\pi_{\underline{a}} : \mathbb{R}^N = \langle \underline{a} \rangle \oplus \langle a \rangle^\perp \to^{\pi_{\underline{a}}} \langle a \rangle^\perp$.

*Receipt: extend $\underline{a} = \underline{a}^{(1)}$ to an orthogonal basis of $\mathbb{R}^N$, $\underline{a}^{(1)}, \ldots, \underline{a}^{(N)}$.*

$$\|\underline{k}\|^2 = \sum_{i=2}^N \frac{(k, a^{(i)})^2}{(a^{(i)}, a^{(i)})}.$$

*Apply this to $\underline{a}^{(1)} = (a, b, a+b)$: $\underline{a}^{(2)} = (1, 1, -1)$.*

*$\underline{k} = (k_1 + \frac{1}{2}, k_2 + \frac{1}{2}, k_3 + \frac{1}{2})$, so $\|\pi_{(a,b,a+b)}(\underline{k})\|^2 \geq \frac{(\underline{k},(1,1,-1))^2}{\|(1,1,-1)\|^2} = \frac{(k_1+k_2-k_3+\frac{1}{2})^2}{3}$.*

*$\frac{-\Theta_a\Theta_b\Theta_{a+b}}{\eta} \in J_{1,m}(\epsilon^8)$ is a quark. The character becomes trivial if we multiply three $Q_1, Q_2, Q_3$ of them. $Q_1Q_2Q_3 \in J_{3,m_1+m_2+m_3}$.*

***Infinite families***

*(i)* $\eta^{-1}\Theta_a\Theta_b\Theta_{a+b}$.

*(ii)* $\eta^{-2}\Theta_a\Theta_b\Theta_{a+b}\Theta_{a-b}$. *Here, $\underline{a}^{(1)} = (a, b, a+b, a-b)$, $\underline{a}^{(2)} = (1, 1, -1, 0)$, and $\underline{a}^{(3)} = (-1, 1, 0, 1)$. So $\|\pi_{\underline{a}(\underline{k})}\|^2 = \frac{(\underline{k}, \underline{a}^{(2)})}{(\underline{a}^{(2)}, \underline{a}^{(2)})} + \cdots \geq \frac{(1/2)^2}{3} + \frac{(1/2)^2}{3}$.*

*(iii)* $\eta^{-4}\Theta_a\Theta_b\Theta_{a+b}\Theta_{a-b}\Theta_{2a-b}\Theta_{a-2b}$.

*(iv)* $\eta^{-3}\Theta_a\Theta_b\Theta_c\Theta_{a+b}\Theta_{b+c}\Theta_{a+b+c}$ *specializes to the above if $c = a - 2b$ (up to the $\eta$ power).*

*(v)* $\eta^{-6}\Theta_a\Theta_b\Theta_c\Theta_d\Theta_{a+b}\Theta_{b+c}\Theta_{c+d}\Theta_{a+b+c}\Theta_{b+c+d}\Theta_{a+b+c+d}$.

*(vi)* $\eta^{-\frac{s(s-1)}{2}}\Theta_{a_1}\cdots\Theta_{a_s}\Theta_{a_1+a_2}\cdots\Theta_{a_1+\cdots+a_s}$. $N=\frac{s(s+1)}{2}$ *and* $M=\frac{s(s-1)}{2}$, *so that* $N-M=s\cong\sqrt{N}$.

*Proof 2: Actually do it!*

**Proposition 1.4.** *For* $a,b\in\mathbb{N}$ *define*

$$\Theta_{a,b}(\tau,z)=\sum_{r,s\in\mathbb{Z}}\big(\frac{s}{3}\big)q^{r^2+rs+\frac{s^2}{3}}\zeta^{(a-b)r+as}\in J_{1,a^2+ab+b^2}(\epsilon^8).$$

*Then* $\Theta_{a,b}=-\frac{\Theta_a\Theta_b\Theta_{a+b}}{\eta}$.

*Proof.* Trick:

$$\forall s\in\mathbb{Z}:\ \big(\frac{-3}{s}\big)=\frac{\sum_{t\in s+3\mathbb{Z}}\big(\frac{-4}{t}\big)q^{t^2/24}}{\eta(\tau)}.$$

Proof:

$$\eta=\frac{1}{2}\overbrace{\big(\frac{12}{n}\big)}^{\left(\frac{-3}{n}\right)\left(\frac{-4}{n}\right)}q^{\frac{n^2}{24}}=\sum_{n\equiv 1\pmod 4}\big(\frac{-3}{n}\big)q^{n^2/24}+\sum_{n\equiv 3\pmod 4}\big(\frac{-4}{n}\big)q^{n^2/24}.$$

This is similar to a trick crucial to tread $L(s,\chi)$ (Gauss)

$$\overline{\chi}(n)=\frac{\sum_{j\pmod f}\chi(j)\underline{e}(\frac{jn}{f})}{\sum_{j\pmod f}\chi(j)\underline{e}(\frac{j}{f})}.$$

Euler:

$$\frac{1}{n^s}=\frac{\int_0^\infty t^{s-1}e^{-nt}\,dt}{\Gamma(s)}.$$

Now consider

$$\{(l,m,n)\in\mathbb{Z}^3\,:\,l\equiv m\equiv n\pmod 2\}\cong\{(r,s,t)\in\mathbb{Z}^3\,:\,s\equiv t\pmod 3\}\quad\text{via}$$
$$(l,m,n)\mapsto\big(\frac{n-m}{2},\frac{l+m}{2}-n,-l-m-n\big)$$

We have

$$(l^2+m^2+n^2)/8=(r^2+rs+s^2/3)+t^2/24.$$

We have

$$-\Theta_a\Theta_b\Theta_{a+b}=\sum_{l,m,n\in\mathbb{Z}}\big(\frac{-4}{lmn}\big)q^{\frac{l^2+m^2+n^2}{8}}\zeta^{\frac{al+bm-(a+b)m}{2}}=\sum_{r,s,t\in\mathbb{Z},\,s\equiv t\pmod 3}\big(\frac{-4}{t}\big)q^{r^2+rs+s^2/3}q^{t^2/24}\zeta^{\frac{al+bm-(a+b)r}{2}}$$

□

## 2. 3. Lecture

$$\underline{a} = (a_1, \dots, a_N) \in \mathbb{Z}^N$$

gives us a theta block

$$\Theta_{\underline{a}}(\tau, z) = \prod_{j=1}^{N} \Theta(\tau, a_i z).$$

Recall

$$M(\underline{a}) = 24 \mathrm{Ord}_\infty(\Theta_{\underline{a}}) = 12 \min_{x \in \mathbb{R}} \sum_{i=1}^{n} d(a_i x, \mathbb{Z} + \frac{1}{2})^2 = \text{biggest } M \text{ s.t. } \eta^{-M}\Theta_{\underline{a}} \text{ is holomorphic.}$$

We find $M = 0$ if and only if all $a_i$ are odd.

$$M(\underline{a}) = 12 \min_{x \in \mathbb{R}} \min_{\underline{k} \in (\mathbb{Z}+\frac{1}{2})^N} \|\underline{a}x - \underline{k}\|^2 = 12 \min_{\underline{k} \in (\mathbb{Z}+\frac{1}{2})^N} \|\pi_{\underline{a}}(\underline{k})\|^2 = 12 d\big(\pi_{\underline{a}}(\frac{1}{2}, \dots, \frac{1}{2}), \pi_{\underline{a}}(\mathbb{Z}^N)\big)^2.$$

We are interested in theta blocks with $N$ as large as possible. There were five families

(i) $\underline{a} = (a, b, a+b)$, $N = 3$, $M(\underline{a}) \geq 1$, $k = 1$.
(ii) $\underline{a} = (a, b, a-b, a+b)$, $N = 4$, $M(\underline{a}) \geq 2$, $k = 1$.
(iii) $\underline{a} = (a, b, b-a, a+b, 2b-a, 2a-b)$, $N = 6$, $M(\underline{a}) \geq 4$, $k = 1$.
(iv) $\underline{a} = (a, b, c, a+b, b+c, a+b+c)$, $N = 6$, $M(\underline{a})\ ge3$, $k = \frac{3}{2}$.
(v) $\underline{a} = (a, b, c, d, a+b, b+c, c+d, a+b+c, b+c+d, a+b+c+d)$, $N = 10$, $M(\underline{a}) \geq 6$, $k = 2$.

For (1) we can form the matrix (rows $\underline{a}$, $\underline{a}'$, $r_1$, and $r_2$)

$$\begin{pmatrix} a & b & a-b & a+b \\ b & -a & b+a & b-a \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}$$

which gives an orthogonal basis of $\mathbb{R}^4$. With $\underline{k} = (\alpha, \beta, \gamma, \delta) \in (\mathbb{Z} + \frac{1}{2})^4$ we have

$$\pi_{\underline{a}}(\underline{k}) = \frac{(k, r_1)^2}{(r_1, r_1)} + \frac{(k, r_2)^2}{(r_2, r_2)} = \frac{(\alpha - \beta - \gamma)^2}{3} + \frac{(\alpha + \beta - \delta)^2}{3} \geq \frac{(1/2)^2}{3} + \frac{(1/2)^2}{3} = \frac{2}{12}.$$

| $\underline{a}$ | $a$ | $b$ | $a-b$ | $a+b$ | $2a-b$ | $a-2b$ |
|---|---|---|---|---|---|---|
| $\underline{a}'$ | | | | | | |
| $r_0$ | $-1$ | $1$ | $1$ | $0$ | $0$ | $0$ |
| $r_1$ | $-1$ | $-1$ | $0$ | $1$ | $0$ | $0$ |
| $r_2$ | $-1$ | $0$ | $-1$ | $0$ | $1$ | $0$ |
| $r_3$ | $0$ | $-1$ | $1$ | $0$ | $0$ | $-1$ |
| $r_4$ | $0$ | $0$ | $0$ | $-1$ | $1$ | $-1$ |

For $\underline{k} \in \langle r_0, \dots, r_4 \rangle$ we get

$$\|\underline{k}\|^2 = \frac{(k, r_0)^2}{3} + \frac{(k, r_1)^2}{4} + \frac{(k, r_2)^2}{4} + \frac{(k, r_3)^2}{4} + \frac{(k, r_4)^2}{4} \geq \frac{(1/2)^2}{3} + 4\frac{(1/2)^2}{4} = \frac{4}{12}.$$

We consider $((n_0, \dots, n_s) \in \mathbb{Z}^{s+1})$

$$\prod_{0 \leq i < j \leq s} \Theta(\tau, (n_i - n_j)z)$$

We have $N = \binom{s+1}{2} = \frac{s(s+1)}{2}$.

Claim: $M = \binom{s}{2} = \frac{s(s-1)}{2}$. We give 3 proofs of this

(i) Ord $\geq \ldots$.

(ii) quotient = explicit $\Theta$ series

a) by comparing transformation properties.

b) explicitly.

Fix $s$

$$V = \{H \in M_{s+1} \,:\, H^{\mathrm{tr}} = -H\}$$
$$\dim V = N = \binom{s+1}{2}.$$

We write $V(\mathbb{R})$, $V(\mathbb{Z})$ for the real and integral points. Orthogonal decomposition $V(\mathbb{R}) = V_0(\mathbb{R}) \oplus V_1(\mathbb{R})$. Set

$$\|H\|^2 = Q(H) = \sum_{0 \leq i < j \leq s} h_{ij}^2 = \frac{1}{2}\sum_{i,j} h_{ij}^2 = -\frac{1}{2}\sum_{i,j} h_{ij}h_{ji} = -\frac{1}{2}\mathrm{tr}(H^2).$$

Notation: $u = (\underbrace{1, \ldots, 1}_{s+1})$. $\beta = (\frac{s}{2}, \frac{s-2}{2}, \ldots, -\frac{s}{2})$, $\beta_i = \frac{s}{2} - i$ for $0 \geq i \geq s$.

$$S = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & & & \vdots \\ \vdots & & & & \\ -1 & \cdots & \cdots & -1 & 0 \end{pmatrix}$$
$$G = \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & s \\ -1 & 0 & 1 & 2 & \cdots & s-1 \\ \vdots & & & & & \\ -s & \cdots & & & & 0 \end{pmatrix}.$$

We have

$$V_0 = \{(y_i - y_j)_{i,j=0,\ldots,s} \,:\, y \in \mathbb{R}^{s+1}/\mathbb{R}\underline{u}\}$$
$$V_1 = \{H \in V \,:\, uH = 0 \text{ or } Hu^{\mathrm{tr}} = 0\}$$
$$V_0 = \{y^{\mathrm{tr}}u - u^{\mathrm{tr}}y\} \quad y = (0 \ldots 1 \ldots 0),$$

where $H = H_0 + H_1 \in V_0 \oplus V_1$. We have $Q(H) = Q_0(H) + Q_1(H)$.

**Proposition 2.1.** *We have* $Q_0(H) = \frac{1}{s+1}\|uH\|^2$, $Q_1(H) = \frac{1}{s+1}\sum_{0 \geq i<j<k \geq s}(h_{ij}+h_{jk}+h_{ki})^2$.

*Proof.* We have $Q_0 = 0$ on $V_1$, $Q_1 = 0$ on $V_0$.

$$6(s+1)Q_1(H) = \sum_{i,j,k=0}^{s} (h_{ij} + h_{jk} + h_{ki})^2$$
$$= 3(s+1)\sum_{i,j} h_{ij}^2 + 6\sum_{i,j,k} h_{ij}h_{jk} = 6(s+1)\|H\|^2 - 6(s+1)Q_0(H).$$

□

**Theorem 2.2.** $\prod_{0 \leq 0 i<j \leq s} \Theta(\tau, (n_i - n_j)z)$ *is divisible by* $\eta^{\binom{s}{2}}$.

*Proof.* $\underline{k} \leftrightarrow K$ is a matrix now.

$$M\big((n_i - n_j)_{0\le i<j\le s}\big) \ge 12 \min_K Q_1(K) = 2(s+1) \sum_{0\le i<j<k\le s} (h_{ij}+h_{jk}+h_{ki})^2 \ge 2(s+1)(1/2)^2 \binom{s+1}{3}.$$

□

Consider $\tau \in \mathbb{H}$, $z \in \mathbb{C}^{s+1}/\mathbb{C}u$

$$P_s(\tau; z_0, \dots, z_s) = \prod_{0\le i<j\le s} \Theta(\tau, z_j - z_i).$$

Claim: $\eta(\tau)^{-\binom{s}{2}} P_s(\tau; \underline{z})$ is holomorphic. If you specialize to $\underline{z} = \underline{n}z$, $z \in \mathbb{C}$, $\underline{n} \in \mathbb{Z}^{s+1}/\mathbb{Z}u$ then this specializes to previous thing.

**Definition 2.3.**

$$Q_s(\tau; \underline{z}) = \sum_{x \in (\mathbb{Z}+\frac{s}{2})^{s+1}_{sum\ 0}} \underbrace{\epsilon(x)}_{\pm 1 \ or\ 0} q^{\frac{\|x\|^2}{2(s+1)}} \zeta_0^{x_0} \cdots \zeta_s^{x_s}.$$

$$\epsilon(x) = \begin{cases} 0 & \textit{if any two } x_i\textit{'s are equal} \pmod{s+1}, \\ \operatorname{sgn}(\pi) & \textit{if } \pi(x) \equiv \beta \pmod{s+1} \textit{ for some permutation } \pi \in \mathfrak{S}_{s+1} \end{cases}.$$

**Theorem 2.4.**

$$P_s(\tau; \underline{z}) = \eta(\tau)^{\binom{s}{2}} Q_s(\tau; \underline{z}).$$

*Proof.*

$$\frac{P_s(\tau; z_0+1, z_1, \dots, z_s)}{P_s(\tau; z_0, \dots, z_s)} = \prod_{i=1}^{s} \frac{\Theta(\tau; z_0 - z_i + 1)}{\Theta(\tau, z_0 - z_i)} = (-1)^s.$$

On the other hand,

$$\frac{Q_s(\tau; z_0+1, z_1, \dots, z_s)}{Q(\tau, \underline{z})} = \underline{e}\Big(\frac{s}{2}\Big) = (-1)^s.$$

$$\frac{P_s(\tau; z_0+\tau, z_1, \dots, z_s)}{P_s(\tau; z_0, \dots, z_s)} = \prod_{i=1}^{s} \frac{\Theta(\tau; z_0 - z_i + \tau)}{\Theta(\tau, z_0 - z_i)} = \prod_{i=1}^{s} \big(-q^{\frac12}\zeta_0^{-1}\zeta_i\big) = (-1)^s q^{-\frac{s}{2}} \zeta_0^{-s} \zeta_1 \cdots \zeta_s.$$

To get the analog behavior for $Q_s$: In the definition of $\epsilon$ the permutation needed changes by a cycle of length $s$, with sign $(-1)^s$.

$$(-1)^s Q_s(\tau, \underline{z}) = \sum_x \epsilon(x) q^{\frac{\|x+A\|^2}{2(s+1)}} \underbrace{\prod \zeta_i^{x_i+A_i}}_{\zeta_0^s\zeta_1^{-1}\cdots\zeta_s^{-1}\prod\zeta_i^{x_i}}$$
$$= \zeta_0^s \zeta_1^{-1} \cdots \zeta_s^{-1} q^{\frac{s}{2}} \underbrace{\sum \epsilon(x) q^{\frac{\|x\|^2}{8}} (q\zeta_0)^{x_0} \zeta_1^{x_1} \cdots \zeta_s^{x_s}}_{P_s(\tau; z_0+\tau, z_1, \dots, z_s)}.$$

Also: $Q_s(\tau, \underline{z})$ is antisymmetric in $z_i$'s, so it vanishes on all diagonals $z_i = z_j$. Thus it is divisible by $P$ as a function of $z$, $\tau$ fixed. We have shown that

$$\frac{Q_s(\tau, \underline{z})}{P_s(\tau, \underline{s})}$$

is holomorphic in $\underline{z}$ and doubly periodic in each $z_i$. That is, it is independent of $\underline{z}$. □

Remember $\beta = (\frac{s}{2}, \frac{s-2}{2}, \ldots, -\frac{s}{2})$. Easy: $\beta$ and its permutations have minimal norm in $(\mathbb{Z} + \frac{s}{2})^{s+1}_{\text{sum } 0}$. For example

- $s = 2$, $\beta = (1, 0, -1)$.
- $s = 3$, $\beta = (\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$.

$$\frac{\|\beta\|^2}{2(s+1)} = \frac{s(s-1)}{2}\frac{1}{24} = \mathrm{Ord}_\infty\big(\eta^{\binom{s}{2}}\big).$$

Coefficient of $q^{\binom{s}{2}/24}$ is

$$\sum_{\pi \in \mathfrak{S}_{s+1},\, x = \pi(\beta)} \mathrm{sgn}(\pi)\zeta_0^{\beta_{\pi(0)}} \cdots \zeta_s^{\beta_{\pi(s)}} = \det(\zeta_i^{\beta_j}) = \prod_{0 \le i < j \le s} \frac{\zeta_i - \zeta_j}{\sqrt{\zeta_i \zeta_j}}.$$

*Proof 2.* Remember

$$P_s(\tau, \underline{z}) = \prod \Theta(\tau, z_j - z_i) = \prod_{i,j} \Big( \sum_n \big(\frac{-4}{n}\big) q^{\frac{n^2}{8}} (\zeta_i/\zeta_j)^{n/2} \Big) = \sum_{\substack{H = -H^{\mathrm{tr}} \\ n_{ij} \text{ odd } (i \ne j)}} \chi(N) q^{\frac{\|H\|^2}{8}} \zeta_0^{x_0} \cdots \zeta_s^{x_s}.$$

Since $2x = uN$:

$$\prod \zeta_i^{x_i} = \prod_{i<j} (\zeta_i/\zeta_j)^{n_{ij}/2} = \prod_{i,j} (\zeta_i/\zeta_j)^{n_{ij}/4} = \prod_{i,j} \zeta_i^{n_{ij}/2} = \prod \zeta_i^{\frac{1}{2}\sum_j n_{ij} = x_j}.$$

$$\chi(H) = \prod_{0 \le i < j \le s} \big(\frac{-4}{h_j}\big) = (-1)^{\sum_{i<j} n_{ij}} = (-1)^{\mathrm{tr}(NS)/2}.$$

Here, $H \in V(\mathbb{Z})$, $N \equiv S \pmod{2V(\mathbb{Z})}$, $H = S + 2\underbrace{N}_{\in V(\mathbb{Z})}$, $h_{ij} = \mathrm{sgn}(j - i) + 2n_{ij}$.

$$\|H\|^2 = Q(H) = Q_0(H) + Q_1(H) = Q_1(H) + \frac{4\|x\|^2}{s+1}.$$

So we can continue

$$P_s(\tau, \underline{z}) = \sum_{x \in (\mathbb{Z} + \frac{s}{2})^{s+1}_{\text{sum } 0}} \underbrace{\Big( \sum_{H = S + 2N,\, uH = 2x} (-1)^{\mathrm{tr}(SN)/2} q^{Q_1(H)/8} \Big)}_{\text{function } \Delta_s \text{ of } \tau \text{ alone}} q^{\frac{\|x\|^2}{2(s+1)}} \zeta_0^{x_0} \cdots \zeta_s^{x_s}$$

Claim: $\Delta_s$ does change for $x' \equiv x \pmod{s+1}$. Simply replace $N$ by $N + (n_i - n_j)_{i,j}$ with same $Q_1(H)$. $\Delta_s$ is antisymmetric.

We get

$$P_s(\tau, \underline{z}) = \Delta_s(\tau) Q_s(\tau, \underline{z})$$

$$\Delta_s(\tau) = \sum_{H = S + 2N,\, uH = 2\beta} (-1)^{\mathrm{tr}(SN)/2} q^{Q_1(H)/8} \stackrel{?}{=} \eta(\tau)^{\binom{s}{2}}.$$

**Example 2.5.** $s = 3$

$$H = \begin{pmatrix} 0 & 1 + 2a - 2b & 1 - 2a + c & 1 + 2b - 2c \\ -1 & 0 & 1 + 2a & 1 - 2b \\ -1 & -1 - 2a & 0 & 1 + 2c \\ -1 & -1 + 2b & -1 - 2c & 0 \end{pmatrix}$$

$$\|M_1\|^2 = 12(a^2 + b^2 + c^2) - 8(ab + ac + bc) + 4(a - b + c) + 1$$

$$3\|M_1\|^2 = (1 + \cdots)^2 + (1 + \cdots)^2 + (1 + \cdots)^2 \quad ?$$

□

## 3. Lecture 4 by Nils Skoruppa

**How many Jacobi forms are theta blocks?** We allow quotients of theta during the whole lecture.

$$\prod_c \prod_{d\,|\,c} \theta_d^{\mu(\frac{c}{d})}$$

are the only admissible theta blocks.

The number of theta block of given index which are holomorphic in $\mathbb{H}\times\mathbb{C}$ (counted up to multiplication by powers of $\eta$) is finite. The lecture will deal with infinite families of theta block, which are the only hope — at least currently — to prove something about this and related questions.

We have two families

$$\big\{\theta(\tau, dz)\big\}_{d\geq 1}, \quad \big\{\theta^*(\tau, dz)\big\}_{d\geq 1},$$

where $\theta^*(\tau, z) = \frac{\theta(\tau,2z)}{\theta(\tau,z)}\eta(\tau) \in J_{\frac{1}{2},\frac{3}{2}}(\epsilon)$.

**Theorem 3.1.** *Every Jacobi form of weight $\frac{1}{2}$, any index, any character is a linear combination of these theta blocks.*

Jacobi forms of matrix index:

$$\underline{L} = (L, \beta),$$

where $\beta$ is a bilinear form $\beta : L \times L \to \mathbb{Z}$ which is non-degenerate. We will write $\mathbb{Q}\otimes L, \mathbb{C}\otimes L$ and then consider $\beta$ as a form on these modules.

Given a lattice $U \subseteq L$, $U^\# = \{y \in \mathbb{Q}\otimes L \,:\, \beta(y, U) \subseteq \mathbb{Z}\}$. We have $L^\# \supseteq L$ of finite index.

We write $\beta(x) = \frac{1}{2}\beta(x, x)$. This induces an element in $\mathrm{Hom}(L, \mathbb{Q}/\mathbb{Z})$ of order 1 or 2. We say $\underline{L}$ is even, if $\beta(x) \in \mathbb{Z}$ f. a. $x \in L$, and $\underline{L}$ is odd otherwise. Set $L_{\mathrm{ev}} := \ker(x \mapsto \beta(x) + \mathbb{Z}) \subseteq L \subseteq L^\# \subseteq_2 L_{\mathrm{ev}}^\#$. We call the set

$$L^\bullet = \big\{r \in L_{\mathrm{ev}}^\# \,:\, \beta(r, x) \equiv \beta(x) \pmod{\mathbb{Z}} \text{ for all } x \in L\big\} \subseteq L_{\mathrm{ev}}^\#$$

the set of special vectors (This is also called "shadow of $\underline{L}$" (Quebbemann, Rains, Sloane, Elkies, etc.)). If $\underline{L}$ is odd, then $L_{\mathrm{ev}}^\# \,/\, L^\# = \{L^\#, L^\bullet\}$.

Recall that we write $\epsilon$ for the multiplier system (not the character!) of $\eta$.

**Definition 3.2.** $J_{k,\underline{L}}(\epsilon^k)$, $k \in \frac{1}{2}\mathbb{Z}$, $h \pmod{24}$, $k \equiv \frac{h}{2} \pmod{\mathbb{Z}}$. $\phi \in J_{k,\underline{L}}$ *if and only if* $\phi : \mathbb{H} \oplus (\mathbb{C}\otimes L) \to \mathbb{C}$ *and*

*(i)* $\phi\big(A\tau, \frac{z}{c\tau+d}\big) = e\big(\frac{c\beta(z)}{c\tau+d}\big)(c\tau+d)^{k-\frac{h}{2}}\epsilon^h(A)\phi(\tau, z)$ *for* $A \in \mathrm{SL}_2(\mathbb{Z})$.
*(ii)* $\phi(\tau, z + x\tau + y) = e(\beta(x+y))e(-\tau\beta(x) - \beta(x, z))\phi(\tau, z)$ *for* $x, y \in L$.
*(iii)* $\phi = \sum_{n\in\frac{h}{2}+\mathbb{Z}} \sum_{r\in L^\bullet,\, n\geq\beta(r)} c(n, r)q^n e(\beta(z, r))$.

**Remarks 3.3.** *(i)* $\underbrace{(x, y)}_{\in L\times L} \mapsto e(\beta(x+y))$ *is a character.*
*(ii)* $\underline{L}$ *not integral, then there are no Jacobi forms.*
*(iii)* $\underline{L}$ *not semi-positive definite, then there are no Jacobi forms.*

*(iv)* $F \in (\frac{1}{2}\mathbb{Z})^{n\times n}$, $F = F^{\mathrm{tr}}$, $F > 0$.

$$J_{k,F}(\epsilon^h) = J_{k,(\mathbb{Z}^n,(x,y)\mapsto x^{\mathrm{tr}}2Fy)}(\epsilon^h)$$

*via* $\mathbb{C}\otimes\mathbb{Z}^n \to^{\cong} \mathbb{C}^n$, $z\otimes x \mapsto zx$.

$$J_{k,m}(\ ) = J_{k,(\mathbb{Z},2mxy)}(\ ).$$

3.1. **Embedding construction.** $\alpha : \underline{L} \to \underline{M}$ isometric gives $\alpha^* : J_{k,\underline{M}}(\epsilon^h) \to J_{k,\underline{L}}(\epsilon^h)$, $\phi \mapsto \alpha^*\phi(\tau,z) = \phi(\tau,\alpha(z))$.

Special case:

$$\alpha : L \to \underline{\mathbb{Z}}^N$$

$$\alpha^* \underbrace{\prod_{j=1}^N \theta(\tau,z_j)}_{\in J_{\frac{N}{2},\underline{\mathbb{Z}}^N}(\epsilon^{3N})} = \underbrace{\prod \theta(\tau,\alpha_j(z))}_{\in J_{\frac{N}{2},\underline{L}}(\epsilon^{3N})}.$$

Given $x_0 \in L$, $m = \beta(x_0)$

$$s_{x_0} : \underline{\mathbb{Z}}(2m) \to \underline{L},\ t \mapsto tx_0$$

$$(s_{x_0}^*\phi)(\tau,w) = \underbrace{\phi(\tau,x_0w)}_{\in J_{k,m}(\epsilon^h)} \quad (w\in\mathbb{C})$$

Infinite family of theta blocks: Let $\phi$ as above and consider $s_{x_0}^*\phi_{x\in L, x\neq 0} = \{\phi(\tau,xw)\}_{x\in L,x\neq 0}$. By what power of $\eta$ can we divide $\phi$? Typically: The power is at most $\eta^{N-n}$, $n = \mathrm{rk}(\underline{L})$, $\underline{L} \to \underline{\mathbb{Z}}^N$. $\eta^{n-N}\alpha^*\phi \in J_{\frac{n}{2},\underline{L}}(\epsilon^h)$ has singular weight, so that we cannot divide by a higher power of $\eta$.

**Definition 3.4** (Eutactic star on $\underline{L}$)**.**

$$S = \{s_j\}_{j=1}^N, \quad 0 \neq s_j \in L^\# \ \textit{s.t.} \quad x = \sum_{j=1}^N \beta(s_j,x)s_j \quad (x \in \mathbb{Q}\otimes L).$$

*We have* $\beta(x,y) = \sum_j \beta(s_j,x)\beta(s_j,y)$ *and* $\beta(x,x) = \sum_j \beta(s_j,x)^2$.

**Remark 3.5.** *We have an embedding*

$$L \ni X \mapsto \big(\beta(s_1,x),\dots,\beta(s_N,x)\big)$$

$$\underline{L} \to \underline{\mathbb{Z}}^N.$$

*Vice verca, if* $\alpha : \underline{L} \to \underline{\mathbb{Z}}^N$ *with all components* $\neq 0$, *then* $\alpha_j(x) = \beta(s_j,x)$ *for a suitable* $s_j \in L^\#$ *and* $\{s_j\}_{j=1}^N$ *is a ES.*

We let $G_S$ be the subgroup of all $g \in \mathrm{O}(\underline{L})$ s.t. there exists a permutation of indices and sign $\epsilon_j = \pm 1$ s.t. $gs_j = \epsilon_j s_{\sigma(j)}$. We set $\delta_S(g) = \prod \epsilon_j$, $\delta_S : G_S \to \{\pm 1\}$.

**Definition 3.6.** $\{s_j\}_{j=1}^N$ *is extremal if* $L^\bullet / L_{\mathrm{ev}}$ *has exactly one* $G_S$*-orbit* $U$ *whose elements have stabilizer in* $\ker\delta_S$.

**Theorem 3.7.** $\underline{L} = (L,\beta)$ *of rank* $n$, $S$ *and extremal ES on* $\underline{L}$ *of length* $N$. *Then*

$$\eta^{n-N}\prod_{j=1}^N \theta(\tau,\beta(s_j,z)) = \gamma \sum_{x\in L_{\mathrm{ev}}} e\big(\beta(w+x)\big) \sum_{g\in G_S} \delta_S(g) e\big(\beta(w+x,gz)\big)$$

*for some constant* $\gamma$. *Here,* $w$ *is a Weyl vector for* $\{s_j\}$. *By definition, we have* $w + L_{\mathrm{ev}} \in U$, $\beta(w)$ *is minimal in* $w+U$.

3.2. **Eutactic stars arising from irreducible root lattices.** Classification: $A_n, B_n, C_n, D_n, E_6, E_7,$ $R$ irreducible root lattice, $R^+$ positive roots.

Let $f_j$ be a $\mathbb{Z}$ basis for $\sum \mathbb{Z} r$.

$$r = \sum_{j=1}^{n} \rho_{r,j} f_j$$

$$\theta_R(\tau, z) = \prod_{r \in R^+} \theta(\tau, \sum_{j=1}^{n} \rho_{r,j} z_j) \in J_{\frac{n}{2}, \underline{R}}(\epsilon^{n+2N}).$$

In the case $A_2$, we get

$$\eta^{-1} \theta(\tau, z) \theta(\tau, z_1 + z_2) \theta(\tau, z_2).$$

For $\underline{A_n} = \{x \in \mathbb{Z}^{n+1} \,:\, x_0 + \cdots + x_n = 0,\ A_n = \{x \in \underline{A_n} \,:\, x^2 = 2\}$, we set $R^+ = \overline{\{(0, \cdots, 1, \cdots, -1, \cdots 0)\}}$, $f_j = (0, \cdots, 1, -1, \cdots, 0)$.

$$\theta_{A_n} = \eta^{-\binom{n}{2}} \prod_{j=1}^{n} \theta(\tau, z_j) \prod_{1 \geq i < j \geq n} \theta(\tau, z_i + z_j).$$

Set

$$h = \sum_{r \in R^+} (r, r)/n,$$

where the scalar product comes from the ambient Euclidean space $E$.

$$L = \text{weight lattice of } R = \{x \in E \,:\, (x, r) \in \mathbb{Z} \text{ for all } r \in R\}.$$

We have

$$h(z, z) = \sum_{r \in R^+} (r, z)^2,$$

and this give the above formula

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