$\boxed{\text{Modular Symbols}}$ Lecture 1 (NS)

proj. lines: $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) = \{(x,y) \in (\mathbb{Z}/N)^2;\ \gcd(x,y,N) = 1\} / (\mathbb{Z}/N)^\times$.

e.g. $|\mathbb{P}^1(\mathbb{Z}/p)| = p+1$

notation: $[x:y] :=$ eq. class of (x,y).

note: $\# \mathbb{P}^1(\mathbb{Z}/N) = N \prod_{p|N} (1 + \frac{1}{p})$.

e.g. for p prime, $\mathbb{P}^1 = \{[a:1], [1:0]\}$.

$$\begin{aligned} \mathbb{Z}/N &\to \mathbb{P}^1(\mathbb{Z}/N) \\ a &\mapsto [a:1]. \end{aligned}$$

$SL_2(\mathbb{Z}) \underset{\text{(right)}}{\circlearrowright} \mathbb{P}^1(\mathbb{Z}/N)$: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$[x:y]A := [ax+cy : bx+dy]$$

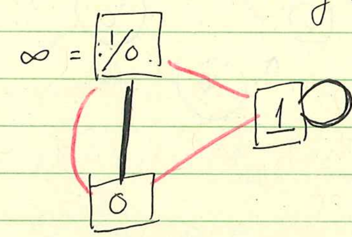
"Number theorists' sudoku": An "N-board" is a colored graph w/ vertices $\leftrightarrow \mathbb{P}^1(\mathbb{Z}/N)$, and $\exists$ a black edge btwn p & q if $p = qS$, and a red edge if $p = qR$ or $p = qR^2$.

$R^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

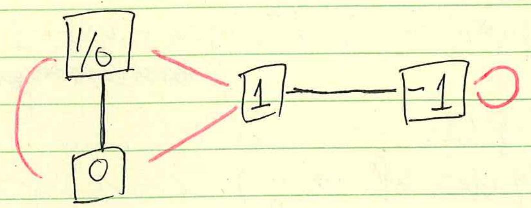
[$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, $R = ST = \begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}$.
($\to R^3 = -1$; up to conj, 1, S & R are the only finite-order elts in $SL_2(\mathbb{Z})$.)]

$\rightsquigarrow$ note $x/y := [x:y] \to$ have $x/y\, S = [y : -x] = -y/x$.
$x/y\, R = [y : y-x] = \frac{y}{y-x}$.

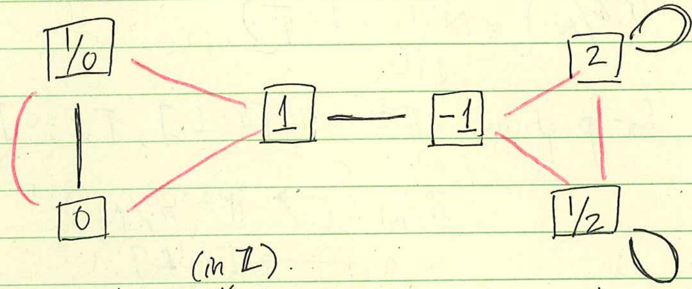
For $N=2$:



For $N=3$:



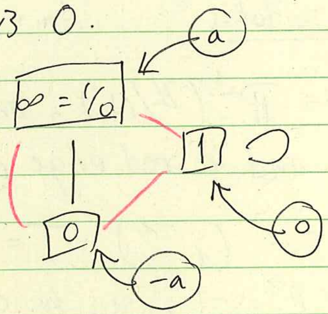
$N=5$:



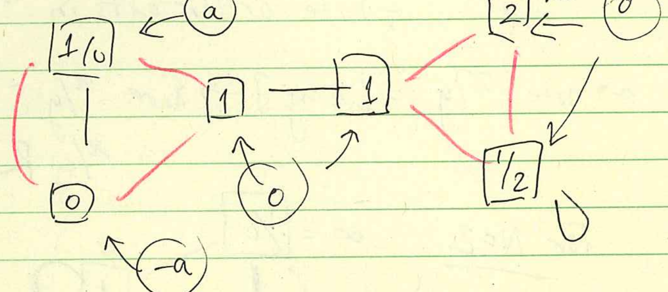
Problem: Assign labels (in $\mathbb{Z}$) to the vertices s.t.

1) the Σ of labels connected by a black edge is 0, and
2) the Σ of labels connected by a red Δ is 0.

e.g. $N=2$:



$N=5$:



Solution: $\mathcal{L}(N) := \{\lambda : \mathbb{P}^1(\mathbb{Z}/N) \to \mathbb{C} :$

$$0 = \lambda(p) + \lambda(pS) = \lambda(p) + \lambda(pR) + \lambda(pR^2)\}$$

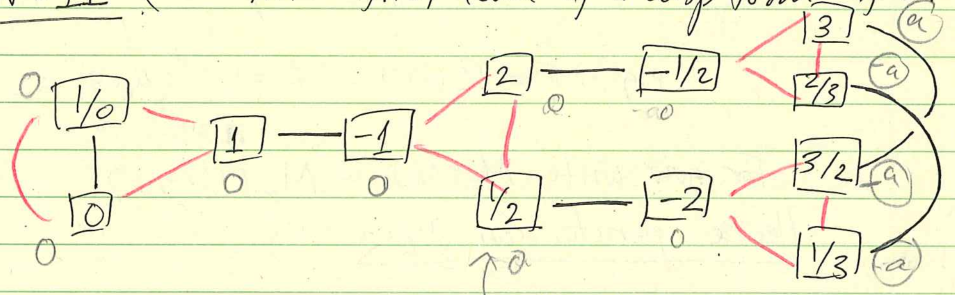
this is (not strictly a $\oplus$ over $\mathbb{Z}$, so we take labels in $\mathbb{C}$ for convenience).

Divide into $\pm$-spaces: $\mathcal{L}(N) = \mathcal{L}(N)^+ \oplus \mathcal{L}(N)$

$$\mathcal{L}(N)^\varepsilon = \{\lambda \in \mathcal{L}(N) : \lambda(-p) = \varepsilon\lambda(p)\}.$$

Write $\mathcal{L}(N)_{\mathbb{Z}}$ for $\mathbb{Z}$-valued labels: then $\mathcal{L}(N) = \mathcal{L}(N)_{\mathbb{Z}} \otimes \mathbb{C}$.

$N=11$ (← Note: first level of a cusp form of wt 2)



taking this as $-a$ forces $\boxed{2} \to a \to \boxed{-1/2} \to a$.
$\cdots \to \boxed{1/2} \to 0$.

$\Rightarrow \dim \mathcal{L}(11)^- = 1$. (and $\mathcal{L}(11) \underset{(1)}{=} \mathcal{L}(11)^-$)

Thm. For every $\lambda \in \mathcal{L}(N)$ and every $[x:y] \in \mathbb{P}^1(\mathbb{Z}/N)$,
(**) the series
$$f_{\lambda;[x:y]} := \sum_{\substack{a,b,c,d \\ a>b\geq 0 \\ d>c\geq 0 \\ \gcd(ax+cy,\, bx+dy,\, N)=1}} \lambda\left([x:y]\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) q^{ad-bc}$$

(for a given value of $ad-bc$, $\exists$ only fin. many $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ s.t. these conditions hold.)

defines an elt of $M_2(N)$ (up to $+$ of a constant), and the set $\{f_{\lambda,[x:y]}\}$ spans $M_2(N)$.

Q's: how do we characterize the subsp of cusp forms? Eis series?

$\boxed{\text{Modular Symbols}}$: lecture 2 (NS)

Theoretical background: $\underset{fin.}{\Gamma \subseteq} SL_2(\mathbb{Z})$ (mostly $\Gamma = \Gamma_0(N)$)

$M_k(\Gamma) :=$ sp of mod forms on Γ, of wt k:

$= \{ f : \mathcal{H} \to \mathbb{C} : f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, + regularity conditions

at the cusps $\}$. ← or write "$a_f(n)$."

e.g. $\Gamma = \Gamma_0(N)$: $f = \sum_{n \geq 0} a_n q^n \leftarrow q = e^{2\pi i z}$.

For now write $M_k(N) := M_k(\Gamma_0(N))$.

Hecke operators on $M_k(N)$:

$T(\ell)$ $(\ell = 1, 2, 3, \ldots)$ (def. omitted)

satisfying: (M.I. = 'magical ID"): $a_{T(\ell) f}(n) = a_{T(n) f}(\ell)$

Mostly we'll take $k = 2$.

Thm. (Basic principle for computing modular forms):

Let X be a Hecke module which is $\cong$ to a sub-Hecke module of $M_k(N)$. Then: for every $\phi \in X^*$, $x \in X$, the series

$$S_\phi(X) := \sum_{\ell \geq 1} \phi(T(\ell)X) q^\ell \quad \text{defines an elt}$$

in M_0 (:= image of M under $f \mapsto f - a_f(0)$).

There exists a ϕ s.t. $S_\phi : X \to M_0$ is an $\cong$ of Hecke modules.

Pf. Let $p : X \to M$ be a Hecke mod. isom.

M^* is generated by $\phi_n : f \mapsto a_f(n)$; likewise X^* is gen. by $\phi_n \circ p$. Can assume $\phi = \phi_n \circ p$;

set $f := p(x)$. $\to S_\phi(x) = \sum_{\ell \geq 1} (\phi_n \circ p)(T(\ell) X) q^\ell$

$= \sum_{\ell \geq 1} \phi_n(T(\ell) f) q^\ell$

$= \sum a_{T(\ell) f}(n) q^\ell \underset{M.I.}{=} \sum a_{T(n) f}(\ell) q^\ell$

$:= T(n) f$.

e.g. $n = 1$: $S_{\phi_1 \circ p}(x) = p(x)$, so $S_{\phi_1 \circ p}$ is an $\cong$.

('deg-0 divisors')

$H(\Gamma) := \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{C})$.

and $H^{Eis}(\Gamma) := \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})], \mathbb{C})$.

$\exists$ a natural (restriction) map $H^{Eis}(\Gamma) \to H(\Gamma)$.

Terms: $\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] = \{ c : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Z} : c(x) = 0$ for $\underset{a.b.f.m.}{}$ $x \in \mathbb{P}^1(\mathbb{Q}) \}$.

Set $e_p := \begin{pmatrix} p \mapsto 1 \\ q \neq p \mapsto 0 \end{pmatrix}$.

$\Rightarrow c = \sum_{p \in \mathbb{P}^1(\mathbb{Q})} c(p) e_p$. $\left(\frac{ap+b}{cp+d}\right)$

$GL_2(\mathbb{Q})$ acts on $\mathbb{P}^1(\mathbb{Q})$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{[x:y]}{p} = [ax+by : cx+dy]$.

... and so it acts on $\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]$:

$$(A, c) \longmapsto Ac = \sum c(p)\, e_{Ap}.$$

Define the degree map: $\deg c = \deg\left(\sum_p c(p)\, e_p\right) := \sum c(p)$

+ def $\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 := \{c \text{ s.t. } \sum c(p) = 0\}$.

Thus: $H(\Gamma) = \{\lambda : \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 \longrightarrow \mathbb{C}$ is a hom. of abgps: $\lambda(Ac) = \lambda(c)\ \forall A \in \Gamma$, + all $c\}$.

We have operators $T(\ell)$ on $H(\Gamma)$ $(\ell = 1, 2, \dots)$:

$$(T(\ell)\lambda)(c) = \sum_{M \in \Gamma_0(N) \backslash G(N)_\ell} \lambda(Mc).$$

(Where $G(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_{2\times 2} : N \mid c, \gcd(a, N) = 1 \right\}$ and

$G(N)_\ell = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(N) : ad - bc = \ell \right\}$)

Let's take $M = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \begin{array}{l} ad = \ell \\ a, d \geq 1, \\ \gcd(a, N) = 1 \end{array} \right.$

$E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E^2 = 1$ acts on $H(N) := H(\Gamma_0(N))$.

via $\lambda \mapsto \lambda'$, $\lambda'(c) = \lambda(Ec)$.

$\Rightarrow E$ defs an involution; $\longrightarrow H(N) = H(N)^+ \oplus H(N)^-$.

Lemma. The $H(N)^\pm$ are invt. under all $T(\ell)$.

Thm. For $\varepsilon = \pm 1$, the space $H(N)^\varepsilon$ is isomorphic (as a Hecke-mod) to $S_2(N) \oplus M_2^{\mathrm{Eis},\varepsilon}(N)$.

(Here $M_2^{\mathrm{Eis},+}(N) \oplus M_2^{\mathrm{Eis},-}(N) = M^{\mathrm{Eis}}(N)$).

* To compute using this thm, we actually don't need an explicit isomorphism.

→ What does this have to do with labelings of the graphs from §1?

"Schreier coset graph"

$$\mathcal{L}(\Gamma) = \{{}_{\Gamma}\backslash SL_2(\mathbb{Z}) \longrightarrow \mathbb{C} : \lambda(x) + \lambda(xS) = \lambda(x) + \lambda(xR) + \lambda(xR^2) = 0\}.$$

$\mathcal{L}(\Gamma_0(N)) \cong \mathcal{L}(N)$: namely

$$\Gamma_0(N)\backslash SL_2(\mathbb{Z}) \xrightarrow{\cong} \mathbb{P}^1(\mathbb{Z}/N)$$

$$\Gamma_0(N) A \longmapsto [0:1]\underset{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{A} = [\tilde{c} : \tilde{d}].$$

Thm. For $\lambda \in H(\Gamma)$, denote by $\tilde{\lambda}$ the map $\tilde{\lambda}$:

$${}_{\Gamma}\backslash SL_2(\mathbb{Z}) \longrightarrow \mathbb{C} \quad \text{s.t. } \tilde{\lambda}(\Gamma A) = \lambda(A(e_\infty - e_0)).$$

(*) $S(e_\infty - e_0) = -(e_\infty - e_0)$; $e_\infty + e_0 + R(e_\infty - e_0) + R^2(e_\infty - e_0) = 0$.

This defines an elt in $\mathcal{L}(\Gamma)$: $(\lambda \mapsto \tilde{\lambda})$ defines an isomorphism

$$\tilde{(\ )} : H(\Gamma) \longrightarrow \mathcal{L}(\Gamma).$$ need to compute it here. (*)

↳ get the conditions in the sum of thm (**)

↑ here, have nice Hecke action

$\boxed{\text{Modular Symbols}}$: Lecture 3 (JC)

Modular Symbols, M-symbols, & homology of mod curves

Idea: Compute homology of modular curves (e.g. $X_0(N)$) in order to compute $S_2(N)$. (or more generally, $S_k(N)$, $k \geq 2$.) with Hecke action $\longrightarrow$ get Hecke e'forms, including rat'l newforms (i.e. those w/ $\mathbb{Q}$-Hecke e'vals)

From $f \rightsquigarrow E_f = \mathbb{C}/\Lambda_f$, where Λ_f = period lattice of f.

Let E_f := ell curve $/\mathbb{Q} \leftrightarrow f$, of conductor N.

Wiles: every $E/\mathbb{Q}$ is isogenous to such an E_f, and $L(E,s) = L(E_f, s)$.

Write: $\Gamma(1) := SL_2(\mathbb{Z})$; $\Gamma \leq \Gamma(1)$ a subgp of fin index, $\Gamma \curvearrowright \mathfrak{h}$; $Y_\Gamma = {}_\Gamma\backslash\mathfrak{h}$; $X_\Gamma = {}_\Gamma\backslash\mathfrak{h}^*$; $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$

(X_Γ is an alg curve $/\overline{\mathbb{Q}}$; for $\Gamma = \Gamma_0(N)$, it's def'd $/\mathbb{Q}$.)

$X_0(N) := X_{\Gamma_0(N)}$, as a curve $/\mathbb{Q}$.

$S_2(\Gamma)$ a $\mathbb{C}$-space of dim $= g_\Gamma$ = genus(X_Γ);

the map $f \longleftrightarrow 2\pi i f(\tau)d\tau$ (holo diff'l on X_Γ).

gives an isom: $S_2(\Gamma) \longrightarrow \Omega^1_{holo}(X_\Gamma)$.

Modular symbols form a space dual to $S_2(\Gamma)$; Hecke ops act compatibly on both.

$\Rightarrow$ use modular symbols to:

1) concretely describe $S_2(\Gamma)$ as a Hecke module $\Rightarrow$ get e'forms + q-expansions for its elts.

* need to be careful about choice of path if α or β is a cusp; e.g. geodesic paths are ok.

2) concretely describe the homology $H_1(X_\Gamma, \mathbb{Z})$ in order to get info about periods of newforms.

Putting these together, we'll find rat'l newforms + $\mathbb{Z}$-bases for their period lattices Λ_f, and then get eqns for $E_f = \mathbb{C}/\Lambda_f$ w/ f a rat'l newform.

Homology & modular symbols: Given $\Gamma \leq \Gamma(1)$, $g := g_\Gamma$,

$\rightarrow H_1(X_\Gamma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Say $\langle \gamma_1, \dots, \gamma_{2g} \rangle_\mathbb{Z}$ is a basis. For any (char 0) ring R, $H_1(X_\Gamma, R) = R^{2g}$.

In particular, $H_1(X_\Gamma, \mathbb{R})$ is an $\mathbb{R}$-~~lattice~~ vs of dim $2g$, in which $H_1(X_\Gamma, \mathbb{Z})$ is a lattice of full rank.

Let $\alpha, \beta \in \mathfrak{h}^*$ w/ $\Gamma\alpha = \Gamma\beta$. Then:

- $\{\alpha, \beta\}$ is a path from α to β in $\mathfrak{h}^*$.
- $\{\alpha, \beta\}_\Gamma$ is the image of this path in X_Γ (a loop), and also the image of the loop in $H_1(X_\Gamma, \mathbb{Z})$.

and $\{\alpha, \beta\}_\Gamma$ determines a $\mathbb{C}$-linear map $S_2(\Gamma) \to \mathbb{C}$ via $f \longmapsto \langle \{\alpha,\beta\}, f \rangle := \int_\alpha^\beta 2\pi i f(\tau)d\tau$.

(More generally, if $\gamma \in H_1(X_\Gamma, \mathbb{Z})$, then $\langle \gamma ; f \rangle := \int_\gamma 2\pi i f(\tau) d\tau$.

The $\{\alpha, \beta\}_\Gamma$ form a lattice of rk $2g$ in $S_2(\Gamma)^*$. Now any elt of $S_2(\Gamma)^*$ is an $\mathbb{R}$-linear combo of the $f \longmapsto \langle \gamma_j ; f \rangle$, & so can be ID'd with a ! elt of $H_1(X_\Gamma, \mathbb{R})$.)

→ Now, for every $\alpha, \beta \in \mathfrak{h}^*$ (not nec. in the same Γ-orbit) we have such an elt (of $S_2(\Gamma)^*$), namely
$f \longmapsto \int_\alpha^\beta 2\pi i f(\tau)\,d\tau$, so we can define $\{\alpha,\beta\}_\Gamma$ to be the assoc. elt of $H_1(X_\Gamma, \mathbb{R})$.

• Can use the cpx structure on $S_2(\Gamma)^*$ to define one on $H_1(X_\Gamma,\mathbb{R})$, & hence we get a perfect pairing (of g-dim'l $\mathbb{C}$-spaces):

$$H_1(X_\Gamma,\mathbb{R})\times S_2(\Gamma)\longrightarrow \mathbb{C}.$$

Properties of the modular symbols $\{\alpha,\beta\}$:

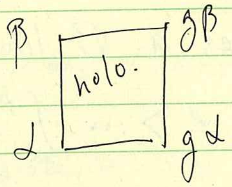
1) $\{\alpha, \alpha\}=0$.

2) $\{\alpha,\beta\}+\{\beta,\alpha\}=0$.

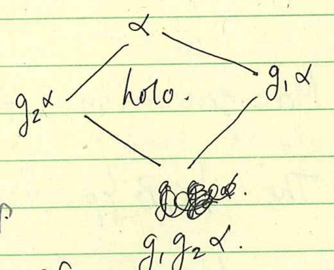
3) $\{\alpha,\beta\}+\{\beta,\gamma\}+\{\gamma,\alpha\}=0$.

4) $\{g\alpha, g\beta\}_\Gamma=\{\alpha,\beta\}_\Gamma$.

5) $\{\alpha, g\alpha\}_\Gamma=\{\beta, g\beta\}_\Gamma \;\forall\, \alpha,\beta$.



6) $\{\alpha, g_1g_2\alpha\}_\Gamma=\{\alpha, g_1\alpha\}_\Gamma + \{\alpha, g_2\alpha\}_\Gamma$



7) $\{\alpha, g\alpha\}_\Gamma \in H_1(X_\Gamma, \mathbb{Z})$ if $g\in\Gamma$.

$\Rightarrow g\mapsto \{\alpha, g\alpha\}_\Gamma$ is a gp hom. $\Gamma\longrightarrow H_1(X_\Gamma,\mathbb{Z})$.

whose kernel contains all commutators & all ell + parabolic elts.

Real structure: The matrix $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ acts on $\mathfrak{h}^*$ by $\tau\mapsto-\bar\tau$;

sends $f=\sum_{n=0}^\infty a_nq^n \longmapsto \sum_{n=0}^\infty \overline{a_n}q^n$; and maps $S_2(\Gamma)$ to itself if $\begin{pmatrix}-1&0\\0&1\end{pmatrix}\Gamma\begin{pmatrix}-1&0\\0&1\end{pmatrix}=\Gamma$. (e.g. for $\Gamma=\Gamma_0(N)$)
Say "Γ has real type" if this is the case.

$\Rightarrow S_2(\Gamma)$ has a basis of mod forms w/ real coeffs;
$S_2(\Gamma)_{\mathbb{R}}=S_2(\Gamma)\cap\mathbb{R}[[q]]$.

Next: $H_1(X_\Gamma,\mathbb{R})=H_1(X_\Gamma,\mathbb{R})^+\oplus H_1(X_\Gamma,\mathbb{R})^-$

where $H_1^\pm(\cdot)=\{\gamma\in H_1 : \gamma^*=\pm\gamma\}$.

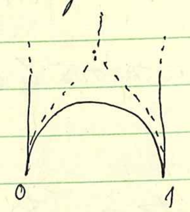
M-symbols

(Give a way of describing modular symbols for $\Gamma_0(N)$.)
To compute $H_1(X_\Gamma)$:

Triangulate X_Γ: first, triangulate $\mathfrak{h}^*$ w/ Δ's of vertices = • cusps, $\mathbb{P}^1(\mathbb{Q})$.
(= 1 $\Gamma(1)$-orbit.)
(for fields $F/\mathbb{Q}$, the # of Γ-orbits in {cusps} is $= Cl(F)$.)

and edges $\{\alpha,\beta\}=\{\frac{b}{d},\frac{a}{c}\}$ w/ $ad-bc=1$
$=\{g(0), g(\infty)\}$, $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma(1)$.

and triangles $=\Gamma(1)$-orbit of the Δ w/ vertices $0, \infty, 1$.



$\to$ edges $\{0,\infty\}$, $\{TS(0), TS(\infty)\}$, $\{(TS)^2(0), (TS)^2(\infty)\}$.

Now replace each edge $\{\alpha,\beta\}$ by $\{\alpha,\beta\}_\Gamma$ to get a triangulation of X_Γ.

Notation: $(g) := \{g(0), g(\infty)\}_\Gamma = g\cdot\{0,\infty\}_\Gamma \quad (g \in \Gamma(1))$

Relns: $(g) = (g'g)$ for $g' \in \Gamma$.

(prop 2 $\to$) $(g) + (gS) = 0$ (path + reverse path).

(prop 3 $\to$) $(g) + (gTS) + (g(TS)^2) = 0$. ($\sum$ over edges of triangle)

These symbols generate $H_1(X_\Gamma, \mathbb{Z}; \text{cusps})$.

To get $H_1(X_\Gamma, \mathbb{Z})$, take the kernel of

$$\delta : \{\alpha,\beta\}_\Gamma \longmapsto [\beta]_\Gamma - [\alpha]_\Gamma$$

$$\delta((g)) = [a/c]_\Gamma - [b/d]_\Gamma.$$

($[\alpha]_\Gamma$ = class of $\alpha \in \mathbb{P}^1(\mathbb{Q})$ in $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$)

$C(\Gamma) := \mathbb{Z}[\Gamma\backslash\Gamma(1)]$.

$\delta \nearrow g(\infty) - g(0) + g(1) - g(\infty) + g(0) - g(1) = 0$

$B(\Gamma) := \langle (g) + (gS),\ (g) + (gTS) + (g(TS)^2) \rangle_{\mathbb{Z}}$.

$B(\Gamma) \subseteq Z(\Gamma) := \ker \delta : C(\Gamma) \longrightarrow \mathbb{Z}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]$

$(g) \longmapsto [g(\infty)]_\Gamma - [g(0)]_\Gamma$.

Then $H_1(X_\Gamma, \mathbb{Z}; \text{cusps}) \cong C(\Gamma)/B(\Gamma) \leftarrow$ (rk $= 2g_\Gamma + g_0$ for some $g_0 \geq 0$)

$H_1(X_\Gamma, \mathbb{Z}) \cong Z(\Gamma)/B(\Gamma) \leftarrow$ (rk $2g_\Gamma$).

$\Rightarrow$ to achieve our goal for $\Gamma = \Gamma_0(N)$, we now just need to find coset reps for Γ in $\Gamma(1)$ ($\Rightarrow$ gens) and see how S & TS permute them ($\Rightarrow$ relns), and turns out this is a matter of computing e.g. graphs from §1 in $\mathbb{P}^1(\mathbb{Z}/N)$.

Def. An M-symbol is an elt of $\mathbb{P}^1(\mathbb{Z}/N)$.

Prop. $\exists$ a bijection $\Gamma_0(N)\backslash SL_2(\mathbb{Z}) \longrightarrow \mathbb{P}^1(\mathbb{Z}/N)$

induced by: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (c:d)$.

(Sometimes $(c:d)$ is viewed as an 'abstract elt' of $\mathbb{P}^1(\mathbb{Z}/N)$; sometimes it's viewed as an elt of $H_1(X_0(N), \mathbb{Z}, \text{cusps})$ via $(c:d) \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \longmapsto \{g(0), g(\infty)\}$.)

Relns: (in $\mathbb{Z}[\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})]$)

$$\begin{cases} (c:d) + (-d:c) = 0. \\ (c:d) + (c+d:-c) + (d:-c-d) = 0. \end{cases}$$

boundary maps: $\partial((c:d)) = [a/c]_{\Gamma_0(N)} - [b/d]_{\Gamma_0(N)}$.

$\to C(N) = \mathbb{Z}[\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})]$; $\nearrow \forall (c:d)$ in $\mathbb{P}^1(\mathbb{Z}/N)$

$B(N) = \langle (c:d) + (-d:c),\ (c:d) + (c+d:-c) + (d:-c-d) \rangle$

and $Z(N) = \ker \delta$.

$\Rightarrow$ get: $C(N)/B(N) \cong H_1(X_0(N), \mathbb{Z}, \text{cusps})$.

$$Z(N)/B(N) \cong H_1(X_0(N), \mathbb{Z}).$$

To obtain $H_1(\cdot)^{\pm}$, add relations $(c:d) = \pm(-c:d)$ to the relns defining $B(N)$ to get $B(N)^{\pm}$.

$\to$ put $Z(N)/B(N)^{\pm} = H_1(\cdot)^{\pm}$.

$C(N) = \mathbb{Z}[\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})]$: free on $\psi(N) = [\Gamma(1) : \Gamma_0(N)]$ generators.

Dual to $C(N)$ is the free $\mathbb{Z}$-module of maps $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \longrightarrow \mathbb{Z}$, and

$$\left(C(N)/B(N)\right)^* = \{\lambda : \mathbb{P}^1(\mathbb{Z}/N) \to \mathbb{Z} \text{ s.t.}$$

$$\lambda((c:d)) + \lambda((-d:c)) = 0 \text{ and}$$

$$0 = \lambda((c:d)) + \lambda((c+d:-c)) + \lambda((d:-c-d))\}$$

i.e., exactly $\mathcal{L}(N)$.

Likewise $\mathcal{L}^{\pm}(N)$ is dual to $C(N)/B(N)^{\pm}$.

So $S_2(N) \overset{\oplus \text{ some Eis part, TBD.}}{\cong} \mathcal{L}(N)^{\pm}_{\mathbb{C}}$ via

$$f \leftrightarrow \left((c:d) \mapsto \int_{b/d}^{a/c} 2\pi i f(\tau)\, d\tau\right).$$

Problems:

1] a) If $\Gamma \leq SL_2(\mathbb{Z})$ has finite index, show $|\Gamma \backslash \mathbb{P}^1(\mathbb{Q})| < \infty$.

b) $\Gamma_0(p) \backslash \mathbb{P}^1(\mathbb{Q}) = \{[0], [\infty]\}$

c) for $\Gamma = \Gamma_0(N)$: find ① & ② s.t. $\frac{p_1}{q_1} \sim \frac{p_2}{q_2} \iff$ ① + ② hold,

and ② is not needed when N is $\square$free.

Suggestion: ① $\gcd(N, q_1) = \gcd(N, q_2)$ $(:= t)$

② $p_1 \cdot \left(\frac{q_1}{t}\right) \equiv p_2 \cdot \left(\frac{q_2}{t}\right) \pmod{\gcd(t, \frac{N}{t})}$

2] Check the 7 relns on modular symbols & show that $\{\alpha, g(\alpha)\} = 0$ when g is elliptic or parabolic.

3] Check that $B(\Gamma) \leq Z(\Gamma)$ and $B(N) \leq Z(N)$.

4] Compute $\mathcal{L}(N)$ &/or $\mathcal{L}^{\pm}(N)$ for $N = 13 \xrightarrow{\dim = 1}, 17 \xrightarrow{\dim = 3}, 17 \xrightarrow{3}, 19 \xrightarrow{3}, 23 \xrightarrow{5} \ldots$

try N not prime, too.

5] Determine $\mathcal{U}$ s.t.

$$0 \to \mathcal{U} \to \mathbb{Z}[SL_2(\mathbb{Z})] \to \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] \to 0.$$
$$A \longmapsto e_{A\infty}$$

6] Determine $\mathcal{U}$ s.t.

$$0 \to \mathcal{U} \to \mathbb{Z}[SL_2(\mathbb{Z})] \to \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 \to 0$$

is exact.

$$A \longmapsto A(e_\infty - e_0) = e_{A\infty} - e_{A0}.$$

Note: $\mathcal{U} \supseteq \mathbb{Z}[G](1+S) + \mathbb{Z}[G](1+R+R^2)$.

Do these generate the whole thing?

$\boxed{\text{Modular Symbols}}$ Lecture 4.

Recall: **Thm.** $H(N)^{\pm} \overset{\text{as Hecke modules}}{\cong} S_2(N) \oplus M_2^{Eis,\pm}(N)$ $\quad (M^{Eis} = M^{Eis,+} \oplus M^{Eis,-})$

Pf. (Sketch) $H^{Eis}(N)/\mathbb{C}\cdot\deg \underset{(HM)}{\cong} M_2^{Eis}(N)$.

$$(T(\ell)\deg)(e_p) = \sum_{M \in G(N)_\ell \backslash \Gamma_0(N)} \deg(Me_p) = \deg(e_p) \sum_{d \mid \ell} d = \deg(e_p)\,\sigma_1(\ell).$$

$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ w/ $ad = \ell$. $b \pmod d$.

$$H^{Eis}(N)/\mathbb{C}\cdot\deg \underset{\text{res}}{\hookrightarrow} H(N)$$

$$S(N) \longrightarrow H(N)^{\pm}$$

$$f \longmapsto \left(C \mapsto \int_{C^{\pm}}^{\text{along geodesic}} f(z)\,dz\right) \quad \text{w/ } C^{\pm} = \tfrac{1}{2}(C \pm E\cdot C)$$

$E = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$.

$$\int_C f(z)\,dz = \sum c(p) \int_{p_0}^{p} f(z)\,dz. \quad \text{and } C = \sum c(p)(e_p - e_{p_0}).$$

$\Rightarrow M^{Eis}(N) \oplus S_2(N) \oplus S_2(N) \underset{\text{H.M.}}{\hookrightarrow} H(N)$. (use e'vals of "Eis $\neq$ cusp")

$\to$ WTS this is surjective.

Thm. The map $\lambda \longmapsto \lambda'$ $\quad (\lambda'(\Gamma A) := \lambda(A(e_\infty - e_0)))$ defines an isomorphism $H(\Gamma) \xrightarrow{\cong} \mathcal{L}(\Gamma)$

(we don't yet ~~ate~~ understand the Hecke actions on the RHS, so can't yet say if the $\cong$ is Hecke-invt.)

Lemma. $\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0$ is a $\mathbb{Z}[SL_2(\mathbb{Z})]$-module of rk 1. More precisely,

$$\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 = SL_2(\mathbb{Z})\cdot(e_\infty - e_0).$$

$\boxed{e_p - e_\infty \in SL_2(\mathbb{Z})(e_\infty - e_0)}$:

Pf. $C \in \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0$, $\quad C = \sum c(p)e_p = \sum c(p)(e_p - e_\infty)$,

$\to$ suff. to show $e_p - e_\infty \in SL_2(\mathbb{Z})(e_\infty - e_0)$.

Use ct'd fraction exp'n of p:

$$p = [a_0, a_1, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}} \sim \tilde{a}_0 - \cfrac{1}{\tilde{a}_1 - 1}$$

$p_0 = \infty, p_1, p_2, \ldots, p_n = p$.

$$e_p - e_\infty = (e_{p_n} - e_{p_{n-1}}) + (e_{p_{n-1}} - e_{p_{n-2}}) + \cdots (e_{p_1} - e_{p_0})$$

$$e_{p_i} - e_{p_{i-1}} = \underbrace{\begin{pmatrix} x_i & x_{i-1} \\ y_i & y_{i-1} \end{pmatrix}}_{\boxed{\det: \pm 1}\ ??}(e_\infty - e_0)$$

$$e_p - e_0 = (A_0 + A_1 + \cdots + A_n)(e_\infty - e_0).$$

$\det = (-1)^{i-1}$

$$A_i \longleftarrow A_i \begin{pmatrix} (-1)^{i-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

$a_0 + \cfrac{1}{a_1 - \cfrac{1}{a_2 + \cfrac{1}{a}}}$

Pf of (*) Lemma implies that "$\lambda \mapsto \lambda'$" is inj.

Lemma: $\dim \mathcal{L}(N) = \dim M_2^{Eis}(N) + 2\cdot\dim S_2(N)$. (afternoon).

$$M_2^{Eis} \oplus S_2 \oplus S_2 \hookrightarrow H(N) \hookrightarrow \mathcal{L}(N)$$

$\dim \longleftarrow \leftarrow (=) \rightarrow \longrightarrow \dim$

Cor. $\Rightarrow \mathcal{L}(N)^{\pm} \cong M^{Eis,\pm}(N) \oplus S_2(N)$.

$S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\infty}} \leftrightarrow ST^{a_0} \ldots ST^{a_1}$

$A_0 = ST^{a_0}$

$A_1 = ST$

$\widetilde{(\cdot)} : H(N) \xrightarrow{\cong} \mathcal{L}(N)$.

Def: ($T(\ell)$ on $\mathcal{L}(N)$): $\lambda = \widetilde{\mathcal{H}}$: $T(\ell)\lambda := \widetilde{T(\ell)\mathcal{H}}$

Calculate: $T(\ell)\lambda(p) = \sum_{M \in \Gamma_0(N)\backslash G(N)_\ell} \mathcal{H}(M\rho(p)(e_\infty - e_0))$.
$p \in \mathbb{P}^1(\mathbb{Z}/N)$.

For $q \in \mathbb{P}^1(\mathbb{Q})$, let $\rho(q) \in SL_2(\mathbb{Z})$ be def'd by:
$q = [0:1]\rho(q)$.

so if, e.g. $q = \frac{x}{y}$, then $\rho(q) =: \begin{pmatrix}a&b\\c&d\end{pmatrix}$; $[0:1]\begin{pmatrix}a&b\\c&d\end{pmatrix} = [c:d] = \frac{c}{d}$. $\Rightarrow \rho(q)$ has 2^{nd} row $(c\ d)$ & $(a\ b)$ are chosen s.t. $\begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL_2\mathbb{Z}$.

(Recall: $\Gamma_0(N)\backslash SL_2(\mathbb{Z}) \xrightarrow[(**)]{\cong} \mathbb{P}^1(\mathbb{Z}/N)$.

$A = \begin{pmatrix}a&b\\c&d\end{pmatrix} \longmapsto [0:1]A = [\tilde{c} : \tilde{d}]$.

Lemma: Let $\mathcal{H}(N) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathbb{Z}^{2\times 2} : \gcd(c,d,N) = 1\right\}$

and let r = inverse of $(**)$. Then, for each $\ell \geq 1$, the map induces a bijection $\mathcal{H}(N)_\ell / SL_2(\mathbb{Z})$ $\xrightarrow{\cong} \Gamma_0(N)\backslash G(N)_\ell$: $M \longmapsto \ell r([0:1]M)M^{-1}$.

pf: well-def'd: ... (check).

$\boxed{\text{reps of LHS: } \begin{pmatrix}d&b\\0&a\end{pmatrix}}$ w/ $ad = \ell$, b mod d, $a > 0$, $(a, N) = 1$.

$\downarrow$

$\boxed{\Gamma_0(N)\begin{pmatrix}a&b\\0&d\end{pmatrix}}$ = reps of RHS.

easy: just see where a rep of the LHS gets sent.

Now look at $T(\ell)\lambda(p)$ again:

$$T(\ell)\lambda(p) = \sum_{M \in \Gamma_0(N)\backslash G(N)_\ell} \mathcal{H}(M\rho(p)(e_\infty - e_0))$$

(Lemma on cosets) $= \sum_{M \in \mathcal{H}(N)_\ell/SL} \mathcal{H}(\ell\rho([0:1]M)M^{-1}\rho(p)(e_\infty - e_0))$

$\overset{\boxed{?}}{\underset{M \leftarrow \rho(p)^{-1}M}{=}} \sum_{M \in \rho(p)^{-1}\mathcal{H}(N)_\ell/SL} \mathcal{H}(\rho([0:1]\rho(p)^{-1}M) \circ M^{-1}(e_\infty - e_0))$

(*)(**) Here C denotes any elt in $\mathbb{Z}[\mathbb{Z}^{2\times 2}_{\det=\ell}]$ such that $\sum_{A \in SL_2(\mathbb{Z})} C(MA)A(e_\infty - e_0) = M^{-1}(e_\infty - e_0)$ for all $M \in \mathbb{Z}^{2\times 2}_\ell$

Lemma: For any $\lambda \in \mathcal{L}(N)$ and $p \in \mathbb{P}^1(\mathbb{Z}/N)$, have a formula for the Hecke action:

$(T(\ell)\lambda)p =$

$$\boxed{?} = \sum_{M \in \rho(p)^{-1}\mathcal{H}(N)_\ell/SL} \sum_A C(MA)\mathcal{H}(\rho([0:1]\rho(p)M) \cdot A \cdot (e_\infty - e_0))$$

note: $p \cdot \rho([0:1]\rho(p))$

move A inside $\mathcal{H}(-)$:

$\Rightarrow \mathcal{H}(\rho([0:1]\rho(p)M)A \cdot (e_\infty - e_0))$
$= \mathcal{H}(\rho(pMA)(e_\infty - e_0))$

$$\boxed{?} = \sum_{M \in \rho(p)^{-1}\mathcal{H}(N)_\ell} C(M)\lambda(pM)$$

$M \in \mathbb{Z}^{2\times 2}_{\det=\ell}$ s.t. pM is well-def'd.

$$\Rightarrow T(\ell)\lambda(p) = \sum_{M \in \mathbb{Z}^{2\times 2}_\ell,\ pM \text{ well-def'd}} C(M)\lambda(pM)$$

For C $(\ast\!\ast\!\ast)$: Choose reps. $M \in \overset{(?)}{\cdots} \in \mathbb{Z}_\ell^{2\times 2}/SL$, and then apply the Manin trick to $M^{-1}(e_\infty - e_0)$ to write $\sum A_j (e_\infty - e_0)$ w/ $A_j \in SL$.

$\downarrow$ this leads to :
$$\boxed{C_{CF} = -\sum_{\substack{ad=\ell \\ a,d>0 \\ 0\le b<d}}} \sum_{j=0}^{n(b/d)} \begin{pmatrix} d & b \\ 0 & a \end{pmatrix} \begin{pmatrix} \varepsilon_i & \\ & 1 \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix}$$
$$\cdot \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (e_\infty - e_0)$$
$$= e_\infty - e_{b/d}.$$

here $\frac{b}{d} = [a_0, \ldots, a_{n(\frac{b}{d})}]$

$\varepsilon = (-1)^{i-1}$.

$\to$ the most general solution to $(\ast\!\ast\!\ast)$ is:
$$C = \boxed{C_{CF}} + \tilde{C} \;;\quad \tilde{C} \in I, \qquad \varepsilon = (-1)^{i-1}$$
$$0 \to I \to \mathbb{Z}[SL_2(\mathbb{Z})] \to \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] \to 0$$
$$I = \mathbb{Z}[SL_2(\mathbb{Z})](1+S) + \mathbb{Z}[SL_2(\mathbb{Z})](1+R+R^2).$$

$\Rightarrow$ "many choices for C!"

Lemma: Set $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : a > b \geq 0,\ d > c \geq 0 \right\}$

$\Rightarrow$ for every $M \in \mathbb{Z}^{2\times 2}_\ell$, one has
$$\sum_{A \in M^{-1}\mathcal{M} \cap SL_2(\mathbb{Z})} A(e_\infty - e_0) = M^{-1}(e_\infty - e_0).$$

Consequence: $(T(\ell)\lambda)(p) = \sum \lambda\left(p\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$.
$$a > b \geq 0,\quad d > c \geq 0,\quad ad - bc = \ell$$
$p\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is well-def'd.

$$M_2(11) = \mathbb{C}\cdot\left(E_2(z) - 11E_2(11\cdot z)\right) \oplus \mathbb{C}\eta(z)^2\eta(11z)^2.$$

$$\boxed{\eta(z)^2\eta(11z)^2} = \sum_{\substack{a>b\geq 0 \\ d>c\geq 0 \\ \frac{3a+c}{3b+d} \equiv 3, \frac{1}{3}\ (11)}} q^{ad-bc} - \sum_{\substack{a>b\geq 0, \\ d>b\geq 0 \\ \frac{3a+c}{3b+d} \equiv \frac{2}{3}, \frac{3}{2}\ (11)}} q^{ad-bc}$$

(more on this ex. in the next lecture)

$$\sum (T(\ell)\lambda)(13:17)_\ell\, q^\ell$$

Generali'n to higher wt, and character:

$$H(N) \xleftarrow{\ \cong \text{ of Hecke modules.}\ } \mathrm{Hom}_\Gamma\left(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0,\ \mathbb{C}[X,Y]_{k-2} \otimes \mathbb{C}(\chi)\right)$$

$\Gamma \leq SL_2(\mathbb{Z})$ fin. ind. ; $\chi : \Gamma \to S^1$;

Γ acts via $(A, f) \mapsto f\left(A^{-1}\begin{bmatrix} X \\ Y \end{bmatrix}\right)$

$$\text{"}\mathcal{L}\text{"} = \{\lambda : SL_2(\mathbb{Z}) \to \mathbb{C}[X,Y]_{k-2} \otimes \mathbb{C}(\chi)\}.$$
$\lambda(Ax) = A\lambda(x)$ for all $x \in SL_2(\mathbb{Z})$, $A \in \Gamma_1(N)$.
$$\lambda(x) + S^{-1}\lambda(xS) = 0$$
$$\lambda(x) + R^{-1}\lambda(xR) + R^{-2}\lambda(xR^2) = 0.$$

$\boxed{\text{Modular Symbols}}$ Lecture 5. (JC)

Constructing modular elliptic curves

Story so far: Given N, we can form M-symbol spaces $H^\pm(N)$ of dim $g^\pm = g + g_0^\pm$.

where $g = \dim S_2(N)$; $g_0^\pm = \dim M_2^{Eis}(N)^\pm$.

- The mod. symbol spaces $H^\pm(N)$ are dual to $S_2(N) \oplus M_2^{Eis}(N)^\pm$.
- Have explicit Hecke action on $H^\pm(N)$, specifically T_p for $p \nmid N$. (Note that formulae from §4 are simpler when l is a prime.) These are given by $g^\pm \times g^\pm$ matrices w/ $\mathbb{Z}$-entries.

Let $\{a_p\}$ = e'vals of T_p. We're interested in $a_p \in \mathbb{Z}$ (b/c we're interested in newforms w/ rat'l e'vals, eventually). Since $|a_p| < 2\sqrt{p}$, there are $< \infty$ of these a_p for given p.

↳ Look for 1-dim'l common e'sp's $\forall\, T_p$, $p \nmid N$, which correspond to Hecke e'forms with rational e'vals

For each sequence of possible e'vals (i.e. seqs $(a_p)_{p\nmid N}$ with $|a_p| < 2\sqrt{p}$ $\forall p$), we can compute the intersection of the kernels of $(T_p - a_p \cdot I_{g^\pm \times g^\pm})$, & this gives a ($\geq 0$) list of e'vecs $\gamma \in H^\pm(N)$ (enough to look in either H^+ or H^-) (we'll look in $H^+(N)$)

s.t. $T_p \gamma = a_p \gamma$ $\forall p$.

Dually, for each e'val seq., we can find a dual e'vec v (a "row vector") s.t $vT_p = a_p v$ $\forall p \nmid N$.

→ from this, we can construct an elliptic curve via period integrals.

Recall: $H_1(X_0(N), \mathbb{Z}) = \langle \gamma_1, \gamma_2, \dots, \gamma_{2g} \rangle_{\mathbb{Z}}$.

$S_2(N) = \langle f_1, \dots, f_g \rangle_{\mathbb{C}}$

→ form the period matrix: $\Omega = (\underbrace{\langle \gamma_i, f_j \rangle})_{i,j} \in M(2g \times g, \mathbb{C})$.

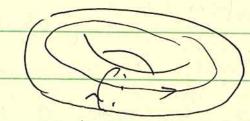
recall: this means $\int_{\gamma_i} 2\pi i f_j(\tau)\, d\tau$.

→ Rows of Ω ($\in \mathbb{C}^g$) span a lattice Λ of rank $2g$ in $\mathbb{C}^g$; $\mathbb{C}^g/\Lambda = \mathrm{Jac}(X_0(N))$ (cpx torus).

For $N=11$: $f = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 + \dots = q\prod_{n\geq 1}(1-q^n)^2(1-q^{11n})^2$

$\gamma_1 = (2:1) = \{0, \frac{1}{2}\}$

$\gamma_2 = (3:1) = \{0, \frac{1}{3}\}$.



$\langle \gamma_1 \rangle = H(11)^+_{cusp}$ since $\gamma_1^* = \gamma_1$, while $\gamma_2^* = \gamma_1 - \gamma_2$.

$\langle \gamma_1 - 2\gamma_2 \rangle = H(11)^-_{cusp}$. So $\langle f, \gamma_1 \rangle = 2x$ ↳ $\langle \gamma_1, f \rangle$?

$(x \in \mathbb{R})$, and $\langle f, \gamma_2 \rangle = x + y i$ $(y \in \mathbb{R})$. ↳ $\langle \gamma_2, f \rangle$?

$\Rightarrow \Lambda = \langle 2x, x + yi \rangle$ (lattice)

Numerically: $x = 1.26921\dots$, $y = 1.4588\dots$

$\Rightarrow$ [$E = \mathbb{C}/\Lambda$: $y^2 + y = x^3 - x^2 - 10x - 20$.

(Get this using classical formula of $c_4(E)$, $c_6(E)$... + do more work to get a minimal model.]

c_4, c_6 are known to be integral (Edix.), so it's enough to compute them approximately & then round.

General case: f an e'form with rational e'vals (a_p), norm'd s.t. $a_1=1$.
Define $\Lambda_f := \{\langle\gamma, f\rangle : \gamma\in H_1(X_0(N),\mathbb{Z})\}$: this is the period lattice of f: a rk 2 lattice $\subseteq \mathbb{C}$.
Then $E_f := \mathbb{C}/\Lambda_f$ is known to be an elliptic curve over $\mathbb{Q}$ of conductor N, and

$$L(E_f,s) = L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s} \text{ where } f=\sum_{n=1} a_n q^n.$$

Moreover, the sign of the FE of $L(E_f,s)$ is $-\varepsilon_N$, where $f\Big|\begin{pmatrix}0&-1\\N&0\end{pmatrix} = \varepsilon_N f$ ($\varepsilon_N=\pm1$).

note: this is specific to E_f, and might be different for an isogenous curve. Our method should give E_f precisely.

We also know $\dfrac{L(E_f,1)}{\Omega_f}\in\mathbb{Q}$ (as an exact value), where $\mathbb{Z}\cdot\Omega_f = \Lambda_f\cap\mathbb{R}$. (eg, $=\frac{1}{5}$ for $N=11$).

How do we find an eqn for E_f? Λ_f is spanned by the $2g$ periods $\langle\gamma_j, f\rangle_{1\le j\le 2g}$, but we only need 2 to generate a lattice. Let $\{\omega_1,\omega_2\}$ be a (unknown so far) $\mathbb{Z}$-basis for Λ_f. For simplicity, assume $\omega_1\in\mathbb{R}$, $\omega_2\in i\mathbb{R}$ (i.e. $\Delta(E_f)>0$)

$\Rightarrow$ to each $\gamma\in H_1(X_0(N),\mathbb{Z})$, $\langle\gamma, f\rangle = n_1(\gamma)\omega_1 + n_2(\gamma)\omega_2$ where $n_j(\gamma)\in\mathbb{Z}$, so f determines 2 maps $n_j: H_f(N)\to\mathbb{Z}$, which are precisely the "dual M-symbols maps" $\lambda^\pm\in\mathcal{L}^\pm(N)$ associated to f.

Think of $\lambda^\pm$ as row vectors of length $2g+g_0$ & γ as a column vector w of length $2g+g_0$.

$\Rightarrow$ $n_1(\gamma) = \lambda^+\cdot w$, $n_2(\gamma)=\lambda^-\cdot w$.

Choose a single γ s.t. $n_1(\gamma)\ne0$, $n_2(\gamma)\ne0$.
$\to$ Compute the single period $\langle f,\gamma\rangle = x_\gamma + iy_\gamma$ (w/ $x_\gamma, y_\gamma\in\mathbb{R}, \ne0$).

+ then $x_\gamma = n_1(\gamma)\cdot\omega_1$, $y_\gamma = n_2(\gamma)\cdot\omega_2$, $\Rightarrow$ $\omega_1 = x_\gamma/n_1(\gamma)$, $\omega_2 = iy_\gamma/n_2(\gamma)$.

(*) How to compute $\langle f,\gamma\rangle = \int_\gamma 2\pi i f(\tau)\,d\tau$ for $\gamma\in H_1(X_P,\mathbb{Z})$?

Say $\gamma = \{\alpha, g(\alpha)\}_\Gamma$; $g\in\Gamma$.

$$\langle f,\gamma\rangle = \int_\alpha^{g(\alpha)} 2\pi i f(\tau)\,d\tau = \int_\alpha^\infty - \int_{\beta=g(\alpha)}^\infty.$$

Let $\alpha\in\mathfrak{H}$, $\alpha = x_0 + y_0 i$, $y_0>0$.

$$\int_\alpha^\infty 2\pi i f(\tau)\,d\tau;\ \tau = x_0+iy \Rightarrow d\tau = i\,dy.$$

$$\hookrightarrow = \int_{y_0}^\infty -2\pi f(x_0+iy)\,dy.$$

(using q-expn of f)

$$= \int_{y_0}^\infty -2\pi\sum_{n=1}^\infty a_n e^{2\pi i n x_0} e^{-2\pi n y}\,dy.$$

$$= \sum_{n=1}^\infty a_n e^{2\pi i n x_0}\cdot\frac{1}{n}\left[e^{-2\pi n y}\right]_{y_0}^\infty$$

$$= (\pm)\sum \frac{a_n}{n} e^{2\pi i n x_0} e^{-2\pi n y_0}$$

($y_0 >> 0$ is best for fast calcs)

One way to choose α: $g = \begin{pmatrix}a&b\\c&d\end{pmatrix}$, $c>0$.
(to make $y_0>>0$)

$$\alpha = \frac{-d+i}{c},\quad g(\alpha) = \frac{a+i}{c}.$$

Q: Once we've gotten a model for E_f: want to know

- Is it integral? (Yes: Edixhoven)
- Is it minimal (probably: Manin's conjecture).

$\rightsquigarrow$ For each N & each rat'l newform $f \in S_2(N)$, we've constructed an ell curve E of conductor N.

From E_1, we can compute the isogeny class $E_1, E_2, \dots, E_m$. ($m \le 8$! (Mazur))
Assume that each E_j is given by a minimal model.
How can we be sure that $E_1 = E_f$ (not just $\cong$)?

Approach #1: approximate closely enough + use integrality.
" #2: use modularity: we know ($\le$) how many isogeny / $\cong$ classes of (modular) ell curves there are of conductor N.

To identify the isogeny class of E_f, compare $L(f,s) = L(E_f,s)$ with $L(E,s)$ $\forall E$ of conductor N.

$\hookrightarrow E_f$ is isogenous to E_1.

$\Rightarrow E_f \cong E_j$ for some $j \le m$.

use:
- Is $j = 1$?
- Is the $\cong$ an $=$?

Modular parametrization $X_0(N) \to E$.
Let ω_{E_1} be the Néron diff'l on E

$$\left(= \frac{dx}{2y + a_1 x + a_3}\right).$$

$$\varphi^* \omega_{E_1} = \boxed{c\,\cdot}\, 2\pi i f(\tau)\, d\tau \text{ for some } c \in \mathbb{Q}^*$$

"Manin's constant".

Manin's Conjecture: $|c| = 1$.
Edixhoven: $c \in \mathbb{Z}$. $\Rightarrow E_f$ has an integral model.

"$c=1$" means that E_f is a minimal model, since the period lattice of $E_1 = \Lambda_1 = c \cdot \Lambda_f$. $\Rightarrow$ easy to check that Λ_f is not homothetic to any Λ_j ($j > 1$), and $c = 1$.

Thm $c = 1$ $\forall N \le 250{,}000$.