Notes on Borcherds Products

Richard Hill

These are the notes for a short course on Borcherds Products held in Aachen on 1st-2nd August 2012. The lectures are intended for someone who does not know what a Borcherds product is, but does know what a modular form is. In preparing these notes I've used the following sources: [1, 2, 3, 4, 5]. All the correct statements in these notes are taken from those sources, and all the mistakes are my own.

Contents

1 Introduction

"Borcherds products" or the "Borcherds lift" is a method for constructing modular forms on the orthogonal groups $SO^+(2, n^-)$. More precisely, we get a map

$$\begin{array}{ccc} \left\{ \begin{array}{c} \textit{weakly holomorphic} \\ \textit{modular forms} \\ \textit{on } \mathrm{SL}_2(\mathbb{Z}) \textit{ with coefficients in } \mathbb{Z} \end{array} \right\} & \to & \left\{ \begin{array}{c} \textit{meromorphic modular} \\ \textit{forms on } O^+(2, n^-) \end{array} \right\} \\ f & \mapsto & \Psi_f. \end{array}$$

The form Ψ_f is called a "Borcherds product". Unlike other lifts in the theory of modular forms, this lift is multiplicative, in the sense that

$$\Psi_{f+g} = \Psi_f \times \Psi_g.$$

The weight of Ψ_f is $c(0)/2$, where $c(0)$ is the constant coefficient of f, and it is easy to describe the zeros and poles of Ψ_f in terms of other Fourier coefficients of f.

Very briefly, a weakly holomorphic modular form is given by an expression of the form

$$f(\tau) = \sum_{n=-N}^{\infty} c(n) q^n, \qquad q = e^{2\pi i \tau}, \quad \tau \in \mathcal{H}.$$

The corresponding Borcherds product is a meromorphic continuation of an expression of the form

$$\Psi(w) = e^{2\pi i (\rho, w)} \prod_{\lambda \in M, (\lambda, w_0) > 0} \left(1 - e^{2\pi i (\lambda, w)}\right)^{c(Q(\lambda))},$$

where

- M is a lattice with a symmetric bilinear form $(-,-)$ of signature $(1, n^- - 1)$ and Q is the associated quadratic form.
- $w \in M \otimes \mathbb{C}$ satisfies $Q(\Im w) > 0$.
- $w_0 \in M$ satisfies $Q(w_0) > 0$.
- $\rho \in M \otimes \mathbb{Q}$ depends additively on f.

In particular, if we restrict to vectors $w = \tau w_0$ for τ in the upper half plane, then we get a meromorphic modular form on the upper half-plane:

$$\Psi(\tau w_0) = q^{(\rho, w_0)} \prod_{\lambda \in M, (\lambda, w_0) > 0} \left(1 - q^{(\lambda, w_0)}\right)^{c(Q(\lambda))}, \qquad q = e^{2\pi i \tau}.$$

We can now see that this formula is similar to the definition of the Dedekind η-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

In this first lecture I hope to define the terms "weakly holomorphic modular form" and "meromorphic modular form on $SO^+(2, n)$", and then give a definition of Ψ_f in terms of f. The definitions and statements given in this lecture are not the most general. I'll give more general definitions in the last lecture.

1.1 Notation

We'll write Γ for the group $\mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{H}$ for the upper half-plane. The symbol τ will mean an element of $\mathcal{H}$ and we'll use the notation $q = e^{2\pi i\tau}$. For an even integer $k \geq 4$, we'll write E_k for the weight k level 1 Eisenstein series:

$$E_k(\tau) = 1 + const \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \qquad const = \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \in \mathbb{Q}^{\times}.$$

1.2 Weakly holomorphic modular forms

Let k be an even integer. By a "weakly holomorphic modular form of weight k", we shall mean a function $f : \mathcal{H} \to \mathbb{C}$ with the following properties:

- f is holomorphic;
- For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have
$$f(\gamma\tau) = (c\tau + d)^k f(\tau).$$
- f has a Fourier expansion of the form
$$f(\tau) = \sum_{n=-N}^{\infty} c(n) q^n, \qquad q = \exp(2\pi i\tau).$$

The word "weakly" just means that we are allowed finitely many negative terms in the Fourier expansion; in other words f is allowed to have a pole at the cusp.

1.2.1 Examples

Any holomorphic form is weakly holomorphic, but there are many more weakly holomorphic forms than holomorphic forms. Indeed there are weakly holomorphic forms of negative weight. For example if we let

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = (E_4(\tau)^3 - E_6(\tau)^2)/1728.$$

Then Δ is a weight 12 cusp form and is nowhere zero on $\mathcal{H}$.

It follows that $\frac{1}{\Delta}$ is a weight -12 weakly modular form. Also note that the j-invariant

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n) q^n$$

is a weight 0 weakly modular form. More generally, if f is a weight k holomorphic modular form then f/Δ^n is a weight $k - 12n$ weakly holomorphic form.

Suppose we have a weakly holomorphic form

$$f = \sum a_n q^n.$$

We define the "principal part" of f to the negative part of the Fourier expansion:

$$\sum_{n<0} a_n q^n.$$

Proposition 1 *Let $g(q) = \sum_{n=-N}^{-1} a_n q^n$ and let $k < 0$. There is a weight k weakly holomorphic form with principal part g if and only if for every weight $2-k$ holomorphic cusp form*

$$h(q) = \sum_{n=1}^{\infty} c_n q^n,$$

we have

$$\sum_{n=1}^{N} c_n a_{-n} = 0.$$

If such a weakly holomorphic form exists then it is unique, and its constant term is given by

$$a_0 = -\sum_{n=1}^{N} d_n a_{-n},$$

where d_n are the coefficients of the weight $2-k$ holomorphic Eisenstein series E_{2-k}.

The proof is an exercise:

Exercise 1

- *Assume there is a weakly holomorphic form f. Prove that $f(\tau)h(\tau)d\tau$ is a Γ-invariant holomorphic differential form on $\mathcal{H}$.*
- *Show that the residue of this form at the cusp $q = 0$ is const $\cdot \sum_{n=1}^{N} c_n a_{-n}$. (Hint: $d\tau =$ const $\cdot\, dq/q$.) Hence deduce that $\sum_{n=1}^{N} c_n a_{-n} = 0$.*
- *Prove the converse to this by Serre duality.*
- *Prove the formula for a_0.*

Exercise 2 *(Ramanujan's congruence) Let $k > 0$. Suppose $N \in \mathbb{N}$ is chosen so that $NE_k = N + \sum d_n q^n$ has coefficients in $\mathbb{Z}$. Then there is a cusp form $f = \sum c_n q^n$ of weight k with coefficients in $\mathbb{Z}$, such that for all n,*

$$c_n \equiv d_n \bmod N.$$

(Hint: subtract $E_4^a E_6^b$ from E_k for suitable a, b).

Exercise 3 *Let $f = \sum a_n q^n$ be a weakly holomorphic form. Show that if $a_n \in \mathbb{Z}$ for all $n < 0$ then $a_0 \in \mathbb{Z}$. (This is equivalent, by the proposition above, to the Ramanujan congruence).*

Exercise 4 *Let $f = \sum a_n q^n$ be a weakly holomorphic form. Show that if $a_n \in \mathbb{Z}$ for all $n < 0$ then $a_n \in \mathbb{Z}$ for all n. (Hint: by induction on n, multiplying by Δ at each step).*

1.3 Modular forms on Orthogonal groups

1.3.1 Orthogonal groups

Let L be a lattice with a non-degenerate symmetric bilinear form $(-,-) : L \times L \to \mathbb{Z}$, such that (v,v) is even for all $v \in L$. We'll write $Q : L \to \mathbb{Z}$ for the corresponding quadratic form, ie

$$Q(v) = \frac{(v,v)}{2}.$$

We let $V = L \otimes \mathbb{R}$, and we'll extend Q and $(-,-)$ to V in the obvious way. Let n^+ be the dimension of a maximal positive definite subspace of V and n^- the dimension of a maximal negative definite subspace. We'll call (n^+, n^-) the signature of L.

The orthogonal group O_V is defined to be the group of linear bijections $V \to V$, which preserve Q (or equivalently, which preserve $(-,-)$). The group O_V is a real Lie group. Suppose we choose a basis $\{b_i\}$ for V and we let J be the matrix whose (i,j)-entry is (b_i, b_j). Then O_V consists of matrices α satisfying

$$\alpha^t J \alpha = J.$$

Every element of O_V has determinant ± 1. We'll also write SO_V for the special orthogonal group, which consists only of the bijections with determinant 1. This is a subgroup of index 2.

If V is positive definite or negative definite, then SO_V is connected. If V is indefinite then SO_V has two connected components. More precisely suppose $V = V^+ \perp V^-$, where V^+ is positive definite and V^- is negative definite. Then $O(V^+) \oplus O(V^-)$ is a maximal compact subgroup of $O(V)$. Like any real reductive group, O_V is homotopic to a maximal compact subgroup. However $O(V^+) \oplus O(V^-)$ has 4 connected components, and hence so does O_V. It follows that SO_V has two connected components. We shall write SO_V^+ for the identity component of SO_V.

We'll also write Γ_L for the group of elements $\alpha \in SO_V^+$, such that $\alpha L = L$. This is an arithmetic subgroup of SO_V^+.

Exercise 5 *Let $G \subset \mathrm{GL}_n(\mathbb{R})$ be a Lie group such that $G = G^t$. Show that $G \cap O(n)$ is a maximal compact subgroup of G.*

1.3.2 The symmetric space for SO_V^+

Let Gr_V be the set of orthogonal decompositions $V = V^+ \perp V^-$, where V^+ and V^- are positive definite and negative definite respectively. There is an obvious action of O_V on Gr_V.

Exercise 6 *Prove that O_V acts transitively on Gr_V. Show that the stabiliser of the point (V^+, V^-) is isomorphic to $O_{V^+} \oplus O_{V^-}$. Hence show that Gr_V may be identified with the quotient*

$$Gr_V \cong O_V/(O_{V^+} \oplus O_{V^-}) \cong SO_V^+/(SO_{V^+} \oplus SO_{V^-}).$$

Exercise 7 *Let V have signature $(1, n^-)$, i.e. $\dim(V^+) = 1$. Choose a vector $v_0 \in V$ such that $Q(v_0) > 0$. Show that there is a natural bijection*

$$Gr_V \cong \{v \in V : Q(v) = 1, (v, v_0) > 0\}.$$

1.3.3 The symmetric space $\mathcal{K}$

We'll now consider in more detail the case that V has signature $(2, n)$. This case is of special interest to us, since Borcherds products are forms on SO_V^+ in this case. We'll introduce a new model of the symmetric space with an obvious complex structure. Let

$$\tilde{\mathcal{K}} = \{v \in V \otimes \mathbb{C} : (v, v) = 0,\ (v, \bar{v}) > 0\}.$$

We can also write this as follows:

$$\tilde{\mathcal{K}} = \{x + iy : x, y \in V,\ x \perp y,\ Q(x) = Q(y) > 0\}.$$

Exercise 8 *Check that these two definitions of $\tilde{\mathcal{K}}$ are equivalent.*

Note that if $v \in \tilde{\mathcal{K}}$ then $c \cdot v \in \tilde{\mathcal{K}}$ for every $c \in \mathbb{C}^\times$. We may therefore define

$$\mathcal{K} = \{[v] \in \mathbb{P}(V \otimes \mathbb{C}) : v \in \tilde{\mathcal{K}}\}.$$

There is an obvious action of O_V on $\mathcal{K}$, and this is compatible with the obvious complex structure ($\mathcal{K}$ is an open subset of a subvariety of $\mathbb{P}(V \otimes \mathbb{C})$).

One can check that the action of the orthogonal group is transitive and the stabilizer of a point of $\mathcal{K}$ is $SO(2) \oplus O(n)$. Therefore $\mathcal{K} \cong O_V/(SO(2) \oplus O(n))$ has two connected components. These components are exchanged by complex conjugation.

Exercise 9 *Check all this.*

Let $\mathcal{K}^+$ be one of these components. It follows that $\mathcal{K}^+$ is a symmetric space of SO_V^+. We can write down an explicit bijection $\mathcal{K}^+ \to Gr_V$ as follows:

$$[v] \mapsto (V^+, V^-), \quad V^+ = \langle \Re v, \Im v \rangle, \quad V^- = (V^+)^\perp.$$

1.3.4 Modular forms on $SO^+(2, n^-)$

We next fix a non-zero vector $\lambda_0 \in L$ such that $Q(\lambda_0) = 0$.

Lemma 1 *For $v \in \tilde{\mathcal{K}}$ we have $(v, \lambda_0) \neq 0$.*

Exercise 10 *Prove the lemma.*

We can now introduce a multiplier system as follows:

$$j(\alpha, [v]) = \frac{(\alpha v, \lambda_0)}{(v, \lambda_0)}, \qquad \alpha \in SO_V^+, \quad [v] \in \mathcal{K}.$$

Lemma 2 *The function j is a multiplier system, i.e. $j(\alpha\beta, [v]) = j(\alpha, \beta[v])j(\beta, [v])$.*

Exercise 11 *Prove the lemma.*

Definition 1 *Let $k \in \mathbb{Z}$ and let Υ be a subgroup of SO_V^+ commensurable with Γ_L. By a modular form of weight $k \in \mathbb{Z}$ and level Υ on the orthogonal group, we shall mean a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, such that for each $\alpha \in \Upsilon$ we have*

$$f(\alpha v) = \chi(\alpha) j(\alpha, v)^k f(v),$$

Where $\chi : \Upsilon \to \mathbb{C}^\times$ is a character with values in a finite subgroup of $\mathbb{C}^\times$. In fact if $n^- \geq 3$ then every character of Υ takes values in a finite subgroup of $\mathbb{C}^\times$.

If k is a half-integer, then we choose for each $\alpha \in \Upsilon$ a square root $j(\alpha, v)^k$ of $j(\alpha, v)^{2k}$, and we require the function χ to satisfy the following relation:

$$\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)\frac{j(\alpha, \beta v)^k j(\beta, v)^k}{j(\alpha\beta, v)^k}.$$

Again χ is required to have values in a finite subgroup of $\mathbb{C}^\times$, and again this condition is redundant if $n^- \geq 3$.

1.3.5 The symmetric space $\mathbb{H}$

Again assume that the lattice L has signature $(2, n^-)$ with $n^- > 0$. We'll choose an orthogonal decomposition of the vector space V as follows:

$$V = W \perp \langle \lambda_0, \lambda_1 \rangle,$$

where $Q(\lambda_0) = Q(\lambda_1) = 0$ and $(\lambda_0, \lambda_1) = 1$. Define

$$\mathbb{H} = \{w \in W \otimes \mathbb{C} : Q(\Im(w)) > 0, (\Im(w), w_0) > 0\}.$$

Here $w_0 \in W$ is chosen to satisfy $Q(w_0) > 0$. Without this final inequality, $\mathbb{H}$ would have two connected components, and the final inequality just means "the component containing iw_0". There is a bijection $\mathbb{H} \to \mathcal{K}^+$ given by

$$w \mapsto [v], \qquad v = w - Q(w)\lambda_0 + \lambda_1.$$

Exercise 12 *Check that this is a bijection between $\mathbb{H}$ and $\mathcal{K}^+$.*

Exercise 13 *Let V have signature $(2, 1)$. Show that there is a natural bijection*

$$\mathbb{H} \cong \mathcal{H},$$

which is compatible with a natural isomorphism

$$SO_V^+ \cong \mathrm{PSL}_2(\mathbb{R}).$$

1.3.6 The Fourier expansion of a modular form on $\mathbb{H}$

For simplicity, we'll assume that our lattice L is unimodular, and we'll assume that the orthogonal decomposition of V comes from an orthogonal decomposition of L:

$$L = M \perp \langle \lambda_0, \lambda_1 \rangle, \qquad W = M \otimes \mathbb{R},$$

where $Q(\lambda_0) = Q(\lambda_1) = 0$ and $(\lambda_0, \lambda_1) = 1$.

Lemma 3 *For any $\lambda \in M$ there is an element $\alpha \in \Gamma_L$ whose action on $\mathbb{H}$ is given by*

$$\alpha w = w + \lambda.$$

We have $j(\alpha, w) = 1$.

Exercise 14 *Prove the lemma.*

Suppose now that $F : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k, level Γ_L, and for simplicity assume that the character χ is trivial. Then we have a Fourier expansion

$$F(w) = \sum_{\lambda \in M} c(\lambda) e^{2\pi i (\lambda, w)}.$$

In fact we'll have $c(\lambda) = 0$ unless $(\lambda, \Im w) \geq 0$ for all $w \in \mathbb{H}$.

1.4 Borcherds products

Assume now that L is the unimodular lattice of signature $(2, n^-)$. We can choose the orthogonal decomposition of V to be given by one of L:

$$L = M \perp \langle \lambda_0, \lambda_1 \rangle,$$

where $Q(\lambda_0) = Q(\lambda_1) = 0$ and $(\lambda_0, \lambda_1) = 1$. Choose a vector $w_0 \in W$ with $Q(w_0) > 0$.

Let $f = \sum c(n) q^n$ be a weakly holomorphic form of weight $\frac{2-n^-}{2}$ with coefficients in $\mathbb{Z}$. For a vector $\rho \in M \otimes \mathbb{Q}$, we define the Borcherds product as follows:

$$\Psi_f(w) = e^{2\pi i (\rho, w)} \prod_{\lambda \in M,\ (\lambda, w_0) > 0} \left(1 - e^{2\pi i (\lambda, w)}\right)^{c(Q(\lambda))}$$

The condition $(\lambda, w_0) > 0$ should be understood as follows: we choose w_0 so that there are no lattice points $\lambda \in M$ orthogonal to w_0 for which $c(Q(\lambda)) \neq 0$. This amounts to saying that $[w_0]$ is not in the Heegner set of signature $(1, n^- - 1)$ in Gr_W.

Theorem 1 *The product Ψ_f converges for $w \in \mathbb{H}$ near $i\infty w_0$. It has a meromorphic continuation to $\mathbb{H}$. There is a unique vector $\rho \in W$ depending additively on f, such that Ψ_f is a modular form of weight $c(0)/2$ of level Γ_L.*

The phrase "near $iw_0\infty$" means that $Q(\Im w)$ is sufficiently large.

1.4.1 Heegner divisors

As before, let L be the unimodular lattice of signature $(2, n^-)$. For a vector $\lambda \in L$ with $Q(\lambda) < 0$ we let

$$\lambda^\perp = \{(V^+, V^-) \in Gr_V : \lambda \perp V^+\} = \{(V^+, V^-) \in Gr_V : \lambda \in V^-\}.$$

In the model $\mathcal{K}$, the subset $\lambda^\perp$ is the orthogonal complement of the vector λ and is therefore a divisor. (It is the divisor of the function $[v] \mapsto (v, \lambda)/(v, \lambda_0)$.)

For a negative integer m, we define the Heegner divisor $H(m)$ on $\mathbb{H}$ as follows:

$$H(m) = \sum_{\lambda \in L/\{1,-1\},\, Q(\lambda)=m} \lambda^\perp.$$

One can check that any compact subset of $\mathbb{H}$ intersects only finitely many $\lambda^\perp$, and so $H(m)$ is also a divisor on $\mathbb{H}$.

Theorem 2 *Let $f = \sum c(n)q^n$ be a weakly holomorphic modular form. The divisor of Ψ_f is the finite sum*

$$\sum_{m<0} c(m)H(m).$$

In particular, f is holomorphic if and only if $c(n) \geq 0$ for all $n < 0$.

2 Sketch proof of Theorems 1 and 2

In this lecture I'll sketch the proofs of theorems 1 and 2. Very briefly, we proceed as follows. Given a weakly holomorphic modular form f, we define a function Φ on $\mathbb{H}$ by

$$\Phi(w) = \int_{\Gamma\backslash\mathcal{H}} f(\tau)\overline{\Theta(\tau, w)} y \frac{dx\, dy}{y^2}.$$

This integral diverges, but nevertheless we can make sense of it. We show that Φ has singularities on the Heegner divisors $H(m)$ where $c(m) \neq 0$, and we calculate the behaviour of Φ near these divisors.

Next we calculate the Fourier expansion of Φ, and show that $\Phi(w) = -4\log|\Psi(w)| + c(0)\log(\Im w, \Im w)$. It follows that Ψ has a meromorphic continuation, and $|\Psi(w)|(\Im w, \Im w)^{c(0)/4}$ is Γ_L-invariant. From this, it follows that Ψ is modular of weight $c(0)/2$.

2.1 The theta correspondence

2.1.1 The theta function

Let's assume from now on that L is a unimodular lattice of arbitrary signature. In particular, this means that the rank of L is even.

Exercise 15 *Show that the rank of L is even. (Hint: consider the bilinear form on $L \otimes \mathbb{F}_2$).*

Exercise 16 *In fact one can show (see chapter 5 of [6]) that $n^+ \equiv n^- \bmod 8$. Give a different proof of this fact using Theorem 2. (You will need the fact that holomorphic modular forms on orthogonal groups have weight ≥ 0, and also that there is no cusp form on Γ, all of whose coefficients are positive.)*

We'll regard an element $p \in Gr_V$ as being given by two orthogonal projection maps $p^+ : V \to V^+$, $p^- : V \to V^-$. For $\tau \in \mathcal{H}$ and $(p^+, p^-) \in Gr_V$ we let

$$\begin{aligned}\Theta(\tau, p) &= \sum_{\lambda\in L} \exp\left(2\pi i \left(\frac{(p_+(\lambda), p_+(\lambda))}{2}\tau + \frac{(p_-(\lambda), p_-(\lambda))}{2}\bar{\tau}\right)\right) &(1)\\ &= \sum_{\lambda\in L} q^{Q(\lambda)}|q|^{-(p_-(\lambda), p_-(\lambda))}. &(2)\end{aligned}$$

Exercise 17 *Show that the two definitions above are equivalent. Show that the sums converge on $\mathcal{H} \times Gr_V$.*

It's easy to see that for any $\alpha \in \Gamma_L$, we have

$$\Theta(\tau, \alpha p) = \Theta(\tau, p).$$

Furthermore, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$\begin{aligned} \Theta(\gamma\tau, p) &= (c\tau + d)^{\frac{n^+}{2}} (c\bar{\tau} + d)^{\frac{n^-}{2}} \Theta(\tau, p) \\ &= |c\tau + d|^{\frac{n^+ + n^-}{2}} \left(\frac{c\tau + d}{c\bar{\tau} + d} \right)^{\frac{n^+ - n^-}{2}} \Theta(\tau, p). \end{aligned}$$

To prove this formula, it's sufficient to prove it for the two generators of Γ. For $\tau \mapsto \tau + 1$ it is trivial, and for $\tau \mapsto \frac{-1}{\tau}$ it follows from the Poisson summation formula.

2.1.2 Divergent integrals

Let L be a lattice of signature (n^+, n^-) and let f be a weakly modular form of weight $k = \frac{n^+ - n^-}{2}$. The function

$$\tau \mapsto f(\tau)\overline{\Theta(\tau, p)} y^{n^+/2}$$

is Γ-invariant, and we define the theta-transform of f as follows:

$$\Phi(p) = \int_{\Gamma\backslash\mathcal{H}} f(\tau)\overline{\Theta(\tau, p)} y^{n^+/2} \frac{dx\ dy}{y^2}$$

(The measure $\frac{dx\ dy}{y^2}$ is $\mathrm{SL}_2(\mathbb{R})$-invariant.) If f is a holomorphic cusp form, then this integral converges. However we'd like to consider the integral when f is weakly holomorphic, and in this case it diverges.

To get around this problem, we define the value of $\Phi(p)$ as follows. Let $\mathcal{F}(Y)$ be the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, truncated at $y = Y$. For $\Re(s)$ sufficiently large, the following limit exists:

$$\Phi(p, s) = \lim_{Y\to\infty} \int_{\mathcal{F}(Y)} f(\tau)\overline{\Theta(\tau, p)} y^{n^+/2 - s} \frac{dx\ dy}{y^2}$$

This function has a meromorphic continuation to $s = 0$, and we define $\Phi(p)$ to be the constant term in the Laurent expansion of $\Phi(p, s)$ at $s = 0$. In practice, this means that we end up with

$$\Phi(p) = \lim_{Y\to\infty} \int_{\mathcal{F}(Y)} \left(f(\tau)\overline{\Theta(\tau, p)} - c(0) \right) y^{n^+/2} \frac{dx\ dy}{y^2} + const.$$

It turns out that Φ has singularities, which we now inventigate. We'll concentrate on the case that L has signature $(2, n^-)$. If we expand $f = \sum c(m) q^m$ and $\overline{\Theta} = \sum q^{-Q(\lambda)} |q|^{(p_+(\lambda), p_+(\lambda))}$, then we obtain terms of the form

$$c(m) \int_{\mathcal{F}} q^{m - Q(\lambda)} |q|^{(p_+(\lambda), p_+(\lambda))} y^{1-s} \frac{dx\ dy}{y^2}.$$

The integral over a truncated domain ($\Im y < 1$) converges for all $s \in \mathbb{C}$. After subtracting this analytic term, we are left with

$$c(m) \int_1^\infty \int_0^1 q^{m - Q(\lambda)} |q|^{(p_+(\lambda), p_+(\lambda))} y^{1-s} \frac{dx\ dy}{y^2}.$$

The x-integral vanishes unless $m = Q(\lambda)$, so we shall make this assumption. Then the integral is

$$c(Q(\lambda)) \int_1^\infty e^{-2\pi(p_+(\lambda),p_+(\lambda))y} y^{-s-1} dy.$$

If $(p_+(\lambda), p_+(\lambda)) > 0$ then the integral converges for all s so there is no problem. We're left with the case that $p_+(\lambda) = 0$. If $\lambda = 0$, then this term is independent of p, and is simply a constant. For other terms, we see that Φ will have singularities on the following sets

$$H(m) = \{p \in Gr : \exists \lambda \in L,\ p_+(\lambda) = 0,\ Q(\lambda) = m\},$$

where $c(m) \neq 0$. These sets are the Heegner divisors introduced earlier.

We now investigate the behaviour of Φ near a Heegner divisor.

Exercise 18 *Show that for* $\epsilon > 0$ *we have* $\int_1^\infty e^{-\epsilon y} \frac{dy}{y} = -\log \epsilon +$ *analytic terms*

From the exercise, it follows that as p approaches a component $\lambda^\perp$ of a Heeger divisor, we have

$$\Phi(p) = -2c(Q(\lambda)) \log(p_+(\lambda), p_+(\lambda)) + \mathit{analytic},$$

if $n^+ = 2$. The factor 2 above comes from the fact that both λ and $-\lambda$ are lattice points.

Exercise 19 *Show that* $(p^+(\lambda), p^+(\lambda)) = \frac{|(\lambda,v)|^2}{(\Im v, \Im v)}$, *where* $(p^+, p^-) \in Gr_V$ *is identified with* $[v] \in \mathcal{K}^+$.

From the exercise, we see that as $[v]$ approaches $\lambda^\perp$ we have

$$\Phi([v]) = -4c(Q(\lambda)) \log \left| \frac{(\lambda, v)}{(\lambda_0, v)} \right| + \mathit{analytic},$$

This looks like $-4 \log |F| + \mathit{analytic}$, where F is a meromorphic function whose divisor is $\sum_{m<0} c(m) H(m)$.

Exercise 20 *Calculate the singularities of* Φ *for lattices with signature not equal to* $(2, n^-)$.

Exercise 21 *Show that* Φ *is real valued. (Hint: use the reflection of* $\mathcal{F}$ *in the line* $x = 0$.)

2.2 The Fourier expansion for lattices of signature (2,n)

We now regard Φ as a function on $\mathbb{H}$. It is clearly invariant under Γ_L, and hence under the map $w \mapsto w + \lambda$ for each $\lambda \in M$. We can therefore write it as a Fourier expansion:

$$\Phi(w) = \sum_{\lambda \in M} c(\lambda, \Im w) e(\lambda, \Re w).$$

The Fourier coefficients $c(\lambda, \Im w)$ can be written as integrals. To cut a long story short, this expansion has the following form

$$\begin{aligned}\Phi(w) &= Y \tilde{\Phi}(\tilde{p}) \\ &\quad + 2Y \sum_{\lambda \in M} \int_{\mathbb{Z}\backslash\mathcal{H}} f(\tau) \sum_{n=1}^\infty \exp\left(-\frac{\pi n^2 Y^2}{y}\right) e(n(\lambda, \Re w)) q^{-Q(\lambda)} |q|^{(\tilde{p}_+(\lambda), \tilde{p}_+(\lambda))} y^{1/2} \frac{dx\, dy}{y^2}.\end{aligned}$$

This needs some explanation.

- $Y = \sqrt{Q(\Im w)}$.

- The point $w \in \mathbb{H}$ corresponds to a point $p = (p^+, p^-)$ in Gr_V. recall that we have a decomposition $V = W \perp \langle \lambda_0, \lambda_1 \rangle$. The point $p \in Gr_V$ gives rise to a point $\tilde{p} = (\tilde{p}^+, \tilde{p}^-)$ in Gr_W. More precisely, $\tilde{p}_+$ is the projection onto the subspace spanned by $\Im w$.

- The function $\tilde{\Phi}(\tilde{p})$ is the integral of f against the theta function for the sublattice M of L. This is a function on Gr_W. Using a similar Fourier expansion of $\tilde{\Phi}$, we can show that the first term $Y\tilde{\Phi}(\tilde{p})$ is a continuous, piecewise linear function of $\Im w$.

- The integrals are over a fundamental domain for $\mathbb{Z}$ in $\mathcal{H}$, where $\mathbb{Z}$ acts by translation. The integral with $\lambda = 0$ needs to be regularized in the same way as before. We define this integral as the constant term at $s = 0$ of the meromorphic continuation of the following:
$$\lim_{Y \to \infty} \int_0^Y \int_0^1 (\cdots) y^{-s} \frac{dx\, dy}{y^2}.$$

The first term $Y\tilde{\Phi}(\tilde{p})$ is a continuous function of the form
$$(\rho([\Im w]), \Im w), \tag{3}$$
where ρ is a piecewise constant function on Gr_W taking values in $W \otimes \mathbb{C}$.

The term with $\lambda = 0$ in the sum above diverges:
$$\begin{aligned}
\int_{\mathbb{Z}\backslash\mathcal{H}} f(\tau) \sum_{n=1}^{\infty} \exp\left(-\frac{\pi n^2 Y^2}{y}\right) y^{1/2} \frac{dx\, dy}{y^2} &= c(0) \int_0^\infty \sum_{n=1}^{\infty} \exp\left(-\frac{\pi n^2 Y^2}{y}\right) y^{-1/2} \frac{dy}{y} \\
&= c(0) \sum_{n=1}^{\infty} \int_0^\infty \exp(-y) \left(\frac{\pi n^2 Y^2}{y}\right)^{-1/2} \frac{dy}{y} \\
&= const \cdot \sum n^{-1}.
\end{aligned}$$
However, after regularizing the integral we get $\frac{-c(0)\log(Y) + const}{Y}$, which gives a contribution to Φ of
$$-2c(0)\log(Y) + const. \tag{4}$$

Exercise 22 *Show that this regularized integral is* $\frac{-c(0)\log(Y)+const}{Y}$ *and find the constant.*

Now consider a term with $\lambda \neq 0$. After expanding $f = \sum c(m) q^m$, we get integrals of the following kind:
$$\int_0^\infty \int_0^1 q^{m - Q(\lambda)} \exp\left(-\frac{\pi n^2 Y^2}{y}\right) |q|^{(\tilde{p}_+(\lambda), \tilde{p}_+(\lambda))} y^{1/2} \frac{dx\, dy}{y^2}.$$
Again the integral vanishes unless $m = Q(\lambda)$. When this is the case, we get
$$\int_0^\infty \exp\left(-\frac{\pi n^2 Y^2}{y} - 2\pi(\tilde{p}_+(\lambda), \tilde{p}_+(\lambda)) y\right) y^{-1/2} \frac{dy}{y}.$$

Exercise 23 *Show that for $A, B > 0$,*

$$\int_0^\infty \exp\left(-\frac{A}{y} - By\right) y^{-1/2}\frac{dy}{y} = \sqrt{\frac{\pi}{A}} e^{-2\sqrt{AB}}.$$

(This can be done by standard integration techniques).

Exercise 24 *Show that* $\sqrt{2(\tilde{p}_+(\lambda)\tilde{p}_+(\lambda))} = \frac{|(\lambda,\Im w)|}{Y}$.

After doing this integral and multiplying by $2Yc(Q(\lambda))e(n(\lambda,\Re w))$, we get the following contribution to Φ:

$$2c(Q(\lambda))\frac{1}{n}e\Big(\ ((\lambda,\Re w) + i|(\lambda,\Im w)|)\ \Big)^n.$$

Summing over n we get

$$-2c(Q(\lambda))\log\Big(1 - e\left((\lambda,\Re w) + i|(\lambda,\Im w)|\right)\Big).$$

Taking this term together with the term for $-\lambda$, we get

$$-4c(Q(\lambda))\log\left|1 - \exp\left(2\pi i(\lambda_+, w)\right)\right|, \tag{5}$$

where λ_+ is $\pm\lambda$ and has positive inner product with $\Im w$.

Adding up all the terms, we get the following:

$$\begin{aligned}\Phi(w) &= (\rho([\Im w]),\Im w) - 2c(0)\log(Y)\\ &\quad -4\sum_{\tilde{\lambda}\in M,\ (\lambda,\Im w)>0} c(Q(\lambda))\log\Big|1-\exp\left(2\pi i(\lambda,w)\right)\Big|\end{aligned}$$

The right hand side is locally away from the singularities of the form $-4\log|\Psi(w)| - 2c(0)\log(Y)$ where Ψ is holomorphic. Furthermore, we've calculated the singularities of Φ. Near $\lambda^\perp$ the singularity is

$$-2c(Q(\lambda))\log(p^+(\lambda), p^+(\lambda)).$$

A short calculation shows that as a function of $[v] \in \mathcal{K}$, this is

$$-4c(Q(\lambda))\log\left|\frac{(\lambda,v)}{(\lambda_0,v)}\right| + \textit{analytic terms}.$$

It follows that there is a meromorphic function Ψ on all of $\mathbb{H}$ such that

$$\Phi(w) = -4\log|\Psi(w)| - 2c(0)\log(Y).$$

We can read off the divisor of Ψ from the singularities of Φ. For $[\Im w]$ near $[w_0]$, we can use the formula

$$\Psi(w) = e^{i(\rho(w_0),w)/4}\prod_{\lambda\in M,\ (\lambda,w_0)>0}\left(1-e^{2\pi i(\lambda,w)}\right)^{c(Q(\lambda))}.$$

To show that Ψ is a meromorphic modular form, we use the following.

Exercise 25 *Let F be a meromorphic modular form on $\mathbb{H}$ of weight k and level Γ. Show that $|F(w)|Y^k$ is Γ-invariant.*
Conversely, let F be a meromorphic function on $\mathbb{H}$ such that $|F(w)|Y^k$ is Γ-invariant. Show that F is a weight k, level Γ meromorphic modular form.

Recall that for $\lambda \in M$, the translations $\alpha(w) = w + \lambda$ are in Γ_L. It follows that

$$\chi(\alpha) = \exp(i(\rho(w_0), \lambda)/4).$$

Since these numbers must all be roots of unity, we find that $i\rho(w_0) \in M \otimes \mathbb{Q}$.

3 Statement of results for more general lattices

Given an even unimodular lattice L of signature $(2, n^-)$, we've seen that we have a lift

$$\{ \text{ weakly holomorphic forms on } \mathrm{SL}_2(\mathbb{Z}) \text{ } \} \to \{ \text{ meromorphic forms on } \Gamma_L \text{ } \}.$$

In this lecture I'll describe how this theory can be generalized to remove the "unimodular" condition and also to deal with weakly holomorphic forms of higher level. In fact, rather than increasing the level, it will be more convenient for us to consider vector valued modular forms.

3.1 The metaplectic group

We'll write $Mp_2(\mathbb{R})$ for the metaplectic group. That is the group of pairs (M, ϕ), where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathcal{H} \to \mathbb{C}^\times$ is a continuous function such that $\phi(\tau)^2 = c\tau + d$. Multiplication in this group is given by

$$(M_1, \phi_1)(M_2, \phi_2) = (M_1 M_2, (\phi_1 \circ M_2) \cdot \phi_2).$$

If we write $\sqrt{z}$ for a complex number z, then the square root will be chosen so that $\arg \sqrt{z} \in (-\pi/2, \pi/2]$. The group $Mp_2(\mathbb{R})$ is a connected Lie group, but it is not the group of real points in an algebraic group. In fact, and homomorphism $Mp_2(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ will have the element $(I_2, -1)$ in its kernel.

There is an obvious surjective homomorphism $Mp_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. The kernel has 2 elements, and is in the centre of $Mp_2(\mathbb{R})$. We'll write $Mp_2(\mathbb{Z})$ for the preimage of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{R})$. The group $Mp_2(\mathbb{Z})$ is generated by the following elements:

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

Exercise 26 *Show that the centre of $Mp_2(\mathbb{R})$ is a cyclic group of order 4, generated by $Z = (-I_2, i)$. Show that $S^2 = Z$, and hence show that S and T generate $Mp_2(\mathbb{Z})$.*

Exercise 27 *Show that $Mp_2(\mathbb{R})$ is connected. (Hint: show that $Mp_2(\mathbb{R})$ is not a split extension of $\mathrm{SL}_2(\mathbb{R})$ by considering the preimage of the centre of $\mathrm{SL}_2(\mathbb{R})$.)*

3.2 Vector valued forms

Let L be a lattice with a non-degenerate symmetric bilinear form $(-,-) : L \times L \to \mathbb{Z}$, such that (v, v) is even for all $v \in L$. As before, we let $V = L \otimes \mathbb{R}$ and we let (n^+, n^-) be the signature. We shall write L' for the dual lattice:

$$L' = \{v \in V : (v, L) \subset \mathbb{Z}\}.$$

(We no longer assume that L is unimodular). The quotient L'/L is finite, and we write $\mathbb{C}[L'/L]$ for the group algebra of this finite group. For $\delta \in L'/L$, we let $[\delta]$ be the corresponding element of the group algebra. There is an action of $Mp_2(\mathbb{Z})$ on the vector space $\mathbb{C}[L'/L]$ defined as follows:

$$\varrho(T)[\delta] = e^{2\pi i Q(\delta)}[\delta], \qquad \varrho(S)[\delta] = \frac{\sqrt{i}^{n^- - n^+}}{\sqrt{|L'/L|}} \sum_{\gamma \in L'/L} e^{-2\pi i(\gamma,\delta)}[\gamma].$$

Exercise 28 *Show that $\varrho(Z)[\delta] = i^{n^- - n^+}[-\delta]$. Hence show that if L has even rank then ϱ reduces to a representation of* $\mathrm{SL}_2(\mathbb{Z})$.

If we regard $\mathbb{C}[L'/L]$ as a Hilbert space, in which the elements $[\delta]$ form an orthonormal basis, then the representation ϱ is unitary. We shall write ϱ^* for the contragredient representation. In matrix terms, with respect to the orthonormal basis, this is just the complex conjugate.

Exercise 29 *Show that ϱ is unitary.*

Exercise 30 *Show that if L is unimodular then ϱ is the trivial representation.*

Let $k \in \frac{1}{2}\mathbb{Z}$ with $k \equiv \frac{\mathrm{rank} L}{2} \bmod \mathbb{Z}$. A modular form of weight k, with values in $\mathbb{C}[L'/L]$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}[L'/L]$, such that

- For all $(M, \phi) \in Mp_2(\mathbb{Z})$,

$$f(M\tau) = \phi(\tau)^{2k} \varrho(M, \phi) f(\tau).$$

- f is "holomorphic at the cusp", i.e. bounded on $\{\tau \in \mathcal{H} : \Im\tau > 1\}$.

Any such function has a Fourier expansion as follows:

$$f(\tau) = \sum_{\delta \in L'/L} \sum_{m \in \mathbb{Z} - Q(\delta)} c(\delta, m) e^{2\pi i m\tau}[\delta],$$

with $c(\delta, m) = 0$ whenever $m < 0$. If $c(\delta, 0) = 0$ for all δ then f is called a cusp form. If we allow finitely many coefficients to be non-zero with $m < 0$ then f is called a weakly holomorphic form.

3.3 Borcherds products

Let L be a lattice of signature $(2, n^-)$ with $n^- \geq 3$. In such an L there is always a primitive vector λ_0 with $Q(\lambda_0) = 0$. We may then choose $\lambda_1 \in L'$ such that $(\lambda_1, \lambda_0) = 1$. We let M be the orthogonal complement of $\langle \lambda_0, \lambda_1 \rangle$. Let $(\lambda_0, L) = N\mathbb{Z}$ with $N > 0$. Choose $\zeta \in L$ with $(\zeta, \lambda_0) = N$. We have a unique decomposition

$$\zeta = \zeta_M + N\lambda_1 + B\lambda_0, \quad \zeta_M \in M', \quad B \in \mathbb{Q}.$$

Define a sublattice:

$$L_0' = \{\lambda \in L' : (\lambda, \lambda_0) \equiv 0 \bmod N\}.$$

We have a map $p : L_0' \to M'$ given by

$$(p(\lambda), \mu) = (\lambda, \mu) - \frac{(\lambda, \lambda_0)}{N}(\zeta_M, \mu), \quad \mu \in M, \quad \lambda \in L_0'.$$

Let f be a weakly holomorphic modular form with values in $\mathbb{C}[L'/L]$ and weight $1 - n^-/2$ and assume that the coefficients $c(\delta, n)$ are in $\mathbb{Z}$ for all $n < 0$. To such an f, we define the corresponding Borcherds product:

$$\Psi(v) = e^{2\pi i(\rho,w)} \prod_{\lambda \in M',\, (w_0,\lambda)>0} \quad \prod_{\delta \in L_0'/L,\, p(\delta)=\lambda+M} \left(1 - e^{2\pi i((\delta,\lambda_1)+(\lambda,w))}\right)^{c(\delta,Q(\lambda))}$$

Theorem 3 *There is a unique choice of $\rho \in M \otimes \mathbb{Q}$, depending additively of f, such that the following are true:*

- *Ψ has a meromorphic continuation to $\mathbb{H}$.*
- *The divisor of Ψ is*

$$\sum_{\lambda \in L'/\{\pm 1\},\, Q(\lambda)<0} c(\lambda, Q(\lambda)) \cdot \lambda^\perp.$$

- *Ψ is a meromorphic modular form of weight $c(0,0)/2$ and level $\Gamma(L, L')$, where*

$$\Gamma(L, L') = \{\alpha \in \Gamma_L : \forall \lambda \in L',\ \alpha\lambda - \lambda \in L\}.$$

3.4 The vector valued theta function

As before, the theorem is proved using the theta correspondence. In this case, we use a vector valued theta function:

$$\Theta_L(\tau, p) = \sum_{\lambda \in L'} e\Big((p^+(\lambda), p^+(\lambda))\tau + (p^-(\lambda), p^-(\lambda))\overline{\tau}\Big) \cdot [\lambda].$$

This is a function $\mathcal{H} \times Gr_V \to \mathbb{C}[L'/L]$. The theta function is invariant under the action of $\Gamma(L, L')$ on Gr_V. Under the action of $Mp_2(\mathbb{Z})$ on τ it has the following behaviour:

$$\Theta_L(\gamma\tau, p) = \phi(\tau)^{n^+}\overline{\phi(\tau)}^{n^-} \varrho(\gamma, \phi)\Theta_L(\tau, v).$$

Using this, we can show that the following function

$$\langle f(\tau), \Theta_L(\tau, p)\rangle \cdot y^{n^+/2}$$

is Γ-invariant. Here f denotes a vector values weakly holomorphic form of weight $(n^+ - n^-)/2$. We may therefore define the theta transform of f:

$$\Phi(p) = \int_{\Gamma\backslash\mathcal{H}} \langle f(\tau), \Theta_L(\tau, p)\rangle \cdot y^{n^+/2}\frac{dx\ dy}{y^2}.$$

Again this diverges, but we can make sense of it in the same way, and we find that

$$\Phi(p) = -4\log|\Psi(p)| - 2c(0,0)\log(Y) + const.$$

Again it follows that Ψ is a weight $c(0,0)/2$ modular form.

References

[1] R. E. Borcherds. Automorphic forms on $O_{s+2,2}(\mathbb{R})^+$ and generalized Kac–Moody algebras. In *Proceedings of the International Congress of Mathematicians, Vol 1, 2 (Zürich, 1994)*, Basel, 1995. Birkhäuser.

[2] R. E. Borcherds. Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products. *Invent. math.*, 120:161–213, 1995.

[3] R. E. Borcherds. Automorphic forms with singularities on grassmannians. *Invent. math.*, 132:491–562, 1998.

[4] J. H. Bruinier. *Borcherds Products on $O(2, l)$ and Chern Classes of Heegner Divisors.* Lecture Notes in Mathematics. Springer Verlag, 2001.

[5] M. Kontsevich. Product formulas for modular forms on $O(2, n)$ [after R. Borcherds]. *Séminaire Bourbaki*, (821):1–16, November 1996.

[6] J. P. Serre. *A course in arithmetic*, Springer-Verlag, New York, 1973.