Generalized Solovay Measures, the HOD Analysis, and the Core Model Induction

by

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Abstract

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This thesis belongs to the field of descriptive inner model theory. Chapter 1 provides a proper context for this thesis and gives a brief introduction to the theory of AD^+, the theory of hod mice, and a definition of $K^J(\mathbb{R})$. In Chapter 2, we explore the theory of generalized Solovay measures. We prove structure theorems concerning canonical models of the theory "AD^+ + there is a generalized Solovay measure" and compute the exact consistency strength of this theory. We also give some applications relating generalized Solovay measures to the determinacy of a class of long games. In Chapter 3, we give a HOD analysis of $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ models below "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular." This is an application of the theory of hod mice developed in [23]. We also analyze HOD of AD^+-models of the form $V = L(\mathbb{R}, \mu)$ where μ is a generalized Solovay measure. In Chapter 4, we develop techniques for the core model induction. We use this to prove a characterization of AD^+ in models of the form $V = L(\mathbb{R}, \mu)$, where μ is a generalized Solovay measure. Using this framework, we also can construct models of "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" from the theory "$\mathsf{ZF} + \mathsf{DC} + \Theta$ is regular + ω_1 is $\mathcal{P}(\mathbb{R})$-supercompact". In fact, we succeed in going further, namely we can construct a model of "$\mathsf{AD}_\mathbb{R} + \Theta$ is measurable" and show that this is in fact, an equiconsistency.

To my parents

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Acknowledgments

I would firstly like to thank my parents for the sacrifice they've made in order to give me and my brother a better future in the U.S. I would also like to thank my aunts, uncles, my grandparents in Vietnam who raised me, and my relatives in the U.S. who helped and supported my family during the first few difficult years in a foreign country.

I would like to thank Grigor Sargsyan, who developed the theory of hod mice, for mentoring me. A large part of this thesis comes from countless conversations with him. I would also like to thank my advisor, John Steel, for many discussions about descriptive inner model theory and for patiently correcting countless mistakes I've made over the years. The big picture about descriptive set theory, inner model theory, and their connection that I've gained comes from those conversations. I would like to thank Hugh Woodin for developing the beautiful theory of AD^+ and the core model induction technique that most of this thesis revolves about and for sharing with me his deep insight and knowledge in set theory. Without him, much of this thesis wouldn't exist. Lastly, I thank Martin Zeman for introducing me to set theory.

Chapter 1

Introduction

In this chapter, we briefly discuss the the general subject of descriptive inner model theory, which provides the context for this thesis; we then summarize basic definitions and facts from the theory of AD^+, the theory of hod mice that we'll need in this thesis, and a definition of $K^J(\mathbb{R})$ for certain mouse operators J.

Descriptive inner model theory (DIMT) is a crossroad between pure descriptive set theory (DST) and inner model theory (IMT) and as such it uses tools from both fields to study and deepen the connection between canonical models of large cardinals and canonical models of determinacy. The main results of this thesis are theorems of descriptive inner model theory.

The first topic this thesis is concerned with is the study of a class of measures called *generalized Solovay measures* (defined in Chapter 2). In [28], Solovay defines a normal fine measure μ_0 on $\mathcal{P}_{\omega_1}(\mathbb{R})$ from $\mathsf{AD}_{\mathbb{R}}$. Martin and Woodin independently prove that determinacy of real games of fixed countable length follows from $\mathsf{AD}_{\mathbb{R}}$ and define a hierarchy of normal fine measures $\langle \mu_\alpha \mid \alpha < \omega_1 \rangle$, where each μ_α is on the set of increasing and continuous funtions from ω^α into $\mathcal{P}_{\omega_1}(\mathbb{R})$ (this set is denoted X_α in Chapter 2). They also define the so-called "ultimate measure" μ_{ω_1} on increasing and continuous functions from α ($\alpha < \omega_1$) into $\mathcal{P}_{\omega_1}(\mathbb{R})$ (this set is denoted X_{ω_1} in Chapter 2) and prove the existence of μ_{ω_1} from $\mathsf{AD}_{\mathbb{R}}$, though not from determinacy of long games. The obvious question that arises is whether the consitency of the theory $(\mathsf{T}_\alpha) \equiv$ "AD + there is a normal fine measure on X_α" (for $\alpha \leq \omega_1$) implies the consistency of the theory $\mathsf{AD}_{\mathbb{R}}$. The answer is "no" and this follows from work of Solovay [28]. Woodin shows furthermore that (T_0) is equiconsistent with "ZFC + there are ω^2 Woodin cardinals", which in turns is much weaker in consistency strength than $\mathsf{AD}_{\mathbb{R}}$. This is the original motivation of this investigation of generalized Solovay measures.

Generalizing Woodin's above result, Chapter 2 computes the exact consistency strength of "AD + there is a normal fine measure on X_α" for all $\alpha > 0$ and shows that these theories are much weaker than $\mathsf{AD}_{\mathbb{R}}$ consistency-wise. Chapter 2 also contains various other results concerning structure theory of AD^+-models of the form $V = L(\mathbb{R}, \mu_\alpha)$, where μ_α is a normal fine measure on X_α (for $\alpha \leq \omega_1$) and its applications.

The second topic of this thesis is the HOD analysis. The HOD analysis is an integral part of descriptive inner model theory as it is the key ingredient in the proof of *the Mouse Set Conjecture* (MSC), which is an important conjecture that provides a connection between canonical models of large cardinals and canonical models of determinacy. We recall a bit of history on the computation of HOD. Under AD, Solovay shows that HOD $\vDash \kappa$ is measurable where $\kappa = \omega_1^V$. This suggests that HOD of canonical models of determinacy (like $L(\mathbb{R})$) is a model of large cardinals. Martin and Steel in [16] essentially show that $\mathrm{HOD}^{L(\mathbb{R})} \vDash \mathsf{CH}$. The methods used to prove the results above are purely descriptive set theoretic. Then Steel, (in [42] or [37]) using inner model theory, shows V_Θ^{HOD} is a fine-structural premouse, which in particular implies $V_\Theta^{\mathrm{HOD}} \vDash \mathsf{GCH}$. Woodin (see [31]), building on Steel's work, completes the full HOD analysis in $L(\mathbb{R})$ and shows HOD $\vDash \mathsf{GCH} + \Theta$ is Woodin and furthermore shows that the full HOD of $L(\mathbb{R})$ is a hybrid mouse that contains some information about a certain iteration strategy of its initial segments. A key fact used in the computation of HOD in $L(\mathbb{R})$ is that if $L(\mathbb{R}) \vDash \mathsf{AD}$ then $L(\mathbb{R}) \vDash \mathsf{MC}$[1]. It's natural to ask whether analogous results hold in the context of $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Recently, Grigor Sargsyan in [23], assuming $V = L(\mathcal{P}(\mathbb{R}))$ and there is no models of "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" (we call this smallness assumption $(*)$ for now), proves Strong Mouse Capturing (SMC) (a generalization of MC) and computes V_Θ^{HOD} for Θ being limit in the Solovay sequence and $V_{\theta_\alpha}^{\mathrm{HOD}}$ for $\Theta = \theta_{\alpha+1}$ in a similar sense as above.

Chapter 3 extends Sargsyan's work to the computation of full HOD under $(*)$. This analysis heavily uses the theory of hod mice developed by Sargsyan in [23]. Chapter 3 also computes full HOD of AD^+-models of the form $L(\mathbb{R}, \mu_\alpha)$ as part of the analysis of the structure theory of these models. This is used to prove (among other things) $\mathsf{AD}_{\mathbb{R},\omega^\alpha}\undertilde{\Pi}^1_1$ implies (and hence is equivalent to) $\mathsf{AD}_{\mathbb{R},\omega^\alpha}$ $<$-ω^2-$\undertilde{\Pi}^1_1$ for $1 \leq \alpha < \omega_1$ (I believe the case $\alpha = 1$ has been known before).

The last topic of this thesis concerns *the core model induction* (CMI). CMI is a powerful technique of descriptive inner model theory pioneered by Woodin and further developed by Steel, Schindler, and others. It draws strength from natural theories such as PFA to inductively construct canonical models of determinacy and large cardinals in a locked-step process. This thesis develops methods for the core model induction to solve a variety of problems. The first of which is a characterization of determinacy in models of the form $L(\mathbb{R}, \mu_\alpha)$ $(\alpha < \omega_1)$. I show that $L(\mathbb{R}, \mu_\alpha) \vDash \mathsf{AD}$ if and only if $L(\mathbb{R}, \mu_\alpha) \vDash \Theta > \omega_2$ (see Section 4.2). Another major application of the core model induction in this thesis is the proof of the equiconsistency of the theories: "$\mathsf{ZF} + \mathsf{DC} + \Theta$ is regular $+ \omega_1$ is $\mathcal{P}(\mathbb{R})$-supercompact" and "$\mathsf{AD}_\mathbb{R} + \Theta$ is measurable" (see Section 4.3).

[1] MC stands for Mouse Capturing, which is the statement that if $x, y \in \mathbb{R}$, then $x \in OD(y) \Leftrightarrow x$ is in a mouse over y.

1.1 AD^+

We start with the definition of Woodin's theory of AD^+. In this thesis, we identify $\mathbb{R}$ with ω^ω. We use Θ to denote the sup of ordinals α such that there is a surjection $\pi : \mathbb{R} \to \alpha$.

Definition 1.1.1. *AD^+ is the theory $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ and*

1. *for every set of reals A, there are a set of ordinals S and a formula φ such that $x \in A \Leftrightarrow L[S,x] \vDash \varphi[S,x]$. (S,φ) is called an ∞-Borel code for A;*

2. *for every $\lambda < \Theta$, for every continuous $\pi : \lambda^\omega \to \omega^\omega$, for every $A \subseteq \mathbb{R}$, the set $\pi^{-1}[A]$ is determined.*

AD^+ is arguably the right structural strengthening of AD. In fact, AD^+ is equivalent to "AD + the set of Suslin cardinals is closed" (see [12]). Another, perhaps more useful, equivalence of AD^+ is "$\mathsf{AD} + \Sigma_1$ statements reflect to Suslin-co-Suslin" (see [40] for a more precise statement).

Recall that Θ is defined to be the supremum of α such that there is a surjection from $\mathbb{R}$ onto α. Under AC, Θ is just $\mathfrak{c}^+$. In the context of AD, Θ is shown to be the supremum of $w(A)$[2] for $A \subseteq \mathbb{R}$. Let $A \subseteq \mathbb{R}$, we let θ_A be the supremum of all α such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto α.

Definition 1.1.2 (AD^+). *The **Solovay sequence** is the sequence $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ where*

1. *θ_0 is the sup of ordinals β such that there is an OD surjection from $\mathbb{R}$ onto β;*

2. *if $\alpha > 0$ is limit, then $\theta_\alpha = \sup\{\theta_\beta \mid \beta < \alpha\}$;*

3. *if $\alpha = \beta + 1$ and $\theta_\beta < \Theta$ (i.e. $\beta < \Omega$), fixing a set $A \subseteq \mathbb{R}$ of Wadge rank θ_β, θ_α is the sup of ordinals γ such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto γ, i.e. $\theta_\alpha = \theta_A$.*

Note that the definition of θ_α for $\alpha = \beta + 1$ in Definition 1.1.2 does not depend on the choice of A. We recall some basic notions from descriptive set theory.

Suppose $A \subseteq \mathbb{R}$ and (N, Σ) is such that N is a transitive model of "$\mathsf{ZFC} - $ *Replacement*" and Σ is an (ω_1, ω_1)-iteration strategy or just ω_1-iteration strategy for N. We use $o(N)$, OR^N, ORD^N interchangably to denote the ordinal height of N. Suppose that δ is countable in V but is an uncountable cardinal of N and suppose that $T, U \in N$ are trees on $\omega \times (\delta^+)^N$. We say (T, U) *locally Suslin captures A at δ* over N if for any $\alpha \leq \delta$ and for N-generic $g \subseteq Coll(\omega, \alpha)$,

$$A \cap N[g] = p[T]^{N[g]} = \mathbb{R}^{N[g]} \backslash p[U]^{N[g]}.$$

[2] $w(A)$ is the Wadge rank of A. We will use either $w(A)$ or $|A|_w$ to denote the Wadge rank of A.

We also say that N locally Suslin captures A at δ. We say that N locally captures A if N locally captures A at any uncountable cardinal of N. We say (N, Σ) *Suslin captures* A at δ, or (N, δ, Σ) *Suslin captures* A, if there are trees $T, U \in N$ on $\omega \times (\delta^+)^N$ such that whenever $i : N \to M$ comes from an iteration via Σ, $(i(T), i(U))$ locally Suslin captures A over M at $i(\delta)$. In this case we also say that $(N, \delta, \Sigma, T, U)$ Suslin captures A. We say (N, Σ) Suslin captures A if for every countable δ which is an uncountable cardinal of N, (N, Σ) Suslin captures A at δ. When δ is Woodin in N, one can perform genericity iterations on N to make various objects generic over an iterate of N. This is where the concept of Suslin capturing becomes interesting and useful. We'll exploit this fact on several occasions.

Definition 1.1.3. *Γ is a* ***good pointclass*** *if it is closed under recursive preimages, closed under $\exists^{\mathbb{R}}$, is ω-parametrized, and has the scale property. Furthermore, if Γ is closed under $\forall^{\mathbb{R}}$, then we say that Γ is* ***inductive-like****.*

Under AD^+, Σ^1_2, Σ^2_1 are examples of good poinclasses. If Γ is a good pointclass, we say (N, Σ) Suslin captures Γ if it Suslin captures every $A \in \Gamma$. The following are two important structure theorems of AD^+ that are used at many places throughout this thesis.

Theorem 1.1.4 (Woodin, Theorem 10.3 of [35])**.** *Assume AD^+ and suppose Γ is a good pointclass and is not the last good pointclass. There is then a function F defined on $\mathbb{R}$ such that for a Turing cone of x, $F(x) = \langle \mathcal{N}^*_x, \mathcal{M}_x, \delta_x, \Sigma_x\rangle$ such that*

1. *$\mathcal{N}^*_x|\delta_x = \mathcal{M}_x|\delta_x$,*
2. *$\mathcal{N}^*_x \vDash$ "$\mathsf{ZF} + \delta_x$ is the only Woodin cardinal",*
3. *Σ_x is the unique iteration strategy of $\mathcal{M}_x$,*
4. *$\mathcal{N}^*_x = L(\mathcal{M}_x, \Lambda)$ where Λ is the restriction of Σ_x to stacks $\vec{\mathcal{T}} \in \mathcal{M}_x$ that have finite length and are based on $\mathcal{M}_x \restriction \delta_x$,*
5. *$(\mathcal{N}^*_x, \Sigma_x)$ Suslin captures Γ,*
6. *for any $\alpha < \delta_x$ and for any $\mathcal{N}^*_x$-generic $g \subseteq Coll(\omega, \alpha)$, $(\mathcal{N}^*_x[g], \Sigma_x)$ Suslin captures $Code((\Sigma_x)_{\mathcal{M}_x \restriction \alpha})$ and its complement at δ_x^+.*

Theorem 1.1.5 (Woodin, unpublished but see [40])**.** *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose A is a set of reals such that there is a Suslin cardinal in the interval $(w(A), \theta_A)$. Then*

1. *The pointclass $\utilde{\Sigma}^2_1(A)$ has the scale property.*
2. *$M_{\utilde{\Delta}^2_1(A)} \prec_{\Sigma_1} L(\mathcal{P}(\mathbb{R}))$.*
3. *$L_\Theta(\mathcal{P}(\mathbb{R})) \prec_{\Sigma_1} L(\mathcal{P}(\mathbb{R}))$.*

Finally, we quote another theorem of Woodin, which will be key in our HOD analysis.

Theorem 1.1.6 (Woodin, see [13])**.** *Assume AD^+. Let $\langle \theta_\alpha \mid \alpha \leq \Omega\rangle$ be the Solovay sequence. Suppose $\alpha = 0$ or $\alpha = \beta + 1$ for some $\beta < \Omega$. Then $HOD \vDash \theta_\alpha$ is Woodin.*

1.2 Hod Mice

In this subsection, we summarize some definitions and facts about hod mice that will be used in our computation. For basic definitions and notations that we omit, see [23]. The formal definition of a hod premouse $\mathcal{P}$ is given in Definition 2.12 of [23]. Let us mention some basic first-order properties of $\mathcal{P}$. There are an ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle(\mathcal{P}(\alpha), \Sigma_\alpha^{\mathcal{P}}) \mid \alpha < \lambda^{\mathcal{P}}\rangle$ and $\langle\delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ such that

1. $\langle\delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \vDash \delta_\alpha^{\mathcal{P}}$ is Woodin;
2. $\mathcal{P}(0) = Lp_\omega(\mathcal{P}|\delta_0)^{\mathcal{P}}$; for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1) = (Lp_\omega^{\Sigma_\alpha^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp_\omega^{\oplus_{\beta<\alpha}\Sigma_\beta^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$;
3. $\mathcal{P} \vDash \Sigma_\alpha^{\mathcal{P}}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))^3$-strategy for $\mathcal{P}(\alpha)$ with hull condensation;
4. if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma_\beta^{\mathcal{P}}$ extends $\Sigma_\alpha^{\mathcal{P}}$.

We will write $\delta^{\mathcal{P}}$ for $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}} = \oplus_{\beta<\lambda^{\mathcal{P}}}\Sigma_{\beta+1}^{\mathcal{P}}$.

Definition 1.2.1. *$(\mathcal{P}, \Sigma)$ is a hod pair if $\mathcal{P}$ is a countable hod premouse and Σ is a $(\omega, \omega_1, \omega_1)$ iteration strategy for $\mathcal{P}$ with hull condensation such that $\Sigma^{\mathcal{P}} \subseteq \Sigma$ and this fact is preserved by Σ-iterations.*

Hod pairs typically arise in AD^+-models, where ω_1-iterability implies $\omega_1 + 1$-iterability. In practice, we work with hod pairs $(\mathcal{P}, \Sigma)$ such that Σ also has branch condensation.

Theorem 1.2.2 (Sargsyan)**.** *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation. Then Σ is pullback consistent, positional and commuting.*

The proof of Theorem 1.2.2 can be found in [23]. Such hod pairs are particularly important for our computation as they are points in the direct limit system giving rise to HOD. For hod pairs $(\mathcal{M}_\Sigma, \Sigma)$, if Σ is a strategy with branch condensation and $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}_\Sigma$ with last model $\mathcal{N}$, $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ is independent of $\vec{\mathcal{T}}$. Therefore, later on we will omit the subscript $\vec{\mathcal{T}}$ from $\Sigma_{N,\vec{\mathcal{T}}}$ whenever Σ is a strategy with branch condensation and $\mathcal{M}_\Sigma$ is a hod mouse.

Definition 1.2.3. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ if there is $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$.*

If $\mathcal{P}$ and $\mathcal{Q}$ are hod premice such that $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ then we say $\mathcal{P}$ is a hod initial segment of $\mathcal{Q}$. If $(\mathcal{P}, \Sigma)$ is a hod pair, and $\mathcal{Q} \trianglelefteq_{hod} \mathcal{P}$, say $\mathcal{Q} = \mathcal{P}(\alpha)$, then we let $\Sigma_{\mathcal{Q}}$ be the strategy of $\mathcal{Q}$ given by Σ. Note that $\Sigma_{\mathcal{Q}} \cap \mathcal{P} = \Sigma_\alpha^{\mathcal{P}} \in \mathcal{P}$.

[3] This just means $\Sigma_\alpha^{\mathcal{P}}$ acts on all stacks of ω-maximal, normal trees in $\mathcal{P}$.

All hod pairs $(\mathcal{P}, \Sigma)$ have the property that Σ has hull condensation and therefore, mice relative to Σ make sense. To state the Strong Mouse Capturing we need to introduce the notion of Γ-fullness preservation. We fix some reasonable coding (we call Code) of $(\omega, \omega_1, \omega_1)$-strategies by sets of reals. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair. Let $I(\mathcal{P}, \Sigma)$ be the set $(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \vec{\mathcal{T}})$ such that $\vec{\mathcal{T}}$ is according to Σ such that $i^{\vec{\mathcal{T}}}$ exists and $\mathcal{Q}$ is the end model of $\vec{\mathcal{T}}$ and $\Sigma_{\mathcal{Q}}$ is the $\vec{\mathcal{T}}$-tail of Σ. Let $B(\mathcal{P}, \Sigma)$ be the set $(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \vec{\mathcal{T}})$ such that there is some $\mathcal{R}$ such that $\mathcal{Q} = \mathcal{R}(\alpha)$, $\Sigma_{\mathcal{Q}} = \Sigma_{\mathcal{R}(\alpha)}$ for some $\alpha < \lambda^{\mathcal{R}}$ and $(\mathcal{R}, \Sigma_{\mathcal{R}}, \vec{\mathcal{T}}) \in I(\mathcal{P}, \Sigma)$.

Definition 1.2.4. *Suppose Σ is an iteration strategy with hull-condensation, a is a countable transitive set such that $\mathcal{M}_\Sigma \in a$[4] and Γ is a pointclass closed under boolean operations and continuous images and preimages. Then $Lp^{\Gamma,\Sigma}_{\omega_1}(a) = \cup_{\alpha<\omega_1} Lp^{\Gamma,\Sigma}_{\alpha}(a)$ where*

1. *$Lp^{\Gamma,\Sigma}_0(a) = a \cup \{a\}$*
2. *$Lp^{\Gamma,\Sigma}_{\alpha+1}(a) = \cup\{\mathcal{M} : \mathcal{M}$ is a sound Σ-mouse over $Lp^{\Gamma,\Sigma}_{\alpha}(a)$[5] projecting to $Lp^{\Gamma,\Sigma}_{\alpha}(a)$ and having an iteration strategy in $\Gamma\}$.*
3. *$Lp^{\Gamma,\Sigma}_{\lambda}(a) = \cup_{\alpha<\lambda} Lp^{\Gamma,\Sigma}_{\alpha}(a)$ for limit λ.*

We let $Lp^{\Gamma,\Sigma}(a) = Lp^{\Gamma,\Sigma}_1(a)$.

Definition 1.2.5 (Γ-Fullness preservation)**.** *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair and Γ is a pointclass closed under boolean operations and continuous images and preimages. Then Σ is a Γ-fullness preserving if whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha + 1 \leq \lambda^{\mathcal{Q}}$ and $\eta > \delta_\alpha$ is a strong cutpoint of $\mathcal{Q}(\alpha+1)$, then*

$$\mathcal{Q}|(\eta^+)^{\mathcal{Q}(\alpha+1)} = Lp^{\Gamma, \Sigma_{\mathcal{Q}(\alpha),\vec{\mathcal{T}}}}(\mathcal{Q}|\eta).$$

and

$$\mathcal{Q}|(\delta_\alpha^+)^{\mathcal{Q}} = Lp^{\Gamma, \oplus_{\beta<\alpha}\Sigma_{\mathcal{Q}(\beta+1),\vec{\mathcal{T}}}}(\mathcal{Q}|\delta_\alpha^{\mathcal{Q}}).$$

When $\Gamma = \mathcal{P}(\mathbb{R})$, we simply say fullness preservation. A stronger notion of Γ-fullness preservation is super Γ-fullness preservation. Similarly, when $\Gamma = \mathcal{P}(\mathbb{R})$, we simply say super fullness preservation.

Definition 1.2.6 (Super Γ-fullness preserving)**.** *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair and Γ is a pointclass closed under boolean operations and continuous images and preimages. Σ is super Γ-fullness preserving if it is Γ-fullness preserving and whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha < \lambda^{\mathcal{Q}}$ and $x \in HC$ is generic over $\mathcal{Q}$, then*

$$Lp^{\Gamma,\Sigma_{\mathcal{Q}(\alpha)}}(x) = \{\mathcal{M} \mid \mathcal{Q}[x] \models \text{“}\mathcal{M} \textit{ is a sound } \Sigma_{\mathcal{Q}(\alpha)}\textit{-mouse over } x \textit{ and } \rho_\omega(\mathcal{M}) = x\text{”}\}.$$

[4] $\mathcal{M}_\Sigma$ is the structure that Σ-iterates.

[5] By this we mean $\mathcal{M}$ has a unique $(\omega, \omega_1 + 1)$-iteration strategy Λ above $Lp^{\Gamma,\Sigma}_{\alpha}(a)$ such that whenever $\mathcal{N}$ is a Λ-iterate of $\mathcal{M}$, then $\mathcal{N}$ is a Σ-premouse.

Moreover, for such an $\mathcal{M}$ as above, letting Λ be the unique strategy for $\mathcal{M}$, then for any cardinal κ of $\mathcal{Q}[x]$, $\Lambda \restriction H_\kappa^{\mathcal{Q}[x]} \in \mathcal{Q}[x]$.

Hod mice that go into the direct limit system that gives rise to HOD have strategies that are super fullness preserving. Here is the statement of the strong mouse capturing.

Definition 1.2.7 (The Strong Mouse Capturing)**.** *The Strong Mouse Capturing (SMC) is the statement: Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and is Γ-fullness preserving for some Γ. Then for any $x, y \in \mathbb{R}$, $x \in OD_\Sigma(y)$ iff x is in some Σ-mouse over $\langle \mathcal{P}, y\rangle$.*

When $(\mathcal{P}, \Sigma) = \emptyset$ in the statement of Definition 1.2.7 we get the ordinary Mouse Capturing (MC). The Strong Mouse Set Conjecture (SMSC) just conjectures that SMC holds below a superstrong.

Definition 1.2.8 (Strong Mouse Set Conjecture)**.** *Assume AD^+ and that there is no mouse with a superstrong cardinal. Then SMC holds.*

Recall that by results of [23], SMSC holds assuming $(*)$. To prove that hod pairs exist in AD^+ models, we typically do a hod pair construction. For the details of this construction, see Definitions 2.1.8 and 2.2.5 in [23]. We recall the Γ-hod pair construction from [23] which is crucial for our HOD analysis. Suppose Γ is a pointclass closed under complements and under continuous preimages. Suppose also that $\lambda^{\mathcal{P}}$ is limit. We let

$$\Gamma(\mathcal{P}, \Sigma) = \{A \mid \exists(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \vec{\mathcal{T}}) \in B(\mathcal{P}, \Sigma)\ A <_w {}^{6} Code(\Sigma_{\mathcal{Q}})\}.$$

$$HP^\Gamma = \{(\mathcal{P}, \Lambda) \mid (\mathcal{P}, \Lambda) \text{ is a hod pair and } Code(\Lambda) \in \Gamma\},$$

and

$$Mice^\Gamma = \{(a, \Lambda, \mathcal{M}) \mid a \in HC,\ a \text{ is self-wellordered transitive, } \Lambda \text{ is an iteration strategy such that } (\mathcal{M}_\Lambda, \Lambda) \in HP^\Gamma,\ \mathcal{M}_\Lambda \in a, \text{ and } \mathcal{M} \trianglelefteq Lp^{\Gamma,\Lambda}(a)\}.$$

If $\Gamma = \mathcal{P}(\mathbb{R})$, we let $HP = HP^\Gamma$ and $Mice = Mice^\Gamma$. Suppose $(\mathcal{M}_\Sigma, \Sigma) \in HP^\Gamma$. Let

$$Mice_\Sigma^\Gamma = \{(a, \mathcal{M}) \mid (a, \Sigma, \mathcal{M}) \in Mice^\Gamma\}.$$

Definition 1.2.9 (Γ-hod pair construction)**.** *Let Γ be an inductive-like pointclass and A_Γ be a universal Γ-set. Suppose (M, δ, Σ) is such that $M \vDash \mathsf{ZFC}$ - Replacement, (M, δ) is countable, δ is an uncountable cardinal in M, Σ is an (ω_1, ω_1)-iteration strategy for M, $\Sigma \cap (L_1(V_\delta))^M \in M$. Suppose M locally Suslin captures A_Γ. Then the Γ-hod pair construction of M below δ is a sequence $\langle \langle \mathcal{N}_\xi^\beta \mid \xi < \delta\rangle, \mathcal{P}_\beta, \Sigma_\beta, \delta_\beta \mid \beta \leq \Omega\rangle$ that satisfies the following properties.*

[6] Wadge reducible to

1. $M \Vdash_{Col(\omega,<\delta)}$ *"for all* $\beta < \Omega, (\mathcal{P}_\beta, \Sigma_\beta)$ *is a hod pair such that* $\Sigma_\beta \in \Gamma$*"*[7]*;*

2. $\langle \mathcal{N}^0_\xi \mid \xi < \delta \rangle$ *are the models of the* $L[\vec{E}]$*-construction of* V^M_δ *and* $\langle \mathcal{N}^\beta_\xi \mid \xi < \delta \rangle$ *are the models of the* $L[\vec{E}, \Sigma_\beta]$*-construction of* V^M_δ*.* δ_0 *is the least* γ *such that* $o(\mathcal{N}^0_\gamma) = \gamma$ *and* $Lp^\Gamma(\mathcal{N}^0_\gamma) \vDash$ *"*γ *is Woodin" and* $\delta_{\beta+1}$ *is the least* γ *such that* $o(\mathcal{N}^{\beta+1}_\gamma) = \gamma$ *and* $Lp^{\Gamma,\Sigma_\beta}(\mathcal{N}^{\beta+1}_\gamma) \vDash$ *"*γ *is Woodin".*

3. $\mathcal{P}_0 = Lp^\Gamma_\omega(\mathcal{N}^0_{\delta_0})$ *and* Σ_0 *is the canonical strategy of* $\mathcal{P}_0$ *induced by* Σ*.*

4. *Suppose* $\delta_{\beta+1}$ *exists,* $\mathcal{N}^{\beta+1}_{\delta_{\beta+1}}$ *doesn't project across* δ_β*. Furthermore, if* $\beta = 0$ *or is successor and* $\mathcal{N}^{\beta+1}_{\delta_{\beta+1}} \vDash$ *"*δ_β *is Woodin" and if* β *is limit then* $(\delta^+_\beta)^{\mathcal{P}_\beta} = (\delta^+_\beta)^{\mathcal{N}^{\beta+1}_{\delta_{\beta+1}}}$*, then* $\mathcal{P}_{\beta+1} = Lp^{\Gamma,\Sigma_\beta}_\omega(\mathcal{N}^{\beta+1}_{\delta_{\beta+1}})$ *and* $\Sigma_{\beta+1}$ *is the canonical strategy* $\mathcal{P}_{\beta+1}$ *induced by* Σ*.*

5. *For limit ordinals* β*, letting* $\mathcal{P}^*_\beta = \cup_{\gamma<\beta}\mathcal{P}_\gamma$*,* $\Sigma^*_\beta = \oplus_{\gamma<\beta}\Sigma_\gamma$*, and* $\delta_\beta = sup_{\gamma<\beta}\delta_\gamma$*, if* $\delta_\beta < \delta$ *then let* $\langle \mathcal{N}^{*,\beta}_\xi \mid \xi < \delta \rangle$ *be the models of the* $L[\vec{E}, \Sigma^*_\beta]$*-construction of* V^M_δ*. If there isn't any* γ *such that* $o(\mathcal{N}^{*,\beta}_\gamma) = \gamma$ *and* $Lp^{\Gamma,\Sigma^*_\beta}(\mathcal{N}^{*,\beta}_\gamma) \vDash$ *"*γ *is Woodin" then we let* $\mathcal{P}_\beta$ *be undefined. Otherwise, let* γ *be the least such that* $o(\mathcal{N}^{*,\beta}_\gamma) = \gamma$ *and* $Lp^{\Gamma,\Sigma^*_\beta}(\mathcal{N}^{*,\beta}_\gamma) \vDash$ *"*γ *is Woodin." If* $\mathcal{N}^{*,\beta}_\gamma$ *doesn't project across* δ_β *then* $\mathcal{P}_\beta = \mathcal{N}^{*,\beta}_\gamma|(\delta^{+\omega}_\beta)^{\mathcal{N}^{*,\beta}_\gamma}$*, and* Σ_β *is the canonical iteration strategy for* $\mathcal{P}_\beta$ *induced by* Σ*. Otherwise, let* $\mathcal{P}_\beta$ *be undefined.*

1.3 A definition of $K^J(\mathbb{R})$

Definition 1.3.1. *Let* $\mathcal{L}_0$ *be the language of set theory expanded by unary predicate symbols* $\dot{E}, \dot{B}, \dot{S}$*, and constant symbols* $\dot{l}$ *and* $\dot{a}$*. Let* a *be a given transitive set. A* ***model with paramemter a*** *is an* $\mathcal{L}_0$*-structure of the form*

$$\mathcal{M} = (M; \in, E, B, \mathcal{S}, l, a)$$

such that M *is a transtive rud-closed set containing* a*, the structure* $\mathcal{M}$ *is amenable,* $\dot{a}^{\mathcal{M}} = a$*,* $\mathcal{S}$ *is a sequence of models with paramemter* a *such that letting* S_ξ *be the universe of* $\mathcal{S}_\xi$

- $\dot{S}^{\mathcal{S}_\xi} = \mathcal{S} \restriction \xi$ *for all* $\xi \in dom(\mathcal{S})$ *and* $\dot{S}^{\mathcal{S}_\xi} \in S_\xi$ *if* ξ *is a successor ordinal;*
- $S_\xi = \cup_{\alpha<\xi} S_\alpha$ *for all limit* $\xi \in dom(\mathcal{S})$*;*
- *if* $dom(\mathcal{S})$ *is a limit ordinal then* $M = \cup_{\alpha \in dom(\mathcal{S})} S_\alpha$ *and* $l = 0$*, and*
- *if* $dom(\mathcal{S})$ *is a successor ordinal, then* $dom(\mathcal{S}) = l$*.*

[7]This means there is a strategy Ψ for $\mathcal{P}_\beta$ extending Σ_β such that $Code(\Psi) \in \Gamma$ and Ψ is locally Suslin captured by M (at δ).

The above definition is due to Steel and comes from [46]. Typically, the predicate $\dot{E}$ codes the top extender of the model; $\dot{S}$ records the sequence of models being built so far. Next, we write down some notations regarding the above definition.

Definition 1.3.2. *Let $\mathcal{M}$ be the model with parameter a. Then $|\mathcal{M}|$ denotes the universe of $\mathcal{M}$. We let $l(\mathcal{M}) = dom(\dot{S}^{\mathcal{M}})$ denote the* ***length of*** *$\mathcal{M}$ and set $\mathcal{M}|\xi = \dot{S}^{\mathcal{M}}_{\xi}$ for all $\xi < l(\mathcal{M})$. We set $\mathcal{M}|l(\mathcal{M}) = \mathcal{M}$. We also let $\rho(\mathcal{M}) \leq l(\mathcal{M})$ be the least such that there is some $A \subseteq M$ definable (from parameters in M) over $\mathcal{M}$ such that $A \cap |\mathcal{M}|\rho(\mathcal{M})| \notin M$.*

Suppose J is a mouse operator that condenses well and relivizes well (in the sense of [26]). The definition of $\mathcal{M}_1^{J,\sharp}$ (more generally, the definition of a J-premouse over a self-wellorderable set) has been given in [26] and [46]. Here we only re-stratify its levels so as to suit our purposes.

Definition 1.3.3. *Let $\mathcal{M}$ be a model with parameter a, where a is self-wellorderable. Suppose J is an iteration strategy for a mouse $\mathcal{P}$ coded in a. Let A be a set of ordinals coding the cofinal branch of $\mathcal{T}$ according to J, where $\mathcal{T}$ is the least (in the canonical well-ordering of $\mathcal{M}$) such that $J(\mathcal{T}) \notin |\mathcal{M}|$ if such a tree exists; otherwise, let $A = \emptyset$. In the case $A \neq \emptyset$, let $A^* = \{o(\mathcal{M}) + \alpha \mid \alpha \in A\}$ and ξ be*

1. *the least such that $\mathcal{J}_\xi(\mathcal{M})[A^*]$ is a $\mathcal{Q}$-structure of $\mathcal{M}|\rho(\mathcal{M})$ if such a ξ exists; or,*
2. *ξ is the least such that $\mathcal{J}_\xi(\mathcal{M})[A^*]$ defines a set not amenable to $\mathcal{M}|\rho(\mathcal{M})$ if such a ξ exists; or else,*
3. *$\xi = sup(A^*)$.*

For $\alpha \leq \xi$, we define $\mathcal{M}_\alpha$. For $\alpha = 0$, let $\mathcal{M}_0 = \mathcal{M}$. For $0 < \alpha < \xi$, suppose $\mathcal{M}_\alpha$ has been defined, we let

$$\mathcal{M}_{\alpha+1} = (|\mathcal{J}(\mathcal{M}_\alpha)[A^*]|; \in, \emptyset, A^* \cap |\mathcal{J}(\mathcal{M}_\alpha)[A^*]|, \dot{S}^\frown\mathcal{M}_\alpha, l(\mathcal{M}_\alpha) + 1, a).$$

For limit α, let $\mathcal{M}_\alpha = \cup_{\beta<\alpha}\mathcal{M}_\beta$. We then let $F_J(\mathcal{M}) = \mathcal{M}_\xi$. In the case $A = \emptyset$, we let

$$F_J(\mathcal{M}) = (|\mathcal{J}(\mathcal{M})|; \in, \emptyset, \emptyset, \dot{S}^\frown\mathcal{M}, l(\mathcal{M}) + 1, a).$$

In the case J is a (hybrid) first-order mouse operator[8]*, we let $J^*(\mathcal{M})$ be the least level of $J(\mathcal{M})$ that is a $\mathcal{Q}$-structure or defines a set not amenable to $\mathcal{M}|\rho(\mathcal{M})$ if it exists; otherwise, $J^*(\mathcal{M}) = J(\mathcal{M})$. We then define $F_J(\mathcal{M})$ as follows. Let $\mathcal{M}_0 = \mathcal{M}$. Suppose for α such that $\omega\alpha < o(J^*(\mathcal{M}))$, we've defined $\mathcal{M}||\alpha$ and maintained that $|\mathcal{M}||\alpha| = |J^*(\mathcal{M})||\alpha|$, let $\mathcal{M}_{\alpha+1} = (|J^*(\mathcal{M})||(\alpha+1)|; \in, \emptyset, \emptyset, \dot{S}^\frown\mathcal{M}_\alpha, l(\mathcal{M}_\alpha) + 1, a)$, where $\dot{S} = \dot{S}^{\mathcal{M}_\alpha}$. If α is limit and $J^*(\mathcal{M})||\alpha$ is passive, let $\mathcal{M}_\alpha = \cup_{\beta<\alpha}\mathcal{M}_\beta$; otherwise, let $\mathcal{M}_\alpha = (\cup_{\beta<\alpha}|\mathcal{M}_\beta|; \in, E, \emptyset, \cup_{\beta<\alpha}\dot{S}^{\mathcal{M}_\beta}, sup_{\beta<\alpha}l(\mathcal{M}_\beta), a)$, where E is $F_\alpha^{J^*(\mathcal{M})}$. Finally,*

[8]This means there is a (hybrid) mouse operator J' that condenses well such that there is a formula ψ in the language of J'-premice and some parameter a such that for every $x \in \mathrm{dom}(J)$, $J(x)$ is the least $\mathcal{M} \lhd Lp^{J'}(x)$ that satisfies $\psi[x, a]$.

$$F_J(\mathcal{M}) = M_\gamma, \textit{ where } \omega\gamma = o(J^*(\mathcal{M})).$$

The rest of the definition of a J-premouse over a self-wellorderable set a is as in [46]. We now wish to extend this definition to non self-wellorderable sets a, and in particular to $\mathbb{R}$. For this, we need to assume that the following absoluteness property holds of the operator J. We then show that if J is a mouse strategy operator for a nice enough strategy, then it does hold.

Definition 1.3.4. *We say J* determines itself on generic extensions (relative to $\mathcal{N} = \mathcal{M}_1^{J,\sharp}$) *iff there are formulas φ, ψ in the language of J-premice such that for any correct, non-dropping iterate $\mathcal{P}$ of $\mathcal{N}$, via a countable iteration tree, any $\mathcal{P}$-cardinal δ, any $\gamma \in OR$ such that $\mathcal{P}|\gamma \vDash \varphi+$ "δ is Woodin", and any g which is set-generic over $\mathcal{P}|\gamma$, then $(\mathcal{P}|\gamma)[g]$ is closed under J and $J \restriction \mathcal{P}[g]$ is defined over $(\mathcal{P}|\gamma)[g]$ by ψ. We say such a pair (φ, ψ)* generically determines *J.*

The model operators that we encounter in the core model induction condense well, relativize well, and determine themselves on generic extensions.

Definition 1.3.5. *We say a (hod) premouse $\mathcal{M}$ is* ***reasonable*** *iff under ZF + AD, $\mathcal{M}$ satisfies the first-order properties which are consequences of $(\omega, \omega_1, \omega_1)$-iterability, or under ZFC, $\mathcal{M}$ satisfies the first-order properties which are consequences of $(\omega, \omega_1, \omega_1+1)$-iterability.*

The following lemma comes from [27].

Lemma 1.3.6. *Let $(\mathcal{P}, \Sigma)$ be such that either (a) $\mathcal{P}$ is a reasonable premouse and Σ is the unique normal OR-iteration strategy for $\mathcal{P}$; or (b) $\mathcal{P}$ is a reasonable hod premouse, $(\mathcal{P}, \Sigma)$ is a hod pair which is fullness preserving and has branch condensation. Assume that $\mathcal{M}_1^\Sigma$ exists and is fully iterable. Then Σ determines itself on generic extensions.*

Let M be a transitive model of some fragment of set theory. Let $\dot{G}$ be the canonical $Col(\omega, M)$-name for the generic $G \subseteq Col(\omega, M)$ and $\dot{x}_{\dot{G}}$ be the canonical name for the real coding $\{(n, m) \mid G(n) \in G(m)\}$, where we identify G with the surjective function from ω onto M that G produces. Let Λ be the strategy for $\mathcal{N} = \mathcal{M}_1^{J,\sharp}$. Using the terminology of [23], we say a tree $\mathcal{T}$ on $\mathcal{N}$ via Λ is the tree for making M *generically generic* if the following holds:

1. $\mathcal{T} \restriction (o(M) + 1)$ is a linear iteration tree obtained by iterating the first total measure of $\mathcal{M}$ and its images $o(M) + 1$ times.
2. For $\alpha \geq o(M) + 1$, $E_\alpha^\mathcal{T}$ is the extender with least index in $\mathcal{M}_\alpha^\mathcal{T}$ such that there is a condition $p \in Col(\omega, M)$ such that $p \Vdash \dot{x}_{\dot{G}}$ does not satisfy an axiom involving $E_\alpha^\mathcal{T}$ from the extender algebra $\mathbb{B}_\delta$, where δ is the Woodin cardinal of $\mathcal{M}_\alpha^\mathcal{T}$.

We denote such a tree $\mathcal{T}_M$. Note that $\mathcal{T}_M \in V$, $\mathcal{T}$ is nowhere dropping, and $lh(\mathcal{T}_M) < |M|^+$. Also note that $\mathcal{T}_M$ does not include the last branch. Given a formula φ, let $\mathcal{T}_M^\varphi = \mathcal{T}_M \restriction \lambda$, where λ is least such that either $\lambda = \mathrm{lh}(\mathcal{T}_M)$ or λ is a limit ordinal and there is $\mathcal{P} \trianglelefteq Q(\mathcal{T}_M \restriction \lambda)$ such that $M(\mathcal{T}_M \restriction \lambda) \trianglelefteq \mathcal{P}$ and $\mathcal{P} \vDash \varphi$. Now suppose there is $\mathcal{P} \triangleleft \mathcal{N}$ such that $\mathcal{N}|\delta^{\mathcal{N}} \trianglelefteq \mathcal{P}$ and $\mathcal{P} \vDash \varphi$. Let $\lambda \leq \mathrm{lh}(\mathcal{T}_M^\varphi)$ be a limit. If $\lambda < \mathrm{lh}(\mathcal{T}_M^\varphi)$ let $Q^\varphi(\mathcal{T}_M \restriction \lambda) = Q(M(\mathcal{T}_M \restriction \lambda))$. Otherwise let $Q^\varphi(\mathcal{T}_M \restriction \lambda) = \mathcal{P}$, where $\mathcal{P}$ is least such that $M(\mathcal{T}_M \restriction \lambda) \trianglelefteq \mathcal{P} \trianglelefteq M^{\mathcal{T}_M}_{\Lambda(\mathcal{T}_M \restriction \lambda)}$ and $\mathcal{P} \vDash \varphi$.

We're ready to define J-premice over an arbitrary transitive set a. The idea that to define a Σ-premouse (over an arbitrary set), it suffices to tell the model branches of trees that make certain levels of the model generically generic comes from [23], where it's used to reorganize hod mice in such a way that S-constructions work.

Definition 1.3.7. *Suppose a is a transitive set coding $\mathcal{M}_1^{J,\sharp}$. Suppose (φ, ψ) generically determines J. Let Λ be the strategy for $\mathcal{M}_1^{J,\sharp}$. We define $F_J^*(a)$ to be a level of a model $\mathcal{M}$ with parameter a with the following properties. There is $\alpha < l(\mathcal{M})$ such that $\mathcal{M}|\alpha \vDash ZF$. Let α be the least such and let ξ be the largest cardinal of $\mathcal{M}|\alpha = \mathcal{J}_\alpha(a)$. Let $\lambda \leq \mathrm{lh}(\mathcal{T}^\varphi_{\mathcal{M}|\alpha})$ be a limit. Let*

$$\mathcal{P}_{\alpha,\lambda} = Q^\varphi(\mathcal{T}_{\mathcal{M}|\alpha} \restriction \lambda).$$

Let $B \subseteq o(\mathcal{P}_{\alpha,\lambda})$ be the standard set coding $\mathcal{P}_{\alpha,\lambda}$. Let $\omega\gamma = o(\mathcal{P}_{\alpha,\lambda})$. Let for $\beta < l(\mathcal{M})$,

$$A_\beta = \{o(\mathcal{M}|\beta) + \eta \mid \eta \in B\} \times \{(\alpha, \lambda)\}.$$

and define

$$F_{J,\alpha,\lambda}(\mathcal{M}|\beta) = \mathcal{J}_\gamma^{A_\beta}(\mathcal{M}|\beta)$$

if no levels of $\mathcal{J}^A_\gamma(\mathcal{M}|\beta)$ is a $\mathcal{Q}$-structure for $(\mathcal{M}|\beta)|\rho(\mathcal{M}|\beta)$ or projects across $\rho(\mathcal{M}|\beta)$; otherwise, let $F_{J,\alpha,\lambda}(\mathcal{M}|\beta) = \mathcal{J}(\mathcal{M}|\beta)$.[9].

Suppose $\mathcal{M}|\beta$ has been defined and there is a λ such that $\mathcal{P}_{\alpha,\lambda}$ is defined, $\mathcal{T}^\varphi_{\mathcal{M}|\alpha} \restriction \lambda \in \mathcal{M}|\beta$, but for no $\beta' < l(\mathcal{M}|\beta)$, $F_{J,\alpha,\lambda}(\mathcal{M}|\beta') \neq \mathcal{J}(\mathcal{M}|\beta')$, we let then $\mathcal{M}|\xi^* = F_{J,\alpha,\lambda}(\mathcal{M}|\beta)$, where $\xi^* = l(F_{J,\alpha,\lambda}(\mathcal{M}|\beta))$ for the least such λ.

*We say that $\mathcal{T}^\varphi_{\mathcal{M}|\alpha}|\lambda$ is **taken care of in** $\mathcal{M}$ if there is a $\beta < l(\mathcal{M})$ such that $F_{J,\alpha,\lambda}(\mathcal{M}|\beta)$ $\triangleleft\mathcal{M}$ and $F_{J,\alpha,\lambda}(\mathcal{M}|\beta) \neq \mathcal{J}(\mathcal{M}|\beta)$. So $\mathcal{M}$ is the least such that for every limit $\lambda \leq lh(\mathcal{T}^\varphi_{\mathcal{M}|\alpha})$, $\mathcal{T}^\varphi_{\mathcal{M}|\alpha} \restriction \lambda$ is taken care of in $\mathcal{M}$.*

Finally, let $F_J^*(a) = \mathcal{M}$ if no levels of $\mathcal{M}$ projects across ξ. Otherwise, let $F_J^*(a) = \mathcal{M}|\beta$, where β is the least such that $\rho_\omega(\mathcal{M}|\beta) < \xi$.

[9]Technically, $F_{J,\alpha,\lambda}(\mathcal{M}|\beta)$ is stratified as a model over a but we suppress the structure for brevity. See Definition 1.3.3 for the stratification.

Definition 1.3.8 (Potential J-premouse over a)**.** *Let a be a transitive structure such that a contains a real coding $\mathcal{N}$. We say that $\mathcal{M}$ is a* ***potential J-premouse over*** *a iff $\mathcal{M}$ is a model with parameter a, and there is an ordinal λ and a increasing, closed sequence $\langle\eta_\alpha\rangle_{\alpha\leq\lambda}$ of ordinals, such that for each $\alpha \leq \lambda$, we have:*

(a) if a is not a self-wellordered set, then $\eta_0 = 1$ and $\mathcal{M}|1 = a$; otherwise, either $\lambda = 0$ and $\mathcal{M} = \mathcal{M}|\eta_0 \trianglelefteq \mathcal{M}_1^{J,\sharp}$ or else $\mathcal{M}|\eta_0 = \mathcal{M}_1^{J,\sharp}$ (in the sense of Definition 1.3.3),

(b) $\eta_\alpha \leq l(\mathcal{M})$,

(c) if $\alpha + 1 < \lambda$, then $\mathcal{M}|\eta_{\alpha+1} = F_J^*(\mathcal{M}|\eta_\alpha)$,

(d) if $\alpha + 1 = \lambda$, then $\mathcal{M} \trianglelefteq F_J^*(\mathcal{M}|\eta_\alpha)$,[10]

(e) $\eta_\lambda = l(\mathcal{M})$,

(f) if $\eta = \eta_\alpha$ and $\dot{E}^{\mathcal{M}|\eta} \neq \emptyset$ (and therefore α is a limit) then $\dot{E}^{\mathcal{M}|\eta}$ codes an extender E that coheres $\mathcal{M}|\eta$ and satisfies the obvious modifications of the premouse axioms (in the sense of Definition 2.2.1 of [46]) and E is $a \times \gamma$-complete for all $\gamma < crt(E)$[11]*.*

We define projecta, standard parameters, solidity, soundness, cores as in section 2.2 of [46].

Definition 1.3.9. *Suppose $\mathcal{M}$ is a potential J-premouse over a. Then we say that $\mathcal{M}$ is a J-premouse over a if for all $\lambda < l(\mathcal{M})$, $\mathcal{M}|\lambda$ is ω-sound.*

Definition 1.3.10. *Suppose $\mathcal{M}$ is a J-premouse over a. We say that $\mathcal{M}$ is active if $\dot{E}^{\mathcal{M}} \neq \emptyset$ or $\dot{B}^{\mathcal{M}} \neq \emptyset$. Otherwise, we say that $\mathcal{M}$ is passive.*

Definition 1.3.11 (J-mouse)**.** *Let $\mathcal{M}, a$ be as in Definition 1.3.9. We say that $\mathcal{N}$ is a J-mouse over a if $\rho_\omega(\mathcal{N}) = a$ and whenever $\mathcal{N}^*$ is a countable transitive J-premouse over some a^* and there is an elementary embedding $\pi : \mathcal{N}^* \to \mathcal{N}$ such that $\pi(a^*) = a$, then $\mathcal{N}^*$ is $\omega_1 + 1$-iterable*[12] *and whenever $\mathcal{R}$ is an iterate of $\mathcal{N}^*$ via its unique iteration strategy, $\mathcal{R}$ is a J-premouse over a^*.*

Suppose $\mathcal{M}$ is a J-premouse over a. We say that $\mathcal{M}$ is *J-complete* if $\mathcal{M}$ is closed under the operator F_J^*. The following lemma is also from [27].

Lemma 1.3.12. *Suppose $\mathcal{M}$ is a J-premouse over a and $\mathcal{M}$ is J-complete. Then $\mathcal{M}$ is closed under J; furthermore, for any set generic extension g of $\mathcal{N}$, $\mathcal{N}[g]$ is closed under J and in fact, J is uniformly definable over $N[g]$ (i.e. there is a $\mathcal{L}_0$-formula ϕ that defines J over any generic extension of N).*

[10] We will also use $\mathcal{M}_\eta$ to denote $\mathcal{M}|\eta$.

[11] This means whenever $\langle X_x \mid x \in a \times \gamma\rangle \in \mathcal{M}|\lambda$ is such that $X_x \in E_b$ for each $x \in a \times \gamma$, where b is a finite subset of $lh(E)$, then $\cap_{x\in a} X_x \in E_b$

[12] Sometimes we need more than just $\omega_1 + 1$-iterability. See Chapter 4.

If a in Definition 1.3.11 is H_{ω_1}, then we define $Lp^J(\mathbb{R})$ to be the union of all J-mice $\mathcal{N}$ over a[13]. In core model induction applications, we typically have a pair $(\mathcal{P}, \Sigma)$ where $\mathcal{P}$ is either a hod premouse and Σ is $\mathcal{P}$'s strategy with branch condensation and is fullness preserving (relative to mice in some pointclass) or $\mathcal{P}$ is a sound (hybrid) premouse projecting to some countable set a and Σ is the unique (normal) strategy for $\mathcal{P}$. Lemma 1.3.6 shows that Σ condenses well and determines itself on generic extension in the sense defined above. We then define $Lp^\Sigma(\mathbb{R})$[14] as above and use a core model induction to prove $Lp^\Sigma(\mathbb{R}) \vDash \mathsf{AD}^+$. What's needed to prove this is the scales analysis of $Lp^\Sigma(\mathbb{R}) \vDash \mathsf{AD}^+$ from the optimal hypothesis similar to those used by Steel to analyze the pattern of scales in $K(\mathbb{R})$.

[13] We'll be also saying J-premouse over $\mathbb{R}$ when $a = H_{\omega_1}$

[14] In this thesis, we use $Lp^\Sigma(\mathbb{R})$ and $K^\Sigma(\mathbb{R})$ interchangably.

Chapter 2

Generalized Solovay Measures

We work under the theory $\mathsf{ZF} + \mathsf{DC}$ unless stated otherwise. For each $\alpha < \omega_1$, for each $f : \alpha \to \mathcal{P}_{\omega_1}(\mathbb{R})$, f is *nice* if for all i, $f(i)$ is coded by an element in $f(i+1)$ (we will abuse notation and write "$f(i) \in f(i+1)$") and if i is limit, $f(i) = \cup_{j<i} f(j)$. Let X_α be the set of all nice $f : \omega^\alpha \to \mathcal{P}_{\omega_1}(\mathbb{R})$. Also let $X_{\omega_1} = \{f : \alpha \to \mathcal{P}_{\omega_1}(\mathbb{R}) \mid f \text{ is nice and } \alpha < \omega_1\}$. For any $f : \alpha \to \mathcal{P}_{\omega_1}(\mathbb{R})$, we let $\mathbb{R}_f = \cup_{\beta<\alpha} f(\beta)$.

Definition 2.0.13 (Fineness)**.** *For $\alpha \leq \omega_1$, μ_α is said to be **fine** if for any $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$, the set of all $g \in X_\alpha$ such that $\sigma \in g(0)$ has μ_α-measure one.*

Definition 2.0.14 (Normality)**.** *For $\alpha < \omega_1$, a measure μ_α on X_α is **normal** if*

1. *(Fodor's property) For any $F : X_\alpha \to \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $\forall^*_{\mu_\alpha} f F(f) \subseteq f(0) \wedge F(f) \neq \emptyset$, there is an $x \in \mathbb{R}$ such that $\forall^*_{\mu_\alpha} f(x \in F(f))$;*
2. *(Shift invariance) If $X \in \mu_\alpha$ and $\beta < \omega^\alpha$, then $\{f^\beta \mid f \in X\} \in \mu_\alpha$ where $f^\beta(i) = f(\beta + i)$.*

*For $\alpha = \omega_1$, a measure μ_α on X_α is **normal** if (1)-(2) hold for μ_α and*

3. *(Idempotence) If $A, B \in \mu_\alpha$, then $A^\frown B = \{f^\frown g \mid f \in A \wedge g \in B \wedge f^\frown g \in X_\alpha\} \in \mu_\alpha$.*

Note that condition (1) of normality is the generalization of the Fodor's property in the ZFC context. This is all that we can demand for the following reasons. For $\alpha = 0$, in the context of DC, the exact statement of Fodor's lemma reduces to countable completeness of μ_0 and this is not sufficient to prove, for example, Los's theorem for ultraproducts using μ_0. Suppose $\alpha > 0$ and consider the function F such that $F(f) = \{x \in \mathbb{R} \mid x \text{ codes } f(0)\}$. There can't be an $x \in \mathbb{R}$ that codes $f(0)$ for μ_α-measure one many f.

Here's an easy lemma that characterizes (1) in terms of diagonal intersection. The proof of the lemma, which does not use the axiom of choice, is easy and we leave it to the reader.

Lemma 2.0.15 ($\mathsf{ZF} + \mathsf{DC}$)**.** *Fix $\alpha < \omega_1$ and suppose μ_α is a fine measure on X_α. The following are equivalent:*

(a) For all $\langle A_x \mid x \in \mathbb{R} \wedge A_x \in \mu_\alpha \rangle$, $\triangle_{x\in\mathbb{R}} = \{f \in X_\alpha \mid f \in \cap_{x\in f(0)} A_x\} \in \mu_\alpha$.

(b) $\forall^*_{\mu_\alpha} f\ (F(f) \subseteq f(0) \rightarrow \exists x \forall^*_{\mu_\alpha} f\ x \in F(f))$.

We need the following (unpublished) theorem of Woodin, which proves the existence of models of "AD^+ + there is a normal fine measure on X_α" for $\alpha \leq \omega_1$ from $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$. A corollary of Theorem 2.0.16 is Theorem 2.0.17, a well-known theorem of Solovay ($\alpha = 0$) and of Martin and Woodin ($\alpha > 0$).

Theorem 2.0.16 (Woodin)**.** *Assume* $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$. *Let* $A \subseteq \mathbb{R}$. *There is a tuple* $(M, \vec{E}, \delta, \Sigma, \tau)$ *such that*

1. $\vec{E}$ *is a weakly coherent extender sequence on* M *in the sense that if* $F \in \vec{E}$ *and* $i_F : M \rightarrow Ult(M, F)$ *is the ultrapower map then* M *agrees with* $Ult(M, F)$ *up to* $lh(F)$*;*

2. $\vec{E}$ *witnesses that* δ *is a measurable limit of Woodin cardinals in* M*;*

3. τ *is a* $Col(\omega, \delta)$*-term in* M *and* Σ *is an iteration strategy for* M *such that if* $i : M \rightarrow N$ *is an iteration map according to* Σ*, then for any* $g \subseteq Col(\omega, i(\delta))$ *generic over* N*,* $A \cap N[g] = i(\tau)_g$.

Theorem 2.0.17 (Martin,Woodin)**.** *Assume* $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$. *Then for all* $\alpha \leq \omega_1$*, there is a normal fine measure* μ_α *on* X_α.

Proof. We'll use Theorem 2.0.16 to show there is a normal fine measure μ_{ω_1} on X_{ω_1}. The measures μ_α can be constructed from μ_{ω_1} as follows. For any $A \subseteq X_\alpha$,

$$A \in \mu_\alpha \Leftrightarrow \{f \in X_{\omega_1} \mid f \restriction \alpha \in A\} \in \mu_{\omega_1}.$$

We proceed to define μ_{ω_1}. Let $A \subseteq X_{\omega_1}$. $A \in \mu_{\omega_1}$ if for all $B \subseteq \mathbb{R}$ coding A[1], letting $(M_B, \vec{E}_B, \delta_B, \Sigma_B, \tau_B)$ be as in Theorem 2.0.16 for B, $g \subseteq Col(\omega, < \delta_B)$ be M_B-generic, δ_α be the α^{th}-limit of Woodin cardinals in M_B, then $\langle \mathbb{R}^{M_B[g\restriction\delta_\alpha]} \mid \delta_\alpha < \delta_B \rangle$[2] $\in (\tau_A)_g$.

Lemma 2.0.18. μ_{ω_1} *is a normal fine measure on* X_{ω_1}.

Proof. We first show μ_{ω_1} is a measure. Suppose not. There is an $A \subseteq X_{\omega_1}$ such that there are $B, C \subseteq \mathbb{R}$ coding A and $(M_B, \vec{E}_B, \delta_B, \Sigma_B, \tau_B)$, $(M_C, \vec{E}_C, \delta_C, \Sigma_C, \tau_C)$ as in Theorem 2.0.16 for B and C respectively and

$$M_B \vDash \emptyset \Vdash_{Col(\omega, <\delta_B)} \dot{f}_{\dot{G}} \in \tau_A,$$

but

$$M_C \vDash \emptyset \Vdash_{Col(\omega, <\delta_C)} \dot{f}_{\dot{G}} \notin \tau_A.$$

[1]The coding is so that if M captures B by τ_B, then M captures A by some τ_A.

[2]From now on, we'll denote this sequence f_g.

We will get a contradiction by a back-and-forth argument that produces iteration maps $i : M_B \to N_B$, $j : M_C \to N_C$ such that there are $g_B \subseteq Col(\omega, < i(\delta_B))$ generic over N_B, $g_C \subseteq Col(\omega, < j(\delta_C))$ generic over N_C such that $f_{g_B} = f_{g_C}$.

We informally describe the first ω steps of this process. Let $\langle \delta_n \mid n < \omega \rangle$ and $\langle \gamma_n \mid n < \omega \rangle$ be the first ω Woodin cardinals of M_B and M_C respectively. Let $M_0 = M_B$ and $N_0 = M_C$. We first iterate M_0 below δ_0 to produce $i_0 : M_0 \to M_1$ and $g_0 \subseteq Col(\omega, i_0(\delta_0))$ such that $N_0|\gamma_0 \in M_1[g_0]$. We then iterate in the window $[\gamma_0, \gamma_1)$ and produce $j_0 : N_0 \to N_1$ and $h_0 \subseteq Col(\omega, j_0(\gamma_1))$ such that $M_1[g_0]|\delta_0 \in N_1[h_0]$. In general, for all $0 < n < \omega$, we produce $i_n : M_n \to M_{n+1}$ in the window $[i_{n-1} \circ \cdots \circ i_0(\delta_{n-1}), i_{n-1} \circ \cdots \circ i_0(\delta_n))$ and $g_n \subseteq Col(\omega, i_n \circ \cdots \circ i_0(\delta_n))$ extending g_{n-1} such that $N_n[h_0, \ldots, h_{n-1}]|(j_{n-1} \circ \cdots \circ j_0(\gamma_{n-1})) \in M_{n+1}[g_n]$ and then $j_n : N_n \to N_{n+1}$ in the window $[j_{n-1} \circ \cdots \circ j_0(\gamma_{n-1}), j_{n-1} \circ \cdots \circ j_0(\gamma_n))$, $h_n \subseteq Col(\omega, j_n \circ \cdots \circ j_0(\gamma_n))$ extending h_{n-1} such that $M_{n+1}[g_n]|(i_n \circ \cdots \circ i_0(\delta_n)) \in N_{i+1}[h_n]$. Let M_ω and N_ω be the direct limits of the M_n's and N_n's respectively. Let $\langle \delta^\omega_n \mid n \leq \omega \rangle$ and $\langle \gamma^\omega_n \mid n \leq \omega \rangle$ be the first ω Woodins and their sup of M_ω and N_ω respectively. Then it's clear from our construction that $\sigma_0 = \{M_\omega[g_n]|\delta^\omega_n \mid n < \omega\} = \{N_\omega[h_n]|\gamma^\omega_n \mid n < \omega\}$ is the symmetric reals at δ^ω_ω of M_ω and γ^ω_ω of N_ω. Let $g_\omega \subseteq Col(\omega, \delta^\omega_\omega)$ be M_ω generic realizing σ_0 as the symmetric reals and $h_\omega \subseteq Col(\omega, \gamma^\omega_\omega)$ be N_ω generic realizing σ_0 as the symmetric reals. We then repeat the back-and-forth process described above using the next ω Woodins. When we use up the Woodins on one side but not the other, we hit the measure of the measurable Woodin cardinal of the shorter side to create more Woodin cardinals and continue the back-and-forth process. The coiteration will stop successfully when we use up the Woodins on both sides. It's easy to see that this process stops successfully and we produce G on the M side and H on the N side such that $f_G = f_H$. Contradiction.

It's easy to see that μ_{ω_1} is fine. To verify property (2) of normality, suppose $X \in \mu_{\omega_1}$ and $\alpha < \omega_1$ is such that $X_\alpha =_{def} \{f^\alpha \mid f \in X\} \notin \mu_{\omega_1}$ where $f^\alpha(i) = f(\alpha + i)$. So there is $B \subseteq \mathbb{R}$ coding X, X_α such that letting $(M_B, \vec{E}_B, \delta_B, \Sigma_B, \tau_B)$ be as in Theorem 2.0.16 for B, we have

$$Col(\omega, < \delta_B) \Vdash^{M_B} \dot{f}_{\dot{G}} \in \tau_X \wedge \dot{f}_{\dot{G}} \notin \tau_{X_\alpha}.$$

Let $M_0 = N_0 = M_B$ and run the back-and-forth argument as above to get a contradiction. The difference here is in the first ω steps of the coiteration: on the M_0 side the iteration uses the first ω Woodins of M_0 and on the N_0 side the iteration ignores the first ω^α Woodins of N_0 and uses the $\omega^\alpha + i^{\text{th}}$ Woodins of N_0. The process stops successfully and results in the end models M_{ω_1} and N_{ω_1}, generics G for M_{ω_1} and H for N_{ω_1} such that $f_G = f^\alpha_H$. But then $M_{\omega_1}[G] \vDash f_G \notin X_\alpha$ while $N_{\omega_1}[H] \vDash f^\alpha_H = f_G \in X_\alpha$. This is a contradiction.

To verify property (1) of normality, suppose F is such that $X = \{f \in X_{\omega_1} \mid F(f) \neq \emptyset \wedge F(f) \subseteq f(0)\} \in \mu_{\omega_1}$ but for all $x \in \mathbb{R}$, $Y_x = \{f \in X \mid x \in F(f)\} \notin \mu_{\omega_1}$. Let $B \subseteq \mathbb{R}$ code $F, X, \{Y_x \mid x \in \mathbb{R}\}$ and let $(M_B, \vec{E}_B, \delta_B, \Sigma_B, \tau_B)$ be as in Theorem 2.0.16 for B. Letting $g \subseteq Col(\omega, \delta_B)$ be generic over M_B, we have

$$M_B[g] \vDash f_g \in \tau_X \wedge \forall x \in \mathbb{R}^{M_B}(f_g \notin \tau_{Y_x}),$$

which means

$$M_B[g] \vDash F(f_g) \neq \emptyset \wedge F(f_g) \subseteq f_g(0) \wedge \forall x \in \mathbb{R}^{M_B}(x \notin F(f_g)).$$

For each $x \in f_g(0)$, let $M_x = M_B[x]$ and $g_x \subseteq Col(\omega, < \delta_B)$ generic over $M_B[x]$ such that $f_g(0) = f_{g_x}(0)$; also, let $M_\emptyset = M_B$. Now use the back-and-forth argument above to coiterate $\{M_x \mid x \in f_g(0)\}$ above the first ω Woodins of each model. The process terminates successfully and produce for each $x \in f_g(0)$ a model M_x^∞, a generic g_x^∞ over M_x^∞ (at the measurable limit of Woodins of the model) such that

1. for all $x, y \in f_g(0)$, $f_{g_x^\infty} = f_{g_y^\infty}$;
2. for all $x \in f_g(0)$, $f_{g_x^\infty}(0) = f_g(0)$;
3. for all $x \in f_g(0)$, $x \notin F(f_{g_x^\infty})$.

(1)-(3) imply $F(f_{g_x^\infty}) = 0$ for any $x \in f_g(0)$. This is a contradiction.

To verify (3), suppose $A, B \in \mu_{\omega_1}$ and let $(M, \vec{E}, \delta, \Sigma, \tau)$ witness this. This means M is both A-iterable and B-iterable via Σ and the term τ computes the terms τ_A and τ_B. Let $g \subseteq Col(\omega, \delta)$ be M-generic and let $f = f_g$. Hence $f \in A$. Now let $i : M \to N$ be the ultrapower map via a measure on δ in M. By coiterating N above δ and M and using a back-and-forth argument similar to the above, we get an iterate P of N (above δ) such that letting $j : N \to P$ be the iteration map, there is a generic $h \subseteq Col(\omega, j(i(\delta)))$ over P such that h extends g and $f_h \backslash f_g \in B$. This means $f_h \in A^\frown B$. Hence we finish verifying (3). □

□

It's not clear that under $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$, the measure μ_{ω_1} defined above is the unique measure satisfying (1)-(3) of Definition 2.0.14. However $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$ implies that the measures μ_α (for $\alpha < \omega_1$) are unique. $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$ implies μ_0 is unique (see [47]) (we just need $\mathsf{DC}_\mathbb{R}$ for the proof of the main theorem in [47] for showing μ_0 is unique). To show uniqueness of μ_α for $\alpha > 0$, we need the following definition. We identify $\mathcal{P}_{\omega_1}(\mathbb{R})$ with X_0.

Definition 2.0.19. *Fix $0 \leq \alpha < \omega_1$. Suppose $A \subseteq X_\alpha$. We say that A is a club*[3] *if there is a function $F : \mathbb{R}^{<\omega} \to \mathbb{R}$ such that $cl_{\alpha,F} = A$ where*

$$cl_{\alpha,F} = \{f \in X_\alpha \mid \forall \beta\ F''f(\beta)^{<\omega} \subseteq f(\beta) \wedge F''f(\beta)^{<\omega} \in f(\beta+1)^4\}.$$

Martin and Woodin actually proves that under $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$, real games of length α for any $\alpha < \omega_1$ are determined. Hence for any $\alpha < \omega_1$, for any $A \subseteq X_\alpha$, either A contains a club or $\mathbb{R}\backslash A$ contains a club. By the same arguments Solovay uses to prove μ_0 is normal under $\mathsf{AD}_\mathbb{R}$, we have that if $\langle A_x \mid A_x \subseteq X_\alpha \wedge x \in \mathbb{R}\rangle$ is a sequence of clubs then the diagonal intersection

[3]For $\alpha = 0$, the notion of a club is just the usual notion of club for subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$. Again we confuse X_0 for $\mathcal{P}_{\omega_1}(\mathbb{R})$.

[4]Recall that this means that there is a real in $f(\beta+1)$ that codes an enumeration of $F''f(\beta)^{<\omega}$

$$\triangle_x A_x =_{def} \{f \in X_\alpha \mid f \in \cap_{x \in f(0)} A_x\}$$

contains a club.

We show μ_1 is unique and the proof of the other cases is similar. So suppose $A \in \mu_1$. It's enough to show A contains a club, that is A contains $cl_{1,F}$ for some F as in Definition 4.3.27. It's easy to check that the following is an equivalent definition of μ_1. We say $A \in \mu_1$ if for all $B \subseteq \mathbb{R}$ coding A, letting $(M_B, \vec{E}_B, \delta_B, \Sigma_B, \tau_B)$ be as in Theorem 2.0.16 for B except that $\vec{E}_B$ witnesses that δ_B is a limit of ω^2 Woodin cardinals in M_B, $g \subseteq Col(\omega, \delta_B)$ be M_B-generic, $f_g \in (\tau_A)_g$. Fix an $(M_A, \vec{E}_A, \delta_A, \Sigma_A, \tau_A)$. Then for a club of $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$, σ is closed under Σ_A. Let F be such that for all $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ closed under F, σ is closed under Σ_A. Using genericity iteration, it's easy to see that for all $f \in \mathrm{cl}_{1,F}$, $f \in A$. This shows that whenever $A \in \mu_1$, then A contains a club. Hence μ_1 is the unique normal fine measure on X_1, since it is just the club filter on X_1. A similar proof works for $2 \leq \alpha < \omega_1$.

2.1 When $\alpha = 0$

2.1.1 The Equiconsistency

We assume familiarity with stationary tower forcing (see [14]) which will be used in the proof of the following theorem of Woodin.

Theorem 2.1.1 (Woodin)**.** *The following are equiconsistent.*

1. *ZFC + there are ω^2 Woodin cardinals.*
2. *There is a filter μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that $L(\mathbb{R}, \mu) \vDash ZF + DC + AD + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.*

We first prove the (1) $\Rightarrow$ (2) direction of Theorem 2.1.1. Assume γ is the sup of ω^2 Woodin cardinals and for each $i < \omega$, let η_i be the sup of the first ωi Woodin cardinals. Suppose $G \subseteq Col(\omega, < \gamma)$ is V-generic and for each i, let $\mathbb{R}^* = \cup_{\alpha<\gamma} \mathbb{R}^{V[G \upharpoonright \alpha]}$ and $\sigma_i = \mathbb{R}^{V[G \upharpoonright Col(\omega, <\eta_i)]}$. We define a filter $\mathcal{F}^*$ as follows: for each $A \subseteq \mathbb{R}^*$ in $V[G]$

$$A \in \mathcal{F}^* \Leftrightarrow \exists n \forall m \geq n (\sigma_m \in A).$$

We call $\mathcal{F}^*$ defined above the *tail filter*.

Lemma 2.1.2. *$L(\mathbb{R}^*, \mathcal{F}^*) \vDash \mathcal{F}^*$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R}^*)$.*

Proof. Suppose not. So this statement is forced by the empty condition in $Col(\omega, < \gamma)$.

Claim. *There is a forcing $\mathbb{P}$ of size less than the first Woodin cardinal such that in $V^{\mathbb{P}}$, $L(\mathbb{R}, \mathcal{C}) \vDash$ "$\mathcal{C}$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$" where $\mathcal{C}$ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.*

Proof. Let κ be the first measurable cardinal and U be a normal measure on κ. Let $j : V \to M$ be the ultrapower map by U. Let $\mathbb{P}_0$ be $Col(\omega, < \kappa)$. Let $G \subseteq \mathbb{P}_0$ be V-generic. $Col(\omega, < j(\kappa)) = j(\mathbb{P}_0)$ is isomorphic to $\mathbb{P}_0 * \mathbb{Q}$ for some $\mathbb{Q}$ and whenever $H \subseteq \mathbb{Q}$ is $V[G]$-generic, then j can be lifted to an elementary embedding $j^+ : V[G] \to M[G][H]$ defined by $j^+(\tau_G) = j(\tau)_{G*H}$. We define a filter $\mathcal{F}^*$ as follows.

$$A \in \mathcal{F}^* \Leftrightarrow \forall H \subseteq \mathbb{Q}(H \text{ is } V[G]\text{-generic} \Rightarrow \mathbb{R}^{V[G]} \in j^+(A)).$$

It's clear from the definition that $\mathcal{F}^* \in V[G]$. Let $\mathbb{R}^* = \mathbb{R}^{V[G]}$ is the symmetric reals. We claim that $L(\mathbb{R}^*, \mathcal{F}^*) \vDash \mathcal{F}^*$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})^*$. Suppose $A \in L(\mathbb{R}^*, \mathcal{F}^*)$ is defined in $V[G]$ by a formula φ from a real $x \in \mathbb{R}^*$ (without loss of generality, we suppress parameters $\{U, s\}$, where $s \in \mathrm{OR}^{<\omega}$ that go into the definition of A); so $\sigma \in A \Leftrightarrow V[G] \vDash \varphi[\sigma, x]$. Let $\alpha < \kappa$ be such that $x \in V[G \restriction \alpha]$ and we let U^* be the canonical extension of U in $V[G \restriction \alpha]$. Then either $\forall^*_{U^*}\beta V[G \restriction \alpha] \vDash \emptyset \Vdash_{Col(\omega,<\beta)} \varphi[\dot{\mathbb{R}}, x]$ or $\forall^*_{U^*}\beta V[G \restriction \alpha] \vDash \emptyset \Vdash_{Col(\omega,<\beta)} \neg\varphi[\dot{\mathbb{R}}, x]$. This easily implies either $A \in \mathcal{F}^*$ or $\neg A \in \mathcal{F}^*$. We leave the proof of normality and fineness to the reader.

Since $\mathcal{P}_{\mathbb{R}^*}$ has size ω_1 in $V[G]$, we can then let $\mathbb{P}_1$ be the iterated club shooting poset defined in 17.2 of [4] to shoot clubs through stationary subsets of $\mathcal{P}_{\mathbb{R}^*}$. By 17.2 of [4], $\mathbb{P}_1$ does not add any ω-sequence of ordinals. Letting $H \subseteq \mathbb{P}_1$ be $V[G]$-generic, in $V[G][H]$, we still have $L(\mathbb{R}^*, \mathcal{F}^*) \vDash \mathcal{F}^*$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})^*$ and furthermore, $\mathcal{F}$ is the restriction of the club filter on $L(\mathbb{R}^*, \mathcal{F}^*)$. Our desirable $\mathbb{P}$ is $\mathbb{P}_0 * \mathbb{P}_1$. $\square$

By the claim, we may assume that in V, the club filter $\mathcal{F}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ has the property that $L(\mathbb{R}, \mathcal{F}) \vDash \mathcal{F}$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Let $\lambda > \gamma$ be inaccessible and let

$$\begin{aligned} S = \{X \prec V_\lambda \quad & \mid X \text{ is countable}, \gamma \in X, \exists \eta \in X \cap \gamma \text{ such that} \\ & \text{for all successor Woodin cardinals } \lambda \in X \cap (\eta, \gamma), \text{ if } D \subseteq \mathbb{Q}_{<\lambda}, D \in X \\ & \text{is predense then } X \text{ captures } D\}. \end{aligned}$$

By lemma 3.1.14 of [14], S is stationary and furthermore, letting $H \subseteq \mathcal{P}(\mathcal{P}_{\omega_1}(V_\lambda))/\mathcal{I}_{NS}$ be generic such that $S \in H$, then for some $\xi < \gamma$, for all $\xi < \delta < \gamma$ and δ is Woodin, $H \cap \mathbb{Q}_{<\delta}$ is V-generic. We may as well assume ξ is less than the first Woodin cardinal and hence for all $\delta < \gamma$, δ is Woodin, $H \cap \mathbb{Q}_{<\delta}$ is V-generic.

Let $j : V \to (M, E)$ be the induced generic embedding given by H. Of course, (M, E) may not be wellfounded but wellfounded at least up to λ because $j''\lambda \in M$. For each $\alpha < \omega^2$, let $j_\alpha : V \to M_\alpha$ be the induced embedding by $H \cap \mathbb{Q}_{<\delta_\alpha}$, let M^* be the direct limit of the M_α's and $j^* : V \to M^*$ be the direct limit map. Note that j_α, j^* factor into j.

Let $\mathbb{R}^* = \mathbb{R}^{M^*}$ and for each $i < \omega$, $\sigma_i = \mathbb{R}^{M_i^*}$ where $M_i^* = lim_n M_{\omega i+n}$. Let $G \subseteq Col(\omega, < \gamma)$ be such that $\cup_{\alpha<\eta_i}\mathbb{R}^{V[G\restriction\alpha]} = \sigma_i$ for all i. Let $\mathcal{F}^*$ be the tail filter defined in $V[G]$. We claim that if $A \in j^*(\mathcal{F})$ then $A \in \mathcal{F}^*$. To see this, let $\pi \in M^*$ witness that A is a club. Let $\alpha < \omega^2$ be such that M_α contains the preimage of π. Then it is clear that $\forall m$ such that $\omega m \geq \alpha$ and $\pi''\sigma_m \subseteq \sigma_m$. This shows $j^*(\mathcal{F}) \subseteq \mathcal{F}^*$ and hence $L_\lambda(\mathbb{R}^*, j^*(\mathcal{F})) = L_\lambda(\mathbb{R}^*, \mathcal{F}^*) \vDash \mathcal{F}^*$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R}^*)$. Since λ can be chosen arbitrarily large, we're done. $\square$

Lemma 2.1.3. $L(\mathbb{R}^*, \mathcal{F}^*) \vDash \mathsf{AD}^+$.

Proof. We use the notation of Lemma 4.4. Note that from the proof of Theorem 4.4, $L(\mathbb{R}^*, \mathcal{F}^*) = L(\mathbb{R}^*, \mathcal{F})$ where $\mathcal{F}$ is the club measure on $\mathcal{P}_{\omega_1}(\mathbb{R}^*)$. We want to show the analogy of Lemma 6.4 in [29], that is

Lemma 2.1.4. *Let $H \subseteq Col(\omega, < \gamma)$ be generic, $\mathbb{R}^*$ be the symmetric reals, $x \in \mathbb{R}^{V[G\restriction\alpha]}$ for some $\alpha < \gamma$, and ψ be a formula in the language of set theory with an additional predicate symbol. Suppose*

$$\exists B \in L(\mathbb{R}^*, \mathcal{F}^*)((HC^*, \in, B) \vDash \psi[x])$$

then

$$\exists B \in Hom_{<\gamma}^{V[G\restriction\alpha]}((HC^{V[G\restriction\alpha]}, \in, B) \vDash \psi[x]).$$

Such a B in Lemma 2.1.4 is called a ψ-witness. Assuming this, the lemma follows from the proof of Theorem 6.1 from Lemma 6.4 in [29]. To see that Lemma 2.1.4 holds, pick the least γ_0 such that some $OD(x)^{L(\mathbb{R}^*,\mathcal{F}^*)}$ ψ-witness B is in $L_{\gamma_0}(\mathbb{R}^*, \mathcal{F}^*)$ and by minimizing the sequence of ordinals in the definition of B, we may assume B is definable (over $L_{\gamma_0}(\mathbb{R}^*, \mathcal{F}^*)$) from x without ordinal parameters. We may as well assume $x \in V$. We want to produce an absolute definition of B as in the proof of Lemma 6.4 in [29]. We do this as follows. First let φ be such that

$$u \in B \Leftrightarrow L_{\gamma_0}(\mathbb{R}^*, \mathcal{F}^*) \vDash \varphi[u, x],$$

and

$$\overline{\psi}(v) = \text{“}v \text{ is a } \psi\text{-witness”}.$$

Let $\mathcal{C}$ denote the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and $\theta(u, v)$ be the natural formula defining B:

$$\begin{aligned}\theta(u,v) \quad = \quad & \text{“}L(\mathbb{R},\mathcal{C}) \vDash \mathcal{C} \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R}) \text{ and } L(\mathbb{R},\mathcal{C}) \vDash \exists B\overline{\psi}[B] \\ & \text{and if } \gamma_0 \text{ is the least } \gamma \text{ such that } L_\gamma(\mathbb{R},\mathcal{C}) \vDash \exists B\overline{\psi}[B] \\ & \text{then } L_{\gamma_0}(\mathbb{R},\mathcal{C}) \vDash \varphi[u,v]\text{”}.\end{aligned}$$

We apply the tree production lemma (see [29]) to the definition $\theta(u, v)$ with parameter $x \in \mathbb{R}^V$. It's clear that stationary correctness holds. To verify generic absolutenss, let $\delta < \gamma$ be a Woodin cardinal; let g be $< \delta$ generic over V and h be $< \delta^+$ generic over $V[g]$. We want to show that if $y \in \mathbb{R}^{V[g]}$

$$V[g] \vDash \theta[y, x] \Leftrightarrow V[g][h] \vDash \theta[y, x].$$

There are $G_0, G_1 \subseteq Col(\omega, < \gamma)$ such that G_0 is generic over $V[g]$ and G_1 is generic over $V[g][h]$ with the property that $\mathbb{R}^*_{G_0} = \mathbb{R}^*_{G_1}$ and furthermore, if $\eta < \gamma$ is a limit of Woodin cardinals above δ, then $\mathbb{R}^*_{G_0} \restriction \eta = \mathbb{R}^*_{G_1} \restriction \eta$[5]. Such G_0 and G_1 exist since h is generic over

[5] $\mathbb{R}^*_{G_0}$ is the symmetric reals defined by G_0 and similarly for $\mathbb{R}^*_{G_1}$. $\mathbb{R}^*_{G_0} \restriction \eta = \mathbb{R}^{V[g][G_0 \cap Col(\omega,<\eta)]}$ and $\mathbb{R}^*_{G_1} \restriction \eta = \mathbb{R}^{V[g][h][G_1 \cap Col(\omega,<\eta)]}$.

$V[g]$ and $\delta < \gamma$. But this means letting $\mathbb{F}_i$ be the tail filter defined from G_i respectively then $L(\mathbb{R}^*_{G_0}, \mathcal{F}_0) = L(\mathbb{R}^*_{G_1}, \mathcal{F}_1)$. The proof of Lemma 4.4 implies that $L(\mathbb{R}, \mathcal{C})^{V[g]}$ is embeddable into $L(\mathbb{R}^*_{G_0}, \mathcal{F}_0)$ and $L(\mathbb{R}, \mathcal{C})^{V[g][h]}$ is embeddable into $L(\mathbb{R}^*_{G_1}, \mathcal{F}_1)$. This proves generic absoluteness. This gives us that $B \cap \mathbb{R}^V \in Hom^V_{<\gamma}$ and $B \cap \mathbb{R}^V$ is a ψ-witness. Hence we're done. $\square$

The proof of the convers of Theorem 2.1.1 is contained in the proof of Theorem 2.1.5, especially that of Lemma 2.1.6.

2.1.2 Structure Theory

We now explore the structure theory of $L(\mathbb{R}, \mu)$ (under determinacy assumption of course). We prove the following theorem, which is also due to Woodin.

Theorem 2.1.5 (Woodin)**.** *The following holds in $L(\mathbb{R}, \mu)$ assuming $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+ + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.*

1. *$(L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})[\mu], \mu \restriction L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})[\mu]) \prec_{\Sigma_1} (L(\mathbb{R}, \mu), \mu)$; furthermore, $\mu \restriction L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})[\mu]$ is contained in the club filter.*

2. *Suppose $L(\mathbb{R}, \mu) \vDash \mu_0, \mu_1$ are normal fine measures on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then $L(\mathbb{R}, \mu) \vDash \mu_0 = \mu_1$.*

To prove (1), we first assume in some generic extension of $L(\mathbb{R}, \mu)$, there is a class model N such that

1. $N \vDash \mathsf{ZFC}+$ there are ω^2 Woodin cardinals;

2. letting λ be the sup of the Woodin cardinals of N, $\mathbb{R}$ can be realized as the symmetric reals over N via $Col(\omega, < \lambda)$;

3. letting $\mathcal{F}$ be the tail filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ in $N[G]$ where $G \subseteq Col(\omega, < \lambda)$ is a generic over N such that $\mathbb{R}$ is the symmetric reals induced by G, $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F})$.

In $N[G]$, let $D = L(\Gamma, \mathbb{R})$ where $\Gamma = \{A \subseteq \mathbb{R} \mid L(A, \mathbb{R}) \vDash \mathsf{AD}^+\}$. Woodin has shown that $D \vDash \mathsf{AD}^+$ and $\Gamma = \mathcal{P}(\mathbb{R})^D$. We claim that $\Gamma = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$. Suppose not, then there is an $A \in D \backslash L(\mathbb{R}, \mu)$. By general theory of AD^+, $\Theta^{L(\mathbb{R},\mu)}$ is a Suslin cardinal in D and $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)} \subseteq Hom^*$ where Hom^* is the pointclass of Suslin co-Suslin sets of D. By the proof of Lemma 4.4, $\mathcal{F} \cap L(\mathbb{R}, \mathcal{F}) = \mathcal{C} \cap L(\mathbb{R}, \mathcal{F})$ where $\mathcal{C}$ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$. This shows $L(\mathbb{R}, \mu)$"$\in$"D and furthermore, $(\mathbb{R}, \mu)^\sharp$ exists in D. This is a contradiction to the fact that D is in a generic extension of $L(\mathbb{R}, \mu)$.

Suppose $(L(\mathbb{R}, \mu), \mu) \vDash \phi$ where ϕ is a Σ_1 statement. Since $L(\mathbb{R}, \mu) \vDash \Theta$ is regular, by a standard argument, $(L_\Theta(\mathbb{R}, \mu), \mu) \prec_1 (L(\mathbb{R}, \mu), \mu)$. This means there is a $\kappa < \Theta$ such that

$(L_\kappa(\mathbb{R},\mu),\mu\cap L_\kappa(\mathbb{R},\mu))\vDash\phi$. There is a set $B\subseteq\mathbb{R}$ in $L(\mathbb{R},\mu)$ such that B codes the structure $(L_\kappa(\mathbb{R},\mu),\mu\cap L_\kappa(\mathbb{R},\mu))$ and hence there is a φ such that

$$(L(\mathbb{R},\mu),\mu)\vDash\phi\Leftrightarrow(HC,\in,B)\vDash\varphi.$$

By the existence of N and the previous section, there is $\alpha<\lambda$ and a $B\in N^{[G\restriction\alpha]}$ such that

$$(HC^{N[G\restriction\alpha]},\in,B)\vDash\varphi.$$

But $(HC^{N[G\restriction\alpha]},\in,B)\prec(HC,\in,B^*)$ where $B^*\in Hom^*$ is the canonical blowup of B. This gives us a $\kappa<\underset{\sim}{\delta}^2_1$ such that $(L_\kappa(\mathbb{R},\mu),\mu\cap L_\kappa(\mathbb{R},\mu))\vDash\phi$. Since ϕ is Σ_1, we have $(L_{\underset{\sim}{\delta}^2_1}(\mathbb{R},\mu),\mu\cap L_{\underset{\sim}{\delta}^2_1}(\mathbb{R},\mu))\vDash\phi$.

Lemma 2.1.6. *There is a forcing notion $\mathbb{P}$ in $L(\mathbb{R},\mu)$ and there is an N in $L(\mathbb{R},\mu)^{\mathbb{P}}$ satisfying (1)-(3) above.*

Proof. Working in $L(\mathbb{R},\mu)$, fix a tree T for a universal Σ^2_1 set. For any real x, by a Σ^2_1 degree d_x, we mean the equivalence class of all y such that $L[T,y]=L[T,x]$. If d_1,d_2 are Σ^2_1 degrees, we say $d_1\leq d_2$ if for any $x\in d_1$ and $y\in d_2$, $x\in L[T,y]$. Let $\mathbb{D}=\{\langle d_i\mid i<\omega\rangle\mid\forall i(d_i\text{ is a }\Sigma^2_1\text{ degree and }d_i\leq d_{i+1})\}$.

Next, we define a measure ν on $\mathbb{D}$. We say $A\in\nu$ iff for any ∞-Borel code S for A, $\forall^*_\mu\sigma\ L[T,S](\sigma)\vDash\mathsf{AD}^++\sigma=\mathbb{R}+\exists(\emptyset,U)\in\mathbb{P}_{\Sigma^2_1}\ (\emptyset,U)\Vdash\dot{G}\in\mathcal{A}_S$. In the definition of ν, $\mathbb{P}_{\Sigma^2_1}$ is the usual Prikry forcing using the Σ^2_1 degrees in $L[T,S](\sigma)$, $\dot{G}$ is the name for the corresponding Prikry sequence, $\mathcal{A}_S$ is the set of reals coded by S. Note that whether $A\in\nu$ does not depend on the choice of S. To see this, let S_0,S_1 be codes for A. Let $T^\infty=\prod_\sigma T$ and $S^\infty_i=\prod_\sigma S_i$ be the ultraproducts by μ. Then since $L[T^\infty,S^\infty_0](\mathbb{R})\cap\mathcal{P}(\mathbb{R})=L[T^\infty,S^\infty_1](\mathbb{R})\cap\mathcal{P}(\mathbb{R})=L(\mathbb{R},\mu)\cap\mathcal{P}(\mathbb{R})$, the $\mathbb{P}_{\Sigma^2_1}$ forcing relations in these models are the same, in particular, $L[T^\infty,S^\infty_0](\mathbb{R})\vDash\exists(\emptyset,U)\in\mathbb{P}_{\Sigma^2_1}\ (\emptyset,U)\Vdash\dot{G}\in\mathcal{A}_{S^\infty_0}$ if and only if $L[T^\infty,S^\infty_1](\mathbb{R})\vDash\exists(\emptyset,U)\in\mathbb{P}_{\Sigma^2_1}\ (\emptyset,U)\Vdash\dot{G}\in\mathcal{A}_{S^\infty_1}$. The claim follows from Los' theorem.

Let $\mathbb{P}$ be the usual Prikry forcing using ν. Conditions in $\mathbb{P}$ are pairs (p,U) where $p=\langle\vec{d^i}\mid i\leq n\wedge\vec{d^i}\in\mathbb{D}\wedge\vec{d^i}\in\vec{d^{i+1}}(0)\rangle$ and U is a ν splitting tree[6] with stem p. $(p,U)\leq_{\mathbb{P}}(q,W)$ if p end extends q and $U\subseteq W$. $\mathbb{P}$ has the usual Prikry property, that is given any condition (p,U), a term τ, a formula $\varphi(x)$, we can find a $(p,U')\leq_{\mathbb{P}}(p,U)$ such that (p,U') decides the value of $\varphi[\tau]$. Let G be $\mathbb{P}$ generic. We identify G with the union of the stems of conditions in G, i.e., G is identified with $\langle\vec{d^i}\mid i<\omega\wedge\exists U(\langle d^j\mid j\leq i\rangle,U)\in G\rangle$. We need some notations before proceeding. We write V for $L(\mathbb{R},\mu)$; for any $g\in\mathbb{D}$, let $\omega^g_1=\sup_i\omega_1^{L[T^\infty,g(i)]}$ and $\delta(g\restriction i)=\omega_2^{L[T^\infty,g\restriction i]}$. To produce a model with ω^2 Woodin cardinals, we use the following theorem.

Theorem 2.1.7 (Woodin)**.** *Assume AD^+. Let R,S be sets of ordinals. Then for a (Turing, Σ^2_1) cone of x, $HOD_R^{L[R,S,x]}\vDash\omega_2^{L[R,S,x]}$ is a Woodin cardinal.*

[6]This means if $q\in U$, then $\forall^*_\nu\vec{d}q^\frown\vec{d}\in U$.

For any countable transitive a which admits a well-ordering rudimentary in a and for any real x coding a, let

$$Q^x_a = \mathrm{HOD}^{L[T^\infty,x]}_{T^\infty,a} \restriction (\delta(x)+1).$$

We now let

$$Q^0_0 = Q^{\vec{d^0}(0)}_{\emptyset},$$

and

$$\delta^0_0 = \delta(\vec{d^0}(0)).$$

For $i < \omega$, let

$$Q^0_{i+1} = Q^{\vec{d^0}(i+1)}_{Q^0_i},$$

and

$$\delta^0_{i+1} = \delta(\vec{d^0}(i+1)).$$

This finishes the first block. Let $Q^0_\omega = \cup_i Q^0_i$. In general, we let

$$Q^{j+1}_0 = Q^{\vec{d^{j+1}}(0)}_{Q^j_\omega},$$

and

$$\delta^{j+1}_0 = \delta(\vec{d^{j+1}}(0)).$$

For $i < \omega$, let

$$Q^{j+1}_{i+1} = Q^{\vec{d^{j+1}}(i+1)}_{Q^{j+1}_i},$$

and

$$\delta^{j+1}_{i+1} = \delta(\vec{d^{j+1}}(i+1)).$$

We observe that the following hold.

1. for all i, $\mathrm{HOD}^{(V[G],V)}_{\{G\}} \cap V[G]_{\omega_1^{G(i)}} = \mathrm{HOD}^V_{\{G\restriction(i+1)\}} \cap V_{\omega_1^{G(i)}}$;

2. for all $a \in \mathbb{D}^{<\omega}$, $\forall^*_\nu g\ \forall i,\ L[T^\infty, a, g] \cap V_{\delta(g\restriction i)+1} = \mathrm{HOD}^V_{\{a,g\}} \cap V_{\delta(g\restriction i)+1} = \mathrm{HOD}^V_{\{a,g\restriction i\}} \cap V_{\delta(g\restriction i)+1} = L[T^\infty, a, g \restriction i] \cap V_{\delta(g\restriction i)+1}$.

3. for any a as above, for a cone of d, $\mathcal{P}(a) \cap Q^d_a \subseteq L[T, a]$.

(1) follows from the Prikry property of $\mathbb{P}$. (2) follows from the definition of ν since $\forall^*_\nu g$, g is a Prikry generic for some local $\mathbb{P}_{\Sigma^2_1}$. To see (3), assume not. For a cone of d, let $b_d =$ the least $b \subseteq a$ such that $b \in Q^d_a \backslash L[T^\infty, a]$. Since a is countable, there is a fixed b such that $b = b_d$ for a cone of d. But then b is $OD(T, a)$ which implies $b \in L[T^\infty, a]$ by a standard arguments. Contradiction.

By (1)-(3) and the above construction, in $\mathrm{HOD}^{(V[G],V)}_{\{G\}}$, the inner model

$$N = L[T^\infty, \langle Q^i_j \mid i,j < \omega\rangle] \vDash \delta^i_j \text{ is a Woodin cardinal for all } i,j < \omega.$$

Letting $\lambda = \sup_{i,j} \delta^i_j$, by Vopenka, there is a $G \subseteq Col(\omega, < \lambda)$ generic over N such that $\mathbb{R}^*_G = \mathbb{R}^V$. In $N[G]$, let $\mathcal{F}$ be the tail filter. It remains to see that $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F})$. Suppose $A \in L(\mathbb{R}, \mathcal{F})$ is such that $A \in \mathcal{F}$ but $\mu(A) = 0$. Let $A^* = \{d \in \mathbb{D} \mid \cup d \in A\}$. Then $\nu(A^*) = 0$. Recall $G = \langle \vec{d^i} \mid i < \omega\rangle$ is our Prikry generic. Since the Prikry forcing is done relative to ν and $\nu(A^*) = 0$, only finitely many $\vec{d^i}$ are in A^*. Since $A \in \mathcal{F}$, $\exists m \forall m \geq n(\vec{d^n} \in A^*)$. Contradiction. Hence we're done. □

The lemma finishes the proof of (1) in Theorem 2.1.5. (2) of Theorem 2.1.5 is also a corollary of the proof of Lemma 2.1.6. One first modifies the definition of $\mathbb{P}$ in Lemma 2.1.6 by redefining the tree U in the condition (p, U) to be ν_0-splitting at the even levels and ν_1-splitting at the odd levels where ν_i is defined from μ_i in the exact way that ν is defined from μ in the proof of Lemma 2.1.6. Everything else in the proof of the lemma stays the same. This implies $L(\mathbb{R}, \mu_0) = L(\mathbb{R}, \mu_1) = L(\mathbb{R}, \mathcal{F})$ and $\mu_0 = \mu_1 = \mathcal{F}$. To see this, just note that since we already know $L(\mathbb{R}, \mathcal{F}) \vDash \mathsf{AD}^+ + \mathcal{F}$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, it suffices to show if $A \in \mathcal{F}$ then $A \in \mu_0$ and $A \in \mu_1$. Suppose there is an $A \in \mathcal{F}$ such that $A \in \mu_0$ and $A \notin \mu_1$ (the cases $A \in \mu_1 \backslash \mu_0$ and $A \notin \mu_0 \cap \mu_1$ are handled similarly). Let A^* be as above. Then $A^* \in \nu_0 \backslash \nu_1$. For any condition (p, U), just shrink U to U^* at the even levels by intersecting with A^* and at the odd levels by intersecting with $\neg A^*$. Then $(p, U^*) \Vdash A \notin \mathcal{F}$. Contradiction. This finishes the proof of Theorem 2.1.5.

Remark: The proof of Theorems 2.1.1 and 2.1.5 also shows that $L(\mathbb{R}, \mu) \vDash \mathsf{AD}$ if and only if $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+$.

2.2 When $\alpha > 0$

2.2.1 The Equiconsistency

This section is devoted to the proof of the following theorem.

Theorem 2.2.1. *1. For $\alpha < \omega$, the theories "$\mathsf{AD}^+ + \mathsf{DC} + \Theta = \theta_0 +$ there is a normal fine measure on X_α" and "$\mathsf{ZFC}+$ there are $\omega^{\alpha+2}$ Woodin cardinals" are equiconsistent. For $\alpha \geq \omega$, the theories "$\mathsf{AD}^+ + \mathsf{DC} +$ there is a normal fine measure on X_α" and "$\mathsf{ZFC}+$ there are $\omega^{\alpha+1}$ Woodin cardinals" are equiconsistent.*

2. *For $\alpha < \omega_1$ limit, the theories "$\mathsf{AD}^+ + \mathsf{DC} + \forall\beta < \alpha\exists\mu_\beta(\mu_\beta$ is a normal fine measure on $X_\beta)$" and "$\mathsf{ZFC}+$ there are ω^α Woodin cardinals" are equiconsistent. For $\alpha = \omega_1$, the theories "$\mathsf{AD}^+ + \mathsf{DC} + \forall\beta < \alpha\exists\mu_\beta(\mu_\beta$ is a normal fine measure on $X_\beta)$" and "$\mathsf{ZFC}+$ there is a κ such that the order type of the set $\{\xi < \kappa \mid \xi$ is Woodin$\}$ is κ are equiconsistent.*

3. *The theories "$\mathsf{AD}^+ + \mathsf{DC}+$ there is a normal fine measure on X_{ω_1}" and "$\mathsf{ZFC} + \exists\kappa$ (κ is mesurable $\wedge\, \kappa$ is a limit of Woodin cardinals)" are equiconsistent.*

We begin with the following defintion.

Definition 2.2.2. *For each $\alpha < \omega_1$, let $\mathbb{D}$ be the set of all Σ^2_1 degrees in $L(\mathbb{R}, \mu_\alpha)$*[7].

$$\mathbb{D}_\alpha = \{g : \omega^{\alpha+1} \to \mathbb{D} \mid g(\beta) < g(\beta+1) \wedge g(\beta) = \sup_{\gamma<\beta} g(\gamma) \text{ for limit } \beta\}.$$

Definition 2.2.3. *Let $\mathbb{D}$ be as in Definition 2.2.2 and let μ be the cone measure on $\mathbb{D}$. Let $\mathbb{P}_{-1}$ be the Prikry forcing relative to μ. Conditions in $\mathbb{P}_{-1}$ are (p, T) where p is an increasing sequence of Σ^2_1 degrees and T is a μ-splitting tree with stem p. The ordering $\leq_{\mathbb{P}_{-1}}$ on $\mathbb{P}_{-1}$ is as follows:*

$$(p, T) \leq_{\mathbb{P}_{-1}} (q, S) \Leftrightarrow p \text{ end extends } q \wedge T \subseteq S.$$

We now prove (1). We first consider the case $\alpha < \omega$. We assume $\alpha > 0$ as the case $\alpha = 0$ is dealt with in the previous subsection. We prove the theorem for $\alpha = 1$. The cases where $1 < \alpha < \omega$ are proved similarly. Suppose $\mathsf{AD}^+ + \mathsf{DC}+$ there is a normal fine measure on X_1. Let μ_1 be such a measure and work in $L(\mathbb{R}, \mu_1)$, which satisfies $\mathsf{AD}^+ + \mathsf{DC} + \mu_1$ is a normal fine measure on X_1. For each $f \in X_1$, let $\mathbb{R}_f = \cup_\beta f(\beta)$ and let $\mathcal{F}_f$ be the tail filter on $\mathbb{R}_f$ defined as follows: for any $A \in \mathcal{P}_{\omega_1}(\mathbb{R}_f)$,

$$A \in \mathcal{F}_f \Leftrightarrow \exists n\forall m \geq n(f(m) \in A).$$

For $f \in X_1$, let $M_f = \mathrm{HOD}^{(L(\mathbb{R},\mu_1),\mu_1)}_{\mathbb{R}_f\cup\{\mathcal{F}_f\}}$. Note that if $f_1, f_2 \in X_1$ are such that $f_1 =^* f_2$, i.e. $\exists\beta_1, \beta_2$ such that $f_1^{\beta_1} = f_1^{\beta_2}$, then $M_{f_1} = M_{f_2}$. Let $M = \prod_f M_f/\mu_1$. Then it's easy to verify that Los theorem holds for this ultraproduct with respect to shift invariant functions, that is if $F(f_1) = F(f_2)$ whenever $f_1 =^* f_2$ and φ is a formula, then $\forall^*_{\mu_1} f M_f \vDash \varphi[F(f)] \Leftrightarrow M \vDash \varphi[[F]_{\mu_1}]$.

Lemma 2.2.4. *For a μ_1-measure one f, the following hold:*

1. *$\mathbb{R}_f = \mathbb{R}^{M_f}$;*

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[7]Working in $L(\mathbb{R}, \mu_\alpha)$, suppose $x, y \in \mathbb{R}$. We say $x \leq y$ if $x \in \mathrm{HOD}^{(L(\mathbb{R},\mu_\alpha),\mu_\alpha)}_y$. We say x and y are Σ^2_1-equivalent, and write $x \equiv y$ if $x \leq y$ and $y \leq x$. Finally, $x < y$ if $x \leq y$ and $y \not\leq x$. A Σ^2_1 degree $\mathbf{d}$ is an equivalence class consisting of reals which are Σ^2_1-equivalent. $\leq$ naturally induces a partial order on Σ^2_1 degrees. Also under AD the cone filter on the Σ^2_1 degree is an ultrafilter.

2. *$M_f \vDash \mathsf{AD}^+ + \mathcal{F}_f$ is a normal fine measure on X_0.*

Proof. Suppose (1) fails, which means $\mathbb{R}^M \neq \mathbb{R}$ (note that the functions $F_1(f) = \mathbb{R}_f$ and $F_2(f) = \mathbb{R}^{M_f}$ are shift invariant). Let $x \in \mathbb{R}\backslash\mathbb{R}^M$. So $x = [\lambda f.x]_{\mu_1}$ and the function $F(f) = x$ for all f is obviously shift-invariant. By fineness, $\forall^*_{\mu_1} f(x \in \mathbb{R}_f)$. This is a contradiction.

We now verify $\forall^*_{\mu_1} f(M_f \vDash \mathcal{F}_f$ is a measure). Suppose not. For each such f, let $F(f) = \{x \in \mathbb{R}_f \mid \exists A \in OD^{M_f}(x, \mathcal{F}_f)(A \notin \mathcal{F}_f \wedge \neg A \notin \mathcal{F}_f)\}$. By normality, we may assume $\exists x \in \mathbb{R}\forall^*_{\mu_1} f\ (x \in F(f) \subseteq f(0))$. For each such f, let A_f be the least $OD^{M_f}(x, \mathcal{F}_f)$ set that is not measured by $\mathcal{F}_f$ and suppose that $\forall^*_{\mu_1} f(f(0) \in A_f)$ (the other case is similar). This implies $\forall^*_{\mu_1} f(f(1) \in A_f)$ because $\forall^*_{\mu_1} f(M_f = M_{f^*})$ where $f^*(\beta) = f(\beta+1)$. This easily gives $A_f \in \mathcal{F}_f$. Contradiction.

Fineness is obvious. It remains to verify normality. By normality of μ_1 and the above argument, we have

$$\forall^*_{\mu_1} f \exists \langle A^f_x \mid x \in \mathbb{R}_f \wedge A^f_x \in \mathcal{F}_f \rangle \in M_f\ (\triangle_x A^f_x \notin \mathcal{F}_f).$$

This means $\forall^*_{\mu_1} f \exists x \in f(0)\ (f(0) \notin A^f_x)$. By normality, we get

$$\exists x \forall^*_{\mu_1} f\ (f(0) \notin A^f_x).$$

Fixing such an x, it's easy to see that for all $n < \omega$, $\forall^*_{\mu_1} f\ (f(n) \notin A^f_x)$. This contradicts the fact that $A^f_x \in \mathcal{F}_f$.

Finally, to show $\forall^*_{\mu_1} f(M_f \vDash \mathsf{AD}^+)$. Suppose $\forall^*_{\mu_1} f(M_f \vDash \neg\mathsf{AD}^+)$. By normality, $\forall^*_{\mu_1} f \exists A_f$ $M_f \vDash A_f$ witnesses $\neg\mathsf{AD}^+$. We may assume that whenever $f_1 =^* f_2$, $A_{f_1} = A_{f_2}$. Hence the function $F(f) = A_f$ is shift invariant. This means $M \vDash [A_f]_{\mu_1}$ witnesses that AD^+ fails. Since $\mathbb{R} \subseteq M \subseteq L(\mathbb{R}, \mu_1) \vDash \mathsf{AD}^+$, $M \vDash \mathsf{AD}^+$. Contradiction. $\square$

Working in $L(\mathbb{R}, \mu_1)$, let T be the tree for a universal Σ^2_1 set and suppose μ_0 is a normal fine measure on X_0. An example of such a μ_0 is the projection of μ_1 to a normal fine measure on X_0. If S is an ω_1-Borel code, we let $\mathcal{A}_S$ be the set interpreted by S. Let ν_0 be a measure on $\mathbb{D}_0$ defined as follows: for any $A \subseteq \mathbb{D}_0$,

$$\begin{aligned} A \in \nu_0 \Leftrightarrow \quad & \text{for any } S\ \omega_1\text{-Borel code of } A\ \forall^*_{\mu_0}\sigma \\ & (L[T,S](\sigma) \vDash \text{“}\mathsf{AD}^+ + \sigma = \mathbb{R} + \exists(\emptyset, U) \in \mathbb{P}_{-1}(\emptyset, U) \vDash \dot{G} \in \mathcal{A}_S\text{”}). \end{aligned}$$

In the above, $\dot{G}$ is the canonical $\mathbb{P}_0$-name for the generic filter. ν_0 is well-defined and is a measure. Let $\mathbb{P}_0$ be the Prikry forcing relative to ν_0. Let ν_1 be the measure on $\mathbb{D}_1$ defined as follows: for any $A \subseteq \mathbb{D}_1$,

$$\begin{aligned} A \in \nu_1 \Leftrightarrow \quad & \text{for any } S\ \infty\text{-Borel code of } A\ \forall^*_{\mu_1} f \\ & (M_f \vDash \mathsf{AD}^+ + \mathbb{R}_f = \mathbb{R} + \exists(\emptyset, U) \in \mathbb{P}_0(\emptyset, U) \vDash \dot{G} \in \mathcal{A}_S), \end{aligned}$$

We rename the filter $\mathcal{F}_f$ defined as above to μ^f_0 to allow for filters of the form μ^f_α for various α that appear later on. The definition of ν_1 makes sense by Lemma 2.2.4. Let $\mathbb{P}_1$ be the

Prikry forcing relative to ν_1. Conditions in $\mathbb{P}_1$ are pairs (p, U) where $p = \langle \vec{d^i} \mid i \leq n \wedge \vec{d^i} \in \mathbb{D}_1 \wedge \vec{d^i} \in d^{\vec{i+1}}(0) \rangle$ and U is a ν_1 splitting tree with stem p. $(p, U) \leq_{\mathbb{P}_1} (q, W)$ if p end extends q and $U \subseteq W$. $\mathbb{P}_0$ and $\mathbb{P}_1$ have the usual Prikry property, that is for $i \in \{0, 1\}$, given any condition (p, U), a term τ, a formula $\varphi(x)$, we can find a $(p, U') \leq_{\mathbb{P}_i} (p, U)$ such that (p, U') decides the value of $\varphi[\tau]$.

Let G be generic for $\mathbb{P}_1$. By a similar proof to that of Lemma 2.1.6, in $L(\mathbb{R}, \mu_1)[G]$, there is an N such that

1. $N \vDash \mathsf{ZFC}+$ there are ω^3 Woodin cardinals;
2. letting γ_k be the limit of the first $\omega^2 k$ Woodin cardinals in N and let $\lambda = \sup_k \gamma_k$, there is $G \subseteq Col(\omega, < \lambda)$ generic over N such that $\mathbb{R}^*_G =_{def} \cup_\alpha \mathbb{R}^{N[G \restriction \alpha]} = \mathbb{R}$;
3. Let G be as above and let $\mathcal{F}$ be defined as in $(*)$ below relative to N, G. Then $L(\mathbb{R}^*_G, Hom^*) = \mathcal{P}(\mathbb{R})^{L(\mathbb{R}^*, \mathcal{F})}$ and $L(\mathbb{R}, \mu_1) = L(\mathbb{R}^*, \mathcal{F})$.

(1) suffices for what we want to prove. We'll use (2) and (3) in the proof of Theorem 2.2.12 and some other occasions.

Now suppose $V \vDash \mathsf{ZFC}+$ there are ω^3 Woodin cardinals. By working in the resulting model of the full background construction $L[E]$, we may assume that letting λ be the sup of the Woodin cardinals, every countable M embeddable into (an sufficiently large initial segment of) V has $Hom_{<\lambda}$ iteration strategy. Let $G \subseteq Col(\omega, < \lambda)$ be V-generic and $\mathbb{R}_G$ be the symmetric reals. Let γ_k be the sup of the first $\omega^2 k$ Woodin cardinals in V. By induction, let $\langle \gamma_0^n \mid n < \omega \rangle$ be the limits of Woodin cardinals below γ_0 and $\langle \gamma_k^n \mid n < \omega \rangle$ be the limits of Woodin cardinals below γ_k and above γ_{k-1}. In $V[G]$, we define the following filter $\mathcal{F}$ as follows

$$A \in \mathcal{F} \Leftrightarrow \exists m \forall n \geq m (\langle \mathbb{R}^{V[G \restriction \gamma_n^k]} \mid k < \omega \rangle \in A). \quad (*)$$

Lemma 2.2.5. *$L(\mathbb{R}_G, \mathcal{F}) \vDash \mathbb{R}_G = \mathbb{R} + \mathsf{AD}^+ + \mathcal{F}$ is a normal fine measure on X_1.*

Proof. That $\mathbb{R}_G = \mathbb{R}^{L(\mathbb{R}_G, \mathcal{F})}$ is clear since $\mathcal{F}$ is definable in $V[G]$ from $\mathbb{R}_G$ and in fact, there is a symmetric term for $\mathcal{F} \restriction V(\mathbb{R}_G)$.

Claim 1. $\mathcal{F}$ is a measure on $L(\mathbb{R}_G, \mathcal{F})$.

Proof. Suppose not. By minimizing counter-examples, we may assume there is an $x \in \mathbb{R}_G$, a $B \in L(\mathbb{R}_G, \mathcal{F})$ not measured by $\mathcal{F}$ such that B is definable from x. By moving to a small generic extension containing x, we may assume $x \in V$; suppose $\langle \mathbb{R}^{V[G \restriction \gamma_0^k]} \mid k < \omega \rangle \in B$ (the $\notin$-case is similar). Hence there is a formula φ such that $\varphi(x, \gamma_0, \lambda)$ holds in $V[G]$. Let M be countable transitive such that there is a $\pi : M \to V$ and $x \in M$. Let Σ be the $Hom_{<\lambda}$ iteration strategy for M. By a standard genericity iteration argument, there is a Σ-iterate M_∞ of M such that there is a $Col(\omega, < \lambda)$ generic H over M_∞ such that $\mathbb{R}_G = \mathbb{R}_H$ and for all $k, n < \omega$, $\mathbb{R}^{V[G\restriction\gamma^n_{k+1}]} = \mathbb{R}^{M_{\omega_1}[H\restriction(\gamma^n_k)^{M_\infty}]}$. We note here that $L(\mathbb{R}_G, \mathcal{F})^{M_\infty[H]} = L(\mathbb{R}_G, \mathcal{F})^{V[G]}$ (up to the ordinal height of M_∞). Since $\varphi(x, \gamma_0, \lambda)$ holds in $M_\infty[H]$, $\langle \mathbb{R}^{V[G\restriction\gamma^n_1]} \mid n < \omega\rangle \in B$. Repeating the argument gives us that for all $k < \omega$, $\langle \mathbb{R}^{V[G\restriction\gamma^n_k]} \mid n < \omega\rangle \in B$, which means $B \in \mathcal{F}$. Contradiction. □

Claim 2. $\mathcal{F}$ is normal fine in $L(\mathbb{R}_G, \mathcal{F})$.

Proof. Fineness is obvious. Let us verify normality. Property (2) of normality follows from the proof of Claim 1. The idea is the following: if $A \in \mathcal{F}$ and assume without loss of generality that $f_{\omega n, \omega(n+1)} =_{def} \langle \mathbb{R}^{V[G\restriction\gamma^k_n]} \mid k < \omega\rangle \in A$ for all n. Fix $m < \omega$. Using the same notation as in Claim 1, we can iterate M to M_∞ so that for all n $(f_{\omega n, \omega(n+1)})^{M_\infty[H]} = f_{\omega(n+m), \omega(n+m+1)}$.

Property (1) of normality is verified as follows. By Lemma 2.0.15, it is enough to verify the following: Suppose $\langle A_x \mid x \in \mathbb{R}_G \wedge A_x \in \mathcal{F}\rangle \in L(\mathbb{R}_G, \mathcal{F})$. Then $\triangle_{x\in\mathbb{R}_G} A_x =_{def} \{f \in X_1 \mid f \in \cap_{x\in f(0)} A_x\} \in \mathcal{F}$. Suppose not. Assume without loss of generality that $f_{0,\omega} \notin \triangle_{x\in\mathbb{R}_G} A_x$, that is $\exists x \in f_{0,\omega}(0) f_{0,\omega} \notin A_x$. Without loss of generality, we may assume $x \in V$. Let M be countable transitive as in the proof of Claim 1 and $x \in M$. By iterating and shifting blocks as in the proof of Claim 1, we have M_∞ such that $(f_{\omega n, \omega(n+1)})^{M_\infty[H]} = f_{\omega(n+1), \omega(n+2)}$. This means $x \in f^{M_{\omega_1}}_{0,\omega}(0)$ and $f^{M_{\omega_1}}_{0,\omega} = f_{\omega, \omega 2} \notin A_x$. Repeating this we get $A_x \notin \mathcal{F}$. Contradiction. □

Claim 3. AD^+ holds in $L(\mathbb{R}_G, \mathcal{F})$.

Proof. It suffices to prove AD holds in $L(\mathbb{R}_G, \mathcal{F})$ since by general AD^+ theory, every set of reals has ∞-Borel code in $L(\mathbb{R}_G, \mathcal{F})$. This is enough to force an N with properties (1)-(3) as above, which will give us AD^+ in $L(\mathbb{R}_G, \mathcal{F})$. Suppose not. Let A be such that A is not determined, A is defined over $L(\mathbb{R}_G, \mathcal{F})$ by (φ, x). Without loss of generality, we may assume $x \in V$. Let

$$\psi(f, x) \equiv Col(\omega, <\lambda) \Vdash L(\mathbb{R}^*_G, \mathcal{F}) \vDash \varphi[f, x].$$

In V, let $B = \{f \mid \psi(f, x)\}$. It's enough to show $B \in Hom^V_{<\lambda}$ since this will imply $A = B^*$, which will give us a contradiction to the fact that A is not determined. To see $B \in Hom^V_{<\lambda}$, it's enough to show B is projective in Σ where Σ is a $Hom_{<\lambda}$ strategy for a countable transitive M containing x and embeddabble into V. But $f \in B$ iff there is a countable iteration tree according to Σ with end model N such that f is generic over N at the first Woodin cardinal of N and $N[f] \vDash \psi[f, x]$. This is because we can further iterate N above the first Woodin to an N_∞ such that there is a generic $H \subseteq Col(\omega, <\lambda^{N_\infty})$ such that $\mathbb{R}^*_H = \mathbb{R}^*_G$ and letting $\mathcal{G}$ be the tail filter defined from (N_∞, H), then $\mathcal{F}$ agrees with $\mathcal{G}$. This finishes the proof of the lemma. □

□

The proof of the second clause of (1) follows from that of the first clause and the proof of (2). We now proceed to the proof of (2). Suppose $\alpha < \omega_1$ is limit. Suppose $V = L(\mathbb{R}, \vec{\mu})$ satisfies AD^+ and $\text{dom}(\vec{\mu}) = \alpha$ and $\forall\beta < \alpha(\vec{\mu}(\beta)$ is a normal fine measure on $X_\beta)$. We first assume

$\alpha = \omega$. Let ν_n be defined from μ_n as above and $\mathbb{P}_n$ be the Prikry forcing associated to ν_n. We define $\mathbb{P}_\omega$, a version of Prikry forcing relative to the sequence of measures $\langle \nu_n \mid n < \omega \rangle$ as follows. $(p, T) \in \mathbb{P}_\omega$ if

- $\exists n < \omega(n = \mathrm{dom}(p))$ and $\forall m < n(p(m) \in \mathbb{D}_m)$;
- $m < n - 1 \to p(m) \in p(m+1)(0)$;
- p is a stem of the tree T;
- $\forall q \in T(m = \mathrm{dom}(q) \to (\forall^*_{\nu_m} f(q \in f(0) \wedge q^\frown f \in T)))$.

Let $G \subseteq \mathbb{P}_\omega$ be generic. Using the construction of Lemma 2.1.6, in $L(\mathbb{R}, \vec{\mu})$, we get a model $N \vDash \mathsf{ZFC} +$ there are ω^ω Woodin cardinals.

Now for the general case of limit $\alpha > \omega$, first fix an $f_\beta : \omega \to \beta$ increasing and cofinal for each limit $\beta \leq \alpha$. We define ν_β from μ_β by induction on $\beta < \alpha$ as above. We define the poset $\mathbb{P}_\alpha$ relative to f_α (a similar comment applies to limits $\beta < \alpha$). This means $(p, T) \in \mathbb{P}_\alpha$ if

- $\exists n < \omega(n = \mathrm{dom}(p))$ and $\forall m < n(p(m) \in \mathbb{D}_{f_\alpha(m)})$;
- $m < n - 1 \to p(m) \in p(m+1)(0)$;
- p is a stem of the tree T;
- $\forall q \in T(m = \mathrm{dom}(q) \to (\forall^*_{\nu_{f_\alpha(m)}} f(q \in f(0) \wedge q^\frown f \in T)))$.

Let $G \subseteq \mathbb{P}_\alpha$ be generic. Again, using the construction of Lemma 2.1.6, in $L(\mathbb{R}, \vec{\mu})$, we get a model $N \vDash \mathsf{ZFC} +$ there are ω^α Woodin cardinals.

For the converse, suppose $V \vDash \mathsf{ZFC} +$ there are ω^α Woodin cardinals. Suppose also that the transitive collapse of a countable elementary substructure of V has $Hom^V_{<\lambda}$-strategy, where λ is the sup of the Woodin cardinals in V. Let $f : \omega^\alpha \to \lambda$ be the increasing and continuous enumeration of the Woodin cardinals and their sups. Fix, for $0 \leq \beta \leq \gamma \leq \alpha$, $f_{\beta,\gamma} : \omega^\beta \to \omega^\gamma$ be increasing and continous (and cofinal if $0 < \beta$) such that each $f(\xi)$ is a limit ordinal. Let $G \subseteq Col(\omega, < \lambda)$ be V-generic and $\mathbb{R}^*_G$ be the symmetric reals. For each $0 \leq \beta < \alpha$, let

$$A \in \mathcal{F}_\beta \Leftrightarrow \exists m \forall n \geq m \langle \mathbb{R}^{V[G \restriction f(f_{\beta, f_{1,\alpha}(n)}(\gamma))]} \mid \gamma < \omega^\beta \rangle \in A$$

Using the techniques developed above, it's not hard to see that $L(\mathbb{R}^*_G, \langle \mathcal{F}_\beta \mid \beta < \alpha \rangle) \vDash \mathsf{AD}^+ + \forall \beta < \alpha(\mathcal{F}_\beta$ is a normal fine measure on $X_\beta)$.

For the second clause of (2). First suppose $V \vDash \mathsf{ZFC} + \lambda = o.t.(\{\delta < \lambda \mid \delta$ is Woodin$\})$. Let κ be the first cardinal such that there is an embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$ and $V_{\kappa+2} \subseteq M$. Fix such a j. Let $\vec{U}$ be a measure sequence on V_κ derived from j such that $\mathrm{cof}(lh(\vec{U})) \geq \kappa^+$ (such a $\vec{U}$ exists by the assumption on κ, see [6]). Let $\mathbb{P}_{\vec{U}}$ be the

Radin forcing defined relative to $\vec{U}$ (see [6]). Let C be a Radin club induced by a $\mathbb{P}_{\vec{U}}$-generic g. By standard theory of Radin forcing, C has order type κ and κ remains regular (hence inaccessible) in $V[g]$ (see Theorem 5.19 in [6]). Let $G \subseteq Col(\omega, < \kappa)$ be $V[g]$ generic. In $V[C,G]$, let $\mathbb{R}^* = \mathbb{R}^{V[C,G]}$ and $\mathcal{F}_\alpha$ be the club filter on X_α in the sense of Definition 4.3.27. The following lemma is key.

Lemma 2.2.6. *In $V[C,G]$, $L(\mathbb{R}^*, \mathcal{F}_\alpha) \vDash \mathcal{F}_\alpha$ is a normal fine measure on X_α. In fact $L(\mathbb{R}^*, \langle \mathcal{F}_\alpha \mid \alpha < \omega_1 \rangle) \vDash \forall \alpha < \omega_1\ \mathcal{F}_\alpha$ is a normal fine measure on X_α.*

Proof. Fix an α. To make the proof simpler notationally, we will assume that $\alpha = \omega^\alpha$. We need the following definition. Suppose $D \subseteq C$ is of order type $\omega\alpha$. D is **good** if D is closed and for any $\beta \in D$, letting β_D^+ be the least element of C bigger than β, we have $(D \upharpoonright \beta + 1, C \upharpoonright \beta + 1) \in V_\gamma[C \upharpoonright \gamma, G \upharpoonright \gamma]$ for some $\gamma < \beta_D^+$. For any limit β of D, let

$$\sigma_\beta = \cup_{\alpha<\beta} \mathbb{R}^{V[C\upharpoonright\alpha, G\upharpoonright\alpha]},$$

and

$$\sigma_D = \langle \sigma_\beta \mid \beta \text{ is a limit in } D \rangle.$$

It's clear that in $V[C,G]$, $\sigma_D \in X_\alpha$. We want to show that the set $A = \{\sigma_D \mid D \text{ is good}\}$ contains a club. To this end, let F be defined as follows. For any $x \in \mathbb{R}^*$, let $\beta \in C$ be the least such that $x \in V[C \upharpoonright \beta, G \upharpoonright \beta]$ and define $F(x)$ to be the first real (in the ordering given by $G \upharpoonright \beta+1$) coding a bijection between ω and $(V_\beta, C \upharpoonright \beta, G \upharpoonright \beta)$. Let $\tau \in \mathrm{cl}_{\alpha,F}$. We want to show that $\tau = \sigma_D$ for some good D. This will show that A contains a club. This is easy. For example, to extract the first ω elements of our desired D, let x_0 be a real that enumerates $(\tau(0), F''\tau(0)^{<\omega})$ in order type ω (we take $x_0 \in \tau(1)$ if $\tau(1)$ exists and x_0 be any such real otherwise). Using x_0 and the fact that $\tau(0)$ is closed under F, we can easily construct an ω-sequence of C coded by x_0 as follows. Let $\langle y_i \mid i < \omega \rangle$ be an enumeration of $\tau(0)$ as coded by x_0. By induction, we construct a sequence $\langle \beta_n^0 \mid n < \omega \rangle$ as follows.

- if $n = 0$, let β_0^0 be the ordinal coded by $F(y_0)$ (note that $\beta_0^0 \in C$);
- if $y_{n+1} \notin V_{\beta_n^0}[C \upharpoonright \beta_n^0, G \upharpoonright \beta_n^0]$, then $\beta_{n+1}^0 > \beta_n^0$ is the ordinal coded by $F(y_{n+1})$; otherwise, $\beta_{n+1}^0 = \beta_n^0$.

We can just repeat this procedure if $\alpha > 1$. For instance, we construct the sequence $\langle \beta_n^1 \mid n < \omega \rangle$ given by $\tau(1)$ as before but we demand that $x_0 \in V_{\beta_0^1}[C \upharpoonright \beta_0^1, G \upharpoonright \beta_0^1]$. It's easy to verify then that the sequence $D = \langle \beta_n^i \mid i < \alpha \wedge n < \omega \rangle \cup \langle \sup_n \beta_n^i \mid n < \omega \wedge i < \alpha \rangle$ is good and furthermore, $\sigma_D = \tau$.

For each $x \in \mathbb{R}^*$, let $A_x = \{\sigma_D \mid D \text{ is good and } x \in \sigma_D(0)\}$. By the discussion above, each A_x contains a club. Let $\mathcal{F}'_\alpha \subseteq \mathcal{F}_\alpha$ be the restriction of $\mathcal{F}_\alpha$ to $\{A_x \mid x \in \mathbb{R}^*\} \cup \{\neg A_x \mid x \in \mathbb{R}^*\}$. It's enough to show that the model $M = L(\mathbb{R}^*, \mathcal{F}'_\alpha) \vDash \mathcal{F}'_\alpha$ is a normal fine measure on X_α. Note that $N \subseteq V(\mathbb{R}^*)$ is a definable class of $V(\mathbb{R}^*)$.

Suppose not. Let A be a counterexample. Without loss of generality, we may assume

A is definable (over $V(\mathbb{R}^*)$ from a real $x \in V$ (otherwise, letting $\beta < \kappa$ be such that $x \in V[C \restriction \beta, G \restriction \beta]$, we can just work in $N = V[C \restriction \beta, G \restriction \beta]$ as the ground model and force with $(\mathbb{P}_{\vec{U}} * Col(\omega, <\kappa))^N$). Suppose without loss of generality that it is forced (by a condition of the form $\langle\langle\emptyset, \vec{U}\rangle, B\rangle$) that $\sigma_D \in A$ where D is just the first $\omega\alpha$ elements of C. Note that D is a good sequence. We want to show that $A \in \mathcal{F}'_\alpha$ by showing that $\sigma_D \in A$ for every good sequence D. Fix such a D. Let $C' = D \cup C \backslash D$. By the basic analysis of Radin forcing, the following hold:

- any closed cofinal subsequence E of C (E need not be in $V[g]$) is Radin generic for $\mathbb{P}_{\vec{U}|\xi}$ over V for some $\xi \leq lh(\vec{U})$;
- if E is a closed cofinal subsequence of C such that there is a $\beta < \kappa$ such that for all $\gamma \geq \beta$, $\gamma \in E \Leftrightarrow \gamma \in C$ then E is V-generic for $\mathbb{P}_{\vec{U}}$.

Hence, C' is Radin generic for $\mathbb{P}_{\vec{U}}$. Let $\gamma = \sup(D)$ and γ^* be the least element of C bigger than γ. Since D is good, $(D, C \restriction \gamma) \in V_\beta[C \restriction \gamma, G \restriction \beta]$ for some $\beta < \gamma^*$. To prove the claim, it's enough to construct (inside $V[C, G]$) a $Col(\omega, \beta)$ generic G^* such that $(C \restriction \gamma, G \restriction \beta) \in V[D, G^*]$. Then, by homogeneity and the Solovay factor lemma for $Col(\omega, <\kappa)$ and nice factoring properties of $\mathbb{P}_{\vec{U}}$, there is a $Col(\omega, <\kappa)$ generic G' extending G^* such that $V[D, G^*][C' \backslash D, G'] = V[C', G']$ and $\mathbb{R}^{V[C,G]} = \mathbb{R}^{V[C',G']}$ which implies $M^{V[C,G]} = M^{V[C',G']}$. Notice then that D is the first $\omega\alpha$ elements of C' so it must be that $\sigma_D \in A$.

We now proceed to the construction of G^*. This is standard. Working in $V[C \restriction \gamma, G \restriction \gamma^*]$, let $\sigma = \mathbb{R}^{V[C \restriction \gamma, G \restriction \beta]}$. Note that σ codes $(C \restriction \gamma, D, G \restriction \gamma)$. Then there is a $G^* \subseteq Col(\omega, <\beta)$ generic over $V[D]$ that realizes σ as the symmetric reals. By absoluteness, there is such a G^* in $V[C \restriction \gamma, G \restriction \gamma^*]$. By the property of σ and G^*, we have $(C \restriction \gamma, G \restriction \gamma) \in V[D, G^*]$. This completes the proof of the lemma. $\square$

The lemma implies that the proof of Theorem 2.1.1 can be used to show that if $H \subseteq Col(\omega, <\lambda)$ is $V[g, G]$-generic, $\mathbb{R}^*$ is the symmetric reals, and μ_α is the tail filter on the X_α of $V[g, G](\mathbb{R}^*)$ for each $\alpha < \lambda$, then in $V[g, G](\mathbb{R}^*)$, the model

$$L(\mathbb{R}^*, \langle\mu_\alpha \mid \alpha < \omega_1\rangle)^8 \vDash \mathsf{AD}^+ + \mu_\alpha \text{ is a normal fine measure on } X_\alpha \text{ for all } \alpha < \omega_1.$$

For the converse, we assume

$$L(\mathbb{R}, \langle\mu_\alpha \mid \alpha < \omega_1\rangle) \vDash \mathsf{AD}^+ + \forall\alpha < \omega_1(\mu_\alpha \text{ is a normal fine measure on } X_\alpha).$$

Working in $L(\mathbb{R}, \langle\mu_\alpha \mid \alpha < \omega_1\rangle)$. We want to force a model of $\mathsf{ZFC}+$ there is a limit of Woodin cardinals κ such that the order type of the set of Woodin cardinals below κ is κ. This is done by defining a Prikry forcing $\mathbb{P}_{\omega_1}$ as follows. Let μ be the club measure on ω_1 and ν_α be the measure defined from μ_α for each $\alpha < \omega_1$. Elements of $\mathbb{P}_{\omega_1}$ are pairs (p, U) where $\exists n$ such that $p = \langle\vec{d^i} \mid i \leq n \wedge \forall i \leq n \exists\alpha_i < \omega_1(\vec{d^i} \in \mathbb{D}_{\alpha_i}) \wedge \forall i < n(\alpha_i < \alpha_{i+1})\rangle$ and U

[8] Note that in $V[g, G](\mathbb{R}^*)$, $\lambda = \omega_1$.

is a tree of height ω with stem p and $\forall q \in U \forall^*_\mu \alpha \forall^*_{\nu_\alpha} f q^\frown f \in U$. It's easy to see $\mathbb{P}_{\omega_1}$ has the Prikry property and forcing with $\mathbb{P}_{\omega_1}$ gives the desired large cardinal property.

We now prove (3). Suppose $V \vDash \mathsf{ZFC} + \exists\kappa(\kappa$ is a measurable cardinal which is a limit of Woodin cardinals). Suppose furthermore that if $\theta >> \kappa$ and $X \prec V_\theta$ is countable and $\pi : M_X \to V_\theta$ is the uncollapse map, then M_X has $Hom_{<\kappa}$ iteration strategy. Let $f : \kappa \to \kappa$ be the increasing continuous enumeration of the set $S = \{\alpha < \kappa \mid \alpha$ is a limit of Woodin cardinals$\}$ and μ a normal measure on κ. Let $G \subseteq Col(\omega, <\kappa)$ be generic over V and for each $\alpha \in S$, $\mathbb{R}_\alpha = \cup_{\beta<\alpha} \mathbb{R}^{V[G\restriction\beta]}$. Also let $\mathbb{R}^*_G = \mathbb{R}^{V[G]}$. We define a filter $\mathcal{F}_G$ as follows. First, let $j : V \to M$ be the ultrapower map by μ and H be $V[G]$-generic such that j lifts to an embedding (which we also denote by j) from $V[G]$ to $M[G][H]$. Say

$$A \in \mathcal{F}_G \Leftrightarrow \langle \mathbb{R}_{j(f)(\alpha)} \mid \kappa \leq \alpha < j(\kappa)\rangle \in j(j)(A).^9$$

It's not hard to see that $L(\mathbb{R}^*_G, \mathcal{F}_G) \vDash$ "$\mathbb{R} = \mathbb{R}^*_G \wedge \mathcal{F}_G$ is a measure on X_{ω_1}".

Lemma 2.2.7. *$L(\mathbb{R}^*_G, \mathcal{F}_G) \vDash \mathcal{F}_G$ is normal and fine.*

Proof. We verify fineness first. Let $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}^*_G)$ and

$$X_\sigma = \{h \in X_{\omega_1} \mid \sigma \in h(0)\}.$$

We want to show $X_\sigma \in \mathcal{F}_G$. Let $x \in \mathbb{R}^*_G$ code σ and $\beta < \kappa$ be such that $x \in V[G \restriction \beta]$. Working in $V[G \restriction \beta]$, let μ^* be the natural extension of μ. Let $j^* : V[G \restriction \beta] \to M[G \restriction \beta]$ be the ultrapower embedding by μ^*. Let $G^* \subseteq Col(\omega, <\kappa)$ be $V[G \restriction \beta]$ generic such that $V[G] = V[G \restriction \beta][G^*]$. Let $\mathcal{F}_{G^*}$ be defined over $V[G \restriction \beta][G^*]$ from μ^* the same way $\mathcal{F}_G$ be defined over $V[G]$ from μ. It's clear that $L(\mathbb{R}^*_G, \mathcal{F}_{G^*}) = L(\mathbb{R}^*_G, \mathcal{F}_G) \vDash \mathcal{F}_G = \mathcal{F}_{G^*}$ and $X_\sigma \in \mathcal{F}_{G^*}$. This means $X_\sigma \in \mathcal{F}_G$.

We now verify normality. (3) of Definition 2.0.14 follows easily from the (equivalent) definition of $\mathcal{F}_G$. To verify (1) of Definition 2.0.14, note that if F is as in the statement of (1), then $j(j)(F)(\langle \mathbb{R}_{j(f)(\alpha)} \mid \kappa \leq \alpha < j(\kappa)\rangle \subseteq \mathbb{R}^*_G$ and is nonempty. Let $x \in j(j)(F)(\langle \mathbb{R}_{j(f)(\alpha)} \mid \kappa \leq \alpha < j(\kappa)\rangle$. Then x witnesses the conclusion of (1). (2) of Definition 2.0.14 is verified using the iterability of M_X in M to shift blocks as before (and note that M_X has $Hom_{<j(\kappa)}$-strategy in M). We leave the details to the reader. □

Lemma 2.2.8. $L(\mathbb{R}, \mathcal{F}_G) \vDash \mathsf{AD}^+$.

Proof. Suppose $B \in L(\mathbb{R}^*_G, \mathcal{F}_G) \cap \mathcal{P}(\mathbb{R})$ is defined by (φ, x) over $L(\mathbb{R}^*_G, \mathcal{F}_G)$. That is,

$$y \in B \Leftrightarrow L(\mathbb{R}^*_G, \mathcal{F}_G) \vDash \varphi[y, x].$$

[9] We abuse the notation a bit here in this definition. $j(j)$ is the ultrapower map from M into $Ult(M, j(\mu)) = N$ and it lifts to $j(j) : M[G][H] \to N[G][H][K]$ for some $K \subseteq Col(\omega, < j(j)(\kappa))$. The definition is equivalent to $A \in \mathcal{F}_G \Leftrightarrow \forall^*_\mu \alpha \forall^*_\mu \beta \Vdash_{Col(\omega,<\beta)} f_{\alpha,\beta} \in A$.

Without loss of generality, we simplify our proof by assuming no ordinal parameters are involved in the definition of B and $x \in \mathbb{R}^V$. Fix $\theta >> \kappa$ and let $X \prec V_\theta$ be such that X is countable and $(x, \mu) \in X$. Let $\pi : M \to V_\theta$ be the uncollapse map and $(\kappa^*, \nu) = \pi^{-1}(\kappa, \mu)$. By our assumption, let Σ be a $Hom^V_{<\kappa}$-iteration strategy for M. Letting δ_0^M be the first Woodin cardinal of M, the set B can be defined as follows

$$\begin{aligned} y \in B \quad \Leftrightarrow \quad & \exists(\mathcal{T}, b) \in \Sigma \exists g (b = \Sigma(\mathcal{T}) \wedge g \subseteq Col(\omega, i_b^{\mathcal{T}}(\delta_0^M)) \wedge y \in \mathcal{M}_b^{\mathcal{T}}[g] \\ & \wedge M_b^{\mathcal{T}}[g] \vDash \emptyset \Vdash_{Col(\omega, < i_b^{\mathcal{T}}(\kappa^*))} L(\dot{R}, \dot{\mathcal{F}}) \vDash \varphi[y, x]). \quad (*) \end{aligned}$$

In $(*)$, $\dot{R}$ is the symmetric term for the symmetric reals for $Col(\omega, < i_b^{\mathcal{T}}(\kappa^*))$ and $\dot{F}$ is the symmetric term for the filter defined from $i_b^{\mathcal{T}}(\nu)$ the same way $\mathcal{F}_G$ is defined from μ. It suffices to verify $(*)$ since this implies $B \in Hom^*$.

We now verify $(*)$. It suffices to show that for any Σ-iterate N of M, we can further iterate N to P such that letting κ^P be the image of κ^*, there is an $H \subseteq Col(\omega, < \kappa^P)$ generic over P such that $\exists \alpha < \kappa$ (α is a limit of Woodin cardinals and for all limit of Woodin cardinals $\beta < \kappa$ $(\mathbb{R}^{P[H \restriction \beta]} = \mathbb{R}^{V[G \restriction (\alpha+\beta)]})$). Without loss of generality, we may assume $N = M$ and $\alpha = 0$. By induction, suppose in $V[G \restriction f(\alpha)]$, there is an Σ-iterate M_α of M via $i_\alpha : M \to M_\alpha$ such that letting $\kappa_\alpha = i_\alpha(\kappa^*)$ and $\nu_\alpha = i_\alpha(\nu)$,

1. $f(\alpha) \leq \kappa_\alpha$;
2. $\exists H_\alpha \subseteq Col(\omega, < f(\alpha))$ such that if α is a successor ordinal, then $\forall \beta \leq \alpha (\mathbb{R}^{V[G \restriction f(\beta)]} = \mathbb{R}^{M_\alpha[H \restriction f(\beta)]})$ and if α is limit, then $\forall \beta < \alpha (\mathbb{R}^{V[G \restriction f(\beta)]} = \mathbb{R}^{M_\alpha[H \restriction f(\beta)]})$.

We describe how to fulfill (1) and (2) for $\alpha + 1$. First inside $V[G \restriction f(\alpha+1)]$, let $M_\alpha^* = M_\alpha$ if $f(\alpha) < \kappa_\alpha$ and $M_\alpha^* = Ult(M_\alpha, \nu_\alpha)$. Let $\langle x_n \mid n < \omega \rangle$ be an enumeration of $\mathbb{R}^{V[G \restriction f(\alpha+1)]}$ inside $V[G \restriction f(\alpha+2)]$. Then let $M_{\alpha+1}$ be a Σ-iterate of M_α^* according to the genericity iteration procedure that successively makes x_n generic at δ_n where $\langle \delta_i \mid i < \omega \rangle$ are the first ω Woodin cardinals of M_α^* above $f(\alpha)$. It's easy to check that (1) and (2) holds for $\alpha + 1$.

Let M_κ be the direct limit of the M_α's for $\alpha < \kappa$. It's easy to check that

- $\kappa_\kappa = \kappa$;
- $C = \{\alpha < \kappa \mid \alpha = \kappa_\alpha\} \in \mu$;
- letting $H \subseteq Col(\omega, < \kappa)$ be generic over M_κ such that $\forall \alpha < \kappa (\mathbb{R}^{M_\kappa[G \restriction f(\alpha)]} = \mathbb{R}^{V[G \restriction f(\alpha)]})$, and $\mathcal{F}_H$ be the filter defined from ν_κ, then $L(\mathbb{R}^*_H, \mathcal{F}_H) = L(\mathbb{R}^*_G, \mathcal{F}_G)$.

The key to verifying the last item is that if $B \in \nu_\kappa$ then B contains a tail of C. This finishes the proof of the lemma. □

For the converse of (3). Suppose

$$L(\mathbb{R}, \mu_{\omega_1}) \vDash \mathsf{ZF} + \mathsf{AD}^+ + \Theta = \theta_0 + \mu_{\omega_1} \text{ is a normal fine measure on } X_{\omega_1}.$$

Working in $L(\mathbb{R}, \mu_{\omega_1})$, let ν_{ω_1} be defined from μ_{ω_1} the same way ν_α is defined from μ_α, that is, letting α_f be the order type of the domain of f for all $f \in X_{\omega_1}$ and μ^f_β be the "tail filter" concentrating on sequences of length $\omega^\beta < \alpha$ defined relative to f[10], we say

$$\begin{aligned} A \in \nu_{\omega_1} \Leftrightarrow \quad & \forall S \forall^*_{\mu_{\omega_1}} f (S \text{ is an } \infty\text{-Borel code for } A \wedge f \in X_{\omega_1} \Rightarrow \\ & \mathrm{HOD}_{\{\langle \mu^f_\beta \mid \beta < \alpha_f \rangle\} \cup \mathbb{R}_f} \vDash \exists (\emptyset, U) \Vdash_{\mathbb{P}_{\omega_1}} \dot{g} \in \mathcal{A}_S). \end{aligned}$$

For the definition of ν_{ω_1} to make sense, we need the following lemma.

Lemma 2.2.9. $\forall^*_{\mu_{\omega_1}} f$ *$HOD_{\mathbb{R}_f \cup \langle \mu^f_\beta \mid \beta < \alpha_f \rangle} \vDash \mathsf{AD}^+ + \mathbb{R} = \mathbb{R}_f + \forall \beta < \alpha_f\ \mu^f_\beta$ is a normal fine measure on X_β.*

Proof. We sketch the proof. The proof of $\forall^*_{\mu_{\omega_1}} f\ M_f =_{def} \mathrm{HOD}_{\mathbb{R}_f \cup \langle \mu^f_\beta \mid \beta < \alpha_f \rangle} \vDash \mathsf{AD}^+ + \mathbb{R} = \mathbb{R}_f$ is similar to that of Lemma 2.2.4. We verify the last clause. Suppose $\forall^*_{\mu_{\omega_1}} f \exists \beta < \alpha_f (M_f \vDash \mu^f_\beta$ is not a measure on $X_\beta)$. By normality, there is a $\beta < \omega_1$ such that $\forall^*_{\mu_{\omega_1}} f M_f \vDash \mu^f_\beta$ is not a measure on X_β. We may as well assume β is the least such. By normality, we can choose $\forall^*_{\mu_{\omega_1}} f$ an A_f witnessing this. The A_f's are chosen so that if f and g agree on a tail, then $A_f = A_g$. Suppose without loss of generality that $\forall^*_{\mu_{\omega_1}} f_{0,\beta} \in A_f$. Then $\forall^*_{\mu_{\omega_1}} f\ f_{\beta, \beta+\beta} \in A_f$ etc. This means $\forall^*_{\mu_{\omega_1}} A_f \in \mu^f_\beta$. Contradiction.

Fineness is obvious. To verify normality, note that property (2) of Definition 2.0.14 is obvious from the fact that the μ^f_β's are tail filters. Properties (1) is verified as in the proof of Lemma 2.2.4. □

Let T be a tree for the universal Σ^2_1 set and let $\mathbb{P}_{\omega_1}$ be the Prikry forcing associated to ν_{ω_1}. Now the proof of Theorem 2.1.5 adapted to $\mathbb{P}_{\omega_1}$, we get the following facts (which will not be used in this proof)

$$(L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}, \mu_{\omega_1}), \mu_{\omega_1}) \prec_{\Sigma_1} (L(\mathbb{R}, \mu_{\omega_1}), \mu_{\omega_1})\ (*)$$

and

$$(L(\mathbb{R}, \mu_{\omega_1}), \mu_{\omega_1}) \vDash \mu_{\omega_1} \text{ is unique } (**).$$

Let $G \subseteq \mathbb{P}_{\omega_1}$ be a Prikry generic and

$$\vec{f}_G = \cup \{ p \mid \exists U ((p, U) \in G) \}.$$

Note that $\vec{f}_G$ can compute G. Let $N_G = \cup_i N_{\vec{f}_G(i)}$[11] be the model associated with the generic G. We know that

[10]Note that $\forall^*_{\mu_{\omega_1}} f$ and for all $\beta < \alpha_f$, $\omega^\beta < \alpha_f$. Furthermore, $\mathrm{HOD}_{\{S, \langle \mu^f_\beta \mid \beta < \alpha_f \rangle\} \cup \mathbb{R}_f} \vDash \alpha_f = \omega_1$.

[11]$N_{\vec{f}_G(i)}$ is defined the same way as in the proof of Lemma 2.1.6. There $N_{\vec{f}_G(i)}(j)$ is denoted as Q^i_j. Also $o(N_{\vec{f}_G(i)})$ is the supremum of $\alpha_{\vec{f}_G(i)}$ many $N_{\vec{f}_G(i)}$-Woodin cardinals.

$$N_G \vDash \omega_1^V \text{ is a limit of Woodins and } o.t.(\{\delta < \omega_1^V \mid N_G \vDash \delta \text{ is Woodin}\}) = \omega_1^V,$$

and furthermore by the Prikry property of $\mathbb{P}_{\nu_{\omega_1}}$, $M =_{def} L[T, N_G]$ does not produce bounded subsets of ω_1^V that are not in N_G.

Lemma 2.2.10. *Suppose $\alpha < \omega_1^V$ and $g : \alpha \to \omega_1^V$ is increasing in $V[G]$. Suppose also that $g \in M$. Then g is not cofinal.*

Proof. We first claim that the sequence $\vec{g} = \langle o(N_{\vec{f}_G(i)}) \mid i < \omega\rangle$ is not in M. This is where we use property (3) of Definition 2.0.14. By (3) for example, the sequence $\vec{f'} =_{def} (\vec{f}_G(0)^\frown \vec{f}_G(1))^\frown\langle \vec{f}_G(i) \mid i \geq 2\rangle$ and the filter G'[12] associated with $\vec{f'}$ is also a Prikry generic and $V[\vec{f}_G] = V[\vec{f'}] = V[G] = V[G']$ and $N_G = N_{G'}$[13]. So it's impossible to define $\vec{g}$ from N_G.

Now suppose $g : \omega \to o(N_G)$ is increasing and cofinal and is in M. Let $\langle (q_n, U_n) \mid n < \omega\rangle$ enumerate in descending order the elements of G. Suppose without loss of generality $(q_0, U_0) \in G$ forces all the relevant facts above g. By induction, we define $g'(n), (p_n, T_n) \in \mathbb{P}_{\omega_1}$ as follows. Let $(p_{-1}, T_{-1}) = (q_0, U_0)$ and set $g(-1) = g'(-1) = 0$. Let $g'(n)$ be the least $\alpha > \max\{g'(n-1), g(n-1)\}$ such that $\alpha =$o.t.$\{\gamma < \alpha \mid \gamma \text{ is Woodin in } N_G\}$. Note that $g' \in L[T, N_G]$. For each n, there is a condition $(q, W) \leq (p_{n-1}, T_{n-1})$ and $(q, W) \leq (q_{k_n}, U_{k_n})$, where k_n is the largest k such that $g'(n) \geq o(N_{q_k})$ that satisfies the following:

- $N_q =_{def} \cup_{i<lh(q)} N_{q(i)} \lhd N_G$;
- $o(N_q) = g'(n)$;
- all Woodin cardinals in N_q are Woodin cardinals in N_G.

Let (p_n, T_n) be such a (q, W). Note that we use DC to construct the sequence $\langle (p_n, T_n) \mid n < \omega\rangle$. Let H be the upward closure of $\{(p_n, T_n) \mid n < \omega\}$. By the construction of H and the fact that g' is cofinal in ω_1^V, we get that H is generic. However, $N_G = N_H$ and $g' \in L[T, N_H]$ give us a contradiction to the previous claim.

Now, we're onto the most general case. Assume $\alpha < \omega_1^V$ and $g : \alpha \to \omega_1^V$ is increasing and cofinal and $g \in M$. We aim to get a contradiction. This is easy. Let $x \in V$ code α and let $g' \in L[T, N_G, x]$ be a cofinal map from ω into ω_1^V defined from g and x. Running the same proof as above (but replacing $L[T, N_G]$ by $L[T, N_G, x]$) gives us a contradiction. $\square$

So we have

$$M \vDash \omega_1^V \text{ is an inaccessible limit of Woodin cardinals.}$$

Let μ_G be a filter on ω_1^V defined (in $V[G]$) as follows.

[12] $G' = \{(p, U) \mid p \lhd \vec{f'} \wedge \vec{f'} \in [U]\}$

[13] Formally speaking, we obtain G' as $\pi''G$ where $\pi : \mathbb{P}_{\omega_1} \to \mathbb{P}_{\omega_1}$ is a projection defined as follows. $\pi((\emptyset, U)) = (\emptyset, U)$, $\pi((p, U)) = (\emptyset, U)$ if $lh(p) = 1$, and $\pi((p, U)) = (p^*, U)$ where if $p = (f_0, f_1, \cdots, f_n)$ then $p^* = (f_0^\frown f_1, \cdots, f_n)$.

$$A \in \mu_G \Leftrightarrow \exists n \forall m \geq n \ (o(N_{\vec{f}_G}(m)) \in A).$$

The following lemma completes the proof.

Lemma 2.2.11. *$L[T, N_G][\mu_G] \vDash \mu_G$ witnesses that ω_1^V is a measurable limit of Woodin cardinals.*

Proof. By idempotence (property (3) of Definition 2.0.14), it's easy to see that μ_G measures all subsets of ω_1^V in $L[T, N_G]$. Indeed, if A is defined by a formula ψ with parameters (T, N_G) in $V[G]$ and $o(N_{\vec{f}(0)}) \in A$. Let $\vec{f}', G'$ be defined as in Lemma 2.2.10, we get that $V[G'] = V[G]$ and $N_G = N_{G'}$ and $o(N_{\vec{f}'(0)}) = o(N_{\vec{f}(1)}) \in A$. Repeating this, we get that $A \in \mu_G$.

Next we want to see that $L[T, N_G][\mu_G] \vDash \omega_1^V$ is inaccessible. By the Prikry property, $L[T, N_G][\mu_G]$ adds no bounded subsets of ω_1^V to N_G. By the same proof as that of Lemma 2.2.10, we get that no cofinal $h : \alpha \to \omega_1^V$ (for $\alpha < \omega_1^V$) can exist in $L[T, N_G][\mu_G]$.

Finally, we need to show $L[T, N_G][\mu_G] \vDash \mu_G$ is a countably complete measure. First we verify that $L[T, N_G][\mu_G] \vDash \mu_G$ is a measure. Again, the proof in the first paragraph generalizes here. The point is that letting $\vec{f}', G'$ be as in the first paragraph, not only $V[G] = V[G']$, $N_G = N_{G'}$ but also $\mu_G = \mu_{G'}$. To verify countable completeness, suppose there exists $\langle A_n \mid n < \omega \wedge A_n \in \mu_G \rangle \in L[T, N_G][\mu_G]$ and $A =_{def} \cap_n A_n \notin \mu_G$. Without loss of generality, we may assume $\vec{f}_G(0) \notin A$. So there is an n such that $\vec{f}_G(0) \notin A_n$. Fix such an n. Let $\vec{f}', G'$ be as above. Then we have $V[G] = V[G'], N_G = N_{G'}, \mu_G = \mu_{G'}$, and $o(N_{\vec{f}'(0)}) = o(N_{\vec{f}(1)}) \notin A_n$. Repeating this, we get $A_n \notin A_G$. Contradiction. □

2.2.2 Structure Theory

We prove the following theorem.

Theorem 2.2.12. *Suppose $\alpha \leq \omega_1$ and $L(\mathbb{R}, \mu_\alpha) \vDash \mathsf{AD}^+ + \mathsf{DC} + \mu_\alpha$ is a normal fine measure on X_α. Then for each $\beta \leq \alpha$ there is a unique normal fine measure on X_β in $L(\mathbb{R}, \mu_\alpha)$. If α is limit, $L(\mathbb{R}, \langle \mu_\beta \mid \beta < \alpha \rangle) \vDash \mu_\beta$ is the unique normal fine measure on X_β. Furthermore, if $\alpha < \omega_1$, then $L(\mathbb{R}, \mu_\alpha) \vDash \Theta = \theta_0$*

Proof. For the first clause of the theorem, we only prove a representative case: $\alpha = 1$. The other cases are similar. So assume $\alpha = 1$. So we have

$$L(\mathbb{R}, \mu_1) \vDash \mathsf{AD}^+ + \mu_1 \text{ is a normal fine measure on } X_1.$$

Let μ_0 be the measure on X_0 defined from μ_1, that is letting $\pi : X_1 \to X_0$ be $\pi(f) = \cup_n f(n)$,

$$A \in \mu_0 \Leftrightarrow \pi^{-1}[A] \in \mu_1.$$

We want to show

$$L(\mathbb{R}, \mu_1) \vDash \forall i \in \{0, 1\} \ \mu_i \text{ is the unique normal fine measure on } X_i.$$

The proof is just a combination of techniques used earlier. To show uniqueness of μ_1, let μ_1' be any normal fine measure on X_1 in $L(\mathbb{R},\mu_1)$. Let ν_1' be defined from μ_1' the same way ν_1 is defined from μ_1. Let $\mathbb{P}_1'$ consist of pairs (p,U) where $p=\langle \vec{d^i} \mid i\leq n \wedge \vec{d^i}\in\mathbb{D}_1 \wedge \vec{d^i}\in \vec{d^{i+1}}(0)\rangle$ and U is a tree with stem p that is ν_1-splitting at the even levels and ν_1'-splitting at the odd levels. $(p,U)\leq_{\mathbb{P}_1'}(q,W)$ if p end extends q and $U\subseteq W$. By the same reasoning as in the proof of Theorem 2.1.5, $L(\mathbb{R},\mu_1)=L(\mathbb{R},\mu_1')$ and $\mu_1=\mu_1'$.

Now let μ_0' be a normal fine measure on X_0 and ν_0' be the measure on $\mathbb{D}_0$ induced by μ_0'. To show $\mu_0=\mu_0'$, we modify the forcing $\mathbb{P}_1'$ as follows. Let $\mathbb{P}_1''$ consist of pairs (p,U) such that $p=\langle \vec{d^i} \mid i\leq n\wedge \vec{d^i}\in \vec{d^{i+1}}(0)\wedge 2i+1\leq n\Rightarrow (\vec{d^{2i}}\in\mathbb{D}_0\wedge \vec{d^{2i+1}}\in\mathbb{D}_1)\rangle$ and U is a tree with stem p such that U is ν_0-splitting at levels of the form $4n$, ν_1-splitting at levels of the form $2n+1$, and ν_0'-splitting at levels of the form $4n+2$. The ordering of the poset is the obvious ordering. Let $\langle \vec{d^i}\mid i<\omega\rangle$ be a generic sequence that gives rise to a model $N\vDash \mathsf{ZFC}+\exists\omega^3$ Woodin cardinals and an N-generic G such that $\mathbb{R}^*_G=\mathbb{R}$. Let $\langle \sigma_0^i,\vec{\sigma}^i=\langle\sigma_j^i\mid 1\leq j<\omega\rangle \mid i<\omega\wedge\sigma^i_j\in\mathcal{P}_{\omega_1}(\mathbb{R})\rangle$ be the sequence given by $\langle\vec{d^i}\mid i<\omega\rangle$. Let $\mathcal{F}_0,\mathcal{F}_1$ be defined as follows.

$$A\in\mathcal{F}_1\Leftrightarrow \exists n\forall m\geq n\ \sigma_0^m{}^\frown\vec{\sigma}^m\in A,$$

and

$$A\in\mathcal{F}_0\Leftrightarrow \exists n\forall m\geq n\ \sigma_0^m\in A.$$

It's easy to see that $L(\mathbb{R},\mu_1)=L(\mathbb{R},\mathcal{F}_1)\vDash \mu_1=\mathcal{F}_1\wedge\mu_0=\mu_0'=\mathcal{F}_0$.

For the second clause, let α be limit and denote $\vec{\mu}=\langle\mu_\beta\mid\beta<\alpha\rangle$. We assume $L(\mathbb{R},\vec{\mu})\vDash \mathsf{AD}^+ +\forall\beta<\alpha(\mu_\beta$ is a normal fine measure on $X_\beta)$. Let $f:\omega\to\alpha$ be increasing and cofinal and ν_β be the measure on $\mathbb{D}_\beta$ induced by μ_β. Let $\mathbb{P}_{f,\alpha}$ be a poset defined as follows. Conditions are pairs (p,U) where $p=\langle\vec{d^i}\mid i\leq n\wedge\forall i(d^i\in\mathbb{D}_{f(i)})\rangle$. U is a tree with stem p that is ν_β-splitting at every level $\beta<\alpha$. The ordering is the usual ordering. So in general if $(p,U)\in\mathbb{P}_{f,\alpha}$, p is finite and U is a tree of height α and the function f guides the extension of the stem p. Let G be a generic and $\langle\vec{d^i}\mid i<\omega\rangle$ be the union of the stems of conditions in G. Let N be the model given by U as above. That is $N\vDash\mathsf{ZFC}+$ there exist $\cup_{\beta<\alpha}\omega^\beta$ Woodin cardinals and there is an N-generic H such that $\mathbb{R}^*_H=\mathbb{R}$. In $N[H]$, for each $\beta<\alpha$, let $\mathcal{F}_\beta$ be the tail filter on X_β. Then $L(\mathbb{R},\vec{\mu})=L(\mathbb{R},\vec{\mathcal{F}})$ where $\mathcal{F}=\langle\mathcal{F}_\beta\mid\beta<\alpha\rangle$. This also proves uniqueness of μ_β for all β.

Finally, we prove that for each $\alpha<\omega_1$,

$$L(\mathbb{R},\mu_\alpha)\vDash\Theta=\theta_0.$$

Suppose not. In $L(\mathbb{R},\mu_\alpha)$, let $M=L(\mathcal{P}(\mathbb{R}))$ and $H=\mathrm{HOD}^M_{\mathbb{R}}$. By a refinement of Theorem 2.0.16, if $A\subseteq X_\alpha$ is a Suslin co-Suslin set, there is a mouse M_A that is (coarsely) A-iterable with Suslin co-Suslin iteration strategy. Furthermore, M_A has $\omega^\alpha+1$ Woodin cardinals. By Theorem 1.3 in [18], the real game with ω^α moves with payoff A is determined. This implies that the club filter is a normal fine measure ν on the Suslin co-Suslin subsets of X_α. Now it's not hard to modify Theorem 1 of [47] to show that ν is unique, hence ν agrees with μ_α on the Suslin co-Suslin sets. Furthermore, we get that ν is OD^M and hence $\nu \restriction \mathcal{P}(\mathbb{R})^H \in H$. This implies $L(\mathbb{R}, \mu_\alpha) \in H$. Contradiction. □

We now summarize more useful facts about the model $M = L(\mathbb{R}, \mu_\alpha)$ $(0 < \alpha < \omega_1)$ assuming $M \vDash \mathsf{AD}^+$ that are parallel to those in the case $\alpha = 0$. The proof of the following theorem combines the techniques developed above along with the proof of the corresponding theorems in the case $\alpha = 0$. Since there is nothing new, we will omit the proof.

Theorem 2.2.13. *Suppose $0 < \alpha < \omega_1$ and $M = L(\mathbb{R}, \mu_\alpha) \vDash \mathsf{AD}^+ + \mu_\alpha$ is a normal fine measure on X_α. The following hold in M assuming $M \vDash \mathsf{AD}^+$.*

- *$\Theta = \theta_0$ and $L(\mathcal{P}(\mathbb{R}))^M \vDash \Theta = \theta_0$;*
- *μ_α is the unique normal fine measure on X_α; furthermore, $\mu_\alpha \restriction \utilde{\Delta}^2_1$ is the club filter;*
- *Letting $M_{\utilde{\delta}^2_1} = L_{\utilde{\delta}^2_1}(\mathbb{R}, \mu_\alpha)$, then $(M_{\utilde{\delta}^2_1}, \mu_\alpha \restriction \utilde{\Delta}^2_1) \prec_1 (M, \mu_\alpha)$.*

We now sketch the proof of the following key theorem which is needed in the proof of Theorems 2.3.10 and 2.3.11.

Theorem 2.2.14. *For $-1 \leq \alpha < \omega_1$, let $M = L(\mathbb{R}, \mu_\alpha)$ and suppose if $\alpha = -1$ then $\mu_\alpha = \emptyset$ and if $\alpha \geq 0$ then $M \vDash \mathsf{AD}^+ + \mu_\alpha$ is a normal fine measure on X_α. Suppose $(\mathbb{R}, \mu_\alpha)^\sharp$ exists. Then there is a premouse $\mathcal{M}$ such that*

- *$\mathcal{M}$ is active;*
- *$\mathcal{M} \vDash$ there are $\omega^{\alpha+2}$ Woodin cardinals if $\alpha < \omega$; otherwise, $\mathcal{M} \vDash$ there are $\omega^{\alpha+1}$ Woodin cardinals.*
- *$\mathcal{M}$ is (ω_1, ω_1)-iterable.*

Remark: The case $\alpha = -1$ has been known to Woodin. For $\alpha \geq 0$, like the $\alpha = -1$ case, the proof makes heavy use of the structure theory of M, especially the HOD analysis.

Proof sketch of Theorem 2.2.14. The sketch makes heavy use of the HOD analysis done in the next section. The reader is advised to consult the relevant results there. Let $\Gamma = (\utilde{\Sigma}^2_1)^M$ and $\Omega = \mathrm{Env}(\Gamma)$. We'll use the following facts.

Theorem 2.2.15. *Assume $M, \mu_\alpha, \Gamma, \Omega$ are defined as above and assume $(\mathbb{R}, \mu_\alpha)^\sharp$ exists. Then*

(a) $\Omega = \mathcal{P}(\mathbb{R})^M$;

(b) there is a sjs[14] $\mathcal{A}$ sealing Ω[15];

[14]sjs stands for self-justifying system, which is a countable collection $\mathcal{A}$ of sets of reals such that if $A \in \mathcal{A}$ then so is $\neg A$ and a scale on A has its norms in $\mathcal{A}$.

[15]This means $\mathcal{A}$ is cofinal in Ω

(c) letting A be the universal $(\Sigma^2_1)^M$-set, then there is a sjs $\mathcal{A}$ sealing Ω containing A.

We sketch the proof of Theorem 2.2.14 for the case $\alpha = 0$. The other cases are similar. First let $\mathcal{A}$ be as in c) of Theorem 4.2.5. Since $\mathcal{A}$ is countable, fix in V an enumeration $\langle A_n \mid n < \omega\rangle$ of $\mathcal{A}$. Hence there is a real x such that for each $n < \omega$, A_n is $OD_x^{(L(\mathbb{R},\mu_0),\mu_0)}$. We assume $x = \emptyset$. The case $x \neq \emptyset$ is just the relativization of the proof for $x = \emptyset$. The proof of Theorem 6.29 in [41] and a reflection argument (as done in the next section, where the HOD computation of $L(\mathbb{R}, \mu_0)$ without assuming $\mathcal{M}^\sharp_{\omega^2}$ exists is discussed) can be used to show that for any $n < \omega$, there is a pair $(\mathcal{P}_n, \Sigma_n)$ such that

(a) $o(\mathcal{P}_n) = Ord$ and there are ω^2 countable ordinals $\langle \delta_\alpha \mid \alpha < \omega^2\rangle$ with sup λ such that for $\alpha < \omega^2$, $\mathcal{P}_n \vDash \delta_\alpha$ is Woodin and furthermore $\mathcal{P}_n = L[\mathcal{P}_n|\lambda]$;

(b) $\forall \alpha < \omega^2$ $\mathcal{P}_n|((\delta_\alpha)^{+\omega})^{\mathcal{P}_n}$ is Γ-suitable (we call such a $\mathcal{P}_n$ ω^2-suitable);

(c) Σ_n is an M-fullness preserving strategy that respects $\oplus_{k<n}A_k$.

We sketch the construction of such a pair $(\mathcal{P}_n, \Sigma_n)$. In fact, we show that for every A which is $OD^{(L(\mathbb{R},\mu_0),\mu_0)}$, there is a pair $(\mathcal{P}_A, \Sigma_A)$ satisfying (a)-(c) for A. Suppose not. Working in $L(\mathbb{R}, \mu_0)$, let ϕ be a formula describing this. By Σ_1-reflection (Theorem 2.1.5), there is a model $N = L_\kappa(\mathbb{R}, \mu_0)$ (for some $\kappa < \underset{\sim}{\delta}^2_1$) such that $N \vDash \mathsf{MC} + \mathsf{AD} + \mathsf{DC} + \mathsf{ZF}^- +\ \Theta = \theta_0 + \phi$. Note that $\mu_0 \cap N$ is the club filter on $\mathcal{P}(X_0)^N$.

Now let $\Omega = \mathcal{P}(\mathbb{R})^N$ and we may assume $\Omega = Env((\underset{\sim}{\Sigma}^2_1)^N)$. We can construct a pair $(\mathcal{N}, \Lambda)$ such that

- there are $\langle \eta_i \mid i < \omega^2\rangle$ such that $\mathcal{N} \vDash \eta_i$ is Woodin for all i;
- letting $\lambda = \sup_i \eta_i$, then $\mathcal{N} = L_\gamma[\mathcal{N}|\lambda]$ for some γ;
- $\rho(\mathcal{N}) < \lambda$ and Λ is N-fullness preserving strategy for $\mathcal{N}$ above $\rho(\mathcal{N})$ and Λ condenses well.

See Subsection 3.2.2 for such a construction. $\mathcal{N}$ is ω^2-suitable relative to N but $\Lambda \notin N$. We may assume $\rho(\mathcal{N}) < \eta_0$. Let $A \in OD^N$ witness ϕ^N. By suitability, there is a term $\tau^{\mathcal{N}}_{A,i} \in \mathcal{N}$ such that for any $g \subseteq Col(\omega, \eta_i)$ generic over $\mathcal{N}$, $A \cap \mathcal{N}[g] = (\tau^{\mathcal{N}}_{A,i})_g$. We will show that for each $i < \omega^2$, a Λ-iterate $\mathcal{Q}_i$ of N is (A, i)-iterable (i.e. for any Λ-iterate $\mathcal{R}$ of $\mathcal{Q}$, letting $j : \mathcal{Q} \to \mathcal{R}$ be the iteration map, $j(\tau^{\mathcal{Q}}_{A,i}) = \tau^{\mathcal{R}}_{A,i}$). By comparing all the $\mathcal{Q}_i$'s, we get a Λ-iterate $\mathcal{Q}$ such that $\mathcal{Q}$ is (A, i)-iterable for all $i < \omega^2$. Contradiction.

To this end, first note that for any Λ-iterate $\mathcal{P}$ of $\mathcal{N}$, there is a further Λ-iterate $\mathcal{P}$' such that the derived model of $\mathcal{P}$'[16] (at the sup of its Woodins) is N. Also for a Λ-iterate $\mathcal{P}$ of $\mathcal{N}$, we'll use $\langle \eta^{\mathcal{P}}_i \mid i < \omega^2\rangle$ to denote the Woodin cardinals of $\mathcal{P}$. Fix an $i < \omega^2$. For notational simplicity, for a Λ-iterate $\mathcal{P}$ of $\mathcal{N}$, we denote $\tau^{\mathcal{P}}_{A,i}$ by $\tau^{\mathcal{P}}_A$. Suppose there is a sequence $\langle \mathcal{Q}_n\ |n < \omega\rangle$ such that

[16]By "derived model", we mean the model of the form $L(\mathbb{R}^*, \mu)$ where $\mathbb{R}^*$ is the symmetric reals for the symmetric collapse at the sup of the Woodins of $\mathcal{P}$' and μ is the tail filter as constructed in [44].

- $\mathcal{Q}_0 = \mathcal{Q}$ and $\mathcal{Q}_{n+1}$ is a Λ-iterate of $\mathcal{Q}_n$;
- letting $i_{k,k+1} : \mathcal{Q}_k \to \mathcal{Q}_{k+1}$ be the iteration map, then $i_{k,k+1}(\tau_A^{\mathcal{Q}_k}) \neq \tau_A^{\mathcal{Q}_{k+1}}$.

For $k < l$, let $i_{k,l} = i_{l-1,l} \circ \cdots i_{k,k+1}$. Let $\mathcal{Q}_\omega$ be the direct limit of the system $(i_{k,l} \mid k < l)$.

We get a contradiction using the definability of A in N and the fact that each $\mathcal{Q}_k$ can be further iterated to realize N as its derived model. Indeed, by a similar construction as that of Theorem 6.29 in [41], we can construct

1. maps $j_k : \mathcal{Q}_k \to \mathcal{Q}_k^\omega$ where $\mathrm{crt}(j_k) > \eta_i^{\mathcal{Q}_k}$ and j_k is an iteration map for $k < \omega$;
2. maps $i_{k,l}^\omega : \mathcal{Q}_k^\omega \to \mathcal{Q}_l^\omega$ such that $i_{k,l}^\omega \circ j_k = j_l \circ i_{k,l}$ for $k < l < \omega$;
3. N can be realized as the derived model of $\mathcal{Q}_k^\omega$ for each $k < \omega$;
4. letting $\mathcal{Q}_\omega^\omega$ be the direct limit of the $\mathcal{Q}_k^\omega$'s under maps $i_{k,l}^\omega$'s, then $\mathcal{Q}_\omega^\omega$ is embedded into a Λ-iterate of $\mathcal{N}$ and hence is well-founded.

This is enough to get a contradiction as in Theorem 6.29 of [41]. This completes our sketch.

Finally, we can then use Corollary 5.4.12 and Theorem 5.4.14 of [26] to get a premouse $\mathcal{P}$ that is ω^2-suitable and an (ω_1, ω_1)-strategy Σ for $\mathcal{P}$ that is M-fullness preserving, condenses well, and is guided by $\mathcal{A}$. Since $(\mathbb{R}, \mu_0)^\sharp$ exists, $\mathcal{P}^\sharp$ exists. Hence we're done. $\square$

2.3 Applications

2.3.1 An Ultra-homogenous Ideal

We prove the following two theorems. The first one uses $\mathbb{P}_{\max}$ forcing over a model of the form $L(\mathbb{R}, \mu)$ as above and the second one is an application of the core model induction. Woodin's book [48] or Larson's handbook article [15] are good sources for $\mathbb{P}_{\max}$.

Theorem 2.3.1. *Suppose $L(\mathbb{R}, \mu) \vDash$ "$\mathsf{AD}^+ + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$" and let G be a $\mathbb{P}_{\max}$ generic over $L(\mathbb{R}, \mu)$. Then in $L(\mathbb{R}, \mu)[G]$, there is a normal fine ideal $\mathcal{I}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that*

1. *letting $\mathcal{F}$ be the dual filter of $\mathcal{I}$ and $A \subseteq \mathbb{R}$ such that A is OD_x for some $x \in \mathbb{R}$, either $A \in \mathcal{F}$ or $\mathbb{R} \backslash A \in \mathcal{F}$;*
2. *$\mathcal{I}$ is precipitous;*
3. *for all $s \in [OR]^\omega$, for all generics $G_0, G_1 \subseteq \mathcal{I}^+$, letting $j_{G_i} : V \to Ult(V, G_i) = M_i$ for $i \in \{0, 1\}$ be the generic embeddings, then $j_{G_0} \restriction HOD_{\{\mathcal{I}, s\}} = j_{G_1} \restriction HOD_{\{\mathcal{I}, s\}}$ and $HOD^{M_0}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} = HOD^{M_1}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} \in V$.*

Proof. For each $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$, let $M_\sigma = \mathrm{HOD}^{(L(\mathbb{R},\mu),\mu)}_{\sigma\cup\{\sigma\}}$. Suppose G is a $\mathbb{P}_{\max}$ generic over $L(\mathbb{R},\mu)$. Note that $L(\mathbb{R},\mu)[G] \vDash ZFC$ since $\mathbb{P}_{\max}$ wellorders the reals. In $L(\mathbb{R},\mu)[G]$, let

$$\mathcal{I} = \{A \mid \exists\langle A_x \mid x \in \mathbb{R}\rangle(A \subseteq \triangledown_{x\in\mathbb{R}} A_x \wedge \forall x\ (\mu(A_x) = 0 \text{ or } A_x = \neg S))\},$$

where $S = \{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid G \cap \sigma \text{ is } \mathbb{P}_{\max} \restriction \sigma\text{-generic over } M_\sigma\}$. It's clear that in $L(\mathbb{R},\mu)[G]$, $\mathcal{I}$ is a normal fine ideal. Let $\mathcal{F}$ be the dual filter of $\mathcal{I}$.

Lemma 2.3.2. *Let $\mathcal{I}^- = \{A \mid \exists\langle A_x \mid x \in \mathbb{R}\rangle(A \subseteq \triangledown_{x\in\mathbb{R}} A_x \wedge \forall x\ \mu(A_x) = 0)\}$. Let $\mathcal{F}^-$ be the dual filter of $\mathcal{I}^-$. Suppose $A \in \mathcal{F}^-$. Then $\exists B, C$ such that $\mu(B) = 1$ and C is a club in $L(\mathbb{R},\mu)[G]$ such that $B \cap C \subseteq A$.*

Proof. Suppose $1 \Vdash_{\mathbb{P}_{\max}} \tau : \mathbb{R} \to \mu$ witnesses $\{\sigma \mid \forall x \in \sigma\ \sigma \in \tau(x)\} \in \mathcal{F}^-$. For each $x \in \mathbb{R}$. let $D_x = \{p \mid p \parallel \tau(x)\}$. It's easy to see that D_x is dense for each x. Furthermore,

$$\forall^*_\mu \sigma \forall x \in \sigma (D_x \cap \sigma \text{ is dense in } \mathbb{P}_{\max} \restriction \sigma \wedge \{q \in D_x \cap \sigma \mid q \Vdash \sigma \in \tau(x)\} \text{ is dense.})$$

For otherwise, $\exists x, q \forall^*_\mu \sigma\ x \in \sigma \wedge q \in D_x \cap \sigma \wedge q \Vdash \sigma \notin \tau(x)$. This contradicts that $q \Vdash \tau(x) \in \mu$. Let B be the set of σ having the property displayed above. $\mu(B) = 1$.

Let $A \subseteq \mathbb{R}$ code the function $x \mapsto D_x$ and let G be a $\mathbb{P}_{\max}$-generic over $L(\mathbb{R},\mu)$. Hence $D = \{\sigma \mid \forall x \in \sigma\ \sigma \in \tau_G(x)\} \in \mathcal{F}^-$. Let $C = \{\sigma \mid (\sigma, A \cap \sigma, G \cap \sigma) \prec (\mathbb{R}, A, G)\}$. Hence C is a club in $L(\mathbb{R},\mu)[G]$ and $B \cap C \subseteq D$. $\square$

Lemma 2.3.3. *Let $\mathcal{I}^-, \mathcal{F}^-$ be as in Lemma 2.3.2. Then $S \notin \mathcal{I}^-$.*

Proof. Suppose not. Then $\neg S \in \mathcal{F}^-$. The following is a Σ_1-statement (with predicate μ) that $L(\mathbb{R},\mu)[G]$ satisfies:

$$\exists B, C(\mu(B) = 1 \wedge C \text{ is a club } \wedge \forall\sigma(\sigma \in B \cap C \Rightarrow \exists D \subseteq \sigma(D \in \mathrm{HOD}^{L(\mathbb{R},\mu)}_{\sigma\cup\{\sigma\}} \wedge G \cap D = \emptyset))).$$

By part (1) of Theorem 2.1.5 and the fact that $\mathbb{P}_{\max}$ is a forcing of size $\mathbb{R}$, $L_{\undertilde{\delta}^2_1}(\mathbb{R},\mu)[G]$ satisfies the same statement. Here μ coincides with the club measure and hence $L_{\undertilde{\delta}^2_1}(\mathbb{R},\mu)[G] \vDash \neg S$ contains a club. Let $\mathcal{C}$ be a club of elementary substructures X_σ containing everything relevant (and a pair of complementing trees for the universal Σ^2_1 set). Then it's easy to see that $\mathcal{C}^* \subseteq S$ where $\mathcal{C}^* = \{\sigma \mid \sigma = \mathbb{R} \cap X_\sigma \wedge X_\sigma \in \mathcal{C}\}$. This is a contradiction. $\square$

We now proceed to characterize $\mathcal{I}$-positive sets in terms of the $\mathbb{P}_{\max}$ forcing relation over $L(\mathbb{R},\mu)$.

Lemma 2.3.4. *Suppose $p \in \mathbb{P}_{\max}$ and τ is a $\mathbb{P}_{\max}$ term for a subset of $\mathcal{P}_{\omega_1}(\mathbb{R})$ in generic extensions of $L(\mathbb{R},\mu)$. Then the following is true in $L(\mathbb{R},\mu)$.*

$$p \Vdash_{\mathbb{P}_{\max}} \tau \text{ is } \mathcal{I}\text{-positive } \Leftrightarrow \forall^*_\mu \sigma\ \forall^* g \subseteq \mathbb{P}_{\max} \restriction \sigma\ (p \in g \Rightarrow \exists q < g\ q \Vdash_{\mathbb{P}_{\max}} \sigma \in \tau).$$

Proof. Some explanations about the notation in the lemma are in order. "$\forall^* g \subseteq \mathbb{P}_{\max} \restriction \sigma$" means "for comeager many filters g over $\mathbb{P}_{\max} \restriction \sigma$"; "$\exists^* g \subseteq \mathbb{P}_{\max} \restriction \sigma$" means "for nonmeager many filters g over $\mathbb{P}_{\max} \restriction \sigma$". These category quantifiers make sense because σ is countable. Also we only force with $\mathbb{P}_{\max}$ here so we'll write " $\Vdash$ " for " $\Vdash_{\mathbb{P}_{\max}}$ " and "$p < q$" for "$p <_{\mathbb{P}_{\max}} q$". Finally, "$q < g$" means "$\forall r \in g\ q < r$".

Claim. Suppose in $L(\mathbb{R}, \mu)$, $\forall \sigma\ X_\sigma$ is comeager in $\mathbb{P}_{\max} \restriction \sigma$. Then $\forall^*_\mu \sigma \forall G_\sigma$ (G_σ is $\mathbb{P}_{\max} \restriction \sigma$-generic over $M_\sigma \Rightarrow G_\sigma \in X_\sigma$).

Proof. Suppose $\sigma \mapsto X_\sigma$ is $OD_{\mu,x}$ for some $x \in \mathbb{R}$. Let $A = \{y \in \mathbb{R} \mid y$ codes (σ, g) where $g \in X_\sigma\}$. Hence A is $OD_{\mu,x}$. Let S be an $OD_{\mu,x}$ ∞-Borel code for A. Hence, $\forall^*_\mu \sigma\ S \in M_\sigma$.

For each such σ, let $G_\sigma \in X_\sigma$ be M_σ-generic and H be $M_\sigma[G_\sigma]$-generic for $Col(\omega, \sigma)$. Then

$$M_\sigma[G_\sigma][H] \vDash (\sigma, G_\sigma) \in X_\sigma.$$

In the above, note that we use $S \in M_\sigma$. Also no $p \in \mathbb{P}_{\max} \restriction \sigma$ can force $(\sigma, \dot{G}) \notin X_\sigma$. Hence we're done. □

Suppose the conclusion of the lemma is false. There are two directions to take care of.
Case 1. $p \Vdash \tau$ is $\mathcal{I}$-positive but $\forall^*_\mu \sigma \exists^* g$ ($p \in q \wedge \forall q < g\ g \Vdash \sigma \notin \tau$).

Extending p if necessary and using normality, we may assume $\forall^*_\mu \sigma \forall^* g(p \in g \wedge \forall q < g\ q \vDash \sigma \notin \tau)$. Let T be the set of such σ. Let G be a $\mathbb{P}_{\max}$ generic and $p \in G$. By the claim and the fact that $S \in \mathcal{F}$, $\tau_G \cap S \cap T \neq \emptyset$. So let $\sigma \in \tau_G \cap S \cap T$ such that $p \in G \cap \sigma$. Then $G \cap \sigma$ is M_σ-generic and $\forall q < G \cap \sigma\ q \Vdash \sigma \notin \tau$. But $\exists q < G \cap \sigma$ such that $q \in G$ by density. This implies $\sigma \notin \tau_G$. Contradiction.
Case 2. $p \Vdash \tau \in \mathcal{I}$ and $\forall^*_\mu \sigma \forall^* g$ ($p \in g \Rightarrow \exists q < g\ q \Vdash \sigma \in \tau$).

Let T be the set of σ as above. Let G be $\mathbb{P}_{\max}$ generic containing p. Hence $T \in \mathcal{F}$. Let $\sigma \in T \cap S \cap \neg\tau_G$ and $p \in G \cap \sigma$. By density, $\exists q < G \cap \sigma\ q \in G \wedge q \Vdash \sigma \in \tau$. Hence $\sigma \in \tau_G$. Contradiction. □

Now suppose $\dot{f}$ is a $\mathbb{P}_{\max}$ name for a function from an $\mathcal{I}$-positive set into OR and let τ be a name for dom$f$ and for simplicity suppose $\emptyset \Vdash \dot{f} : \tau \to \check{\text{OR}}$. Let $F : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \text{OR}$ be defined as follows:

$$F(\sigma) = \begin{cases} \alpha_\sigma \text{ where } \alpha_\sigma \text{ is the least } \alpha \text{ such that} \\ \forall^* g \subseteq \mathbb{P}_{\max} \restriction \sigma\ \exists q < g\ q \Vdash \check{\sigma} \in \tau \wedge \dot{f}(\check{\sigma}) = \check{\alpha_\sigma}, \text{ and} \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, $F \in L(\mathbb{R}, \mu)$. It's easy to see using Lemma 2.3.4 that if G is $\mathbb{P}_{\max}$-generic, then in $L(\mathbb{R}, \mu)[G]$, F agrees with $\dot{f}_G$ on an $\mathbb{I}$-positive set. We summarize this fact.

Lemma 2.3.5. *Suppose G is a $\mathbb{P}_{\max}$ generic over $L(\mathbb{R},\mu)$. Suppose $f : A \to OR$ in $L(\mathbb{R},\mu)[G]$ and A is $\mathcal{I}$-positive. Then there is a function $F : \mathcal{P}_{\omega_1}(\mathbb{R}) \to OR$ in $L(\mathbb{R},\mu)$ such that $\{\sigma \mid F(\sigma) = f(\sigma)\}$ is $\mathcal{I}$-positive.*

Working in $L(\mathbb{R},\mu)[G]$, let $H \subseteq \mathcal{I}^+$ be generic. We show that (1)-(3) hold. Let $A \subseteq \mathbb{R}$ be OD_x for some $x \in \mathbb{R}$. By countable closure and homogeneity of $\mathbb{P}_{\max}$, $x \in L(\mathbb{R},\mu)$ and hence $A \in L(\mathbb{R},\mu)$. Since $\mathcal{F} \restriction L(\mathbb{R},\mu) = \mu$, we obtain (1)[17]. Lemma 2.3.5 implies $\forall s \in \mathrm{OR}^\omega$ $\mathrm{HOD}_s \in V$ and is independent of H. To see this, note that $s \in L(\mathbb{R},\mu)$ as $\mathbb{P}_{\max}$ is countably closed and $L(\mathbb{R},\mu) \vDash DC$; furthermore, by homogeneity of $\mathbb{P}_{\max}$, $\mathrm{HOD}_s \subseteq \mathrm{HOD}_s^{L(\mathbb{R},\mu)}$ and there is a bijection between OR and HOD_s in $L(\mathbb{R},\mu)$. So Lemma 2.3.5 applies to functions $f : S \to \mathrm{HOD}_s$ where S is $\mathcal{I}$-positive. This implies $j_H \restriction \mathrm{HOD}_s = j_\mu \restriction \mathrm{HOD}_s$, which also shows (2).

To show $j_H \restriction \mathrm{HOD}_{\mathcal{I}}$ is independent of H, first note that $\mathcal{I}$ is generated by μ and $\mathcal{A} =_{\mathrm{def}} \{T \subseteq \mathcal{P}_{\omega_1}(\mathbb{R}) \mid \exists C(C \text{ is a club and } T \cap C = S \cap C\}$. Note that $\mathcal{A}$ is definable in $L(\mathbb{R},\mu)[G]$ (from no parameters). To see this, suppose G_0, G_1 are two $\mathbb{P}_{\max}$ generics (in $L(\mathbb{R},\mu)[G]$) and let S_{G_i} be defined relative to G_i ($i \in \{0,1\}$) the same way S is defined relative to G. Also let $A_{G_i} \subseteq \omega_1$ be the generating set for G_i. Let $p \in G_0 \cap G_1$ and $a_0, a_1 \in \mathcal{P}(\omega_1)^p$ be such that $j_i(a_i) = A_i$ where j_i are unique iteration maps of p. The proof of homogeneity of $\mathbb{P}_{\max}$ gives a bijection π from $\{q \mid q < p\}$ to itself. It's easy to see that

$$C = \{\sigma \mid (\sigma, \mathbb{P}_{\max} \restriction \sigma, \pi \restriction \sigma) \prec (\mathbb{R}, \mathbb{P}_{\max}, \pi)\},$$

is a club and $S_{G_0} \cap C = S_{G_1} \cap C$. By homogeneity of $\mathbb{P}_{\max}$, there is a bijection (definable over) $L(\mathbb{R},\mu)$ from OR onto $\mathrm{HOD}_{\mathcal{I}}$. So the ultraproduct $[\sigma \mapsto \mathrm{HOD}_{\mathcal{I}}]_H$ using functions in $L(\mathbb{R},\mu)[G]$ is just $[\sigma \mapsto \mathrm{HOD}_{\mathcal{I}}]_\mu$ using functions in $L(\mathbb{R},\mu)$.

Finally, to see $\mathrm{HOD}^{M_0}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} = \mathrm{HOD}^{M_1}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} \in V$, note that for any generic H, letting $V = L(\mathbb{R},\mu)[G]$, $\mathrm{HOD}^{Ult(V,H)}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}}$ is represented by $\sigma \mapsto \mathrm{HOD}^V_{\sigma \cup \{\sigma\}}$. Let f be such that $\mathrm{dom}(f) = S$ where S is $\mathcal{I}$-positive and $\forall \sigma \in S, f(\sigma) \in \mathrm{HOD}^V_{\sigma \cup \{\sigma\}}$. By normality, shrinking S if necessary, we may assume $\exists x \in \mathbb{R} \forall \sigma \in S, f(\sigma) \in \mathrm{HOD}^V_{\{x,\sigma\}}$ and Lemma 2.3.5 can be applied to this f. We finished the proof of Theorem 2.3.1. $\square$

A normal fine ideal $\mathcal{I}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is **ultra-homogenenous** if it satisfies (1)-(3) in Theorem 2.3.1. The next theorem then gives the equiconsistency of the conclusion of Theorem 2.3.1 with ω^2 Woodin cardinals. It is technically an application of the core model induction but the core model induction argument has been carried out in [46] so we will not reproduce it here.

Theorem 2.3.6 (ZFC). *Suppose there is an ultra-homogeneous normal fine ideal $\mathcal{I}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then in $V[G]$ for some (possibly trivial) G, there is a filter μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such*

[17]The proof of (1) in fact shows more. It shows that if $A \subseteq \mathbb{R}$ is OD_s for some $s \in \mathrm{OR}^\omega$, then $A \in \mathcal{F}$ or $\mathbb{R}\backslash A \in \mathcal{F}$

that

$$L(\mathbb{R}, \mu) \vDash \text{“}\mathsf{AD} + \mu \textit{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R})\text{”}.$$

Proof. Suppose not. By Lemma 4.5.1 of [46], the existence of a normal fine ideal $\mathcal{I}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that $\mathcal{I}$ is precipituous and for all generics $G_0, G_1 \subseteq \mathcal{I}^+$, $s \in \mathrm{OR}^\omega$, $j_{G_0} \restriction \mathrm{HOD}_s = j_{G_1} \restriction \mathrm{HOD}_s \in V$ and $\mathrm{HOD}^{Ult(V,G_0)}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} = \mathrm{HOD}^{Ult(V,G_1)}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} \in V$ implies $\mathsf{AD}^{K(\mathbb{R})}$. Let $M = K(\mathbb{R})$. Let $\mathcal{F}$ be the directed system in M, $\mathcal{M}_\infty = \mathrm{dirlim}\mathcal{F}$ in M. Fix a generic $G \subseteq \mathcal{I}^+$ and let $j = j_G$ be the generic embedding. To prove the theorem, we consider two cases.

Case 1. $\Theta^{K(\mathbb{R})} < \mathfrak{c}^+$.

We first observe that the argument giving Theorem 4.6.6 in [46] can be carried out in this case. This is because hypothesis of Case 1 replaces the strength of the ideal in the hypothesis of Theorem 4.6.6 in [46] and ultra-homogeneity implies pseudo-homogeneity. Hence we get a model N containing $\mathbb{R} \cup \mathrm{OR}$ such that $N \vDash \mathsf{AD}^+ + \Theta > \theta_0$.

In N, one can easily show that the club filter μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ restricted to $\mathcal{P}_{\Theta_0}(\mathbb{R})$ is in $\mathrm{HOD}_\mathbb{R}$ and is a normal fine measure there. This implies

$$L(\mathbb{R}, \mu) + \mathsf{AD} + \mu \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R}).$$

This finishes the proof of the theorem in Case 1.

Case 2. $\Theta^{K(\mathbb{R})} \geq \mathfrak{c}^+$

Recall that $\mathcal{F}$ is the dual filter to $\mathcal{I}$. Let $\mu = \mathcal{F} \cap M$. First we observe by (1) that μ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})^M$. Next, we need to see that μ doesn't construct sets of reals beyond M. This is the content of the next lemma.

Lemma 2.3.7. *$L(\mathbb{R}, \mu) \subseteq M$.*

Proof. We first prove the following claim.

Claim. μ is amenable to M in that if $\langle A_x \mid x \in \mathbb{R} \wedge A_x \in \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))^M \rangle \in M$ then $\langle A_x \mid x \in \mathbb{R} \wedge \mu(A_x) = 1 \rangle \in M$.

Proof. Fix a sequence $\mathcal{C} = \langle A_x \mid x \in \mathbb{R} \wedge A_x \in \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))^M \rangle \in M$ and fix an ∞-Borel code S for the sequence. Let T be the tree for a universal $(\Sigma^2_1)^M$ set. We may assume $S \in OD^M$ and is a bounded subset of Θ^M. We also assume S codes T. By MC and the definition of T, S in M, it's easy to see that in M,

$$\forall^*_\mu \sigma (\mathcal{P}(\sigma) \cap L(S, \sigma) = \mathcal{P}(\sigma) \cap L(T, \sigma) = \mathcal{P}(\sigma) \cap Lp(\sigma)).$$

Let $S^* = [\sigma \mapsto S]_\mu$ and $T^* = [\sigma \mapsto T]_\mu$ where the ultraproducts are taken with functions in M. Now, S^*, T^* may not be in M but

$$\mathcal{P}(\mathbb{R}) \cap L(S^*, \mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap L(T^*, \mathbb{R}) = \mathcal{P}(\mathbb{R})^M.$$

This implies $\mathcal{C} \in L(S^*, \mathbb{R})$. For each $x \in \mathbb{R}$,

$$\begin{aligned} A_x \in \mu &\Leftrightarrow (\forall^*_\mu \sigma)(\sigma \in A_x \cap \mathcal{P}_{\omega_1}(\sigma)) \\ &\Leftrightarrow (\forall^*_\mu \sigma)(L(S,\sigma) \vDash \emptyset \Vdash_{Col(\omega,\sigma)} \sigma \in A_x \cap \mathcal{P}_{\omega_1}(\sigma)) \\ &\Leftrightarrow L(S^*, \mathbb{R}) \vDash \emptyset \Vdash_{Col(\omega,\mathbb{R})} \mathbb{R} \in A_x. \end{aligned}$$

The above shows $\mu \restriction \mathcal{C} \in L(S^*, \mathbb{R})$. Since $\mu \restriction \mathcal{C}$ can be coded by a set of reals in $L(S^*, \mathbb{R})$, $\mu \restriction \mathcal{C} \in M$. This finishes the proof of the claim. □

Using the claim, we finish the proof of the lemma as follows. Suppose α is least such that $\exists A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ $A \in L_{\alpha+1}(\mathbb{R})[\mu] \backslash L_\alpha(\mathbb{R})[\mu]$ and $A \notin M$. By properties of α and condensation of μ, there is a definable over $L_\alpha(\mathbb{R})[\mu]$ surjection of $\mathbb{R}$ onto $L_\alpha(\mathbb{R})[\mu]$. This implies $\alpha < \mathfrak{c}^+$. Also by minimality of α, $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subseteq M$.

Now, if $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subsetneq \mathcal{P}(\mathbb{R})^M$, then Lemma 4.3.24 gives us $\mu \cap L_\alpha(\mathbb{R})[\mu] \in M$ which implies $A \in M$. Contradiction. So we may assume $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] = \mathcal{P}(\mathbb{R})^M$. This means $\Theta^{L_\alpha(\mathbb{R})[\mu]} = \Theta^M \geq \mathfrak{c}^+$. This contradicts the fact that $\alpha < \mathfrak{c}^+$. □

Lemma 4.3.24 implies $L(\mathbb{R}, \mu) \vDash \mathsf{AD} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. This finishes the proof of the theorem. □

2.3.2 Determinacy of Long Games

Definition 2.3.8. *Let β be an ordinal and $A \subseteq \omega^\omega$. A is β-$\undertilde{\Pi}^1_1$ if there exist $\undertilde{\Pi}^1_1$ sets A_α for $\alpha \leq \beta$ such that $A_\beta = \emptyset$ and*

$$A = \{x \in \omega^\omega \mid \exists \alpha \leq \beta (\alpha \text{ is odd} \wedge x \in \cap_{\gamma<\alpha} A_\gamma \backslash A_\alpha)\}.$$

We also say A is $<\beta$-$\undertilde{\Pi}^1_1$ if A is α-$\undertilde{\Pi}^1_1$ for some $\alpha < \beta$.

We define the class of long game determinacy we're interested in.

Definition 2.3.9. *Let $\alpha < \omega_1$. Then we say*

1. *$\mathsf{AD}_{\omega,\alpha}\undertilde{\Pi}^1_1$ ($\mathsf{AD}_{\omega,\alpha}$<-$\omega^2$-$\undertilde{\Pi}^1_1$) holds if all integer games of length α with $\undertilde{\Pi}^1_1$-payoff (<-ω^2-$\undertilde{\Pi}^1_1$-payoff, respectively) are determined.*
2. *$\mathsf{AD}_{\mathbb{R},\alpha}\undertilde{\Pi}^1_1$ ($\mathsf{AD}_{\mathbb{R},\alpha}$<-$\omega^2$-$\undertilde{\Pi}^1_1$) holds if all real games of length α with $\undertilde{\Pi}^1_1$-payoff (<-ω^2-$\undertilde{\Pi}^1_1$-payoff, respectively) are determined.*

As a warm-up for the main theorem (Theorem 2.3.11), we prove the following theorem. One can think of Theorem 2.3.10 as a special case (when $\alpha = 1$) of Theorem 2.3.11 where the relevant model here is $L(\mathbb{R})$ (as opposed to $L(\mathbb{R}, \mu_\alpha)$). The structure theory of $L(\mathbb{R})$ is well-known.

Theorem 2.3.10. *The following statements are equivalent:*

1. $\mathsf{AD}_{\mathbb{R},\omega}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\mathbb{R},\omega}{<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$, *and "*$\mathbb{R}^\sharp$ *exists."*

2. $\mathsf{AD}_{\omega,\omega^2}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\omega,\omega^2}{<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$, *and "*$\mathsf{AD}^{L(\mathbb{R})}$ *and* $\mathbb{R}^\sharp$ *exists."*

Theorem 2.3.11. *1. For* $1 < \alpha < \omega$ *the following statements are equivalent:* $\mathsf{AD}_{\omega,\omega^{1+\alpha}}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\omega,\omega^{1+\alpha}}{<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\mathbb{R},\omega^\alpha}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\mathbb{R},\omega^\alpha}{<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$, $L(\mathbb{R},\mu_{\alpha-2}) \vDash$ *"*$\mathsf{AD}^+ + \mu_{\alpha-2}$ *is a normal fine measure on* $X_{\alpha-2}$*" and* $(\mathbb{R},\mu_{\alpha-2})^\sharp$ *exists, where* $\mu_{\alpha-2}$ *is the club filter on* $X^V_{\alpha-2}$*.*

2. *For* $\omega \leq \alpha < \omega_1$ *the following statements are equivalent:* $\mathsf{AD}_{\omega,\omega^{\alpha+1}}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\omega,\omega^{\alpha+1}}{<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\mathbb{R},\omega^{\alpha+1}}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\mathbb{R},\omega^{\alpha+1}}$ ${<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$, $L(\mathbb{R},\mu_\alpha) \vDash$ *"*$\mathsf{AD}^+ + \mu_\alpha$ *is a normal fine measure on* X_α*" and* $(\mathbb{R},\mu_\alpha)^\sharp$ *exists, where* μ_α *is the club filter on* X^V_α*.*

3. *For* $1 \leq \alpha < \omega_1$ *such that* α *is a limit ordinal, the following statements are equivalent:* $\mathsf{AD}_{\omega,\omega^\alpha}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\omega,\omega^\alpha}{<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\mathbb{R},\omega^\alpha}\underset{\sim}{\Pi}^1_1$, $\mathsf{AD}_{\mathbb{R},\omega^\alpha}$ ${<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$, $L(\mathbb{R},\langle\mu_\beta \mid \beta<\alpha\rangle) \vDash$ *"*$\mathsf{AD}^+ + \mu_\beta$ *is a normal fine measure on* X_β *for all* $\beta < \alpha$*" and* $(\mathbb{R},\langle\mu_\beta \mid \beta<\alpha\rangle)^\sharp$ *exists, where* μ_β *is the club filter on* X^V_β*.*

From the above theorems, we easily get the following.

Corollary 2.3.12. *For* $\alpha < \omega_1$*, the following hold.*

1. *If* $\alpha = 1$, $\mathsf{AD}_{\mathbb{R},\omega^\alpha}\underset{\sim}{\Pi}^1_1$ *is (logically) strictly weaker than* $\mathsf{AD}_{\omega,\omega^{\alpha+1}}\underset{\sim}{\Pi}^1_1$*.*

2. *If* $1 < \alpha < \omega$, $\mathsf{AD}_{\mathbb{R},\omega^\alpha}\underset{\sim}{\Pi}^1_1$ *is equivalent to* $\mathsf{AD}_{\omega,\omega^{\alpha+1}}\underset{\sim}{\Pi}^1_1$*.*

3. *If* $\omega \leq \alpha < \omega_1$, $\mathsf{AD}_{\mathbb{R},\omega^\alpha}\underset{\sim}{\Pi}^1_1$ *is equivalent to* $\mathsf{AD}_{\omega,\omega^\alpha}\underset{\sim}{\Pi}^1_1$*.*

Proof of Theorem 2.3.10. We start with the proof of part 1.
"$\mathbb{R}^\sharp$ exists" implies $\mathsf{AD}_{\mathbb{R},\omega}{<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$: The proof of this is just Martin's proof that "$0^\sharp$ exists" implies $\mathsf{AD}_{\omega,\omega}{<}\text{-}\omega^2\text{-}\Pi^1_1$ (cf. [5]).
$\mathsf{AD}_{\mathbb{R},\omega}{<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$ implies $\mathsf{AD}_{\mathbb{R},\omega}\underset{\sim}{\Pi}^1_1$: This is clear since the poinclass $\underset{\sim}{\Pi}^1_1$ is contained in the pointclass ${<}\text{-}\omega^2\text{-}\underset{\sim}{\Pi}^1_1$.
$\mathsf{AD}_{\mathbb{R},\omega}\underset{\sim}{\Pi}^1_1$ implies "$\mathbb{R}^\sharp$ exists": This is the main implication. We will follow Woodin's proof of Harrington's theorem that $\mathsf{AD}_{\omega,\omega}\Pi^1_1$ implies "$0^\sharp$ exists."

Let γ be a (large) regular cardinal. Let $N = V_\gamma$. We want to show that $N^{Col(\omega,\mathbb{R})} \vDash \mathbb{R}^{\mathbb{V}\sharp}$ exists. Then by absoluteness, $\mathbb{R}^\sharp$ exists in V. To this end, we play the following game G_0. G_0 consists of ω rounds. At round k, player I plays x_{2k}, n_{2k} and player II responds with x_{2k+1}, n_{2k+1}, where $x_{2k}, x_{2k+1} \in \mathbb{R}$ and $n_{2k}, n_{2k+1} \in \omega$. The game stops after ω rounds are finished. Let $\sigma = \{x_k \mid k < \omega\}$, $y_I = \langle n_{2k} \mid k<\omega\rangle$ and $y_{II} = \langle n_{2k+1} \mid k<\omega\rangle$.

Before describing the winning condition for the players in the game G_0, let us fix some notation. For an $x \in \omega^\omega$[18], let $E_x = \{\langle n,m\rangle \mid x(n)=m\}$ be the ordering coded by x, let M_x be the transitive collapse of the structure (ω, E_x) if E_x is wellfounded, and let $|x|$ be the rank of E_x if E_x is wellfounded. Finally, if M is an ω-model, let std(M) be the standard part of M. Using the notation introduced, player II wins the play if

[18] In this paper, $\mathbb{R}$ is the same as ω^ω

(a) E_{y_I} is not a total order which is a wellorder; or

(b) $M_{y_{II}}$ is an ω-model of "$\mathsf{ZF} \setminus$ Powerset $+ V = L(\sigma)$" such that $|E_{y_I}| < \mathrm{std}(M_{y_{II}})$.

$$\begin{array}{c|cccc} \mathrm{I} & x_0, n_0 & & x_2, n_2 & \cdots \\ \hline \mathrm{II} & & x_1, n_1 & & \cdots \end{array}$$

The game G_0

It's clear that the above condition is Σ^1_1 so G_0 is determined.

Lemma 2.3.13. *Player I cannot have a winning strategy in G_0.*

Proof. Suppose τ is a winning strategy for I. If (ρ, y) is a play by II and $\tau(\rho, y) = (\rho^*, y^*)$, then we let $\rho^* = \tau(\rho, y)_0$ and $y^* = \tau(\rho, y)_1$. For a club of $M^* \prec N$, $\tau \in M^*$. Let M be the transitive collapse of M^* and $\sigma = \mathbb{R} \cap M$. Then we have $\tau[\sigma] \subseteq \sigma$ and $\tau \restriction \sigma \in M$. Let $g \subseteq Col(\omega, \sigma)$ be M-generic and $g \in V$. Working in $M[g]$, let

$$A = \{|\tau(\sigma, y_{II})_1| \mid (\sigma, y_{II}) \text{ is a play by II in } G_0\}.$$

By Σ^1_1-boundedness, $\sup(A) < \omega_1$. So II can defeat $\tau \restriction \sigma$ by playing (σ, y_{II}) such that $y_{II} \in M[g]$ and $M_{y_{II}}$ is $L_\beta(\sigma)$ such that $\beta \geq \sup(A)$. Contradiction. $\square$

Now let τ be II's winning strategy. Again, let $M^* \prec N$ be countable such that $\tau \in M^*$. Let M be the transitive collapse of M^* and $\sigma = \mathbb{R}^M$. Let $g \subseteq Col(\omega, \sigma)$ be M-generic and $g \in V$. We claim that $\sigma^\sharp$ exists in $M[g]$. By the above discussion, this finishes the proof of part 1.

Since from now on, we'll work inside $M[g]$, we will rename $\tau \restriction \sigma$ to τ and denote G_σ to be the game G_0 where players are only allowed to play reals $x_i \in \sigma$ in their moves. Let $x \in \mathbb{R}^{M[g]}$ code σ, τ. What we will show is that in $M[g]$, for any $\gamma \geq \Theta^{L(\sigma)}$, if γ is admissible relative to x, then γ is a cardinal in $L(\sigma)$. This implies that $\sigma^\sharp$ exists (see for example [7]).

Suppose α is a counterexample. By moving to a generic extension of $M[g]$ where α is countable if necessary, we may assume $\alpha < \omega_1^{M[g]}$. Let $z \in \mathbb{R}$ be such that α is the least admissible relative to (x, z). Such a z exists. Since α is not a cardinal of $L(\sigma)$, let $\Theta^{L(\sigma)} \leq \gamma < \alpha$ be the largest cardinal of $L(\sigma)$ below α. The next lemma is the key lemma and is the only place where the existence of τ is used.

Lemma 2.3.14. *There is an $X \subseteq \mathcal{P}(\gamma) \cap L_\alpha[x, z]$ such that if (P, E) is an ω-model such that*

1. *$P \vDash \mathsf{ZFC} \setminus$ Powerset;*

2. *$x, z \in P$;*

3. *$std(P) = \alpha$;*

4. $P \models \tau$ is a winning strategy for II in G_σ and x enumerates σ, τ;

then $X = (\mathcal{P}(\gamma) \cap L(\sigma))^P$.

Proof. Let $(P_0, E_0), (P_1, E_1)$ be two ω-models satisfying (1)-(4). We will occasionally confuse the set P_i with the structure (P_i, E_i). We further assume that $\mathbb{R}^{P_0} \cap \mathbb{R}^{P_1} = \mathbb{R}^{L_\alpha[x,z]}$ and $\mathcal{P}(\gamma)^{P_0} \cap \mathcal{P}(\gamma)^{P_1} = \mathcal{P}(\gamma)^{L_\alpha[x,z]}$. Given any ω-model (P_0, E_0) satisfying (1)-(4), we can always find an ω-model (P_1, E_1) satisfying (1)-(4) and is "divergent" from (P_0, E_0) in the above sense. We want to show $(P(\gamma) \cap L(\sigma))^{P_0} = (P(\gamma) \cap L(\sigma))^{P_1}$. Let $A \in (P(\gamma) \cap L(\sigma))^{P_0}$. We show that $A \in (P(\gamma) \cap L(\sigma))^{P_1}$. Of course, the converse is symmetrical.

Let $a_0 \in Ord^{P_0}$ be such that $a > \alpha = \mathrm{std}(P_0)$, a is additively closed[19], and $(P_0, E_0) \models$ "$A \in L_{a_0}(\sigma)$." Let $a_1 \in P_1$ be such that $a_1 > \alpha$ and a_1 is additively closed. The following fact is key.

Claim 1: $(\{b \mid bE_0a_0\}) \cong (\{b \mid bE_1a_1\}) \cong \alpha + \alpha\mathbb{Q}$.

Proof. We prove $(\{b \mid bE_0a_0\}) \cong \alpha + \alpha\mathbb{Q}$; the other equality is similar. We define an equivalence $\sim$ on a_0 as follows. Let $a, b \in Ord^{P_0}$ be such that aE_0a_0 and bE_0a, we say

$$a \sim b \Leftrightarrow \exists\beta < \alpha(a = \beta + b \vee b = \beta + a).$$

Clearly, $\sim$ is an equivalence relation on the set $\{b \in Ord^{P_0} \mid bE_0a\}$. We claim that for all bE_0a_0, $[b]_\sim$ has an E_0-least element. To see this, first note that for any $\beta < \alpha$, $[\beta]_\sim = [0]_\sim$ has an E_0-least element, namely 0. Now suppose $b \notin \mathrm{std}(P_0)$. Note then that the set

$$\{b - a \mid aE_0b\}$$

is finite in P_0 and hence is actually finite since P_0 is an ω-model and hence it is correct about finiteness. This in turns implies the set

$$\{b - \beta \mid \beta E_0b \wedge \beta < \alpha\}$$

is non-empty, finite, and is bounded in α. But this implies $[b]_\sim$ has an E_0-least element as claimed.

Now let $c, d \notin \mathrm{std}(P_0)$ be such that $cE_0dE_0a_0$ and $c \nsim d$. We want to show that there is an e such that cE_0eE_0d, $c \nsim e$, and $e \nsim d$. To this end, let $d^* = \min([d]_\sim)$. By the claim above, d^* exists and cE_0d^* and $c \nsim d^*$. But this means $\{c + \beta \mid \beta < \alpha\}$ is not cofinal in d^*, so we can choose an e such that $cE_0eE_0d^*$ as required.

The above shows that $\{[c]_\sim \mid cE_0a_0\} \cong 1 + \mathbb{Q}$, which implies $\{c \mid cE_0a_0\} \cong \alpha + \alpha\mathbb{Q}$. □

Claim 1 implies that $Col(\omega, a_0)^{P_0} \cong Col(\omega, a_1)^{P_1} \cong Col(\omega, \alpha + \alpha\mathbb{Q})$. Next we build two generics G_0, G_1 as follows:

(a) G_i is P_i generic for $Col(\omega, a_i)$ where $i = 0, 1$;

[19]This means for all bE_0a_0, $b + a_0 = a_0$.

(b) $\{(i,j) \mid G_0(i)E_0G_0(j)\} = \{(i,j) \mid G_1(i)E_1G_1(j)\}$;

(c) $P_0 \cap P_0[G_0] = P_0$, $P_1 \cap P_1[G_1] = P_1$, $P_0 \cap P_1[G_1] = P_0 \cap P_1$, and $P_1 \cap P_0[G_0] = P_0 \cap P_1$;

(d) $P_i[G_i] \vDash \tau$ is a winning strategy for II in G_σ.

The requirement (b) can be ensured by a standard back-and-forth argument and the requirement (c) can be ensured by choosing sufficiently generic G_0 and G_1. The requirement (d) is just by Σ^1_1-absoluteness.

Now let $h \in \mathbb{R}$ be such that $E_h = \{(i,j) \mid G_0(i)E_0G_0(j)\}$.

Claim 2: $A \in P_1[G_1]$.

Proof. First mote that $h \in P_0[G_0] \cap P_1[G_1]$ and hence $\tau(\sigma, h) \in P_0[G_0] \cap P_1[G_1]$. We then have $P_0[G_0] \vDash$ "h gives a wellordering of length a_0 and $\tau(\sigma,h)$ codes a model of $V = L(\sigma)$ with standard part $> a_0$." This means $P_0[G_0] \vDash$ "A is coded by $\tau(\sigma,h)$." Since $\tau(\sigma,h) \in P_1[G_1]$, $A \in P_1[G_1]$. □

Claim 2 implies $A \in P_1[G_1] \cap P_0$. c) implies that $A \in P_1$. Hence we're done. □

Using Lemma 2.3.14, we get.

Lemma 2.3.15. *Let $R_\gamma \subseteq \gamma$ and $R_\gamma \in L_{\gamma+1}(\sigma)$ be a coding of the canonical bijection $e : \gamma \times \gamma \to \gamma$. Let X be as in Lemma 2.3.14, then $X \subseteq L(\sigma)$ and in fact,*

$$(X, \gamma, R_\gamma, \in) \prec (P(\gamma) \cap L(\sigma), \gamma, R_\gamma, \in).$$

Proof. We prove $X \subseteq L(\sigma)$, the proof of that

$$(X, \gamma, R_\gamma, \in) \prec (P(\gamma) \cap L(\sigma), \gamma, R_\gamma, \in)$$

is very similar. Let $A \in X$. By definition of X, $A \in L_\alpha[x,z]$. Suppose $A \notin L(\sigma)$. This is a fact in $M[g]$ about A, x, z, α, σ; hence, there is an ω-model P in $M[g]$ such that

- $P \vDash \mathsf{ZFC} \setminus$ Powerset;
- $x, z, \sigma, z \in P$ and $P \vDash \tau$ is II's winning strategy for G_σ and x enumerates σ, τ;
- $\alpha = \mathrm{std}(P)$;
- $P \vDash A \notin L(\sigma)$.

This contradicts the Lemma 2.3.14. □

Let $\kappa = (\gamma^+)^{L(\sigma)}$. Let X be as in Lemmata 2.3.14 and 2.3.15 and let $L_\eta(\sigma)$ be the transitive collapse of X.

Claim: $\eta < \alpha$.

Proof. Fix a P as in Lemma 2.3.14. η cannot be bigger than α since then $\alpha \in \mathrm{std}(P)$. $\alpha = \eta$ is also impossible since then α is definable in P, which in turns implies $\alpha \in \mathrm{std}(P)$. $\square$

Lemmata 2.3.14 and 2.3.15 and the claim give us a contradiction since they together imply that $\kappa < \alpha$, which contradicts the definition of γ.

Now we are onto the proof of part 2. The implications that are proved are indicated in Figure 2.1. The theory (T) in Figure 2.1 is: $\mathsf{AD}^{L(\mathbb{R},\mu_0)} + (\mathbb{R},\mu_0)^\sharp$ exists.

"$\mathsf{AD}^{L(\mathbb{R})}$ and $\mathbb{R}^\sharp$ exists" implies $\mathsf{AD}_{\omega,\omega^2}$<-$\omega^2$-$\utilde{\Pi}^1_1$: By Woodin's theorem (mentioned in the Remark after Theorem 2.2.14), there is a premouse $\mathcal{M}$ such that

- $\mathcal{M}$ is active, $\rho(\mathcal{M}) = \omega$, and $\mathcal{M}$ is sound;
- $\mathcal{M} \vDash$ there are ω Woodin cardinals;
- $\mathcal{M}$ is (ω_1, ω_1)-iterable.

The existence of $\mathcal{M}$ is what we need to run the proof in Chapter 2 of [19] to obtain the conclusion.

$\mathsf{AD}_{\omega,\omega^2}$<-$\omega^2$-$\utilde{\Pi}^1_1$ implies $\mathsf{AD}_{\omega,\omega^2}\utilde{\Pi}^1_1$: This is clear.

$\mathsf{AD}_{\omega,\omega^2}\utilde{\Pi}^1_1$ implies "$\mathsf{AD}^{L(\mathbb{R})}$ and $\mathbb{R}^\sharp$ exists": By part 1, $\mathsf{AD}_{\mathbb{R},\omega}\utilde{\Pi}^1_1$ implies "$\mathbb{R}^\sharp$ exists." $\mathsf{AD}_{\omega,\omega^2}\utilde{\Pi}^1_1$ implies $\mathsf{AD}_{\mathbb{R},\omega}\utilde{\Pi}^1_1$ since any real game of length ω can be simulated (in an obvious way) by an integer game of length ω^2. It remains to prove $\mathsf{AD}^{L(\mathbb{R})}$.

Suppose not. Let γ be the least such that there is a formula ϕ and a real x that defines over $L_\gamma(\mathbb{R})$ a nondetermined set, that is, there is a set $A \in L(\mathbb{R})$ such that A is not determined and

$$y \in A \Leftrightarrow L_\gamma(\mathbb{R}) \vDash \phi[x,y].$$

Let G_1 be the following game. G_1 is a game on integers with ω many rounds $\langle R_i \mid i < \omega\rangle$ and each round is of length ω. In round R_0, the players take turns playing integers $\langle n_i \mid i < \omega\rangle$. In round R_i for $i > 0$, player I starts by playing a pair of integers (x_0^i, m_i), player II responds with an integer x_1^i, player I responds with an integer x_2^i, player II plays x_3^i etc. At the end of the play, the players have played: a real $y = \langle n_i \mid i < \omega\rangle$, a countable set of

$$\begin{tikzcd}
\mathsf{AD}_{\omega,\omega^3}\text{<-}\omega^2\text{-}\utilde{\Pi}^1_1 \arrow[r] \arrow[d] & \mathsf{AD}_{\omega,\omega^3}\utilde{\Pi}^1_1 \arrow[r] & \mathsf{AD}_{\mathbb{R},\omega^2}\utilde{\Pi}^1_1 \arrow[d] \\
\mathsf{AD}_{\mathbb{R},\omega^2}\text{<-}\omega^2\text{-}\utilde{\Pi}^1_1 \arrow[urr] & & (\mathrm{T}) \arrow[ull]
\end{tikzcd}$$

Figure 2.1: Implications in the proof of part 2 of Theorem 2.3.11

reals $\sigma = \{x^i \mid 0 \leq i < \omega\}$, where $x^i = \langle x^i_k \mid k < \omega\rangle$, and player I also plays a real $z = \langle m_i \mid 0 \leq i < \omega\rangle$. If $x \notin \sigma$, I loses. If z does not code a wellordering of ω (i.e. $z \notin WO$) then I loses. Otherwise, let γ_σ be the ordinal z codes, then

I wins the play of G_1 iff $L_{\gamma_\sigma}(\sigma) \vDash \mathsf{AD}$ and $L_{\gamma_\sigma}(\sigma) \vDash \phi[y,x]$ and (ϕ,x) defines over $L_{\gamma_\sigma}(\sigma)$ a nondetermined set.

$$\begin{array}{c|ccccccccccc} \mathrm{I} & n_0 & & n_2 & \cdots & x^0_0, m_0 & & x^0_2 & \cdots & x^1_0, m_1 & & x^1_2 \;\; \cdots \\ \hline \mathrm{II} & & n_1 & & \cdots & & x^0_1 & & \cdots & & x^1_1 & \;\;\cdots \end{array}$$

The game G_1

Clearly, the winning condition for player I is $\underset{\sim}{\Pi}^1_1(x)$. By our assumption, the game G_1 is determined. We proceed to get a contradiction from this.

Suppose player I wins via strategy τ. Let $\kappa >> \gamma$ be regular and let $M^* \prec V_\kappa$ be countable such that $\tau, x \in M^*$. Let M be the transitive collapse of M^* and let $\sigma = \mathbb{R}^M$. Let $\pi : M \to V_\kappa$ be the uncollapse map and $\pi(\sigma, \gamma_\sigma) = (\mathbb{R}, \gamma)$. Note that σ is closed under τ. By elementarity, in M, γ_σ is the least ordinal ξ in $L(\sigma)$ such that there is a nondetermined set definable over $L_\xi(\sigma)$ and (ϕ,x) defines such a set. So we can via II's moves play out all the reals in σ and τ is forced to build the model $L_{\gamma_\sigma}(\sigma)$. But then τ when restricted to R_0 is the winning strategy for player I in the nondetermined game with payoff set defined by (ϕ,x) over $L_{\gamma_\sigma}(\sigma)$. Contradiction.

Suppose now player II wins via strategy τ. Let M, σ, γ_σ be as the previous paragraph. Again, σ is closed under τ. By a similar reasoning, we can via I's moves, force τ to build the model $L_{\gamma_\sigma}(\sigma)$ and hence τ restricted to R_0 is II's winning strategy for the nondetermined game defined over $L_{\gamma_\sigma}(\sigma)$ via (ϕ,x). Contradiction. □

Proof of Theorem 2.3.11. We only prove part 1) of the theorem. Part 2) and 3) are similar. We prove the case $\alpha = 2$ for part 1). The proof of the other cases is the same. Let μ_0 be the club filter on X_0 in V.

$\mathsf{AD}_{\omega,\omega^3}$ $<$-ω^2-$\underset{\sim}{\Pi}^1_1$ implies $\mathsf{AD}_{\omega,\omega^3}\underset{\sim}{\Pi}^1_1$: This is clear.

$\mathsf{AD}_{\omega,\omega^3}\underset{\sim}{\Pi}^1_1$ implies $\mathsf{AD}_{\mathbb{R},\omega^2}\underset{\sim}{\Pi}^1_1$:[20] This is also clear since every real game of length ω^2 with $\underset{\sim}{\Pi}^1_1$ payoff can be simulated by an integer game of length ω^3 with the same payoff.

$\mathsf{AD}_{\mathbb{R},\omega^2}\underset{\sim}{\Pi}^1_1$ implies $L(\mathbb{R},\mu_0) \vDash$ "$\mathsf{AD}^+ + \mu_0$ is a normal fine measure on X_0" + $(\mathbb{R},\mu_0)^\sharp$ exists: Assume $\mathsf{AD}_{\mathbb{R},\omega^2}\underset{\sim}{\Pi}^1_1$. We first show $L(\mathbb{R},\mu_0) \vDash$ "$\mathsf{AD}^+ + \mu_0$ is a normal fine measure on X_0." To this end, suppose not. Let γ be the least such that $L_\gamma(\mathbb{R},\mu_0)$ defines a set that is either not determined or not measured by μ_0. To be more precise, let (ϕ,x) be such that ϕ is a formula and $x \in \mathbb{R}$ and

(a) either there is an $A \subseteq \mathbb{R}$ such that A is not determined and $\forall y \in \mathbb{R}\ y \in A \Leftrightarrow L_\gamma(\mathbb{R},\mu_0) \vDash \phi[y,x]$, or

[20] One can also prove directly that $\mathsf{AD}_{\omega,\omega^3}\underset{\sim}{\Pi}^1_1 \Leftrightarrow \mathsf{AD}_{\mathbb{R},\omega^2}\underset{\sim}{\Pi}^1_1$ and $\mathsf{AD}_{\omega,\omega^3}$ $<$-ω^2-$\underset{\sim}{\Pi}^1_1 \Leftrightarrow \mathsf{AD}_{\mathbb{R},\omega^2}$ $<$-ω^2-$\underset{\sim}{\Pi}^1_1$ by modifying the argument in [2].

(b) there is an $A \subseteq X_0$ such that $A \notin \mu_0$ and $\neg A \notin \mu_0$ and $\forall \sigma \in X_0\ \sigma \in A \Leftrightarrow L_\gamma(\mathbb{R}, \mu_0) \vDash \phi[\sigma, x]$.

Suppose (a) holds. Let G_2 be the following game. G_2 consists of ω many rounds $\langle R_i \mid i < \omega \rangle$ and each round is of length ω. In round R_0, the players take turns playings integers $\langle n_i \mid i < \omega \rangle$. In round R_i for $i > 0$, player I starts by playing a pair (x_0^i, m_i), where $x_0^i \in \mathbb{R}$ and $m_i \in \omega$, player II responds with a real x_1^i, player I responds with a real x_2^i, player II plays a real x_3^i etc. At the end of the play, the players have played: a real $y = \langle n_i \mid i < \omega \rangle$, a countable set of reals $\sigma = \cup_{k \geq 1} \sigma_k$, where $\sigma_k = \{x_i^k \mid i < \omega\}$, and player I also plays a real $z = \langle m_i \mid 1 \leq i < \omega \rangle$. If $x \notin \sigma$, I loses. If z does not code a wellordering of ω (i.e. $z \notin WO$) then I loses. Otherwise, let γ_σ be the ordinal z codes and C_σ be the tail filter generated by the sequence $\langle \sigma_k \mid k \geq 1 \rangle$[21], then

I wins the play of G_2 iff $L_{\gamma_\sigma}(\sigma, C_\sigma) \vDash$ "$\mathsf{AD} + \mathrm{C}_\sigma$ is a normal fine measure on X_0" and $L_{\gamma_\sigma}(\sigma, C_\sigma) \vDash \phi[y, x]$,where (ϕ, x) defines over $L_{\gamma_\sigma}(\sigma, C_\sigma)$ a nondetermined set. $(*)$

I	n_0		n_2	$\cdots$	x_0^0, m_0		x_2^0	$\cdots$	x_0^1, m_1		x_2^1	$\cdots$
II		n_1		$\cdots$		x_1^0		$\cdots$		x_1^1		$\cdots$

The game G_2

The game G_2 can be considered a real game of length ω^2 with $\Pi_1^1(x)$-payoff. So by our assumption, G_2 is determined. Suppose I wins G_2 via strategy τ. Let $\kappa >> \gamma$ be regular and $\langle M_k \mid k \leq \omega \rangle$ be such that

- $M_k \prec V_\kappa$ for all $k \leq \omega$;
- $x, \tau \in M_0$;
- $M_k \subsetneq M_{k+1}$ for $k < \omega$ and $M_\omega = \cup_{k<\omega} M_k$.

Let $\sigma_k = \mathbb{R}^{M_k}$ for $k < \omega$ and $\sigma = \cup_{k<\omega} \sigma_k = \mathbb{R}^M$. Let $\pi : N \to M_\omega$ be the uncollapse map and let $\pi(\gamma_\sigma, \mu_0^\sigma) = (\gamma, \mu_0)$. Note that for all $k < \omega$, σ_k is closed under τ and hence so is σ. We defeat $\tau \restriction \sigma$ as follows. For $i < \omega$, let player II play out σ_i in round R_{i+1}; then I is forced to construct (via τ) the model $L_{\gamma_\sigma^*}(\sigma, C_\sigma)$ satisfying $(*)$.

Lemma 2.3.16. *$L_{\gamma_\sigma}(\sigma, \mu_0^\sigma) = L_{\gamma_\sigma}(\sigma, C_\sigma)$ and $\gamma_\sigma = \gamma_\sigma^*$.*

Proof. Note that by elementarity, in N, $L_{\gamma_\sigma}(\sigma, \mu_0^\sigma)$ is the least level over which (ϕ, x) defines a nondetermined set. It's enough to prove that $\mu_0^\sigma \cap L_{\gamma_\sigma}(\sigma, \mu_0^\sigma) \subseteq C_\sigma$. This easily implies $L_{\gamma_\sigma}(\sigma, \mu_0^\sigma) = L_{\gamma_\sigma^*}(\sigma, C_\sigma)$. Let $A \subseteq X_0^N$ be a club in N and let $f : \sigma^{<\omega} \to \sigma$ witness this. By definition of N and σ, $\exists n \forall m \geq n\ f[\sigma_m^{<\omega}] \subseteq \sigma_m$. This means $\exists n \forall m \geq n\ \sigma_m \in A$, which implies $A \in C_\sigma$. This finishes the proof of the lemma. $\square$

[21] $B \in C_\sigma \Leftrightarrow \exists n \forall m \geq n\ \sigma_m \in B$.

By the lemma, in N, $\tau \upharpoonright \sigma$ when restricted to R_0 of G_2 is I's winning strategy for the nondetermined game defined by (ϕ, x) over $L_{\gamma_\sigma}(\sigma, \mu_0^\sigma)$. Contradiction. A similar (and a bit easier) reasoning gives a contradiction from the assumption that II has a winning strategy in G_2.

Now suppose (b) holds. Then (ϕ, x) defines over $L_\gamma(\mathbb{R}, \mu_0)$ a set A such that $A \notin \mu_0$ and $\neg A \notin \mu_0$. Let G_3 be the following game. G_3 has ω rounds $\{R_i \mid i < \omega\}$ and each round is of length ω. For $i < \omega$, player I starts round R_i by playing a pair (x_0^i, n_i), where $x_0^i \in \mathbb{R}$ and $n_i \in \omega$, II responds with a real x_1^i, I then responds with a ral x_2^i, II plays a real x_3^i in response etc. Let $\sigma_i = \{x_k^i \mid k < \omega\}$. Additionally, if $i > 0$, we require that the real x_0^i must code σ_{i-1}[22] (otherwise, I loses). At the end of the play, let $\sigma = \cup_i \sigma_i$, $y = \langle n_i \mid i < \omega \rangle$, and let C_σ be the tail filter defined by the sequence $\langle \sigma_i \mid i < \omega \rangle$. If $y \notin WO$, I loses. Otherwise, letting γ_σ be the ordinal coded by y, then

$$\text{I wins the play of } G_3 \Leftrightarrow x \in \sigma_0 \text{ and } L_{\gamma_\sigma}(\sigma, C_\sigma) \vDash \text{“}\mathsf{AD} + C_\sigma \text{ is a normal fine measure on } X_0\text{”}$$
$$\text{and } (\phi, x) \text{ defines over } L_{\gamma_\sigma}(\sigma, C_\sigma) \text{ a set not measured by } C_\sigma \text{ and } \sigma_0 \in C_\sigma. \quad (**)$$

I	x_0^0, n_0		x_2^0	$\cdots$	x_0^1 (codes σ_0), n_1		x_2^1	$\cdots$
II		x_1^0		$\cdots$		x_1^1		$\cdots$

The game G_3

The winning condition for player I in G_3 is $\Pi_1^1(x)$ so G_3 is determined. First let τ be I's winning strategy. Let $\kappa >> \gamma$ be regular and $\langle M_k \mid k \leq \omega \rangle$ be such that

- $M_k \prec V_\kappa$ for all $k \leq \omega$;
- $x, \tau \in M_0$;
- $M_k \subsetneq M_{k+1}$ for $k < \omega$ and $M_\omega = \cup_{k<\omega} M_k$.

Let $\sigma_k = \mathbb{R}^{M_k}$ for $k < \omega$ and $\sigma = \cup_{k<\omega} \sigma_k = \mathbb{R}^M$. Let $\pi : N \to M_\omega$ be the uncollapse map and let $\pi(\gamma_\sigma, \mu_0^\sigma) = (\gamma, \mu_0)$. Note that for all $k < \omega$, σ_k is closed under τ and hence so is σ. We defeat $\tau \upharpoonright \sigma$ as follows. We consider the following sequence $\langle P_n \mid n < \omega \rangle$, where each P_n is a complete play of G_3 according τ. We describe how II plays in P_n. In round R_i, II plays out σ_{i+n} (since σ_{i+n} is closed under τ, at the end of round R_i, the players play out the countable set of reals σ_{i+n}). At the end of the play P_n, the players play out the sequence $S_n = \langle \sigma_{i+n} \mid i < \omega \rangle$ (note that $\cup S_n = \sigma$) and τ constructs a model $L_{\gamma^n}(\sigma, C_n)$ satisfying $(**)$, where C_n is the tail filter given by S_n. Note that $C_n = C_m$ for all n, m. Let C be C_n for some (any) n. Hence by the fact that $L_{\gamma^n}(\sigma, C_n)$ satisfies $(**)$, for all n, m, $L_{\gamma^n}(\sigma, C_n) = L_{\gamma^m}(\sigma, C_m)$. Let $\xi = \gamma_n$ for some (any) n. Let A be the set not measured by C defined by (ϕ, x) over $L_\xi(\sigma, C)$. Recall that for each n, the countable set of reals played out in R_0 of P_n is σ_n, we have by the fact that τ is I's winning strategy that $\sigma_n \in A$. So $A \in C$

[22] This just means that x_0^i can compute an enumeration in order type ω of σ_{i-1}

after all. Contradiction.

Assume now II has a winning strategy in G_3. The same reasoning as above gives us a contradiction. The only difference is we get that for all n, $\sigma_n \notin A$ so $\neg A \in C$.

Now it remains to prove $(\mathbb{R}, \mu_0)^\sharp$ exists. The proof is a combination of techniques used in the proof that $\mathsf{AD}_{\mathbb{R},\omega^2}\underset{\sim}{\Pi}^1_1 \Rightarrow L(\mathbb{R}, \mu_0) \vDash$ "$\mathsf{AD}^+ + \mu_0$ is a normal fine measure on X_0" and $\mathsf{AD}_{\mathbb{R},\omega}\underset{\sim}{\Pi}^1_1 \Rightarrow \mathbb{R}^\sharp$ exists. We sketch only the proof here.

Recall that at this point, we have proved $L(\mathbb{R}, \mu_0) \vDash \mathsf{AD}^+ + \mu_0$ is a normal fine measure on X_0. We describe a real game G of length ω^2 with Π^1_1 payoff. A typical play of G is as follows. At round R_i, player I starts the round by playing (x^i_0, n_i) where $x^i_0 \in \mathbb{R}$, $n_i \in \omega$, player II responds with (x^i_1, m_i) where $x^i_1 \in \mathbb{R}$, $m_i \in \omega$, player I responds with x^i_2, player II plays x^i_3 in response. In general, for $k \leq 1$, player I plays a real x^i_{2k} while player II plays a real x^i_{2k+1}. At the end of the play, the players play a sequence $\langle \sigma_i \mid i < \omega \rangle$, where $\sigma_i = \{x^i_j \mid j < \omega\}$, player I plays a real $x = \langle n_i \mid i < \omega \rangle$, and player II plays a real $y = \langle m_i \mid i < \omega \rangle$. Let C be the tail filter generated by the sequence $\langle \sigma_i \mid i < \omega \rangle$ and $\sigma = \cup_i \sigma_i$.

Using the notation introduced in the proof of $\mathsf{AD}_{\mathbb{R},\omega}\underset{\sim}{\Pi}^1_1 \Rightarrow \mathbb{R}^\sharp$, Player II wins the play if

(a) E_x is not a total order which is a wellorder; or

(b) M_y is an ω-model of "$\mathsf{ZF} \setminus$ Powerset $+ V = L(\sigma, C) + \mathsf{AD}^+ + C$ is a normal fine measure on X_0" such that $|E_x| < \mathrm{std}(M_y)$.

Again, G is determined and player I cannot have a winning strategy in G. Hence II has a winning strategy τ. Let κ be a large, regular cardinal in V, and $M^* \prec V_\kappa$ be countable such that M^* is the union of an increasing sequence $\langle M^*_n \mid n < \omega \rangle$ of countable elementary substructures of V_κ and $\tau \in M^*_0$. Let M be the transitive collapse of M^* and π is the uncollapse map. Let $\pi(\sigma, \mu^\sigma_0) = (\mathbb{R}, \mu_0)$. Let $g \subseteq Col(\omega, \sigma)$ be M-generic and $g \in V$. We prove that $\sigma^\sharp$ exists in $M[g]$ by proving in $M[g]$, there is a real x such that for every $\gamma \geq \Theta^{L(\sigma, \mu^\sigma_0)}$, if γ is admissible relative to x then γ is a cardinal in $L(\sigma, \mu^\sigma_0)$. The rest of the proof is similar to that of the corresponding part in the proof of $\mathsf{AD}_{\mathbb{R},\omega}\underset{\sim}{\Pi}^1_1 \Rightarrow \mathbb{R}^\sharp$. We leave the details to the reader.

$L(\mathbb{R}, \mu_0) \vDash$ **"$\mathsf{AD}^+ + \mu_0$ is a normal fine measure on X_0" $+$ $(\mathbb{R}, \mu_0)^\sharp$ exists implies $\mathsf{AD}_{\omega,\omega^3}$ $<$-ω^2-$\underset{\sim}{\Pi}^1_1$:** Theorem 4.2.5 implies there is a premouse $\mathcal{M}$ such that

- $\mathcal{M}$ is active, $\rho(\mathcal{M}) = \omega$, and $\mathcal{M}$ is sound;
- $\mathcal{M} \vDash$ there are ω^2 Woodin cardinals;
- $\mathcal{M}$ is (ω_1, ω_1)-iterable.

The existence of $\mathcal{M}$ is what we need to run the proof in Chapter 2 of [19] to obtain the conclusion.

$\mathsf{AD}_{\omega,\omega^3}$ $<$-ω^2-$\underset{\sim}{\Pi}^1_1$ implies $\mathsf{AD}_{\mathbb{R},\omega^2}$ $<$-ω^2-$\underset{\sim}{\Pi}^1_1$: This is again clear since every real game of length ω^2 with $<$-ω^2-$\underset{\sim}{\Pi}^1_1$ payoff can be simulated by an integer game of length ω^3 with the

same payoff.
$\mathsf{AD}_{\mathbb{R},\omega^2}$ $<$-ω^2-$\underset{\sim}{\Pi}^1_1$ implies $\mathsf{AD}_{\mathbb{R},\omega^2}\underset{\sim}{\Pi}^1_1$: This is also clear.
The above implications cover all the equivalences that need to be proved. Hence we're done. $\square$

Chapter 3

HOD Analysis

3.1 When $V = L(\mathcal{P}(\mathbb{R}))$

3.1.1 The Successor Case

3.1.1.1 Preliminaries

Let $(\mathcal{P}, \Sigma)$ be a hod pair such that Σ has branch condensation. We let $K^{\Sigma,-}(\mathbb{R})$ be the union of all $\mathcal{N}$ such that $\mathcal{N}$ is a Σ-premouse over $\mathbb{R}$ (in the sense of [27])[1], $\rho_\omega(\mathbb{R}) = \mathbb{R}$, and whenever $\mathcal{M}$ is countable transitive such that $\mathcal{P} \in \mathcal{M}$ and $\mathcal{M}$ elementarily embeds into $\mathcal{N}$, then $\mathcal{M}$ has a unique $\omega_1 + 1$-iteration strategy Λ such that whenever $\mathcal{Q}$ is a Λ-iterate of $\mathcal{M}$, then $\mathcal{Q}$ is also a Σ-premouse. We say $\mathcal{M}$ is Σ-iterable. Finally, let $K^{\Sigma}(\mathbb{R}) = L(K^{\Sigma,-}(\mathbb{R}))$.

Let $(\mathcal{P}, \Sigma)$ be as above. We briefly describe a notion of Prikry forcing that will be useful in our HOD computation. The forcing $\mathbb{P}$ described here is defined in $K^{\Sigma}(\mathbb{R})$ and is a modification of the forcing defined in Section 6.6 of [41]. All facts about this forcing are proved similarly as those in Section 6.6 of [41] so we omit all proofs.

First, let T be the tree of a $\Sigma^2_1(\Sigma)$ scale on a universal Σ^2_1 set U. Write $\mathcal{P}_x$ for the Σ-premouse coded by the real x. Let a be countable transitive, $x \in \mathbb{R}$ such that a is coded by a real recursive in x. A normal iteration tree $\mathcal{U}$ on a 0-suitable Σ-premouse $\mathcal{Q}$ (see Definition 3.1.20, where $(\mathcal{Q}, \Sigma)$ is defined to be 0-suitable) is short if for all limit $\xi \leq lh(\mathcal{U})$, $Lp^{\Sigma}(\mathcal{M}(\mathcal{U}|\xi)) \vDash \delta(\mathcal{U}|\xi)$ is not Woodin. Otherwise, we say that $\mathcal{U}$ is maximal. We say that a 0-suitable $\mathcal{P}_z$ is short-tree iterable by Λ if for any short tree $\mathcal{T}$ on $\mathcal{P}_z$, $b = \Lambda(\mathcal{T})$ is such that $\mathcal{M}^{\mathcal{T}}_b$ is 0-suitable, and b has a Q-structure $\mathcal{Q}$ such that $\mathcal{Q} \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$. Put

$$\mathcal{F}^x_a = \{\mathcal{P}_z \mid z \leq_T x, \mathcal{P}_z \text{ is a short-tree iterable 0-suitable } \Sigma\text{-premouse over } a\}$$

For each a, for x in the cone in the previous claim, working in $L[T, x]$, we can simultaneously compare all $\mathcal{P}_z \in \mathcal{F}^x_a$ (using their short-tree iteration strategy) while doing the genericity

[1] Here we assume Σ is sufficiently iterable that the definition of Σ-premouse over $\mathbb{R}$ makes sense; see [27] for more details.

iterations to make all y such that $y \leq_T x$ generic over the common part of the final model $\mathcal{Q}_a^{x,-}$. This process (hence $\mathcal{Q}_a^{x,-}$) depends only on the Turing degree of x. Put

$$\mathcal{Q}_a^x = Lp_\omega^\Sigma(\mathcal{Q}_a^{x,-}), \text{ and } \delta_a^x = o(\mathcal{Q}_a^{x,-}).$$

By the above discussion, $\mathcal{Q}_a^x, \delta_a^x$ depend only on the Turing degree of x. Here are some properties obtained from the above process.

1. $\mathcal{F}_a^x \neq \emptyset$ for x of sufficiently large degree;
2. $\mathcal{Q}_a^{x,-}$ is full (no levels of $\mathcal{Q}_a^x$ project strictly below δ_a^x);
3. $\mathcal{Q}_a^x \vDash \delta_a^x$ is Woodin;
4. $\mathcal{P}(a) \cap \mathcal{Q}_a^x = \mathcal{P}(a) \cap OD_T(a \cup \{a\})$ and $\mathcal{P}(\delta_a^x) \cap \mathcal{Q}_a^x = \mathcal{P}(\delta_a^x) \cap OD_T(Q_a^{x,-} \cup \{Q_a^{x,-}\})$;
5. $\delta_a^x = \omega_1^{L[T,x]}$.

Now for an increasing sequence $\vec{d} = \langle d_0, ..., d_n\rangle$ of Turing degrees, and a countable transitive, set

$$\mathcal{Q}_0(a) = \mathcal{Q}_a^{d_0} \text{ and } \mathcal{Q}_{i+1}(a) = \mathcal{Q}_{\mathcal{Q}_i(a)}^{d_{i+1}} \text{ for } i < n$$

We assume from here on that the degrees d_{i+1}'s are such that $\mathcal{Q}_{\mathcal{Q}_i(a)}^{d_{i+1}}$ are defined. For $\vec{d}$ as above, write $\mathcal{Q}_i^{\vec{d}}(a) = \mathcal{Q}_i(a)$ even though $\mathcal{Q}_i(a)$ only depends on $\vec{d}|(i+1)$. Let μ be the cone measure on the Turing degrees. We can then define our Prikry forcing $\mathbb{P}$ (over $L(T,\mathbb{R})$) as follows. A condition $(p,S) \in \mathbb{P}$ just in case $p = \langle \mathcal{Q}_0^{\vec{d}}(a), ..., \mathcal{Q}_n^{\vec{d}}(a)\rangle$ for some $\vec{d}$, $S \in L(T,\mathbb{R})$ is a "measure-one tree" consisting of stems q which either are initial segments or end-extensions of p and such that $(\forall q = \langle \mathcal{Q}_0^{\vec{e}}(a), ..., \mathcal{Q}_k^{\vec{e}}(a)\rangle \in S)(\forall_\mu^* d)$ let $\vec{f} = \langle \vec{e}(0), ..., \vec{e}(k), d\rangle$, we have $\langle \mathcal{Q}_0^{\vec{f}}(a), ..., \mathcal{Q}_{(k+1)}^{\vec{f}}(a)\rangle \in S$. The ordering on $\mathbb{P}$ is defined as follows.

$$(p,S) \preccurlyeq (q,W) \text{ iff } p \text{ end-extends } q,\ S \subseteq W, \text{ and } \forall n \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)\ (p|(n+1) \in W).$$

$\mathbb{P}$ has the Prikry property in $K^\Sigma(\mathbb{R})$. Let G be a $\mathbb{P}$-generic over $K^\Sigma(\mathbb{R})$, $\langle \mathcal{Q}_i \mid i < \omega\rangle = \cup\{p \mid \exists \vec{X}(p,\vec{X}) \in G\}$ and $\mathcal{Q}_\infty = \bigcup_i \mathcal{Q}_i$. Let δ_i be the largest Woodin cardinal of $\mathcal{Q}_i$. Then

$$P(\delta_i) \cap L[T, \langle \mathcal{Q}_i \mid i < \omega\rangle] \subseteq \mathcal{Q}_i,$$

and

$$L[T, \mathcal{Q}_\infty] = L[T, \langle \mathcal{Q}_i \mid i < \omega\rangle] \vDash \delta_i \text{ is Woodin.}$$

Definition 3.1.1 (Derived models)**.** *Suppose $M \vDash$ ZFC and $\lambda \in M$ is a limit of Woodin cardinals in M. Let $G \subseteq Col(\omega, < \lambda)$ be generic over M. Let $\mathbb{R}_G^*$ (or just $\mathbb{R}^*$) be the symmetric reals of $M[G]$ and Hom_G^* (or just Hom^*) be the set of $A \subseteq \mathbb{R}^*$ in $M(\mathbb{R}^*)$ such that there is a tree T such that $A = p[T] \cap \mathbb{R}^*$ and there is some $\alpha < \lambda$ such that*

$$M[G \restriction \alpha] \vDash \text{“}T \textit{ has a } {<}\text{-}\lambda\text{-complement”}.$$

By the ***old derived model*** *of M at λ, denoted by $D(M,\lambda)$, we mean the model $L(\mathbb{R}^*, Hom^*)$. By the* ***new derived model*** *of M at λ, denoted by $D^+(M,\lambda)$, we mean the model $L(\Gamma, \mathbb{R}^*)$, where Γ is the closure under Wadge reducibility of the set of $A \in M(\mathbb{R}^*) \cap \mathcal{P}(\mathbb{R}^*)$ such that $L(A, \mathbb{R}^*) \vDash \mathsf{AD}^+$.*

Theorem 3.1.2 (Woodin)**.** *Let M be a model of ZFC and $\lambda \in M$ be a limit of Woodin cardinals of M. Then $D(M,\lambda) \vDash \mathsf{AD}^+$, $D^+(M,\lambda) \vDash \mathsf{AD}^+$. Furthermore, Hom^* is the pointclass of Suslin co-Suslin sets of $D^+(M,\lambda)$.*

Using the proof of Theorem 3.1 from [36] and the definition of $K^\Sigma(\mathbb{R})$ from [27], we get that in $K^\Sigma(\mathbb{R})[G]$, there is a Σ-premouse $\mathcal{Q}^+_\infty$ extending $\mathcal{Q}_\infty$ such that $K^\Sigma(\mathbb{R})$ can be realized as a (new) derived model of $\mathcal{Q}^+_\infty$ at ω_1^V, which is the limit of Woodin cardinals of $\mathcal{Q}^+_\infty$. Roughly speaking, the Σ-premouse $\mathcal{Q}^+_\infty$ is the union of Σ-premice $\mathcal{R}$ over $\mathcal{Q}_\infty$, where $\mathcal{R}$ is an S-translation of some $\mathcal{M} \lhd K^\Sigma(\mathbb{R})$ (see [23] for more on S-translations).

3.1.1.2 When $\alpha = 0$

We recall some basic notions from descriptive inner model theory. All the notions and notations used in this section are standard. See [26] and [38] for full details. Here, we'll try to stay as close notationally as possible to those used in [26] and [38]. The reader who are familiar with basic descriptive inner model theory can skip ahead to the actual computation and come back to this when necessary.

Definition 3.1.3 (k-suitable premouse)**.** *Let $0 \leq k < \omega$ and Γ be an inductive-like pointclass. A premouse $\mathcal{N}$ is k-suitable with respect to Γ iff there is a strictly increasing sequence $\langle \delta_i \mid i \leq k\rangle$ such that*

1. *for all δ, $\mathcal{N} \vDash$ “δ is Woodin” iff $\delta = \delta_i$ for some $i < 1 + k$;*
2. *$OR^{\mathcal{N}} = sup(\{(\delta_k^{+n})^N | n < \omega\})$;*
3. *$Lp^\Gamma(\mathcal{N}|\xi) = \mathcal{N}|(\xi^+)^+$ for all cutpoints ξ of $\mathcal{N}$ where $Lp^\Gamma(\mathcal{N}|\xi) = \cup\{\mathcal{M} \mid \mathcal{N}|\xi \unlhd \mathcal{M} \wedge \rho(\mathcal{M}) = \xi \wedge \mathcal{M}$ has iteration strategy in $\Gamma\}$;*
4. *if $\xi \in OR \cap \mathcal{N}$ and $\xi \neq \delta_i$ for all i, then $Lp^\Gamma(\mathcal{N}|\xi) \vDash$ ”ξ is not Woodin.”*

Definition 3.1.4. *Let $\mathcal{N}$ be as above and $A \subseteq \mathbb{R}$. Then $\tau^{\mathcal{N}}_{A,\nu}$ is the unique standard term $\sigma \in \mathcal{N}$ such that $\sigma^g = A \cap \mathcal{N}[g]$ for all g generic over $\mathcal{N}$ for $Col(\omega,\nu)$, if such a term exists. We say that $\mathcal{N}$ term captures A iff $\tau^{\mathcal{N}}_{A,\nu}$ exists for all cardinals ν of $\mathcal{N}$.*

If $\mathcal{N}, \Gamma$ are as in Definition 2.1 and $A \in \Gamma$, then [26] shows that $\mathcal{N}$ term captures A. Later on, if the context is clear, we'll simply say capture instead of term capture or Suslin capture. For a complete definition of “$\mathcal{N}$ is A-iterable”, see [38]. Roughly speaking, $\mathcal{N}$ is A-iterable if $\mathcal{N}$ term captures A and

1. for any maximal tree $\mathcal{T}$ (or stack $\vec{\mathcal{T}}$) on $\mathcal{N}$, there is a cofinal branch b such that the branch embedding $i_b^{\mathcal{T}} =_{def} i$ moves the term relation for A correctly i.e., for any κ cardinal in $\mathcal{N}$, $i(\tau^{\mathcal{N}}_{A,\kappa}) = \tau^{\mathcal{M}_b^{\mathcal{T}}}_{A,i(\kappa)}$;

2. if $\mathcal{T}$ on $\mathcal{N}$ is short, then there is a branch b such that $\mathcal{Q}(b,\mathcal{T})$[2] exists and $\mathcal{Q}(b,\mathcal{T}) \trianglelefteq Lp^{\Gamma}(\mathcal{M}(\mathcal{T}))$[3]; we say that $\mathcal{T}^\frown b$ is Γ-guided.

This obviously generalizes to define $\vec{A}$-iterability for any finite sequence $\vec{A}$.

Definition 3.1.5. *Let $\mathcal{N}$ be k-suitable with respect to Σ^2_1 and $k < \omega$. Let $\vec{A} = \langle A_i \mid i \leq n\rangle$ be a sequence of OD sets of reals and $\nu = (\delta_k^{+\omega})^{\mathcal{N}}$. Then*

1. $\gamma^{\mathcal{N}}_{\vec{A}} = \sup(\{\xi | \xi$ *is definable over* $(\mathcal{N}|\nu, \tau^{\mathcal{N}}_{A_0,\delta_k}, ..., \tau^{\mathcal{N}}_{A_n,\delta_k})\} \cap \delta_0)$;

2. $H^{\mathcal{N}}_{\vec{A}} = Hull^{\mathcal{N}}(\gamma^{\mathcal{N}}_{\vec{A}} \cup \{\tau^{\mathcal{N}}_{A_0,\delta_k}, ..., \tau^{\mathcal{N}}_{A_n,\delta_k}\})$, *where we take the full elementary hull without collapsing.*

From now on, we will write $\tau^{\mathcal{N}}_A$ without further clarifying that this stands for $\tau^{\mathcal{N}}_{A,\delta}$ where δ is the largest Woodin cardinal of $\mathcal{N}$. We'll also write $\tau^{\mathcal{N}}_{A,l}$ for $\tau^{\mathcal{N}}_{A,\delta_l^{\mathcal{N}}}$ for $l \leq k$. Also, we'll occasionally say k-suitable without specifying the pointclass Γ.

Definition 3.1.6. *Let $\mathcal{N}$ be k-suitable with respect to some pointclass Γ and $A \in \Gamma$. $\mathcal{N}$ is strongly A-iterable if $\mathcal{N}$ is A-iterable and for any suitable $\mathcal{M}$ such that if $i, j : \mathcal{N} \to \mathcal{M}$ are two A-iteration maps then $i \restriction H^{\mathcal{N}}_A = j \restriction H^{\mathcal{M}}_A$.*

Definition 3.1.7. *Let Γ be an inductive-like pointclass and $\mathcal{N}$ be k-suitable with respect to Γ for some k. Let $\mathcal{A}$ be a countable collection of sets of reals in $\Gamma \cup \breve{\Gamma}$. We say $\mathcal{A}$ guides a strategy for $\mathcal{N}$ below $\delta_0^{\mathcal{N}}$ if whenever $\mathcal{T}$ is a countable, normal iteration tree on $\mathcal{N}$ based on $\delta_0^{\mathcal{N}}$ of limit length, then*

1. *if $\mathcal{T}$ is short, then there is a unique cofinal branch b such that $\mathcal{Q}(b,\mathcal{T})$ exists and $\mathcal{Q}(b,\mathcal{T}) \trianglelefteq Lp^{\Gamma}(\mathcal{M}(\mathcal{T}))$[4], and*

2. *if $\mathcal{T}$ is maximal, then there is a unique nondropping branch b such that $i_b^{\mathcal{T}}(\tau^{\mathcal{N}}_{A,\mu}) = \tau^{\mathcal{M}_b^{\mathcal{T}}}_{A,i_b(\mu)}$ for all $A \in \mathcal{A}$ and cardinals $\mu \geq \delta_k^{\mathcal{N}}$ of $\mathcal{N}$ and $\delta(\mathcal{T}) = \sup\{\gamma^{\mathcal{M}_b^{\mathcal{T}}}_{A,0} \mid A \in \mathcal{A}\}$ where $\delta = i_b^{\mathcal{T}}(\delta_0)$.*

[2] $\mathcal{Q}(b,\mathcal{T})$ is called the $\mathcal{Q}$-structure and is defined to be the least initial segment of $\mathcal{M}_b^{\mathcal{T}}$ that defines the failure of Woodinness of $\delta(\mathcal{T})$.

[3] This implicitly assumes that $\mathcal{Q}(b,\mathcal{T})$ has no extenders overlapping $\delta(\mathcal{T})$. We're only interested in trees $\mathcal{T}$ arising from comparisons between suitable mice and for such trees, $\mathcal{Q}$ structures have no extenders overlapping $\delta(\mathcal{T})$.

[4] Again we disregard the case where $\mathcal{Q}$-structures have overlapping extenders.

We can also define an $\mathcal{A}$-guided strategy that acts on finite stacks of normal trees in a similar fashion.

The most important instance of the above definition used in this paper is when $\mathcal{A}$ is a self-justifying-system that seals a Σ_1 gap. A strategy guided by such an $\mathcal{A}$ has many desirable properties.

We state a theorem of Steel's which essentially says that Mouse Capturing implies Mouse Capturing for mice over $\mathbb{R}$.

Theorem 3.1.8 (Steel, see [35]). *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R})) + \Theta = \theta_0 + \mathsf{MC}$*[5]*. Then every set of reals is in a (countably iterable) mouse over $\mathbb{R}$ projecting to $\mathbb{R}$. In other words, $V = K(\mathbb{R})$, where*

$$K(\mathbb{R}) = L(\cup\{M \mid M \text{ is } \mathbb{R}\text{-sound},\ \rho_\omega(M) = \mathbb{R},\ \text{and } M \text{ is countably iterable}\}).$$

Next, we prove the following theorem of Woodin's which roughly states that HOD is coded into a subset of Θ.

Theorem 3.1.9 (Woodin). *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Then $HOD = L[P]$ for some $P \subseteq \Theta$ in HOD.*

Proof. First, let

$$\mathbb{P} = \{(\vec{\alpha}, \vec{a}) \mid \vec{\alpha} = \langle \alpha_0, \alpha_1, ..., \alpha_n\rangle \in \Theta^{<\omega}, \vec{a} = \langle a_0, a_1, ..., a_n\rangle, \forall i \leq n(a_i \subseteq \alpha_i)\}.$$

$\mathbb{P}$ is a poset with the (obvious) order by extension. If g is a $\mathbb{P}$-generic over V then g induces an enumeration of order type ω of $(\Theta, \cup_{\gamma<\Theta}\mathcal{P}(\gamma))$. Now let

$$\mathbb{Q}^* = \{(\vec{\alpha}, A) \mid \vec{\alpha} = \langle \alpha_0, \alpha_1, ..., \alpha_n\rangle \in \Theta^{<\omega}, A \subseteq \mathcal{P}(\alpha_0) \times \mathcal{P}(\alpha_1) \times ... \times \mathcal{P}(\alpha_n), A \in OD\}.$$

The ordering on $\mathbb{Q}^*$ is defined as follows:

$$(\vec{\alpha}, A) \leq (\vec{\beta}, B) \Leftrightarrow \forall i < \mathrm{dom}(\vec{\alpha})\vec{\alpha}(i) = \vec{\beta}(i), B|(\mathcal{P}(\vec{\alpha}(0)) \times ... \times \mathcal{P}(\vec{\alpha}(\mathrm{dom}(\alpha) - 1)))^6 \subseteq A.$$

There is a poset $\mathbb{Q} \in \mathrm{HOD} \cap \mathcal{P}(\Theta)$ that is isomorphic to $\mathbb{Q}^*$ via an OD map π. For our convenience, whenever $p \in \mathbb{Q}$, we will write p^* for $\pi(p)$. Furthermore, we can define π so that elements of $\mathbb{Q}$ have the form $(\vec{\alpha}, A)$ whenever $p^* = (\vec{\alpha}, A^*)$. In other words, we can think of π as a bijection of Θ and the set of OD subsets of $\mathcal{P}(\alpha_0) \times \mathcal{P}(\alpha_1) \times ... \times \mathcal{P}(\alpha_n)$ for $\alpha_0, \alpha_1, ..., \alpha_n < \Theta$. For notational simplicity, if $p^* = (\vec{\alpha}, A^*)$, we write $o(p^*)$ for $\vec{\alpha}$ and $s(p^*)$ for A^*.

Claim 1. *Let g be $\mathbb{P}$-generic over V. Then g induces a $\mathbb{Q}$-generic G_g over HOD. In fact, for any condition $q \in \mathbb{Q}$, we can find a $\mathbb{P}$-generic g over V such that $q \in G_g$ and G_g is a $\mathbb{Q}$-generic over HOD.*

[5] MC stands for Mouse Capturing, which states that if $x, y \in \mathbb{R}$ and $x \in OD(y)$ then x is in a sound mouse over y projecting to y.

Proof. As mentioned above, g induces a generic enumeration f of $(\Theta, \cup_{\gamma<\Theta}\mathcal{P}(\gamma))$ of order type ω. Furthermore, for each $n < \omega$, $f(n)_0 < \Theta$ and $f(n)_1 \subseteq f(n)_0$. Let

$$G = \cup_{n<\omega}\{(\langle f(0)_0, ..., f(n)_0\rangle, A) \in \mathbb{Q} \mid \langle f(0)_1, ..., f(n)_1\rangle \in A^*\}.$$

We claim that G is $\mathbb{Q}$-generic over HOD. To see this, let $D \subseteq \mathbb{Q}$, $D \in$ HOD be a dense set. Let $p = f|(n+1)$ for some n. It's enough to find a $q = (\langle\alpha_0, ..., \alpha_m\rangle, \langle a_0, ..., a_m\rangle) \in \mathbb{P}$ extending p such that $D_q \cap D \neq \emptyset$ where

$$D_q = \{(\langle\alpha_0, ..., \alpha_m\rangle, A) \mid \langle a_0, ..., a_m\rangle \in A^*\}.$$

If no such q exists, let $r = (\langle f(0)_0, ..., f(n)_0\rangle, B)$, where

$$b \in B^* \Leftrightarrow \forall t \in D \forall c (b^\frown c \notin s(t)).$$

Then r is a condition in $\mathbb{Q}$ with no extension in D. Contradiction. $\square$

For each $\alpha < \Theta$, $n < \omega$, and $\langle\alpha_0, ..., \alpha_n\rangle$, let $A_{\alpha,\langle\alpha_0,...,\alpha_n\rangle} = (\langle\alpha_0, ..., \alpha_n\rangle, A)$ such that $\forall a \in A^*(\alpha \in a(n))$. We can then define a canonical term in HOD for a generic enumeration of $\cup_{\gamma<\Theta}\mathcal{P}(\gamma)$. For each $n < \omega$, let $\sigma_n = \{(p, \check{\alpha}) \mid p \in \mathbb{Q}, p \leq A_{\alpha,\langle\alpha_0,...,\alpha_n\rangle}$ for some $\langle\alpha_0, ..., \alpha_n\rangle \in \Theta^{n+1}\}$; let $\tau = \{(A, \sigma_n) \mid n < \omega, A \in \mathbb{Q}\}$. Then it's easy to see that whenever G is $\mathbb{P}$-generic over HOD induced by a $\mathbb{P}$-generic over V, τ_G enumerates $\cup_{\gamma<\Theta}\mathcal{P}(\gamma)$ in order type ω. This means we can recover $\mathcal{P}(\mathbb{R})^V$ in the model $L[\mathbb{Q}, \tau][G]$ by AD^+ (here we only use the fact that every set of reals has an ∞-Borel code which is a bounded subset of Θ).

To sum up, we have $L[\mathbb{Q}, \tau] \subseteq \text{HOD} \subseteq L[\mathbb{Q}, \tau][G]$ for some $\mathbb{Q}$-generic G over HOD. By a standard argument, this implies that $L[\mathbb{Q}, \tau] = \text{HOD}$. $\square$

Now let $\mathcal{F} = \{(\mathcal{M}, \vec{A}) \mid \vec{A}$ is a finite sequence of OD sets of reals and $\mathcal{M}$ is k-suitable for some k and is strongly $\vec{A}$-iterable$\}$. We say $(\mathcal{M}, \vec{A}) \leq_{\mathcal{F}} (\mathcal{N}, \vec{B})$ if $\vec{A} \subseteq \vec{B}$ and $\mathcal{M}$ iterates to a suitable initial segment of $\mathcal{N}$, say $\mathcal{N}^-$, via its iteration strategy that respects $\vec{A}$. We then let $\pi_{(\mathcal{M},\vec{A}),(\mathcal{N},\vec{B})} : H^{\mathcal{M}}_{\vec{A}} \to H^{\mathcal{N}^-}_{\vec{B}}$ be the unique map. That is, given any two different iteration maps $i_0, i_1 : \mathcal{M} \to \mathcal{N}^-$ according to $\mathcal{M}$'s iteration strategy, by strong $\vec{A}$-iterability, $i_0 \restriction H^{\mathcal{M}}_{\vec{A}} = i_1 \restriction H^{\mathcal{M}}_{\vec{A}}$, so the map $\pi_{(\mathcal{M},\vec{A}),(\mathcal{N},\vec{B})}$ is well-defined. The following theorem is basically due to Woodin. We just sketch the proof and give more details in the proof of Proposition 3.1.12.

Theorem 3.1.10. *Assume $V = L(\mathcal{P}(\mathbb{R})) + \mathsf{AD}^+ + \mathsf{MC} + \Theta = \theta_0$. Given any OD set of reals A and any $n \in \omega$, there is an n-suitable M that is strongly A-iterable. The same conclusion holds for any finite sequence $\vec{A}$ of OD sets of reals.*

Proof. We'll prove the theorem for $n = 1$. The other cases are similar. So suppose not. By Theorem 3.1.8, $V = K(\mathbb{R})$. Then $V \vDash \phi$ where $\phi = (\exists\alpha)\ (K(\mathbb{R})|\alpha \vDash$ "$\mathsf{ZF}^- + \Theta$ exists + $(\exists A)$ (A is OD and there is no 1-suitable strongly A-iterable mouse))".

Let $\gamma < \underset{\sim}{\delta}^2_1$ be least such that $K(\mathbb{R})|\gamma \vDash \phi$. Such a γ exists by Σ_1-reflection, i.e. Theorem 1.1.5. Then it is easy to see that γ ends a proper weak gap, say $[\overline{\gamma}, \gamma]$ for some $\overline{\gamma} < \gamma$. Fix the least such A as above. By [39] and the minimality of γ, we get a self-justifying-system (sjs) $\langle A_i \mid i < \omega\rangle$ of $\mathrm{OD}^{K(\mathbb{R})|\gamma}$ sets of reals in $K(\mathbb{R})|\gamma$ that seals the gap[7]. We may and do assume $A = A_0$. Let $\Gamma = \Sigma_1^{K(\mathbb{R})|\overline{\gamma}}$ and Ω a good pointclass beyond $K(\mathbb{R})|(\gamma+1)$, i.e. $\mathcal{P}(\mathbb{R})^{K(\mathbb{R})|(\gamma+1)} \subsetneq \underset{\sim}{\Delta}_\Omega$. Ω exists because $\gamma < \underset{\sim}{\delta}^2_1$. Let N^* be a coarse Ω-Woodin, fully iterable mouse. Such an N^* exists by [35] or by Theorem 1.1.4. In fact by Theorem 1.1.4, one can choose N^* that Suslin captures Ω and the sequence $\langle A_i \mid i < \omega\rangle$. Also by [35], there are club-in-OR^{N^*} many Γ-Woodin cardinals in N^*. It can be shown that the $L[E]$-construction done inside N^* reaches a $\mathcal{P}$ such that $\mathcal{P}$ is 1-suitable with respect to Γ (hence has canonical terms for the A_i's) and $\mathcal{P} \vDash$ "δ_0 and δ_1 are Woodin cardinals" where δ_0 and δ_1 are the first two Γ-Woodin cardinals in N^*. Let Σ be the strategy for $\mathcal{P}$ induced by that of N^*. By lifting up to the background strategy and using term condensation for the self-justifying-system, we get that Σ is guided by $\langle A_i \mid i < \omega\rangle$, hence $(\mathcal{P}, \Sigma)$ is strongly A-iterable. But then $K(\mathbb{R})|\gamma \vDash$"$\mathcal{P}$ is strongly A-iterable." This is a contradiction. $\square$

The theorem implies $\mathcal{F} \neq \varnothing$. Moreover, we have that $\mathcal{F}$ is a directed system because given any $(\mathcal{M}, \vec{A}), (\mathcal{N}, \vec{B}) \in \mathcal{F}$, we can do a simultaneous comparison of $(\mathcal{M}, \vec{A}), (\mathcal{N}, \vec{B})$, and some $(\mathcal{P}, \vec{A} \oplus \vec{B}) \in \mathcal{F}$ using their iteration strategies to obtain some $(\mathcal{Q}, \vec{A} \oplus \vec{B}) \in \mathcal{F}$ such that $(\mathcal{M}, \vec{A}), (\mathcal{N}, \vec{B}) \leq_{\mathcal{F}} (\mathcal{Q}, \vec{A} \oplus \vec{B})$. We summarize facts about $\mathcal{M}_\infty$ proved in [26] and [41]. These results are due to Woodin.

Lemma 3.1.11. *1. $\mathcal{M}_\infty$ is wellfounded.*

2. $\mathcal{M}_\infty$ has ω Woodin cardinals ($\delta_0^{\mathcal{M}_\infty}, \delta_1^{\mathcal{M}_\infty}$...) cofinal in its ordinals.

3. $\theta_0 = \delta_0^{\mathcal{M}_\infty}$ and $HOD|\theta_0 = \mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty}$.

We'll extend this computation to the full HOD. Now we define a strategy Σ_∞ for $\mathcal{M}_\infty$. For each $A \in OD \cap \mathcal{P}(\mathbb{R})$, let $\tau^{\mathcal{M}_\infty}_{A,k}$ = common value of $\pi_{(\mathcal{P},A),\infty}(\tau^{\mathcal{P}}_{A,k})$ where $\pi_{(\mathcal{P},A),\infty}$ is the direct limit map and $\tau^{\mathcal{P}}_{A,k}$ is the standard term of $\mathcal{P}$ that captures A at δ^P_k. Σ_∞ will be defined (in V) for (finite stacks of) trees on $\mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty}$ in $\mathcal{M}_\infty$. For $k \geq n$, $\mathcal{M}_\infty \vDash$ "$Col(\omega, \delta_n^{\mathcal{M}_\infty}) \times Col(\omega, \delta_k^{\mathcal{M}_\infty}) \Vdash (\tau^{\mathcal{M}_\infty}_{A,n})_g = (\tau^{\mathcal{M}_\infty}_{A,k})_h \cap \mathcal{M}_\infty[g]$" where g is $Col(\omega, \delta_n^{\mathcal{M}_\infty})$ generic and h is $Col(\omega, \delta_k^{\mathcal{M}_\infty})$ generic. This is just saying that the terms cohere with one another.

Let G be $Col(\omega, <\lambda^{\mathcal{M}_\infty})$ generic over $\mathcal{M}_\infty$ where $\lambda^{\mathcal{M}_\infty}$ is the sup of Woodin cardinals in $\mathcal{M}_\infty$. Then $\mathbb{R}^*_G$ is the symmetric reals and $A^*_G := \cup_k (\tau^{\mathcal{M}_\infty}_{A,k})_{G|\delta_k^{\mathcal{M}_\infty}}$.

Proposition 3.1.12. *For all $A \subseteq \mathbb{R}$, A is OD, $L(A^*_G, \mathbb{R}^*_G) \vDash \mathsf{AD}^+$.*

[7]This means that for all i, $\neg A_i$ and a scale for A_i are in $\langle A_i \mid i < \omega\rangle$. Furthermore, the A_i's are cofinal in the Wadge hierarchy of $K(\mathbb{R})|\gamma$.

Proof. Suppose not. Using Σ_1-reflection, there is an N, which is a level of $K(\mathbb{R})$ below $\underset{\sim}{\delta}_1^2$ satisfying the statement (T) $\equiv$ "$\mathsf{AD}^+ + \mathsf{ZF}^- + \mathsf{DC} + \mathsf{MC} + \exists A(A$ is OD and $L(A_G^*, \mathbb{R}_G^*) \nvDash \mathsf{AD}^+))$". We may assume N is the first such level. Let

$$U = \{(x, \mathcal{M}) : \mathcal{M} \text{ is a sound } x\text{-mouse}, \rho_\omega(\mathcal{M}) = \{x\}, \text{ and has an iteration strategy in } N\}.$$

Since MC holds in N, U is a universal $(\Sigma_1^2)^N$-set. Let $A \in N$ be an OD set of reals witnessing ϕ. We assume that A has the minimal Wadge rank among the sets witnessing ϕ. Using the results of [46], we can get a $\vec{B} = \langle B_i : i < \omega\rangle$ which is a self-justifying-system (sjs) such that $B_0 = U$ and each $B_i \in N$. Furthermore, we may assume that each B_i is OD in N.

Because MC holds and $\Gamma^* =_{def} \mathcal{P}(\mathbb{R})^N \subsetneq \underset{\sim}{\Delta}_1^2$, there is a real x such that there is a sound mouse $\mathcal{M}$ over x such that $\rho(\mathcal{M}) = x$ and $\mathcal{M}$ doesn't have an iteration strategy in N. Fix then such an $(x, \mathcal{M})$ and let Σ be the strategy of $\mathcal{M}$. Let Γ be a good pointclass such that $Code(\Sigma), \vec{B}, U, U^c \in \underset{\sim}{\Delta}_\Gamma$. Let F be as in Theorem 1.1.4 and let z be such that $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$ Suslin captures $Code(\Sigma), \vec{B}, U, U^c$.

We let $\Phi = (\Sigma_1^2)^N$. We have that Φ is a good pointclass. Because $\vec{B}$ is Suslin captured by $\mathcal{N}_z^*$, we have $(\delta_z^+)^{\mathcal{N}_z^*}$-complementing trees $T, S \in \mathcal{N}_z^*$ which capture $\vec{B}$. Let κ be the least cardinal of $\mathcal{N}_z^*$ which, in $\mathcal{N}_z^*$ is $< \delta_z$-strong.

Claim 1. *$\mathcal{N}_z^* \vDash$ "κ is a limit of points η such that $Lp^{\Gamma^*}(\mathcal{N}_z^*|\eta) \vDash$ "η is Woodin".*

Proof. The proof is an easy reflection argument. Let $\lambda = \delta_z^+$ and let $\pi : M \to \mathcal{N}_z^*|\lambda$ be an elementary substructure such that

1. $T, S \in ran(\pi)$,
2. if $\mathrm{cp}(\pi) = \eta$ then $V_\eta^{\mathcal{N}_z^*} \subseteq M$, $\pi(\eta) = \delta_z$ and $\eta > \kappa$.

By elementarity, we have that $M \vDash$ "η is Woodin". Letting $\pi^{-1}(\langle T, S\rangle) = \langle \bar{T}, \bar{S}\rangle$, we have that $(\bar{T}, \bar{S})$ Suslin captures the universal Φ set over M at $(\eta^+)^M$. This implies that M is Φ-full and in particular, $Lp^{\Gamma^*}(\mathcal{N}_z^*|\eta) \in M$. Therefore, $Lp^{\Gamma^*}(\mathcal{N}_z^*|\eta) \vDash$ "η is Woodin". The claim then follows by a standard argument. □

Let now $\langle \eta_i : i < \omega\rangle$ be the first ω points $< \kappa$ such that for every $i < \omega$, $Lp^{\Gamma^*}(\mathcal{N}_z^*|\eta_i) \vDash$ "η_i is Woodin". Let now $\langle \mathcal{N}_i : i < \omega\rangle$ be a sequence constructed according to the following rules:

1. $\mathcal{N}_0 = L[\vec{E}]^{\mathcal{N}_z^*|\eta_0}$,
2. $\mathcal{N}_{i+1} = (L[\vec{E}][\mathcal{N}_i])^{\mathcal{N}_z^*|\eta_{i+1}}$.

Let $\mathcal{N}_\omega = \cup_{i<\omega}\mathcal{N}_i$.

Claim 2. *For every $i < \omega$, $\mathcal{N}_\omega \vDash$ "η_i is Woodin" and $\mathcal{N}_\omega|(\eta_i^+)^{\mathcal{N}_\omega} = Lp^{\Gamma^*}(\mathcal{N}_i)$.*

Proof. It is enough to show that

1. $\mathcal{N}_{i+1} \vDash$ "η_i is Woodin",
2. $\mathcal{N}_i = V_{\eta_i}^{\mathcal{N}_{i+1}}$,
3. $\mathcal{N}_{i+1}|(\eta_i^+)^{\mathcal{N}_{i+1}} = Lp^{\Gamma^*}(\mathcal{N}_i)$.

To show 1-3, it is enough to show that if $\mathcal{W} \trianglelefteq \mathcal{N}_{i+1}$ is such that $\rho_\omega(W) \leq \eta_i$ then the fragment of $\mathcal{W}$'s iteration strategy which acts on trees above η_i is in Γ^*. Fix then i and $\mathcal{W} \trianglelefteq \mathcal{N}_{i+1}$ is such that $\rho_\omega(W) \leq \eta_i$. Let ξ be such that the if $\mathcal{S}$ is the ξ-th model of the full background construction producing $\mathcal{N}_{i+1}$ then $\mathbb{C}(\mathcal{S}) = \mathcal{W}$. Let $\pi : \mathcal{W} \to \mathcal{S}$ be the core map. It is a fine-structural map but that it irrelevant and we surpass this point. The iteration strategy of $\mathcal{W}$ is the π-pullback of the iteration strategy of $\mathcal{S}$. Let then $\nu < \eta_{i+1}$ be such that $\mathcal{S}$ is the ξ-th model of the full background construction of $\mathcal{N}_x^*|\nu$. To determine the complexity of the induced strategy of $\mathcal{S}$ it is enough to determine the strategy of $\mathcal{N}_x^*|\nu$ which acts on non-dropping stacks that are completely above η_i. Now, notice that by the choice of η_{i+1}, for any non-dropping tree $\mathcal{T}$ on $\mathcal{N}_x^*|\nu$ which is above η_i and is of limit length, if $b = \Sigma(\mathcal{T})$ then $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T})$ has no overlaps, and $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq Lp^{\Gamma^*}(\mathcal{M}(\mathcal{T}))$. This observation indeed shows that the fragment of the iteration strategy of $\mathcal{N}_x^*|\nu$ that acts on non-dropping stack that are above η_i is in Γ^*. Hence, the strategy of $\mathcal{W}$ is in Γ^*. $\square$

We now claim that there is $\mathcal{W} \trianglelefteq Lp(\mathcal{N}_\omega)$ such that $\rho(W) < \eta_\omega$. To see this suppose not. It follows from MC that $Lp(\mathcal{N}_\omega)$ is Σ_1^2-full. We then have that x is generic over $Lp(\mathcal{N}_\omega)$ at the extender algebra of $\mathcal{N}_\omega$ at η_0. Because $Lp(\mathcal{N}_\omega)[x]$ is Σ_1^2-full, we have that $\mathcal{M} \in Lp(\mathcal{N}_\omega)[x]$ and $Lp(\mathcal{N}_\omega) \vDash$ "$\mathcal{M}$ is η_ω-iterable" by fullness of $Lp(\mathcal{N}_\omega)$. Let $\mathcal{S} = (L[\vec{E}][x])^{\mathcal{N}_\omega[x]|\eta_2}$ where the extenders used have critical point $> \eta_0$. Then working in $\mathcal{N}_\omega[x]$ we can compare $\mathcal{M}$ with $\mathcal{S}$. Using standard arguments, we get that $\mathcal{S}$ side doesn't move and by universality, $\mathcal{M}$ side has to come short (see [23]). This in fact means that $\mathcal{M} \trianglelefteq \mathcal{S}$. But the same argument used in the proof of Claim 2 shows that every $\mathcal{K} \trianglelefteq \mathcal{S}$ has an iteration strategy in Γ^*, contradiction!

Let now $\mathcal{W} \trianglelefteq Lp(\mathcal{N}_\omega)$ be least such that $\rho_\omega(\mathcal{W}) < \eta_\omega$. Let k, l be such that $\rho_l(\mathcal{W}) < \eta_k$. We can now consider $\mathcal{W}$ as a $\mathcal{W}|\eta_k$-mouse and considering it such a mouse we let $\mathcal{N} = \mathbb{C}_l(\mathcal{W})$. Thus, $\mathcal{N}$ is sound above η_k. We let $\langle \gamma_i : i < \omega \rangle$ be the Woodin cardinals of $\mathcal{N}$ and $\gamma = \sup_{i<\omega} \gamma_i$.

Let Λ be the strategy of $\mathcal{N}$. We claim that Λ is Γ^*-fullness preserving above γ_k. To see this fix $\mathcal{N}^*$ which is a Λ-iterate of $\mathcal{N}$ such that the iteration embedding $i : \mathcal{N} \to \mathcal{N}^*$ exists. If $\mathcal{N}^*$ isn't Γ^*-full then there is a strong cutpoint ν of $\mathcal{N}^*$ and a $\mathcal{N}^*|\nu$-mouse $\mathcal{W}$ with iteration strategy in Γ^* such that $\rho_\omega(\mathcal{W}) = \nu$ and $\mathcal{W} \ntrianglelefteq \mathcal{N}^*$. If $\mathcal{N}^*$ is not sound above ν then $\mathcal{N}^*$ wins the coiteration with $\mathcal{W}$; but this then implies $\mathcal{W} \ntriangleleft \mathcal{N}^*$, which contradicts our assumption. Otherwise, $\mathcal{N}^* \triangleleft \mathcal{W}$, which is also a contradiction. Hence Λ is Γ^*-fullness preserving.

Now it's not hard to see that $\mathcal{N}$ has the form $\mathcal{J}_{\xi+1}^{\vec{E}}(\mathcal{N}|\gamma)$ and $\mathcal{J}_{\xi}^{\vec{E}}(\mathcal{N}|\gamma)$ satisfies "my derived model at γ satisfies (T)." This is basically the content of Lemma 7.5 of [41]. The argument is roughly that we can iterate $\mathcal{N}$ to an $\mathcal{R}$ such that $\mathcal{R} = \mathcal{J}(\mathcal{Q}_\infty^+)$, where $\mathcal{Q}_\infty^+$ is discussed in the previous subsection and the Prikry forcing is done inside N.

Now let $\mathcal{N}^*$ be the transitive collapse of the pointwise definable hull of $\mathcal{N}|\xi$. We can then realize N as a derived model of a Λ-iterate $\mathcal{R}$ of $\mathcal{N}^*$ such that $\mathcal{R}$ extends a Prikry generic over N (the Prikry forcing is discussed in the previous subsection and $\mathcal{R}$ is in fact the $\mathcal{Q}^+_\infty$, where $\mathcal{Q}^+_\infty$ is as in the previous subsection). We can then use Lemmas 7.6, 7.7, and 6.51 of [41] to show that $\mathcal{M}^N_\infty$ is a Λ-iterate of $\mathcal{N}^*$.

In N, let $A \subseteq \mathbb{R}$ be the least OD set such that $L(A^*_G, \mathbb{R}^*_G) \nvDash \mathsf{AD}$. Then there is an iterate $\mathcal{M}$ of $\mathcal{N}^*$ having preimages of all the terms $\tau^{\mathcal{M}_\infty}_{A,k}$. We may assume $\mathcal{M}$ has new derived model N (this is possible by the above discussion) and suitable initial segments of $\mathcal{M}$ are points in the HOD direct limit system of N. Since $N \vDash \mathsf{AD}^+$, $\mathcal{M}$ thinks that its derived model satisfies that $L(A, \mathbb{R}) \vDash \mathsf{AD}^+$. Now iterate $\mathcal{M}$ to $\mathcal{P}$ such that $\mathcal{M}_\infty$ is an initial segment of $\mathcal{P}$. By elementarity $L(A^*_G, \mathbb{R}^*_G) \vDash \mathsf{AD}^+$. This is a contradiction. $\square$

Definition 3.1.13 (Σ_∞)**.** *Given a normal tree $\mathcal{T} \in \mathcal{M}_\infty$ and $\mathcal{T}$ is based on $\mathcal{M}_\infty|\theta_0$. $\mathcal{T}$ is by Σ_∞ if the following hold (the definition is similar for finite stacks):*

- *If $\mathcal{T}$ is short then Σ picks the branch guided by $\mathcal{Q}$-structure (as computed in $\mathcal{M}_\infty$).*
- *If $\mathcal{T}$ is maximal then $\Sigma_\infty(\mathcal{T})$ = the unique cofinal branch b which moves $\tau^{\mathcal{M}_\infty}_{A,0}$ correctly for all $A \in OD \cap \mathcal{P}(\mathbb{R})$ i.e. for each such A, $i_b(\tau^{\mathcal{M}_\infty}_{A,0}) = \tau^{\mathcal{M}^{\mathcal{T}}_b}_{A^*,0}$.*

Lemma 3.1.14. *Given any such $\mathcal{T}$ as above, $\Sigma_\infty(\mathcal{T})$ exists.*

Proof. Suppose not. By reflection (Theorem 1.1.5), there is a (least) $\gamma < \underset{\sim}{\delta}^2_1$ such that $N =_{def} K(\mathbb{R})|(\gamma) \vDash \phi$ where ϕ is the statement "$\mathsf{ZF}^- + \mathsf{DC} + \mathsf{MC} + \exists\mathcal{T}(\Sigma_\infty(\mathcal{T})$ doesn't exist)". We have a self-justifying-system $\vec{B}$ for $\Gamma^* = \mathcal{P}(\mathbb{R})^N$. By the construction of Proposition3.1.12, there exists a mouse $\mathcal{N}$ with ω Woodin cardinals which has strategy Γ guided by $\vec{B}$.

By reflecting to a countable hull, it's easy to see that $\mathcal{M}^N_\infty$ is a Γ-tail of $\mathcal{N}$ (the reflection is just to make all relevant objects countable). Note that by Theorem 3.1.10, for every A, which is OD in N, there is a Γ-iterate of $\mathcal{N}$ that is strongly A-iterable. Let Σ^N_∞ be the strategy of $\mathcal{M}^N_\infty$ given by Γ. It follows then that for any tree $\mathcal{T}$, $\Sigma^N_\infty(\mathcal{T})$ is the limit of all branches b_{A^*}, where A is OD in N and b_{A^*} moves the term relation for A^* correctly. This fact can be seen in N. This gives a contradiction. $\square$

It is evident that $L(\mathcal{M}_\infty, \Sigma_\infty) \subseteq$ HOD. Next, we show $\mathcal{M}_\infty$ and Σ_∞ capture all of HOD. In $L(\mathcal{M}_\infty, \Sigma_\infty)$, first construct (using Σ_∞) a mouse $\mathcal{M}^+_\infty$ extending $\mathcal{M}_\infty$ such that o$(\mathcal{M}_\infty)$ is the largest cardinal of $\mathcal{M}^+_\infty$ as follows:

1. Let $\mathbb{R}^*_G$ be the symmetric reals obtained from a generic $G \subseteq Col(\omega, <\lambda^{\mathcal{M}_\infty})$ over $L(\mathcal{M}_\infty)$.
2. For each A^*_G (defined as above where $A \in \mathcal{P}(\mathbb{R}) \cap \mathrm{OD}^{K(\mathbb{R})}$) (we know $L(\mathbb{R}^*_G, A^*_G) \vDash \mathsf{AD}^+$), S-translate the $\mathbb{R}^*_G$-mice in this model to mice $\mathcal{S}$ extending $\mathcal{M}_\infty$ with the derived model of $\mathcal{S}$ at $\lambda^{\mathcal{M}_\infty}$ $D^+(\mathcal{S}, \lambda^{\mathcal{M}_\infty}) = L(\mathbb{R}^*_G, A^*_G)$. This is again proved by a reflection argument similar to that in Proposition 3.1.12.

3. Let $\mathcal{M}_\infty^+ = \cup_{\mathcal{S}} \mathcal{S}$ for all such $\mathcal{S}$ as above. It's easy to see that $\mathcal{M}_\infty^+$ is independent of G. By a reflection argument like that in Proposition 3.1.12, we get that mice over $\mathcal{M}_\infty$ are all compatible, no levels of $\mathcal{M}_\infty^+$ projects across $o(\mathcal{M}_\infty)$.

Remark 3.1.15. $\delta_0^{\mathcal{M}_\infty}$ is not collapsed by Σ_∞ because it is a cardinal in HOD. Σ_∞ is used to obtain the A_G^* above by moving correctly the $\tau_{A,0}^{\mathcal{M}_\infty}$ in genericity iterations. $L(\mathcal{M}_\infty)$ generally does not see the sequence $\langle \tau_{A,k}^{\mathcal{M}_\infty} \mid k \in \omega\rangle$ hence can't construct A_G^*; that's why we need Σ_∞. Since Σ_∞ collapses $\delta_1^{\mathcal{M}_\infty}, \delta_2^{\mathcal{M}_\infty}$... by genericity iterating $\mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty}$ to make $\mathcal{M}_\infty|\delta_i^{\mathcal{M}_\infty}$ generic for $i > 0$, it doesn't make sense to talk about $D(L(\mathcal{M}_\infty, \Sigma_\infty))$.

Lemma 3.1.16. *HOD* $\subseteq L(\mathcal{M}_\infty, \Sigma_\infty)$

Proof. Using Theorem 3.1.9, we know HOD $= L[P]$ for some $P \subseteq \Theta$. Therefore, it is enough to show P $\in L(\mathcal{M}_\infty, \Sigma_\infty)$. Let ϕ be a formula defining P, i.e.

$$\alpha \in P \Leftrightarrow K(\mathbb{R}) \vDash \phi[\alpha].$$

Here we suppress the ordinal parameter. Now in $L(\mathcal{M}_\infty, \Sigma_\infty)$ let $\pi : \mathcal{M}_\infty|((\delta_0^{\mathcal{M}_\infty})^{++})^{\mathcal{M}_\infty} \to (\mathcal{M}_\infty)^{D(\mathcal{M}_\infty^+, \lambda^{\mathcal{M}_\infty})}$ where π is according to Σ_∞. We should note that Σ_∞-iterates are cofinal in the directed system $\mathcal{F}$ defined in $D(\mathcal{M}_\infty^+, \lambda^{\mathcal{M}_\infty})$ by the method of generic comparisons (see [23] for more on this).

Claim: $K(\mathbb{R}) \vDash \phi[\alpha] \Leftrightarrow D(M_\infty^+, \lambda^{M_\infty}) \vDash \phi[\pi(\alpha)]$ $(**)$

Proof. Otherwise, reflect the failure of $(**)$ as before to the least $K(\mathbb{R})|\gamma$ and get a self-justifying-system $\vec{B}$ of OD sets along with an ω-suitable mouse $\mathcal{N}$ with $\vec{B}$-guided iteration strategy Γ. By genericity iteration above its first Woodin, we may assume $D(\mathcal{N}, \lambda^{\mathcal{N}}) = K(\mathbb{R})|\gamma$. Fix an α witnessing the failure of $(**)$. Let $\sigma : \mathcal{N}|((\delta_0^{\mathcal{N}})^{++})^{\mathcal{N}} \to (\mathcal{M}_\infty)^{D(\mathcal{N}, \lambda^{\mathcal{N}})}$ be the direct limit map by Γ (by taking a countable hull containing all relevant objects, we can assume σ exists). We may assume there is an $\overline{\alpha}$ such that $\sigma(\overline{\alpha}) = \alpha$. Notice here that $\Sigma_\infty^{K(\mathbb{R})|\gamma}$ is a tail of Γ as $\Sigma_\infty^{K(\mathbb{R})|\gamma}$ moves all the term relations for $OD^{K(\mathbb{R})|\gamma}$ sets of reals correctly and Γ is guided by the self-justifying system $\vec{B}$, which is cofinal in $\mathcal{P}(\mathbb{R}) \cap OD^{K(\mathbb{R})|\gamma}$. It then remains to see that:

$$D(\mathcal{M}_\infty^+, \lambda^{\mathcal{M}_\infty}) \vDash \phi[\pi(\alpha)] \Leftrightarrow D(\mathcal{N}, \lambda^{\mathcal{N}}) \vDash \phi[\sigma(\overline{\alpha})] \quad (***).$$

To see that $(***)$ holds, we need to see that the fragment of Γ that defines $\sigma(\overline{\alpha})$ can be defined in $D(\mathcal{N}, \lambda^{\mathcal{N}})$. This then will give the equivalence in $(***)$. Because $\alpha < \delta_0^{\mathcal{M}_\infty^{K(\mathbb{R})|\gamma}} = \delta_0^{\mathcal{M}_\infty^{D(\mathcal{N},\lambda^{\mathcal{N}})}}$, pick an $A \in \vec{B}$ such that $\gamma_{A,0}^{D(\mathcal{N},\lambda^{\mathcal{N}})} > \alpha$. Then the fragment of Γ that defines $\sigma(\overline{\alpha})$ is definable from A (and $\mathcal{N}|(\delta_0^N)$) in $D(\mathcal{N}, \lambda^{\mathcal{N}})$, which is what we want.

The equivalence $(***)$ gives us a contradiction. $\square$

The claim finishes the proof of $P \in L(\mathcal{M}_\infty, \Sigma_\infty)$ because the right hand side of the equivalence $(**)$ can be computed in $L(\mathcal{M}_\infty, \Sigma_\infty)$. This then implies $\mathrm{HOD} = L[P] \subseteq L(\mathcal{M}_\infty, \Sigma_\infty)$. □

Remark 3.1.17. Woodin (unpublished) has also computed the full HOD for models satisfying $V = L(\mathcal{P}(\mathbb{R})) + \mathsf{AD}^+ + \Theta = \theta_0$. To the best of the author's knowledge, here's a very rough idea of his computation. Let $\mathcal{M}_\infty, \Sigma_\infty, P$ be as above. For each $\alpha < \Theta$, let Σ_α be the fragment of Σ_∞ that moves α along the good branch of a maximal tree. Woodin shows that the structure $(\mathbb{R}^*_G, \langle \Sigma_\alpha \mid \alpha < \Theta \rangle)$ can compute the set P. This then gives us that $\mathrm{HOD} \subseteq L(\mathcal{M}_\infty, \Sigma_\infty)$.

3.1.1.3 When $\Theta = \theta_{\alpha+1}$

Recall that in this section we assume $\Theta = \theta_{\alpha+1}$ for some α. We state without proof the theorem that will be important for the computation in this section.

Theorem 3.1.18 (Sargsyan, Steel). *Assume $\mathsf{AD}^+ + \mathsf{SMC}$. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair below $\mathsf{AD}_\mathbb{R}$ + "Θ is regular"[8] such that Σ has branch condensation and is fullness preserving. Then*

$$\{A \subseteq \mathbb{R} : A \in OD_\Sigma(y) \text{ for some real } y\} = \mathcal{P}(\mathbb{R}) \cap K^\Sigma(\mathbb{R}).$$

First we need to compute V_Θ^{HOD}. Here's what is done in [23] regarding this computation.

Theorem 3.1.19 (Sargsyan, see Section 4.3 in [23]). *Let $\Gamma = \{A \subseteq \mathbb{R} \mid w(A) < \theta_\alpha\}$. Then there is a hod pair $(\mathcal{P}, \Sigma)$ such that*

1. *Σ is fullness preserving and has branch condensation;*
2. *$\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)|\theta_\alpha = V_{\theta_\alpha}^{HOD}$, where $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$ is the direct limit of all Σ-iterates of $\mathcal{P}$.*

It is clear that there is no hod pair $(\mathcal{P}, \Sigma)$ satisfying Theorem 3.1.19 with Γ replaced by $\mathcal{P}(\mathbb{R})$ as this would imply that $\Sigma \notin V$. So to compute V_Θ^{HOD}, we need to mimic the computation in the previous subsection. For a more detailed discussion regarding Definitions 3.1.20, 3.1.21, 3.1.22, and 3.1.23, see Section 3.1 of [23].

Definition 3.1.20 (n-suitable pair). *$(\mathcal{P}, \Sigma)$ is an n-suitable pair (or $\mathcal{P}$ is an n-suitable Σ-premouse) if there is δ such that $(\mathcal{P}|(\delta^{+\omega})^\mathcal{P}, \Sigma)$ is a hod pair and*

1. *$\mathcal{P} \vDash \mathsf{ZFC}$ - Replacement + "there are n Woodin cardinals, $\delta_0 < \delta_1 < ... < \delta_n$ above δ";*
2. *$o(\mathcal{P}) = sup_{i<\omega}(\delta_n)^{+i^P}$;*
3. *$\mathcal{P}$ is a Σ-premouse over $\mathcal{P}|\delta$;*

[8] i.e., there are no inaccessible limit of Woodins in $\mathcal{P}$.

4. *for any $\mathcal{P}$-cardinal $\eta > \delta$, if η is a strong cutpoint then $\mathcal{P}|(\eta^+)^{\mathcal{P}} = Lp^{\Sigma}(\mathcal{P}|\eta)$.*

Sometimes, we just refer to $\mathcal{P}$ as being n-suitable. For $\mathcal{P}, \delta$ as in the above definition, let $\mathcal{P}^- = \mathcal{P}|(\delta^{+\omega})^{\mathcal{P}}$ and

$$\mathbb{B}(\mathcal{P}^-, \Sigma) = \{B \subseteq \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \quad | \quad B \text{ is OD, and for any } (\mathcal{Q}, \Lambda) \text{ iterate of } (\mathcal{P}^-, \Sigma), \text{ and for any } (x, y) \in B_{(\mathcal{Q},\Lambda)}, x \text{ codes } \mathcal{Q}\}.$$

Suppose $B \in \mathbb{B}(P^-, \Sigma)$ and $\kappa < o(\mathcal{P})$. Let $\tau^{\mathcal{P}}_{B,\kappa}$ be the canonical term in $\mathcal{P}$ that captures B at κ i.e. for any $g \subseteq Col(\omega, \kappa)$ generic over $\mathcal{P}$

$$B_{(\mathcal{P}^-,\Sigma)} \cap \mathcal{P}[g] = (\tau^{\mathcal{P}}_{B,\kappa})_g.$$

For each $m < \omega$, let

$$\gamma^{\mathcal{P},\Sigma}_{B,m} = \sup(Hull^{\mathcal{P}}(\tau^{\mathcal{P}}_{B,(\eta^{+m}_{n-1})^{\mathcal{P}}}) \cap \eta_0),$$

$$H^{\mathcal{P},\Sigma}_{B,m} = Hull^{\mathcal{P}}(\gamma^{\mathcal{P},\Sigma}_{B,m} \cup \{\tau^{\mathcal{P}}_{B,(\eta^{+m}_{n-1})^{\mathcal{P}}}\}),$$

$$\gamma^{\mathcal{P},\Sigma}_{B} = \sup\nolimits_{m<\omega} \gamma^{\mathcal{P},\Sigma}_{B,m},$$

and

$$H^{\mathcal{P},\Sigma}_{B} = \cup_{m<\omega} H^{\mathcal{P},\Sigma}_{B,m}.$$

Similar definitions can be given for $\gamma^{\mathcal{P},\Sigma}_{\vec{B},m}, H^{\mathcal{P},\Sigma}_{\vec{B},m}, \gamma^{\mathcal{P},\Sigma}_{\vec{B}}, H^{\mathcal{P},\Sigma}_{\vec{B}}$ for any finite sequence $\vec{B} \in \mathbb{B}(\mathcal{P}^-, \Sigma)$. One just needs to include relevant terms for each element of $\vec{B}$ in each relevant hull. Now we define the notion of B-iterability.

Definition 3.1.21 (B-iterability)**.** *Let $(\mathcal{P}, \Sigma)$ be an n-suitable pair and $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$. We say $(\mathcal{P}, \Sigma)$ is B-**iterable** if for all $k < \omega$, player II has a winning quasi-strategy for the game $G^{(P,\Sigma)}_{B,k}$ defined as follows. The game consists of k rounds. Each round consists of a main round and a subround. Let $(\mathcal{P}_0, \Sigma_0) = (\mathcal{P}, \Sigma)$. In the main round of the first round, player I plays countable stacks of normal nondropping trees based on $\mathcal{P}^-_0$ or its images and player II plays according to Σ_0 or its tails. If the branches chosen by player II are illfounded or do not move some term for B correctly, he loses. Player I has to exit the round at a countable stage; otherwise, he loses. Suppose $(\mathcal{P}^*, \Sigma^*)$ is the last model after the main round is finished. In the subround, player I plays a normal tree above $(\mathcal{P}^*)^-$ or its images based on (δ, γ), where δ and γ are two successive Woodin cardinals of $\mathcal{P}^*$. If the tree is short, player II plays the unique cofinal branch given by the $\mathcal{Q}$-structure. Otherwise, Player II plays a cofinal nondropping branch that moves all terms for B correctly. Player II loses if the branch model he plays is illfounded or the branch embedding (if in the case the tree is maximal) does not move some terms for B correctly. Suppose $(\mathcal{P}_1, \Sigma_1)$ is the last model of the subround. If II hasn't lost, the next round proceeds the same way as the previous one but for the pair $(\mathcal{P}_1, \Sigma_1)$. If the game lasts for k rounds, II wins.*

Definition 3.1.22 (Strong B-iterability)**.** *Let $(\mathcal{P},\Sigma)$ be an n-suitable pair and $B\in\mathbb{B}(\mathcal{P}^-,\Sigma)$. We say $(\mathcal{P},\Sigma)$ is **strongly** B**-iterable** if $(\mathcal{P},\Sigma)$ is B-iterable and if r_1 is a run of $G^{\mathcal{P},\Sigma}_{B,n_1}$ and r_2 is a run of $G^{\mathcal{P},\Sigma}_{B,n_2}$ for some $n_1,n_2<\omega$ according to some (any) B-iterability quasi-strategy of $\mathcal{P}$ and the runs produce the same end model $\mathcal{Q}$ then the runs move the hull $H^{\mathcal{P},\Sigma}_B$ the same way. That is if i_1 and i_2 are B-iteration maps accoring to r_1 and r_2 respectively then $i_1\restriction H^{\mathcal{P},\Sigma}_B=i_2\restriction H^{\mathcal{P},\Sigma}_B$.*

Definition 3.1.23 (Strong B-condensation)**.** *Let $(\mathcal{P},\Sigma)$ be an n-suitable pair and $B\in\mathbb{B}(\mathcal{P}^-,\Sigma)$. Suppose Σ has branch condensation and is fullness preserving. Suppose $(\mathcal{P},\Sigma)$ is strongly B-iterable as witnessed by Λ. We say Λ has **strong** B**-condensation** if whenever $(\mathcal{Q},\Sigma_\mathcal{Q},\Lambda_\mathcal{Q})$ is a Λ-iterate of $(\mathcal{P},\Sigma,\Lambda)$, for every $\Lambda_\mathcal{Q}$-iterate $(\mathcal{R},\Sigma_\mathcal{R},\Lambda_\mathcal{R})$ of $(\mathcal{Q},\Sigma_\mathcal{Q},\Lambda_\mathcal{Q})$, suppose $i:\mathcal{Q}\to\mathcal{S}$ is such that there is some $k:\mathcal{S}\to\mathcal{R}$ such that $j=k\circ i$ where $j:\mathcal{Q}\to\mathcal{R}$ is according to $\Lambda_\mathcal{Q}$, then $\mathcal{R}$ is n-suitable and furthermore, $k^{-1}(\tau^\mathcal{R}_{B,k(\kappa)})=\tau^\mathcal{S}_{B,\kappa}$ for all $\mathcal{S}$-cardinal κ above $\delta^\mathcal{S}$.*

Now we're ready to define our direct limit system. Let

$$\mathcal{F}=\{(P,\Sigma,\vec{B})\quad|\quad \vec{B}\in\mathbb{B}(P^-,\Sigma)^{<\omega},(P^-,\Sigma)\text{ satisfies Theorem 3.1.19},(P,\Sigma)\text{ is }n\text{-suitable for some }n,\text{ and }(P,\Sigma)\text{ is strongly }\vec{B}\text{-iterable}\}.$$

The ordering on $\mathcal{F}$ is defined as follows:

$$(\mathcal{P},\Sigma,\vec{B})\preccurlyeq(\mathcal{Q},\Lambda,\vec{C})\quad\Leftrightarrow\quad \vec{B}\subseteq\vec{C},\exists k\exists r(r\text{ is a run of }G^{\mathcal{P},\Sigma}_{B,k}\text{ with the last model }\mathcal{P}^*\text{ such that }(\mathcal{P}^*)^-=\mathcal{Q}^-,\ \Sigma_{(\mathcal{P}^*)^-}=\Lambda,\mathcal{P}^*=\mathcal{Q}|(\eta^{+\omega})^\mathcal{Q}\text{ where }\mathcal{Q}\vDash\eta>o(Q^-)\text{ is Woodin}).$$

Suppose $(P,\Sigma,\vec{B})\preccurlyeq(Q,\Lambda,\vec{C})$ then there is a unique map $\pi^{(\mathcal{P},\Sigma),(\mathcal{Q},\Lambda)}_{\vec{B}}:H^{\mathcal{P},\Sigma}_{\vec{B}}\to H^{\mathcal{Q},\Lambda}_{\vec{B}}$. $(\mathcal{F},\preccurlyeq)$ is then directed. Let

$$\mathcal{M}_\infty=\text{direct limit of }(\mathcal{F},\preccurlyeq)\text{ under maps }\pi^{(\mathcal{P},\Sigma),(\mathcal{Q},\Lambda)}_{\vec{B}}.$$

Also for each $(\mathcal{P},\Sigma,\vec{B})\in\mathcal{F}$, let

$$\pi^{(\mathcal{P},\Sigma),\infty}_{\vec{B}}:H^{\mathcal{P},\Sigma}_{\vec{B}}\to\mathcal{M}_\infty$$

be the natural map.

Clearly, $\mathcal{M}_\infty\subseteq\mathrm{HOD}$. But first, we need to show $\mathcal{F}\neq\emptyset$. In fact, we prove a stronger statement.

Theorem 3.1.24. *Suppose $(\mathcal{P},\Sigma)$ satisfies Theorem 3.1.19. Let $B\in\mathbb{B}(\mathcal{P},\Sigma)$. Then for each $1\le n<\omega$, there is a $\mathcal{Q}$ such that $\mathcal{Q}^-$ is a Σ-iterate of $\mathcal{P}^-$, $(\mathcal{Q},\Sigma_{\mathcal{Q}^-})$ is n-suitable, $(\mathcal{Q},\Sigma_{\mathcal{Q}^-},B)\in\mathcal{F}$ as witnessed by Λ; furthermore, Λ has strong B-condensation.*

Proof. Suppose not. By Σ_1-reflection (Theorem 1.1.5), there is a transitive model N coded by a Suslin, co-Suslin set of reals such that $Code(\Sigma) \in \mathcal{P}(\mathbb{R})^N$ and

$$\begin{aligned} N \ \vDash \ & \mathsf{ZF}\backslash\{\text{Powerset}\} + \mathsf{AD}^+ + \mathsf{SMC} + \text{“}\Theta \text{ exists and is successor in the Solovay sequence”} \\ & + \text{“}\exists n \exists B \in \mathbb{B}(\mathcal{P}, \Sigma) \nexists \mathcal{Q} \ ((\mathcal{Q}, \Sigma_{\mathcal{Q}^-}) \text{ is } n\text{-suitable and } (\mathcal{Q}, \Sigma_{\mathcal{Q}^-}, B) \in \mathcal{F})\text{”}. \end{aligned}$$

We take a minimal such N (so N is pointwise definable from $\mathbb{R}\cup\{\Sigma\}$) and fix a $B \in \mathbb{B}(\mathcal{P}, \Sigma)^N$ witnessing the failure of the Theorem in N. Using Theorem 1.1.4 and the assumption on N, there is an $x \in \mathbb{R}$ and a tuple $\langle N_x^*, \delta_x, \Sigma_x\rangle$ satisfying the conclusions of Theorem 1.1.4 relative to Γ, a good pointclass containing $(\mathcal{P}(\mathbb{R})^N, \cup_{n<\omega}\Sigma_n^N(\mathbb{R} \cup \{\Sigma\}))$. Futhermore, let's assume that N_x^* Suslin captures $\langle A \mid A \text{ is projective in } \Sigma\rangle)$. Let $\Omega = \mathcal{P}(\mathbb{R})^N$. For simplicity, we show that in N, there is a Σ-iterate $(\mathcal{R}, \Sigma_{\mathcal{R}})$ such that there is a 1-suitable $(\mathcal{S}, \Sigma_{\mathcal{R}})$ such that $(\mathcal{S}, \Sigma_{\mathcal{R}}, B) \in \mathcal{F}$ and $(\mathcal{S}, \Sigma_{\mathcal{R}})$ has strong B-condensation.

By the assumption on N, $N \vDash V = K^{\Sigma}(\mathbb{R})$. Now N_x^* has club many $(\Sigma_1^2(\Sigma))^{\Omega}$ Woodins below δ_x by a standard argument (see [30]). Hence, the $(\Sigma_1^2(\Sigma))^{\Omega}$-hod pair construction done in N_x^* will reach a model having ω Woodins (these are $\Sigma_1^2(\Sigma)^N$-full Woodins). Let $\mathcal{Q}$ be the first model in the construction with that property and with the property that the (new) derived model of $\mathcal{Q}$ is elemetarily equivalent to N. So $(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$ is an iterate of $(\mathcal{P}, \Sigma)$; to simplify the notation, we assume $(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-}) = (\mathcal{P}, \Sigma)$. We may also assume that $\mathcal{Q}$ is pointwise definable from $\Sigma_{\mathcal{Q}^-}$ and $\mathcal{Q} \vDash \mathsf{ZFC}^-$ (this is because N is pointwise definable from $\{\Sigma\} \cup \{\mathbb{R}\}$ and derived model of $\mathcal{Q}$ is elementarily equivalent to N). Let $\langle \delta_i^{\mathcal{Q}} \mid i < \omega\rangle$ be the first ω Woodins of $\mathcal{Q}$ above $o(\mathcal{P})$. A similar self-explanatory notation will be used to denote the Woodins of any Λ-iterate of $\mathcal{Q}$. Let Λ (which extends Σ) be the strategy of $\mathcal{Q}$ induced from the background universe. Λ is Ω-fullness preserving. At this point it's not clear that Λ has strong B-condensation. The proof of Theorem 3.1.10 doesn't generalize as it's not clear what the corresponding notion of a self-justifying-system for sets in $\mathbb{B}(\mathcal{P}, \Sigma)$ is.

We now show that an iterate $(\mathcal{R}, \Lambda_{\mathcal{R}})$ of $(\mathcal{Q}, \Lambda)$ is strongly B-iterable and in fact $\Lambda_{\mathcal{R}}$ has strong B-condensation. Once we prove $\Lambda_{\mathcal{R}}$ is B-iterable, we automatically get strong B-iterability by the Dodd-Jensen property of $\Lambda_{\mathcal{R}}$ (recall $\mathcal{Q}$ is sound and Λ is $\mathcal{Q}$'s unique strategy so Λ has the Dodd-Jensen property and iteration maps according to Λ is fully elementary because $\mathcal{Q} \vDash \mathsf{ZFC}^-$). Once we have this pair $(\mathcal{R}, \Lambda_{\mathcal{R}})$, we can just let our desired $\mathcal{S}$ to be $\mathcal{R}|((\delta_0^{\mathcal{R}})^{+\omega})^{\mathcal{R}}$.

Suppose no such $(\mathcal{R}, \Lambda_{\mathcal{R}})$ exists. Using the property of $\mathcal{Q}$ and the relativized (to Σ) Prikry forcing in N (see [36]), we get that for any n, there is an iterate $\mathcal{R}$ of $\mathcal{Q}$ (above $\delta_0^{\mathcal{Q}}$) extending a Prikry generic and having N as the (new) derived model (computed at the sup of the first ω Woodins above $o(\mathcal{P})$). Furthermore, this property holds for any Λ iterate of $\mathcal{Q}$. As before, without going further into details of the techniques used in [36], we remark that if $\mathcal{R}$ is an $\mathbb{R}$-genericity iterate of $\mathcal{Q}$, then the new derived model of $\mathcal{R}$ is N. In other words, once we know one such $\mathbb{R}$-genericity iterate of $\mathcal{Q}$ realizes N as its derived model then all $\mathbb{R}$-genericity iterates of $\mathcal{Q}$ do. Let (ϕ, s) define B over N, i.e.

$$(\mathcal{R}, \Psi, x, y) \in B \Leftrightarrow N \vDash \phi[((\mathcal{R}, \Psi, x, y)), s].^{9}$$

The following argument mirrors that of Lemma 3.2.15 in [23] though it's not clear to the author who this argument is orginially due to. The process below is described in Figure 3.1. From now to the end of the proof, all stacks on $\mathcal{Q}$ or its iterates thereof are below the $\delta_0^{\mathcal{Q}}$ or its image. By our assumption, there is $\langle \vec{\mathcal{T}}_i, \vec{\mathcal{S}}_i, \mathcal{Q}_i, \mathcal{R}_i, \pi_i, \sigma_i, j_i \mid i < \omega \rangle \in N$ such that

1. $\mathcal{Q}_0 = \mathcal{Q}$; $\vec{\mathcal{T}}_0$ is a stack on $\mathcal{Q}$ according to Λ with last model $\mathcal{Q}_1$; $\pi_0 = i^{\vec{\mathcal{T}}_0}$; $\vec{\mathcal{S}}_0$ is a stack on $\mathcal{Q}$ with last model $\mathcal{R}_0$; $\sigma_0 = i^{\vec{\mathcal{S}}_0}$; and $j_0 : \mathcal{R}_0 \to \mathcal{Q}_1$.
2. $\vec{\mathcal{T}}_i$ is a stack on $\mathcal{Q}_i$ according to Λ with last model $\mathcal{Q}_{i+1}$; $\pi_i = i^{\vec{\mathcal{T}}_i}$; $\vec{\mathcal{S}}_i$ is a stack on $\mathcal{Q}_i$ with last model $\mathcal{R}_i$; $\sigma_i = i^{\vec{\mathcal{S}}_i}$; $j_0 : \mathcal{R}_i \to \mathcal{Q}_{i+1}$.
3. for all k, $\pi_k = j_k \circ \sigma_k$.
4. for all k, $\pi_k(\tau^{\mathcal{Q}_k}_{B,\delta_0^{\mathcal{Q}_k}}) \neq \tau^{\mathcal{Q}_{k+1}}_{B,\delta_0^{\mathcal{Q}_{k+1}}}$ or $j_k(\tau^{\mathcal{R}_k}_{B,\delta_0^{\mathcal{R}_k}}) \neq \tau^{\mathcal{Q}_{k+1}}_{B,\delta_0^{\mathcal{Q}_{k+1}}}$.

Let $\mathcal{Q}_\omega$ be the direct limit of the $\mathcal{Q}_i$'s under maps π_i's. We rename the $\langle \mathcal{Q}_i, \mathcal{R}_i, \pi_i, \sigma_i, j_i \mid i < \omega \rangle$ into $\langle \mathcal{Q}_i^0, \mathcal{R}_i^0, \pi_i^0, \sigma_i^0, j_i^0 \mid i < \omega \rangle$. We then assume that N is countable (by working with a countable elementary substructure of N) and fix (in V) $\langle x_i \mid i < \omega \rangle$- a generic enumeration of $\mathbb{R}$. Using our assumption on $\mathcal{Q}$, we get $\langle \mathcal{Q}_i^n, \mathcal{R}_i^n, \pi_i^n, \sigma_i^n, j_i^n, \tau_1^n, k_i^n \mid n, i \leq \omega \rangle$ such that

1. $\mathcal{Q}_i^\omega$ is the direct limit of the $\mathcal{Q}_i^n$'s under maps τ_i^n's for all $i \leq \omega$.
2. $\mathcal{R}_i^\omega$ is the direct limit of the $\mathcal{R}_i^n$'s under maps k_i^n's for all $i < \omega$.
3. $\mathcal{Q}_\omega^n$ is the direct limit of the $\mathcal{Q}_i^n$'s under maps π_i^n's.
4. for all $n \leq \omega$, $i < \omega$, $\pi_i^n : \mathcal{Q}_i^n \to \mathcal{Q}_{i+1}^n$; $\sigma_i^n : \mathcal{Q}_i^n \to \mathcal{R}_i^n$; $j_i^n : \mathcal{R}_i^n \to \mathcal{Q}_{i+1}^n$ and $\pi_i^n = j_i^n \circ \sigma_i^n$.
5. Derived model of the $\mathcal{Q}_i^\omega$'s, $\mathcal{R}_i^\omega$'s is N.

Then we start by iterating $\mathcal{Q}_0^0$ above $\delta_0^{\mathcal{Q}_0^0}$ to $\mathcal{Q}_0^1$ to make x_0-generic at $\delta_1^{\mathcal{Q}_0^1}$. During this process, we lift the genericity iteration tree to all $\mathcal{R}_n^0$ for $n < \omega$ and $\mathcal{Q}_n^0$ for $n \leq \omega$. We pick branches for the tree on $\mathcal{Q}_0^0$ by picking branches for the lift-up tree on $\mathcal{Q}_\omega^0$ using $\Lambda_{\mathcal{Q}_\omega^0}$. Let $\tau_0^0 : \mathcal{Q}_0^0 \to \mathcal{Q}_0^1$ be the iteration map and $\mathcal{W}$ be the end model of the lift-up tree on $\mathcal{Q}_\omega^0$. We then iterate the end model of the lifted tree on $\mathcal{R}_0^0$ to $\mathcal{R}_0^1$ to make x_0 generic at $\delta_1^{\mathcal{R}_0^1}$ with branches being picked by lifting the iteration tree onto $\mathcal{W}$ and using the branches according

[9] As pointed out by J. Steel, the argument can be simplified by choosing a B that is simply definable, i.e. $s = \emptyset$. Since we know the new derived model of $\mathcal{Q}$ is elementarily equivalent to N, Λ automatically witnesses that $(\mathcal{Q}, \Sigma_{\mathcal{Q}^-})$ is B-iterable and it's also not hard to see that Λ has strong B-condensation. The dovetailing argument at the end of the proof is not needed.

to $\Lambda_{\mathcal{W}}$. Let $k_0^0 : \mathcal{R}_0^0 \to \mathcal{R}_0^1$ be the iteration embedding, $\sigma_0^1 : \mathcal{Q}_0^1 \to R_0^1$ be the natural map, and $\mathcal{X}$ be the end model of the lifted tree on the $\mathcal{W}$ side. We then iterate the end model of the lifted stack on $\mathcal{Q}_1^0$ to $\mathcal{Q}_1^1$ to make x_0 generic at $\delta_1^{\mathcal{Q}_1^1}$ with branches being picked by lifting the tree to $\mathcal{X}$ and using branches picked by $\Lambda_{\mathcal{X}}$. Let $\tau_1^0 : \mathcal{Q}_1^0 \to \mathcal{Q}_1^1$ be the iteration embedding, $j_0^1 : \mathcal{R}_0^1 \to \mathcal{Q}_1^1$ be the natural map, and $\pi_0^1 = j_0^1 \circ \sigma_0^1$. Continue this process of making x_0 generic for the later models $\mathcal{R}_n^0$'s and $\mathcal{Q}_n^0$'s for $n < \omega$. We then let $\mathcal{Q}_\omega^1$ be the direct limit of the $\mathcal{Q}_n^1$ under maps π_n^1's. We then start at $\mathcal{Q}_0^1$ and repeat the above process to make x_1 generic appropriate iterates of $\delta_2^{\mathcal{Q}_0^1}$ etc. This whole process define models and maps $\langle \mathcal{Q}_i^n, \mathcal{R}_i^n, \pi_i^n, \sigma_i^n, j_i^n, \tau_1^n, k_i^n \mid n, i \leq \omega\rangle$ as described above. See Figure 3.1.

Note that by our construction, for all $n < \omega$, the maps π_n^0's and τ_ω^n's are via Λ or its appropriate tails; furthermore, $\mathcal{Q}_\omega^\omega$ is wellfounded and full (with respect to mice in N). This in turns implies that the direct limits $\mathcal{Q}_n^\omega$'s and $\mathcal{R}_n^\omega$'s are wellfounded and full. We must then have that for some k, for all $n \geq k$, $\pi_n^\omega(s) = s$. This implies that for all $n \geq k$

$$\pi_n^\omega(\tau^{\mathcal{Q}_n^\omega}_{B,\delta_0^{\mathcal{Q}_n^\omega}}) = \tau^{\mathcal{Q}_{n+1}^\omega}_{B,\delta_0^{\mathcal{Q}_{n+1}^\omega}}.$$

We can also assume that for all $n \geq k$, $\sigma_n^\omega(s) = s, j_n^\omega(s) = s$. Hence

$$\sigma_n^\omega(\tau^{\mathcal{Q}_n^\omega}_{B,\delta_0^{\mathcal{Q}_n^\omega}}) = \tau^{\mathcal{R}_n^\omega}_{B,\delta_0^{\mathcal{R}_n^\omega}}.;$$

$$j_n^\omega(\tau^{\mathcal{R}_n^\omega}_{B,\delta_0^{\mathcal{R}_n^\omega}}) = \tau^{\mathcal{Q}_{n+1}^\omega}_{B,\delta_0^{\mathcal{Q}_{n+1}^\omega}}.;$$

This is a contradiction, hence we're done. □

Remark 3.1.25. The proof of Theorem 3.1.24 also shows that if $(\mathcal{P}, \Sigma)$ is n-suitable and $(\mathcal{P}, \Sigma, B) \in \mathcal{F}$ and $C \in \mathbb{B}(\mathcal{P}^-, \Sigma)$ then there is a B-iterate $\mathcal{Q}$ of $\mathcal{P}$ such that $(\mathcal{Q}, \Sigma_{\mathcal{Q}^-}, B \oplus C) \in \mathcal{F}$; in fact, $(\mathcal{Q}, \Sigma_{\mathcal{Q}^-}, B \oplus C) \in \mathcal{F}$ is witnessed by a quasi-strategy Λ that has strong $B \oplus C$-condensation.

It is easy to see that $\mathcal{M}_\infty | \theta_\alpha = V_{\theta_\alpha}^{\text{HOD}}$. Let $\langle \eta_i \mid i < \omega\rangle$ be the increasing enumeration of Woodin cardinals in $\mathcal{M}_\infty$ larger than θ_α. Theorem 3.1.24 is used to show that $\mathcal{M}_\infty$ is large enough in that

Lemma 3.1.26. *1. $\mathcal{M}_\infty$ is well-founded.*

2. $\mathcal{M}_\infty | \eta_0 = V_\Theta^{HOD}$. In particular, $\eta_0 = \Theta$.

Proof. We prove (1) and (2) simultaneously. For a similar argument, see Lemma 3.3.2 in [23]. Toward a contradiction, suppose not. By Σ_1-reflection (Theorem 1.1.5), there is a transitive model N coded by a Suslin, co-Suslin set of reals such that $Code(\Sigma) \in \mathcal{P}(\mathbb{R})^N$ and

$$N \vDash \mathsf{ZF} \backslash \{\text{Powerset}\} + \mathsf{DC} + \mathsf{SMC} + \text{“}\Theta \text{ exists and is successor in the Solovay sequence”} + \text{“(1) and (2) do not both hold”}.$$

$$
\begin{tikzcd}
\mathcal{Q}^0_0 \arrow[rr, "\pi^0_0"] \arrow[dr, "\sigma^0_0"'] \arrow[dd, "\tau^0_0"'] & & \mathcal{Q}^0_1 \arrow[rr, "\pi^0_1"] \arrow[dr, "\sigma^0_1"'] \arrow[dd, "\tau^0_1"'] & & \mathcal{Q}^0_2 \arrow[r, "\pi^0_2"] \arrow[dd, "\tau^0_2"'] & \cdots & \mathcal{Q}^0_\omega \arrow[dd, "\tau^0_\omega"'] \\
& \mathcal{R}^0_0 \arrow[ur, "j^0_0"'] \arrow[dd, "k^0_0"] & & \mathcal{R}^0_1 \arrow[ur, "j^0_1"'] \arrow[dd, "k^0_1"] & & & \\
\mathcal{Q}^1_0 \arrow[rr, "\pi^1_0"] \arrow[dr, "\sigma^1_0"'] \arrow[dd, "\tau^1_0"'] & & \mathcal{Q}^1_1 \arrow[rr, "\pi^1_1"] \arrow[dr, "\sigma^1_1"'] \arrow[dd, "\tau^1_1"'] & & \mathcal{Q}^1_2 \arrow[r, "\pi^1_2"] \arrow[dd, "\tau^1_2"'] & \cdots & \mathcal{Q}^1_\omega \arrow[dd, "\tau^1_\omega"'] \\
& \mathcal{R}^1_0 \arrow[ur, "j^1_0"'] \arrow[dd, "k^1_0"] & & \mathcal{R}^1_1 \arrow[ur, "j^1_1"'] \arrow[dd, "k^1_1"] & & & \\
\mathcal{Q}^2_0 \arrow[rr, "\pi^2_0"] \arrow[dr, "\sigma^2_0"'] \arrow[dd, "\tau^2_0"'] & & \mathcal{Q}^2_1 \arrow[rr, "\pi^2_1"] \arrow[dr, "\sigma^2_1"'] \arrow[dd, "\tau^2_1"'] & & \mathcal{Q}^2_2 \arrow[r, "\pi^2_2"] \arrow[dd, "\tau^2_2"'] & \cdots & \mathcal{Q}^2_\omega \arrow[dd, "\tau^2_\omega"'] \\
& \mathcal{R}^2_0 \arrow[ur, "j^2_0"'] \arrow[dd, "k^2_0"] & & \mathcal{R}^2_1 \arrow[ur, "j^2_1"'] \arrow[dd, "k^2_1"] & & & \\
\vdots & & \vdots & & \vdots & & \vdots \\
& \vdots & & \vdots & & & \\
\mathcal{Q}^\omega_0 \arrow[rr, "\pi^\omega_0"] \arrow[dr, "\sigma^\omega_0"'] & & \mathcal{Q}^\omega_1 \arrow[rr, "\pi^\omega_1"] \arrow[dr, "\sigma^\omega_1"'] & & \mathcal{Q}^\omega_2 \arrow[r, "\pi^\omega_2"] & \cdots & \mathcal{Q}^\omega_\omega \\
& \mathcal{R}^\omega_0 \arrow[ur, "j^\omega_0"'] & & \mathcal{R}^\omega_1 \arrow[ur, "j^\omega_1"'] & & &
\end{tikzcd}
$$

Figure 3.1: The process in Theorem 3.1.24

As in the previous lemma, we take a minimal such N and let $\Omega = \mathcal{P}(\mathbb{R})^N$. We then get $N \vDash V = K^\Sigma(\mathbb{R})$ and a $(\mathcal{Q}, \Lambda)$ with the property that $\mathcal{Q} \vDash \mathsf{ZFC}\backslash\{\text{Powerset}\}$, $\mathcal{Q}$ is pointwise definable from $\Sigma_{\mathcal{Q}^-}$, has ω Woodin cardinals above $\mathcal{Q}^-$. Furthermore, Λ is Ω-fullness preserving and for all $B \in \mathbb{B}(P, \Sigma)^N$, there is a Λ iterate $(\mathcal{R}, \Lambda_{\mathcal{R}})$ of $\mathcal{Q}$ such that $\Lambda_{\mathcal{R}}$ has strong B-condensation. $(\mathcal{Q}, \Lambda)$ also has the property that any Λ iterate $\mathcal{R}$ of $\mathcal{Q}$ can be further iterated by $\Lambda_{\mathcal{R}}$ to $\mathcal{S}$ such that N is the derived model of $\mathcal{S}$.

Fix $\langle \alpha_i \mid i < \omega \rangle$ a cofinal in Θ^Ω sequence of ordinals. Such a sequence exists since $\Omega = Env((\Sigma^2_1)^N)$. For each n, let

$$\begin{aligned} D_n = \{(\mathcal{R}, \Psi, x, y) \quad | \quad & (\mathcal{R}, \Psi) \text{ is a hod pair equivalent to } (\mathcal{P}, \Sigma),\ x \text{ codes } \mathcal{R}, \\ & y \in \text{ the least } OD^N_\Psi \text{ set of reals with Wadge rank } \geq \alpha_n\} \end{aligned}$$

Clearly, for all n, $D_n \in \mathbb{B}(\mathcal{P}, \Sigma)^N$. By replacing $(\mathcal{Q}, \Lambda)$ by its appropriate iterate, we may assume Λ has strong D_n-condensation for all n. Let $\vec{D} = \langle D_n \mid n < \omega \rangle$. Before proving the next claim, let us introduce some notation. First let for a set $A (A \subseteq \mathbb{R}$ or $A \in \mathbb{B}(\mathcal{P}, \Sigma))$, $\tau^{Q,0}_{A,m}$ be the canonical capturing term for A in $\mathcal{Q}$ at $(\delta_0^{+m})^{\mathcal{Q}}$. Set

$$\gamma^{\mathcal{Q},0}_{D_i,m} = sup\{H^{\mathcal{Q}}_1(P \cup \{\tau^{\mathcal{Q},0}_{D_i,m}\}) \cap \delta_0\};$$
$$\gamma^{\mathcal{Q},0}_{D_i} = sup_{m<\omega}\gamma^{\mathcal{Q},0}_{D_i,m}.$$

Claim 1. *For any Λ-iterate $(\mathcal{S}, \Upsilon)$ of $\mathcal{Q}$. Suppose $i : \mathcal{Q} \to \mathcal{S}$ is the iteration map. Then*

$$i(\delta_0) = sup_{i<\omega}\gamma^{\mathcal{S},0}_{D_i}.$$

Proof. Working in N, let $\langle A_i \mid i < \omega \rangle$ be a sequence of OD_Σ sets such that A_0 is a universal $\Sigma^2_1(\Sigma)$ set; $A_1 = \mathbb{R}\backslash A_0$; the $\langle A_i \mid i \geq 2 \rangle$ is a semiscale on A_1 (note that $\langle A_i \mid i \geq 2 \rangle$ is Wadge cofinal in Ω). Suppose ϕ_i and $s_i \in \mathrm{OR}^{<\omega}$ are such that

$$x \in A_i \;\Leftrightarrow\; N \vDash \phi_i[\Sigma, s_i, x]$$

Now for each i, let

$$\begin{aligned} A^*_i = \{(\mathcal{R}, \Psi, x, y) \quad | \quad & (\mathcal{R}, \Psi) \text{ is a hod pair equivalent to } (\mathcal{P}, \Sigma),\ x \text{ codes } \mathcal{R}, \\ & N \vDash \phi_i[\Psi, s_i, y]\} \end{aligned}$$

By the choice of the A_i's (or just by the fact that $\mathcal{Q}$ is pointwise definable (over $\mathcal{Q}$) from $\{\Sigma_{\mathcal{Q}^-}\}$), we get

$$\delta_0 = sup_{i<\omega}\gamma^{\mathcal{Q},0}_{A^*_i}.$$

By arguments in [26], we get that Λ is guided by $\langle A_i \mid i < \omega \rangle$ for stacks above $\mathcal{Q}^-$ and below δ_0. This is just a straightforward adapatation of the proof of a similar fact in the case $\Theta = \theta_0$. This fact in turns implies

$$\delta_0 = sup_{i<\omega}\gamma^{\mathcal{Q},0}_{D_i}.$$

To see this, fix an A_i^*. We'll show that there is a j such that $\gamma_{D_j}^{\mathcal{Q},0} \geq \gamma_{A_i^*}^{\mathcal{Q},0}$. Well, fix a real coding $\mathcal{P}$ and let j be such that

$$w(A_i) = w((A_i^*)_{(\mathcal{P},\Sigma,x)}) \leq w((D_j)_{(\mathcal{P},\Sigma,x)}).$$

Let z be a real witnessing the reduction. Then there is a map $i : \mathcal{Q} \to \mathcal{R}$ such that

1. i is according to Λ and the iteration is above $\mathcal{Q}^- = \mathcal{P}$;

2. z is generic for the extender algebra $\mathbb{A}$ of $\mathcal{R}$ at $\delta^{\mathcal{R}}$.

Note that $i(\tau_{A_i^*}^{\mathcal{Q}}) = \tau_{A_i^*}^{R}$, $i(\tau_{D_j}^{\mathcal{Q}}) = \tau_{D_j}^{\mathcal{R}}$, and $\mathcal{R}[z] \vDash \tau_{A_i^*} \leq_w \tau_{D_j}$ via z. Hence $\tau_{A_i^*}^{\mathcal{R}} \in X = \{\tau \in \mathcal{R}^{\mathbb{A}} \mid (\exists p \in \mathbb{A})(p \Vdash_R \tau \leq_w \tau_{D_j} \text{ via } \dot{z})\}$ and $|X|^{\mathcal{R}} < \delta^{\mathcal{R}}$ (by the fact that the extender algebra $\mathbb{A}$ is $\delta^{\mathcal{R}}$-cc). But X is definable over $\mathcal{R}$ from $\tau_{D_j}^{R}$, hence $|X|^{\mathcal{R}} < \gamma_{D_j}^{R,0}$. Since $\tau_{A_i^*}^{R} \in X$, $\gamma_{A_i^*}^{\mathcal{R},0} \leq \gamma_{D_j}^{\mathcal{R},0}$ which in turns implies $\gamma_{A_i^*}^{\mathcal{Q},0} \leq \gamma_{D_j}^{\mathcal{Q},0}$.

Now to finish the claim, let $(\mathcal{S}, \Upsilon)$ be a Λ iterate of $\mathcal{Q}$. Suppose $i : \mathcal{Q} \to \mathcal{S}$ is the iteration map. Let $\mathcal{R} = i(\mathcal{P})$ and $\Sigma_{\mathcal{Q}}$ be the tail of Σ under the iteration. We claim that

$$i(\delta_0) = sup_{i<\omega} \gamma_{D_i}^{S,0}.(*)$$

This is easily seen to finish the proof of Claim 1. To see $(*)$, we repeat the proof of the previous part applied to $(\mathcal{S}, \Upsilon)$ and $\langle B_i \mid i < \omega\rangle$ where B_0 is a universal $\Sigma_1^2(\Sigma_{\mathcal{Q}})$; $B_1 = \mathbb{R}\backslash B_0$; $\langle B_i \mid i \geq 2\rangle$ is a semiscale on B_1. We may assume $(\mathcal{S}, \Upsilon)$ is guided by $\langle B_i \mid i < \omega\rangle$ for stacks above R and below $i(\delta_0)$. Now we are in the position to apply the exact same argument as above and conclude that $(*)$ holds. Hence we're done. □

Since Λ has the Dodd-Jensen property, the direct limit $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ of Λ-iterates of $\mathcal{Q}$ below $\delta_0^{\mathcal{Q}}$ is defined and is wellfounded (note that we allow iterations based on $\mathcal{Q}^-$). Let $\langle \delta_i^{\mathcal{Q}} \mid i < \omega\rangle$ be the first ω Woodins of $\mathcal{Q}$ above $\mathcal{Q}^-$ and $i_{\mathcal{Q},\infty}^{\mathcal{Q},\Lambda} : \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ be the direct limit embedding. For $(\mathcal{R}, \Lambda_{\mathcal{R}})$, an iterate of $(\mathcal{Q}, \Lambda)$ below $\delta_0^{\mathcal{Q}}$, let $i_{\mathcal{R},\infty}^{\mathcal{R},\Lambda_{\mathcal{R}}}$ have the obvious meaning and $i_{\mathcal{Q},\mathcal{R}}^{\mathcal{Q},\Lambda}$ be the iteration map according to Λ. By a similar argument as in the computation of $\mathrm{HOD}^{L(\mathbb{R})}$, we get $\langle \eta_n \mid n < \omega\rangle = \langle i_{\mathcal{Q},\infty}^{\mathcal{Q},\Lambda}(\delta_i^{\mathcal{Q}}) \mid i < \omega\rangle$. Also in N, $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)|\eta_n = \mathcal{M}_\infty|\eta_n$ for all n and hence $\mathcal{M}_\infty = \mathcal{M}_\infty(\mathcal{Q}, \Lambda)|\mathrm{sup}_{n<\omega}\eta_n$. In particular, $\mathcal{M}_\infty$ is wellfounded.

Working in N, we first claim that

Claim 2. $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)|\eta_0 = V_{\eta_0}^{\mathrm{HOD}}$. $(*)$

Proof. To show $(*)$, it is enough to show that if $A \subseteq \alpha < \eta_0$ and A is OD then $A \in \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$. To see this, let i be such that $\gamma_{D_i}^{\mathcal{M}_\infty(\mathcal{Q},\Lambda),0} > \alpha$ (such an i exists by the proof of Claim 1). Let

$$\begin{aligned} C = \{(\mathcal{R}, \Psi, x, y) \quad | \quad & (\mathcal{R}, \Psi) \text{ is a hod pair equivalent to } (\mathcal{P}, \Sigma),\ x \text{ codes } \mathcal{R},\ y \text{ codes } (N, \gamma) \\ & \text{such that } (\mathcal{N}, \Psi) \text{ is 1-suitable, } \mathcal{N} \text{ is strongly } D_i \text{ iterable via a} \\ & \text{quasi-strategy } \Phi \text{ extending } \Psi,\ \gamma < \gamma_{D_i}^{\mathcal{N},0},\ \pi_{D_i}^{(\mathcal{N},\Psi),\infty}(\gamma) \in A\}. \end{aligned}$$

By replacing $\mathcal{Q}$ by an iterate we may assume $(\mathcal{Q}, \Lambda)$ is C-iterable. Let $\tau_C^{\mathcal{Q}} = \tau^{\mathcal{Q}}_{C,(\delta_0^{+\omega})^{\mathcal{Q}}}$ and $\tau_C = i^{(\mathcal{Q},\Lambda)}_{\mathcal{Q},\infty}(\tau_C^{\mathcal{Q}})$. The following equivalence is easily shown by a standard computation:

$$\xi \in A \; iff \; \mathcal{M}_\infty(\mathcal{Q},\Lambda) \vDash\Vdash_{Col(\omega,\eta_0^{+\omega})} \text{"if } x \text{ codes } i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\infty}(\mathcal{P}), y \text{ codes } (\mathcal{M}_\infty(\mathcal{Q},\Lambda)|\eta_0^{+\omega}, \xi) \text{ then } (x,y) \in \tau_C\text{"}.$$

For the reader's convenience, we'll show why the above equivalence holds. First suppose $\xi \in A$. Let $(\mathcal{S}, \Xi) \in I(\mathcal{Q}, \Lambda)$ be such that there is a $\gamma < \gamma^{\mathcal{S},0}_{D_i}$ and $i^{\mathcal{S},\Xi}_{\mathcal{S},\infty}(\gamma) = \xi$. Then we have (letting $\nu = i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\delta_0)$)

$$\mathcal{S} \vDash\Vdash_{Col(\omega,\nu^{+\omega})} \text{"if } x \text{ codes } i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\mathcal{P}), y \text{ codes } (S|\nu^{+\omega}, \gamma) \text{ then } (x,y) \in i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\tau_C^{\mathcal{Q},\Lambda})\text{"}.$$

By applying $i^{\mathcal{S},\Xi}_{\mathcal{S},\infty}$ to this ,we get

$$\mathcal{M}_\infty(\mathcal{Q},\Lambda) \vDash\Vdash_{Col(\omega,\eta_0^{+\omega})} \text{"if } x \text{ codes } i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\infty}(\mathcal{P}), y \text{ codes } (\mathcal{M}_\infty(\mathcal{Q},\Lambda)|\eta_0^{+\omega}, \xi) \text{ then } (x,y) \in \tau_C\text{"}.$$

Now to show $(\Leftarrow)$, let $(\mathcal{S}, \Xi) \in I(\mathcal{Q}, \Lambda)$ be such that for some $\gamma < \gamma^{\mathcal{S},0}_{D_i}$, $\xi = i^{\mathcal{S},\Xi}_{\mathcal{S},\infty}(\gamma)$. Let $\nu = i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\delta_0)$, we have

$$\mathcal{S} \vDash\Vdash_{Col(\omega,\nu^{+\omega})} \text{"if } x \text{ codes } i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\mathcal{P}), y \text{ codes } (\mathcal{S}|\nu^{+\omega}, \gamma) \text{ then } (x,y) \in i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\tau_C^{\mathcal{Q},\Lambda})\text{"}.$$

This means there is a quasi-strategy Ψ on $\mathcal{S}(0)$ ($\mathcal{S}(0) = \mathcal{S}|(\nu^{+\omega})^{\mathcal{S}}$) such that $(\mathcal{S}(0), i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\Sigma))$ is 1-suitable, Ψ extends $i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\Sigma))$, and Ψ is D_i-iterable. We need to see that $\pi_{D_i}^{(\mathcal{S}(0), i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\Sigma)),\infty}(\gamma) = \xi$. But this is true by the choice of D_i, $\xi = i^{\mathcal{S},\Xi}_{S,\infty}(\gamma)$, and the fact that Ψ agrees with Ξ on how ordinals below $\gamma^{\mathcal{S},0}_{D_i}$ are mapped.

The equivalence above shows $A \in \mathcal{M}_\infty(Q, \Lambda)$, hence completes the proof of $(*)$. □

$(*)$ in turns shows that η_0 is a cardinal in HOD and $\eta_0 \leq \Theta$ (otherwise, $\text{HOD}|\eta_0 = \mathcal{M}_\infty(Q,\Lambda)|\eta_0 \vDash \Theta$ is not Woodin while $\text{HOD} \vDash \Theta$ is Woodin). Next we show

Claim 3. $\eta_0 = \Theta$. $(**)$

Proof. Suppose toward a contradiction that $\eta_0 < \Theta$. Let $\mathcal{Q}(0) = \mathcal{Q}|(\delta_0^{+\omega})^{\mathcal{Q}}$, $\Lambda_0 = \Lambda|\mathcal{Q}(0)$, and $\mathcal{M}_\infty(\mathcal{Q},\Lambda)(0) = \mathcal{M}_\infty(\mathcal{Q},\Lambda)|(\eta_0^{+\omega})^{\mathcal{M}_\infty(\mathcal{Q},\Lambda)}$. Let $\pi = i \restriction \mathcal{Q}(0)$; so π is according to Λ_0. By the Coding Lemma and our assumption that $\eta_0 < \Theta$, $\pi, \mathcal{M}_\infty(Q,\Lambda)(0) \in N$. From this, we can show $\Lambda_0 \in N$ by the following computation: $\Lambda_0(\vec{\mathcal{T}}) = b$ if and only if

1. the part of $\vec{\mathcal{T}}$ based on P is according to Σ;

2. if $i_b^{\vec{\mathcal{T}}}$ exists then there is a $\sigma : \mathcal{M}_b^{\vec{\mathcal{T}}} \to \mathcal{M}_\infty(Q,\Lambda)(0)$ such that $\pi = \sigma \circ i_b^{\vec{\mathcal{T}}}$;

3. $\vec{\mathcal{T}}^\frown\mathcal{M}_b^{\vec{\mathcal{T}}}$ is Q-structure guided.

By branch condensation of Λ_0, (1),(2), and (3) indeed define Λ_0 in N. This means Λ_0 is OD^N from Σ (and some real x); hence $\Lambda_0 \in N$. So suppose $\gamma = w(Code(\Lambda_0)) < \Theta^\Omega$. In N, let

$$\begin{aligned} B = \{(\mathcal{R}, \Psi, x, y) \quad | \quad & (\mathcal{R}, \Psi) \text{ is a hod pair equivalent to } (\mathcal{P}, \Sigma),\ x \text{ codes } \mathcal{R},\ y \in A_{\mathcal{R}} \\ & \text{where } A_{\mathcal{R}} \text{ is the least } OD(Code(\Psi)) \text{ set such that } w(A_{\mathcal{R}}) > \gamma \\ & \text{and } A_{\mathcal{R}} \text{ is not projective in any set with Wadge rank } \gamma\} \end{aligned}$$

Then $B \in \mathbb{B}(\mathcal{P}, \Sigma)^N$. We may assume Λ_0 respects B. It is then easy to see that whenever $(\mathcal{R}, \Lambda_{\mathcal{R}}) \in I(Q(0), \Lambda_0)$ (also let $\mathcal{S} \lhd \mathcal{R}$ be the iterate of $\mathcal{P}$), $A_{\mathcal{R}}$ is projective in $Code(\Lambda_{\mathcal{R}})$ (and hence $A_{\mathcal{R}}$ is projective in $Code(\Lambda_0)$) because $\Lambda_{\mathcal{R}}$ can compute membership of $A_{\mathcal{R}}$ by performing genericity iterations (above $\mathcal{S}$) to make reals generic. This contradicts the choice of $A_{\mathcal{R}}$. □

Claims 2 and 3 complete the proof of the lemma. □

Now we define a strategy Σ_∞ for $\mathcal{M}_\infty$ extending the strategy Σ_∞^- of $\mathcal{M}_\infty^- = V_{\theta_\alpha}^{\text{HOD}}$. Let $(\mathcal{P}, \Sigma, A) \in \mathcal{F}$ and suppose $\mathcal{P}$ is n-suitable with $\langle \delta_i \mid i < n \rangle$ being the sequence of Woodins of $\mathcal{P}$ above $\mathcal{P}^-$, let $\tau_{A,k}^{\mathcal{M}_\infty}$ = common value of $\pi_{\vec{B},\infty}^{\mathcal{P},\Sigma}(\tau_{A,\delta_k}^{\mathcal{P}})$. Σ_∞ will be defined (in V) for trees on $\mathcal{M}_\infty|\eta_0$ in $\mathcal{M}_\infty$. For $k \geq n$, $\mathcal{M}_\infty \vDash$ "$Col(\omega, \eta_n) \times Col(\omega, \eta_k) \Vdash (\tau_{A,n}^{\mathcal{M}_\infty})_g = (\tau_{A,k}^{M_\infty})_h \cap \mathcal{M}_\infty[g]$" where g is $Col(\omega, \eta_n)$ generic and h is $Col(\omega, \eta_k)$ generic and $(\tau_{A,n}^{\mathcal{M}_\infty})_g$ is understood to be $A_{(\mathcal{M}_\infty^-, \Sigma_\infty^-)} \cap \mathcal{M}_\infty[g]$. This is just saying that the terms cohere with one another.

Let $\lambda^{\mathcal{M}_\infty} = sup_{i<\omega}\eta_i$. Let G be $Col(\omega, \lambda^{\mathcal{M}_\infty})$ generic over $\mathcal{M}_\infty$. Then $\mathbb{R}_G^*$ is the symmetric reals and $A_G^* := \cup_k (\tau_{A,k}^{\mathcal{M}_\infty})_{G|\eta_k}$.

Proposition 3.1.27. *For all $A \in \mathbb{B}(\mathcal{M}_\infty^-, \Sigma_\infty^-)$, $L(A_G^*, \mathbb{R}_G^*) \vDash \mathsf{AD}^+$*

Proof. We briefly sketch the proof of this since the techniques involved have been fully spelled out before. If not, reflect the situation down to a model N coded by a Suslin co-Suslin set. Next get a "next mouse" $\mathcal{N}$ with ω Woodin cardinals that iterates out to (possibly a longer mouse than) $\mathcal{M}_\infty^N$. $\mathcal{N}$ also has the property that its derived model is elementarily equivalent to N.

Let $A \subseteq \mathbb{B}(\mathcal{M}_\infty^-, \Sigma_\infty^-)$ be the least OD set such that $L(A_G^*, \mathbb{R}_G^*) \nvDash \mathsf{AD}^+$. Then there is an iterate $\mathcal{M}$ of $\mathcal{N}$ having preimages of all the terms $\tau_{A,k}^{M_\infty}$. We may assume $\mathcal{M}$ has derived model $K^\Sigma(\mathbb{R})$ and suitable initial segments of $\mathcal{M}$ are in $\mathcal{F}^N$. Since we have AD^+, $\mathcal{M}$ thinks that its derived model (in this case is $K^\Sigma(\mathbb{R})$) satisfies that $L(A_{(\mathcal{P},\Sigma)}, \mathbb{R}) \vDash \mathsf{AD}^+$, where we reuse $(\mathcal{P}, \Sigma)$ for an equivalent (but possibly different) hod pair from the original one. Now iterate $\mathcal{M}$ to $\mathcal{Q}$ such that $\mathcal{M}_\infty$ is an initial segment of $\mathcal{Q}$. By elementarity $L(A_G^*, \mathbb{R}_G^*) \vDash \mathsf{AD}^+$. This is a contradiction. □

Definition 3.1.28. *Given a normal tree $\mathcal{T} \in \mathcal{M}_\infty$ and $\mathcal{T}$ is based on $\mathcal{M}_\infty|\theta_0$. $\mathcal{T}$ is by Σ_∞ if the following hold (the definition is similar for finite stacks):*

- *If $\mathcal{T}$ is short then Σ picks the branch guided by Q-structure (as computed in $\mathcal{M}_\infty$).*
- *If $\mathcal{T}$ is maximal then $\Sigma_\infty(\mathcal{T})$ = the unique cofinal branch b which moves $\tau_{A,0}^{\mathcal{M}_\infty}$ correctly for all $A \in OD$ such that there is some $(\mathcal{P}, \Sigma, A) \in \mathcal{F}$ i.e. for each such A, $i_b(\tau_{A,0}^{\mathcal{M}_\infty}) = \tau_{A^*,0}^{\mathcal{M}_b^\mathcal{T}}$.*

Lemma 3.1.29. *Given any such $\mathcal{T}$ as above, $\Sigma_\infty(\mathcal{T})$ exists.*

The proof of the lemma is similar to the proof of Lemma 3.1.14. So we omit it.

It is evident that $L(\mathcal{M}_\infty, \Sigma_\infty) \subseteq$ HOD. Next, we show $\mathcal{M}_\infty$ and Σ_∞ capture all unbounded subsets of Θ in HOD. In $L(\mathcal{M}_\infty, \Sigma_\infty)$, first construct (using Σ_∞) a mouse $\mathcal{M}_\infty^+$ extending $\mathcal{M}_\infty$ such that $\mathrm{o}(\mathcal{M}_\infty)$ is the largest cardinal of $\mathcal{M}_\infty^+$ as follows:

1. Let $\mathbb{R}_G^*$ be the symmetric reals obtained from a generic G over $\mathcal{M}_\infty$ of $\mathrm{Col}(\omega, < \lambda^{\mathcal{M}_\infty})$.
2. For each A_G^* (defined as above) (we know $L(\mathbb{R}_G^*, A_G^*) \vDash \mathsf{AD}^+$), pull back the hybrid mice over $\mathbb{R}_G^*$ in this model to hybrid mice $\mathcal{S}$ extending $\mathcal{M}_\infty$ with $D(S, \lambda^{M_\infty}) = L(\mathbb{R}_G^*, A_G^*)$.
3. Let $\mathcal{M}_\infty^+ = \cup_\mathcal{S} \mathcal{S}$ for all such $\mathcal{S}$ above. $\mathcal{M}_\infty^+$ is independent of G. By a reflection argument (and Prikry-like forcing) as above, the translated mice over $\mathcal{M}_\infty$ are all compatible, no levels of $\mathcal{M}_\infty^+$ projects across $o(\mathcal{M}_\infty)$, and $\mathcal{M}_\infty^+$ contains as its initial segments all translation of $\mathbb{R}_G^*$-mice in $D(\mathcal{M}_\infty^+, \lambda^{\mathcal{M}_\infty})$. This is just saying that $\mathcal{M}_\infty^+$ contains enough mice to compute HOD.

Remark 3.1.30. Θ is not collapsed by Σ_∞ as it is a cardinal in HOD. Σ_∞ is used to obtain the A_G^* above by moving correctly the $\tau_{A,0}^{M_\infty}$ in genericity iterations. $L(\mathcal{M}_\infty)$ does not see the sequence $\langle \tau_{A,k}^{\mathcal{M}_\infty} | k \in \omega \rangle$ hence can't construct A_G^*. Also since Σ_∞ collapses $\delta_1^{M_\infty}, \delta_2^{\mathcal{M}_\infty}...$, it doesn't make sense to talk about $D(L(\mathcal{M}_\infty, \Sigma_\infty))$.

Lemma 3.1.31. *HOD $\subseteq L(\mathcal{M}_\infty, \Sigma_\infty)$*

Proof. Using Theorem 3.1.9, we know HOD $= L[P]$ for some $P \subseteq \Theta$. Therefore, it is enough to show $P \in L(\mathcal{M}_\infty, \Sigma_\infty)$. Let ϕ be a formula defining P, i.e.

$$\alpha \in P \Leftrightarrow V \vDash \phi[\alpha].$$

We suppress the ordinal parameter here. Now in $L(\mathcal{M}_\infty, \Sigma_\infty)$ let $\pi : \mathcal{M}_\infty|(\eta_0^{++})^{\mathcal{M}_\infty} \to (\mathcal{M}_\infty)^{(\mathcal{M}_\infty^+, \lambda^{\mathcal{M}_\infty})}$ where π is according to Σ_∞.

Claim: $\alpha \in P \Leftrightarrow D(\mathcal{M}_\infty^+, \lambda^{M_\infty}) \vDash \phi[\pi(\alpha)]$. $(*)$

Proof. Otherwise, reflect the failure of $(*)$ as before to get a model N coded by a Suslin co-Suslin set, a hod pair $(\mathcal{P},\Sigma)$ giving us $\mathrm{HOD}|\theta_\alpha$ such that

$$N \vDash \mathsf{ZF}+\mathsf{DC}+\mathsf{AD}^+ + V = K^\Sigma(\mathbb{R}) + (\exists\alpha)(\phi[\alpha] \not\Leftrightarrow D(\mathcal{M}^+_\infty,\Sigma_\infty)\vDash\phi[\pi(\alpha)]).$$

Fix such an α. As before, let $\mathcal{N}$ be the next mouse (i.e. $\mathcal{N}$ has ω Woodins $\langle\delta_i \mid i<\omega\rangle$ on top of $\mathcal{P}$) with $\rho(\mathcal{N}) < sup_i\delta_i)$ with strategy Λ extending Σ and Λ has branch condensation and is Ω-fullness preserving, where $\Omega = (\Sigma^2_1)^N$. We may assume Λ is guided by $\vec{D}$ where $\vec{D} = \langle D_n \mid n<\omega\rangle$ is defined as in Lemma 3.1.26. As before, we may assume $\mathcal{N}$ has derived model N. Let $\sigma : N|((\delta_0^N)^{++})^N \to (M_\infty)^{D(N,\lambda^N)}$ be the direct limit map by Λ. We may assume $\sigma(\overline{\alpha}) = \alpha$ for some $\overline{\alpha}$. Working in N, it then remains to see that:

$$D(\mathcal{M}^+_\infty,\lambda^{\mathcal{M}_\infty})\vDash\phi[\pi(\alpha)] \Leftrightarrow D(N,\lambda^N)\vDash\phi[\sigma(\overline{\alpha})] \quad (**).$$

To see that $(**)$ holds, we need to see that the fragment of Λ that defines $\sigma(\overline{\alpha})$ can be defined in $D(N,\lambda^N)$. This then will give the equivalence in $(**)$. Because $\alpha<\eta_0$, $\overline{\alpha}<\delta_0$, pick an n such that such that $\gamma^{N,0}_{D_n,0} > \overline{\alpha}$. Then the fragment of Λ that defines $\sigma(\overline{\alpha})$ is definable from D_n (and $N|(\delta_0^N)$) in $D(N,\lambda^N)$, which is what we want.

The equivalence $(**)$ gives us a contradiction. □

The claim finishes the proof of $P\in L(\mathcal{M}_\infty,\Sigma_\infty)$. This then implies $\mathrm{HOD} = L[P] \subseteq L(\mathcal{M}_\infty,\Sigma_\infty)$. □

Lemma 3.1.31 implies $\mathrm{HOD} = L(\mathcal{M}_\infty,\Sigma_\infty)$, hence completes our computation.

3.1.2 The Limit Case

There are two cases: the easier case is when HOD $\vDash$ "$cof(\Theta)$ is not measurable", and the harder case is when HOD $\vDash$ "$cof(\Theta)$ is measurable".

Here's the direct limit system that gives us V_Θ^{HOD}.

$\mathcal{F} = \{(\mathcal{Q},\Lambda) \mid (\mathcal{Q},\Lambda)$ is a hod pair; Λ is fullness preserving and has branch condensation$\}$.

The order on $\mathcal{F}$ is given by

$$(\mathcal{Q},\Lambda) \le^{\mathcal{F}} (\mathcal{R},\Psi) \;\Leftrightarrow\; \mathcal{Q} \text{ iterates to a hod initial segment of } \mathcal{R}.$$

By Theorem 1.2.2, $\le^{\mathcal{F}}$ is directed and we can form the direct limit of $\mathcal{F}$ under the natural embeddings coming from the comparison process. Let $\mathcal{M}_\infty$ be the direct limit. By the computation in [23],

$$|\mathcal{M}_\infty| = V_\Theta^{\mathrm{HOD}}.$$

$\mathcal{M}_\infty$ as a structure also has a predicate for its extender sequence and a predicate for a sequence of strategies.

We quote a theorem from [23] which will be used in the upcoming computation. For unexplained notations, see [23].

Theorem 3.1.32 (Sargsyan, Theorem 4.2.23 in [23]). *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and is fullness preserving. There is then $\mathcal{Q}$ a Σ-iterate of $\mathcal{P}$ such that whenever $\mathcal{R}$ is a $\Sigma_{\mathcal{Q}}$-iterate of $\mathcal{Q}$, $\alpha < \lambda^{\mathcal{R}}$, and $B \in (\mathbb{B}(\mathcal{R}(\alpha), \Sigma_{\mathcal{R}(\alpha)}))^{L(\Gamma(\mathcal{R}(\alpha+1), \Sigma_{\mathcal{R}(\alpha+1)}))}$*

1. *$\Sigma_{\mathcal{R}(\alpha+1)}$ is super fullness preserving and is strongly guided by some*

$$\vec{B} = \langle B_i \mid i < \omega \rangle \subseteq (\mathbb{B}(\mathcal{R}(\alpha), \Sigma_{\mathcal{R}(\alpha)}))^{L(\Gamma(\mathcal{R}(\alpha+1), \Sigma_{\mathcal{R}(\alpha+1)}))};$$

2. *there is a $(\mathcal{S}, \Sigma_{\mathcal{S}}) \in I(\mathcal{R}(\alpha+1), \Sigma_{\mathcal{R}(\alpha+1)})$ such that $\Sigma_{\mathcal{S}}$ has strong B-condensation.*

We deal with the easy case first.

3.1.2.1 Nonmeasurable Cofinality

The following theorem is the full HOD computation in this case.

Theorem 3.1.33. *$HOD = L(\mathcal{M}_\infty)$*

Proof. To prove the theorem, suppose the equality is false. Then by Theorem 3.1.9, there is an $A \subseteq \Theta$ such that $A \in \mathrm{HOD} \backslash L(\mathcal{M}_\infty)$ (the fact that $L(\mathcal{M}_\infty) \subseteq \mathrm{HOD}$ is obvious). By Σ_1-reflection (i.e. Theorem 1.1.5), there is a transitive N coded by a Suslin co-Suslin set such that

$$\begin{aligned} N \ \vDash\ & \mathsf{ZF}^- + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R})) + \mathsf{SMC} + \Theta \text{ exists and is limit in the Solovay sequence} \\ & +\mathrm{HOD} \vDash \text{“}cof(\Theta) \text{ is not measurable ”} + \text{“}\exists B \subseteq \Theta(B \in \mathrm{HOD}\backslash L(\mathcal{M}_\infty))\text{”}. \end{aligned}$$

Take N to be the minimal such and let B witness the failure of the theorem in N. Let ϕ define B (for simplicity, we suppress the ordinal parameter) i.e.

$$\alpha \in B \ \Leftrightarrow\ N \vDash \phi[\alpha]$$

Let $\Omega = \mathcal{P}(\mathbb{R})^N$. There is a pair $(\mathcal{P}, \Sigma)$ such that:

1. $\mathcal{P} = L_\beta(\cup_{\gamma<\lambda^P} P_\gamma)$ for some λ^P;
2. for all $\gamma < \lambda^{\mathcal{P}}$, $\mathcal{P}_\beta$ is a hod mouse whose strategy $\Sigma_\gamma \in \Omega$ is Ω-fullness preserving, has branch condensation, and $\lambda^{\mathcal{P}_\beta} = \beta$;
3. if $\gamma < \eta < \lambda^{\mathcal{P}}$, $\mathcal{P}_\gamma \trianglelefteq_{hod} \mathcal{P}_\eta$;
4. β is least such that $\rho_\omega(L_\beta(\cup_{\gamma<\lambda^{\mathcal{P}}} \mathcal{P}_\gamma)) < o(\cup_{\gamma<\lambda^{\mathcal{P}}} \mathcal{P}_\gamma))$;
5. $\mathcal{P} \vDash cof(\lambda^{\mathcal{P}})$ is not measurable;
6. Σ has branch condensation and extends $\oplus_{\gamma<\lambda^{\mathcal{P}}} \Sigma_\gamma$;

Such a $(\mathcal{P}, \Sigma)$ can be obtained by performing a Ω-hod pair construction (see Definition 1.2.9) inside some N_x^* capturing a good pointclass beyond Ω. We may and do assume that $(\cup_{\gamma<\lambda^{\mathcal{P}}}\mathcal{P}_\gamma, \oplus_{\gamma<\lambda^{\mathcal{P}}}\Sigma_\gamma)$ satisfies Theorem 3.1.32 applied in N. This implies that the direct limit $\mathcal{M}_\infty^+$ of all Σ-iterates of $\mathcal{P}$ is a subset of HOD^N. Let $j : \mathcal{P} \to \mathcal{M}_\infty^+$ be the natural map. Then in N, $\mathcal{M}_\infty^+|j(\lambda^{\mathcal{P}}) = \mathcal{M}_\infty$.

Now pick a sequence $\langle \gamma_i \mid i < \omega\rangle$ cofinal in $\lambda^{\mathcal{P}}$ such that $\delta_{\lambda^{\mathcal{P}_{\gamma_i}}}$ is Woodin in $\mathcal{P}$, an enumeration $\langle x_i \mid i < \omega\rangle$ of $\mathbb{R}$ and do a genericity iteration of $\mathcal{P}$ to successively make each x_i generic at appropriate image of $\delta_{\lambda^{\mathcal{P}_{\gamma_i}}}$. Let $\mathcal{Q}$ be the end model of this process and $i : \mathcal{P} \to \mathcal{Q}$ be the iteration embedding. Then by assumption (5) above, we have that N is the derived model of $\mathcal{Q}$ at $i(\lambda^P)$.

In N, let D be the derived model of $\mathcal{M}_\infty^+$ at Θ and

$$\pi_\infty : \mathcal{M}_\infty \to (\mathcal{M}_\infty)^D$$

be the direct limit embedding given by the join of the strategies of $\mathcal{M}_\infty$'s hod initial segments. Then by the same argument as that given in Lemma 3.1.16, we have

$$\alpha \in B \ \Leftrightarrow\ D \vDash \phi[\pi_\infty(\alpha)].$$

The proof of Lemma 3.1.16 also gives that $B \in (L(\mathcal{M}_\infty))^N$, which contradicts our assumption. Hence we're done. □

Remark 3.1.34. *It's not clear that in the statement of Theorem 3.1.33, "$\mathcal{M}_\infty$" can be replaced by "V_Θ^{HOD}".*

3.1.2.2 Measurable Cofinality

Suppose $\mathrm{HOD} \vDash cof(\Theta)$ is measurable. We know by [23] that V_Θ^{HOD} is $|\mathcal{N}_\infty|$ where $\mathcal{N}_\infty$ is the direct limit (under the natural maps) of $\mathcal{F}$, where $\mathcal{F}$ is introduced at the beginning of this section. Let

$$\mathcal{M}_\infty = Ult_0(\mathrm{HOD}, \mu)|\Theta,$$

where μ is the order zero measure on $cof^{\mathrm{HOD}}(\Theta)$. Let $f : \mathrm{cof}^{\mathrm{HOD}}(\Theta) =_{def} \alpha \to \Theta$ be a continuous and cofinal function in HOD. For notational simplicity, for each $\beta < \alpha$, let Λ_β be the strategy of $\mathcal{M}_\infty(f(\beta))$ and Σ_β be the strategy of $\mathcal{N}_\infty(f(\beta))$. Let

$$\mathcal{M}_\infty^+ = Ult_0(\mathrm{HOD}, \mu)|(\Theta^+)^{Ult_0(\mathrm{HOD},\mu)},$$

and

$$\mathcal{N}_\infty^+ = \cup\{\mathcal{M} \mid \mathcal{N}_\infty \trianglelefteq \mathcal{M},\ \rho(\mathcal{M}) = \Theta, \mathcal{M} \text{ is a hybrid mouse satisfying property } (*)\}.$$

Here a mouse $\mathcal{M}$ satisfies property $(*)$ if whenever $\pi : \mathcal{M}^* \to \mathcal{M}$ is elementary, $\mathcal{M}^*$ is countable, transitive, and $\pi(\Theta^*) = \Theta$, then $\mathcal{M}^*$ is a $\oplus_{\xi<\Theta^*}\Sigma^*_\xi$-mouse for stacks above Θ^*, where Σ_ξ^* is the strategy for the hod mouse $\mathcal{M}^*(\xi)$ obtained by the following process: let $(\mathcal{P}, \Sigma) \in \mathcal{F}$ and $i : \mathcal{P} \to \mathcal{M}_\infty$ be the direct limit embedding such that the range of i contains the range of $\pi \restriction \mathcal{M}^*(\xi)$; Σ_ξ^* is then defined to be the $\pi \circ i^{-1}$-pullback of Σ. It's easy to see that the strategy Σ_ξ^* as defined doesn't depend on the choice of $(\mathcal{P}, \Sigma)$. This is because if $(\mathcal{P}_0, \Sigma_0, i_0)$ and $(\mathcal{P}_1, \Sigma_1, i_1)$ are two possible choices to define Σ_ξ^*, we can coiterate $(\mathcal{P}_0, \Sigma_0)$ against $(\mathcal{P}_1, \Sigma_1)$ to a pair $(\mathcal{R}, \Lambda)$ and let $i_i : \mathcal{P}_i \to \mathcal{R}$ be the iteration maps and let $i_2 : \mathcal{R} \to \mathcal{M}_\infty$ be the direct limit embedding. Then $\Sigma_0 = \Lambda^{i_0}$ and $\Sigma_1 = \Lambda^{i_1}$; hence the $\pi \circ i_0^{-1}$-pullback of Σ_0 is the same as the $\pi \circ i_1^{-1}$-pullback of Σ_1 because both are the same as the $\pi \circ i_2^{-1}$-pullback of Λ.

We give two characterizations of HOD here: one in terms of $\mathcal{M}_\infty^+$ and the other in terms of $\mathcal{N}_\infty^+$. The first one is easier to see.

Theorem 3.1.35. *1. $HOD = L(\mathcal{N}_\infty, \mathcal{M}_\infty^+)$.*

2. $HOD = L(\mathcal{N}_\infty^+)$.

Proof. To prove (1), first let $j_\mu : \mathrm{HOD} \to Ult_0(\mathrm{HOD}, \mu)$ be the canonical ultrapower map. Let $A \in \mathrm{HOD}$, $A \subseteq \Theta$. By the computation of HOD below Θ, we know that for each limit $\beta < \alpha$,

$$A \cap \theta_{f(\beta)} \in |\mathcal{N}_\infty(f(\beta))|.$$

This means

$$j_\mu(A) \cap \Theta \in \mathcal{M}_\infty^+.$$

We then have

$$\gamma \in A \Leftrightarrow j_\mu(\gamma) \in j_\mu(A) \cap \Theta.$$

Since $j_\mu|\Theta$ agrees with the canonical ultrapower map $k : \mathcal{N}_\infty \to Ult_0(\mathcal{N}_\infty, \mu)$ on all ordinals less than Θ, the above equivalence shows that $A \in L(\mathcal{N}_\infty, \mathcal{M}_\infty^+)$. This proves (1).

Suppose the statement of (2) is false. There is an $A \subseteq \Theta$ such that $A \in \mathrm{HOD} \backslash \mathcal{N}_\infty^+$. By Σ_1-reflection (i.e. Theorem 1.1.5), there is a transitive N coded by a Suslin co-Suslin set such that

$$\begin{aligned} N \ \vDash\ & \mathsf{ZF}^- + \mathsf{DC} + V = L(\mathcal{P}(\mathbb{R})) + \mathsf{SMC} + \text{“}\Theta \text{ exists and is limit in the Solovay sequence ”} \\ & + \text{“HOD} \vDash cof(\Theta) = \alpha \text{ is measurable as witnessed by } f\text{”} \\ & + \text{“}\exists A \subseteq \Theta (A \in \mathrm{HOD} \backslash \mathcal{N}_\infty^+)\text{”}. \end{aligned}$$

Take N to be the minimal such and let A witness the failure of (2) in N. Let μ, j_μ, $\mathcal{M}_\infty$, $\mathcal{M}_\infty^+$, $\mathcal{N}_\infty$, $\mathcal{N}_\infty^+$ be as above but relativized to N. Working in N, there is a sequence $\langle \mathcal{M}_\beta \mid \beta < \alpha, \beta \text{ is limit} \rangle \in \mathrm{HOD}$ such that for each limit $\beta < \alpha$, $\mathcal{M}_\beta$ is the least hod initial segment of $\mathcal{N}_\infty | \theta_{f(\beta)}$ such that $A \cap \theta_{f(\beta)}$ is definable over $\mathcal{M}_\beta$.

Let $\Omega = \mathcal{P}(\mathbb{R})^N$. Fix an N_x^* capturing a good pointclass beyond Ω. Now, we again do the Ω-hod pair construction in N_x^* to obtain a pair $(\mathcal{Q}, \Lambda)$ such that

1. there is a limit ordinal $\lambda^{\mathcal{Q}}$ such that for all $\gamma < \lambda^{\mathcal{Q}}$, $\mathcal{Q}_\beta$ is a hod mouse with $\lambda^{\mathcal{Q}_\beta} = \beta$ and whose strategy $\Psi_\gamma \in \Omega$ is Ω-fullness preserving, has branch condensation;

2. if $\gamma < \eta < \lambda^{\mathcal{Q}}$, $\mathcal{Q}_\gamma \trianglelefteq_{hod} \mathcal{Q}_\eta$;

3. $\mathcal{Q}$ is the first sound mouse from the $L[E, \oplus_{\gamma<\lambda^{\mathcal{Q}}}\Psi_\gamma][\cup_{\gamma<\lambda^{\mathcal{Q}}}\mathcal{Q}_\gamma]$-construction done in N^*_x that has projectum $\leq o(\cup_{\gamma<\lambda^{\mathcal{Q}}}\mathcal{Q}_\gamma)$ and extends $Lp^{\Omega,\oplus_{\gamma<\lambda^{\mathcal{Q}}}\Psi_\gamma}(\cup_{\gamma<\lambda^{\mathcal{Q}}}\mathcal{Q}_\gamma)$ [10] and Λ be the induced strategy of $\mathcal{Q}$.

From the construction of $\mathcal{Q}$ and the properties of N, it's easy to verify the following:

1. Let $\delta_{\lambda^{\mathcal{Q}}} = o(\cup_{\gamma<\lambda^{\mathcal{Q}}}\mathcal{Q}_\gamma)$ and $\eta = o(Lp^{\Omega,\oplus_{\gamma<\lambda^{\mathcal{Q}}}\Psi_\gamma}(\cup_{\gamma<\lambda^{\mathcal{Q}}}\mathcal{Q}_\gamma))$. Then $\eta = (\delta^+_{\lambda^{\mathcal{Q}}})^{\mathcal{Q}}$.

2. $\Lambda \notin \Omega$.

3. $\mathcal{Q} \vDash \delta_{\lambda^{\mathcal{Q}}}$ has measurable cofinality.

Let $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ be the direct limit (under natural embeddings) of Λ-iterates of $\mathcal{Q}$.

Lemma 3.1.36. *$\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ exists.*

Proof. First note that Λ is Ω-fullness preserving. To see this, suppose not. Let $k : \mathcal{Q} \to \mathcal{R}$ be according to Λ witnessing this. It's easy to see that the tail $\Lambda_{\mathcal{R}}$ of Λ acting on $\mathcal{R}|k(\eta)$ is not in Ω (otherwise, $\Lambda^k_{\mathcal{R}} = \Lambda$ by hull condensation and hence $\Lambda \in \Omega$. Contradiction.) However, $\oplus_{\gamma<\lambda^{\mathcal{R}}}\Psi_{\mathcal{R}(\gamma)} \in \Omega$ since the iterate of N^*_x by the lift-up of k thinks that the fragment of its strategy inducing $\oplus_{\gamma<\lambda^{\mathcal{R}}}\Psi_{\mathcal{R}(\gamma)}$ is in Ω. Now suppose $\mathcal{M}$ is a $\oplus_{\gamma<\lambda^{\mathcal{R}}}\Psi_{\mathcal{R}(\gamma)}$-mouse projecting to $\delta_{\lambda^{\mathcal{R}}}$ with strategy Ξ in Ω and $\mathcal{M} \ntrianglelefteq \mathcal{R}$ (again, Ξ acts on trees above $\delta_{\lambda^{\mathcal{R}}}$ and moves the predicates for $\oplus_{\gamma<\lambda^{\mathcal{R}}}\Psi_{\mathcal{R}(\gamma)}$ correctly). We can compare $\mathcal{M}$ and $\mathcal{R}$ (the comparison is above $\delta_{\lambda^{\mathcal{R}}}$). Let $\overline{\mathcal{M}}$ be the last model on the $\mathcal{M}$ side and $\overline{\mathcal{R}}$ on the $\mathcal{R}$ side. Then $\overline{\mathcal{R}} \lhd \overline{\mathcal{M}}$. Let $\pi : \mathcal{R} \to \overline{\mathcal{R}}$ be the iteration map from the comparison process and Σ be the $\pi \circ k$-pullback of the strategy of $\overline{\mathcal{R}}$. Hence $\Sigma \in \Omega$ since $\Xi \in \Omega$. Σ acts on trees above $\delta_{\lambda^{\mathcal{Q}}}$ and moves the predicate for $\oplus_{\gamma<\lambda^{\mathcal{Q}}}\Psi_\gamma$ correctly by by our assumption on Ξ and branch condensation of $\oplus_{\gamma<\lambda^{\mathcal{Q}}}\Psi_\gamma$. These properties of Σ imply that $\mathcal{Q} \lhd Lp^{\Omega,\oplus_{\gamma<\lambda^{\mathcal{Q}}}\Psi_\gamma}(\cup_{\gamma<\lambda^{\mathcal{Q}}}\mathcal{Q}_\gamma)$. Contradiction. For the case that there are $\alpha < \lambda^{\mathcal{R}}$, $\delta^{\mathcal{R}}_\alpha \leq \eta < \delta^{\mathcal{R}}_{\eta+1}$, and η is a strong cutpoint of $\mathcal{R}$, and $\mathcal{M}$ is a sound $\Psi_{\mathcal{R}(\alpha)}$-mouse projecting to η with iteration strategy in Ω, the proof is the same as that of Theorem 3.7.6 in [23].

Now we show Λ has branch condensation (see Figure 3.2). The proof of this comes from private conversations between the author and John Steel. We'd like to thank him for this. For notational simplicity, we write Λ^- for $\oplus_{\gamma<\lambda^{\mathcal{Q}}}\Psi_\gamma$. Hence, $\Lambda \notin \Omega$ and $\Lambda^- \in \Omega$. Suppose Λ does not have branch condensation. We have a minimal counterexample as follows: there are an iteration $i : \mathcal{Q} \to \mathcal{R}$ by Λ, a normal tree $\mathcal{U}$ on $\mathcal{R}$ in the window $[\xi, \gamma)$ where $\xi < \gamma$

[10] If $\mathcal{M} \lhd Lp^{\Omega,\oplus_{\gamma<\lambda^{\mathcal{Q}}}\Psi_\gamma}(\cup_{\gamma<\lambda^{\mathcal{Q}}}\mathcal{Q}_\gamma)$ and $\mathcal{M}$ extends $\cup_{\gamma<\lambda^{\mathcal{Q}}}\mathcal{Q}_\gamma$ then $\mathcal{M}$ is a mouse in N in the sense that N knows how to iterate $\mathcal{M}$ for stacks above $o(\cup_{\gamma<\lambda^{\mathcal{Q}}}\mathcal{Q}_\gamma)$.

are two consecutive Woodins in $\mathcal{R}$ such that $\sup i''\delta_{\lambda^{\mathcal{Q}}} \leq \xi$, two distinct cofinal branches of $\mathcal{U}$: b and $c = \Lambda_{\mathcal{R}}(\mathcal{U})$, an iteration map $j : \mathcal{Q} \to \mathcal{S}$ by Λ, and a map $\sigma : \mathcal{M}^{\mathcal{U}}_b \to \mathcal{S}$ such that $j = \sigma \circ i^{\mathcal{U}}_b \circ i$. We may also assume that if $\overline{\mathcal{R}}$ is the first model along the main branch of the stack from $\mathcal{Q}$ to $\mathcal{R}$ giving rise to i and $i_{\overline{\mathcal{R}},\mathcal{R}} : \overline{\mathcal{R}} \to \mathcal{R}$ be the natural map such that $i_{\overline{\mathcal{R}},\mathcal{R}}(\overline{\xi}) = \xi$ and $i_{\overline{\mathcal{R}},\mathcal{R}}(\overline{\gamma}) = \gamma$, then the extenders used to get from $\mathcal{Q}$ to $\overline{\mathcal{R}}$ have generators below $\overline{\xi}$. This gives us $\sup(Hull^{\mathcal{R}}(\xi \cup \{p\}) \cap \gamma) = \gamma$ where p is the standard parameter of $\mathcal{R}$. Let $\Phi = \Lambda^{\sigma}_{\mathcal{S}}$ and $\Phi^- = \oplus_{\xi<\lambda^{\mathcal{M}^{\mathcal{U}}_b}} \Phi_{\mathcal{M}^{\mathcal{U}}_b(\xi)}$. It's easy to see that $\Phi^- \in \Omega$. By the same proof as in the previous paragraph, Φ is Ω-fullness preserving. This of course implies that $\mathcal{M}^{\mathcal{U}}_b$ is Ω-full and $\Phi \notin \Omega$.

Now we compare $\mathcal{M}^{\mathcal{U}}_b$ and $\mathcal{M}^{\mathcal{U}}_c$. First we line up the strategies of $\mathcal{M}^{\mathcal{U}}_b|\delta(\mathcal{U})$ and $\mathcal{M}^{\mathcal{U}}_c|\delta(\mathcal{U})$ by iterating them into the (Ω-full) hod pair construction of some N^*_y (where y codes $(x, \mathcal{M}^{\mathcal{U}}_c, \mathcal{M}^{\mathcal{U}}_b)$). This can be done because the strategies of $\mathcal{M}^{\mathcal{U}}_b|\delta(\mathcal{U})$ and of $\mathcal{M}^{\mathcal{U}}_c|\delta(\mathcal{U})$ have branch condensation by Theorems 2.7.6 and 2.7.7 of [23][11]. This process produces a single normal tree $\mathcal{W}$. Let $a = \Phi(\mathcal{W})$ and $d = \Lambda_{\mathcal{M}^{\mathcal{U}}_c}(\mathcal{W})$. Let $X = Hull^{\mathcal{R}}(\xi \cup \{p\}) \cap \gamma$. Note that $(i^{\mathcal{W}}_a \circ i^{\mathcal{U}}_b)$"X $\subseteq \delta(\mathcal{W})$ and $i^{\mathcal{W}}_d \circ i^{\mathcal{U}}_c$"X $\subseteq \delta(\mathcal{W})$. Now continue lining up $\mathcal{M}^{\mathcal{W}}_a$ and $\mathcal{M}^{\mathcal{W}}_d$ above $\delta(\mathcal{W})$ (using the same process as above). We get $\pi : \mathcal{M}^{\mathcal{W}}_a \to \mathcal{K}$ and $\tau : \mathcal{M}^{\mathcal{W}}_d \to \mathcal{K}$ (we indeed end up with the same model $\mathcal{K}$ by our assumption on the pair (Λ, Λ^-)). But then

$$(\pi \circ i^{\mathcal{W}}_a \circ i^{\mathcal{U}}_b)\text{"X} = (\tau \circ i^{\mathcal{W}}_d \circ i^{\mathcal{U}}_c)\text{"X}.$$

But by the fact that $(i^{\mathcal{W}}_a \circ i^{\mathcal{U}}_b)$"X $\subseteq \delta(\mathcal{W})$ and $i^{\mathcal{W}}_d \circ i^{\mathcal{U}}_c$"X $\subseteq \delta(\mathcal{W})$ and π agrees with τ above $\delta(\mathcal{W})$, we get

$$(i^{\mathcal{W}}_a \circ i^{\mathcal{U}}_b)\text{"X} = (i^{\mathcal{W}}_d \circ i^{\mathcal{U}}_c)\text{"X}.$$

This gives $ran(i^{\mathcal{W}}_a) \cap ran(i^{\mathcal{W}}_d)$ is cofinal in $\delta(\mathcal{W})$, which implies $a = d$. This in turns easily implies $b = c$. Contradiction. Finally, let $\mathcal{R}$ and $\mathcal{S}$ be Λ-iterates of $\mathcal{Q}$ and let $\Lambda_{\mathcal{R}}$ and $\Lambda_{\mathcal{S}}$ be the tails of Λ on $\mathcal{R}$ and $\mathcal{S}$ respectively. We want to show that $\mathcal{R}$ and $\mathcal{S}$ can be further iterated (using $\Lambda_{\mathcal{R}}$ and $\Lambda_{\mathcal{S}}$ respectively) to the same model. To see this, we compare $\mathcal{R}$ and $\mathcal{S}$ against the Ω-full hod pair construction of some N^*_y (for some y coding $(x, \mathcal{R}, \mathcal{S})$). Then during the comparison, only $\mathcal{R}$ and $\mathcal{S}$ move (to say $\mathcal{R}^*$ and $\mathcal{S}^*$). It's easy to see that $\mathcal{R}^* = \mathcal{S}^*$ and their strategies are the same (as the induced strategy of N^*_y on its appropriate background construction). □

By the properties of $(\mathcal{Q}, \Psi)$ and Λ, we get that $\rho(\mathcal{M}_\infty(\mathcal{Q}, \Lambda)) \leq \Theta$ and $(\mathrm{HOD}|\Theta)^N = \mathcal{M}_\infty(\mathcal{Q}, \Lambda)|\Theta$. Let k be the least such that $\rho_{k+1}(\mathcal{Q}) \leq \delta_{\lambda^{\mathcal{Q}}}$.

Claim. $\mathcal{M}_\infty(\mathcal{Q}, \Lambda) \notin N$

[11]We note here that suppose $(\mathcal{P}, \Sigma)$ is a hod pair and $\mathcal{P} \vDash \delta^{\mathcal{P}}$ has measurable cofinality. Then knowing that all "lower level" strategies of all iterates of $(\mathcal{P}, \Sigma)$ has branch condensation does not tell us that Σ itself has branch condensation.

$$
\begin{tikzcd}
\mathcal{Q} \arrow[r, "i"] \arrow[ddrr, "j"'] & \mathcal{R} \arrow[r, "\mathcal{U}, c"] \arrow[dr, "\mathcal{U}, b"'] & \mathcal{M}^{\mathcal{U}}_c \arrow[r, "\mathcal{W}, d"] & \mathcal{M}^{\mathcal{W}}_d \arrow[r, "\tau"] & \mathcal{K} \\
 & & \mathcal{M}^{\mathcal{U}}_b \arrow[r, "\mathcal{W}, a"] \arrow[d, "\sigma"] & \mathcal{M}^{\mathcal{W}}_a \arrow[ur, "\pi"] & \\
 & & \mathcal{S} & &
\end{tikzcd}
$$

Figure 3.2: The proof of branch condensation of Λ in Lemma 3.1.36

Proof. Suppose not. Let $i : \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ be the direct limit map according to Λ. By an absoluteness argument (i.e. using the absoluteness of the illfoundedness of the tree built in $N[g]$ for $g \subseteq Col(\omega, |\mathcal{M}_\infty(\mathcal{Q}, \Lambda)|)$ generic over N of approximations of a embedding from $\mathcal{Q}$ into $\mathcal{M}_\infty(\mathcal{Q}, \Xi)$ extending the iteration embedding according to $\oplus_{\beta<\lambda^{\mathcal{Q}}}\Psi_\beta$ on $\mathcal{Q}|\delta_{\lambda^{\mathcal{Q}}})$, we get a map π such that

1. $\pi \in N$
2. $\pi : \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$;
3. for each $\beta < \lambda^{\mathcal{Q}}$, $\pi|Q(\beta)$ is according to Ψ_β.
4. $\pi(p) = i(p)$ where $p = p_k(\mathcal{Q})$.

This implies that $\pi = i \in N$ since $\mathcal{Q}$ is $\delta_{\lambda^{\mathcal{Q}}}$-sound and $\rho(\mathcal{Q}) \leq \delta^{\lambda^{\mathcal{Q}}}$. But this map determines Λ in N as follows: let $\mathcal{T} \in N$ be countable and be according to Λ, N can build a tree searching for a cofinal branch b of $\mathcal{T}$ along with an embedding $\sigma : \mathcal{M}^{\mathcal{T}}_b \to \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ such that $\pi = \sigma \circ i^{\mathcal{T}}_b$. Using the fact that Λ has branch condensation, we easily get that $\Lambda \in N$. But this is a contradiction. □

Returning to the proof of (2), let $j =_{def} j_\mu : \text{HOD} \to Ult_0(\text{HOD}, \mu)$ and $\mathcal{W} = j(\langle \mathcal{M}_\beta \mid \beta < \alpha, \beta \text{ is limit}\rangle)(\alpha)$. Let $i : \mathcal{M}_\infty(\mathcal{Q}, \Lambda) \to Ult_k(\mathcal{M}_\infty(\mathcal{Q}, \Lambda), \mu)$ be the canonical map. Note that $A \notin \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$. To see this, assume not, let $\mathcal{R} \lhd \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ be the first level $\mathcal{S}$ of $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ such that A is definable over $\mathcal{S}$.

We claim that $\mathcal{R} \in N$. Recall that $\mathcal{W}$ is the first level of $\mathcal{M}^+_\infty$ such that $j(A) \cap \Theta$ is definable over $\mathcal{W}$. Now let

$$k : \mathcal{R} \to Ult_0(\mathcal{R}, \mu) =_{def} \mathcal{R}^*$$

be the Σ_0-ultrapower map. By the definition of $\mathcal{W}$ and $\mathcal{R}^*$ and the fact that they are both countably iterable, we get that $\mathcal{W} = \mathcal{R}^* \in N$. Let p be the standard parameters for $\mathcal{R}$. In

N, we can compute $Th_0^{\mathcal{R}}(\Theta \cup p)$ as follows: for a formula ψ in the language of hod premice and $s \in \Theta^{<\omega}$,

$$(\psi, s) \in Th_0^{\mathcal{R}}(\Theta \cup p) \Leftrightarrow (\psi, j(s)) \in Th_0^{\mathcal{R}^*}(\Theta \cup k(s)).$$

Since $Th_0^{\mathcal{R}^*}(\Theta \cup k(s)) = Th_0^{\mathcal{W}}(\Theta \cup k(s)) \in N$, $j|\Theta \in N$, and $k(s) \in \mathcal{W} \in N$, we get $Th_0^{\mathcal{R}}(\Theta \cup p) \in N$. This shows $\mathcal{R} \in N$.

To get a contradiction, we show $\mathcal{R} \lhd \mathcal{N}_\infty^+$ by showing $\mathcal{R}$ is satisfies property $(*)$ in N. Let $\mathcal{K}$ be a countable mouse embeddable into $\mathcal{R}$ by a map $k \in N$. Then we can compare $\mathcal{K}$ and $\mathcal{Q}$ against the Ω-full hod pair construction of some N_y^* just like in the argument on the previous page; hence we may assume $\mathcal{K} \lhd \mathcal{Q}$ ($\mathcal{Q} \unlhd \mathcal{K}$ can't happen because then $\Lambda \in N$). The minimality assumption on $\mathcal{Q}$ easily implies $\mathcal{K} \lhd Lp^{\Omega, \oplus_{\gamma<\lambda^{\mathcal{Q}}}\Psi_\gamma}(\mathcal{Q}|\delta_{\lambda^{\mathcal{Q}}})$. But then N can iterate $\mathcal{K}$ for stacks on $\mathcal{K}$ above $\delta_{\lambda^{\mathcal{Q}}} = \delta_{\lambda^{\mathcal{K}}}$, which is what we want to show. The fact that $\mathcal{R} \lhd \mathcal{N}_\infty^+$ contradicts $A \notin \mathcal{N}_\infty^+$.

Next, we note that $Ult_0(\mathrm{HOD}, \mu)|\Theta = Ult_k(\mathcal{M}_\infty(\mathcal{Q}, \Lambda), \mu)|\Theta$ and $i|\Theta = j|\Theta$. Let $\mathcal{R} = Th^{\mathcal{M}_\infty(\mathcal{Q},\Lambda)}(\Theta \cup \{p\})$ where $p = p_k(\mathcal{M}_\infty(\mathcal{Q}, \Lambda))$ and $\mathcal{S} = Th^{Ult_k(\mathcal{M}_\infty(\mathcal{Q},\Lambda),\mu)}(\Theta \cup \{i(p)\})$. We have that $\mathcal{M}_\alpha$ and $\mathcal{S}$ are sound hybrid mice in the same hierarchy, hence by countable iterability, we can conclude either $\mathcal{M}_\alpha \lhd \mathcal{S}$ or $\mathcal{S} \unlhd \mathcal{M}_\alpha$.

If $\mathcal{M}_\alpha \lhd \mathcal{S}$, then $\mathcal{M}_\alpha \in Ult_k(\mathcal{M}_\infty(\mathcal{Q}, \Lambda), \mu)$. This implies $A \in \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ by a computation similar to that in the proof of (1), i.e.

$$\beta \in A \Leftrightarrow \mathcal{M}_\infty(\mathcal{Q}, \Lambda) \vDash (i|\Theta)(\beta) \in \mathcal{M}_\alpha.$$

This is a contradiction to the fact that $A \notin \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$. Now suppose $\mathcal{S} \unlhd \mathcal{M}_\alpha$. This then implies $\mathcal{S} \in Ult_0(\mathrm{HOD}, \mu)$, which in turns implies $\mathcal{M}_\infty(\mathcal{Q}, \Lambda) \in \mathrm{HOD}$ by the following computation: for any formula ϕ and $s \in \Theta^{<\omega}$,

$$(\phi, s) \in \mathcal{R} \Leftrightarrow \mathrm{HOD} \vDash (\phi, (j|\Theta)(s)) \in \mathcal{S}.$$

This is a contradiction to the claim. This completes the proof of (2). □

Theorem 3.1.35 completes our analysis of HOD for determinacy models of the form "$V = L(\mathcal{P}(\mathbb{R}))$ below "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular."

3.2 When $V = L(\mathbb{R}, \mu)$

We prove the case $\alpha = 0$. The other cases are similar. Throughout this section, we assume $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+ + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. We'll be using the following theorem.

Theorem 3.2.1 (Woodin). *Suppose $L(\mathbb{R}, \mu) \vDash AD^+ + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then in $L(\mathbb{R}, \mu)$, there is a set $A \subseteq \Theta$ such that $HOD = L[A]$.*

3.2.1 $\mathrm{HOD}^{L(\mathbb{R},\mu)}$ with $\mathcal{M}^\sharp_{\omega^2}$

We assume μ comes from the club filter, $\mathcal{M}^\sharp_{\omega^2}$ exists and has Hom_∞ iteration strategy. We'll show how to get rid of these assumptions later on. We first show how to iterate $\mathcal{M}_{\omega^2}$ to realize μ as the tail filter.

Lemma 3.2.2. *There is an iterate $\mathcal{N}$ of $\mathcal{M}_{\omega^2}$ such that letting λ be the limit of $\mathcal{N}$'s Woodin cardinals, $\mathbb{R}$ can be realized as the symmetric reals over $\mathcal{N}$ at λ and letting $\mathcal{F}$ be the tail filter over $\mathcal{N}$ at λ, $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F})$.*

Proof. Let δ_i be the sup of the first ωi Woodin cardinals of $\mathcal{M}_{\omega^2}$ and $\gamma = \sup_i \delta_i$. Let $\xi \geq \omega_1$ be such that $H(\xi) \vDash \mathsf{ZFC}^-$. In $V^{Col(\omega,H(\xi))}$, let $\langle X_i \mid i < \omega\rangle$ be an increasing and cofinal chain of countable (in V) elementary substructures of $H(\xi)$ and $\sigma_i = \mathbb{R}\cap X_i$. To construct the $\mathcal{N}$ as in the statement of the lemma, we do an $\mathbb{R}$-genericity iteration (in $V^{Col(\omega,H(\xi))}$) as follows. Let $\mathcal{P}_0 = \mathcal{M}^\sharp_{\omega^2}$ and assume $\mathcal{P}_0 \in X_0$. For $i > 0$, let $\mathcal{P}_i$ be the result of iterating $\mathcal{P}_{i-1}$ in X_{i-1} in the window between the $\omega(i-1)^{th}$ and ωi^{th} Woodin cardinals of $\mathcal{P}_{i-1}$ to make σ_{i-1} generic. We can make sure that each finite stage of the iteration is in X_{i-1}. Let $\mathcal{P}_\omega$ be obtained from the direct limit of the $\mathcal{P}_i$'s and iterating the top extender out of the universe. Let λ be the limit of Woodin cardinals in $\mathcal{P}_\omega$. It's clear that there is a $G \subseteq Col(\omega, <\lambda)$ generic over $\mathcal{P}_\omega$ such that $\mathbb{R} =_{def} \mathbb{R}^V$ is the symmetric reals over $\mathcal{P}_\omega$ and $L(\mathbb{R},\mu)$ is in $\mathcal{P}_\omega[G]$. Let $\mathcal{F}$ be the tail filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ defined over $\mathcal{P}_\omega[G]$. By section 2, $L(\mathbb{R},\mathcal{F}) \vDash \mathcal{F}$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

We want to show $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F})$. To show this, it's enough to see that if $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ is in $L(\mathbb{R},\mu)$ and A is a club then $A \in \mathcal{F}$. Let $\pi : \mathbb{R}^{<\omega} \to \mathbb{R} \in V$ witness that A is a club. By the choice of the X_i's, there is an n such that for all $m \geq n$, $\pi \in X_m$ and hence $\pi''\sigma_m^{<\omega} \subseteq \sigma_m$. This shows $A \in \mathcal{F}$. This in turns implies $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F})$. □

Let $\mathcal{M}^+_\infty$ be the direct limit of all iterates of $\mathcal{M}_{\omega^2}$ below the first Woodin cardinal and $\mathcal{H}^+$ be the corresponding direct limit system. We'll define a direct limit system $\mathcal{H}$ in $L(\mathbb{R},\mu)$ that approximates $\mathcal{H}^+$. Working in $L(\mathbb{R},\mu)$, we say $\mathcal{P}$ is suitable if it is full (with respect to mice), has only one Woodin cardinal $\delta^\mathcal{P}$ and $\mathcal{P} = Lp_\omega(\mathcal{P}|\delta^\mathcal{P})$. The following definition comes from Definition 6.21 in [41].

Definition 3.2.3. *Working in $L(\mathbb{R},\mu)$, we let $\mathcal{O}$ be the collection of all functions f such that f is an ordinal definable function with domain the set of all countable, suitable $\mathcal{P}$, and $\forall \mathcal{P} \in \mathrm{dom}(f)(f(\mathcal{P}) \subseteq \delta^\mathcal{P})$.*

Definition 3.2.4. *Suppose $\vec{f} \in \mathcal{O}^{<\omega}$, $\mathcal{P}$ is suitable, and $\mathrm{dom}(\vec{f}) = n$. Let*

$$\gamma_{(\mathcal{P},\vec{f})} = \sup\{Hull^\mathcal{P}(\vec{f}(0)(\mathcal{P}),\cdots,\vec{f}(n-1)(\mathcal{P}))\cap\delta^\mathcal{P}\},$$

and

$$H_{(\mathcal{P},\vec{f})} = Hull^\mathcal{P}(\gamma_{(\mathcal{P},\vec{f})}\cup\{\vec{f}(0)(\mathcal{P}),\cdots,\vec{f}(n-1)(\mathcal{P})\}).$$

We refer to reader to Section 6.3 of [41] for the definitions of $\vec{f}$-iterability, strong $\vec{f}$-iterability. The only difference between our situation and the situation in [41] is that our notions of "suitable", "short", "maximal", "short tree iterable" etc. are relative to the pointclass $(\Sigma_1^2)^{L(\mathbb{R},\mu)}$ instead of $(\Sigma_1^2)^{L(\mathbb{R})}$ as in [41].

Now, let $(\mathcal{P},\vec{f}) \in \mathcal{H}$ if $\mathcal{P}$ is strongly $\vec{f}$-iterable. The ordering on $\mathcal{H}$ is defined as follows:

$$(\mathcal{P},\vec{f}) \leq_{\mathcal{H}} (\mathcal{Q},\vec{g}) \Leftrightarrow \vec{f} \subseteq \vec{g} \wedge \mathcal{Q} \text{ is a psuedo-iterate of } \mathcal{P}^{12}.$$

Note that if $(\mathcal{P},\vec{f}) \leq_{\mathcal{H}} (\mathcal{Q},\vec{g})$ then there is a natural embedding $\pi_{(\mathcal{P},\vec{f}),(\mathcal{Q},\vec{g})} : H_{\mathcal{P},\vec{f}} \rightarrow H_{\mathcal{Q},\vec{g}}$. We need to see that $\mathcal{H} \neq \emptyset$.

Lemma 3.2.5. *Let $\vec{f} \in \mathcal{O}^{<\omega}$. Then there is a $\mathcal{P}$ such that $(\mathcal{P},\vec{f}) \in \mathcal{H}$.*

Proof. For simplicity, assume $\mathrm{dom}(\vec{f}) = 1$. The proof of this lemma is just like the proof of Theorem 6.29 in [41]. We only highlight the key changes that make that proof work here.

First let $\nu, \mathbb{P}$ be as in the proof of Lemma 2.1.6. Let a be a countable transitive self-wellordered set and x be a real that codes a. We need to modify the Q_a^x defined in the proof of Lemma 2.1.6 to the structure defined along the line of Subsection 3.1.1.1. Fix a coding of relativized premice by reals and write $\mathcal{P}_z$ for the premouse coded by z. Then let

$$\mathcal{F}_a^x = \{\mathcal{P}_z \mid z \leq_T x \text{ and } \mathcal{P}_z \text{ is a suitable premouse over } a \text{ and } \mathcal{P}_z \text{ is short-tree iterable}\}.$$

Let

$$\mathcal{Q}_a^x = Lp(\mathcal{Q}_a^{x,-}),$$

where $\mathcal{Q}_a^{x,-}$ is the direct limit of the simultaneous comparison and $\{y \mid y \leq_T x\}$-genericity iteration of all $\mathcal{P} \in \mathcal{F}_a^x$. The definition of $\mathcal{Q}_a^x$ comes from Section 6.6 of [41]. As in the proof of Lemma 2.1.6, we have:

1. letting $\langle \vec{d^i} \mid i < \omega \rangle$ be the generic sequence for $\mathbb{P}$ and $\langle \mathcal{Q}_j^i \mid i,j < \omega \rangle$ be the sequence of models associated to $\langle \vec{d^i} \mid i < \omega \rangle$ as defined in the proof of Lemma 2.1.6, we have that the model $N = L[T^\infty, \mathcal{M}^{\langle \vec{d^i} \rangle_i}] \vDash$ "there are ω^2 Woodin cardinals", where $\mathcal{M}^{\langle \vec{d^i} \rangle_i} = L[\cup_i \cup_j \mathcal{Q}_j^i]$;

2. letting λ be the sup of the Woodin cardinals of N, there is a $G \subseteq Col(\omega, <\lambda)$, G is N-generic such that letting $\mathbb{R}_G^*$ be the symmetric reals of $N[G]$ and $\mathcal{F}$ be the tail filter defined over $N[G]$, then $L(\mathcal{R}_G^*, \mathcal{F}) = L(\mathbb{R},\mu)$ and $\mathcal{F} \cap L(\mathbb{R},\mu) = \mu$.

The second key point is that whenever $\mathcal{P}$ is an iterate of $\mathcal{M}_{\omega^2}$, we can then iterate $\mathcal{P}$ to $\mathcal{Q}$ (above any Woodin cardinal of $\mathcal{P}$) so that $\mathbb{R}^V$ can be realized as the symmetric reals for some $G \subseteq Col(\omega, <\delta_{\omega^2}^{\mathcal{Q}})$ and $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F})$ and $\mu \cap L(\mathbb{R},\mu) = \mathcal{F} \cap L(\mathbb{R},\mu)$, where $\mathcal{F}$ is the tail filter defined over $\mathcal{Q}[G]$. This is proved in Lemma 3.2.2.

[12] See definition 6.20 of [41] for the definition of psuedo-iterate.

We leave it to the reader to check that the proof of Theorem 6.29 of [41] goes through for our situation. This completes our sketch. □

Remark: The lemma above obviously shows $\mathcal{H} \neq \emptyset$. Its proof also shows for any $\vec{f} \in \mathcal{O}^{<\omega}$ and any $(\mathcal{P}, \vec{g}) \in \mathcal{H}$, there is a $\vec{g}$-iterate $\mathcal{Q}$ of $\mathcal{P}$ such that $\mathcal{Q}$ is $(\vec{f} \cup \vec{g})$-strongly iterable.

Now we outline the proof that $\mathcal{M}_\infty^+ \subseteq \mathrm{HOD}^{L(\mathbb{R},\mu)}$. We follow the proof in Section 6.7 of [41]. Suppose $\mathcal{P}$ is suitable and $s \in [\mathrm{OR}]^{<\omega}$, let $\mathcal{L}_{\mathcal{P},s}$ be the language of set theory expanded by constant symbols c_x for each $x \in \mathcal{P}|\delta^{\mathcal{P}} \cup \{\mathcal{P}\}$ and d_x for each x in the range of s. Since s is finite, we can fix a coding of the syntax of $\mathcal{L}_{\mathcal{P},s}$ such that it is definable over $\mathcal{P}|\delta^{\mathcal{P}}$ and the map $x \mapsto c_x$ is definable over $\mathcal{P}|\delta^{\mathcal{P}}$. We continue to use $\mathbb{P}$ to denote the Prikry forcing in Lemma 2.1.6.

Definition 3.2.6. *Let $\mathcal{P}$ be suitable and $s = \{\alpha_1, \cdots, \alpha_n\}$. We set*

$$T_s(\mathcal{P}) = \{\phi \in \mathcal{L}_{\mathcal{P},s} \mid \exists p \in \mathbb{P}\ (p = (\emptyset, X) \wedge p \Vdash (\mathcal{M}^{\vec{d}_{\dot{G}}}, \alpha_1, \cdots, \alpha_n, x)_{x \in \mathcal{P}|\delta^{\mathcal{P}}} \vDash \phi\}.$$

In the above definition, $\mathcal{M}^{\vec{d}_{\dot{G}}}$ is the canonical name for the model $\mathcal{M}^{\langle \vec{d^i} \rangle_i}$ defined in Lemma 3.2.5 where $\langle \vec{d^i} \rangle_i$ is the Prikry sequence given by a generic $G \subseteq \mathbb{P}$. Note that $T_s(\mathcal{P})$ is a complete, consistent theory of $\mathcal{L}_{\mathcal{P},s}$ and if $s \subseteq t$, we can think of $T_s(\mathcal{P})$ as a subtheory of $T_t(\mathcal{P})$ in a natural way (after appropriately identifying the constant symbols of one with those of the other). Furthermore, $T_s \in \mathcal{O}$ for any $s \in [\mathrm{OR}]^{<\omega}$.

Let $\mathcal{N}_\infty$ be the direct limit of $\mathcal{H}$ under maps $\pi_{(\mathcal{P},\vec{f}),(\mathcal{Q},\vec{g})}$ for $(\mathcal{P}, \vec{f}) \leq_{\mathcal{H}} (\mathcal{Q}, \vec{g})$. Let $\pi_{(\mathcal{P},\vec{f}),\infty} : H_{\mathcal{P},\vec{f}} \to \mathcal{N}_\infty$ be the direct limit map. For each $s \in [\mathrm{OR}]^{<\omega}$ and $\mathcal{P}$ which is strongly T_s-iterable, we let

$$T_s^* = \pi_{(\mathcal{P},T_s),\infty}(T_s(\mathcal{P})).$$

Again, $s \subseteq t$ implies $T_s^* \subseteq T_t^*$, so we let

$$T^* = \bigcup\{T_s^* \mid s \in [\mathrm{OR}]^{<\omega}\}.$$

We have that T^* is a complete, consistent, and Skolemized[13] theory of $\mathcal{L}$, where $\mathcal{L} = \bigcup\{\mathcal{L}_{\mathcal{N}_\infty,s} \mid s \in [\mathrm{OR}]^{<\omega}\}$. We note that T^* is definable in $L(\mathbb{R}, \mu)$ because the map $s \mapsto T_s^*$ is definable in $L(\mathbb{R}, \mu)$.

Let $\mathcal{A}$ be the unique pointwise definable $\mathcal{L}$-structure such that $\mathcal{A} \vDash T^*$. We show $\mathcal{A}$ is wellfounded and let $\mathcal{N}_\infty^+$ be the transitive collapse of $\mathcal{A}$, restricted to the language of premice.

Lemma 3.2.7. $\mathcal{N}_\infty^+ = \mathcal{M}_\infty^+$

[13] This is because of the Prikry property of $\mathbb{P}$.

Proof. We sketch the proof which completely mirrors the proof of Lemma 6.51 in [41]. Let Σ be the iteration strategy of $\mathcal{M}_{\omega^2}$ and $\Sigma_{\mathcal{P}}$ be the tail of Σ for a Σ-iterate $\mathcal{P}$ of $\mathcal{M}_{\omega^2}$. We will also use $\langle\delta^{\mathcal{P}}_\alpha \mid \alpha < \omega^2\rangle$ to denote the Woodin cardinals of a Σ-iterate $\mathcal{P}$ of $\mathcal{M}_{\omega^2}$. We write $\mathcal{P}^- = \mathcal{P}|((\delta^{\mathcal{P}}_0)^{+\omega})^{\mathcal{P}}$. Working in $V^{Col(\omega,\mathbb{R})}$, we define sequences $\langle\mathcal{N}_k \mid k < \omega\rangle$, $\langle\mathcal{N}^\omega_k \mid k < \omega\rangle$, $\langle j_{k,l} \mid k \leq l \leq \omega\rangle$, $\langle i_k \mid k < \omega\rangle$, $\langle G_k \mid k < \omega\rangle$, and $\langle j^\omega_{k,l} \mid k \leq l \leq \omega\rangle$ such that

(a) $\mathcal{N}_k \in \mathcal{H}^+$ for all k;

(b) for all k, $\mathcal{N}_{k+1}$ is a $\Sigma_{\mathcal{N}_k}$-iterate of $\mathcal{N}_k$ (below the first Woodin cardinal of $\mathcal{N}_k$) and the corresponding iteration map is $j_{k,k+1}$;

(c) the $\mathcal{N}_k$'s are cofinal in $\mathcal{H}^+$;

(d) $i_k : \mathcal{N}_k \to \mathcal{N}^\omega_k$ is an iteration map according to $\Sigma_{\mathcal{N}_k}$ with critical point $> \delta^{\mathcal{N}_k}_0$;

(e) G_k is generic over $\mathcal{N}^\omega_k$ for the symmetric collapse up to the sup of its Woodins and $\mathbb{R}^*_{G_k} = \mathbb{R}^V$;

(f) $\mathcal{N}^\omega_k = \mathcal{M}^{\langle\vec{d^i}\rangle_i}$ for some $\langle\vec{e^i}\rangle_i$ which is $\mathbb{P}$-generic over $L(\mathbb{R},\mu)$ such that $(\mathcal{N}^\omega_k)^-$ is coded by a real in $\vec{e}^0(0)$;

(g) $j^\omega_{k,k+1} : \mathcal{N}^\omega_k \to \mathcal{N}^\omega_{k+1}$ is the iteration map;

(h) for $k < l$, $j^\omega_{k,l} \circ i_k = i_l \circ j_{k,l}$, where $j_{k,l} : \mathcal{N}_k \to \mathcal{N}_l$ and $j^\omega_{k,l} : \mathcal{N}^\omega_k \to \mathcal{N}^\omega_l$ are natural maps;

(i) $j_{k,k+1}|\mathcal{N}^-_k = j^\omega_{k,k+1}|(\mathcal{N}^\omega_k)^-$;

(j) the direct limit $\mathcal{N}^\omega_\omega$ of the $\mathcal{N}^\omega_k$ under maps $j^\omega_{k,l}$'s embeds into a $\Sigma_{\mathcal{M}^+_\infty}$-iterate of $\mathcal{M}^+_\infty$;

(k) for each $s \in [\mathrm{OR}]^{<\omega}$, for all sufficiently large k,

$$\mathcal{N}^\omega_k \vDash \phi[x,s] \Leftrightarrow \exists p \in \mathbb{P}\ (p = (\emptyset, X) \wedge p \Vdash (\mathcal{M}^{\vec{d}_{\dot{G}}} \vDash \phi[x,s]),$$

for $x \in \mathcal{N}^\omega_k|\delta^{\mathcal{N}^\omega_k}_0$.

Everything except for (f) is as in the proof of Lemma 6.51 of [41]. To see (f), fix a $k < \omega$. We fix a Prikry sequence $\langle\vec{d^i}\rangle_i$ such that $\mathcal{N}^\omega_k$ is coded into $\vec{d}^0(0)$ and letting $\sigma_i = \{y \in \mathbb{R}^V \mid y$ is recursive in $\vec{d^i}(j)$ for some $j < \omega\}$, then for each i, σ_i is closed under the iteration strategy $\Sigma_{\mathcal{N}^\omega_k}$. We then (inductively) for all i, construct a sequence $\langle\vec{e^i} \mid i < \omega\rangle$ such that $\vec{e^i}$ is a Prikry generic subsequence of $\vec{d^i}$ such that $M^{\langle\vec{e^i}\rangle_i}$ is an iterate of $\mathcal{M}_{\omega^2}$ (see Lemma 6.49 of [41]). The sequence $\langle\vec{e^i}\rangle_i$ satisfies (f) for $\mathcal{N}^\omega_k$.

Having constructed the above objects, the proof of Lemma 6.51 in [41] adapts here to give an isomorphism between $\mathcal{A}$ (viewed as a structure for the language of premice) and $\mathcal{M}^+_\infty$. The isomorphism is the unique extension to all of $\mathcal{A}$ of the map σ, where $\sigma(c^{\mathcal{A}}_x) = x$

(for $x \in \mathcal{M}_\infty^+|\delta_0^{\mathcal{M}_\infty^+}$) and $\sigma(d_\alpha^{\mathcal{A}}) = j_{k,\omega}^\omega(\alpha)$ for k large enough such that $j_{l,l+1}^\omega(\alpha) = \alpha$ for all $l \geq k$. This completes our sketch. □

Now we continue with the sketch of the proof that $\mathrm{HOD}^L(\mathbb{R}, \mu)$ is a strategy mouse in the presence of $\mathcal{M}_{\omega^2}^\sharp$. Let Λ_∞ be the supremum of the Woodin cardinals of $\mathcal{M}_\infty^+$. Let $\mathbb{R}^*$ be the reals of the symmetric collapse of a $G \subseteq Col(\omega, < \lambda_\infty)$ generic over $\mathcal{M}_\infty^+$ and $\mathcal{F}^*$ be the corresponding tail filter defined in $\mathcal{M}_\infty^+[G]$. Since $L(\mathbb{R}^*, \mathcal{F}^*) \equiv L(\mathbb{R}, \mu)$, its has its own version of $\mathcal{H}$ and $\mathcal{N}_\infty^+$, so we let

$$\mathcal{H}^* = \mathcal{H}^{L(\mathbb{R}^*, \mathcal{F}^*)} \text{ and } (\mathcal{N}_\infty^+)^* = (\mathcal{N}_\infty^+)^{L(\mathbb{R}^*, \mathcal{F}^*)}.$$

Let Λ be the restriction of $\Sigma_{\mathcal{M}_\infty^+}$ to stacks $\vec{\mathcal{T}} \in \mathcal{M}_\infty^+|\lambda_\infty$, where

- $\vec{\mathcal{T}}$ is based on $\mathcal{M}_\infty^+|\delta_0^{\mathcal{M}_\infty^+}$;
- $L(\mathbb{R}^*, \mathcal{F}^*) \vDash \vec{\mathcal{T}}$ is a finite full stack[14].

We show $L[\mathcal{M}_\infty^+, \Lambda] = \mathrm{HOD}^{L(\mathbb{R},\mu)}$ through a sequence of lemmas. For an ordinal α, put

$$\alpha^* = d_\alpha^{\mathcal{A}},$$

and for $s = \{\alpha_1, \cdots, \alpha_n\}$ a finite set of ordinals, put

$$s^* = \{\alpha_1^*, \cdots, \alpha_n^*\}.$$

Lemma 3.2.8 (Derived model resemblance)**.** *Let $(\mathcal{P}, \vec{f}) \in \mathcal{H}$ and $\overline{\eta} < \gamma_{(\mathcal{P},\vec{f})}$, and $\eta = \pi_{(\mathcal{P},\vec{f}),\infty}(\overline{\eta})$. Let $s \in [\mathrm{OR}]^{<\omega}$, and $\phi(v_0, v_1, v_2)$ be a formula in the language of set theory; then the following are equivalent*

(a) $L(\mathbb{R}^*, \mathcal{F}^*) \vDash \phi[\mathcal{M}_\infty, \eta, s^*]$;

(b) $L(\mathbb{R}, \mu) \vDash$ "there is an $(\mathcal{R}, \vec{f}) \geq_{\mathcal{F}} (\mathcal{P}, \vec{f})$ such that whenever $(\mathcal{Q}, \vec{f}) \geq_{\mathcal{H}} (\mathcal{R}, \vec{f})$, then $\phi(\mathcal{Q}, \pi_{(\mathcal{P},\vec{f}),(\mathcal{Q},\vec{f})}(\overline{\eta}), s)$".

The proof of this lemma is almost exactly like the proof of Lemma 6.54 of [41], so we omit it. The only difference is in Lemma 6.54 of [41], the proof of Lemma 6.51 of [41], here we use Lemma 4.2.18.

Lemma 3.2.9. *Λ is definable over $L(\mathbb{R}, \mu)$, and hence $L[\mathcal{M}_\infty^+, \Lambda] \subseteq HOD^{L(\mathbb{R},\mu)}$*

Proof. Suppose $f \in \mathcal{O}$ is definable in $L(\mathbb{R}, \mu)$ by a formula ψ and $s \in [\mathrm{OR}]^{<\omega}$, then we let $f^* \in \mathcal{O}^{L(\mathbb{R}^*, \mathcal{F}^*)}$ be definable in $L(\mathbb{R}^*, \mathcal{F}^*)$ from ψ and s^*.

Sublemma 3.2.10. *Let $\vec{\mathcal{T}}$ be a finite full stack on $\mathcal{M}_\infty^+|\delta_0^{\mathcal{M}_\infty^+}$ in $L(\mathbb{R}^*, \mathcal{F}^*)$ and let $\vec{b} = \Sigma_{\mathcal{M}_\infty^+}(\vec{\mathcal{T}})$. Then $\vec{b}$ respects f^*, for all $f \in \mathcal{O}$.*

[14] See Definition 6.20 of [41] for the precise definition of finite full stacks.

The proof of Sublemma 3.2.10 is just like that of Claim 6.57 in [41] (with appropriate use of the proof of Lemma 4.2.18. Sublemma 3.2.10 implies $\mathcal{M}_\infty$ is strongly f^*-iterable in $L(\mathbb{R}^*, \mathcal{F}^*)$ for all $f \in \mathcal{O}$. Sublemma 3.2.10 also gives the following.

Sublemma 3.2.11. *Suppose $\mathcal{Q}$ is a psuedo-iterate*[15] *of $\mathcal{M}_\infty$ and $\mathcal{T}$ is a maximal tree on $\mathcal{Q}$ in the sense of $L(\mathbb{R}^*, \mathcal{F}^*)$. Let $b = \Lambda(\mathcal{T})$; then for all $\eta < \delta^{\mathcal{Q}}$, the following are equivalent:*

(a) $i_b^{\mathcal{T}}(\eta) = \xi$;

(b) there is some $f \in \mathcal{O}$ such that $\eta < \gamma_{(\mathcal{Q}, f^*)}$ and exists some branch choice[16] *of $\mathcal{T}$ that respects f^* and $i_c^{\mathcal{T}}(\eta) = \xi$.*

Since the $\gamma_{(\mathcal{Q}, f^*)}$'s sup up to $\delta^{\mathcal{Q}}$ and i_b is continuous at $\delta^{\mathcal{Q}}$, clause (b) defines Λ over $L(\mathbb{R}, \mu)$. $\square$

We have an iteration map

$$\pi_\infty : \mathcal{N}_\infty \to \mathcal{N}_\infty^*$$

which is definable over $L(\mathbb{R}^*, \mathcal{F}^*)$ by the equality

$$\pi_\infty = \cup_{f \in \mathcal{O}} \pi^{\mathcal{H}^*}_{(\mathcal{N}_\infty, f^*), \infty}.$$

By Boolean comparison, $\pi_\infty \in L[\mathcal{M}_\infty^+, \Lambda]$. This implies $\mathcal{N}_\infty^*$ is the direct limit of all Λ-iterates of $\mathcal{N}_\infty$ which belong to $\mathcal{M}_\infty^+$ and π_∞ is the canonical map into the direct limit. Lemma 3.2.8 also gives us the following.

Lemma 3.2.12. *For all $\eta < \delta_0^{\mathcal{M}_\infty^+}$, $\pi_\infty(\eta) = \eta^*$.*

Finally, we have

Theorem 3.2.13. *Suppose $\mathcal{M}_{\omega^2}^\sharp$ exists and is $(\omega, \mathrm{OR}, \mathrm{OR})$-iterable. Suppose μ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+ + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then the following models are equal:*

1. *$HOD^{L(\mathbb{R}, \mu)}$,*
2. *$L[\mathcal{M}_\infty^+, \pi_\infty]$,*
3. *$L[\mathcal{M}_\infty^+, \Lambda]$.*

Proof. Since $\pi_\infty \in L[\mathcal{M}_\infty^+, \Lambda]$, $L[\mathcal{M}_\infty^+, \pi_\infty] \subseteq L[\mathcal{M}_\infty^+, \Lambda]$. Lemma 3.2.9 implies $L[\mathcal{M}_\infty^+, \Lambda] \subseteq \mathrm{HOD}^{L(\mathbb{R}, \mu)}$. It remains to show $\mathrm{HOD}^{L(\mathbb{R}, \mu)} \subseteq L[\mathcal{M}_\infty, \pi_\infty]$. By Theorem 3.2.1, in $L(\mathbb{R}, \mu)$, there is some $A \subseteq \Theta$ such that $\mathrm{HOD} = L[A]$. Let ϕ define A. By Lemma 3.2.8

$$\alpha \in A \Leftrightarrow L[\mathcal{M}_\infty^+, \pi_\infty] \vDash \mathcal{M}_\infty^+ \vDash (1 \Vdash L(\mathbb{R}^*, \mathcal{F}^*) \vDash \phi[\alpha^*]).$$

By Lemma 3.2.12, $\alpha^* = \pi_\infty(\alpha)$ and hence the above equivalence defines A over $L[\mathcal{M}_\infty^+, \pi_\infty]$. This completes the proof of the theorem. $\square$

[15] See Definition 6.13 of [41].
[16] See Definition 6.23 of [41].

3.2.2 $\mathrm{HOD}^{L(\mathbb{R},\mu)}$ without $\mathcal{M}^{\sharp}_{\omega^2}$

We now describe how to compute HOD just assuming $V = L(\mathbb{R},\mu)$ satisfying AD^+. Let $\mathcal{H}$ be as above. The idea is that we use Σ_1 reflection to reflect a "bad" statement φ (like "$\mathcal{N}^+_\infty$ is illfounded" or "HOD $\neq L(\mathcal{N}^+_\infty,\Lambda)$") to a level $L_\kappa(\mathbb{R},\mu)$ where $\kappa < \underset{\sim}{\delta}^2_1$ (i.e. we have that $L_\kappa(\mathbb{R},\mu) \vDash \varphi$). But then since $\mu \cap L_\kappa(\mathbb{R},\mu)$ comes from the club filter, all we need to compute HOD in $L_\kappa(\mathbb{R},\mu)$ is to construct a mouse $\mathcal{N}$ related to N just like $M^{\sharp}_{\omega^2}$ related to $L(\mathbb{R},\mu)$. Once the mouse $\mathcal{N}$ is constructed, we sucessfully compute HOD of $L_\kappa(\mathbb{R},\mu)$ and hence show that $L_\kappa(\mathbb{R},\mu) \vDash \neg\varphi$. This gives us a contradiction.

We now proceed to construct $\mathcal{N}$. To be concrete, we fix a "bad" statement φ (like "HOD is illfounded") and let $N = L_\kappa(\mathbb{R},\mu)$ be least such that $N \vDash (T)$ where $(T) \equiv$ "$\mathsf{MC} + \mathsf{AD}^+ + \mathsf{DC} + \mathsf{ZF}^- + \Theta = \theta_0 + \varphi$". Let $\Phi = (\Sigma^2_1)^N$, $\Gamma^* = \mathcal{P}(\mathbb{R})^N$ and U be the universal Φ-set. We have that Φ is a good pointclass and $Env(\underset{\sim}{\Phi}) = \Gamma^*$ by closure of N. Let $\vec{B} = \langle B_i \mid i < \omega\rangle$ be a sjs sealing $Env(\underset{\sim}{\Phi})$ with each $B_i \in N$ and $B_0 = U$. Such a $\vec{B}$ exists (see Section 4.1 of [46]).

Because MC holds and $\Gamma^* \subsetneq \underset{\sim}{\Delta}^2_1$, there is a real x such that there is a sound mouse $\mathcal{M}$ over x such that $\rho(\mathcal{M}) = x$ and $\mathcal{M}$ doesn't have an iteration strategy in N. Fix then such an $(x,\mathcal{M})$ and let Σ be the strategy of $\mathcal{M}$. Let $\Gamma \subsetneq \underset{\sim}{\Delta}^2_1$ be a good pointclass such that $Code(\Sigma), \vec{B}, U, U^c \in \underset{\sim}{\Delta}_\Gamma$. By Theorem 10.3 in [35], there is a z such that $(\mathcal{N}^*_z, \delta_z, \Sigma_z)$ Suslin captures $Code(\Sigma), \vec{B}, U, U^c$.

Because $\vec{B}$ is Suslin captured by $\mathcal{N}^*_z$, we have $(\delta_z^+)^{\mathcal{N}^*_z}$-complementing trees $T, S \in \mathcal{N}^*_z$[17] which capture $\vec{B}$. Let κ be the least cardinal of $\mathcal{N}^*_z$ which, in $\mathcal{N}^*_z$ is $< \delta_z$-strong.

Claim 1. $\mathcal{N}^*_z \vDash$ "κ is a limit of points η such that $Lp^{\Gamma^*}(\mathcal{N}^*_z|\eta) \vDash$ "η is Woodin".

Proof. The proof is an easy reflection argument. Let $\lambda = \delta_z^+$ and let $\pi : M \to \mathcal{N}^*_z|\lambda$ be an elementary substructure such that

1. $T, S \in ran(\pi)$,
2. if $\mathrm{cp}(\pi) = \eta$ then $V_\eta^{\mathcal{N}^*_z} \subseteq M$, $\pi(\eta) = \delta_z$ and $\eta > \kappa$.

By elementarity, we have that $M \vDash$ "η is Woodin". Letting $\pi^{-1}(\langle T, S\rangle) = \langle \bar{T}, \bar{S}\rangle$, we have that $(\bar{T}, \bar{S})$ Suslin captures $\vec{B}$ over M at η. This implies that M is Φ-full and in particular, $Lp^{\Gamma^*}(\mathcal{N}^*_z|\eta) \in M$. Therefore, $Lp^{\Gamma^*}(\mathcal{N}^*_z|\eta) \vDash$ "η is Woodin". The claim then follows by a standard argument. □

Let now $\langle \eta_i : i < \omega^2\rangle$ be the first ω^2 points $< \kappa$ such that for every $i < \omega$, $Lp^{\Gamma^*}(\mathcal{N}^*_z|\eta_i) \vDash$ "η_i is Woodin". Let now $\langle \mathcal{N}_i : i < \omega^2\rangle$ be a sequence constructed according to the following rules:

[17] This means that whenever g is $< (\delta_z^+)^{\mathcal{N}^*_z}$-generic over $\mathcal{N}^*_z$, then in $\mathcal{N}^*_z[g]$, $p[T]$ and $p[S]$ project to complements.

1. $\mathcal{N}_0 = L[\vec{E}]^{\mathcal{N}_z^*|\eta_0}$,
2. if i is limit, $\mathcal{N}_i' = \cup_{j<i}\mathcal{N}_i$ and $\mathcal{N}_i = (L[\vec{E}][\mathcal{N}_i'])^{\mathcal{N}_z^*|\eta_i}$,
3. $\mathcal{N}_{i+1} = (L[\vec{E}][\mathcal{N}_i])^{\mathcal{N}_z^*|\eta_{i+1}}$.

Let $\mathcal{N}_{\omega^2} = \cup_{i<\omega^2}\mathcal{N}_i$.

Claim 2. For every $i < \omega^2$, $\mathcal{N}_{\omega^2} \vDash$ "η_i is Woodin" and $\mathcal{N}_{\omega^2}|(\eta_i^+)^{\mathcal{N}_\omega} = Lp^{\Gamma^*}(\mathcal{N}_i)$.

Proof. It is enough to show that

1. $\mathcal{N}_{i+1} \vDash$ "η_i is Woodin",
2. $\mathcal{N}_i = V_{\eta_i}^{\mathcal{N}_{i+1}}$,
3. $\mathcal{N}_{i+1}|(\eta_i^+)^{\mathcal{N}_{i+1}} = Lp^{\Gamma^*}(\mathcal{N}_i)$,
4. if i is limit, then $\mathcal{N}_i|((\sup_{j<i}\eta_j^+)^{\mathcal{N}_i}) = Lp^{\Gamma^*}(\mathcal{N}_i')$.

To show 1-4, it is enough to show that if $\mathcal{W} \trianglelefteq \mathcal{N}_{i+1}$ is such that $\rho_\omega(W) \leq \eta_i$ or if i is limit and $\mathcal{W} \triangleleft \mathcal{N}_i$ is such that $\rho_\omega(W) \leq \sup_{j<i}\eta_j$ then the fragment of $\mathcal{W}$'s iteration strategy which acts on trees above η_i ($\sup_{j<i}\eta_j$ respectively) is in Γ^*. Suppose first that i is a successor and $\mathcal{W} \trianglelefteq \mathcal{N}_{i+1}$ is such that $\rho_\omega(W) \leq \eta_i$. Let ξ be such that the if $\mathcal{S}$ is the ξth model of the full background construction producing $\mathcal{N}_{i+1}$ then $\mathbb{C}(\mathcal{S})$[18] $= \mathcal{W}$. Let $\pi : \mathcal{W} \to \mathcal{S}$ be the core map. It is a fine-structural map but that it irrelevant and we surpass this point. The iteration strategy of $\mathcal{W}$ is the π-pullback of the iteration strategy of $\mathcal{S}$. Let then $\nu < \eta_{i+1}$ be such that $\mathcal{S}$ is the ξth model of the full background construction of $\mathcal{N}_x^*|\nu$. To determine the complexity of the induced strategy of $\mathcal{S}$ it is enough to determine the strategy of $\mathcal{N}_x^*|\nu$ which acts on non-dropping stacks that are completely above η_i. Now, notice that by the choice of η_{i+1}, for any non-dropping tree $\mathcal{T}$ on $\mathcal{N}_x^*|\nu$ which is above η_i and is of limit length, if $b = \Sigma(\mathcal{T})$ then $\mathcal{Q}(b,\mathcal{T})$ exists and $\mathcal{Q}(b,\mathcal{T})$ has no overlaps, and $\mathcal{Q}(b,\mathcal{T}) \trianglelefteq Lp^{\Gamma^*}(\mathcal{M}(\mathcal{T}))$. This observation indeed shows that the fragment of the iteration strategy of $\mathcal{N}_x^*|\nu$ that acts on non-dropping stack that are above η_i is in Γ^*. Hence, the strategy of $\mathcal{W}$ is in Γ^*.

In the case $i < \omega^2$ is limit, the argument in the previous section that an iterate of $\mathcal{M}_{\omega^2}$ extends a Prikry generic shows that $\mathcal{W}$ cannot project across $\sup_{j<i}\eta_j$ and that $\mathcal{W} \triangleleft Lp^{\Gamma^*}(\mathcal{N}_i')$. This completes the proof of the claim. □

Working in $L(\mathbb{R},\mu)$, we now claim that there is $\mathcal{W} \trianglelefteq Lp(\mathcal{N}_{\omega^2})$ such that $\rho(W) < \eta_{\omega^2}$. To see this suppose not. It follows from MC that $Lp(\mathcal{N}_{\omega^2})$ is Σ_1^2-full. We then have that x is generic over $Lp(\mathcal{N}_{\omega^2})$ at the extender algebra of $\mathcal{N}_{\omega^2}$ at η_0. Because $Lp(\mathcal{N}_{\omega^2})[x]$ is Σ_1^2-full, we have that $\mathcal{M} \in Lp(\mathcal{N}_{\omega^2})[x]$ and $Lp(\mathcal{N}_{\omega^2})[x] \vDash$ "$\mathcal{M}$ is η_{ω^2}-iterable" by fullness of $Lp(\mathcal{N}_{\omega^2})[x]$.

[18] $\mathbb{C}(\mathcal{S})$ denotes the core of $\mathcal{S}$.

Let $\mathcal{S} = (L[\vec{E}][x])^{\mathcal{N}_{\omega^2}[x]|\eta_2}$ where the extenders used have critical point $> \eta_0$. Then working in $\mathcal{N}_{\omega^2}[x]$ we can compare $\mathcal{M}$ with $\mathcal{S}$. Using standard arguments, we get that $\mathcal{S}$ side doesn't move and by universality, $\mathcal{M}$ side has to come short (see [23]). This in fact means that $\mathcal{M} \trianglelefteq \mathcal{S}$. But the same argument used in the proof of Claim 2 shows that every $\mathcal{K} \trianglelefteq \mathcal{S}$ has an iteration strategy in Γ^*, contradiction!

Let $\eta_{\omega^2} = \sup_{i<\omega^2} \eta_i$ and $\mathcal{W} \trianglelefteq Lp(\mathcal{N}_{\omega^2})$ be least such that $\rho_\omega(\mathcal{W}) < \eta_{\omega^2}$.We can show the following.

Lemma 3.2.14. *$\mathcal{W} = \mathcal{J}_{\xi+1}(\mathcal{N}_{\omega^2})$ where ξ is least such that for some τ, $\mathcal{J}_\xi(\mathcal{N}_{\omega^2}) \vDash$ "$\mathsf{ZF}^- + \tau$ is a limit of Woodin cardinals + (T) holds in my derived model below τ[19]."*

Since the proof of this lemma is almost the same as that of Claim 7.5 in [41], we will not give it here. However, we have a few remarks regarding the proof:

- we typically replace N by a countable transitive $\overline{N}$ elementarily embeddable into N since the strategy of $\mathcal{W}$ is not known to extend to $V^{Col(\omega,\mathbb{R})}$. Having said this, we will confuse our N with its countable copy.
- We can then do an $\mathbb{R}^N$-iteration of $\mathcal{W}$ to "line up" its iterate with a $\mathbb{P}^N$-generic.

Asides from these remarks, everything else can just be transferred straightforwardly from the proof of Lemma 7.5 in [41] to the proof of Lemma 3.2.14. Now we just let $\mathcal{N}$ be the pointwise definable hull of $\mathcal{W}|\xi$. Letting $\mathcal{N}$'s unique iteration strategy be Λ, we can show Λ is Φ-fullness preserving and for any $\vec{f} \in (\mathcal{O}^{<\omega})^N$, there is a strongly $\vec{f}$-iterable, N-suitable $\mathcal{P}$ (in fact, $\mathcal{P} = \mathcal{Q}^-$ for some Λ-iterate $\mathcal{Q}$ of $\mathcal{N}$). We leave the rest of the details to the reader.

3.2.3 HOD$^{L(\mathbb{R},\mu_\alpha)}$ for $\alpha > 0$

For $0 < \alpha < \omega_1$, the HOD computation for AD^+-models of the form $L(\mathbb{R}, \mu_\alpha)$ is parallel to that of the case $\alpha = 0$. Therefore, we just state the results here. The notations used here are from the previous sections or are just the obvious adaptation of the notations used before.

Theorem 3.2.15. *Suppose $\alpha < \omega_1$ and suppose $V = L(\mathbb{R}, \mu_\alpha) \vDash$ "$\mathsf{AD}^+ + \mu_\alpha$ is a normal fine measure on X_α." Then $HOD = L(\mathcal{M}_\infty, \Lambda)$, where $\mathcal{M}_\infty$ is a premouse with $\omega^{\alpha+2}$ Woodin cardinals if $\alpha < \omega$ and $\omega^{\alpha+1}$ Woodin cardinals if $\alpha \geq \omega$ and Λ is the strategy acting on finite stacks of normal trees based on $\mathcal{M}|\Theta$ in $\mathcal{M}_\infty$ which picks the unique branch respecting f^* for all $f \in \mathcal{O}$. Here Θ is the first Woodin cardinal of $\mathcal{M}_\infty$ and $o(\mathcal{M}_\infty)$ is the sup of the Woodin cardinals of $\mathcal{M}_\infty$.*

[19]Here "derived model" means the model $L(\mathbb{R}^*, \mathcal{F}^*)$ where $\mathbb{R}^*$ is the symmetric reals for the Levy collapse at τ and $\mathcal{F}^*$ is the corresponding tail filter.

There's not much we can say about HOD of AD^+-models of the form $L(\mathbb{R}, \mu_{\omega_1})$ at this point.

Question. What is HOD when $V = L(\mathbb{R}, \mu_{\omega_1})$ satisfies "$\mathsf{AD}^+ + \mu_{\omega_1}$ is a normal fine measure on X_{ω_1}"? Is $\mathrm{OR}^{\mathcal{M}_\infty}$ the supremum of a measurable limit of Woodin cardinals in $L[\mathcal{M}_\infty]$?

Chapter 4

The Core Model Induction

4.1 Framework for the Induction

This section is an adaptation of the framework for the core model induction developed in [22], which in turns builds on earlier formulations of the core model induction in [26]. For basic notions such as model operators, mouse operators, F-mice, Lp^F, Lp^Γ, condenses well, relativizes well, the envelope of an inductive-like pointclass Γ (denoted $\mathrm{Env}(\Gamma)$), iterability, quasi-iterability, see [46].

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation. We assume here that $\mathcal{P}$ is countable and Σ is at least $(\omega, \mathfrak{c}^+ + 1)$-iterable. Hod pairs constructed in this thesis satisfy this kind of iterability. One purpose of demanding this is to make sense of the structure $K^\Sigma(\mathbb{R})$.

A *plain* Σ*-mouse operator* J is defined as follows. There is a formula φ in the language of Σ-premice and a parameter a such that for each $x \in \mathrm{dom}(J)$ (x is transitive containing a), $J(x)$ is the least $\mathcal{M} \lhd Lp^\Sigma(x)$ satisfying $\varphi[x, a]$.

If Γ is inductive-like such that $\Sigma \in \underset{\sim}{\Delta}_\Gamma$ and $\mathcal{A} = \{A_n \mid n < \omega\}$ is a self-justifying system in $Env(\Gamma(x))$ for some $x \in \mathbb{R}$ such that A_0 is the universal Γ set, then $J^\Gamma_{\Sigma,\mathcal{A}}$[1] is the mouse operator defined as: $J^\Gamma_{\Sigma,\mathcal{A}}(\mathcal{M}) = (\mathcal{M}^+, \in, B)$, where $\mathcal{M}^+ = Lp^{\Gamma,\Sigma}_\omega(\mathcal{M})$ and B is the term relation for $\mathcal{A}$ (see Definition 4.3.9 of [46]). $J_{\Sigma,\mathcal{A}}$ is called a *term relation hybrid mouse operator*.

Typically, mouse operators defined above have domain (a cone of) H_{ω_1}. In this thesis, we'll need them to be defined on a bigger domain. For a mouse operator J as above, we let F_J be the corresponding model operator as defined in Definition 2.1.8 of [46]. F_J is just the stratified version of J. We are now in a position to introduce the core model induction operators that we will need in this thesis. These are particular kinds of model operators that are constructed during the course of the core model induction. These model operators come from mouse operators that condense well and relativize well and determine themselves on

[1] When Γ is clear, we just write $J_{\Sigma,\mathcal{A}}$.

generic extensions.

Definition 4.1.1 (Core model induction operators)**.** *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation. We say F is a Σ core model induction operator or just Σ-cmi operator if one of the following holds:*

1. *For some $\alpha \in \mathrm{OR}$, letting $M = K^\Sigma(\mathbb{R})|\alpha$, $M \vDash \mathsf{AD}^+ + \mathsf{MC}(\Sigma)$*[2]*, one of the following holds:*
   (a) *$F = F_J$ (as in the notation of Definition 2.1.8 of [46]), where $J \in \mathcal{J}(\mathcal{M})$ is a plain Σ-model operator which condenses, relativizes well, and determines itself on generic extensions.*
   (b) *For some swo $b \in HC$*[3] *and F is a Σ-model operator F_J coding the Σ-$\mathcal{A}$-mouse operator $J = J_{\Sigma,\mathcal{A}}$ defined on a cone above b and $\mathcal{A} = (A_i : i < \omega)$ is a self-justifying-system such that $\mathcal{A} \in \mathrm{OD}^M_{b,\Sigma,x}$ for some $x \in b$ and α ends either a weak or a strong gap in the sense of [37] and $\mathcal{A}$ seals a gap that ends at α*[4]*.*
   (c) *For some H, H satisfies a or b above and for some $n < \omega$, F is the $x \to \mathcal{M}_n^{\#,H}(x)$ operator or for some $b \in HC$, F is the ω_1-iteration strategy of $\mathcal{M}_n^{\#,H}(b)$.*
2. *For some $a \in HC$ and $\mathcal{M} \lhd Lp^\Sigma(a)$, letting Λ be $\mathcal{M}$'s unique strategy, the above conditions hold for F with $L^\Lambda(\mathbb{R})$ replacing M*[5]*.*

As mentioned above, the Σ-cmi operators all condense, relativize well, and determine themselves on generic extension. When $\Sigma = \emptyset$ then we omit it from our notation. Recall that under AD, if X is any set then θ_X is the least ordinal which isn't a surjective image of $\mathbb{R}$ via an OD_X function. The following is the core model induction theorem that we will use.

Theorem 4.1.2. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and for every Σ-cmi operator F, $\mathcal{M}_1^{F,\sharp}$ exists. Then $K^\Sigma(\mathbb{R}) \vDash \mathsf{AD}^+ + \theta_\Sigma = \Theta$.*

The proof of Theorem 4.1.2 is very much like the proof of the core model induction theorems in [20], [26] (see Chapter 7) and [34]. To prove the theorem we have to use the scales analysis for $K^\Sigma(\mathbb{R})$. The readers familiar with the scales analysis of $K(\mathbb{R})$ as developed by Steel in [37] and [38] should have no problem seeing how the general theory should be developed. However, there is a point worth going over.

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[2] $\mathsf{MC}(\Lambda)$ stands for the Mouse Capturing relative to Λ which says that for $x, y \in \mathbb{R}$, x is $\mathrm{OD}(\Lambda, y)$ iff x is in some Λ-mouse over y.

[3] b is self well-ordered, i.e. there is a well-order of b in $L_1[b]$.

[4] This means that $\mathcal{A}$ is cofinal in $\underset{\sim}{Env}(\Gamma)$, where $\Gamma = \Sigma_1^M$. Note that $\underset{\sim}{Env}(\Gamma) = \mathcal{P}(\mathbb{R})^M$ if α ends a weak gap and $\underset{\sim}{Env}(\Gamma) = \mathcal{P}(\mathbb{R})^{K^\Sigma(\mathbb{R})|(\alpha+1)}$ if α ends a strong gap.

[5] Here $L^\Lambda(\mathbb{R})$ is constructed up to γ_Λ, where γ_Λ is the largest γ such that Λ is a γ-iteration strategy. In this thesis, all Λ that we encounter have $\gamma_\Lambda > \Theta$ so we can confuse $L^\Lambda_{\gamma_\Lambda}(\mathbb{R})$ with $L^\Lambda(\mathbb{R})$.

Suppose we are doing the core model induction to prove Theorem 4.1.2. During this core model induction, we climb through the levels of $K^{\Sigma}(\mathbb{R})$ some of which project to $\mathbb{R}$ but do not satisfy that "$\Theta = \theta_\Sigma$". It is then the case that the scales analysis of [37], [38], and [27] cannot help us in producing the next "new" set. However, such levels can never be problematic for proving that AD^+ holds in $K^{\Sigma}(\mathbb{R})$. This follows from the following lemma.

Lemma 4.1.3. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and for every Σ-cmi operator F, $\mathcal{M}_1^{F,\sharp}$ exists. Suppose $\mathcal{M} \lhd K^{\Sigma}(\mathbb{R})$ is such that $\rho(\mathcal{M}) = \mathbb{R}$ and $\mathcal{M} \vDash$ "$\Theta \neq \theta_\Sigma$". Then there is $\mathcal{N} \lhd K^{\Sigma}(\mathbb{R})$ such that $\mathcal{M} \lhd \mathcal{N}$, $\mathcal{N} \vDash$ "$\mathsf{AD}^+ + \Theta = \theta_\Sigma$".*

Proof. Since $\mathcal{M} \vDash$ "$\Theta \neq \theta_\Sigma$" it follows that $\mathcal{P}(\mathbb{R})^{\mathcal{M}} \cap (K^{\Sigma}(\mathbb{R}))^{\mathcal{M}} \neq \mathcal{P}(\mathbb{R})^{\mathcal{M}}$. It then follows that there is some $\alpha < o(\mathcal{M})$ such that $\rho(\mathcal{M}|\alpha) = \mathbb{R}$ but $\mathcal{M}|\alpha \ntrianglelefteq (K^{\Sigma}(\mathbb{R}))^{\mathcal{M}}$. Let $\pi : \mathcal{N} \to \mathcal{M}|\alpha$ be such that $\mathcal{N}$ is countable and its corresponding ω_1-iteration strategy (coded as a set of reals) is not in $\mathcal{M}$. Let Λ be the iteration strategy of $\mathcal{N}$. Then a core model induction through $L^{\Lambda}(\mathbb{R})$ shows that $L^{\Lambda}(\mathbb{R}) \vDash \mathsf{AD}^+$ (this is where we needed clause 2 of Definition 4.1.1). However, it's clear that $L^{\Lambda}(\mathbb{R}) \vDash$ "$\Theta = \theta_\Sigma$". It then follows from an unpublished result of Sargsyan and Steel that $L^{\Lambda}(\mathbb{R}) \vDash \mathcal{P}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap K^{\Sigma}(\mathbb{R})$. Let then $\mathcal{K} \trianglelefteq (K^{\Sigma}(\mathbb{R}))^{L^{\Lambda}(\mathbb{R})}$ be such that $\rho(\mathcal{K}) = \mathbb{R}$, $\mathcal{K} \vDash \Theta = \theta_\Sigma$ and $\Lambda \restriction HC \in \mathcal{K}$ (there is such a $\mathcal{K}$ by an easy application of $\Sigma_1^2(\Sigma)$ reflection). Since countable submodels of $\mathcal{K}$ are iterable , we have that $\mathcal{K} \trianglelefteq K^{\Sigma}(\mathbb{R})$. Also we cannot have that $\mathcal{K} \lhd \mathcal{M}$ as otherwise $\mathcal{N}$ would have a strategy in $\mathcal{M}$. Therefore, $\mathcal{M} \trianglelefteq \mathcal{K}$. $\square$

We can now do the core model induction through the levels of $K^{\Sigma}(\mathbb{R})$ as follows. If we have reached a gap satisfying "$\Theta = \theta_\Sigma$" then we can use the scales analysis of [27] to go beyond. If we have reached a level that satisfies "$\Theta \neq \theta_{\Sigma^g}$" then using Lemma 4.1.3 we can skip through it and go to the least level beyond it that satisfies "$\Theta = \theta_\Sigma$". We leave the rest of the details to the reader. This completes our proof sketch of Theorem 4.1.2. In Subsection 4.2.5, we will outline a different (and somewhat simpler) approach from [46] to doing the core model induction.

Finally, let us remark that many results using the core model induction seem to be relying on the following conjecture.

Conjecture 4.1.4 (Quasi-iterability conjecture)**.** [6]*Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation. Let Γ be an inductive-like pointclass such that $\underset{\sim}{\Delta}_\Gamma$ is determined, $\Sigma \in \underset{\sim}{\Delta}_\Gamma$ and Γ-$\mathsf{MC}(\Sigma)$*[7] *holds. Then for every $A \in Env(\Gamma)$, there is a strongly A-quasi-iterable Σ-premouse.*

The above conjecture is particularly useful in core model induction arguments carried out in some generic extension $V[g]$. It implies that in 1b) of Definition 4.1.1, there is (in

[6]A version of this has been proved by Woodin, where "strongly A-quasi iterable" is replaced by "strongly A-iterable".

[7]This means for every transitive $a \in HC$, and every $b \in C_\Gamma(a)$, we have $b \in Lp^{\Gamma,\Sigma}(a)$, which consists of Σ mice $\mathcal{M}$ over a such that $\mathcal{M}$ is sound, $\rho_\omega(\mathcal{M}) = a$, and $\mathcal{M}$'s unique iteration strategy is in Γ.

$V[g]$) a Γ-suitable Σ-premouse $\mathcal{N}$ where $\Gamma = \Sigma_1^M$ and an (ω_1, ω_1) strategy Σ of $\mathcal{N}$ guided by $\mathcal{A}$, hence Σ condenses well. By the method of boolean comparison, we can get a pair $(\mathcal{N}, \Sigma) \in V$. One can then use the strength of the hypothesis satisfied by V to continue with the core model induction. In this thesis, we don't assume Conjecture 4.1.4 as core model inductions are carried out in V and term relation hybrids are sufficient.

4.2 $\Theta > \omega_2$ Can Imply AD^+

4.2.1 Introduction

It is well-known that the existence of an $L(\mathbb{R}, \mu)$[8] that satisfies $\mathsf{ZF} + \mathsf{DC} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$[9] is equiconsistent with that of a measurable cardinal. The model $L(\mathbb{R}, \mu)$ obtained from standard proofs of the equiconsistency satisfies Θ[10] $= \omega_2$ and hence fails to satisfy AD. So it is natural to consider the situations where $L(\mathbb{R}, \mu) \vDash \Theta > \omega_2$ and try to understand how much determinacy holds in this model.

To analyze the sets of reals that are determined in such a model, which we will call V, we run the core model induction in a certain submodel of V that agrees with V on all bounded subsets of Θ. This model will be defined in the next section. What we'll show is that $K(\mathbb{R}) \vDash \mathsf{AD}^+$. We will then show $\Theta^{K(\mathbb{R})} = \Theta$ by an argument like that in Chapter 7 of [26]. Finally, we prove that

$$\mathcal{P}(\mathbb{R}) \cap K(\mathbb{R}) = \mathcal{P}(\mathbb{R}),$$

which implies $L(\mathbb{R}, \mu) \vDash \mathsf{AD}$. We state the main result of Section 4.2.

Theorem 4.2.1. [11] *Suppose $V = L(\mathbb{R}, \mu) \vDash \mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2 + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+$.*

Woodin has shown the following.

Theorem 4.2.2 (Woodin)**.** *Suppose $L(\mathbb{R}, \mu) \vDash \mathsf{AD} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+ + \mu$ is unique.*

Combining the results in Theorem 4.2.2 and Theorem 4.2.1, we get the following.

Corollary 4.2.3. *Suppose $V = L(\mathbb{R}, \mu) \vDash \mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2 + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+ + \mu$ is unique.*

[8] By $L(\mathbb{R}, \mu)$ we mean the model constructed from the reals and using μ as a predicate. We will also use the notation $L(\mathbb{R})[\mu]$ and $L_\alpha(\mathbb{R})[\mu]$ in various places in the paper.

[9] A measure μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is fine if for all $x \in \mathbb{R}$, $\mu(\{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid x \in \sigma\}) = 1$. μ is normal if for all functions $F : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $\mu(\{\sigma \mid F(\sigma) \subseteq \sigma\}) = 1$, there is an $x \in \mathbb{R}$ such that $\mu(\{\sigma \mid x \in F(\sigma)\}) = 1$.

[10] Θ is the sup of all α such that there is a surjection from $\mathbb{R}$ onto α

[11] The theorem is due independently to W. H. Woodin and the author.

The equiconsistency result we get from this analysis is the following.

Theorem 4.2.4. *The following theories are equiconsistent.*

1. ZFC + *There are* ω^2 *Woodin cardinals.*

2. $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+$ + *There is a normal fine measure on* $\mathcal{P}_{\omega_1}(\mathbb{R})$.

3. $\mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2$ + *There is a normal fine measure on* $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Proof. The equiconsistency of (1) and (2) is a theorem of Woodin (see [48] for more information). Theorem 4.2.1 immediately implies the equiconsistency of (2) and (3). □

4.2.2 Basic setup

In this section we prove some basic facts about V assuming $V = L(\mathbb{R}, \mu) \vDash \mathsf{ZF} + \mathsf{DC} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. First note that we cannot well-order the reals hence full AC fails in this model. Secondly, ω_1 is regular; this follows from DC. Now μ induces a countably complete nonprincipal ultrafilter on ω_1; hence, ω_1 is a measurable cardinal. DC also implies that $\mathrm{cof}(\omega_2) > \omega$. We collect these facts into the following lemma, whose easy proof is left to the reader.

Lemma 4.2.5. *Suppose* $V = L(\mathbb{R}, \mu) \vDash \mathsf{ZF} + \mathsf{DC} + \mu$ *is a normal fine measure on* $\mathcal{P}_{\omega_1}(\mathbb{R})$. *Then*

1. ω_1 *is regular and in fact measurable;*

2. $\mathit{cof}(\omega_2) > \omega$;

3. AC *fails and in fact, there cannot be an* ω_1*-sequence of distinct reals.*

Lemma 4.2.6. Θ *is a regular cardinal.*

Proof. Suppose not. Let $f : \mathbb{R} \to \Theta$ be a cofinal map. Then there is an $x \in \mathbb{R}$ such that f is $\mathrm{OD}(\mu, x)$. For each $\alpha < \Theta$, there is a surjection $g_\alpha : \mathbb{R} \to \alpha$ such that g_α is $\mathrm{OD}(\mu)$ (we may take g_α to be the least such). We can get such a g_α because we can "average over the reals." Now define a surjection $g : \mathbb{R} \to \Theta$ as follows

$$g(y) = g_{f(y_0)}(y_1) \text{ where } y = \langle y_0, y_1 \rangle.$$

It's easy to see that g is a surjection. But this is a contradiction. □

Lemma 4.2.7. ω_1 *is inaccessible in any (transitive) inner model of choice containing* ω_1.

Proof. This is easy. Let N be such a model. Since $P = L(N, \mu)$ is also a choice model and ω_1 is measurable in P, hence ω_1 is inaccessible in P. This gives ω_1 is inaccessible in N. □

Next, we define two key models that we'll use for our core model induction. Let

$$M = \Pi_{\sigma\in\mathcal{P}_{\omega_1}(\mathbb{R})} M_\sigma/\mu \text{ where } \ M_\sigma = \mathrm{HOD}_{\sigma\cup\{\sigma,\mu\}}$$

and,

$$H = \Pi_{\sigma\in\mathcal{P}_{\omega_1}(\mathbb{R})} H_\sigma/\mu \text{ where } \ H_\sigma = \mathrm{HOD}^{\mathrm{M}_\sigma}_{\{\sigma,\mu\}}.$$

We note that in the definition of M_σ and H_σ above, ordinal definability is with respect to the structure $(L(\mathbb{R},\mu),\mu)$.

Lemma 4.2.8. *Lós theorem holds for both of the ultraproducts defined above.*

Proof. We do this for the first ultraproduct. The proof is by induction on the complexity of formulas. It's enough to show the following. Suppose $\phi(x, y)$ is a formula and f is a function such that $\forall^*_\mu \sigma M_\sigma \vDash \exists x\phi[x, f(\sigma)]$. We show that $M \vDash \exists x\phi[x, [f]_\mu)$.

Let $g(\sigma) = \{x \in \sigma \mid (\exists y \in \mathrm{OD}(\mu, x))(M_\sigma \vDash \phi[y, f(\sigma)])\}$. Then $\forall^*_\mu \sigma g(\sigma)$ is a non-empty subset of σ. By normality of μ, there is a fixed real x such that $\forall^*_\mu \sigma x \in g(\sigma)$. Hence we can define $h(\sigma)$ to be the least y in $\mathrm{OD}(\mu, x)$ such that $M_\sigma \vDash \phi[y, f(\sigma)]$. It's easy to see then that $M \vDash \phi[[h]_\mu, [f]_\mu]$. □

By Lemma 4.2.8, M and H are well-founded so we identify them with their transitive collapse. First note that $M \vDash \mathsf{ZF} + \mathsf{DC}$ and $H \vDash \mathsf{ZFC}$. We then observe that $\Omega = [\lambda\sigma.\omega_1]_\mu$ is measurable in M and in H. This is because ω_1 is measurable in M_σ and H_σ for all σ. Note also that $\Omega > \Theta$ as $\forall^*_\mu \sigma$, Θ^{M_σ} is countable and $\mathcal{P}(\omega_1)^{M_\sigma}$ is countable. The key for this is just an easy fact stated in Lemma 4.2.5: There are no sequences of ω_1 distinct reals. By a standard Vopenka argument, for any set of ordinals $A \in M$ of size less than Ω, there is an H-generic G_A (for a forcing of size smaller than Ω) such that $A \in H[G_A] \subseteq M$ and Ω is also measurable in $H[G_A]$.

Lemma 4.2.9. $\mathcal{P}(\mathbb{R}) \subseteq M$.

Proof. Let $A \subseteq \mathbb{R}$. Then there is an $x \in \mathbb{R}$ such that $A \in \mathrm{OD}(x, \mu)$. By fineness of μ, $(\forall^*_\mu \sigma)(x \in \sigma)$ and hence $(\forall^*_\mu \sigma)(A \cap \sigma \in \mathrm{OD}(x, \mu, \sigma))$. So we have $(\forall^*_\mu \sigma)(A \cap \sigma \in M_\sigma)$. This gives us that $A = [\lambda\sigma.A \cap \sigma]_\mu \in M$. □

Lemma 4.2.9 implies that M contains all bounded subsets of Θ.

4.2.3 Getting one more Woodin

By the discussion of the last section, to show AD holds in $K(\mathbb{R})$, it is enough to show that if F is a cmi operator (that is defined on a cone on H_{ω_1} above some $a \in H_{\omega_1}$, i.e. $a \in L_1[x]$[12]), then $\mathcal{M}_1^{\sharp,F}(x)$ exists (and is (ω_1, ω_1)-iterable) for all $x \in H_{\omega_1}$ coding a. We may

[12] This is another way of saying F is defined on the cone above a.

as well assume F is a model operator that relativizes, condenses well, and determines itself on generic extensions. Recall that by the discussion of Subsection 4.2.2, since we can lift F to the cone on H_Ω^M above a, we can then lift the $\mathcal{M}_1^{\sharp,F}$-operator to the cone on H_Ω^M above a and for each $x \in H_\Omega^M$ coding a, we can also show $M_1^{\sharp,F}(x)$ has an (Ω,Ω)-iteration strategy (in M).[13]

Theorem 4.2.10. *Suppose F is a model operator that relativizes, condenses well, and determines itself on generic extensions. Suppose F is defined on the cone on H_{ω_1} above some $a \in H_{\omega_1}$ (and hence defined on the cone on H_Ω^M above a). Then $\mathcal{M}_1^{\sharp,F}(x)$ exists for all $x \in H_{\omega_1}$ coding a. Furthermore, $\mathcal{M}_1^{\sharp,F}(x)$ is (ω_1,ω_1)-iterable, hence (Ω,Ω)-iterable in M.*

Proof. To simplify the notation, we will prove that $\mathcal{M}_1^\sharp(x)$ exists for all $x \in H_{\omega_1}$. The proof relativizes to any model operator F as in the hypothesis of the lemma.

To start off, it's easy to see that in M, the #-operator is total on H_Ω. This is because Ω is measurable in M. The same conclusion holds for H and any generic extension J of H by a forcing of size smaller than Ω and $J \subseteq M$.

First, we prove

Lemma 4.2.11. *For each $x \in \mathbb{R}$, $\mathcal{M}_1^\#(x)$ exists.*

Proof. This is the key lemma. For brevity, we just show $\mathcal{M}_1^\#$ exists. The proof relativizes trivially to any real. Suppose not. Then in H, K (built up to Ω) exists and is $\Omega+1$ iterable. This is because in H, the #-operator is total on H_Ω. Let $\kappa = \omega_1$. By Lemma 4.2.7, κ is inaccessible in H and in any set generic extension J of H and $J \subseteq M$. By [33], $K^H = K^{H[G]}$ for any H-generic G for a poset of size smaller than Ω. We use K to denote K^H.

Claim. $(\kappa^+)^K = (\kappa^+)^H$.

Proof. The proof follows that of Theorem 3.1 in [24]. Suppose not. Let $\lambda = (\kappa^+)^K$. Hence $\lambda < (\kappa^+)^H$. Working in H, let N be a transitive, power admissible set such that ${}^\omega N \subseteq N$, $V_\kappa \cup \mathcal{J}_{\lambda+1}^K \subseteq N$, and $\mathrm{card}(N) = \kappa$. We then choose $A \subseteq \kappa$ such that $N \in L[A]$ and $K^{L[A]}|\lambda = K|\lambda$, $\lambda = (\kappa^+)^{K^{L[A]}}$, and $\mathrm{card}(N)^{L[A]} = \kappa$. Such an A exists by Lemma 3.1.1 in [24] and the fact that $\lambda < (\kappa^+)^H$.

Now, since $A^\#$ exists in H, $(\kappa^+)^{L[A]} < (\kappa^+)^H$. By GCH in $L[A]$, $\mathrm{card}^H(\mathcal{P}(\kappa) \cap L[A]) = \kappa$. So in M, there is an $L[A]$-ultrafilter U over κ that is nonprincipal and countably complete (in M and in V). This is because such a U exists in V as being induced from μ and since U can be coded as a subset of $\omega_1^V = \kappa$, $U \in M$. Let J be a generic extension of H (of size smaller than Ω) such that $U \in J$. From now on, we work in J. Let

$$j : L[A] \to Ult(L[A], U) = L[j(A)]$$

[13] We'll sometimes say a model operator F is total to mean that it's defined on a cone above a certain a when we don't need to specify what a is.

be the ultrapower map. Then j is well-founded, $\text{crit}(j) = \kappa$, $A = j(A) \cap \kappa \in L[j(A)]$. So $L[A] \subseteq L[j(A)]$. The key point here is that $\mathcal{P}(\kappa) \cap K^{L[A]} = \mathcal{P}(\kappa) \cap K^{L[j(A)]}$. To see this, first note that the $\subseteq$ direction holds because any κ-strong mouse in $L[A]$ is a κ-strong mouse in $L[j(A)]$ as $\mathbb{R} \cap L[A] = \mathbb{R} \cap L[j(A)]$ and $L[A]$ and $L[j(A)]$ have the same $< \kappa$-strong mice. To see the converse, suppose not. Then there is a sound mouse $\mathcal{M} \lhd K^{L[j(A)]}$ such that $\mathcal{M}$ extends $K^{L[A]}|\lambda$ and $\mathcal{M}$ projects to κ. The iterability of $\mathcal{M}$ is absolute between J and $L[j(A)]$, by the following folklore result

Lemma 4.2.12. *Assume ZFC + "there is no class model with a Woodin." Let M be a transitive class model that satisfies ZFC$^-$+ "there is no class inner model of a Woodin". Futhermore, assume that $\omega_1 \subseteq M$. Let $\mathcal{P} \in M$ be a premouse with no definable Woodin. Then*

$$\mathcal{P} \text{ is a mouse } \Leftrightarrow M \vDash \mathcal{P} \text{ is a mouse.}$$

For a proof of this, see [25]. By a theorem of Ralf Schindler which essentially states that K is just a stack of mice above ω_2 (here $\omega_2^J < \kappa$), we have $\mathcal{M} \lhd K^J = K$. But $\lambda = (\kappa^+)^K$ and $\mathcal{M} \lhd K|\lambda$. Contradiction.

Now the rest of the proof is just as in that of Theorem 3.1 in [24]. Let E_j be the superstrong extender derived from j. Since $\text{card}(N) = \kappa$ and $\lambda < \kappa^+$, a standard argument (due to Kunen) shows that $F, G \in L[j(A)]$ where

$$F = E_j \cap ([j(\kappa)]^{<\omega} \times K^{L[A]})$$

and,

$$G = E_j \cap ([j(\kappa)]^{<\omega} \times N).$$

The key is $\text{card}(N)^{L[A]} = \kappa$ and $\text{card}(K \cap \mathcal{P}(\kappa))^{L[A]} = \kappa$. We show $F \in L[j(A)]$. The proof of $G \in L[j(A)]$ is the same. For $a \in [j(\kappa)]^{<\omega}$, let $\langle B_\alpha \mid \alpha < \kappa \rangle \in L[A]$ be an enumeration of $\mathcal{P}([\kappa]^{|a|}) \cap K^{L[A]} = \mathcal{P}([\kappa]^{|a|}) \cap K^{L[j(A)]}$ and

$$E_a = \{B_\alpha \mid \alpha < \kappa \wedge a \in j(B_\alpha)\}.$$

Then $E_a \in L[j(A)]$ because $\langle j(B_\alpha) \mid \alpha < \kappa \rangle \in L[j(A)]$.

Hence $(K^{L[j(A)]}, F)$ and (N, G) are elements of $L[j(A)]$. In $L[j(A)]$, for cofinally many $\xi < j(\kappa)$, $F|\xi$ coheres with K and (N, G) is a weak $\mathcal{A}$-certificate for $(K, F \restriction \xi)$ (in the sense of [24]), where

$$\mathcal{A} = \bigcup_{n<\omega} \mathcal{P}([\kappa]^n)^K.$$

By Theorem 2.3 in [24], those segments of F are on the extender sequence of $K^{L[j(A)]}$. But then κ is Shelah in $K^{L[j(A)]}$, which is a contradiction. $\square$

The proof of the claim also shows that $(\kappa^+)^K = (\kappa^+)^J$ for any set (of size smaller than Ω) generic extension J of H. In particular, since any $A \subseteq \omega_1^V = \kappa$ belongs to a set generic extension of H of size smaller than Ω, we immediately get that $(\kappa^+)^K = \omega_2$. This is impossible in the presence of μ.[14] To see this, let $\vec{C} = \langle C_\alpha \mid \alpha < \omega_2 \rangle$ be the canonical $\square_\kappa$-sequence in K. Working in V, let ν be the measure on $\mathcal{P}_{\omega_1}(\omega_2)$ induced by μ defined as follows. First, fix a surjection $\pi : \mathbb{R} \rightarrow \omega_2$. Then π trivially induces a surjection from $\mathcal{P}_{\omega_1}(\mathbb{R})$ onto $\mathcal{P}_{\omega_1}(\omega_2)$ which we also call π. Then our measure ν is defined as

$$A \in \nu \Leftrightarrow \pi^{-1}[A] \in \mu.$$

Now consider the ultrapower map $j : K \rightarrow Ult(K, \nu) = K^*$ (where the ultrapower uses all functions in V). An easy calculation gives us that $j''\omega_2 = [\lambda\sigma.\sigma]_\nu$ and $A \in \nu \Leftrightarrow j''\omega_2 \in j(A)$. So let $\gamma = \sup j''\omega_2$ and $\vec{D} = j(\vec{C}) \in K^*$. Note that $(\kappa^+)^{K^*} = \omega_2$ and since $K^* \vDash \mathsf{ZFC}$, ω_2 is regular in K^*. Also $\gamma < j(\omega_2^V)$. Now consider the set D_γ. By definition, D_γ is an club in γ so it has order type at least ω_2. However, let $C = \langle \alpha < \omega_2 \mid \mathrm{cof}(\alpha) = \omega \rangle$. Then $j(C) = j''C$ is an ω-club in γ. Hence $E = D_\gamma \cap j(C)$ is an ω-club in γ. For each $\alpha \in \lim E \subseteq E$, $D_\alpha = D_\gamma \cap \alpha$ and D_α has order type strictly less than ω_1 (this is because $\mathrm{cof}(\alpha) = \omega$). This implies that every proper initial segment of D_γ has order type strictly less than ω_1 which is a contradiction. $\square$

The lemma shows that the $\mathcal{M}_1^{\#}$-operator is total on H_κ (in M as well as in V). This implies that $\forall^*_\mu \sigma$, M_σ is closed under the $\mathcal{M}_1^{\#}$-operator on H_κ. By Ló*s*, M is closed under the $\mathcal{M}_1^{\#}$-operator on H_Ω^M. Similar conclusions hold for H as well as its generic extensions in M. $\square$

The above proof relativizes to any model operator F in a straightforward way with only obvious modifications. In particular, we replace $\mathcal{M}_1^\sharp$ by $\mathcal{M}_1^{\sharp,F}$, K now is obtained from the $K^{c,F}$-construction, $L[A]$ is replaced by $L^F[A]$, and the model M in Lemma 4.2.12 is required to be closed under F. By the discussion in Section 4.1, we have proved.

Lemma 4.2.13. $M_0 =_{def} K(\mathbb{R}) \vDash AD^+ + \Theta = \theta_0$.

Remark. $\mathsf{AD}^{K(\mathbb{R})}$ is the most amount of determinacy one could hope to prove. This is because if μ comes from the Solovay measure (derived from winning strategies of real games) in an $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R} + \mathsf{SMSC}$ universe, call it V (any $\mathsf{AD}_\mathbb{R} + V = L(\mathcal{P}(\mathbb{R}))$-model below "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" would do here), then $L(\mathbb{R}, \mu)^V \cap \mathcal{P}(\mathbb{R}) \subseteq K(\mathbb{R})^V$. This is because μ is OD hence $\mathcal{P}(\mathbb{R}) \cap L(\mu, \mathbb{R}) \subseteq \mathcal{P}_{\theta_0}(\mathbb{R})$. Since $\mathsf{AD}^+ + \mathsf{SMC}$ gives us that any set of reals of Wadge rank $< \theta_0$ is contained in an $\mathbb{R}$-mouse (by an unpublished result of Sargsyan and Steel but see [35]), we get that $\mathcal{P}(\mathbb{R}) \cap L(\mu, \mathbb{R}) \subseteq K(\mathbb{R})$ (it is conceivable that the inclusion is strict). By Theorem 4.2.2, $L(\mathbb{R}, \mu) \vDash \Theta = \theta_0$, which implies $L(\mathbb{R}, \mu) \vDash V = K(\mathbb{R})$. Putting all of this together, we get $L(\mathbb{R}, \mu) \vDash K(\mathbb{R}) = L(\mathcal{P}(\mathbb{R})) + \mathsf{AD}^+$.

The above remark suggests that we should try to show that every set of reals in $V = L(\mathbb{R}, \mu)$ is captured by an $\mathbb{R}$-mouse, which will prove Theorem 4.2.1. This is accomplished in the next three sections.

[14] The argument we're about to give is based on Solovay's proof that square fails above a supercompact cardinal.

4.2.4 $\Theta^{K(\mathbb{R})} = \Theta$

Suppose for contradiction that $\Theta^{K(\mathbb{R})} < \Theta$. For simplicity, we first get a contradiction from the smallness assumption that "there is no model containing $\mathbb{R} \cup \mathrm{OR}$ that satisfies $\mathsf{AD}^+ + \Theta > \theta_0$". The argument will closely follow the argument in Chapter 7 of [26]. All of our key notions and notations come from there unless specified otherwise. Let $\Theta^* = \Theta^{K(\mathbb{R})}$. Let $\mathcal{M}_\infty$ be $\mathrm{HOD}^{\mathrm{K}(\mathbb{R})} \restriction \Theta^*$. Then $\mathcal{M}_\infty = \mathcal{M}_\infty^+ \restriction \Theta^*$ where $\mathcal{M}_\infty^+$ is the limit of a directed system (the hod limit system) indexed by pairs $(\mathcal{P}, \vec{A})$ where $\mathcal{P}$ is a suitable premouse, $\vec{A}$ is a finite sequence of OD sets of reals, and $\mathcal{P}$ is strongly $\vec{A}$-quasi-iterable in $K(\mathbb{R})$. For more details on how the direct limit system is defined, the reader should consult Chapter 7 of [26]. Let Γ be the collection of $\mathrm{OD}^{K(\mathbb{R})}$ sets of reals. For each $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $Lp(\sigma) \vDash \mathsf{AD}^+$, let $\mathcal{M}_\infty^\sigma$ and Γ^σ be defined the same as $\mathcal{M}_\infty$ and Γ but in $Lp(\sigma)$. Let $\Theta^\sigma = o(\mathcal{M}_\infty^\sigma)$. By $\mathsf{AD}^{K(\mathbb{R})}$ and $\Theta^* < \Theta$, we easily get

Lemma 4.2.14. $\forall_\mu^* \sigma(Lp(\sigma) \vDash \mathsf{AD}^+$, *and there is an elementary map* $\pi_\sigma : (Lp(\sigma), \mathcal{M}_\infty^\sigma, \Gamma^\sigma) \to (K(\mathbb{R}), \mathcal{M}_\infty, \Gamma)$.)

Proof. First, it's easily seen that $K(\mathbb{R}) \vDash \mathsf{AD}^+$ implies $\forall_\mu^* \sigma Lp(\sigma) \vDash \mathsf{AD}^+$. We also have that letting ν be the induced measure on $\mathcal{P}_{\omega_1}(K(\mathbb{R}))$

$$\forall_\nu^* X \ X \prec K(\mathbb{R}).$$

The second clause of the lemma follows by transitive collapsing the X's above. Note that $\forall_\mu^* \sigma \ Lp(\sigma)$ is the uncollapse of some countable $X \prec K(\mathbb{R})$ such that $\mathbb{R}^X = \sigma$. This is because if $\mathcal{M}$ is an $\mathbb{R}$-mouse then $\forall_\nu^* X \ \mathcal{M} \in X$. The π_σ's are just the uncollapse maps. □

We may as well assume $(\forall_\mu^* \sigma)(Lp(\sigma) = Lp(\sigma)^{K(\mathbb{R})})$ as otherwise, fix a σ such that $Lp(\sigma) \vDash \mathsf{AD}^+$ and $\mathcal{M} \lhd Lp(\sigma)$ a sound mouse over σ, $\rho_\omega(\mathcal{M}) = \sigma$ and $\mathcal{M} \notin Lp(\sigma)^{K(\mathbb{R})}$. Let Λ be the strategy of $\mathcal{M}$. Then by a core model induction as above, we can show that $L^\Lambda(\mathbb{R}) \vDash \mathsf{AD}^+ + \Theta > \theta_0$. Since this is very similar to the proof of PD, we only mention a few key points for this induction. First, Λ is a $\omega_1 + 1$ strategy with condensation and $\forall_\mu^* \sigma \ \Lambda \restriction M_\sigma \in M_\sigma$ and $\forall_\mu^* \sigma \ \Lambda \restriction H_\sigma[\mathcal{M}] \in H_\sigma[\mathcal{M}]$. This allows us to lift Λ to a $\Omega + 1$ strategy in M and construct K^Λ up to Ω inside $\prod_\sigma H_\sigma[\mathcal{M}]$. This is a contradiction to our smallness assumption.

Lemma 4.2.15. $\forall_\mu^* \sigma$ $\mathcal{M}_\infty^\sigma$ *is full in* $K(\mathbb{R})$ *in the sense that* $Lp(\mathcal{M}_\infty^\sigma) \vDash \Theta^\sigma$ *is Woodin.*

Proof. First note that $Lp_2(\sigma) =_{def} Lp(Lp(\sigma)) \vDash \mathsf{AD}^+ + \Theta = \theta_0$ because $\mathcal{P}(\mathbb{R})^{Lp_2(\sigma)} = \mathcal{P}(\mathbb{R})^{Lp(\sigma)}$. So suppose $\mathcal{N}^\sigma \rhd \mathcal{M}_\infty^\sigma$ is the Q-structure. It's easy to see that $\mathcal{N}^\sigma \in Lp_2(\sigma)$ and is in fact OD there.

Next we observe that in $Lp_2(\sigma)$, $\Theta = \Theta^\sigma$. By a Theorem of Woodin, we know $\mathrm{HOD}^{\mathrm{Lp_2}(\sigma)} \vDash \Theta^\sigma$ is Woodin (see Theorem 5.6 of [13]). But this is a contradiction to our assumption that $\mathcal{N}^\sigma$ is a Q-structure for Θ^σ. □

The last lemma shows that for a typical σ, $Lp_\omega(\mathcal{M}_\infty^\sigma)$ is suitable in $K(\mathbb{R})$. Let $\mathcal{M}_\infty^{\sigma,+}$ be the hod limit computed in $Lp(\sigma)$. Let $(\Gamma^\sigma)^{<\omega} = \{\vec{A_n} \mid n < \omega\}$ and for each $n < \omega$, let $\mathcal{N}_n$ be such that $\mathcal{N}_n$ is strongly $\vec{A_n}$-quasi-iterable in $Lp(\sigma)$ such that $\mathcal{M}_\infty^{\sigma,+}$ is the quasi-limit of the $\mathcal{N}_n$'s in $Lp(\sigma)$. Let $\mathcal{M}_\infty^{\sigma,*}$ be the quasi-limit of the $\mathcal{N}_n$'s in $K(\mathbb{R})$. We'll show that $\pi_\sigma''\Gamma^\sigma$ is cofinal in Γ, $\mathcal{M}_\infty^{\sigma,+} = \mathcal{M}_\infty^{\sigma,*} = Lp_\omega(\mathcal{M}_\infty^\sigma)$ and hence $\mathcal{M}_\infty^{\sigma,+}$ is strongly A-quasi-iterable in $K(\mathbb{R})$ for each $A \in \pi_\sigma''\Gamma^\sigma$. From this we'll get a strategy Σ_σ for $\mathcal{M}_\infty^{\sigma,+}$ with weak condensation. This proceeds much like the proof in Chapter 7 of [26].

Let T be the tree for a universal $(\Sigma_1^2)^{K(\mathbb{R})}$-set; let $T^* = \prod_\sigma T$ and $T^{**} = \prod_\sigma T^*$. To show $(\forall_\mu^*\sigma)(\pi_\sigma''\Gamma^\sigma$ is cofinal in $\Gamma)$ we first observe that

$$(\forall_\mu^*\sigma)(L[T^*, \mathcal{M}_\infty^\sigma]|\Theta^\sigma = \mathcal{M}_\infty^\sigma),$$

that is, T^* does not create Q-structures for $\mathcal{M}_\infty^\sigma$. This is because $\mathcal{M}_\infty^\sigma$ is countable, ω_1^V is inaccessible in any inner model of choice, $L[T^*, \mathcal{M}_\infty^\sigma]|\omega_1^V = L[T, \mathcal{M}_\infty^\sigma]|\omega_1^V$, and $L[T, \mathcal{M}_\infty^\sigma]|\Theta^\sigma = \mathcal{M}_\infty^\sigma$ by Lemma 4.6. Next, let E_σ be the extender derived from π_σ with generators in $[\gamma]^{<\omega}$, where $\gamma = \sup\pi_\sigma''\Theta^\sigma$. By the above, E_σ is a pre-extender over $L[T^*, \mathcal{M}_\infty^\sigma]$.

Lemma 4.2.16. *$(\forall_\sigma^*\sigma)(Ult(L[T^*, \mathcal{M}_\infty^\sigma], E_\sigma)$ is wellfounded).*

Proof. The statement of the lemma is equivalent to

$$Ult(L[T^{**}, \mathcal{M}_\infty], \Pi_\sigma E_\sigma/\mu) \text{ is wellfounded.} \quad (*)$$

To see $(*)$, note that

$$\prod_\sigma E_\sigma = E_\mu$$

where E_μ is the extender from the ultrapower map j_μ by μ (with generators in $[\xi]^{<\omega}$, where $\xi = \sup j_\mu''\Theta^*$). This uses normality of μ. We should metion that the equality above should be interpreted as saying: the embedding by $\Pi_\sigma E_\sigma/\mu$ agrees with j_μ on all ordinals (less than Θ).

Since μ is countably complete and DC holds, we have that $Ult(L[T^{**}, \mathcal{M}_\infty], E_\mu)$ is wellfounded. Hence we're done. $\square$

Theorem 4.2.17. *1. $(\forall_\mu^*\sigma)(\pi_\sigma$ is continuous at $\theta^\sigma)$. Hence $cof(\Theta^{K(\mathbb{R})}) = \omega$.*

2. If $i : \mathcal{M}_\infty^\sigma \to S$, and $j : S \to \mathcal{M}_\infty$ are elementary and $\pi_\sigma = j \circ i$ and S is countable in $K(\mathbb{R})$, then S is full in $K(\mathbb{R})$. In fact, if W is the collapse of a hull of S containing $rng(i)$, then W is full in $K(\mathbb{R})$.

Proof. The keys are Lemma 4.2.16 and the fact that the tree T^*, which enforces fullness for $\mathbb{R}$-mice, does not generate Q-structures for $\mathcal{M}_\infty^\sigma$. To see (1), suppose not. Fix a typical σ for which (1) fails. Let $\gamma = \sup\pi_\sigma''\Theta^\sigma < \Theta^*$. Let E_σ be the extender derived from π_σ with generators in $[\gamma]^{<\omega}$ and consider the ultrapower map

$$\tau : L[T^*, \mathcal{M}_\infty^\sigma] \to N_\sigma =_{def} Ult(L[T^*, \mathcal{M}_\infty^\sigma], E_\sigma).$$

We may as well assume N_σ is transitive by Lemma 4.2.16. We have that τ is continuous at Θ^σ and $N_\sigma \vDash o(\tau(\mathcal{M}^\sigma_\infty))$ is Woodin. Since $o(\tau(\mathcal{M}^\sigma_\infty)) = \gamma < \Theta^*$, there is a Q-structure $\mathcal{Q}$ for $o(\tau(\mathcal{M}^\sigma_\infty))$ in $K(\mathbb{R})$. But $\mathcal{Q}$ can be constructed from T^*, hence from $\tau(T^*)$. To see this, suppose $\mathcal{Q} = \Pi_\sigma \mathcal{Q}_\sigma/\mu$ and $\gamma = \Pi_\sigma \gamma_\sigma/\mu$. Then $\forall^*_\mu \sigma\ \mathcal{Q}_\sigma$ is the Q-structure for $\mathcal{M}^\sigma_\infty|\gamma_\sigma$ and the iterability of $\mathcal{Q}_\sigma$ is certified by T. This implies the iterability of $\mathcal{Q}$ is certified by T^*. But $\tau(T^*) \in N_\sigma$, which does not have Q-structures for $\tau(\mathcal{M}^\sigma_\infty)$. Contradiction.

(1) shows then that $\pi''_\sigma\Gamma^\sigma$ is cofinal in Γ. The proof of (2) is similar. We just prove the first statement of (2). The point is that i can be lifted to an elementary map

$$i^* : L[T^*, \mathcal{M}^\sigma_\infty] \to L[\overline{T}, S]$$

for some $\overline{T}$ and j can be lifted to

$$j^* : L[\overline{T}, S] \to N_\sigma$$

by the following definition

$$j^*(i^*(f)(a)) = \tau(f)(j(a))$$

for $f \in L[T^*, \mathcal{M}^\sigma_\infty]$ and $a \in [o(\mathcal{S})]^{<\omega}$. By the same argument as above, $\overline{T}$ certifies iterability of mice in $K(\mathbb{R})$ and hence enforces fullness for S in $K(\mathbb{R})$. This is what we want. $\square$

We can define a map $\tau : \mathcal{M}^{\sigma,+}_\infty \to \mathcal{M}^{\sigma,*}_\infty$ as follows. Let $x \in \mathcal{M}^{\sigma,+}_\infty$. There is an $i < \omega$ and a y such that in $Lp(\sigma)$, $x = \pi^{A_i}_{\mathcal{N}_i,\infty}(y)$, where $\pi^{A_i}_{\mathcal{N}_i,\infty}$ is the direct limit map from $H^{\mathcal{N}_i}_{A_i}$ into $\mathcal{M}^{\sigma,+}_\infty$ in $Lp(\sigma)$. Let

$$\tau(x) = \pi^{A_i}_{\mathcal{N}_i,\mathcal{M}^{\sigma,*}_\infty}(y),$$

where $\pi^{A_i}_{\mathcal{N}_i,\mathcal{M}^{\sigma,*}_\infty}$ witnesses $(\mathcal{N}_i, A_i) \preceq (\mathcal{M}^{\sigma,*}_\infty, A_i)$ in the hod direct limit system in $K(\mathbb{R})$.

Lemma 4.2.18. *1. $\mathcal{M}^{\sigma,*}_\infty = H^{\mathcal{M}^{\sigma,*}_\infty}_{\pi''_\sigma\Gamma_\sigma}$; furthermore, for any quasi-iterate $\mathcal{Q}$ of $\mathcal{M}^{\sigma,*}_\infty$, $\mathcal{Q} = H^{\mathcal{Q}}_{\pi''_\sigma\Gamma_\sigma}$ and $\pi^{\pi''_\sigma\Gamma_\sigma}_{\mathcal{M}^{\sigma,*}_\infty,\mathcal{Q}}(\tau^{\mathcal{M}^{\sigma,+}_\infty}_A) = \tau^{\mathcal{Q}}_A$ for all $A \in \pi''_\sigma\Gamma_\sigma$.*

2. $\tau = id$ and $\mathcal{M}^{\sigma,+}_\infty = \mathcal{M}^{\sigma,*}_\infty$.

3. $\pi_\sigma = \pi^{\pi''_\sigma\Gamma_\sigma}_{\mathcal{M}^{\sigma,+}_\infty,\infty}$.

Proof. The proof is just that of Lemmata 7.8.7 and 7.8.8 in [26]. We first show (1). In this proof, "suitable" means suitable in $K(\mathbb{R})$. The key is for any quasi-iterate $\mathcal{Q}$ of $\mathcal{M}^{\sigma,*}_\infty$, we have

$$\pi_\sigma|\mathcal{M}^{\sigma,+}_\infty = \pi^{\pi''_\sigma\Gamma_\sigma}_{\mathcal{Q},\infty} \circ \pi^{\pi''_\sigma\Gamma_\sigma}_{\mathcal{M}^{\sigma,*}_\infty,\mathcal{Q}} \circ \tau. \quad (*)$$

Using this and Theorem 4.2.17, we get $H^{\mathcal{Q}}_{\pi''_\sigma\Gamma_\sigma} = \mathcal{Q}$ for any quasi-iterate $\mathcal{Q}$ of $\mathcal{M}^{\sigma,*}_\infty$. To see this, first note that $\mathcal{Q}$ is suitable; Theorem 4.2.17 implies the collapse $\mathcal{S}$ of $H^{\mathcal{Q}}_{\pi''_\sigma\Gamma_\sigma}$ must be suitable. This means, letting δ be the Woodin of $\mathcal{Q}$, $H^{\mathcal{Q}}_{\pi''_\sigma\Gamma_\sigma}|(\delta+1) = \mathcal{Q}|(\delta+1)$. Next, we show $H^{\mathcal{Q}}_{\pi''_\sigma\Gamma_\sigma}|((\delta^+)^{\mathcal{Q}}) = \mathcal{Q}|((\delta^+)^{\mathcal{Q}})$. The proof of this is essentially that of Lemma 4.35 in [11]. We sketch the proof here. Suppose not. Let $\pi : \mathcal{S} \to \mathcal{Q}$ be the uncollapse map. Note that $\mathrm{crt}(\pi) = (\delta^+)^{\mathcal{S}}$ and $\pi((\delta^+)^{\mathcal{S}}) = (\delta^+)^{\mathcal{Q}}$. Let $\mathcal{R}$ be the result of first moving the least measurable of $\mathcal{Q}|((\delta^+)^{\mathcal{Q}})$ above δ and then doing the genericity iteration (inside $\mathcal{Q}$) of the resulting model to make $\mathcal{Q}|\delta$ generic at the Woodin of $\mathcal{R}$. Let $\mathcal{T}$ be the resulting tree. Then $\mathcal{T}$ is maximal with $lh(\mathcal{T}) = (\delta^+)^{\mathcal{Q}}$; $\mathcal{R} = Lp(\mathcal{M}(\mathcal{T}))$; and the Woodin of $\mathcal{R}$ is $(\delta^+)^{\mathcal{Q}}$. Since $\{\gamma^{\mathcal{R}}_A \mid A \in \pi''_\sigma\Gamma_\sigma\}$ are definable from $\{\tau^{\mathcal{Q}}_{A,(\delta^+)^{\mathcal{Q}}} \mid A \in \pi''_\sigma\Gamma_\sigma\}$, they are in $\mathrm{rng}(\pi)$. This gives us that $\sup H^{\mathcal{Q}}_{\pi''_\sigma\Gamma_\sigma} \cap (\delta^+)^{\mathcal{Q}} = (\delta^+)^{\mathcal{Q}}$, which easily implies $(\delta^+)^{\mathcal{Q}} \subseteq H^{\mathcal{Q}}_{\pi''_\sigma\Gamma_\sigma}$. The proof that $(\delta^{+n})^{\mathcal{Q}} \subseteq H^{\mathcal{Q}}_{\pi''_\sigma\Gamma_\sigma}$ for $1 < n < \omega$ is similar and is left for the reader.

(2) easily follows from (1). (3) follows using $(*)$ and $\tau = id$. □

For each σ such that Theorem 4.2.17 and Lemma 4.2.18 hold for σ, let Σ_σ be the canonical strategy for $\mathcal{M}^\sigma_\infty$ as guided by $\pi''_\sigma\Gamma^\sigma$. Recall $\pi''_\sigma\Gamma^\sigma$ is a cofinal collection of $\mathrm{OD}^{K(\mathbb{R})}$ sets of reals. The existence of Σ_σ follows from Theorem 7.8.9 in [26]. Note that Σ_σ has weak condensation, i.e., suppose $\mathcal{Q}$ is a Σ_σ iterate of $\mathcal{M}^{\sigma,+}_\infty$ and $i : \mathcal{M}^{\sigma,+}_\infty \to \mathcal{Q}$ is the iteration map, and suppose $j : \mathcal{M}^{\sigma,+}_\infty \to \mathcal{R}$ and $k : \mathcal{R} \to \mathcal{Q}$ are such that $i = k \circ j$ then $\mathcal{R}$ is suitable (in the sense of $K(\mathbb{R})$).

Definition 4.2.19 (Branch condensation)**.** *Let $\mathcal{M}^{\sigma,+}_\infty$ and Σ_σ be as above. We say that Σ_σ has branch condensation if for any Σ_σ iterate $\mathcal{Q}$ of $\mathcal{M}^{\sigma,+}_\infty$, letting $k : \mathcal{M}^{\sigma,+}_\infty \to \mathcal{Q}$ be the iteration map, for any maximal tree $\mathcal{T}$ on $\mathcal{M}^{\sigma,+}_\infty$, for any cofinal non-dropping branch b of $\mathcal{T}$, letting $i = i^{\mathcal{T}}_b$, $j : \mathcal{M}^{\mathcal{T}}_b \to \mathcal{P}$, where $\mathcal{P}$ is a Σ_σ iterate of $\mathcal{M}^\sigma_\infty$ with iteration embedding k, suppose $k = j \circ i$, then $b = \Sigma_\sigma(\mathcal{T})$.*

Theorem 4.2.20. $(\forall^*_\mu\sigma)$(*A tail of Σ_σ has branch condensation.*)

Proof. The proof is like that of Theorem 7.9.1 in [26]. We only mention the key points here. We assume that $\forall^*_\mu\sigma$ no Σ_σ-tails have branch condensation. Fix such a σ. First, let $X_\sigma = \mathrm{rng}(\pi_\sigma \restriction \mathcal{M}^{\sigma,+}_\infty)$ and

$$H = \mathrm{HOD}_{\{\mu,\mathcal{M}^{\sigma,+}_\infty,\mathcal{M}_\infty,\pi_\sigma,\mathrm{T}^*,\mathrm{X}_\sigma,\mathrm{x}_\sigma\}},$$

where x_σ is a real enumerating $M^{\sigma,+}_\infty$. So $H \vDash \mathsf{ZFC} +$ "$\mathcal{M}^\sigma_\infty$ is countable and ω_1^V is measurable."

Next, let $\overline{H}$ be a collapse of a countable elementary substructure of a sufficiently large rank-initial segment of H. Let $(\gamma, \rho, \mathcal{N}, \nu)$ be the preimage of $(\omega_1^V, \pi_\sigma, \mathcal{M}_\infty, \mu)$ under the uncollapse map, call it π. We have that $\overline{H} \vDash \mathsf{ZFC}^- +$ "γ is a measurable cardinal as witnessed by ν." This $\overline{H}$ will replace the countable iterable structure obtained from the hypothesis HI(c) in Chapter 7 of [26]. Now, in $K(\mathbb{R})$, the following hold true:

1. There is a term $\tau \in \overline{H}$ such that whenever g is a generic over $\overline{H}$ for $Col(\omega, < \gamma)$, then τ^g is a $(\rho, \mathcal{M}^{\sigma,+}_\infty, \mathcal{N})$-certified bad sequence. See Definitions 7.9.3 and 7.9.4 in [26] for the notions of a bad sequence and a $(\rho, \mathcal{M}^{\sigma,+}_\infty, \mathcal{N})$-certified bad sequence respectively.

2. Whenever $i : \overline{H} \to J$ is a countable linear iteration map by the measure ν and g is J-generic for $Col(\omega, < i(\gamma))$, then $i(\tau)^g$ is truly a bad sequence.

The proof of (1) and (2) is just like that of Lemma 7.9.7 in [26]. The key is that in (1), any $(\rho, \mathcal{M}_\infty^{\sigma,+}, \mathcal{N})$-certified bad sequence is truly a bad sequence from the point of view of $K(\mathbb{R})$ and in (2), any countable linear iterate J of $\overline{H}$ can be realized back into H by a map ψ in such a way that $\pi = \psi \circ i$.

Finally, using (1), (2), the iterability of $\overline{H}$, and an AD^+-reflection in $K(\mathbb{R})$ like that in Theorem 7.9.1 in [26], we get a contradiction. □

Theorem 4.2.20 allows us to run the core model induction in $L(\Sigma_\sigma, \mathbb{R})$ and show that $L(\Sigma_\sigma, \mathbb{R}) \vDash \mathsf{AD}$. This along with the fact that $\Sigma_\sigma \notin K(\mathbb{R})$ imply

$$L(\Sigma_\sigma, \mathbb{R}) \vDash \Theta > \theta_0.$$

This is a contradiction to our smallness assumption.

4.2.5 An alternative method

In this section, we outline an alternative method for proving AD holds in $K(\mathbb{R})$ and the existence of a model of "$\mathsf{AD}^+ + \Theta > \theta_0$" containing all the reals and ordinals (if $\Theta^{K(\mathbb{R})} < \Theta$). The method comes from [46]. We note that this method does not seem to work in all known core model induction arguments. This method does apply in our case because of our specific hypothesis (namely the existence of a supercompact measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ in this section and the existence of a supercompact measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$ in the next section).

We will use the terminology and notation from [46] for the argument we're about to give. The only notational difference is [46] uses $Lp(\mathbb{R})$ to denote what we call $K(\mathbb{R})$. Finally, in this subsection, we work under the smallness assumption that there is no model containing $\mathbb{R} \cup \mathrm{OR}$ that satisfies "$\mathsf{AD}^+ + \Theta > \theta_0$" since otherwise, we can use the proof in the next subsection to get a contradiction (namely $L(\mathbb{R}, \mu)$"$\in$"$L(\mathbb{R}, \mu)$, see the remark at the end of next subsection).

Let α be the strict supremum of the ordinals γ such that

1. the coarse mouse witness condition $W^*_{\gamma+1}$ holds;
2. γ is a critical ordinal in $K(\mathbb{R})$, that is, there is some $U \subset \mathbb{R}$ such that U and $\mathbb{R} \backslash U$ have scales in $K(\mathbb{R})|(\gamma + 1)$ but not in $K(\mathbb{R})|\gamma$; and
3. $\gamma + 1$ begins a Σ_1-gap in $K(\mathbb{R})$.

The proof of getting $\mathcal{M}_1^{\sharp,F}$ for any model operator F we encounter in the core model induction has been done in Subsection 4.2.3 and easily implies that α is a limit ordinal. By results in Section 2.6 of [46] (which in turns uses the smallness assumption), the pointclass $\Gamma = \Sigma_1^{K(\mathbb{R})|\alpha}$

is inductive-like and $\underset{\sim}{\Delta}_\Gamma = \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})|\alpha$. Since Γ is inductive-like and $\underset{\sim}{\Delta}_\Gamma$ is determined, $\underset{\sim}{Env}(\Gamma)$ is determined by Theorem 3.2.4 of [46]. Since whenever γ is a critical ordinal in $K(\mathbb{R})$ and $W^*_{\gamma+1}$ holds then AD holds in $K(\mathbb{R})|(\gamma+1)$, we have that AD holds in $K(\mathbb{R})|\alpha$. The following lemma mirrors Lemma 4.5.1 of [46].

Lemma 4.2.21. *Assume there is no model of "$\mathsf{AD}^+ + \Theta > \theta_0$". Let α be the strict supremum of ordinals γ such that $W^*_{\gamma+1}$ holds, γ is a critical ordinal in $K(\mathbb{R})$, and $\gamma+1$ begins a Σ_1-gap in $K(\mathbb{R})$. Suppose there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, then $\underset{\sim}{Env}(\Gamma) = \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$. Hence $K(\mathbb{R}) \vDash \mathsf{AD}^+$.*

Proof. We first show $\underset{\sim}{Env}(\Gamma) \subseteq \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$. Let $A \in \underset{\sim}{Env}(\Gamma)$, say $A \in Env(\Gamma)(x)$ for some $x \in \mathbb{R}$. By definition of Env, for each countable $\sigma \subseteq \mathbb{R}$, $A \cap \sigma = A' \cap \sigma$ for some A' that is Δ_1-definable over $K(\mathbb{R})|\alpha$ from x and some ordinal parameter. Hence for μ-almost all σ, $x \in \sigma$ and $A \cap \sigma \in C_\Gamma(\sigma)$. By mouse capturing in $K(\mathbb{R})|\alpha$ (which follows from the coarse mouse conditions W^*_γ's for $\gamma < \alpha$), for μ-almost all σ, $A \cap \sigma \in Lp(\sigma)$. But then $A \in K(\mathbb{R})$.

Now assume toward a contradiction that $\underset{\sim}{Env}(\Gamma) \subsetneq \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$. Hence $\alpha < \Theta^{K(\mathbb{R})}$. Let β^* be the end of the gap starting at α in $K(\mathbb{R})$. Let $\beta = \beta^*$ if the gap is weak and $\beta = \beta^* + 1$ if the gap is strong. Note that $\alpha \leq \beta$, $\mathcal{P}(\mathbb{R})^{K(\mathbb{R})|\beta} = \underset{\sim}{Env}(\Gamma)^{K(\mathbb{R})} \subseteq \underset{\sim}{Env}(\Gamma) \subsetneq \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$. Hence $\beta < \Theta^{K(\mathbb{R})}$ and $K(\mathbb{R})|\beta$ projects to $\mathbb{R}$. Furthermore, $K(\mathbb{R})|\beta \vDash \mathsf{AD} + \Gamma\text{-}\mathsf{MC}$. $K(\mathbb{R})|\beta \vDash \Gamma\text{-}\mathsf{MC}$ is clear; if $\beta = \beta^*$, $K(\mathbb{R})|\beta \vDash \mathsf{AD}$ by the fact that $[\alpha, \beta^*]$ is a Σ_1-gap; otherwise, $K(\mathbb{R}) \vDash \mathsf{AD}$ by Kechris-Woodin transfer theorem (see [10]). Since $K(\mathbb{R})|\beta$ projects to $\mathbb{R}$, every countable sequence from $\underset{\sim}{Env}(\Gamma)^{K(\mathbb{R})}$ is in $K(\mathbb{R})|(\beta+1)$. The scales analysis of [37] and [38], Theorem 4.3.2 and Corollary 4.3.4 of [46] together imply that there is a self-justifying-system $\mathcal{A} = \{A_i \mid i < \omega\} \subseteq \underset{\sim}{Env}(\Gamma)^{K(\mathbb{R})}$ containing a universal Γ set.

Let $U \in \mathcal{A}$ be a universal $\check{\Gamma}$ set and say $(A_{n_i} \mid i < \omega) \subseteq \mathcal{A}$ is a scale on U. So U has a scale in $K(\mathbb{R})|(\beta+1)$ but U cannot have a scale in $K(\mathbb{R})|\beta$ because $K(\mathbb{R})|\beta \cap \mathcal{P}(\mathbb{R}) \subseteq \underset{\sim}{Env}(\Gamma)^{K(\mathbb{R})}$. Hence β is a critical ordinal in $K(\mathbb{R})$.

From the self-justifying system $\mathcal{A}$ we can get a sequence of model operators $(F_n : n < \omega)$ where each F_n is in $K(\mathbb{R})|(\beta+1)$. Namely, let $F_0 = F_{\mathcal{A}}$ be the term relation hybrid operator corresponding to $\mathcal{A}$ (see Definition 4.6.1 of [46]), and let $F_{n+1} = \mathcal{M}_1^{F_n,\sharp}$ be the F_n-Woodin model operator whose existence comes from the proof in Subsection 4.2.3. Each model operator F_n condenses and relativizes well and determines itself on generic extensions. These model operators are all projective in $\mathcal{A}$ and are cofinal in the projective-like hierarchy containing $\mathcal{A}$, or equivalently in the Levy hierarchy of sets of reals definable from parameters over $K(\mathbb{R})|\beta$. Together these model operators can be used to establish the coarse mouse witness condition $W^*_{\beta+1}$. Therefore $\beta < \alpha$ by the definition of α, which is a contradiction. $\square$

The following lemma completes our outline.

Lemma 4.2.22. *Suppose AD^+ holds in $K(\mathbb{R})$ and $\Theta^{K(\mathbb{R})} < \Theta$. Assume $V = L(\mathbb{R}, \mu)$, where μ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then there is a model of "$\mathsf{AD}^+ + \Theta > \theta_0$" containing all the reals and ordinals.*

Proof. The reader should consult Section 4.6 of [46] for the notations used in this proof. Suppose toward a contradiction that the conclusion of the theorem is false. Let $\Gamma = \Sigma_1^{K(\mathbb{R})|\alpha}$, where α is defined at the beginnning of the section. Then Γ is inductive-like and $\underset{\sim}{\Delta}_\Gamma$ is determined. Since $\underset{\sim}{Env}(\Gamma) = \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$ and $\Theta^{K(\mathbb{R})} < \Theta$, there is a surjection from $\mathbb{R}$ onto $\underset{\sim}{Env}(\Gamma)$. This means there is a normal fine measure on $\mathcal{P}_{\omega_1}(\underset{\sim}{Env}(\Gamma))$. Theorems 4.1.4, 4.3.2 and Corollary 4.3.4 of [46] imply that there is a self-justifying-system $\mathcal{A} = \{A_i \mid i < \omega\}$ sealing $\underset{\sim}{Env}(\Gamma)$ and A_0 is a universal Γ-set[15].

Using the proof and notations of Section 4.6 in [46] (with $Lp^{\vec{A}}(\mathbb{R})$ there being $K^{\mathcal{A}}(\mathbb{R})$ here) and the proof of Lemma 4.2.21, we get that $M =_{\text{def}} K^{\mathcal{A}}(\mathbb{R}) \vDash \mathsf{AD}^+$. We claim that $\mathcal{A}$ is not OD_x in $K^{\mathcal{A}}(\mathbb{R})$ for any $x \in \mathbb{R}$. Suppose not. Then there is some ω-sound premouse $\mathcal{M}$ over $\mathbb{R}$ projecting to $\mathbb{R}$ such that $\mathcal{M}$ is countably iterable in M and $\mathcal{A} \in \mathcal{M}$ (this is because $M \vDash \mathsf{MC}$). Since $\mathcal{M} \lhd K(\mathbb{R})$ and $\mathcal{A}$ is cofinal in $\mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$, we have a contradiction. □

4.2.6 AD in $L(\mathbb{R}, \mu)$

Now we know $\Theta^{K(\mathbb{R})} = \Theta$. We want to show $\mathcal{P}(\mathbb{R}) \cap K(\mathbb{R}) = \mathcal{P}(\mathbb{R})$.

Lemma 4.2.23. *$\mathcal{P}(\mathbb{R}) \cap L(T^*, \mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$.*

Proof. By MC in $K(\mathbb{R})$, we have

$$(\forall^*_\mu \sigma)(\mathcal{P}(\sigma) \cap L(T, \sigma) = Lp(\sigma) \cap \mathcal{P}(\sigma)).$$

This proves the lemma. □

We now show that μ is amenable to $K(\mathbb{R})$ in the sense that μ restricting to any Wadge initial segment of $\mathcal{P}(\mathbb{R})^{K(\mathbb{R})}$ is in $K(\mathbb{R})$. The following lemma is due to Woodin.

Lemma 4.2.24. *Suppose $S = \{(x, A_x) \mid x \in \mathbb{R} \ \wedge \ A_x \in \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))\} \in K(\mathbb{R})$. Then $\mu \restriction S = \{(x, A_x) \mid \mu(A_x) = 1\} \in K(\mathbb{R})$.*

Proof. Let A_S be an ∞-Borel code[16] for S in $K(\mathbb{R})$. We may pick A_S such that it is a bounded subset of Θ^*. We may as well assume that A_S is $\text{OD}^{K(\mathbb{R})}$ and A_S codes T. This gives us

$$(\forall^*_\mu \sigma)(\mathcal{P}(\sigma) \cap L(A_S, \sigma) = \mathcal{P}(\sigma) \cap L(T, \sigma)),$$

or equivalently letting $A^*_S = \prod_\sigma A_S$,

$$\mathcal{P}(\mathbb{R}) \cap L(A^*_S, \mathbb{R}) = L(T^*, \mathbb{R}).$$

[15]In fact, a fine measure on $\mathcal{P}_{\omega_1}(\underset{\sim}{Env}(\Gamma))$ suffices.

[16]If $S \subseteq \mathbb{R}$, A_S is an ∞-Borel code for S if $A_S = (T, \psi)$ where T is a set of ordinals and ψ is a formula such that for all $x \in \mathbb{R}$, $x \in S \Leftrightarrow L[T, x] \vDash \psi[T, x]$.

We have the following equivalences:

$$\begin{aligned}(x, A_x) \in \mu \restriction S &\Leftrightarrow (\forall^*_\mu \sigma)(\sigma \in A_x \cap \mathcal{P}_{\omega_1}(\sigma))\\ &\Leftrightarrow (\forall^*_\mu \sigma)(L(A_S, \sigma) \models \emptyset \Vdash_{Col(\omega,\sigma)} \sigma \in A_x \cap \mathcal{P}_{\omega_1}(\sigma))\\ &\Leftrightarrow L(A^*_S, \mathbb{R}) \models \emptyset \Vdash_{Col(\omega,\mathbb{R})} \mathbb{R} \in A_x.\end{aligned}$$

The above equivalences show that $\mu \restriction S \in L(S^*, \mathbb{R})$. But by Lemma 4.2.23 and the fact that $\mu \restriction S$ can be coded as a set of reals in $L(S^*, \mathbb{R})$, hence $\mu \restriction S \in L(T^*, \mathbb{R})$, we have that $\mu \restriction S \in K(\mathbb{R})$. $\square$

Lemma 4.2.25. $\mathcal{P}(\mathbb{R}) \cap K(\mathbb{R}) = \mathcal{P}(\mathbb{R})$. *Hence* $L(\mathbb{R}, \mu) \models \mathsf{AD}$.

Proof. First we observe that if α is such that there is a new set of reals in $L_{\alpha+1}(\mathbb{R})[\mu]\backslash L_\alpha(\mathbb{R})[\mu]$ then there is a surjection from $\mathbb{R}$ onto $L_\alpha(\mathbb{R})[\mu]$. This is because the predicate μ is a predicate for a subset of $\mathcal{P}(\mathbb{R})$, which collapses to itself under collapsing of hulls of $L_\alpha(\mathbb{R})[\mu]$ that contain all reals. With this observation, the usual proof of condensation (for L) goes through with one modification: one must put all reals into hulls one takes.

Now suppose for a contradiction that there is an $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \mu)$ such that $A \notin K(\mathbb{R})$. Let α be least such that $A \in L_{\alpha+1}(\mathbb{R})[\mu]\backslash L_\alpha(\mathbb{R})[\mu]$. We may assume that $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subseteq K(\mathbb{R})$. By the above observation, $\alpha < \Theta = \Theta^{K(\mathbb{R})}$ because otherwise, there is a surjection from $\mathbb{R}$ on Θ, which contradicts the definition of Θ. Now if $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subsetneq \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$, then by Lemma 4.3.24, $\mu \restriction \mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \in K(\mathbb{R})$. But this means $A \in K(\mathbb{R})$. So we may assume $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] = \mathcal{P}(\mathbb{R}) \cap K(\mathbb{R})$. But this means that we can in $L_\Theta(\mathbb{R})[\mu]$ use $\mu \restriction \mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu]$ compute $\Theta^{K(\mathbb{R})}$ and this contradicts the fact that $\Theta^{K(\mathbb{R})} = \Theta$. $\square$

Lemma 4.2.25 along with Theorem 4.2.2 imply Theorem 4.2.1 assuming the smallness assumption in the previous section. We now show how to get rid of it.

Recall that we have shown $\mathsf{AD}^{K(\mathbb{R})}$. The proof of this section shows that if $\Theta^{K(\mathbb{R})} = \Theta$ then $L(\mathbb{R}, \mu) \models \mathsf{AD}$, which proves Theorem 4.2.1. So suppose $\Theta > \Theta^{K(\mathbb{R})}$. Then the proof of Section 4.2.4 produces a strategy Σ with branch condensation such that Σ is fullness preserving with respect to mice in $K(\mathbb{R})$ and $\Sigma \notin K(\mathbb{R})$. By a similar core model induction to that of getting $\mathsf{AD}^{K(\mathbb{R})}$, we get $\mathsf{AD}^{K^\Sigma(\mathbb{R})}$. Now $K^\Sigma(\mathbb{R})$ is the maximal model of $\mathsf{AD}^+ + \Theta = \theta_1$.

Let $M = K^\Sigma(\mathbb{R})$ and $H = \mathrm{HOD}^{\mathrm{M}}_{\mathbb{R}}$. Note that $\mathcal{P}(\mathbb{R})^H = \mathcal{P}(\mathbb{R})^{K(\mathbb{R})} = \mathcal{P}_{\theta_0}(\mathbb{R})^M$. We aim to show that $L(\mathbb{R}, \mu) \subseteq H$, which is a contradiction. By the proof of Theorem 4.3.24, we get that $\nu =_{\mathrm{def}} \mu \restriction \mathcal{P}(\mathbb{R})^H \in M$. Let $\pi : \mathbb{R}^\omega \to \mathcal{P}_{\omega_1}(\mathbb{R})$ be the canonical map, i.e. $\pi(\vec{x}) = \mathrm{rng}(\vec{x})$. Let $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ be in H. There is a natural interpretation of A as a set of Wadge rank less than θ_0^M, that is the preimage $\vec{A}$ of A under π has Wadge rank less than θ_0^M. Fix such an A; note that $\vec{A}$ is invariant in the sense that whenever $\vec{x} \in \vec{A}$ and $\vec{y} \in \mathbb{R}^\omega$ and $\mathrm{rng}(\vec{x}) = \mathrm{rng}(\vec{y})$ then $\vec{y} \in \vec{A}$. Let $G_{\vec{A}}$ and G_A be the Solovay games corresponding to $\vec{A}$ and A respectively. In these games, players take turns and play finite sequences of reals and suppose $\langle x_i \mid i < \omega \rangle \in \mathbb{R}^\omega$ is the natural enumeration of the reals played in a typical play in either game, then the payoff is as follows:

$$\text{Player I wins the play in } G_{\vec{A}} \text{ if } \langle x_i \mid i < \omega\rangle \in \vec{A},$$

and

$$\text{Player I wins the play in } G_A \text{ if } \{x_i \mid i < \omega\} \in A.$$

Lemma 4.2.26. *G_A is determined.*

Proof. For each $\vec{x} \in \mathbb{R}^\omega$, let $\sigma_{\vec{x}} = \mathrm{rng}(\vec{x})$. Consider the games $G^{\vec{x}}_{\vec{A}}$ and $G^{\sigma_{\vec{x}}}_A$ which have the same rules and payoffs as those of $G_{\vec{A}}$ and G_A respectively except that players are required to play reals in $\sigma_{\vec{x}}$. Note that these games are determined and Player I wins the game $G^{\vec{x}}_{\vec{A}}$ iff Player I wins the corresponding game $G^{\sigma_{\vec{x}}}_A$.

Without loss of generality, suppose $\nu(\{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid \text{Player I wins } G^\sigma_A\}) = 1$. For each such σ, let τ_σ be the canonical winning strategy for Player I given by the Moschovakis's Third Periodicity Theorem. We can easily integrate these strategies to construct a strategy τ for Player I in G_A. We show how to define $\tau(\emptyset)$ and it'll be clear that the definition of τ on finite sequences is similar. Let ρ be the restriction of μ on the Suslin co-Suslin sets of M. Note that $\rho \in M$. We know

$$\forall^*_\rho \sigma\ \tau_\sigma(\emptyset) \in \sigma.$$

We have to use ρ since the set displayed above in general does not have Wadge rank less than θ_0 in M. Normality of ρ implies

$$\exists x \in \mathbb{R}\ \forall^*_\rho \sigma\ \tau_\sigma(\emptyset) = x.$$

Let $\tau(\emptyset) = x$ where x is as above. It's easy to show τ is a winning strategy for Player I in G_A. □

The lemma and standard results of Woodin (see [47]) show that ρ (as defined in the previous lemma) is the unique normal fine measure on the Suslin co-Suslin sets of M and hence $\rho \in \mathrm{OD}^M$. This means $\rho \restriction \mathcal{P}(\mathbb{R})^H = \nu$ is OD in M. This implies $L(\mathbb{R}, \nu) \subseteq H$. But $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mu)$. Contradiction.

Remark: Using a local version of Theorem 2.0.16 and the proof of Theorem 2.0.17, we can get that whenever N is a model of "$\mathsf{AD}^+ + \Theta > \theta_0$" then N satisfies "the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is the (unique) normal fine measure on the Suslin co-Suslin sets." This means that if there is such a model N in $L(\mathbb{R}, \mu)$, then μ restricted to the Suslin co-Suslin sets of N (call this ν) is in N and is the club filter there. Hence $L(\mathbb{R}, \nu)$"$\in$"N and $\mathcal{P}(\mathbb{R}, \nu) \subseteq \mathcal{P}_{\theta_0}(\mathbb{R})^N$, but $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mu)$. Contradiction. This shows there cannot be such a model N in $L(\mathbb{R}, \mu)$.

Question: Suppose $0 < \alpha \leq \omega_1$ and $L(\mathbb{R}, \mu_\alpha) \vDash$ "$\mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2 + \mu_\alpha$ is a normal fine measure on X_α". Is it the case that $L(\mathbb{R}, \mu_\alpha) \vDash \mathsf{AD}^+$?

4.3 ZF + DC + Θ is regular + ω_1 is $\mathcal{P}(\mathbb{R})$-supercompact

In this section, we study the consistency strength of the theory (T) $\equiv$ "$\mathsf{ZF}+\mathsf{DC}+\Theta$ is regular + ω_1 is $\mathcal{P}(\mathbb{R})$-supercompact". In the first subsection, we discuss how to construct models of theory (T) from strong determinacy hypotheses and from the derived model construction. The second construction is of interest since it shows that it's possible to put a normal fine measure on top of a derived model without adding new sets of reals. The next three subsections go through various stages of the proof that Con(T) is equivalent to Con(S) where (S) $\equiv$ "$\mathsf{AD}_{\mathbb{R}}$ + there is an $\mathbb{R}$-complete measure on Θ."

4.3.1 Digression: upper bound for consistency strength

In this section, we discuss how to construct models of the theory "$\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular + ω_1 is $\mathcal{P}(\mathbb{R})$-supercompact", hence of the theory "$\mathsf{ZF}+\mathsf{DC}+\Theta$ is regular + ω_1 is $\mathcal{P}(\mathbb{R})$-supercompact". The first construction is done inside an AD^+-model and the second construction shows that we can get models of this theory by adjoining the club filter to the derived model. We merely want to illustrate the methods used to construct such models; the hypotheses used in Theorems 4.3.1 and 4.3.3 are by no means optimal.

Theorem 4.3.1. *Assume $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}} + \Theta = \theta_{\alpha+\omega}$ where α is a limit ordinal and θ_α is regular in HOD_Γ where $\Gamma = \{A \subseteq \mathbb{R} \mid w(A)$[17] $< \theta_\alpha\}$. Let μ be a measure on $\mathcal{P}_{\omega_1}(\Gamma)$ coming from the Solovay measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Let $M = \mathrm{HOD}_{\Gamma\cup\{\mu\}}$. Then $\mathcal{P}(\mathbb{R})^M = \Gamma$ and $M \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular $+\,\mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$.*

Proof. By [47], μ is unique and hence OD. This implies $\mathcal{P}(\mathbb{R})^M = \Gamma$ and hence $M \vDash \mathrm{AD}_{\mathbb{R}} + \Theta = \theta_\alpha$. The key point is $\mathcal{P}_{\omega_1}(\Gamma)^M = \mathcal{P}_{\omega_1}(\Gamma)$ since by definition, Γ is closed under ω-sequences. This implies $M \vDash \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$. Finally, $M \vDash \Theta = \theta_\alpha$ is regular since if $f : \gamma \to \theta_\alpha$ for some $\gamma < \theta_\alpha$ is in M, then f is $\mathrm{OD}_{A,\mu}$ for some $A \in \Gamma$, hence f is OD_A since μ is OD. This means $f \in \mathrm{HOD}_\Gamma$ to begin with. $\square$

We remark that the hypothesis of the theorem is consistent. For example, it follows from the theory "$\mathsf{AD}_{\mathbb{R}}+\Theta$ is Mahlo" or "$\mathsf{AD}^+ + \Theta = \theta_{\alpha+1}$" where θ_α is the largest Suslin cardinal.

Definition 4.3.2. *Let Γ_{ub} [18] be the collection of universally Baire sets and δ is a Woodin cardinal. We say that δ is **good** if whenever g is a $<\delta$-generic over V and G is a stationary tower $\mathbb{Q}^{V[g]}_{<\delta}$ generic over $V[g]$, then letting $j : V[g] \to M \subseteq V[g][G]$ be the associated embedding, $j(\Gamma^{V[g]}_{ub}) = \Gamma^{V[g][G]}_{ub}$.*

The following theorem comes from conversations between the author and G. Sargsyan.

[17] Recall $w(A)$ is the Wadge rank of A.

[18] We will also use Hom_∞ to denote the collection of universally Baire sets.

Theorem 4.3.3. *Suppose there is a proper class of Woodin cardinals. Suppose there is a cardinal δ_0 which is a supercompact cardinal. Suppose $\langle \delta_i \mid 1 \leq i < \omega\rangle$ is an increasing sequence of good Woodin cardinals above δ_0 which are also strong cardinals. Let $G \subseteq Col(\omega, <\delta_0)$ be V-generic. Then in $V[G]$, there is a class model M containing $\mathbb{R}^{V[G]}$ such that $M \vDash$ $\mathsf{AD}_{\mathbb{R}}+$ there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$ + Θ is regular.*

Remark: The hypothesis of the theorem follows from the existence of a proper class of Woodin cardinals and two supercompact cardinals (see the proof of Theorem 3.4.17 in [14]).

Proof. We assume basic facts about universally Baire sets from [29] and [14]. Let Γ_{ub} denote the collection of universally Baire sets. Let $G \subseteq Col(\omega, <\delta_0)$ be V-generic. In $V[G]$, let $\mathbb{R}^* = \mathbb{R}^{V[G]}$ and

$$\begin{aligned} Hom^* = \{A \subseteq \mathbb{R}^* \mid \quad & A \in V(\mathbb{R}^*) \wedge \exists \alpha < \delta_0 \exists T \in V[G|\alpha]\ (V[G|\alpha] \vDash \\ & \text{“}T \text{ is } \delta_0\text{-complemented”} \wedge p[T] \cap \mathbb{R}^* = A)\}. \end{aligned}$$

For more on Hom^*, see [29]. In $V[G]$, we claim that $Hom^* = \Gamma_{ub}$. Since δ_0 is a limit of strong cardinals, it's easy to see that $Hom^* \trianglelefteq \Gamma_{ub}$. To see the reverse inclusion, let $A \in \Gamma_{ub}$. Let σ be a (countable) homogeneity system witnessing this. We may assume the measures in σ have additivity κ for some $\kappa >> \delta_0$. But then any $\mu \in \sigma$ is a canonical extension of some $\nu \in V$ ($A \in \mu \Leftrightarrow \exists B \in \nu\ B \subseteq A$). This easily implies $A \in Hom^*$.

Let $j : V \to M$ witness δ_0 is measurable. We define a filter on $\mathcal{P}_{\omega_1}(Hom^*)^{V(\mathbb{R}^*)}$ as follows.

$$A \in \mathcal{F} \Leftrightarrow V[G] \Vdash_{Col(\omega,<j(\delta_0))} j^+[Hom^*] \in j^+(A)^{19}.$$

It's easy to see that $\mathcal{F} \in V[G]$ and

$$L(Hom^*, \mathcal{F}) \vDash \text{“}\mathcal{F} \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(Hom^*).\text{”}$$

Note that in the construction above (and in the proof that $Hom^* = \Gamma_{ub}^{V[G]}$), we only use that δ_0 is a measurable limit of strong and Woodin cardinals. The assumption that δ_0 is supercompact is only used at the very end of the proof to conclude the model $L(Hom^*, \mathcal{F}) \vDash$ $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

Now we claim that $|Hom^*| = \omega_1$ in $V[G]$. This is because Hom^* is determined (in $V[G]$) by V_{δ_0} and the sequence $\langle G|\alpha \mid \alpha < \delta_0\rangle$. Since $|Hom^*| = \omega_1$ in $V[G]$, we can use the club-shooting construction in [4] to get a $V[G]$-generic G' such that in $V[G][G']$, we have

- $(\mathrm{OR}^\omega)^{\mathrm{V[G]}} = (\mathrm{OR}^\omega)^{\mathrm{V[G][G']}}$, hence in particular, $\mathbb{R}^* = \mathbb{R}^{V[G][G']}$.
- $Hom^* = \Gamma_{ub}^{V[G]} = \Gamma_{ub}^{V[G][G']}$.
- In $V[G][G']$, $L(Hom^*, \mathcal{F}) \vDash$ “$\mathcal{F}$ is a normal fine measure on $\mathcal{P}_{\omega_1}(Hom^*)$” and $\mathcal{F}$ comes from the club filter.

[19] $j^+ : V[G] \to M[G][H]$ for $H \subseteq Col(\omega, <j(\delta_0))$ being $V[G]$-generic is the lift-up of j.

To ease the notation, we rename $V[G][G']$ to $V[G]$. It remains to prove the following

Lemma 4.3.4. $\mathcal{P}(\mathbb{R})^{L(Hom^*,\mathcal{F})} = Hom^*$.

Proof. Let δ be the limit of the δ_i's. Let $H \subseteq \mathbb{Q}^{V[G]}_{<\delta}$ be $V[G]$-generic and

$$j : V[G] \to M \subseteq V[G][H]$$

be the associated embedding. Let $\mathbb{R}^{**} = \mathbb{R}^{V[G][H]}$ and Hom^{**} be defined in $V(\mathbb{R}^{**})$ the same way Hom^* is defined in $V(\mathbb{R}^*)$. By our assumption on δ,

$$j(Hom^*) = Hom^{**}.$$

There is a $K \subseteq Col(\omega, <\delta)$ be $V[G]$-generic such that $\mathbb{R}^{V[G][K]} = \mathbb{R}^{**}$. Let $\mathcal{C}$ be the club filter on $\mathcal{P}_{\omega_1}(Hom^{**})^{V(\mathbb{R}^{**})}$ in $V[G][H]$, then we claim that

$$L(Hom^{**}, j(\mathcal{F})) = L(Hom^{**}, \mathcal{C}) \vDash \mathcal{C} = j(\mathcal{F}) \wedge \mathcal{C} \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(Hom^{**}).$$

We can choose H so that for all $1 \leq n < \omega$, $H \cap \mathbb{Q}_{\delta_n}$ is $V[G]-$generic. Let $j_n : V[G] \to M_n$ be the induced embedding by $H \cap \mathbb{Q}_{<\delta_n}$, hence M is the direct limit of the M_n's. Note that the j_n's factor into j via map k_n (i.e. $j = k_n \circ j_n$). Also for $n \leq k$, let $j_{n,k} : M_n \to M_k$ be the natural embedding so that k_n is the limit of the $j_{n,k}$'s.

For each $i < \omega$, let $\sigma_i = (Hom^*)^{M_i} = \Gamma^{V[H|\delta_i]}_{ub}$. Let $\mathcal{F}^*$ be the "tail filter" defined in $V[G][H]$ as follows: $A \in \mathcal{F}^* \Leftrightarrow \exists n \forall m \geq n\ k''_m \sigma_m \in A$. We claim that if $A \in j(\mathcal{C}^{V[G]})$ then $A \in \mathcal{F}^*$. To see this, let $n < \omega$ such that M_n contains the preimage of A, say $k_n(A_n) = A$. Then A_n is a club in M_n. We claim that $\forall m \geq n\ k''_m \sigma_m \in A$. We prove this for the case $m = n$. The other cases are similar. Since $k_n = k_{n+1} \circ j_{n,n+1}$, it suffices to show $j''_{n,n+1}\sigma_n \in j_{n,n+1}(A_n)$. We have that in M_n, $\sigma_n = \cup_{\alpha<\omega_1} \tau_\alpha$ where $\tau_\alpha \in A_n$. In M_{n+1}, $\{j_{n,n+1}(\tau_\alpha) \mid \alpha < \omega_1^{M_n}\}$ is a countable subset of $j_{n,n+1}(A_n)$ whose union is $j''_{n,n+1}\sigma_n$. Since $j_{n,n+1}(A_n)$ is a club in M_{n+1}, $j''_{n,n+1}\sigma_n \in j_{n,n+1}(A_n)$. Hence we're done with the claim. The claim proves that $L(Hom^{**}, \mathcal{C}^{V[G][H]}) = L(Hom^{**}, j(\mathcal{C}^{V[G]})) = L(Hom^{**}, \mathcal{F}^*) \vDash$ "$\mathcal{C}^{V[G][H]} = j(\mathcal{C}^{V[G]}) = \mathcal{F}^*$ is a normal fine measure", where $\mathcal{C}$ is the club filter. Let us note that this argument also shows that if $K \subseteq Col(\omega, <\delta)$ is $V[G]$-generic and $\mathbb{R}^{V[G|\delta_i]} = \mathbb{R}^{V[G][H|\delta_i]}$ (hence by choice of δ_i, $(Hom^*)^{V[G|\delta_i]} = \sigma_i$) and $\mathcal{G}$ is the "tail filter" defined in $V[G][K]$ from the sequence $\langle \sigma_i \mid i < \omega \rangle$, then $L((Hom^*)^{V[G]}, \mathcal{F})$ embeds into $L((Hom^*)^{V[G][K]}, \mathcal{G})$.

Now it suffices to show $L(Hom^{**}, \mathcal{C}) \vDash \mathsf{AD}^+$. This then will imply $\mathcal{P}(\mathbb{R})^{L(Hom^{**},\mathcal{C})} = Hom^{**}$ since otherwise, there is $A \in \mathcal{P}(\mathbb{R})^{L(Hom^{**},\mathcal{C})} \backslash Hom^{**}$ such that $L(A, \mathbb{R}^{**}) \vDash \mathsf{AD}^+$. By the choice of δ and a theorem of Woodin, $Hom^{**} = \{A \subseteq \mathbb{R}^{**} \mid A \in V(\mathbb{R}^{**}) \wedge L(A, \mathbb{R}^{**}) \vDash \mathsf{AD}^+\}$[20]. This is a contradiction. By elementarity, $\mathcal{P}(\mathbb{R})^{L(Hom^*,\mathcal{F})} = Hom^*$ and hence the lemma follows.

To show $L(Hom^{**}, \mathcal{C}) \vDash \mathsf{AD}^+$, we use the tree production lemma. Suppose not. Let $x \in \mathbb{R}^{**}$, $T \in V[G][H|\alpha]$ for some $\alpha < \delta$ be a δ-complemented tree, γ be least such that

[20]In fact, this equality holds for δ being limit of Woodin and $< -\delta$-strong cardinals.

there is a counter-example of AD^+ $B \in L(Hom^{**}, \mathcal{C})$ definable over $L_\gamma(Hom^{**}, \mathcal{C})$ from $(\varphi, x, p[T] \cap \mathbb{R}^{**})$ i.e.

$$y \in B \Leftrightarrow L_\gamma(Hom^{**}, \mathcal{C}) \vDash \varphi[y, p[T] \cap \mathbb{R}^{**}, x].$$

Let $\theta(u, v)$ be the natural formula defining B (where $\mathcal{C}$ is the club filter):

$\theta(u, v) \quad = \quad$ "$L(\Gamma_{ub}, \mathcal{C}) \vDash \mathcal{C}$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\Gamma_{ub})$ and $L(\Gamma_{ub}, \mathcal{C}) \vDash \exists B(\mathsf{AD}^+$ fails for $B)$ and if γ_0 is the least γ such that $L_\gamma(\Gamma_{ub}, \mathcal{C}) \vDash \exists B(\mathsf{AD}^+$ fails for $B)$then $L_{\gamma_0}(\Gamma_{ub}, \mathcal{C}) \vDash \varphi[u, p[T] \cap \mathbb{R}, v]$".

We verify that the tree production lemma holds for $\theta(-, x)$. This gives $B \in Hom^{**}$. Without loss of generality, let $g \in HC^{V[H]}$ be such that $(G, H|\alpha, x, T) \in V[g]$ and $(Hom^*)^{V[g]} = \Gamma_{ub}^{V[g]}$ and $L((Hom^*)^{V[g]}, \mathcal{C}) \vDash \mathcal{C}$ is a normal fine measure on $\mathcal{P}_{\omega_1}((Hom^*)^{V[g]})$ where $\mathcal{C}$ is the club filter in $V[g]$. We can make this assumption about g because δ is a limit of measurable cardinals which are limits of Woodin and strong cardinals. Let $\xi < \delta$ be a good Woodin cardinal. We first verify stationary correctness. Let $K \subseteq \mathbb{Q}_{<\xi}^{V[g]}$ be $V[g]$-generic, and

$$k : V[g] \to N \subseteq V[g][K]$$

be the associated embedding. By the property of ξ, $k(\Gamma_{ub}^{V[g]}) = \Gamma_{ub}^N = \Gamma_{ub}^{V[g][K]}$. Furthermore, $\mathcal{C}^N \subseteq \mathcal{C}^{V[g][K]}$ (here $\mathcal{C}$ denotes the club filter in the relevant universe) and by elementarity, $L(\Gamma_{ub}^N, \mathcal{C}^N) \vDash \mathcal{C}^N$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\Gamma_{ub})$. This implies $L(\Gamma_{ub}, \mathcal{C})^N = L(\Gamma_{ub}, \mathcal{C})^{V[g][K]}$. Hence we're done.

To verify generic absoluteness at ξ. We rename $V[g]$ to V. Let g be $< \xi$-generic over V and h be $< \xi^+$-generic over $V[g]$. Let $y \in \mathbb{R}^{V[g]}$. We want to show

$$V[g] \vDash \theta[y, x] \Leftrightarrow V[g][h] \vDash \theta[y, x].$$

There are $G_0, G_1 \subseteq Col(\omega, < \delta)$ such that G_0 is generic over $V[g]$ and G_1 is generic over $V[g][h]$ with the property that $\mathbb{R}^{V[G_0|\delta_i]} = \mathbb{R}^{V[G_1|\delta_i]}$ for all $\delta_i > \xi$. Also, $(Hom^*)^{V[G_0|\delta_i]} = (Hom^*)^{V[G_1|\delta_i]} = \Gamma_{ub}^{V[G_0|\delta_i]} = \Gamma_{ub}^{V[G_1|\delta_i]}$. Let us denote this σ_i. Such G_0 and G_1 exist since h is generic over $V[g]$ and $\xi < \delta$. So we get that $(Hom^*)^{V[g][G_0]} = (Hom^*)^{V[g][h][G_1]}$. By the discussion above, $L(Hom^*, \mathcal{C})^{V[g]}$ is embeddable into $L(Hom^*, \mathcal{G})^{V[g][G_0]}$ and $L(Hom^*, \mathcal{C})^{V[g][h]}$ is embeddable into $L(Hom^*, \mathcal{G})^{V[g][h][G_1]}$ and $L(Hom^*, \mathcal{G})^{V[g][G_0]} = L(Hom^*, \mathcal{G})^{V[g][h][G_1]}$, where $\mathcal{G}$ is the "tail filter"[21] defined from the sequence $\langle \sigma_i \mid i < \omega \rangle$. This proves generic absoluteness. $\square$

Lemma 4.3.4 completes the proof of the theorem since by the derived model theorem (cf. [29]), $L(Hom^*, \mathbb{R}^*) \vDash \mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$ and $Hom^* = \mathcal{P}(\mathbb{R})^{L(Hom^*, \mathbb{R}^*)}$, hence $M = L(Hom^*, \mathbb{R}^*, \mathcal{F}) \vDash \mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$. Finally, Woodin (unpublished) has shown that $L(Hom^*, \mathbb{R}^*)$ $\vDash \mathsf{AD}_\mathbb{R} + \Theta$ is regular and in fact, since M is very close to $L(Hom^*, \mathbb{R}^*)$ the same proof also shows $M \vDash \mathsf{AD}_\mathbb{R} + \Theta$ is regular. This finishes the proof of the theorem. $\square$

[21] This piece of the proof was pointed out by the author's advisor, Prof. John Steel. The author would like to thank him for this.

4.3.2 $\Theta > \theta_0$

Without loss of generality, we assume $V = L(\mathcal{P}(\mathbb{R}), \mu)$[22] and Θ is regular and μ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$. For each $\sigma \in \mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$, let $M_\sigma = \mathrm{HOD}_{\sigma\cup\{\sigma\}}$ and $H_\sigma = \mathrm{HOD}^{\mathrm{M}_\sigma}_{\{\sigma\}}$. Let $M = \prod_\sigma M_\sigma/\mu$ and $H = \prod_\sigma H_\sigma/\mu$.

Lemma 4.3.5. *Let $\psi(v_0, v_1)$ be a formula in the language of set theory and $[f]_\mu \in M$. Then $\exists x\ (M \vDash \psi[x, [f]_\mu])$ if and only if $\forall^*_\mu \sigma \exists x\ (M_\sigma \vDash \psi[x, f(\sigma)])$.*

Proof. We prove the lemma by induction on the complexity of the formula ψ. It's enough to show that if $\forall^*_\mu \sigma \exists x\ (M_\sigma \vDash \psi[x, f(\sigma)])$ then $\exists x\ (M \vDash \psi[x, [f]_\mu])$. For a typical σ, let $A_\sigma = \{x \in \sigma \mid \exists y \in OD(x)\ (M_\sigma \vDash \psi[y, f(\sigma)])\}$. By our hypothesis, the function $F(\sigma) = A_\sigma$ is such that $\forall^*_\mu \sigma\ (F(\sigma) \neq \emptyset \wedge F(\sigma) \subseteq \sigma)$. By normality of μ, there is an x such that $\forall^*_\mu \sigma(x \in F(\sigma))$. Fix such an x and let $g(\sigma)$ be the least $\mathrm{OD}(x)$ set y such that $M_\sigma \vDash \psi[y, f(\sigma)]$ if such a y exists and $\emptyset$ otherwise. Then $M \vDash \psi[[g]_\mu, [f]_\mu]$. $\square$

By the lemma, we can identify M and H with their transitive collapse. Let $\Omega = [\lambda\sigma.\omega_1]_\mu$.

Lemma 4.3.6. *1. $\mathcal{P}(\mathbb{R}) \subseteq M$.*

2. $\Omega > \Theta$.

Proof. For each $\sigma \in \mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$, let $\mathbb{R}_\sigma = \mathbb{R} \cap \sigma$ and $\Theta^\sigma = \Theta^{M_\sigma}$. To prove (1), note first that $\mathbb{R} = [\lambda\sigma.\mathbb{R}_\sigma]_\mu = \mathbb{R}^M$. Let $A \subseteq \mathbb{R}$. By fineness of μ, $\forall^*_\mu \sigma\ (A \in \sigma)$. This means $\forall^*_\mu \sigma\ (A \cap \mathbb{R}_\sigma \in M_\sigma)$ and $A = [\lambda\sigma.A \cap \sigma] \in M$ (by Lemma 4.3.5). This finishes the proof of (1).

For (2), it's clear that $\Theta \leq \Omega$ since $\forall^*_\mu \sigma(\mathbb{R}_\sigma = \mathbb{R}^{M_\sigma})$ is countable. Using the easily verified fact that there are no sequences of ω_1^V distinct reals, we get $\mathcal{P}(\mathbb{R}_\sigma)^{M_\sigma}$ is countable, which implies $\Theta^\sigma < \omega_1^V$. $\square$

We then observe that Ω is measurable in M and in H. This is because ω_1 is measurable in M_σ and H_σ for all σ. Using the fact that there are no sequences of ω_1^V distinct reals and Lemma 4.3.5, we get, by a standard Vopenka argument, for any set of ordinals $A \in M$ of size less than Ω, there is an H-generic G_A (for a forcing of size smaller than Ω) such that $A \in H[G_A] \subseteq M$ and Ω is also measurable in $H[G_A]$.

With the set-up above, using a similar argument as in the previous section, we get a model of "$\mathsf{AD}^+ + \Theta > \theta_0$". The complete proof of this will be written up in [45]. Let us mention one key point in adapting the proof of Section 4.2 in this situation. The proof of Section 4.2 basically shows that if $\Theta^{K(\mathbb{R})} < \Theta$ and there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ then there is a model of "$\mathsf{AD}^+ + \Theta > \theta_0$". Since we have a normal fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$, we can get a model of "$\mathsf{AD}^+ + \Theta > \theta_0$" without needing to check in advance whether $\Theta^{K(\mathbb{R})} < \Theta$.

[22] V is a structure of the language of set theory with an extra predicate for μ.

4.3.3 $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular

Let $\Gamma_\alpha = \mathcal{P}(\mathbb{R})^{M_\alpha}$ and $\Gamma_{\omega_1} = \cup_{\alpha<\omega_1}\Gamma_\alpha$. We assume throughout this section that there are no class models M containing $\mathbb{R}$ such that $M \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

Theorem 4.3.7. *$L(\Gamma_{\omega_1}, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \Gamma_{\omega_1}$, hence $L(\Gamma_{\omega_1}, \mathbb{R}) \vDash AD_{\mathbb{R}} + DC$. Furthermore, there is a hod pair $(\mathcal{P}, \Sigma)$ such that Σ is Γ-fullness preserving, $\Sigma \notin \Gamma$, and $L(\Sigma, \mathbb{R}) \vDash \mathrm{AD}^+$.*

Proof. Let $\mathcal{H} = \mathrm{HOD}^\Gamma$ and $\theta = \Theta^\Gamma$. Note that $\theta < \Theta$ and $o(\mathcal{H}) = \theta$. It's enough to construct a hod pair $(\mathcal{P}, \Sigma)$ as stated in the theorem since then we have $L(\Sigma, \mathbb{R}) \vDash \Theta = \theta_{\omega_1+1}$ and $\Gamma_{\omega_1} = \mathcal{P}_{\theta_{\omega_1}}(\mathbb{R})^{L(\Sigma,\mathbb{R})}$ by the maximality of Γ_{ω_1}. This shows that Γ_{ω_1} satisfies the first clause of the theorem.

Suppose no such pair $(\mathcal{P}, \Sigma)$ exists. Write Γ for Γ_{ω_1}. Let μ_Γ be the normal fine measure on $\mathcal{P}_{\omega_1}(\Gamma)$ induced by μ. $\forall^*_{\mu_\Gamma}\sigma \prec \Gamma$, let $\mathcal{H}_\sigma = \mathrm{HOD}^\sigma$. We then let

$$\mathcal{H}^+ = \Pi_\sigma Lp^{\bigoplus_{\alpha<\Theta^\sigma}\Sigma^{\mathcal{H}_\sigma}_\alpha}(\mathcal{H}_\sigma)/\mu_\Gamma. \tag{4.1}$$

Let M be a structure of $\mathsf{ZF}^- + \mathsf{DC}$ such that $\mathcal{P}(\mathbb{R}) \twoheadrightarrow M$ and $\mathcal{H}^+, \Gamma \in M$. Let μ_M be the normal fine measure on $\mathcal{P}_{\omega_1}(M)$ induced by μ. It's easy to see that

$$\forall^*_{\mu_M}\sigma\ (\sigma \prec M).$$

For each such σ, let M_σ be the transitive collapse of σ and $\pi_\sigma : M_\sigma \to M$ be the uncollapse map. Let $\pi_\sigma(\mathcal{H}^{\sigma,+}, \omega_1^\sigma, \theta^\sigma, \Gamma^\sigma) = (\mathcal{H}^+, \omega_1, \theta, \Gamma)$. Then by the definition of $\mathcal{H}^+$, $\mathcal{H}^{\sigma,+} = Lp^{\oplus_{\alpha<\omega_1^\sigma}\Sigma^{\mathcal{H}^\sigma}_\alpha}(\mathcal{H}^\sigma) = \cup\{\mathcal{M} \mid \mathcal{H}^\sigma \lhd \mathcal{M} \wedge \rho(\mathcal{M}) \leq o(\mathcal{H}^\sigma) \wedge \mathcal{M}$ is a $\oplus_{\alpha<\omega_1^\sigma}\Sigma^{\mathcal{H}^\sigma}$-mouse in $\Gamma\}$. We first prove the following.

Lemma 4.3.8. *No level $\mathcal{M}$ of $\mathcal{H}^+$ is such that $\rho(\mathcal{M}) < \theta$.*

Proof. We start with the following.
Claim: For measure one many σ, for any $\beta < \omega_1^\sigma$, $\Sigma^{\mathcal{H}^\sigma}_\beta$ is Γ-fullness preserving.

Proof. Fix a $\beta < \omega_1^\sigma$. By the HOD analysis in Γ^σ, there is a hod pair $(\mathcal{P}, \Sigma)$ such that

- Σ is Γ^σ-fullness preserving and has branch condensation;
- $\mathcal{H}^\sigma(\beta)$ is an iterate of Σ.

Using π_σ, we get that $\pi_\sigma(\Sigma)$ is an (ω_1, ω_1) strategy for $\mathcal{P}$ that is Γ-fullness preserving and has branch condensation. Since $\Sigma = \pi_\sigma(\Sigma) \restriction \Gamma^\sigma$, $\Sigma^{\mathcal{H}^\sigma}_\beta$ is an iterate of $\pi_\sigma(\Sigma)$ and hence satisfies the conclusion of the claim. □

We are now ready to finish the proof of the lemma. Fix a σ in the claim. Let $\mathcal{H}^{\sigma,*}$ be the least level of $\mathcal{H}^+_\sigma$ that projects across θ^σ. We may assume $\mathcal{H}^*_\sigma \in M_\sigma$; otherwise, let $\mathcal{H}^* = \Pi_\sigma\mathcal{H}^*_\sigma/\mu_M$ and choose a transitive model N of $\mathsf{ZF}^- + \mathsf{DC}$ such that $\mathcal{P}(\mathbb{R}) \cup \mathcal{H}^* \cup \{\mathcal{H}^*\} \subseteq N$

and $\mathcal{P}(\mathbb{R}) \twoheadrightarrow N$. We can then let μ_N be the supercompact measure on $\mathcal{P}_{\omega_1}(N)$ given by μ and work with the pair (N, μ_N) instead of (M, μ_M).

Let Σ_σ be the natural strategy of $\mathcal{H}^*_\sigma$ defined from π_σ (see page 237 of [21]). The important properties of Σ_σ are:

1. Σ_σ extends $\Sigma^-_\sigma =_{def} \oplus_{\alpha<\omega_1^\sigma} \Sigma_{\mathcal{H}^\sigma(\alpha)}$ and Σ_σ is $\mathrm{OD}_{\{\sigma\}}$;

2. whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{H}^*_\sigma, \Sigma_\sigma)$, for all $\alpha < \lambda^{\mathcal{Q}}$, $\Sigma_{\mathcal{T},\mathcal{Q}(\alpha)}$ is the pullback of some strategy of a hod pair $(\mathcal{R}, \Lambda)$ such that Λ has branch condensation and is Γ-fullness preserving and hence by Theorem 2.7.6 of [23], $\Sigma_{\mathcal{T},\mathcal{Q}(\alpha)}$ has branch condensation;

3. Σ_σ is $\Gamma(\mathcal{H}^*_\sigma, \Sigma_\sigma)$-fullness preserving by (2) and Theorem 2.7.6 of [23];

4. a tail $(\mathcal{Q}, \Lambda)$ of $(\mathcal{H}^*_\sigma, \Sigma_\sigma)$ has branch condensation by (1)-(3) and Theorem 2.8.1 of [23].

If $\Lambda \notin \Gamma$, where Λ is as in (4) above, then since Λ has branch condensation, we can run a core model induction argument like above to prove $L(\Lambda, \mathbb{R}) \vDash \mathsf{AD}^+$. This pair $(\mathcal{Q}, \Lambda)$ witnesses the second clause of the theorem. This also implies the first clause of the theorem since Γ_{ω_1} is then equal to $\{A \in L(\Lambda, \mathbb{R}) \mid w(A) < \theta^{L(\Lambda,\mathbb{R})}_{\omega_1}\}$ and hence is constructibly closed. This completes the proof of Theorem 4.3.7 in this case.

Suppose then that $\Lambda \in \Gamma$. We can define a direct limit system $\mathcal{F} = \{(\mathcal{Q}', \Lambda') \mid (\mathcal{Q}', \Lambda') \equiv (\mathcal{Q}, \Lambda)\}$ in Γ[23] (this uses that $\Lambda \in \Gamma$). Let $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$. Hence $\mathcal{M}_\infty \in \mathrm{HOD}^\Gamma$, $\mathrm{HOD}^\Gamma|\gamma_\sigma$[24] $\lhd \mathcal{M}_\infty$ by fullness preservation of Σ^-_σ, and $\rho_1(\mathcal{M}_\infty) < \gamma_\sigma$. This means $\mathcal{M}_\infty$ constructs a bounded subset of γ_σ in HOD^Γ but not in $\mathrm{HOD}^\Gamma|\gamma_\sigma$. This contradicts the fact that $\mathrm{HOD}^\Gamma|\gamma_\sigma = V^{\mathrm{HOD}^\Gamma}_{\gamma_\sigma}$ and γ_σ is a strong limit cardinal in HOD^Γ. □

Lemma 4.3.9. *$\forall^*_{\mu_M}\sigma$, there is a Γ-fullness preserving strategy Σ_σ of $\mathcal{H}^+_\sigma$.*

Proof. First let $\forall^*_{\mu_M}\sigma$, Σ_σ be the natural strategy of $\mathcal{H}^+_\sigma$ defined using π_σ (on page 237 of [21]). This strategy is mentioned in the proof of the previous lemma. There it was a strategy for $\mathcal{H}^*_\sigma$. We still have properties (1)-(4) of Σ_σ as described in Lemma 4.3.8. Fix such a σ. We may assume $\Sigma_\sigma \in \Gamma$ as otherwise, the argument in Lemma 4.3.8 will finish the proof of this lemma. Note that since Σ_σ is $OD_{\{\sigma\}}$, $\Sigma_\sigma \cap (N_\sigma =_{def} \mathrm{HOD}_{\sigma\cup\{\sigma\}}) \in N_\sigma$. By (4) in Lemma 4.3.8, some tail of Σ_σ has branch condensation and in fact, since N_σ is closed under Σ_σ, in N_σ there is a Σ_σ-iterate $(\mathcal{Q}, \Lambda)$ such that Λ has branch condensation. Let $i : \mathcal{H}^+_\sigma \to \mathcal{Q}$ be the iteration map and $\Lambda_\sigma = \Lambda^i$ is the pull-back of Λ to $\mathcal{H}^+_\sigma$. Since $i \in N_\sigma$, $\Lambda_\sigma \in OD_{\sigma\cup\{\sigma\}}$ and note that Λ_σ has hull condensation.

What we've shown is that $\forall^*_{\mu_M}\sigma$, there is a strategy Λ_σ for $\mathcal{H}^+_\sigma$ such that Λ_σ is $OD_{\sigma\cup\{\sigma\}}$, has hull condensation, and satisfies properties (1)-(4) in Lemma 4.3.8. By normality, there

[23]This means these hod pairs are Dodd-Jensen equivalent. The fine structural details involved in the comparison process is described in [23].

[24]$\gamma_\sigma = \sup(\pi_\sigma[\theta^\sigma]) = \sup(i^{\Sigma^-_\sigma}_{\mathcal{H}^{\sigma,+},\infty}[\theta^\sigma]) < \theta$ since $\mathrm{cof}(\theta)$ is uncountable.

is an x such that $\forall^*_{\mu_M}\sigma\ x \in \sigma$ and there is Λ_σ with hull condensation, is $\Lambda_\sigma \in OD_{x\cup\{\sigma\}}$, and satisfies (1)-(4) in Lemma 4.3.8. So $\forall^*_{\mu_M}\sigma$, let Λ_σ be the least $OD_{x\cup\{\sigma\}}$ such strategy for $\mathcal{H}^+_\sigma$. We claim then that Λ_σ is Γ-fullness preserving. Suppose not. Let $(\mathcal{U},\mathcal{M})$ witness this, that is, $\mathcal{U}$ is a normal tree (or stack of normal trees) according to Λ_σ such that letting $i:\mathcal{H}^+_\sigma \to \mathcal{M}$ be the iteration map, then $\mathcal{M}$ is not Γ-full.

Next, let $\mathcal{P}_\sigma$ be a 2-Λ_σ-suitable premouse over $\mathcal{H}^+_\sigma$. Suppose there is $\mathcal{N}_\sigma \lhd \mathcal{P}_\sigma$ such that $\rho(\mathcal{N}_\sigma) \leq \theta^\sigma$ and $\mathcal{N}_\sigma \notin \mathcal{H}^+_\sigma$, then we let $\mathcal{N} = \prod_\sigma \mathcal{N}_\sigma/\mu_M$ and run a similar proof to that of Lemma 4.3.8 to get a contradiction. Here are some details.

First suppose $\rho(\mathcal{N}_\sigma) < \theta^\sigma$. Let $\xi = \max\{\rho(\mathcal{N}_\sigma),\kappa\}$, where κ is the cofinality in $\mathcal{H}^+_\sigma$ of θ^σ. We let $\Sigma^{-,\xi}_\sigma$ be the fragment of Σ^-_σ for stacks on $\mathcal{H}^+_\sigma$ above γ, where γ is the least α such that $\delta^{\mathcal{H}^+_\sigma}_\alpha \geq \xi$. Recall we have assumed without loss of generality that $\mathcal{N}_\sigma \in M_\sigma$, which means π_σ acts on $\mathcal{N}_\sigma$, which in turns implies that whenever $i:\mathcal{H}^+_\sigma \to \mathcal{P}$ is according to $\Sigma^{-,\xi}_\sigma$, then $\mathrm{Ult}(\mathcal{N}_\sigma, E_i)$ is wellfounded and in fact factors into $\mathcal{N}$ via some k such that $\pi_\sigma \restriction \mathcal{N}_\sigma = k \circ i^*$, where $i^*: \mathcal{N}_\sigma \to \mathrm{Ult}(\mathcal{N}_\sigma, E_i)$ is the ultrapower map. We then consider the system $\mathcal{F}$ of tuples $(\mathcal{R},\mathcal{P},\Sigma)$ where $(\mathcal{P},\Sigma) \equiv_{DJ} (\mathcal{H}^+_\sigma, \Sigma^{-,\xi}_\sigma)$ and if $i:\mathcal{P}\to\mathcal{S}$ and $j:\mathcal{H}^+_\sigma \to \mathcal{R}$ are comparison maps, then $\mathrm{Ult}(\mathcal{N}_\sigma, E_j) = \mathrm{Ult}(\mathcal{R}, E_i)$, where E_j is the $(\mathrm{crt}(j), \delta_{\lambda^{\mathcal{R}}})$-extender derived from j and likewise for i. To see that this is a direct limit system, it's enough to see that whenever $(\mathcal{R}_0,\Sigma_0)$ and $(\mathcal{R},\Sigma_1)$ are $\Sigma^{-,\xi}_\sigma$-iterates of $\mathcal{H}^+_\sigma$, then letting $j_i:\mathcal{H}^+_\sigma \to \mathcal{R}_i$ be the iteration maps for $i\in\{0,1\}$, letting $\mathcal{S}_i = \mathrm{Ult}(\mathcal{N}_\sigma, E_{j_i})$, then letting $k_i:\mathcal{R}_i \to \mathcal{W}_i$ be comparison maps and $\mathcal{Y}_i = \mathrm{Ult}(\mathcal{S}_i, E_{k_i})$, then $\mathcal{Y}_0 = \mathcal{Y}_1$. This is easy to see since $k_0\circ j_0 = k_1 \circ j_1$ by the Dodd-Jensen property of $\Sigma^{-,\xi}_\sigma$. This means $\mathcal{Y}_0 = \mathrm{Ult}(\mathcal{N}_\sigma, E_{k_0\circ j_0}) = \mathrm{Ult}(\mathcal{N}_\sigma, E_{k_1\circ j_1}) = \mathcal{Y}_1$. By a similar argument as that of Lemma 4.3.8, we get that $\mathcal{M}_\infty =_{def} \mathrm{dirlim}(\mathcal{F})$ under iteration maps via $\Sigma^{-,\xi}_\sigma$ is in $\mathrm{HOD}^\Gamma_{\Sigma_{\sigma,\mathcal{H}_\sigma(\gamma)}}$. Furthermore, letting $\gamma = \sup(i^{\Sigma^{-,\xi}_\sigma}_{\mathcal{H}^+_\sigma,\infty}[\theta^\sigma])$, then γ is a limit of Woodin cardinals in $\mathrm{HOD}^\Gamma_{\Sigma_{\sigma,\mathcal{H}_\sigma(\gamma)}}$ and $\mathcal{M}_\infty|\gamma = \mathrm{HOD}^\Gamma_{\Sigma_{\sigma,\mathcal{H}_\sigma(\gamma)}}|\gamma$. Yet $\rho(\mathcal{M}_\infty) < \gamma$. Contradiction.

Now suppose $\rho(\mathcal{N}_\sigma) = \theta^\sigma$. The idea of the following argument goes back to [23]. Let $f:\kappa \to \theta^\sigma$ be an increasing and cofinal map in $\mathcal{H}^+_\sigma$. We construe $\mathcal{N}_\sigma$ as the sequence $g = \langle \mathcal{N}_\alpha \mid \alpha < \kappa\rangle$, where $\mathcal{N}_\alpha = \mathcal{N}_\sigma \cap \delta^{\mathcal{H}^+_\sigma}_{f(\alpha)}$. Note that $\mathcal{N}_\alpha \in \mathcal{H}^+_\sigma$ for each $\alpha < \kappa$. Now let $\mathcal{R}_0 = \mathrm{Ult}_0(\mathcal{H},\mu)$, $\mathcal{R}_1 = \mathrm{Ult}_\omega(\mathcal{N}_\sigma,\mu)$, where $\mu \in \mathcal{H}^+_\sigma$ is the measure on κ with Mitchell order 0. Let $i_0:\mathcal{H}^+_\sigma \to \mathcal{R}_0$, $i_1:\mathcal{N}_\sigma \to \mathcal{R}_1$ be the ultrapower maps. Letting $\delta = \delta_{\lambda^{\mathcal{H}^+_\sigma}}$, it's easy to see that $\mathcal{P}(\delta)^{\mathcal{R}_0} = \mathcal{P}(\delta)^{\mathcal{R}_1}$. This means $\langle i_1(\mathcal{N}_\alpha) \mid \alpha < \kappa\rangle \in \mathcal{R}_0$. By fullness of $\mathcal{H}^+_\sigma$ in Γ, $\langle i_1(\mathcal{N}_\alpha) \mid \alpha<\kappa\rangle \in \mathcal{H}^+_\sigma$. Using i_0, $\langle i_1(\mathcal{N}_\alpha)\mid\alpha<\kappa\rangle \in \mathcal{H}^+_\sigma$, and the fact that $i_0 \restriction \mathcal{H}^+_\sigma|\theta^\sigma = i_1 \restriction \mathcal{N}_\sigma|\theta^\sigma \in \mathcal{H}^+_\sigma$, we can get $\mathcal{N}_\sigma \in \mathcal{H}^+_\sigma$ as follows. For any $\alpha,\beta < \theta^\sigma$, $\alpha \in \mathcal{N}_\beta$ if and only if $i_0(\alpha) \in i_1(\mathcal{N}_\beta) = i_0(\mathcal{N}_\beta)$. Since $\mathcal{H}^+_\sigma$ can compute the right hand side of the equivalence, it can compute the sequence $\langle \mathcal{N}_\alpha \mid \alpha<\kappa\rangle$. Contradiction. Hence we have $\mathcal{P}(\theta^\sigma)\cap\mathcal{H}^+_\sigma = \mathcal{P}(\theta^\sigma)\cap\mathcal{P}_\sigma$.

We may assume now that π_σ acts on $\mathcal{P}_\sigma$ and $\forall^*_{\mu_M}\tau$, $\pi_{\sigma,\tau}(\mathcal{P}_\sigma) = \mathcal{P}_\tau$. Suppose Λ_σ is not Γ-fullness preserving. By $\Sigma^2_1(\Lambda_\sigma)$-absoluteness and the fact that $\mathcal{P}_\sigma$ is Λ_σ-full, $\mathcal{P}_\sigma[g] \vDash$ "there is some $(\mathcal{T},\mathcal{R},\Sigma_{\mathcal{R}}) \in I(\mathcal{H}^+_\sigma,\Lambda_\sigma)$ such that $\mathcal{R}$ is not full" where $g \subseteq Col(\omega,\mathcal{H}^+_\sigma)$ is generic over $\mathcal{P}_\sigma$. So there is a strong cutpoint $\xi \in [\delta^{\mathcal{R}}_\alpha, \delta^{\mathcal{R}}_{\alpha+1})$ of $\mathcal{R}$ for some $\alpha < \lambda^{\mathcal{R}}$ such that there is an $\mathcal{N}$ which is a sound $\Sigma_{\mathcal{R}(\alpha)}$-mouse projecting to ξ but $\mathcal{N}\notin\mathcal{R}$. $\mathcal{N}$ is iterable in $\mathcal{P}[g]$.

Let $i : \mathcal{P}_\sigma \to \mathcal{Q}$ be the lift of $i^\mathcal{T}$ to all of $\mathcal{P}_\sigma$. Note that $\mathcal{Q}$ is wellfounded since $i^\mathcal{T}$ factors into π_σ. $\mathcal{P}_\sigma[g]$ can compare $\mathcal{N}$ and $\mathcal{Q}$ and sees that $\mathcal{N}$ wins the comparison. But this means $\mathcal{N}$ collapses cardinals of $\mathcal{P}_\sigma[g]$ above $\mathcal{H}^+$. Contradiction. □

Now, by [23], a tail $(\mathcal{P}, \Sigma)$ of $(\mathcal{H}^+_\sigma, \Sigma_\sigma)$ has branch condensation. Let $k : \mathcal{H}^+_\sigma \to \mathcal{P}$ be the iteration map. By the property of Σ_σ, there is a map $l : \mathcal{P} \to \mathcal{H}^+$ such that $\pi_\sigma | \mathcal{H}^+_\sigma = l \circ k$. Finally, we claim that $\Sigma \notin \Gamma$ and since Σ has branch condensation, we can show $L(\Sigma, \mathbb{R}) \vDash \mathsf{AD}^+$.

It remains to prove the claim. First suppose:

(T) $=_{def}$ For any Σ-iterate $(\mathcal{Q}, \Lambda)$ of $(\mathcal{P}, \Sigma)$, letting $i_{\mathcal{P},\mathcal{Q}} : \mathcal{P} \to \mathcal{Q}$ be the iteration map, there is $\tau_\mathcal{Q} : \mathcal{Q} \to \mathcal{H}^+$ be such that $l = \tau_\mathcal{Q} \circ i_{\mathcal{P},\mathcal{Q}}$ such that $\tau_\mathcal{Q}$ agrees with the iteration map via Λ up to $\lambda^\mathcal{Q}$.

Let $\mathcal{M}_\infty$ be the direct limit of Σ-iterates of $\mathcal{P}$. If $\mathcal{M}_\infty = \mathcal{H}^+$, then it's easy to see that $\Sigma \notin \Gamma$. By the disussion of the last paragraph, we're done. Now suppose $\mathcal{M}_\infty = \mathcal{H}^+(\alpha)$ for some α such that $\delta^{\mathcal{H}^+}_\alpha < \Theta^\Gamma$. By (T), there is an elementary map $m : \mathcal{M}_\infty \to \mathcal{H}^+$ such that $\mathrm{crt}(m) = \delta^{\mathcal{M}_\infty}$. The map m is defined as follows: for any $x \in \mathcal{M}_\infty$, let $(\mathcal{R}, \Sigma_\mathcal{R})$ be a Σ-iterate of $\mathcal{P}$ such that $\Sigma_\mathcal{R}$ has branch condensation and there is a $y \in \mathcal{R}$ such that $i^{\Sigma_\mathcal{R}}_{\mathcal{R},\infty}(y) = x$, then $m(x) = \tau_\mathcal{R}(y)$. It's easy to see that m is well-defined and is elementary. By a standard argument (see [3]), noting that $\delta^{\mathcal{M}_\infty}$ is on the Solovay sequence of Γ, $L(\Gamma | \delta^{\mathcal{M}_\infty}, \mathbb{R}) \vDash \mathsf{AD}_\mathbb{R} + \Theta$ is regular. This contradicts our smallness assumption.

Now we prove that (T) holds. We accomplish this by proving Lemmas 4.3.11 and 4.3.13, whose key ideas are due to G. Sargsyan. But first, let us start with a definition.

Definition 4.3.10 (Sargsyan). *Suppose $\pi_\sigma : \mathcal{H}^+_\sigma \to \mathcal{H}^+$ is as above and $A \in \mathcal{H}^+_\sigma \cap \mathcal{P}(\theta^\sigma)$. We say that π_σ has **A-condensation** if whenever $\mathcal{Q}$ is such that there are elementary embeddings $\upsilon : \mathcal{H}^+_\sigma \to \mathcal{Q}$, $\tau : \mathcal{Q} \to \mathcal{H}^+$ such that $\pi_\sigma = \tau \circ \upsilon$, then $\upsilon(T^{\mathcal{H}^+_\sigma}_A) = T_{\mathcal{Q},\tau,A}$, where*

$$T^{\mathcal{H}^+_\sigma}_A = \{(\phi, s) \mid s \in [\theta^\sigma]^{<\omega} \wedge \mathcal{H}^+_\sigma \vDash \phi[s, A]\},$$

and

$$T_{\mathcal{Q},\tau,A} = \{(\phi, s) \mid s \in [\delta^\mathcal{Q}_\alpha]^{<\omega} \text{ for some } \alpha < \lambda^\mathcal{Q} \wedge \mathcal{H}^+ \vDash \phi[i^{\Sigma^{\tau,-}_\mathcal{Q}}_{\mathcal{Q}(\alpha),\infty}(s), \pi_\sigma(A)]\}.$$

Lemma 4.3.11. *$\forall^*_{\mu_M}\sigma$ $\forall A \in \mathcal{H}^+_\sigma \cap \mathcal{P}(\theta^\sigma)$ π_σ has A-condensation.*

Proof. Suppose not. By normality of μ_M, there is an A such that $\forall^*_{\mu_M}\sigma$, letting $A_\sigma = \pi^{-1}_\sigma(A)$, then π_σ does not have A_σ-condensation. Note that for σ, τ as above such that $\pi_{\sigma,\tau}$ exists, $\pi_{\sigma,\tau}(A_\sigma) = A_\tau$. We also let $A = \pi_\sigma(A_\sigma)$ for such a σ.

Fix such a σ_0 and let $\tau_0 = \pi_{\sigma_0} \upharpoonright \mathcal{H}^+_{\sigma_0}$ and $\mathcal{P}_0 = \mathcal{H}^+_{\sigma_0}$. Let also $A_0 = A_{\sigma_0}$. Hence, there is a tuple $(\mathcal{Q}_0, \pi_0, \sigma_0)$ such that $\pi_0 : \mathcal{P}_0 \to \mathcal{Q}_0$ and $\sigma_0 : \mathcal{Q}_0 \to \mathcal{H}^+$ such that $\tau_0 = \sigma_0 \circ \pi_0$ and $\pi_0(T^{\mathcal{P}_0}_{A_0}) \neq T_{\mathcal{Q}_0,\sigma_0,A_0}$. Let σ_1 be such that $\pi_{\sigma_1} =_{def} \tau_1$ does not have $A_{\sigma_1} =_{def} A_1$-condensation and there is a map $\xi_0 : \mathcal{Q}_0 \to \mathcal{P}_1$ such that $\pi_{\sigma_0,\sigma_1} = \pi_0 \circ \xi_0$. We write ϕ_0 for π_{σ_0,σ_1}. By induction and using DC, we can construct a sequence $\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \pi_i, \xi_i, \sigma_i, \phi_i \mid i < \omega\rangle$ such that

1. $\tau_i =_{def} \pi_{\sigma_i} : \mathcal{H}^+_{\sigma_i} =_{def} \mathcal{P}_i \to \mathcal{H}^+$;
2. $(\mathcal{Q}_i, \pi_i, \sigma_i)$ witnesses that τ_i does not have $A_i =_{def} A_{\sigma_i}$-condensation;
3. $\phi_i =_{def} \pi_{\sigma_i,\sigma_{i+1}} = \pi_i \circ \xi_i$;
4. $\sigma_i = \tau_{i+1} \circ \xi_i$.

Let τ be such that letting $X = \cup_i \tau_i[\mathcal{P}_i] \cup \sigma_i[\mathcal{Q}_i]$, then $X \subseteq \pi_\tau[\mathcal{H}^+_\tau]$. Such a τ exists since Θ^Γ has uncountable cofinality. We can define A_i-condensation relative to $\tau^*_i = m_\tau^{-1} \circ \tau_i$, where m_τ is the natural map such that letting $i^{\Sigma_\tau}_{\mathcal{H}^+_\tau,\infty} : \mathcal{H}^+_\tau \to \mathcal{M}_{\infty,\tau}$ be the direct limit map, $m_\tau : \mathcal{M}_{\infty,\tau} \to \mathcal{H}^+$, and $\pi_\tau \restriction \mathcal{H}^+_\tau = m_\tau \circ i^{\Sigma_\tau}_{\mathcal{H}^+_\tau,\infty}$. Let $A^*_\tau = m_\tau^{-1}(A)$. We have a sequence $\langle \mathcal{P}_i, \mathcal{Q}_i, \tau^*_i, \pi_i, \xi_i, \sigma^*_i =_{def} m_\tau^{-1} \circ \sigma_i, \phi_i \mid i < \omega\rangle$ such that

5. $(\mathcal{Q}_i, \pi_i, \sigma^*_i)$ witnesses that τ^*_i does not have $A_i =_{def} A_{\sigma_i}$-condensation, i.e. $\pi_i(T^{\mathcal{P}_i}_{A_i}) \neq T_{\mathcal{Q}_i,\sigma^*_i,A_i}$;
6. writing Σ_i for Σ_{σ_i}, we have $(\phi, s) \in T^{\mathcal{P}_i}_{A_i} \Leftrightarrow \mathcal{M}_{\infty,\tau} \vDash \phi[i^{\Sigma_i^-}_{\mathcal{P}(\alpha),\infty}(s), A^*_\tau]$, where α is least such that $s \in [\delta^{\mathcal{P}_i}_\alpha]^{<\omega}$;
7. $\phi_i =_{def} \pi_{\sigma_i,\sigma_{i+1}} = \pi_i \circ \xi_i$;
8. $\sigma^*_i = \tau^*_{i+1} \circ \xi_i$.

(6) follows immediately from the definition of $T^{\mathcal{P}_i}_{A_i}$ and τ^*_i. We call the sequence $\langle \mathcal{P}_i, \mathcal{Q}_i, \tau^*_i, \pi_i, \xi_i, \sigma^*_i, \phi_i \mid i < \omega\rangle$ *A-bad*. We also assume the whole situation is inside an M_ϵ where $\pi_{\tau,\epsilon}$ exists (there are μ_M-measure one many such ϵ). We fix some notation. $\forall^*_{\mu_M}\sigma$, we let $\Sigma^-_\sigma = \bigoplus_{\alpha<\lambda^{\mathcal{H}^+_\sigma}} \Sigma_{\mathcal{H}^+_\sigma(\alpha)}$.

Lemma 4.3.12. *There is a pair (σ_0, ϵ) such that there is a hod pair $(\mathcal{W}, \Pi)$ over $(\mathcal{H}^+_\epsilon, \Sigma^-_\epsilon)$ such that $\mathcal{W} \in M_\epsilon$ and a hod pair $(\mathcal{W}^*, \Pi^*)$ over $(\mathcal{P}_0, \Sigma^-_0)$ such that $\mathcal{W}^* \in M_{\sigma_0}$ and $\Pi^* = \Pi^{\pi_{\sigma_0,\epsilon}}$. Furthermore, $\Gamma(\mathcal{W}^*, \Pi^*)$ witnesses that the sequence $\langle \mathcal{P}_i, \mathcal{Q}_i, \tau^*_i, \pi_i, \xi_i, \sigma^*_i, \phi_i \mid i < \omega\rangle$ is A-bad.*

Proof. For a typical σ, let $\Omega_\sigma \lhd \Gamma$ be the minimal pointclass that witnesses there is an A-bad sequence starting with $\mathcal{H}_\sigma$, i.e. $\mathcal{H}_\sigma$ is the $\mathcal{P}_0$ of the sequence in the notation above. Also, let D_σ be the collection of hod mice $\mathcal{Q}$ over $(\mathcal{H}_\sigma, \Sigma^-_\sigma)$ such that there is a strategy Λ for $\mathcal{Q}$ with branch condensation and is Γ-fullness preserving such that $\Omega_\sigma \trianglelefteq \Gamma(\mathcal{Q}, \Lambda)$; note that D_σ is $OD_{\{\sigma\}}$. Let $D = \Pi_\sigma D_\sigma \backslash \mu_M$. Expanding M if necessary, we may assume $M \cap D \in M$ and $M \cap D \neq \emptyset$ (regularity of Θ and the fact that Θ^Γ is singular allow us to do this).

Fix $\mathcal{W} \in M \cap D$ and for a typical σ, let $\mathcal{W}_\sigma = \pi_\sigma^{-1}(\mathcal{W})$. Fix a sequence $\langle \sigma_n \mid n < \omega \rangle$ such that for each n, there is an A-bad sequence S_n starting with $\mathcal{H}_{\sigma_n}$ in $M_{\sigma_{n+1}}$. Fix such an S_n for each n and fix also strategy Λ_n for $\mathcal{W}_{\sigma_n}$ such that $\Gamma(\mathcal{W}_{\sigma_n}, \Lambda_n)$ witnesses the sequence S_n is A-bad. It's easy to see that there are $n < m$ such that $\Gamma(\mathcal{W}_{\sigma_n}, \Lambda_n) = \Gamma(\mathcal{W}_{\sigma_n}, \Lambda_m^{\pi_{\sigma_n,\sigma_m}})$. We can just let $\sigma_n = \sigma_0$, $\sigma_m = \epsilon$ as in the lemma and let $(\mathcal{W}, \Pi) = (\mathcal{W}_{\sigma_m}, \Lambda_m)$ and $(\mathcal{W}^*, \Pi^*) = (\mathcal{W}_{\sigma_n}, \Lambda_m^{\pi_{\sigma_n,\sigma_m}})$. □

Now let $(\mathcal{W}^*, \Pi^*)$ and $(\mathcal{W}, \Pi)$ be as in Lemma 4.3.12. Since we assume $\Gamma(\mathcal{W}^*, \Pi^*)$ is large enough, there is a finite sequence of ordinals t and a formula $\theta(u, v)$ such that in $\Gamma(\mathcal{W}^*, \Pi^*)$

9. for every $i < \omega$, $(\phi, s) \in T_{A_i}^{\mathcal{P}_i} \Leftrightarrow \theta[i_{\mathcal{P}_i(\alpha),\infty}^{\Sigma_i^-}, t]$, where α is least such that $s \in [\delta_\alpha^{\mathcal{P}_i}]^{<\omega}$;

10. for every i, there is $(\phi_i, s_i) \in T_{\pi_i(A_i)}^{\mathcal{Q}_i}$ such that $\neg\theta[i_{\mathcal{Q}_i(\alpha)}^{\Psi_i^-}(s_i), t]$ where $\Psi_i = \Sigma_{i+1}^{\pi_i}$ and α is least such that $s_i \in [\delta_\alpha^{\mathcal{Q}_i}]^{<\omega}$.

The pair (θ, t) essentially defines a Wadge-initial segment of $\Gamma(\mathcal{W}^*, \Pi^*)$ that can define the pair $(\mathcal{M}_{\infty,\tau}, A_\tau^*)$.

Let E_i be the $(\mathrm{crt}(\pi_i), \delta^{\mathcal{Q}_i})$-extender derived from π_i and F_i be the $(\mathrm{crt}(\xi_i), \delta^{\mathcal{P}_{i+1}})$-extender derived from ξ_i. Let $\mathcal{P}_0^+ = \mathcal{W}^*$, $\mathcal{Q}_0^+ = \mathrm{Ult}(\mathcal{P}_0^+, E_0)$, $\mathcal{P}_1^+ = \mathrm{Ult}(\mathcal{Q}_0, F_0)$. We inductively define $\mathcal{P}_i^+$, $\mathcal{Q}_i^+$ for all i. Let $\pi_i^+ : \mathcal{P}_i^+ \to \mathcal{Q}_i^+$, $\xi_i^+ : \mathcal{Q}_i \to \mathcal{P}_{i+1}$ be the ultrapower maps and $\phi_i^+ = \pi_i^+ \circ \xi_i^+$. These maps in turn commute with the map $\xi : \mathcal{P}_0 \to \mathcal{W}$ and hence we can let Π_i^+ be the strategy of $\mathcal{P}_i^+$ and Ψ_i^+ be the strategy of $\mathcal{Q}_i^+$ obtained from pulling back the strategy Π under the appropriate maps. We note that $\mathcal{P}_i^+$ is a Σ_i-premouse and $\mathcal{Q}_i^+$ is a Ψ_i-premouse for each i.

By a similar argument as in Theorem 3.1.24, we can use the strategies Π_i^+'s and Ψ_i^+'s to simultanously execute a $\mathbb{R}^{M_\epsilon *}$-genericity iterations, where M_ϵ^* contains everything considered so far and is closed under Π. The process yields a sequence of models $\langle \mathcal{P}_{i,\omega}^+, \mathcal{Q}_{i,\omega}^+ \mid i < \omega \rangle$ and maps $\pi_{i,\omega}^+ : \Pi_{i,\omega}^+ \to \mathcal{Q}_{i,\omega}^+$, $\xi_{i,\omega}^+ : \mathcal{Q}_{i,\omega}^+ \to \mathcal{P}_{i+1,\omega}^+$, and $\phi_{i,\omega}^+ = \xi_{i,\omega}^+ \circ \pi_{i,\omega}^+$. Furthermore, each $\mathcal{P}_{i,\omega}^+$, $\mathcal{Q}_{i,\omega}^+$ embeds into a Π-iterate of $\mathcal{W}$ and hence the direct limit $\mathcal{P}_\infty$ of $(\mathcal{P}_{i,\omega}^+, \mathcal{Q}_{j,\omega}^+ \mid i, j < \omega)$ under maps $\pi_{i,\omega}^+$'s and $\xi_{i,\omega}^+$'s is wellfounded. We note that $\mathcal{P}_{i,\omega}^+$ is a Σ_i-premouse and $\mathcal{Q}_{i,\omega}^+$ is a Ψ_i-premouse. Let C_i be the derived model of $\mathcal{P}_{i,\omega}^+$, D_i be the derived model of $\mathcal{Q}_{i,\omega}^+$ (at the sup of the Woodin cardinals of each model), then $\mathbb{R}^{M_\epsilon *} = \mathbb{R}^{C_i} = \mathbb{R}^{D_i}$. Furthermore, $C_i \cap \mathcal{P}(\mathbb{R}) \subseteq D_i \cap \mathcal{P}(\mathbb{R}) \subseteq C_{i+1} \cap \mathcal{P}(\mathbb{R})$ for all i.

(9), (10) and the construction above give us that there is a $t \in [\mathrm{OR}]^{<\omega}$, a formula $\theta(u, v)$ such that

11. for each i, in C_i, for every (ϕ, s) such that $s \in \delta^{\mathcal{P}_i}$, $(\phi, s) \in T_{A_i}^{\mathcal{P}_i} \Leftrightarrow \theta[i_{\mathcal{P}_i(\alpha),\infty}^{\Sigma_i^-}(s), t]$ where α is least such that $s \in [\delta_\alpha^{\mathcal{P}_i}]^{<\omega}$.

Let n be such that for all $i \geq n$, $\pi_{i,\omega}^+(t) = t$. Such an n exists because the direct limit $\mathcal{P}_\infty$ is wellfounded. By elementarity of $\pi_{i,\omega}^+$ and the fact that $\pi_{i,\omega}^+ \restriction \mathcal{P}_i = \pi_i$,

12. for all $i \geq n$, in D_i, for every (ϕ, s) such that $s \in \delta^{\mathcal{Q}_i}$, $(\phi, s) \in T^{\mathcal{Q}_i}_{\pi_i(A_i)} \Leftrightarrow \theta[i^{\Psi_i^-}_{\mathcal{Q}_i(\alpha),\infty}(s), t]$ where α is least such that $s \in [\delta^{\mathcal{Q}_i}_\alpha]^{<\omega}$.

However, using (10), we get

13. for every i, in D_i, there is a formula ϕ_i and some $s_i \in [\delta^{\mathcal{Q}_i}]^{<\omega}$ such that $(\phi_i, s_i) \in T^{\mathcal{Q}_i}_{\pi_i(A_i)}$ but $\neg\phi[i^{\Psi_i^-}_{\mathcal{Q}_i(\alpha),\infty}(s_i), t]$ where α is least such that $s \in [\delta^{\mathcal{Q}_i}_\alpha]^{<\omega}$.

Clearly (12) and (13) give us a contradiction. This completes the proof of the lemma. □

Suppose $(\mathcal{Q}, \vec{\mathcal{T}}) \in I(\mathcal{H}^+_\sigma, \Sigma_\sigma)$ is such that $i^{\vec{\mathcal{T}}} : \mathcal{H}^+_\sigma \to \mathcal{Q}$ exists. Let $\gamma^{\vec{\mathcal{T}}}$ be the sup of the generators of $\vec{\mathcal{T}}$. For each $x \in \mathcal{Q}$, say $x = i^{\vec{\mathcal{T}}}(f)(s)$ for $f \in \mathcal{T}^+_\sigma$ and $s \in [\delta^{\mathcal{Q}}_\alpha]^{<\omega}$, where $\delta^{\mathcal{Q}}_\alpha \leq \gamma^{\vec{\mathcal{T}}}$ is least such, then let $\tau_{\mathcal{Q}}(x) = \pi_\sigma(f)(i^{\Sigma^-_{\mathcal{Q},\vec{\mathcal{T}}}}_{\mathcal{Q}(\alpha),\infty}(s))$. By Lemma 4.3.10, $\tau_{\mathcal{Q}}$ is elementary. Note also that $\tau_{\mathcal{H}^+_\sigma} = \pi_\sigma \restriction \mathcal{H}^+_\sigma$.

Lemma 4.3.13. *Suppose $(\mathcal{Q}, \vec{\mathcal{T}}) \in I(\mathcal{H}^+_\sigma, \Sigma_\sigma)$ and $(\mathcal{R}, \vec{\mathcal{U}}) \in I(\mathcal{Q}, \Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$ are such that $i^{\vec{\mathcal{T}}}, i^{\vec{\mathcal{U}}}$ exist and $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ and $\Sigma_{\mathcal{R},\vec{\mathcal{U}}}$ have branch condensation. Then $\tau_{\mathcal{Q}} = \tau_{\mathcal{R}} \circ i^{\vec{\mathcal{U}}}$.*

Proof. Let $x \in \mathcal{Q}$. There are some $f \in \mathcal{H}^+_\sigma$ and $s \in [\gamma^{\vec{\mathcal{T}}}]^{<\omega}$ such that $x = i^{\vec{\mathcal{T}}}(f)(s)$. So $\tau_{\mathcal{Q}}(x) = \pi_\sigma(f)(i^{\Sigma^-_{\mathcal{Q},\vec{\mathcal{T}}}}_{\mathcal{Q},\infty}(s))$. On the other hand, $\tau_{\mathcal{R}} \circ i^{\vec{\mathcal{U}}}(x) = \tau_{\mathcal{R}} \circ i^{\vec{\mathcal{U}}}(i^{\vec{\mathcal{T}}}(f)(s)) = \pi_\sigma(f)(i^{\Sigma^-_{\mathcal{R},\vec{\mathcal{U}}}}_{\mathcal{R},\infty} \circ i^{\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}}(s)) = \pi_\sigma(f)(i^{\Sigma^-_{\mathcal{Q},\vec{\mathcal{T}}}}_{\mathcal{Q},\infty}(s)) = \tau_{\mathcal{Q}}(x)$. □

The lemmas imply that (T) holds and finish the proof of the claim.

Remark: The proof above works more generally for any Γ such that $\mathcal{H}^+ \vDash \mathrm{cof}(\Theta)$ is measurable. If $\Gamma = \Gamma_{\omega_1}$, it's automatic that $\mathcal{M}_\infty \notin \mathrm{HOD}^\Gamma$, so the proof above is superfluous. □

The proof of Theorem 4.3.7 gives the following.

Theorem 4.3.14. *Suppose there is no Γ such that $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and $\mathrm{L}(\Gamma, \mathbb{R}) \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. Suppose $\Gamma = \Gamma_\alpha$ is the current maximal pointclass such that $\Theta^\Gamma < \Theta$. Then there is a hod pair $(\mathcal{P}, \Sigma)$ such that $\Sigma \notin \Gamma$ and is Γ-fullness preserving and has branch condensation. In particular, $\Gamma_{\alpha+1}$ exists.*

Proof. Using the same notations as in the proof of Theorem 4.3.7. We have three cases.

Case 1: $\mathrm{cof}(\Theta^\Gamma) = \omega$. This case is easy. First we let $\langle \alpha_n \mid n < \omega \rangle$ be an increasing and cofinal sequence in Θ^Γ, where each α_n is a member of the Solovay sequence of Γ. Using DC, we can choose for each n, a sjs $\mathcal{A}_n$ at α_n (i.e. members of $\mathcal{A}_n$ are Wadge cofinal in $\Gamma|\alpha_n$). We then let $\mathcal{A} = \bigoplus_n \mathcal{A}_n$ be the amalgamation of the $\mathcal{A}_n$'s. $\mathcal{A}$ defines a model operator F that condenses and relativizes well. We can then prove $L^F(\mathbb{R}) \vDash \mathsf{AD}^+ + \Theta = \theta_{\alpha+1}$. Then a hod pair $(\mathcal{P}, \Sigma)$ as in the conclusion of the theorem exists in $L^F(\mathbb{R})$.

Case 2: cof$(\Theta^\Gamma) > \omega$ and $\mathcal{H}^+ \vDash \Theta^\Gamma$ is singular. This is the main case and the proof is just that of Theorem 4.3.8.

Case 3: cof$(\Theta^\Gamma) > \omega$ and $\mathcal{H}^+ \vDash \Theta^\Gamma$ is regular. We claim that this case cannot occur. For a typical σ, $\mathcal{H}^+_\sigma \vDash \Theta^{\Gamma^\sigma}$ is regular. Set $\Sigma_\sigma = \oplus_{\alpha<\Theta^\gamma}\Sigma^{\mathcal{H}_\sigma}_\alpha$ to be the join of the lower level strategies. Then for a typical σ, the iteration map (via Σ_σ) $\pi^{\Sigma_\sigma}_{\mathcal{H}^+_\sigma,\infty}$ in the direct limit system of Γ maps Θ^σ to γ_σ where $\gamma_\sigma < \Theta^\Gamma$[25] is a member of the Solovay sequence of Γ, say $\gamma_\sigma = \theta^\Gamma_\alpha$. But this means $\mathcal{H}(\alpha) \vDash \theta^\Gamma_\alpha$ is regular and this implies $L(\Gamma|\theta^\Gamma_\alpha, \mathbb{R}) \vDash \Theta$ is regular (see [3] for a proof). This contradicts the smallness assumption of the theorem. □

The following completes the proof of the theorem.

Theorem 4.3.15. *There is a pointclass Γ such that $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and $L(\Gamma, \mathbb{R}) \vDash$ $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.*

Proof. Suppose not. By Theorem 4.3.14, we can run the core model induction to produce a pointclass Γ such that $\Theta^\Gamma = \Theta$. Since Θ is regular, by a standard argument, $\Gamma = L(\Gamma, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$. We give this argument below.

Suppose not. Let α be the least such that $\rho(L_\alpha(\Gamma, \mathbb{R})) = \mathbb{R}$. Hence $\alpha \geq \Theta$. Let $f : \alpha \times \Gamma \twoheadrightarrow L_\alpha(\Gamma, \mathbb{R})$ be a surjection that is definable over $L_\alpha(\Gamma, \mathbb{R})$ (from parameters). We define a transitive model M such that

1. $\mathbb{R} \subseteq M$;
2. there is an elementary embedding from M into $L_\alpha(\Gamma, \mathbb{R})$.

We first define a sequence $\langle H_i \mid i < \omega\rangle$ as follows. Let $H_0 = \mathbb{R}$. By induction, suppose H_n is defined and there is a surjection from $\mathbb{R} \twoheadrightarrow H_n$. Suppose (ψ, a) is such that $a \in H_n$ and $L_\alpha(\Gamma, \mathbb{R}) \vDash \exists x\psi[x, a]$. Let $(\gamma_{a,\psi}, \beta_{a,\psi})$ be the $<_{lex}$-least pair such that there is a $B \in \Gamma$ with Wadge rank $\beta_{a,\psi}$ such that

$$L_\alpha(\Gamma, \mathbb{R}) \vDash \psi[f(\gamma_{a,\psi}, B), a].$$

Let then $H_{n+1} = H_n \cup \{f(\gamma_{a,\psi}, B) \mid L_\alpha(\Gamma, \mathbb{R}) \vDash \exists x\psi[x, a] \wedge w(B) = \beta_{a,\psi} \wedge a \in H_n\}$. It's easy to see that there is a surjection from $\mathbb{R} \twoheadrightarrow H_{n+1}$. This uses the fact that Θ is regular, which implies $\sup\{\beta_{a,\psi} \mid a \in H_n \wedge L_\alpha(\Gamma, \mathbb{R}) \vDash \exists x\psi[x, a]\} < \Theta$. Let $H = \cup_n H_n$. By construction, $H \prec L_\alpha(\Gamma, \mathbb{R})$. Finally, let M be the transitive collapse of H. M clearly satisfies properties 1 and 2.

Say $M = L_\beta(\Gamma^*, \mathbb{R})$. By the above properties of M, $\Gamma^* = \Gamma|\theta^\Gamma_\gamma$ for some γ such that $\theta^\Gamma_\gamma < \Theta$. But then $\rho(L_\beta(\Gamma^*, \mathbb{R})) = \mathbb{R}$. This contradicts that Γ^* is constructibly closed. This gives $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. □

[25] This is because $\mathrm{cof}(\gamma_\sigma) = \omega$ while $\mathrm{cof}(\Theta^\Gamma) > \omega$.

4.3.4 $\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable

The main result of this subsection is the proof that the theory $(\mathrm{T}_1) \equiv$ "$\mathsf{AD}_{\mathbb{R}}$ + DC + there is an $\mathbb{R}$-complete measure on Θ" is equiconsistent with $(\mathrm{T}_2) \equiv$ "ZF + DC + there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$ + Θ is regular." The proof of the main theorem also shows that (T_1), (T_2) are both equiconsistent with the theory $(\mathrm{T}_3) \equiv$ "ZF + DC + there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ + there is an $\mathbb{R}$-complete measure on Θ."

Theorem 4.3.16. *Con(T_1) $\Leftrightarrow$ Con(T_2). Furthermore, these theories are equiconsistent with (T_3).*

Woodin (unpublished) has shown that Con(P) follows from Con(ZFC + there is a proper class of Woodin limits of Woodin cardinals), where (P)$\equiv$ "ZF + DC + ω_1 is supercompact". We conjecture that a (closed to optimal) lower-bound consistency strength for the theory (P) is "ZFC + there is a Woodin limit of Woodin cardinals". The methods developed in this thesis, in particular the proof of Theorem 4.3.16, gives us the following.

Corollary 4.3.17. *Con(ZF + DC + ω_1 is $\mathcal{P}(\mathcal{P}(\mathbb{R}))$-supercompact) $\Rightarrow$ Con(T_1). In particular, Con(P) $\Rightarrow$ Con(T_1).*

The outline of the proof is as follows. We first show in subsection 4.3.4.1 that (T_1) implies (T_2). Subsections 4.3.4.3 proves the converse of the theorem. We use the core model induction (developed in the previous sections) to construct pointclass Γ of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" such that $L(\Gamma,\mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \Gamma$. We then define a certain model $\mathcal{H}^+$ extending HOD^Γ and a normal measure ν on Θ over $\mathcal{H}^+$. Finally, using Theorem 4.3.19 in subsection 4.3.4.2, we show $L[\mathcal{H}^+,\nu](\Gamma) \vDash (\mathrm{T}_2)$.

4.3.4.1 $(\mathrm{T}_1) \Rightarrow (\mathrm{T}_2)$

Suppose $V \vDash (\mathrm{T}_1)$. The hypothesis implies there is a $\mathbb{R}$-complete and normal measure on Θ by a standard argument (see Theorem 10.20 of [9] and note that DC is enough for the proof of the theorem). Let ν be such a measure. For each $\alpha < \Theta$, let μ_α be the normal fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}_\alpha(\mathbb{R}))$ derived from the Solovay measure μ_0 on $\mathcal{P}_{\omega_1}(\mathbb{R})$ (i.e. we first fix a surjection $\pi : \mathbb{R} \to \mathcal{P}_\alpha(\mathbb{R})$; then we let $\pi^* : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathcal{P}_{\omega_1}(\mathcal{P}_\alpha(\mathbb{R}))$ be the surjection induced from π and let $A \in \mu_\alpha \Leftrightarrow (\pi^*)^{-1}[A] \in \mu_0$). It's worth noting that by [47], μ_α are unique for all $\alpha < \Theta$. We derive from ν a measure μ on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$ as follows. Let $A \subseteq \mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$, then

$$A \in \mu \Leftrightarrow \forall^*_\nu \alpha\ A \restriction \mathcal{P}_\alpha(\mathbb{R}) =_{def} \{\sigma \in A \mid \sigma \in \mathcal{P}_{\omega_1}(\mathcal{P}_\alpha(\mathbb{R}))\} \in \mu_\alpha.$$

It's clear that μ is a measure. It's also clear that μ is fine since the measures μ_α's are fine. It remains to show normality of μ. We first need an alternative formulation of normality.

Lemma 4.3.18 (ZF+DC)**.** *Suppose μ is a fine measure on $\mathcal{P}_{\omega_1}(X)$. The following are equivalent.*

1. *μ is normal.*

2. *Suppose we have $\langle A_x \mid x \in X \wedge A_x \in \mu\rangle$. Then $\triangle_{x\in X} A_x =_{def} \{\sigma \mid \sigma \in \cup_{x\in\sigma} A_x\} \in \mu$.*

The proof of the lemma is standard and we leave it to the reader. We proceed with the proof. Suppose μ is not normal. By Lemma 4.3.18, there is a sequence $\langle A_x \mid x \in \mathcal{P}(\mathbb{R}) \wedge A_x \in \mu\rangle$ but $\triangle_{x\in\mathcal{P}(\mathbb{R})} A_x \notin \mu$. This means

$$\forall^*_\nu \alpha \forall^*_{\mu_\alpha} \sigma \exists x \in \sigma\ \sigma \notin A_x.$$

By normality of μ_α, we then have

$$\forall^*_\nu \alpha \exists x \forall^*_{\mu_\alpha} \sigma\ x \in \sigma \wedge \sigma \notin A_x. \tag{4.2}$$

We now define a regressive function $F : \Theta \to \Theta$ as follows. Let $F(\alpha)$ be the least $\beta < \alpha$ such that there is an $x \in \mathcal{P}(\mathbb{R})$ such that $w(x) = \beta$[26] and $\forall^*_{\mu_\alpha} \sigma\ \sigma \notin A_x$; otherwise, let $F(\alpha) = 0$. By 4.2, $\forall^*_\nu \alpha\ 0 < F(\alpha) < \alpha$. By normality of ν, there is a β such that $\forall^*_\nu \alpha\ F(\alpha) = \beta$.

For each x such that $w(x) = \beta$, let

$$B_x = \{\alpha < \Theta \mid \forall^*_{\mu_\alpha} \sigma\ \sigma \notin A_x\}.$$

Note that $\cup_x B_x \in \nu$. Since there are only $\mathbb{R}$-many such x, by $\mathbb{R}$-completeness of ν, there is an x such that $B_x \in \nu$. Fix such an x. We then have

$$\forall^*_\nu \alpha \forall^*_{\mu_\alpha} \sigma\ \sigma \notin A_x. \tag{4.3}$$

The above equation implies $A_x \notin \mu$. Contradiction.

4.3.4.2 A Vopenka Forcing

We prove a theorem of Woodin concerning a variation of the Vopenka algebra. This theorem will play an important role in the next subsection. Suppose Γ is such that $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$ and $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$. Let $\mathcal{H}$ be $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$. Woodin has shown that $\mathcal{H} = L[A]$ for some $A \subseteq \Theta^\Gamma$ (see [43]). We write Θ for Θ^Γ. Let $\mathcal{H}^+$ be a ZFC model such that $A \in \mathcal{H}^+$ and $V^{\mathcal{H}}_\Theta = V^{\mathcal{H}^+}_\Theta$.

Theorem 4.3.19 (Woodin)**.** *There is a forcing $\mathbb{P} \in \mathcal{H}$ such that*

1. *$\mathbb{P}$ is homogeneous;*

2. *there is a $G \subseteq \mathbb{P}$ generic over $\mathcal{H}^+$ such that $\mathcal{H}^+(\Gamma)$[27] is the symmetric part of $\mathcal{H}^+[G]$.*

3. *$\mathcal{P}(\mathbb{R}) \cap \mathcal{H}^+(\Gamma) = \Gamma$.*

[26] $w(x)$ denotes the Wadge rank of x.

[27] $\mathcal{H}^+(\Gamma)$ is the minimal ZF model containing $\mathcal{H}^+$ and Γ

In particular, $\mathcal{H}^+(\Gamma) \vDash \mathsf{AD}_{\mathbb{R}}$.

Proof. First, we define a forcing $\mathbb{Q} \in L(\Gamma, \mathbb{R})$. A condition $q \in \mathbb{Q}$ if $q : n_q \to \mathcal{P}(\alpha_q)$ for some $n_q < \omega$ and $\alpha_q < \Theta$. The ordering $\leq_{\mathbb{Q}}$ is as follows:

$$q \leq_{\mathbb{Q}} r \Leftrightarrow n_r \leq n_q \wedge \alpha_r \leq \alpha_q \wedge \forall i < n_r \; q(i) \cap \alpha_r = r(i).$$

Now we define

$$\mathbb{P}^* = \{A \mid \exists \alpha_A < \Theta \exists n_A < \omega \; A \subseteq \mathcal{P}(\alpha_A)^{n_A} \wedge A \in OD^{L(\Gamma,\mathbb{R})}\}.$$

The ordering $\leq_{\mathbb{P}^*}$ is defined as follows:

$$A \leq_{\mathbb{P}^*} B \Leftrightarrow n_B \leq n_A \wedge \alpha_B \leq \alpha_A \wedge A \cap \mathcal{P}(\alpha_B)^{n_B} \subseteq B.$$

It's easy to see that there is a partial order $(\mathbb{P}, \leq_{\mathbb{P}}) \in \mathcal{H}$ isomorphic to $(\mathbb{P}^*, \leq_{\mathbb{P}^*})$ and in $\mathcal{H}$, $(\mathbb{P}, \leq_{\mathbb{P}})$ has size Θ. Let $\pi : (\mathbb{P}, \leq_{\mathbb{P}}) \to (\mathbb{P}^*, \leq_{\mathbb{P}^*})$ be the isomorphism. We will occasionally confuse these two partial orders. $(\mathbb{P}, \leq_{\mathbb{P}})$ is the direct limit of the directed system of complete boolean algebras $\mathbb{P}_{\alpha,n}$ in $\mathcal{H}$, where $\mathbb{P}^*_{\alpha,n}$ is the Vopenka algebra on $\mathcal{P}(\alpha)^n$ and the maps from $\mathbb{P}_{\alpha,n}$ into $\mathbb{P}_{\beta,m}$ for $\alpha \leq \beta$ and $n \leq m$ are the natural maps. It's clear that $\mathbb{P}$ is homogeneous. Similarly, $\mathbb{Q}$ is homogeneous and is a natural direct limit of the partial orders $\{\mathbb{Q}_\alpha = \mathbb{Q} \restriction \alpha \mid \alpha < \Theta\}$.

Now let $g \subseteq \mathbb{Q}$ be $L(\Gamma, \mathbb{R})$-generic. Let $h \subseteq \mathbb{P}$ be defined as follows:

$$p \in h \Leftrightarrow (g \restriction n_p) \cap \mathcal{P}(\alpha_p)^{n_p} \in \pi(p).$$

Lemma 4.3.20. *Write the filter h above h_g. Then h_g is generic over $\mathcal{H}$ and $L(\Gamma, \mathbb{R})$ is the symmetric part of $\mathcal{H}[h_g]$. In fact, for any condition $p \in \mathbb{P}$, there is a generic filter h over $\mathcal{H}$ such that $p \in h$ and $L(\Gamma, \mathbb{R})$ is the symmetric part of $\mathcal{H}[h]$.*

Proof. Suppose h_g is not generic over $\mathcal{H}$. Then there is an open dense set $D \subseteq \mathbb{P}$ in $\mathcal{H}$ such that $h_g \cap D = \emptyset$. Fix a condition $p \in g$ which forces this. For each $i < \omega$, let p_i be the join in $\mathbb{P}_{\alpha_p,i}$ of all b which can be refined in $\mathbb{P}$ to an element of D by not increasing i but (possibly) increasing α_p, that is there is a $\beta \geq \alpha_p$ and a $d \in \mathbb{P}_{\beta,i}$ such that $d \in D$ and $d \restriction \alpha_p = b$.

Since D is open dense, the set $\{p_i \mid i < \omega\}$ is predense in the limit $\mathbb{P}_{\alpha_p}$ of the $\mathbb{P}_{\alpha_p,i}$'s. Since $g \restriction \alpha_p =_{def} \langle g(n) \restriction \alpha_p \mid n < \omega \rangle$ is generic for $\mathbb{Q} \restriction \alpha_p$, there must be some $i \geq n_p$ and $\beta \geq \alpha_p$ such that there is some $b \in \mathbb{P}_{\beta,i} \cap D$ such that $(g \restriction \alpha_p) \restriction i \in \pi(b) \restriction \mathcal{P}(\alpha_p)^i$. But this means we can easily refine p to a condition q such that $q \Vdash \dot{h} \cap D \neq \emptyset$. Just take q to be a thread in b extending $(g \restriction \alpha_p) \restriction i$.

In fact, we just proved that given an open dense set $D \subseteq \mathbb{P}$ in $\mathcal{H}$, for any condition $p \in \mathbb{Q}$, there is a $q \leq_{\mathbb{Q}} p$ such that $q \Vdash_{\mathbb{Q}} \dot{h} \cap D \neq \emptyset$. Given g and h_g as above, we also can define g from h_g in a simple way. Let $b \subseteq \alpha$ for some $\alpha < \Theta$ such that $b \in L(\Gamma, \mathbb{R})$. Let $p_{b,\alpha,n} \in \mathbb{P}$ be such that $n_{p_{b,\alpha,n}} = n + 1$ and $\alpha_{p_{b,\alpha,n}} = \alpha$ and $b \in \pi(p_{b,\alpha,n})(n)$. We can pick a map $\langle b, \alpha, n \rangle \mapsto p_{b,\alpha,n}$ in $\mathcal{H}$. Then

$$b = h(\alpha, n) \Leftrightarrow A_{b,\alpha,n} \in g_h.$$

We then can define symmetric $\mathbb{P}$-terms for $h(\alpha, n)$ and $ran(h)$ by

$$\sigma_{\alpha,n} = \{\langle p, \check{b}\rangle \mid b \subseteq \alpha \wedge p \leq_{\mathbb{P}} p_{b,\alpha,n}\},$$

and

$$\dot{R} = \{\langle p, \sigma_{\alpha,n}\rangle \mid p \in \mathbb{P} \wedge \alpha < \Theta \wedge n < \omega\}.$$

By the proof above, we have the following.

Lemma 4.3.21. *1. For any $h \subseteq \mathbb{Q}$ generic over $L(\Gamma, \mathbb{R})$, $\sigma_{\alpha,n}^{g_h} = h(\alpha, n)$ for all α, n and $\dot{R}^{g_h} = ran(h) = \mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$.*

2. For any condition $p \in \mathbb{P}$, there is an $\mathcal{H}$-generic g such that $p \in g$ and $\dot{R}^g = \mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$.

Since $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$, $L(\Gamma, \mathbb{R})$ can be recovered over $\mathcal{H}$ from $\mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$. This and Lemma 4.3.21 prove Lemma 4.3.20. □

Now work in $L(\mathcal{H}^+, g)$ for a generic g over $\mathcal{H}$ such that $\dot{R}^g = \mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$. It makes sense then to talk about the forcing $\mathbb{Q}$ in the model $L(\mathcal{H}^+, g)$. Also, note that $\mathbb{P} \in \mathcal{H}^+$. The following lemma is the key lemma.

Lemma 4.3.22. *There is a $\mathbb{P}$-generic g^* over $\mathcal{H}^+$ such that*

1. $\dot{R}^{g^*} = \dot{R}^g = \mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$.

2. $\mathcal{H}^+(\dot{R}^{g^*}) \cap \mathcal{P}_\Theta(\Theta) = \mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$ and $\mathcal{H}^+(\dot{R}^{g^*}) \cap \mathcal{P}(\mathbb{R}) = \Gamma$.

Proof. Let $h^* \subseteq \mathbb{Q}$ be $L(\mathcal{H}^+, g)$-generic. As mentioned above, $\mathbb{Q} \in L(\mathcal{H}^+, g)$ since $\dot{R}^g = \mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$. Now, let $g^* = g_{h^*}$. Using the proof of Lemma 4.3.20 and the fact that $V_\Theta^{\mathcal{H}^+} = V_\Theta^{\mathcal{H}}$, we get that g^* is generic over $\mathcal{H}^+$ and $\dot{R}^{g^*} = \dot{R}^g$.

Now we want to verify clause (2) of the lemma. For the first equality, it's clear that the $\supseteq$-direction holds. For the converse, if A is a bounded subset of Θ in $\mathcal{H}^+(\dot{R}^{g^*})$, then using the automorphisms of $\mathbb{P}$ that are in $\mathcal{H}$, it's easy to see that there are some $\alpha < \Theta$ such that $A \in \mathcal{H}^+[g^* \restriction \alpha]$. The idea is that if $p_0, p_1 \in \mathbb{P}$ decide differently the statement "$\check{\beta} \in \dot{A}$", then there is an automorphism in $\mathcal{H}$ that maps p_0 to p_0' compatible with p_1. This is a contradiction. Now since $\mathbb{P} \restriction \alpha$ is Θ-c.c. and $V_\Theta^{\mathcal{H}^+} = V_\Theta^{\mathcal{H}}$, $A \in \mathcal{H}(\dot{R}^g)$, and hence $A \in \mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$.

Note that the first equality of (2) shows that $\mathbb{R} \cap \mathcal{H}^+(\dot{R}^{g^*}) = \mathbb{R}^V$. Now we're onto the second equality of (2). The $\supseteq$-direction holds since $\mathcal{H}(\mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}) = L(\Gamma, \mathbb{R}) \subseteq \mathcal{H}^+(\dot{R}^{g^*})$. Let $A \subseteq \mathbb{R}^V$ be in $\mathcal{H}^+(\dot{R}^{g^*})$. First we assume A is definable in $\mathcal{H}^+(\dot{R}^{g^*})$ from an element $a \in \mathcal{H}^+$, via a formula ψ. Let $\dot{x}$ be a $\mathbb{P} \restriction \omega$-name for a real. The statement $\psi(\dot{x}, \check{a})$ is decided by $\mathbb{P} \restriction \omega$ by homogeneity of $\mathbb{P}$ (i.e. $\mathcal{H}^+ \vDash$ "$\emptyset \Vdash_{\mathbb{P}\restriction\omega} \psi[\dot{x}, \check{a}] \vee \emptyset \Vdash_{\mathbb{P}\restriction\omega} \neg\psi[\dot{x}, \check{a}]$"). Again, by the fact that $\mathbb{P} \restriction \omega$ is Θ-c.c., we get that $A \in \mathcal{H}[g^* \restriction \omega]$, hence $A \in \Gamma$. Now suppose A is definable in $\mathcal{H}^+(\dot{R}^{g^*})$ from an $a \in \mathcal{H}^+$ and a $b \in \mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$. By a basic Vopenka argument, there is a $< \Theta$-generic G_b over $\mathcal{H}$ and $\mathcal{H}^+$ such that $\mathrm{HOD}_b^{L(\Gamma,\mathbb{R})} = \mathcal{H}[G_b] \subseteq \mathcal{H}^+[G_b]$. Let us use $\mathcal{H}_b$ to denote $\mathcal{H}[G_b]$ and $\mathcal{H}_b^+$ to denote $\mathcal{H}^+[G_b]$. Now in $\mathcal{H}_b$, we can define the poset $\mathbb{P}_b$ the same way that $\mathbb{P}$ defined but we replace OD by $OD(b)$. Let g_b be $\mathbb{P}_b$-generic over $\mathcal{H}_b$ such that the symmetric part of $\mathcal{H}_b[g_b]$ is $\mathcal{H}_b(\mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})})$. Now we get a generic g_b^* over $\mathcal{H}_b^+$ from g_b as before. A is then definable over $\mathcal{H}_b^+(\mathcal{P}_\Theta(\Theta)^{L(\Gamma,\mathbb{R})})$ from parameters in $\mathcal{H}_b^+$. Now, we just have to repeat the argument above. This completes the proof of Lemma 4.3.22. □

Lemmata 4.3.20, 4.3.21, and 4.3.22 together prove Theorem 4.3.19. □

4.3.4.3 Con(T$_2$) ⇒ Con(T$_1$)

The main result of this section owes much to conversations between the author and G. Sargsyan when the author visited him in Rutgers in Fall 2012. The author would like to thank him. Recall the following result of the previous subsections.

Theorem 4.3.23. *Suppose (T$_2$) holds. Then there is a $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ such that $L(\Gamma,\mathbb{R}) \vDash \mathsf{AD}_\mathbb{R} + \Theta$ is regular.*

Suppose there is no model of (T$_1$) and let Γ be from Theorem 4.3.23 and be the maximal such pointclass. We define a model $\mathcal{H}^+$ as follows. Let μ be a normal fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$ witnessing (T$_2$). Let μ_Γ be the normal fine measure on $\mathcal{P}_{\omega_1}(\Gamma)$ induced by μ. $\forall^*_{\mu_\Gamma}\sigma \prec \Gamma$, let $\mathcal{H}_\sigma = \mathrm{HOD}^\sigma$. We then let

$$\mathcal{H}^+ = \Pi_\sigma Lp^{\oplus_{\alpha<\Theta^\sigma}\Sigma_\alpha^{\mathcal{H}_\sigma}}(\mathcal{H}_\sigma)/\mu_\Gamma.$$

We define a measure ν on Θ^Γ over $\mathcal{H}^+$ as follows. Let $A \in \mathcal{H}^+$. Then

$$A \in \nu \Leftrightarrow \forall^*_{\mu_\Gamma}\sigma \ sup(\sigma \cap \Theta^\Gamma) \in A. \tag{4.4}$$

The definition only makes sense if $\mathrm{cof}(\Theta^\Gamma) > \omega$. In fact, no Γ with $\mathrm{cof}(\Theta^\Gamma) = \omega$ can satisfy our hypothesis. Note also that the above definition makes sense for all $A \in V$ but we only care about those A's in $\mathcal{H}^+$. First we show the following.

Lemma 4.3.24. *ν is amenable to $\mathcal{H}^*$.*

Proof. Let $\mathcal{M} \lhd \mathcal{H}^+$ be sound and $\rho(\mathcal{M}) = \Theta^\Gamma$ (note that $\mathcal{H}^+$ is the union of such $\mathcal{M}$'s). Let $\nu_\mathcal{M} = \nu \restriction \mathcal{M}$. It's enough to show $\nu_\mathcal{M} \in \mathcal{H}^+$. Let N be a transitive model of $\mathsf{ZF}^- + \mathsf{DC}$ such that there is a surjection of Γ onto N and $\mathcal{M}, \Gamma \in N$. Let μ_N be the normal fine measure induced by μ. $\forall^*_{\mu_N}\sigma$, let $\pi_\sigma : M_\sigma \to \sigma$ be the uncollapse map. Let $\pi_\sigma(\Theta_\sigma, \mathcal{M}_\sigma, \nu_\sigma, \mathcal{H}_\sigma) = (\Theta^\Gamma, \mathcal{M}, \nu_\mathcal{M}, \mathrm{HOD}^\Gamma)$. It's easy to see that

$$\Pi_\sigma \nu_\sigma/\mu_N = \mu_\mathcal{M}.$$

Now let $\Sigma_\sigma = \oplus_{\alpha<\Theta^\sigma}\Sigma_\alpha^{\mathcal{H}_\sigma}$ and $\mathcal{H}_\sigma^+ = Lp^{\Sigma_\sigma}(\mathcal{H}_\sigma)$. It's also clear that

$$\Pi_\sigma \mathcal{H}_\sigma^+/\mu_N = \mathcal{H}^+$$

and

$$\forall^*_{\mu_N}\sigma\ \mathcal{M}_\sigma \lhd \mathcal{H}^+_\sigma.$$

We want to show $\forall^*_{\mu_N}\sigma\ \nu_\sigma \in \mathcal{H}^+_\sigma$. Fix such a σ, let $N_\sigma = \mathrm{HOD}^\Gamma_{\mathcal{H}^+_\sigma,\Sigma_\sigma}$. Note that $\mathcal{P}(\Theta_\sigma)\cap N_\sigma = \mathcal{P}(\Theta_\sigma)\cap\mathcal{H}^+_\sigma$ since $\mathcal{H}^+_\sigma$ is full in Γ. Let $\vec{A} = \langle A_\alpha \mid \alpha < \Theta_\sigma\rangle$ be a definable over $\mathcal{M}_\sigma$ enumeration of $\mathcal{P}(\Theta_\sigma)\cap\mathcal{M}_\sigma$. We want to show $\langle\alpha \mid A_\alpha \in \nu_\sigma\rangle \in N_\sigma$ which in turns implies $\langle\alpha \mid A_\alpha \in \nu_\sigma\rangle \in \mathcal{H}^+_\sigma$.

Let $\gamma_\sigma = \sup(\pi_\sigma[\Theta_\sigma])$ (note that $\pi_\sigma[\Theta_\sigma]$ coincides with the the iteration embedding via Σ_σ and it's part of our assumption that $\gamma_\sigma < \Theta^\Gamma$). Note that

$$\forall\alpha < \Theta_\sigma\ A_\alpha \in \nu_\sigma \Leftrightarrow \gamma_\sigma \in \pi_\sigma(A)|(\gamma_\sigma+1) \tag{4.5}$$

and

$$\langle\pi_\sigma(A_\alpha)|(\gamma_\sigma+1) \mid \alpha < \Theta_\sigma\rangle \in N_\sigma. \tag{4.6}$$

4.6 is true because $\langle\pi_\sigma(A_\alpha) \mid \alpha < \Theta_\sigma\rangle \in \mathcal{H}^+$. Hence $\langle\pi_\sigma(A_\alpha)|(\gamma_\sigma+1) \mid \alpha < \Theta_\sigma\rangle \in \mathcal{H}^+|\Theta = \mathrm{HOD}^\Gamma$. Since $\mathrm{HOD}^\Gamma \subseteq N_\sigma$, we have 4.6.

By Equations 4.5 and 4.6, we have $\langle\alpha \mid A_\alpha \in \nu_\sigma\rangle \in N_\sigma$. This completes the proof of the lemma. □

Now we want to show that ν is normal and $\mathcal{P}(\Theta^\Gamma)\cap L[\mathcal{H}^+,\nu] = \mathcal{P}(\Theta^\Gamma)\cap\mathcal{H}^+$. Let $\mathcal{M}\lhd\mathcal{H}^+$ be sound and $\rho(\mathcal{M}) = \Theta^\Gamma$. From now to the end of Lemma 4.3.31, we will write Θ interchangably with Θ^Γ.

Lemma 4.3.25. *$\nu_\mathcal{M} =_{def} \nu\restriction\mathcal{M}$ is normal.*

Proof. Suppose not. Let $N \vDash \mathsf{ZF}^- + \mathsf{DC}$ be such that

1. there is a surjection $\pi : \Gamma \twoheadrightarrow N$;
2. $\Gamma, \mu_\Gamma, \mathcal{M}, \nu_\mathcal{M} \in N$;
3. N is transitive.

We may also assume that N sees a surjection from Γ onto $\mathcal{M}$. Working in V, let μ_N be the measure on $\mathcal{P}_{\omega_1}(N)$ induced by μ_Γ.

$\forall^*_{\mu_N}\sigma$, let $\pi_\sigma : M_\sigma \to \sigma$ be the uncollpase map and $(\mathcal{M}_\sigma,\Theta_\sigma) = \pi_\sigma^{-1}(\mathcal{M},\Theta)$. We define a measure ν_σ on Θ_σ over $\mathcal{M}_\sigma$ as follows.

$$A \in \nu_\sigma \Leftrightarrow \gamma_\sigma =_{def} \sup(\pi_\sigma[\Theta_\sigma]) \in \pi_\sigma(A). \tag{4.7}$$

It's easy to see that

$$\Pi_\sigma\nu_\sigma/\mu_N = \nu_\mathcal{M}. \tag{4.8}$$

By the assumption on $\nu_{\mathcal{M}}$, we have that $\forall^*_{\mu_N}\sigma\ \nu_\sigma$ is not normal. This means

$$\{\pi_\sigma(f)(\gamma_\sigma) \mid f \in \mathcal{M}_\sigma\} \cap \gamma_\sigma \neq \sigma \cap \gamma_\sigma. \tag{4.9}$$

In other words, $\forall^*_{\mu_N}\sigma \exists f \in \sigma\ f(\gamma_\sigma) \notin \sigma \cap \gamma_\sigma \wedge f(\gamma_\sigma) < \gamma_\sigma$. By normality of μ_N, $\exists f \forall^*_{\mu_N}\sigma\ f(\gamma_\sigma) \notin \sigma \cap \gamma_\sigma \wedge f(\gamma_\sigma) < \gamma_\sigma$. Fix such an $f \in \mathcal{M}$ and let

$$A' = \{\sigma \mid f(\gamma_\sigma) \notin \sigma \cap \gamma_\sigma \wedge f(\gamma_\sigma) < \gamma_\sigma\}. \tag{4.10}$$

Note that $A' \in \mu_N$. This implies that $B \in \nu_{\mathcal{M}}$ where

$$B = \{\gamma \mid f(\gamma) < \gamma\}. \tag{4.11}$$

We may assume f is regressive everywhere. First we show that $\nu_{\mathcal{M}}$ is weakly normal in the following sense.

Lemma 4.3.26. *There is an $\eta < \Theta^\Gamma$ such that $\forall^*_{\mu_N}\sigma\ f(\gamma_\sigma) \leq \eta$.*

Proof. $\forall^*_{\mu_N}\sigma$, let $i_\sigma : \mathcal{M}_\sigma \to \mathcal{N}_\sigma$ be the iteration map according to $\Sigma_\sigma =_{def} \oplus_{\alpha<\Theta_\sigma}\Sigma_{\mathcal{H}_\sigma}(\alpha)$. Note that i_σ acts on $\mathcal{M}_\sigma$. Let $\nu^*_\sigma = i_\sigma(\nu_\sigma)$ and $(f_\sigma, B_\sigma) = (\pi^{-1}_\sigma(f), \pi^{-1}_\sigma(B))$. We have then that $\forall^*_{\mu_N}\sigma\ B_\sigma \in \nu_\sigma$, which implies that $i_\sigma(B_\sigma) \in \nu^*_\sigma$. By normality of ν^*_σ, $\exists \eta^*_\sigma < \gamma_\sigma \forall^*_{\nu^*_\sigma}\alpha\ i_\sigma(f_\sigma)(\alpha) = \eta^*_\sigma$. Let $\eta_\sigma < \Theta_\sigma$ be largest such that $i_\sigma(\eta_\sigma) \leq \eta^*_\sigma$. Then it's easy to see that $\forall^*_{\nu_\sigma}\alpha\ f_\sigma(\alpha) \leq \eta_\sigma$. Let $\eta = \Pi_\sigma \eta_\sigma/\mu_N < \Theta$. Note that $\forall^*_{\mu_N}\ \pi_\sigma(\eta_\sigma) = \eta$. This means $\forall^*_{\mu_N}\sigma \forall^*_{i_\sigma(\nu_\sigma)}\alpha\ i_\sigma(f_\sigma)(\alpha) \leq \eta$. Hence $\forall^*_{\mu_N}\sigma\ f(\gamma_\sigma) \leq \eta$. □

Let now $A = \{\sigma \mid f(\gamma_\sigma) \leq \eta\}$. By the previous lemma, $A \in \mu_N$.

Definition 4.3.27. *Suppose $A \subseteq \mathcal{P}_{\omega_1}(N)$. We say that A is **unbounded** if for all $\sigma \in \mathcal{P}_{\omega_1}(N)$, there is a $\tau \in A$ such that $\sigma \subseteq \tau$. We say that A is a **strong club (scub)** if A is unbounded and $\forall \sigma \in \mathcal{P}_{\omega_1}(N) \forall \tau \subseteq \sigma$, if whenever τ is finite, then there is a $\tau' \in A$ such that $\tau \subseteq \tau' \subseteq \sigma$, then $\sigma \in A$. A is a **weak club (wcub)** if A is unbounded and whenever $\langle \sigma_n \mid n < \omega\rangle$ is a $\subseteq -$increasing sequence of elements of A then $\cup_n \sigma_n \in A$.*

Clearly, a strong club is a weak club. This is a special case of the notion of scub introduced in [1].

Lemma 4.3.28. *Suppose $B \in \mu_N$. Then B meets every strong club. In particular, A meets every strong club.*

Proof. Suppose $C \subseteq \mathcal{P}_{\omega_1}(N)$ is a strong club and $C \cap B = \emptyset$. Let F be defined as follows. $F(\sigma) = \sigma \backslash \cup\{\tau \mid \tau \subsetneq \sigma \wedge \tau \in C\}$. By our assumption that C is a strong club and $C \cap B = \emptyset$, $\forall^*_{\mu_N}\sigma\ F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset$. This means $\exists x \forall^*_{\mu_N}\sigma\ \sigma \in B \backslash C \wedge x \in F(\sigma)$.

We claim that this is a contradiction. Fix such an x. Since C is a strong club, there is a $\sigma^* \in C$ such that $x \in \sigma^*$. By fineness and countable completeness of μ_N, the set $\{\sigma \in B \mid \sigma^* \subsetneq \sigma\} \in \mu_N$. This contradicts the definition of F. □

Note also that the above lemma implies that if C is a strong club, then $\mu_N(C) = 1$. Now let $\mathbb{P}$ be the natural forcing that shoots a weak club through A. Conditions in $\mathbb{P}$ are countable $W \subseteq A$ such that whenever $\langle \sigma_n \mid n < \omega \wedge \sigma_n \in W \rangle$ is $\subseteq$ −increasing then $\cup_n \sigma_n \in W$. $\forall C_0, C_1 \in \mathbb{P}$, $C_0 \leq_{\mathbb{P}} C_1$ iff $C_1 \subseteq C_0$.

Lemma 4.3.29. *$\mathbb{P}$ is (ω_1, ∞)−distributive.*

Proof. Fix a condition $C_0 \in \mathbb{P}$ and a sequence $\vec{D} = \langle D_i \mid i < \omega \rangle$ of open dense sets in $\mathbb{P}$. We want to find a condition $C \leq_{\mathbb{P}} C_0$ such that $C \in D_i$ for all i.
Claim: The set $D = \{\sigma \mid \sigma \prec N\}$ contains a strong club.

Proof. D is certainly unbounded. Now let $\sigma \in \mathcal{P}_{\omega_1}(N)$ and suppose for all finite $\tau \subseteq \sigma$, there is $\tau' \in D$ such that $\tau \subseteq \tau' \subseteq \sigma$. We want to show $\sigma \in D$. We prove by induction that for any n, for any finite $\tau \subseteq \sigma$, whenever $\tau \subseteq \tau' \subseteq \sigma$ and $\tau' \in D$ then $\tau' \prec_{\Sigma_n} \sigma \prec_{\Sigma_n} N$.

This clearly holds for $n = 0$. Now suppose the claim holds for n and let Ψ be a Π_n formula, $\tau \subseteq \sigma$ be finite such that $N \vDash \exists x\ \Psi[x, \tau]$. By our assumption, there is a $\tau' \in D$ such that $\tau \subseteq \tau' \subseteq \sigma$. By definition of D, $\tau' \prec N$, hence $\tau' \vDash \exists x\ \Psi[x, \tau]$. Let $x \in \tau'$ be a witness. We have then $\tau' \vDash \Psi[x, \tau]$. But $x \in \sigma$ and Ψ is Π_n; by the induction hypothesis, $\sigma \vDash \Psi[x, \tau']$. This proves the claim. □

Let N' be a transitive model of $\mathsf{ZF}^- + \mathsf{DC}$ such that $\mathcal{P}(\mathbb{R}) \twoheadrightarrow N'$ and $N, \mathbb{P}, \vec{D} \in N'$ and let N'' be a countable elementary submodel of N' such that $\mathbb{P}, \vec{D} \in N'' \cap N \in D$. Such an N'' exists by the claim. By a standard argument, we can build a $\leq_{\mathbb{P}}$ −descending chain of conditions $\langle C_n \mid n < \omega \rangle$ such that

1. $C_{n+1} \in D_n$;
2. $C_n \in N''$ for all n;
3. $\cup_n C_n = N'' \cap N$.

Let $C = \cup_n C_n \cup \{N'' \cap N\}$. Then $C \in \mathbb{P}$ and $C \leq_{\mathbb{P}} C_n$ for all n. This means $C \in D_n$ for all n. Hence we're done. □

Let $G \subseteq \mathbb{P}$ be V-generic. In $V[G]$, DC holds and there is a weak club $C \subseteq A$. Let then

$$C^* = \{\gamma_\sigma \mid \sigma \in C\}.$$

Then C^* contains an ω−club in $V[G]$. Now we proceed to derive a contradiction. Now we use an abstract pointclass argument to generalize Solovay's proof that ω_1 is measurable under AD to show the following.

Lemma 4.3.30. *There is a $\kappa < \Theta^\Gamma$ such that:*

1. *the ω−club filter on κ is an η^+-complete ultrafilter on $\mathcal{P}(\kappa) \cap \Gamma$;*

2. *the set* $\{\sigma \in A \mid \gamma_\sigma < \kappa\}$ *is unbounded in* $\mathcal{P}_{\omega_1}(\Gamma|\kappa)$[28]*; in particular,* $\{\gamma_\sigma \mid \sigma \in A\}$ *is unbounded in* κ*;*

3. $\forall \xi < \eta$*, the set of* $\sigma \in A$ *such that* $\xi \in \sigma$ *and* $\gamma_\sigma < \kappa$ *is unbounded in* $\mathcal{P}_{\omega_1}(\Gamma|\kappa)$*.*

Proof. Since Solovay's proof is well-known, we only highlight the necessary changes needed to run that proof in this situation. Working in $L(\Gamma, \mathbb{R})$, let $\eta^+ < \rho_1 < \rho_2 < \Theta$ where ρ_1, ρ_2 are regular Suslin cardinals. Furthermore, we assume that there is a prewellordering of length η in $S(\rho_1)$[29]. Fix a prewellordering $\leq$ of length η such that $\leq \in S(\rho_1)$ and let $f : \mathbb{R} \twoheadrightarrow \eta$ be the natural function induced from $\leq$.

We claim that there is a κ which is a limit of Suslin cardinals of cofinality ρ_2 (in Γ) and κ satisfies clauses (2) and (3) of the lemma. To see such a κ exists, first note that by Theorem 4.3.19, $\mathcal{H}^+(\Gamma) \cap \mathcal{P}(\mathbb{R}) = \Gamma$; since $\mathcal{H}^+(\Gamma)$ is the symmetric part of some homogeneous forcing and $\mathcal{H}^+ \vDash \Theta^\Gamma$ is regular, $\mathcal{H}^+(\Gamma) \vDash \mathsf{AD}_\mathbb{R} + \Theta$ is regular. Now the set Y of $\sigma \cap \mathcal{H}^+(\Gamma)$ such that Σ_σ is Γ-fullness preserving is in $\mathcal{H}^+(\Gamma)$ (note that γ_σ is a limit of Suslin cardinals and $\mathrm{cof}(\gamma_\sigma) = \omega$ in $\mathcal{H}^+(\Gamma)$); also, for each $\xi < \eta$, the set Y_ξ of $\sigma \in Y$ such that $\xi \in \sigma$ is in $\mathcal{H}^+(\Gamma)$. From these facts and the regularity of Θ^Γ in $\mathcal{H}^+(\Gamma)$, we easily get such a κ.

Fix such a κ. We show that κ satisfies (1) as well. Let Ω be the Steel pointclass at κ (see [32] or [8] for the definition of the Steel pointclass). The properties we need for Ω are:

1. $\exists^\mathbb{R} \utilde{\Delta}_\Omega \subseteq \utilde{\Delta}_\Omega$ (in fact, $\utilde{\Delta}_\Omega = \{Y \mid w(Y) < \kappa\}$);

2. Ω is closed under $\cap, \cup$ with $S(\rho_1)$-sets.

Let Z be an Ω-universal set and $\pi : Z \twoheadrightarrow \kappa$ be an Ω-norm.

For each $A \in \mathcal{P}(\kappa) \cap \Gamma$, we define the Solovay game G_A as follows. Players I and II take turns to play natural numbers. After ω many moves, say player I plays a real x and player II plays a real y. I wins the run of G_A iff either there is an i such that either $x_i \notin Z$ or $y_i \notin Z$ and letting j be the least such then $y_j \notin Z$ or $\sup\{\pi(x_i), \pi(y_j) \mid i, j < \omega\} \in A$.

Now we're ready to prove the $\omega-$club filter at κ $\mathcal{U}_\kappa$ is an η^+-complete ultrafilter. Note that $\mathcal{U}_\kappa$ is an ultrafilter follows from AD and in fact, $A \in \mathcal{U}_\kappa$ iff player I has a winning strategy in the game G_A. Fix a sequence $\langle A_\alpha \mid \alpha < \eta \wedge A_\alpha \in \mathcal{U}_\kappa\rangle$. We want to show $\cap_\alpha A_\alpha \in \mathcal{U}_\kappa$. Since $A_\alpha \in \mathcal{U}_\kappa$, player I has a winning strategy for the game G_{A_α}. Let $g : \eta \to \mathcal{P}(\mathbb{R})$ be such that for all $\xi < \eta$, $g(\xi) \subseteq \{\tau \mid \tau$ is a winning strategy for player I in $G_{A_\xi}\}$ and furthermore $Code(g, \leq) = \{(x, \tau) \mid \tau \in g(f(x))\} \in S(\rho_1)$. Such a g exists by the coding lemma.

For each $\xi < \kappa$, let $Y_\xi = \{(\tau[y])_n \mid n < \omega \wedge \exists x(x, \tau) \in Code(g, \leq) \wedge \forall i(\pi(y_i) < \xi)\}$. It's easy to see from the fact that π is Ω-norm, Ω is closed under intersection with $S(\rho_1)-$sets that $Y_\xi \in \utilde{\Delta}_\Omega$. By boundedness, $g(\xi) = \sup\{\pi(z) \mid z \in Y_\xi\} < \kappa$ for all ξ. This easily implies that I has a winning strategy in the game $G_{\cap_\alpha A_\alpha}$, which in turns implies $\cap_\alpha A_\alpha \in \mathcal{U}_\kappa$. □

[28]This is the collection of $X \in \Gamma$ such that $w(X) < \kappa$.

[29]For a Suslin cardinal ξ, $S(\xi)$ is the pointclass of $\xi-$Suslin sets.

Fix a κ as in Lemma 4.3.30 and let $\mathcal{U}_\kappa$ be the $\omega-$club filter in Γ. Note that κ has uncountable cofinality in $V[G]$. Let $D = \{\gamma \mid f(\gamma) \leq \eta\} \in \nu_{\mathcal{M}}$. By the coding lemma, $D \cap \kappa \in \Gamma$. We claim that $D \cap \kappa \in \mathcal{U}_\kappa$. Otherwise, $D \cap \kappa$ is disjoint from an $\omega-$club $E \in \Gamma$. But in $V[G]$, $D \cap \kappa$ contains an $\omega-$club, namely $C^* \cap \kappa$. In $V[G]$, E remains an $\omega-$club, hence has nonempty intersection with $C^* \cap \kappa$. This is a contradiction.

Finally, since $D \cap \kappa \in \mathcal{U}_\kappa$ and $\mathcal{U}_\kappa$ is η^+-complete (in $\mathcal{H}^+$), there is a $\xi \leq \eta$ such that $D_\xi = \{\gamma < \kappa \mid f(\gamma) = \xi\} \in \mathcal{U}_\kappa$. But then there is a $\sigma \in C$ such that $\gamma_\sigma < \kappa$, $\xi \in \sigma$, and $f(\gamma_\sigma) = \xi$. This contradicts the fact that $\forall \sigma \in C\ f(\gamma_\sigma) \notin \sigma$. This completes the proof of Lemma 4.3.25. $\square$

Lemma 4.3.31. *Let $\mathcal{H}^{++} = Lp^{\oplus_{\alpha<\Theta^\Gamma}\Sigma^{\mathcal{H}}_\alpha}(\mathcal{H}^+)$. Then $\mathcal{P}(\Theta) \cap (\mathcal{H}^{++}, \nu)$[30] $= \mathcal{P}(\Theta) \cap \mathcal{H}^+$.*

Proof. Suppose not. Then there is an $\mathcal{M}^* \trianglelefteq (\mathcal{H}^{++}, \mu)$ such that $\rho(\mathcal{M}^*) \leq \Theta$ and $\mathcal{M}^*$ defines a set not in $\mathcal{H}^+$. We may assume $\mathcal{M}^*$ is minimal and $\rho_1(\mathcal{M}^*) \leq \Theta$. Let $\mathcal{M}$ be the transitive collapse of $Hull_1^{\mathcal{M}^*}(\Theta \cup p_1^{\mathcal{M}^*})$. Then $\mathcal{M}$ is transitive and $\mathcal{M}$ Σ_1-defines a set not in $\mathcal{H}^+$.

Let $N \vDash \mathsf{ZF}^-$ be transitive such that $\Gamma \twoheadrightarrow N$ and $\mathcal{M}, \Gamma \in N$. Let μ_N be the supercompact measure on $\mathcal{P}_{\omega_1}(N)$ induced by μ_Γ. $\forall^*_{\mu_N}\sigma$, let $\pi_\sigma : M_\sigma \to N$ be the uncollapse map. Let $\pi_\sigma(\mathcal{M}_\sigma, \mathcal{H}_\sigma, \Theta_\sigma) = (\mathcal{M}, \mathrm{HOD}^\Gamma, \Theta)$.

Lemma 4.3.32. *There is a strategy Σ^+_σ for $\mathcal{M}_\sigma$ with the following properties:*

1. *Σ^+_σ is a π_σ-realizable strategy that extends Σ_σ[31]. This means that whenever $\vec{\mathcal{T}}$ is a stack according to Σ^+_σ, letting $i : \mathcal{M}_\sigma \to \mathcal{P}$ be the iteration embedding, then there is a map $k : \mathcal{P} \to \mathcal{M}$ such that $\pi_\sigma = k \circ i$.*

2. *Whenever $(\mathcal{Q}, \Lambda) \in I(\mathcal{M}_\sigma, \Sigma^+_\sigma)$, $\forall \alpha < \lambda^{\mathcal{Q}}$, $\Lambda_{\mathcal{Q}(\alpha)}$ is $\Gamma(\mathcal{M}_\sigma, \Sigma^+_\sigma)$-fullness preserving and has branch condensation. Hence Σ^+_σ is $\Gamma(\mathcal{M}_\sigma, \Sigma^+_\sigma)$-fullness preserving.*

Proof. We prove (1) (see Diagram 4.1. The proof of (2) is just the proof of Theorem 2.7.6 of [23] so we omit it; we just mention the key point in proving (2) is that $\Lambda_{\mathcal{Q}(\alpha)}$ for $\alpha < \lambda^{\mathcal{Q}}$ is a pullback of a strategy that is Γ-fullness preserving and has branch condensation.

Fix a σ. Let $\nu_\sigma = \pi_\sigma^{-1}(\nu_{\mathcal{M}})$. Suppose $i : \mathcal{M}_\sigma, \nu_\sigma \to \mathcal{P}, \nu_{\mathcal{P}}$ is the ultrapower map using ν_σ. We describe how to obtain a π_σ-realizable strategy $\Sigma_{\mathcal{P}(\alpha)}$ for $\alpha < \lambda^{\mathcal{P}}$. We then let $\Sigma^-_{\mathcal{P}} = \oplus_{\alpha<\lambda^{\mathcal{P}}}\Sigma_{\mathcal{P}(\alpha)}$ and $\vec{\mathcal{T}}$ be a stack on $\mathcal{P}$ according to $\Sigma^-_{\mathcal{P}}$ with end model $\mathcal{Q}$. Let $j : \mathcal{P}, \nu_{\mathcal{P}} \to \mathcal{Q}, \nu_{\mathcal{Q}}$ be the iteration map and $k : \mathcal{Q} \to \mathcal{R}$ be the ultrapower map by $\nu_{\mathcal{Q}}$. We describe how to obtain π_σ-realizable strategy $\Sigma_{\mathcal{Q}(\alpha)}$ for all $\alpha < \lambda^{\mathcal{Q}}$ and a π_σ-realizable strategy $\Sigma_{\mathcal{R}(\alpha)}$ for all $\alpha < \lambda^{\mathcal{R}}$. The construction of the strategy for this special case has all the ideas needed to construct the full strategy.

Let $\tau \prec N$ be such that $\sigma, \vec{\mathcal{T}} \in \tau$ and are countable there. μ_N-allmost-all τ have this property. Let $\pi_{\sigma,\tau} = \pi_\tau^{-1} \circ \pi_\sigma$, where $\pi_\tau : N_\tau \to N$ is the uncollapse map. Let $\pi_\tau(\mathcal{M}_\tau, \nu_\tau, \mathcal{H}_\tau) =$

[30] We identify ν with the top extender indexed at $o(\mathcal{H}^{++})$ according to the rule of the fine-extender sequence in [17].

[31] Recall that $\Sigma_\sigma = \oplus_{\alpha<\Theta_\sigma}\Sigma^{\mathcal{H}_\sigma}_\alpha$ is the join of the strategies of the $\mathcal{H}_\sigma(\alpha)$'s

$$
\begin{tikzcd}
 & & & \mathcal{M},\nu \\
 & & \mathcal{M}_\tau,\nu_\tau \arrow[ur, "\pi_\tau"] \arrow[r, "h"] & Ult(\mathcal{M}_\tau,\nu_\tau) \arrow[u, "h^*"'] \\
\mathcal{M}_\sigma,\nu_\sigma \arrow[urr, "\pi_{\sigma,\tau}"] \arrow[r, "i"'] & \mathcal{P},\nu_{\mathcal{P}} \arrow[ur, "i^*"'] \arrow[r, "\vec{\mathcal{T}},j"'] & \mathcal{Q},\nu_{\mathcal{Q}} \arrow[u, "j^*"] \arrow[r, "k"'] & \mathcal{R},\nu_{\mathcal{R}} \arrow[u, "k^*"]
\end{tikzcd}
$$

Figure 4.1: The construction of Σ_σ^+

$(\mathcal{M},\nu_{\mathcal{M}},\mathcal{H})$. Working in N_τ, let $\Sigma^-_{\mathcal{M}_\sigma} = \oplus_{\alpha<\lambda^{\mathcal{M}_\sigma}}\Sigma_{\mathcal{M}_\sigma(\alpha)}$ and $\gamma_0 = i^{\Sigma^-_{\mathcal{M}_\sigma}}_{\mathcal{M}_\sigma,\infty}(\lambda^{\mathcal{M}_\sigma})$. Let $i^* : \mathcal{P} \to \mathcal{M}_\tau$ be such that $i^*(i(f)(\lambda^{\mathcal{M}_\sigma})) = \pi_{\sigma,\tau}(f)(\gamma_0)$. It's easy using the fact that ν_σ is normal to show i^* is elementary and $\pi_{\sigma,\tau} = i^* \circ i$ (so $\pi_\sigma = \pi_\tau \circ i^* \circ i$). Note also that $i^*(\nu_{\mathcal{P}}) = \nu_\tau$. Now, let $(\mathcal{N},\Lambda)$ be a point in the direct limit system giving rise to $\mathcal{H}_\tau$ be such that $ran(i^* \restriction \lambda^{\mathcal{P}}) \subseteq ran(i^\Lambda_{\mathcal{N},\infty})$. There is some $s : \mathcal{P}|\lambda^{\mathcal{P}} \to \mathcal{N}$ such that $i^\Lambda_{\mathcal{N},\infty} \circ s = i^* \restriction \lambda^{\mathcal{P}}$. Then $\Sigma^-_{\mathcal{P}}$ is simply the s-pullback of Λ. Note that Λ can be extended to a fullness preserving strategy with branch condensation in Γ. It's not hard to show that the definition of $\Sigma^-_{\mathcal{P}}$ doesn't depend on the choice of $(\mathcal{N},\Lambda)$ and the choice of τ.

Now every element of $\mathcal{Q}$ has the form $j(f)(a)$ for some $f \in \mathcal{P}$ and $a \in \alpha(\vec{\mathcal{T}})^{<\omega}$, where $\alpha(\vec{\mathcal{T}})$ is the sup of the all the generators used along $\vec{\mathcal{T}}$. We let $j^* : \mathcal{Q} \to \mathcal{M}_\tau$ be such that $j^*(j(f)(a)) = i^*(f)(i_F(i^{\Sigma^-_{\mathcal{Q}}}_{\mathcal{Q},\infty}(a)))$. Hence $i^* = j^* \circ j$ and $\pi_\sigma = j^* \circ j \circ i$.

Finally, every element of $\mathcal{R}$ has the form $k(f)(\lambda^{\mathcal{Q}})$ for some $f \in \mathcal{Q}$. Let $h : \mathcal{M}_\tau \to Ult(\mathcal{M}_\tau,\nu_\tau)$ be the ultrapower map and $h^* : Ult(\mathcal{M}_\tau,\nu_\tau) \to \mathcal{M}$ be such that $\pi_\tau = h^* \circ h$. Then let $k^* : \mathcal{Q} \to Ult(\mathcal{M}_\tau,\nu_\tau)$ be such that $k^*(k(f)(\lambda^{\mathcal{Q}})) = h(j^*(f))(\lambda^{\mathcal{M}_\tau})$. It's easy to see that $h \circ j^* = k^* \circ k$. We can now derive the strategy $\Sigma^-_{\mathcal{R}}$ using $h^* \circ k^* \restriction \lambda^{\mathcal{R}}$ the same way we used $i^* \restriction \lambda^{\mathcal{P}}$ to derive the strategy $\Sigma^-_{\mathcal{P}}$. Again, it's easy to show that $\Sigma^-_{\mathcal{R}}$ is a π_σ-realizable strategy. □

By a ZFC-comparison argument (Theorem 2.3.2 of [23]) and the fact that Σ^+_σ is $\Gamma(\mathcal{M}_\sigma,\Sigma^+)$ -fullness preserving, an iterate of Σ^+_σ has branch condensation. Without loss of generality, we may assume Σ^+_σ has branch condensation.

Now by the maximality of Γ, $\Sigma^+_\sigma \in \Gamma$. Otherwise, by a core model induction using Σ^+_σ has branch condensation, we get $L(\Sigma^+_\sigma,\mathbb{R}) \vDash \mathsf{AD}^+$ and the argument for getting a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" gives a pointclass Γ' strictly extending Γ such that $L(\Gamma',\mathbb{R}) \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. We proceed to derive a contradiction from the assumption that $\Sigma^+_\sigma \in \Gamma$.

First assume $\rho_1(\mathcal{M}_\sigma) < \Theta_\sigma$. In Γ, we can define a direct limit system $\mathcal{F} = \{(\mathcal{Q},\Lambda) \mid (\mathcal{Q},\Lambda)$

$\equiv (\mathcal{M}_\sigma, \Sigma_\sigma^+)\}$[32] (this uses that $\Sigma_\sigma^+ \in \Gamma$). Let $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$. Hence $\mathcal{M}_\infty \in \mathrm{HOD}^\Gamma$, $\mathrm{HOD}^\Gamma|\gamma_\sigma \lhd \mathcal{M}_\infty$ by fullness preservation of Σ_σ, and $\rho_1(\mathcal{M}_\infty) < \gamma_\sigma$. This means $\mathcal{M}_\infty$ constructs a bounded subset of γ_σ in HOD^Γ but not in $\mathrm{HOD}^\Gamma|\gamma_\sigma$. This contradicts the fact that $\mathrm{HOD}^\Gamma|\gamma_\sigma = V_{\gamma_\sigma}^{\mathrm{HOD}^\Gamma}$ and γ_σ is a strong limit cardinal in HOD^Γ.

Now assume $\rho_1(\mathcal{M}_\sigma) = \Theta_\sigma$ and let $A \subseteq \Theta_\sigma$ be a set Σ_1 definable over $\mathcal{M}_\sigma$ but not in $Lp^{\Sigma_\sigma}(\mathcal{H}_\sigma)$ (A exists by our assumption). Say

$$\alpha \in A \Leftrightarrow \mathcal{M}_\sigma \vDash \psi[\alpha, s, p_1^{\mathcal{M}_\sigma}], \tag{4.12}$$

for some $s \in \Theta_\sigma^{<\omega}$. Recall that $\mathcal{M}_\sigma \vDash \Theta_\sigma$ is measurable as witnessed by ν_σ. We define $\mathcal{F}$ as above[33]. Let $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$ and let $i_{\mathcal{M}_\sigma,\infty} : \mathcal{M}_\sigma \to \mathcal{M}_\infty$ be the iteration embedding. We have that $\mathrm{HOD}^\Gamma|\gamma_\sigma \lhd \mathcal{M}_\infty \in \mathrm{HOD}^\Gamma$ and $\rho_1(\mathcal{M}_\infty) = \gamma_\sigma$. Let A_∞ be defined over $\mathcal{M}_\infty$ the same way A is defined over $\mathcal{M}_\sigma$, i.e.

$$\alpha \in A_\infty \Leftrightarrow \mathcal{M}_\infty \vDash \psi[\alpha, i_{\mathcal{M}_\sigma,\infty}(s), p_1^{\mathcal{M}_\infty}]. \tag{4.13}$$

Since A_∞ is OD^Γ, A is ordinal definable from $(\mathcal{H}_\sigma, \Sigma_\sigma)$ in Γ. By mouse capturing in Γ, $A \in Lp^{\Sigma_\sigma}(\mathcal{H}_\sigma)$. Contradiction. $\square$

Lemma 4.3.33. *Suppose $\Gamma = L(\Gamma, \mathbb{R}) \cap \Gamma$. Then $L[\mathcal{H}^+, \nu](\Gamma) \cap \mathcal{P}(\mathbb{R}) = \Gamma$ and $L[\mathcal{H}^+, \nu](\Gamma) \vDash$ $\mathsf{AD}_{\mathbb{R}}+$ there is an $\mathbb{R}$-complete measure on Θ.*

Proof. The equality of in the conclusion of the lemma follows from Theorem 4.3.19 with HOD^Γ playing the role of $\mathcal{H}$ and $(\mathcal{H}^{++}, \nu)$ playing the role of $\mathcal{H}^+$. Hence we also get $(\mathcal{H}^{++}, \nu)(\Gamma) \vDash \mathsf{AD}_{\mathbb{R}}$. The $\mathbb{R}$-complete measure on Θ in $(\mathcal{H}^{++}, \nu)(\Gamma)$ comes from ν from the proof of Theorem 2.4 in [3]. The proof uses the fact that every $A \in \Gamma$ can be added to $(\mathcal{H}^{++}, \nu)$ via a forcing of size $< \Theta$. $\square$

This completes the proof of Theorem 4.3.16.

[32] This means these hod pairs are Dodd-Jensen equivalent. The fine structural details involved in the comparison process is described in [23].

[33] We take Σ_0-ultrapowers for extenders with critical points $\geq$ the image of Θ_σ under iteration embeddings by Σ_σ and Σ_1-ultrapowers otherwise

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