Scales in hybrid mice over $\mathbb{R}$

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Abstract

We analyze scales in $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F}{\restriction}\mathrm{HC})$, the stack of projecting, Θ-g-organized $\mathcal{F}$-mice over $\mathcal{F}{\restriction}\mathrm{HC}$, for operators $\mathcal{F}$ with nice condensation properties. This builds on Steel's analysis of scales in $\mathrm{Lp}(\mathbb{R})$ in [17] and [20]. As in [20], we work from optimal determinacy hypotheses. One of the main applications of our work is in the core model induction.

1 Introduction

There has been significant progress made in the core model induction in recent years. Pioneered by W. H. Woodin and further developed by J. R. Steel, R. D. Schindler and others, it is a powerful method for obtaining lower-bound consistency strength for a large class of theories. One of the key ingredients is the scales analysis in $L(\mathbb{R})$ ([18]) and in $\mathrm{Lp}(\mathbb{R})$ (that is, $K(\mathbb{R})$; see [17] and [20]). Applications include Woodin's proof of $\mathsf{AD}^{L(\mathbb{R})}$ from an ω_1-dense ideal on ω_1 and Steel's proof that PFA implies $\mathsf{AD}^{L(\mathbb{R})}$, amongst many others.

To obtain lower-bound consistency strength stronger than $\mathsf{AD}^{L(\mathbb{R})}$ - for example, to construct models of "$\mathsf{AD}^+ + \Theta > \Theta_0$" - one would like to have the scales analysis of $\mathrm{Lp}^{\mathcal{F}}(\mathbb{R})$ (the stack of projecting $\mathcal{F}$-mice over $\mathbb{R}$) for various operators $\mathcal{F}$. Unfortunately, if $\mathcal{F}$ is an operator coding an iteration strategy Σ, the usual definition[1] of "$\mathcal{F}$-premouse over $\mathbb{R}$" doesn't make sense, because $\mathbb{R}$ is not wellordered. One might try to get around this particular issue by arranging $\mathcal{F}$-premice by simultaneously feeding in multiple branches instead of feeding them in one by one. But it seems difficult to define an amenable predicate achieving

[1] Roughly, that is: Given $\mathcal{F}$-premice $\mathcal{N} \trianglelefteq \mathcal{M}$, with $\mathcal{N}$ reasonably closed, and letting $\mathcal{T}$ be the $<_{\mathcal{N}}$-least iteration tree for which $\mathcal{N}$ lacks instruction regarding the branch $b = \Sigma(\mathcal{T})$, then b is the next piece of information fed in to $\mathcal{M}$ after $\mathcal{N}$. See §3 for details.

this,[2] and even if one could do so, the scale constructions in [17] and [20] do not appear to generalize well with such an approach, because of their dependence on the close relationship between a mouse over $\mathbb{R}$ and its HOD. These problems are solved by using the hierarchy of *Θ-g-organized $\mathcal{F}$-premice*, which are a certain kind of strategy premouse $\mathcal{M}$ built over $(\mathrm{HC}^{\mathcal{M}}, X)$, where X is *self-scaled* in $\mathcal{M}$ (see 4.22; this holds for $X = \emptyset$). The definition is a simple variant of *g-organization*, which is essentially due to Sargsyan; its main point is contained within his notion of reorganized hod premice, [6, §3.7]. However, in our presentation some of the details are a little different. For the precise definitions see 4.15, 4.17, and 4.23. We only define (Θ-)g-organization for *nice* operators $\mathcal{F}$ (niceness demands both a degree of *condensation* and of *generic determination* of $\mathcal{F}$; see 4.1). Given a nice $\mathcal{F}$ and self-scaled $X \subseteq \mathrm{HC}$, we define $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$ as the stack of all sound, countably iterable Θ-g-organized $\mathcal{F}$-premice over (HC, X), projecting to $\mathbb{R}$. We will analyze scales in this structure. If $X = \mathcal{F}\restriction\mathrm{HC}$, the analysis can be done from optimal determinacy assumptions. We remark that when $\mathrm{Lp}^{\mathcal{F}}(\mathbb{R}, X)$ is actually well-defined (such as when $\mathcal{F}$ is a mouse operator), we usually have $\mathrm{Lp}^{\mathcal{F}}(\mathbb{R}, X) \neq \mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$, but the two hierarchies agree on their $\mathcal{P}(\mathbb{R})$, and actually have identical extender sequences (see 5.5).[3]

The scale constructions themselves are mostly a fairly straightforward generalization of Steel's work in [18], [17] and [20]; the reader should be familiar with these.[4] Let $\mathcal{F}, X$ be as above, and let $\mathcal{M}$ end a weak gap of $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$. The construction of new scales over such $\mathcal{M}$ breaks into three cases, covered in Theorems 6.9, 6.16 and 6.20; these are analogous to [17, Theorems 4.16, 4.17] and [20, Theorem 0.1] respectively. Thus, for the first we must assume that $\mathcal{J}_1(\mathcal{M}) \models \mathsf{AD}$. In the context of our primary application (core model induction), this assumption *will* hold *if* $\mathcal{F}\restriction\mathrm{HC} \notin \mathcal{M}|\alpha$ and there are *no divergent* AD *pointclasses*; see 6.52. For the latter two we require that $\mathcal{M} \models \mathsf{AD}$, along with further assumptions. If X is the code-set for $\mathcal{F}\restriction\mathrm{HC}$ then the latter two theorems cover all weak gaps, and so one never requires that $\mathcal{J}_1(\mathcal{M}) \models \mathsf{AD}$.

We won't reproduce all the details of the proofs in [17] and [20], but will focus on the new features (and fill in some omissions). The most significant of these are as follows. First, we must generalize the local HOD analysis of a level $\mathcal{M}$ of $\mathrm{Lp}(\mathbb{R})$ to that of a level $\mathcal{M}$ of $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$. As in [17], we establish a level-by-level fine-structural correspondence between $\mathcal{H}$, the local HOD of $\mathcal{M}$, and $\mathcal{M}$ itself, above $\Theta^{\mathcal{M}}$. The fact that we are using Θ-g-organized $\mathcal{F}$-premice is very important in establishing this correspondence (and as for $\mathrm{Lp}(\mathbb{R})$, the

[2] See the remarks in Appendix B.

[3] There have been recent works that make use of methods and results from this paper, for example [22], [3], and [5].

[4] One needs familiarity with said papers for §§5,6 of this paper. If the reader has familiarity with just [18], one might read the present paper, referring to [17] and [20] as (will be) necessary.

correspondence itself is very important in the scales analysis). Second, an issue not dealt with in [20], but with which we deal here, is that a short tree $\mathcal{T}$ on a k-suitable $\mathcal{F}$-premouse $\mathcal{N}$ may introduce Q-structures with extenders overlapping $\delta(\mathcal{T})$ (since nontame $\mathcal{F}$-mice may exist). (However, such Q-structures *do not occur* in genericity iterations and in comparisons of suitable g-organized $\mathcal{F}$-mice.)

The paper is organized as follows. In §2 we first cover some background material, filling in some gaps in the literature. We discuss operators $\mathcal{F}$, and $\mathcal{F}$-premice. We define when $\mathcal{F}$ *condenses finely*, showing that this property ensures that $L^{\mathcal{F}}[\mathbb{E}]$-constructions run smoothly. In §3 we discuss *strategy premice* in detail, give a new presentation of these, and prove some condensation properties thereof, assuming that the strategy itself has good condensation properties. In §4 we define g-organized and Θ-g-organized $\mathcal{F}$-premice, and prove related condensation. In §5 we analyse the local HOD of $\mathcal{M} \triangleleft \mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$ when $\mathcal{M} \models$"Θ exists". In §6 we analyse the scales pattern in $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$. In the appendices we explain why we have used the notion of *condenses finely* in place of notions used by others, and the advantages in the presentation of strategy premice in §3.

Definitions and Notation. We work under ZF+"ω_1 is regular" throughout the paper.

For a set X, we write $\mathrm{card}(X)$ for the cardinality of X. For an ordinal θ, we write $\mathcal{P}(<\theta)$ for the set of bounded subsets of θ and $\mathscr{H}_\theta$ for the set of sets hereditarily of size $<\theta$. For M a transitive structure we write $\mathrm{o}(M)$ for the ordinal height of M. We write $\mathrm{trancl}(X)$ for the transitive closure of X. We use $a\frown b$ to denote the concatenation of a and b.

Given a transitive set X, possibly with some additional structure, we write $\mathcal{J}_\alpha(X)$ for the α^{th} step in Jensen's $\mathcal{J}$-hierarchy over X (so for example, $\mathcal{J}_1(X)$ is the rudimentary closure of $X \cup \{X\}$). Given a transitive set X and predicates $A_i \subseteq X$, and $\mathcal{M} = (X, A_1, \ldots)$, we write $\lfloor \mathcal{M} \rfloor$ for the universe X of $\mathcal{M}$.

A premouse $\mathcal{M}$ is as in [21]; in particular $\mathcal{M}$ is a $\mathcal{J}$-structure of the form $\mathcal{M} = (\mathcal{J}_\alpha[E], \in, E^{\mathcal{M}}, F^{\mathcal{M}})$, where $E = E^{\mathcal{M}}$ is a fine extender sequence and $F = F^{\mathcal{M}}$ is the (amenable code for the) top extender of $\mathcal{M}$. We write $\mathbb{E}_+(\mathcal{M})$ for $E \cup \{F\}$ and $\mathbb{E}(\mathcal{M})$ for E. For $\gamma \leq \alpha$, we write $\mathcal{M}|\gamma$ for $(\mathcal{J}_\gamma[E\restriction\gamma], \in, E\restriction\gamma, E(\gamma))$, and write $\mathcal{M}||\gamma$ for $(\mathcal{J}_\gamma[E\restriction\gamma], \in, E\restriction\gamma, \emptyset)$. So $\mathcal{M}|\gamma = \mathcal{M}||\gamma$ if and only if $E(\gamma) = \emptyset$. If $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ with successor length, we write $\mathcal{N}^{\mathcal{T}}$ for the last model of $\mathcal{T}$. We also apply the preceding terminology and notation to Y-premice over X for various X, Y; see 2.3, 2.10, 2.11 and 2.13 for some clarification. We use certain notions from [6].[5] Other terminology is mostly as in [21].

[5]Starting in §3, the reader should know the definitions of *hull condensation* and *branch condensation*. For a complete understanding of this article, one should also know the definitions of *hod premouse* and *hod pair* and some related material. However, everything that we do in relation to hod premice, we also do in relation to standard premice, and so the main ideas in this article can be understood without knowing the definition of *hod premouse*.

2 $\mathcal{F}$-premice

Definition 2.1. Let $\mathcal{L}_0$ be the language of set theory expanded by unary predicate symbols $\dot{E}, \dot{B}, \dot{S}$, and constant symbols $\dot{a}$, $\dot{\mathfrak{P}}$. Let $\mathcal{L}_0^- = \mathcal{L}_0 \backslash \{\dot{E}, \dot{B}\}$.

Let a be transitive. Let $\varrho : a \to \operatorname{rank}(a)$ be the rank function. We write $\hat{a} = \operatorname{trancl}(\{(a, \varrho)\})$. Let $\mathfrak{P} \in \mathcal{J}_1(\hat{a})$.

A **$\mathcal{J}$-structure over a (with parameter $\mathfrak{P}$) (for $\mathcal{L}_0$)** is a structure $\mathcal{M}$ for $\mathcal{L}_0$ such that $a^{\mathcal{M}} = a$, ($\mathfrak{P}^{\mathcal{M}} = \mathfrak{P}$), and there is $\lambda \in [1, \mathrm{Ord})$ such that $\lfloor \mathcal{M} \rfloor = \mathcal{J}_\lambda^{S^{\mathcal{M}}}(\hat{a})$.

Here we also let $l(\mathcal{M})$ denote λ, the **length** of $\mathcal{M}$, and let $\hat{a}^{\mathcal{M}}$ denote $\hat{a}$.

For $\alpha \in [1, \lambda]$ let $\mathcal{M}_\alpha = \mathcal{J}_\alpha^{S^{\mathcal{M}}}(\hat{a})$. We say that $\mathcal{M}$ is **acceptable** iff for each $\alpha < \lambda$ and $\tau < \mathrm{o}(\mathcal{M}_\alpha)$, if

$$\mathcal{P}(\tau^{<\omega} \times \hat{a}^{<\omega}) \cap \mathcal{M}_\alpha \neq \mathcal{P}(\tau^{<\omega} \times \hat{a}^{<\omega}) \cap \mathcal{M}_{\alpha+1},$$

then there is a surjection $\tau^{<\omega} \times \hat{a}^{<\omega} \to \mathcal{M}_\alpha$ in $\mathcal{M}_{\alpha+1}$.

A **$\mathcal{J}$-structure (for $\mathcal{L}_0$)** is a $\mathcal{J}$-structure over a, for some a. $\dashv$

As all $\mathcal{J}$-structures we consider will be for $\mathcal{L}_0$, we will omit the phrase "for $\mathcal{L}_0$". We also often omit the phrase "with parameter $\mathfrak{P}$". Note that if $\mathcal{M}$ is a $\mathcal{J}$-structure over a then $\lfloor \mathcal{M} \rfloor$ is transtive and rud-closed, $\hat{a} \in M$ and $\mathrm{Ord} \cap M = \operatorname{rank}(M)$. This last point is because we construct from $\hat{a}$ instead of a.

$\mathcal{F}$-premice will be $\mathcal{J}$-structures of the following form.

Definition 2.2. A **$\mathcal{J}$-model over a (with parameter $\mathfrak{P}$)** is an acceptable $\mathcal{J}$-structure over a (with parameter $\mathfrak{P}$), of the form

$$\mathcal{M} = (M; E, B, S, a, \mathfrak{P})$$

where $\dot{E}^{\mathcal{M}} = E$, etc, and letting $\lambda = l(\mathcal{M})$, the following hold.

1. $\mathcal{M}$ is amenable.

2. $S = \langle S_\xi \mid \xi \in [1, \lambda) \rangle$ is a sequence of $\mathcal{J}$-models over a (with parameter $\mathfrak{P}$).

3. For each $\xi \in [1, \lambda)$, $\dot{S}^{S_\xi} = S \restriction \xi$.

4. Suppose $E \neq \emptyset$. Then $B = \emptyset$ and there is an extender F over $\mathcal{M}$ which is $\hat{a} \times \gamma$-complete for all $\gamma < \operatorname{crit}(F)$ and such that the premouse axioms [23, Definition 2.2.1] hold for $(\mathcal{M}, F)$, and E codes $\tilde{F} \cup \{G\}$ where: (i) $\tilde{F} \subseteq M$ is the amenable code for F

(as in [21]); and (ii) if F is not type 2 then $G = \emptyset$, and otherwise G is the "longest" non-type Z proper segment of F in $\mathcal{M}$.[6] $\dashv$

Note that with notation as above, if λ is a successor ordinal then $M = \mathcal{J}(S^{\mathcal{M}}_{\lambda-1})$, and otherwise, $M = \bigcup_{\alpha<\lambda} \lfloor S_\alpha \rfloor$. The predicate $\dot{B}$ will be used to code extra information. Suppose $E^{\mathcal{M}}$ codes an extender F. Clearly $\mathrm{rank}(a) < \mathrm{crit}(F)$. Note that, in accordance with [23, Definition 2.2.1], but as opposed to [21, Definition 2.4], *we allow F to be of superstrong type* (see below).[7] Next, we describe some terminology and notation regarding the above definition.

Definition 2.3. Let $\mathcal{M}$ be a $\mathcal{J}$-model with parameter a. Let $E^{\mathcal{M}}$ denote $\dot{E}^{\mathcal{M}}$, etc. Let $\lambda = l(\mathcal{M})$, $S^{\mathcal{M}}_0 = a$, $S^{\mathcal{M}}_\lambda = \mathcal{M}$, and $\mathcal{M}|\xi = S^{\mathcal{M}}_\xi$ for all $\xi \leq \lambda$. An **(initial) segment** of $\mathcal{M}$ is just a structure of the form $\mathcal{M}|\xi$ for some $\xi \in [1, \lambda]$. We write $\mathcal{P} \trianglelefteq \mathcal{M}$ iff $\mathcal{P}$ is a segment of $\mathcal{M}$, and $\mathcal{P} \triangleleft \mathcal{M}$ iff $\mathcal{P} \trianglelefteq \mathcal{M}$ and $\mathcal{P} \neq \mathcal{M}$. Let $\mathcal{M}||\xi$ be the structure having the same universe and predicates as $\mathcal{M}|\xi$, except that $E^{\mathcal{M}||\xi} = \emptyset$. We say that $\mathcal{M}$ is E**-active** iff $E^{\mathcal{M}} \neq \emptyset$, and B**-active** iff $B^{\mathcal{M}} \neq \emptyset$. **Active** means either E-active or B-active; E**-passive** means not E-active; B**-passive** means not B-active; and **passive** means not active. Also, $\mathcal{M}$ is **type 0** iff $\mathcal{M}$ is passive, **type 4** iff $\mathcal{M}$ is B-active, and **type 1, 2** or **3** iff $\mathcal{M}$ is E-active, with the usual numerology. If $\mathcal{M}$ is E-active with extender F, we say $\mathcal{M}$, or F, is **superstrong** iff $i_F(\mathrm{crit}(F)) = \nu(F)$. We say that $\mathcal{M}$ is **super-small** iff $\mathcal{M}$ has no superstrong initial segment.

If $\mathcal{M}$ is not type 3, we define the fine-structural notions (i.e. projecta, parameters, solidity, soundness, cores) precisely as for passive premice in [1], using the language[8] $\mathcal{L}_0 \cup \hat{a}$, where $\hat{a}$ consists of constant symbols.[9] If $\mathcal{M}$ is type 3, we define the **squash** $\mathcal{M}^{\mathrm{sq}}$ of $\mathcal{M}$ as in [1], and fine-structure is defined over $\mathcal{M}^{\mathrm{sq}}$, still using the same language as in the previous case.

The classes of **Q-formulas** and **P-formulas** in the language $\mathcal{L}_0$, are defined analogously to in [1, §§2,3] (but with Σ_1 in place of the rΣ_1 of [1]).

[6]We use G explicitly, instead of the code $\gamma^{\mathcal{M}}$ used for G in [1, §2], because G does not depend on which (if there is any) wellorder of $\mathcal{M}$ we use. This ensures that certain pure mouse operators are *forgetful*.

[7]The main point of permitting superstrong type extenders is that it simplifies certain things. However, the cost is that it complicates others. If the reader prefers, one could instead require, as in [21], that F *not* be of superstrong type, but various statements throughout the paper regarding condensation would need to be modified, along the lines of [1, Lemma 3.3].

[8]So even if $E^{\mathcal{M}} \neq \emptyset$, we do not include constants analogous to those used in [1]. The interpretations of these constants are all encoded into $E^{\mathcal{M}}$.

[9]So $\mathfrak{C}_0(\mathcal{M}) = \mathcal{M}$. We only define the ρ_{k+1}, p_{k+1}, etc, given that $\mathfrak{C}_k(\mathcal{M})$ is a k-sound model over a. In any case, it certainly makes sense to ask whether "$\mathcal{M}$ is 1-solid" or "$\mathcal{M}$ is 1-sound", and to define $\mathfrak{C}_1(\mathcal{M})$, for example. We set ρ_1 to be the least ordinal ρ such that $\rho \geq \mathrm{rank}(a)$ and $\rho \geq \omega$ and there is some $A \subseteq \rho^{<\omega} \times \hat{a}^{<\omega}$ which is $\Sigma_1^{\mathcal{M}}(\mathcal{M})$, but $A \notin \mathcal{M}$. We say "$\rho_1 = a$" to mean "$\rho_1 = \max(\omega, \mathrm{rank}(a))$". Etc.

Let $\rho(\mathcal{M})$ be the least $\rho \leq \lambda$ such that there is some $A \subseteq \mathcal{M}$ such that A is $\Sigma_\omega^{\mathcal{M}}(\mathcal{M})$ and $A \cap \lfloor \mathcal{M}|\rho \rfloor \notin \mathcal{M}$.

An a-**cardinal** of $\mathcal{M}$ is an ordinal $\gamma < \mathrm{o}(\mathcal{M})$ such that in $\mathcal{M}$ there is no surjection $\hat{a}^{<\omega} \times \eta^{<\omega} \to \gamma$ with $\eta < \gamma$. We write $\Theta^{\mathcal{M}}$ for the supremum of all $\gamma < \mathrm{o}(\mathcal{M})$ such that in $\mathcal{M}$ there is a surjection $\hat{a}^{<\omega} \to \gamma$.

Let $\mathcal{M}$ be a $\mathcal{J}$-model and $\mathcal{N} \trianglelefteq \mathcal{M}$. We say that $\mathcal{N}$ is a **(strong) cutpoint** of $\mathcal{M}$ iff for all $\mathcal{P} \trianglelefteq \mathcal{M}$, if $\mathcal{N} \triangleleft \mathcal{P}$ and $E^{\mathcal{P}} \neq \emptyset$ then $\mathrm{o}(\mathcal{N}) \leq \mathrm{crit}(E^{\mathcal{P}})$ $(\mathrm{o}(\mathcal{N}) < \mathrm{crit}(E^{\mathcal{P}}))$.

Given a $\mathcal{J}$-model $\mathcal{M}_1$ over b and a $\mathcal{J}$-model $\mathcal{M}_2$ over $\mathcal{M}_1$, we write $\mathcal{M}_2{\downarrow}b$ for the $\mathcal{J}$-model $\mathcal{M}$ over b, such that $\mathcal{M}$ is "$\mathcal{M}_1 \frown \mathcal{M}_2$", if this is well-defined. That is, $\mathcal{M}_2{\downarrow}b$ is the unique $\mathcal{J}$-model $\mathcal{M}$ such that $\lfloor \mathcal{M} \rfloor = \lfloor \mathcal{M}_2 \rfloor$, $a^{\mathcal{M}} = b$, $E^{\mathcal{M}} = E^{\mathcal{M}_2}$, $B^{\mathcal{M}} = B^{\mathcal{M}_2}$, and $\mathcal{P} \triangleleft \mathcal{M}$ iff $\mathcal{P} \trianglelefteq \mathcal{M}_1$ or there is $\mathcal{Q} \triangleleft \mathcal{M}_2$ such that $\mathcal{P} = \mathcal{Q}{\downarrow}b$, when such an $\mathcal{M}$ exists. (Existence depends only on whether the $\mathcal{J}$-structure $\mathcal{M}$ described here is acceptable.)

Inverting this, given a $\mathcal{J}$-model $\mathcal{M}$ over b and $\mathcal{M}_1 \triangleleft \mathcal{M}$ such that $\mathcal{M}_1$ is a strong cutpoint of $\mathcal{M}$, we write $\mathcal{M}{\downarrow}\mathcal{M}_1$ for the $\mathcal{J}$-model $\mathcal{M}_2$ over $\mathcal{M}_1$ such that $\mathcal{M}_2{\downarrow}b = \mathcal{M}$. $\dashv$

Lemma 2.4. *The natural adaptations of Lemmas 2.4, 2.5, 3.2, 3.3 of* [1] *hold,*[10] *and in fact, in adapting conclusion* (b) *of* [1, Lemma 3.3], *we can omit the clause "or $\mathcal{N}$ is of superstrong type".*[11]

In fact, we can strengthen a little Lemmas 2.4 and 3.2 of [1].

Definition 2.5. Let $\mathcal{N}$ be a $\mathcal{J}$-structure with $E^{\mathcal{N}} \neq \emptyset$. If $E^{\mathcal{N}}$ is a set of partial extenders over $\mathcal{N}$, all with the same critical point μ, then we define $\mu(E^{\mathcal{N}}) = \mu$.

Let $\mathcal{M}$ be a $\mathcal{J}$-model. Let $\mathcal{R}$ be an $\mathcal{L}_0$-structure (possibly illfounded). If $\mathcal{M}$ is type 3 then let $\pi : \mathcal{R} \to \mathcal{M}^{\mathrm{sq}}$, and otherwise let $\pi : \mathcal{R} \to \mathcal{M}$.

We say that π is a **weak** 0**-embedding** iff π is Σ_0-elementary (therefore $\mathcal{R}$ is extensional and wellfounded, so we assume $\mathcal{R}$ is transitive) and there is an $\in$-cofinal set $X \subseteq \mathcal{R}$ such that π is Σ_1-elementary on elements of X, and if $\mathcal{M}$ is type 1 or 2, then (by the proof of 2.6 it follows that $\mu = \mu(E^{\mathcal{R}})$ is defined) there is an $\in \times \in$-cofinal set $Y \subseteq \mathcal{R}|(\mu^+)^{\mathcal{R}} \times \mathcal{R}$ such that π is Σ_1-elementary on elements of Y.

More generally, we define **(weak, near)** k**-embedding** analogously to [1]. Let $\mathcal{M}$ be a $\mathcal{J}$-model of type 3. A (weak, near) k-embedding $\pi : \mathcal{N} \to \mathcal{M}$ literally has domain $\lfloor \mathcal{N}^{\mathrm{sq}} \rfloor$ and codomain $\lfloor \mathcal{M}^{\mathrm{sq}} \rfloor$ and the elementarity of π is with regard to $\mathcal{N}^{\mathrm{sq}}, \mathcal{M}^{\mathrm{sq}}$. Here either $\mathcal{N}$ is a $\mathcal{J}$-model of type 3 (so $\mathcal{N}^{\mathrm{sq}} \neq \mathcal{N}$) or $\mathcal{N}$ is a $\mathcal{J}$-structure which we are already considering "at the squashed level" (for example, $\mathcal{N} = \mathrm{Ult}(\mathcal{Q}^{\mathrm{sq}}, E^{\mathcal{Q}})$ for some $\mathcal{J}$-model $\mathcal{Q}$ of type 3), in which case $\mathcal{N}^{\mathrm{sq}}$ denotes $\mathcal{N}$ itself. $\dashv$

[10] Note that for type 1 or 2 $\mathcal{J}$-models, the $\mu^{\mathcal{M}}$ and $\nu^{\mathcal{M}}$ (with notation as in [1, Lemma 2.5]) are in fact computable from any element of $E^{\mathcal{M}}$, and so we don't really need constant symbols for them.

[11] Because we allow superstrong extenders.

Lemma 2.6. *Let $\pi, \mathcal{R}, \mathcal{M}$ be as in 2.5, with π a weak* 0*-embedding.*[12]

(*1*) *$\mathcal{R}$ is a $\mathcal{J}$-structure.*

(*2*) *Suppose $\mathcal{M}$ is not type 3. Then for any Q-formula ψ and $z \in \mathcal{R}$, if $\mathcal{M} \models \psi(\pi(z))$ then $\mathcal{R} \models \psi(z)$. Therefore $\mathcal{R}$ is a $\mathcal{J}$-model of the same type as $\mathcal{M}$.*

(*3*) *Suppose $\mathcal{M}$ is type 3. Then for any P-formula ψ and $z \in \mathcal{R}$, if $\mathcal{M}^{\mathrm{sq}} \models \psi(\pi(z))$ then $\mathcal{R} \models \psi(z)$. Let $\mathcal{U} = \mathrm{Ult}(\mathcal{R}, E^{\mathcal{R}})$, $\gamma = \mathrm{o}(\mathcal{R})$ and $\lambda = (\gamma^+)^{\mathcal{U}}$. If $\mathcal{U}|\lambda$ is wellfounded then $\mathcal{R} = \mathcal{N}^{\mathrm{sq}}$ for some $\mathcal{J}$-model $\mathcal{N}$ of type 3.*

Proof. Let X, and Y if $\mathcal{M}$ is type 1 or 2, witness that π is a weak 0-embedding.

We first prove (1). Given $x \in \mathcal{R}$, let $y \in X$ with $x \in y$. Since $\mathcal{M} \models$"$\pi(y) \in \mathcal{S}_\alpha^{S^{\mathcal{M}}}(\hat{a}^{\mathcal{M}})$ for some ordinal α", therefore $\mathcal{R} \models$"$y \in \mathcal{S}_\alpha^{S^{\mathcal{R}}}(\hat{a}^{\mathcal{R}})$ for some ordinal α". This suffices.

We now prove (2) assuming that $\mathcal{M}$ is type 1 or 2. The function f is $\Sigma_1^{\mathcal{R}}$, where $f : \mathcal{R} \to \mathcal{R}$ and $f : y \mapsto \mathcal{S}_\alpha^{S^{\mathcal{R}}}(\hat{a}^{\mathcal{R}})$ where α is least such that $y \in \mathcal{S}_\alpha^{S^{\mathcal{R}}}(\hat{a}^{\mathcal{R}})$. Therefore we may and do assume that $X \subseteq \mathrm{rg}(f)$ and $Y \subseteq \mathrm{rg}(f) \times \mathrm{rg}(f)$.

Now by Σ_1-elementarity without parameters, $E^{\mathcal{R}} \neq \emptyset$ and $\gamma^{\mathcal{R}}$ is defined, and since π is Σ_0-elementary, π"$E^{\mathcal{R}} \subseteq E^{\mathcal{M}}$ and $\pi(\gamma^{\mathcal{R}}) = \gamma^{\mathcal{M}}$. Therefore $\mu = \mu(E^{\mathcal{R}})$ is defined.

Now for simplicity assume that ψ has only $n = 1$ free variable. Suppose

$$\psi(z) \iff \forall x \forall \theta < (\mu^+) \exists y \exists \nu [x \subseteq y \;\&\; \theta \leq \nu < (\mu^+) \;\&\; \varphi(z, y, \nu)]$$

where φ is Σ_1. Let $z \in \mathcal{R}$ be such that $\mathcal{M} \models \psi(\pi(z))$. Let $x \in \mathcal{R}$ and $\theta < (\mu^+)^{\mathcal{R}}$. Let $x \in x' \in \mathcal{R}$ and $\theta \in t \in \mathcal{R}|(\mu^+)^{\mathcal{R}}$ be such that $(x', t) \in Y$. Let $\theta' = \mathrm{o}(t)$. Then

$$\mathcal{M} \models \exists y \exists \nu [\pi(x') \subseteq y \;\&\; \pi(\theta') \leq \nu \;\&\; \mathrm{card}(\theta') = \mathrm{card}(\nu) \;\&\; \varphi(\pi(z), y, \nu)],$$

and this statement pulls back under π, which completes the proof.

We leave the remaining cases to the reader. □

Definition 2.7. We say that X is **explicitly swo'd (self-wellordered)** iff $X = x \cup \{x, <\}$ for some transitive set x, and wellorder $<$ of x. In this situation, $<_X$ denotes the wellorder of X extending $<$, and with last two elements x and $<$. We say that $\mathcal{M}$ is **implicitly swo'd** iff either $\mathcal{M}$ is explicitly swo'd, or $\mathcal{M}$ is a $\mathcal{J}$-model with parameter X for some explicitly swo'd X. In the latter case, $<_{\mathcal{M}}$ denotes the natural wellorder of $\lfloor \mathcal{M} \rfloor$ extending $<_X$. We may identify an implicitly swo'd $\mathcal{M}$ with the explicitly swo'd $\lfloor \mathcal{M} \rfloor \cup \{\mathcal{M}, <_{\mathcal{M}}\}$.

[12] In case of any confusion in relation to the last paragraph of 2.5, let us clarify that here if $\mathcal{M}$ is type 3 then we are considering $\mathcal{R}$ "at the squashed level".

We say that a set or class $\mathscr{B}$ is an **operator background** iff (i) $\mathscr{B}$ is transitive, rudimentarily closed and $\omega \in \mathscr{B}$, (ii) for all $x \in \mathscr{B}$ and all y, f, if $f\colon x^{<\omega} \to \mathrm{trancl}(y)$ is a surjection then $y \in \mathscr{B}$, and (iii) for every transitive $x \in \mathscr{B}$ and $a \subseteq x$ there are club many countable elementary substructures of (x, a). (So $\mathrm{o}(\mathscr{B}) = \mathrm{rank}(\mathscr{B})$ is a cardinal; if $\omega < \kappa \leq \mathrm{Ord}$ then $\mathscr{H}_\kappa$ is an operator background, and under ZFC these are the only operator backgrounds.)

Let $\mathscr{B}$ be an operator background. A set C is a **cone of** $\mathscr{B}$ iff there is $a \in \mathscr{B}$ such that C is the set of all $x \in \mathscr{B}$ such that $a \in \mathcal{J}_1(\hat{x})$. With a, C as such, we say C is **the cone above** a. If $b \in \mathcal{J}_1(a)$ we say C is **above** b. A set D is an **swo'd cone of** $\mathscr{B}$ iff $D = C \cap S$, for some cone C in $\mathscr{B}$, and where S is the class of explicitly swo'd sets. Here D is **(the swo'd cone) above** a iff C is (the cone) above a. A **cone** is a cone of $\mathscr{B}$ for some operator background $\mathscr{B}$. Likewise for **swo'd cone**. $\dashv$

We will deal with $\mathcal{F}$-premice where $\mathcal{F}$ is some *operator*. As in [15], there are two main classes of operators we have in mind: mouse operators and (iteration) strategy operators. We will now give some abstract framework for this, and will discuss the specific types of operators later in detail later. In the definition of *pre-operator* below, the reason we incorporate the variable i is as follows. Suppose we want to build a *strategy premouse* $\mathcal{N}$, i.e. a $\mathcal{J}$-model in which the B-predicates are used to code some fragment of an iteration strategy Σ (see 3.7 for a precise definition). Suppose we feed Σ is fed into $\mathcal{N}$ by always providing $b = \Sigma(\mathcal{T})$, for the $<_{\mathcal{N}}$-least tree $\mathcal{T}$ for which this information is required. So given a reasonably closed level $\mathcal{P} \triangleleft \mathcal{N}$, the choice of which tree $\mathcal{T}$ should be processed next will usually depend on the information regarding Σ already encoded in $\mathcal{P}$ (its *history*). Using an operator $\mathcal{F}$ to build $\mathcal{N}$, then $\mathcal{F}(i, \mathcal{P})$ will be a structure extending $\mathcal{P}$ and over which $b = \Sigma(\mathcal{T})$ is encoded. The variable i should be interpreted as follows. When $i = 1$, we respect the history of $\mathcal{P}$ when selecting $\mathcal{T}$. When $i = 0$ we ignore history when selecting $\mathcal{T}$.

Definition 2.8. Let $\mathscr{B}$ be an operator background. A **pre-operator over** $\mathscr{B}$ **with domain** D is a function $\mathcal{F} : D \to \mathscr{B}$ where for some (maybe swo'd) cone $C = C_{\mathcal{F}}$ of $\mathscr{B}$:

– $D \subseteq \{0, 1\} \times C$,

– for all $X \in C$ we have $(0, X) \in D$,

– for all $(1, X) \in D$, X is a $\mathcal{J}$-model over some $X_1 \in C$,

and for each $(i, X) \in D$, $\mathcal{F}_i(X) = \mathcal{F}(i, X)$ is a $\mathcal{J}$-model over X such that for each $\mathcal{P} \trianglelefteq \mathcal{F}_i(X)$, $\mathcal{P}$ is fully sound. (Note that $\mathcal{P}$ is a $\mathcal{J}$-model over X, so soundness is in this sense.)

Let $\mathcal{F}, D, \mathscr{B}$ be as above. For $a \in \mathscr{B}$ we say that a is a **base for** $\mathcal{F}$ iff $C_{\mathcal{F}}$ contains the (swo'd) cone above a. We say $\mathcal{F}$ is **forgetful** iff $\mathcal{F}_0(X) = \mathcal{F}_1(X)$ whenever $(0, X), (1, X) \in D$, and whenever X is a $\mathcal{J}$-model over X_1, and X_1 is a $\mathcal{J}$-model over $X_2 \in C_{\mathcal{F}}$ and $X{\downarrow}X_2$ is acceptable, $\mathcal{F}_1(X) = \mathcal{F}_1(X{\downarrow}X_2)$. Otherwise we say $\mathcal{F}$ is **historical**. We say $\mathcal{F}$ is **basic** iff for all $(i, X) \in D$ and $\mathcal{P} \trianglelefteq \mathcal{F}_i(X)$, we have $E^{\mathcal{P}} = \emptyset$. We say $\mathcal{F}$ is **projecting** iff for all $(i, X) \in D$, we have $\rho_\omega^{\mathcal{F}_i(X)} = X$. ⊣

At times we write $\mathcal{F}(X)$ instead of $\mathcal{F}_i(X)$. Note that $\mathscr{B}, C_{\mathcal{F}}$ are determined by $\mathrm{dom}(\mathcal{F})$. Here are some examples of the above terminology. Strategy operators (to be explained in more detail later) are basic, and as usually defined, projecting and historical. The operator $\mathcal{F}(X) = X^{\#}$ is forgetful and projecting, and not basic.

Definition 2.9. For any P and any ordinal $\alpha \geq 1$, the (pre-)operator $\mathcal{J}_\alpha^{\mathsf{m}}(\,\cdot\,; P)$ is defined as follows.[13] For X such that $P \in \mathcal{J}_1(\hat{X})$, let $\mathcal{J}_\alpha^{\mathsf{m}}(X; P)$ be the $\mathcal{J}$-model $\mathcal{M}$ over X, with parameter P, such that $\lfloor \mathcal{M} \rfloor = \mathcal{J}_\alpha(\hat{X})$ and for each $\beta \in [1, \alpha]$, $\mathcal{M}|\beta$ is passive. Clearly $\mathcal{J}_\alpha^{\mathsf{m}}(\,\cdot\,; P)$ is basic and forgetful. If $P = \emptyset$ or we wish to supress P, we just write $\mathcal{J}_\alpha^{\mathsf{m}}(\,\cdot\,)$. ⊣

Definition 2.10 (Potential $\mathcal{F}$-premouse)**.** Let $\mathcal{F}$ be a pre-operator and $b \in C_{\mathcal{F}}$. A **potential $\mathcal{F}$-premouse over** b is a $\mathcal{J}$-model $\mathcal{M}$ over b such that there is an ordinal $\iota > 0$ and an increasing, closed sequence $\langle \zeta_\alpha \rangle_{\alpha \leq \iota}$ of ordinals such that for each $\alpha \leq \iota$, we have:

1. $0 = \zeta_0 \leq \zeta_\alpha \leq \zeta_\iota = l(\mathcal{M})$ (so $\mathcal{M}|\zeta_0 = b$ and $\mathcal{M}|\zeta_\iota = \mathcal{M}$).
2. If $1 < \iota$ then $\mathcal{M}|\zeta_1 = \mathcal{F}_0(b)$.
3. If $1 = \iota$ then $\mathcal{M} \trianglelefteq \mathcal{F}_0(b)$.
4. If $1 < \alpha + 1 < \iota$ then $\mathcal{M}|\zeta_{\alpha+1} = \mathcal{F}_1(\mathcal{M}|\zeta_\alpha){\downarrow}b$.
5. If $1 < \alpha + 1 = \iota$, then $\mathcal{M} \trianglelefteq \mathcal{F}_1(\mathcal{M}|\zeta_\alpha){\downarrow}b$.
6. If α is a limit then $\mathcal{M}|\zeta_\alpha$ is B-passive.

We say that $\mathcal{M}$ is **($\mathcal{F}$-)whole** iff, if $\iota = \alpha + 1$ then $\mathcal{M} = \mathcal{F}_1(\mathcal{M}|\zeta_\alpha){\downarrow}b$.

A **(potential) $\mathcal{F}$-premouse** is a (potential) $\mathcal{F}$-premouse over b, for some b. ⊣

Note that if $\mathcal{F}$ is over $\mathscr{B}$ and $\mathcal{M}$ is a potential $\mathcal{F}$-premouse then $\mathrm{o}(\mathcal{M}) \leq \mathrm{o}(\mathscr{B})$.

Definition 2.11. Let $\mathcal{F}$ be a pre-operator and $b \in C_{\mathcal{F}}$. Let $\mathcal{N}$ be a whole $\mathcal{F}$-premouse over b. A **potential continuing $\mathcal{F}$-premouse over** $\mathcal{N}$ is a $\mathcal{J}$-model $\mathcal{M}$ over $\mathcal{N}$ such that $\mathcal{M}{\downarrow}b$ is a potential $\mathcal{F}$-premouse over b. (Therefore $\mathcal{N}$ is a whole strong cutpoint of $\mathcal{M}$.)

We say that $\mathcal{M}$ (as above) is **whole** iff $\mathcal{M}{\downarrow}b$ is whole.

A **(potential) continuing $\mathcal{F}$-premouse** is a (potential) continuing $\mathcal{F}$-premouse over b, for some b. ⊣

[13] The "m" is for "model".

Definition 2.12. An **operator over** $\mathscr{B}$ is a pre-operator $\mathcal{F}$ over $\mathscr{B}$ such that for every sound whole $\mathcal{F}$-premouse $\mathcal{M} \in \mathscr{B}$, $(1, \mathcal{M}) \in \mathrm{dom}(\mathcal{F})$.

We say that an operator $\mathcal{F}$ is **uniformly** Σ_1 iff there are Σ_1 formulas φ_1 and φ_2 in $\mathcal{L}_0^-$ such that for all (continuing) $\mathcal{F}$-premice $\mathcal{M}$, then the set of whole proper segments of $\mathcal{M}$ is defined over $\mathcal{M}$ by φ_1 (φ_2). For such an operator $\mathcal{F}$, let $\varphi_{\mathrm{wh}}^{\mathcal{F}}$ denote the least such φ_1.

Given a $\mathcal{J}$-model $\mathcal{R}$ and φ in $\mathcal{L}_0^-$-Σ_1 and $\mathcal{P} \triangleleft \mathcal{R}$, we say that $\mathcal{P}$ is φ**-putatively whole (for** $\mathcal{R}$**)** iff $\mathcal{R} \models \varphi(\mathcal{P})$. ⊣

From now on we will deal exclusively with operators (as opposed to the more general pre-operators).

Definition 2.13. Let $\mathcal{F}$ be an operator over $\mathscr{B}$ and $\mathcal{M}$ a (continuing) $\mathcal{F}$-premouse.

If $E^{\mathcal{M}} \neq \emptyset$ we say that $E^{\mathcal{M}}$ is **non-**$\mathcal{F}$ **(for** $\mathcal{M}$**)** iff $\mathcal{M}$ is a limit of whole proper segments. Otherwise we say that $E^{\mathcal{M}}$ is an $\mathcal{F}$**-extender (for** $\mathcal{M}$**)**.

($\mathcal{F}$**-)Iteration trees**, **(**$\mathcal{F}$**-)iterability** and **countable (**$\mathcal{F}$**-)iterability**[14] for (continuing) $\mathcal{F}$-premice over a are defined as for standard premice, with the conditions that for $\mathcal{T}$ to be an $\mathcal{F}$-iteration tree, (i) for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, $E_\alpha^{\mathcal{T}} = E(\mathcal{M}_\alpha^{\mathcal{T}}|\gamma)$ for some γ, and $E_\alpha^{\mathcal{T}}$ is non-$\mathcal{F}$ for $\mathcal{M}_\alpha^{\mathcal{T}}$; (ii) for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, $M_\alpha^{\mathcal{T}}$ is a (continuing) $\mathcal{F}$-premouse over a; (iii) if $\mathrm{lh}(\mathcal{T}) = \alpha + 1$ then $M_\alpha^{\mathcal{T}}$ is wellfounded and $M_\alpha^{\mathcal{T}}|\mathrm{o}(\mathscr{B})$ is a (continuing) $\mathcal{F}$-premouse. In the iteration game, the first player to break any rule loses, and if no rules are broken player II wins.[15] When there is no risk of ambiguity, we will drop the prefix "$\mathcal{F}$-".[16]

We define the term k**-maximal**, regarding $\mathcal{F}$-iteration trees $\mathcal{T}$, as in [21, Definition 3.4], except that for $\alpha + 1 < \beta + 1 < \mathrm{lh}(\mathcal{T})$, we only require that $\mathrm{lh}(E_\alpha^{\mathcal{T}}) \leq \mathrm{lh}(E_\beta^{\mathcal{T}})$, instead of requiring that $\mathrm{lh}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\beta^{\mathcal{T}})$. ⊣

Remark 2.14. This modification to *k-maximality* is non-trivial because we are permitting premice with superstrong extenders. For example, we might have that $E_0^{\mathcal{T}}$ is type 2 and $E_1^{\mathcal{T}}$ is superstrong with $\mathrm{crit}(E_1^{\mathcal{T}})$ the largest cardinal of $\mathcal{M}_0^{\mathcal{T}}|\mathrm{lh}(E_0^{\mathcal{T}})$, in which case $\mathcal{M}_2^{\mathcal{T}}$ is active but $\mathrm{o}(\mathcal{M}_2^{\mathcal{T}}) = \mathrm{lh}(E_1^{\mathcal{T}})$, and therefore we might have $\mathrm{lh}(E_2^{\mathcal{T}}) = \mathrm{lh}(E_1^{\mathcal{T}})$.

The preceding example is essentially general. It is easy to show that if $\mathcal{T}$ is k-maximal and $\alpha + 1 \leq \beta < \mathrm{lh}(\mathcal{T})$ then either $\mathrm{lh}(E_\alpha^{\mathcal{T}}) < \mathrm{o}(M_\beta^{\mathcal{T}})$ and $\mathrm{lh}(E_\alpha^{\mathcal{T}})$ is a cardinal of $M_\beta^{\mathcal{T}}$, or $\beta = \alpha + 1$ and $\mathrm{lh}(E_\alpha^{\mathcal{T}}) = \mathrm{o}(M_{\alpha+1}^{\mathcal{T}})$ and $E_\alpha^{\mathcal{T}}$ is superstrong and $M_{\alpha+1}^{\mathcal{T}}$ is type 2. Therefore if $\alpha + 1 < \beta + 1 < \mathrm{lh}(\mathcal{T})$ then $\nu(E_\alpha^{\mathcal{T}}) < \nu(E_\beta^{\mathcal{T}})$, and if $\alpha + 1 \leq \beta < \mathrm{lh}(\mathcal{T})$ then $E_\alpha^{\mathcal{T}} \restriction \nu(E_\alpha^{\mathcal{T}})$ is not an initial segment of any extender on $\mathbb{E}_+(M_\beta^{\mathcal{T}})$.

[14] The latter is ω_1-iterability (and $\omega_1 + 1$-iterability if AD fails) for countable substructures; the iterability might literally be, say, (k, ω_1)-iterability.

[15] Therefore, if, for example, $\mathscr{B} = \mathscr{H}_{\omega_1}$ and $\mathcal{T}$ is an $\mathcal{F}$-iteration tree of length $\omega_1 + 1$ and $M_0^{\mathcal{T}}$ is countable, then player I cannot make any move extending $\mathcal{T}$ without losing, as $\mathrm{o}(M_{\omega_1}^{\mathcal{T}}) > \omega_1$ and therefore $M_{\omega_1}^{\mathcal{T}}$ is not an $\mathcal{F}$-premouse, so any extension of $\mathcal{T}$ made by player I would violate rule (ii).

[16] We will consider distinct operators $\mathcal{F}$, Y, such that every $\mathcal{F}$-premouse is also a Y-premouse.

The comparison algorithm needs to be modified slightly. Say we are comparing models $\mathcal{M}, \mathcal{N}$, via padded k-maximal trees $\mathcal{T}, \mathcal{U}$, respectively. Say we have produced $\mathcal{T}\upharpoonright\alpha + 1$ and $\mathcal{U}\upharpoonright\alpha + 1$. Let γ be least such that $\mathcal{M}^{\mathcal{T}}_\alpha|\gamma \neq \mathcal{M}^{\mathcal{U}}_\alpha|\gamma$. If only one of these models is active, then we use that active extender next. Suppose both are active. If one active extender is type 3 and one is type 2, then we use only the type 3 extender next. Otherwise we use both extenders next. With this modification, and with the remarks in the preceding paragraph, the usual proof that comparison succeeds goes through.

The reader might wonder why we code $\mathcal{F}$-extenders with $\dot{E}$ instead of $\dot{B}$. The problem with using $\dot{B}$ is that we need to consider fine structure, including taking cores and forming fine-structural ultrapowers, of arbitrary segments of $\mathcal{F}$-premice, even non-whole segments. We will also have occasion to form iteration trees on $\mathcal{F}$-premice which use $\mathcal{F}$-extenders. Thus, if we had $\dot{B}^{\mathcal{M}}$ code a type 3 extender, it would be natural to treat the fine structure of $\mathcal{M}$ at the squashed level. This would complicate our presentation of fine structure for $\mathcal{J}$-models. So it seems to make more organizational sense to have $\mathcal{F}$-extenders coded with $\dot{E}$. This could in general make it difficult to distinguish between the $\mathcal{F}$- and non-$\mathcal{F}$ extenders of an $\mathcal{F}$-premouse, but this distinction is easy when $\mathcal{F}$ is uniformly Σ_1.

The following lemma was stated in [17] in the case that $a = \mathbb{R}$.

Lemma 2.15. *Let $\mathcal{M}$ be an acceptable $\mathcal{J}$-structure over a. Let $\lambda \in \mathrm{o}(\mathcal{M})$. Then λ is an a-cardinal of $\mathcal{M}$ iff $\lambda \geq \Theta^{\mathcal{M}}$ and λ is a cardinal of $\mathcal{M}$.*

Proof Sketch. We write $\mathcal{M}_\alpha = \mathcal{J}^{S^{\mathcal{M}}}_\alpha(\hat{a})$. Assume $\theta = \Theta^{\mathcal{M}} < \mathrm{o}^{\mathcal{M}}$, and let $\lambda \geq \theta$ and $g : \hat{a}^{<\omega} \times \eta^{<\omega} \to \lambda$ witness that λ is not an a-cardinal in $\mathcal{M}$. For each $\vec{\beta} \in \eta^{<\omega}$, let $g_{\vec{\beta}}(\vec{x}) = g(\vec{x}, \vec{\beta})$. Let $\leq_{\vec{\beta}}, \varphi_{\vec{\beta}}$ be the prewellorder (of $\hat{a}$) and norm determined by $\mathrm{g}_{\vec{\beta}}$. Then $\leq_{\vec{\beta}}, \varphi_{\vec{\beta}} \in \mathcal{M}_\theta$, and moreover, the function $\vec{\beta} \mapsto \varphi_{\vec{\beta}}$ is definable over $\mathcal{M}_\alpha$, given g is definable over $\mathcal{M}_\alpha$. It is easy to use this to show that λ is not a cardinal in $\mathcal{M}$. $\square$

The following lemma is an easy enough consequence:

Lemma 2.16. *Let $\mathcal{F}$ be a projecting, uniformly Σ_1 operator and let $b \in C_{\mathcal{F}}$. Let $\mathcal{M}$ be an $\mathcal{F}$-premouse. Let $0 < \eta < l(\mathcal{M})$ be such that $\mathcal{M}|\eta$ is whole and let $\gamma \in [\Theta^{\mathcal{M}}, \mathrm{o}(\mathcal{M}|\eta))$ be a cardinal of $\mathcal{M}$. Then $\gamma \leq \eta$ and $\mathrm{o}(\mathcal{M}|\gamma) = \gamma$ and $\mathcal{M}|\gamma$ is a limit of whole proper segments of $\mathcal{M}$.*

Definition 2.17. Let x be transitive. We say that **countable x-based hulls are club** iff for all $a \subseteq \hat{x}^{<\omega}$, there are club many countable elementary substructures of $(\hat{x}^{<\omega}, a)$.

Let $\mathcal{F}$ be an operator over $\mathscr{B}$ with a base in HC. (Therefore if $x \in C_{\mathcal{F}}$ then for club many countable hulls $\bar{x}$ of x, $\bar{x} \in C_{\mathcal{F}}$.)

Let $\mathcal{M}$ be an $\mathcal{F}$-premouse over a and let $n \leq \omega$ (and $\eta \leq \mathrm{o}(\mathcal{M})$). We say that $\mathcal{M}$ is **countably (above-η) n-$\mathcal{F}$-iterable** iff for club many countable substructures $\bar{\mathcal{M}}$ of $\mathcal{M}$, $\bar{\mathcal{M}}$ is an (above-$\bar{\eta}$) (n, ω_1+1)-$\mathcal{F}$-iterable $\mathcal{F}$-premouse (where $\bar{\eta}$ is the collapse of η).

Let $x \in C_{\mathcal{F}}$ and assume that countable x-based hulls are club. Then $\mathrm{Lp}^{\mathcal{F}}(x)$ denotes the stack of all countably ω-$\mathcal{F}$-iterable $\mathcal{F}$-premice $\mathcal{M}$ over x such that $\mathcal{M}$ is fully sound and projects to x.[17] Assuming that $\mathbb{R} \in \mathscr{B}$, for $X \subseteq \mathrm{HC}$, $\mathrm{Lp}^{\mathcal{F}}(\mathbb{R}, X)$ denotes $\mathrm{Lp}^{\mathcal{F}}((\mathrm{HC}, X))$.[18]

Let $\mathcal{N}$ be a whole $\mathcal{F}$-premouse in $\mathscr{B}$. Then $\mathrm{Lp}^{\mathcal{F}}_{+}(\mathcal{N})$ denotes the stack of all continuing $\mathcal{F}$-premice $\mathcal{M}$ over $\mathcal{N}$ such that $\mathcal{M}$ is fully sound, $\rho_\omega^{\mathcal{M}} = \mathcal{N}$ and $\mathcal{M}{\downarrow}a^{\mathcal{N}}$ is countably above-$\mathrm{o}(\mathcal{N})$ (ω, ω_1+1)-$\mathcal{F}$-iterable, if there is any such $\mathcal{M}$; otherwise $\mathrm{Lp}^{\mathcal{F}}_{+}(\mathcal{N}) = \mathcal{N}$. ⊣

From now on, whenever we refer (implicitly) to $\mathrm{Lp}^{\mathcal{F}}$ or $\mathrm{Lp}^{\mathcal{F}}_{+}$, we are making the assumptions above. Note that if x is countable then countable x-based hulls are club. We can now describe the kinds of non-basic operators we will be interested in:

Definition 2.18 (Mouse operator)**.** Let Y be a basic, projecting, uniformly Σ_1 operator, over $\mathscr{B}$.

A **lower Y-mouse operator** $\mathcal{F}$ is an operator over $\mathscr{B}$ such that for each (i, X) in its domain, $\mathcal{F}_i(X) \trianglelefteq \mathrm{Lp}^Y(X)$.

A **continuing Y-mouse operator** $\mathcal{F}$ is an operator over $\mathscr{B}$ with domain D such that for each $(0, X) \in D$, $\mathcal{F}_0(X) \trianglelefteq \mathrm{Lp}^Y(X)$, and for each $(1, X) \in D$, X is a sound whole Y-premouse and $X \triangleleft \mathcal{F}_1(X) \trianglelefteq \mathrm{Lp}^Y_{+}(X)$.

Let $\mathcal{F}$ be a continuing Y-mouse operator. We say that $\mathcal{F}$ is **whole** iff for all $(0, X) \in D$, $\mathcal{F}_0(X)$ is Y-whole, and for all $(1, X) \in D$, either $\mathcal{F}_1(X)$ is Y-whole, or $\mathcal{F}_1(X){\downarrow}a^X$ is not sound (and therefore $\mathcal{F}_1(X) = \mathrm{Lp}^Y_{+}(X)$). ⊣

The next lemma is easy:

Lemma 2.19. *Let $\mathcal{F}$ be a whole continuing Y-mouse operator. Then every $\mathcal{F}$-premouse is a Y-premouse.*

We now describe background extender constructions to build $\mathcal{F}$-mice.

Definition 2.20. Let $\mathcal{N}$ be an $\mathcal{F}$-premouse and $k \leq \omega$. Then $\mathcal{N}$ is **k-$\mathcal{F}$-solid** iff $\mathcal{N}$ is k-solid, and for each $i \leq k$, $\mathfrak{C}_k(\mathcal{N})$ is an $\mathcal{F}$-premouse. ⊣

[17] Our assumptions ensure that $\mathrm{Lp}^{\mathcal{F}}(x)$ is indeed a stack. For assume that x is infinite and let $\mathcal{M}_1, \mathcal{M}_2$ be $\mathcal{J}$-models meeting the criteria. We can code $\mathcal{M}_1 \oplus \mathcal{M}_2$ with some structure $(\hat{x}^{<\omega}, a)$ with $a \subseteq \hat{x}^{<\omega}$. Taking a countable hull, we get $\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2$ over $\bar{x}$, which we can compare, to deduce that $\bar{\mathcal{M}}_1 = \bar{\mathcal{M}}_2$, as usual; for the latter we just need iterability, the ISC and fine structure. (Because all models which appear during the comparison are $\mathcal{F}$-premice, all extenders used are non-$\mathcal{F}$.) If x is finite, it is easier.

[18] Since $\mathbb{R}$ is not transitive, this is not an abuse of notation.

Definition 2.21. Given a $\mathcal{J}$-model $\mathcal{N}$ over a, and $\mathcal{M} \triangleleft \mathcal{N}$ such that $\mathcal{M}$ is fully sound, the **$\mathcal{M}$-drop-down sequence** of $\mathcal{N}$ is the sequence of pairs $\langle(\mathcal{Q}_n, m_n)\rangle_{n<k}$ of maximal length such that $\mathcal{Q}_0 = \mathcal{M}$ and $m_0 = \omega$ and for each $n+1 < k$:

1. $\mathcal{M} \triangleleft \mathcal{Q}_{n+1} \trianglelefteq \mathcal{N}$ and $\mathcal{Q}_n \trianglelefteq \mathcal{Q}_{n+1}$,
2. every proper segment of $\mathcal{Q}_{n+1}$ is fully sound,
3. $\rho_{m_n}(\mathcal{Q}_n)$ is an a-cardinal of $\mathcal{Q}_{n+1}$,
4. $0 < m_{n+1} < \omega$,
5. $\mathcal{Q}_{n+1}$ is $(m_{n+1}-1)$-sound,
6. $\rho_{m_{n+1}}(\mathcal{Q}_{n+1}) < \rho_{m_n}(\mathcal{Q}_n) \leq \rho_{m_{n+1}-1}(\mathcal{Q}_{n+1})$. ⊣

Definition 2.22. Let $\mathcal{F}$ be an operator over $\mathscr{B}$ and let C be some class of E-active $\mathcal{F}$-premice. Let $b \in C_{\mathcal{F}}$ and $\chi \leq \mathrm{o}(\mathscr{B})+1$. A **($C$-certified) $L^{\mathcal{F}}[\mathbb{E}, b]$-construction (of length χ)** is a sequence $\langle\mathcal{N}_\alpha\rangle_{\alpha<\chi}$ with the following properties.

We have $\mathcal{N}_0 = b$ and $\mathcal{N}_1 = \mathcal{F}(0, b)$.

Let $0 < \alpha < \chi$. Then $\alpha \leq \mathrm{o}(\mathscr{B})$ and $\mathcal{N}_\alpha$ is an $\mathcal{F}$-premouse over b. If α is a limit then $\mathcal{N}_\alpha = \liminf_{\beta<\alpha} \mathcal{N}_\beta$. Now suppose that $\alpha+1 < \chi$. Then either:

(i) $\mathcal{N}_\alpha$ is a passive limit of whole proper segments and $\mathcal{N}_{\alpha+1} = (\mathcal{N}_\alpha, G)$ for some extender G (with $\mathcal{N}_{\alpha+1} \in C$); or

(ii) $\mathcal{N}_\alpha$ is ω-$\mathcal{F}$-solid. Let $\mathcal{M}_\alpha = \mathfrak{C}_\omega(\mathcal{N}_\alpha)$. Let $\mathcal{M}$ be the largest whole segment of $\mathcal{M}_\alpha$. So either $\mathcal{M}_\alpha = \mathcal{M}$ or $\mathcal{M}_\alpha{\downarrow}\mathcal{M} \triangleleft \mathcal{F}_1(\mathcal{M})$. Let $\mathcal{N} \trianglelefteq \mathcal{F}_1(\mathcal{M})$ be least such that either $\mathcal{N} = \mathcal{F}_1(\mathcal{M})$ or for some $k < \omega$, $(\mathcal{N}{\downarrow}b, k+1)$ is on the $\mathcal{M}_\alpha$-drop-down sequence of $\mathcal{N}{\downarrow}b$. Then $\mathcal{N}_{\alpha+1} = \mathcal{N}{\downarrow}b$. Note that $\mathcal{M}_\alpha \triangleleft \mathcal{N}_{\alpha+1}$. ⊣

We now proceed to describe some consendation properties for operators $\mathcal{F}$ under which together whether sufficient iterability ensure that $L^{\mathcal{F}}[\mathbb{E}]$-constructions do not break down.

Definition 2.23. Let Y be an operator. We say that Y **condenses coarsely** iff for all $i \in \{0,1\}$ and $(i, \bar{X}), (i, X) \in \mathrm{dom}(Y)$, and all $\mathcal{J}$-models $\mathcal{M}^+$ over $\bar{X}$, if $\pi : \mathcal{M}^+ \to Y_i(X)$ is fully elementary, then

– if $i = 0$ then $\mathcal{M}^+ \trianglelefteq Y_0(\bar{X})$; and

– if $i = 1$ and X is a sound whole Y-premouse, then $\mathcal{M}^+ \trianglelefteq Y_1(\bar{X})$. ⊣

Lemma 2.24. *Let Y be a uniformly Σ_1 operator which condenses coarsely and let $\mathcal{M}$ be an E-passive whole Y-premouse. Let $\pi : \mathcal{N} \to \mathcal{M}$ be fully elementary, where $a^{\mathcal{N}} \in C_Y$. Then* (*a*) *$\mathcal{N}$ is a Y-premouse and for all $\mathcal{P} \triangleleft \mathcal{N}$, $\mathcal{P}$ is whole iff $\pi(\mathcal{P})$ is whole. Moreover,* (*b*) *if $\mathcal{M}$ is sound or a limit of whole proper segments then $\mathcal{N}$ is whole.*

Proof. If $\mathcal{M} = Y_0(a^{\mathcal{M}})$ then a slight variant of the argument to follow shows that $\mathcal{N} = Y_0(a^{\mathcal{N}})$, which suffices. So assume that $Y_0(a^{\mathcal{M}}) \triangleleft \mathcal{M}$. We claim then that for all $\mathcal{P} \triangleleft \mathcal{N}$, $\mathcal{N} \models \varphi_Y(\mathcal{P})$ iff $\mathcal{P}$ is a whole Y-premouse. This can be proved by induction on $\mathcal{P}$. We again skip the argument as it is similar to the one to follow.

It now follows easily that if $\mathcal{Q} \trianglelefteq \mathcal{N}$ and $\mathcal{Q}$ is a limit of whole proper segments $\mathcal{P}$, then $B^{\mathcal{Q}} = \emptyset$ and so $\mathcal{Q}$ is a (whole) Y-premouse. In particular, if $\mathcal{M}$ is a limit of whole proper segments then $\mathcal{N}$ is likewise and the lemma follows easily. So suppose instead that $\mathcal{M}$ has a largest whole proper segment; then this is $\pi(\mathcal{P})$ where $\mathcal{P}$ is the largest whole proper segment of $\mathcal{N}$. Now $\mathcal{M} = Y_1(\pi(\mathcal{P}))\downarrow a^{\mathcal{M}}$. So by coarse condensation, $\mathcal{N}\downarrow\mathcal{P} \trianglelefteq Y_1(\mathcal{P})$, so $\mathcal{N}$ is a Y-premouse, giving (a). Now suppose that $\mathcal{M}$ is sound but $\mathcal{N}\downarrow\mathcal{P} \triangleleft Y_1(\mathcal{P})$. Then there is a Y-premouse $\mathcal{M}'$ such that $\mathcal{M} \triangleleft \mathcal{M}'$ and $l(\mathcal{M}') = l(\mathcal{M}) + 1$. Because Y is uniformly Σ_1, $\mathcal{M}' \models \varphi_Y(\mathcal{M})$. But $\varphi_Y \in \mathcal{L}_0^-$, so by elementarity, $\mathcal{N}$ is sound and $(\mathcal{J}_1^{\mathsf{m}}(\mathcal{N};\mathfrak{P}^{\mathcal{N}})\downarrow a^{\mathcal{N}}) \models \varphi_Y(\mathcal{N})$ and there is a Y-premouse $\mathcal{N}'$ such that $\mathcal{N} \triangleleft \mathcal{N}'$ and $\lfloor \mathcal{N}' \rfloor = \mathcal{J}_1(\mathcal{N})$. But then $\mathcal{N}' \models \varphi_Y(\mathcal{N})$, so $\mathcal{N}$ is whole, contradiction. This proves (b). $\square$

Lemma 2.25. *Let Y be a uniformly Σ_1 operator which condenses coarsely and let $\mathcal{M}$ be an E-active Y-premouse. Let $\pi : \mathcal{N} \to \mathcal{M}$ be a weak* 0*-embedding, where $a^{\mathcal{N}} \in C_Y$. If $\mathcal{M}$ is type 3, suppose also that* $\mathrm{Ult}(\mathcal{M}^{\mathrm{sq}}, E^{\mathcal{M}})$ *is a Y-premouse. Then $\mathcal{N}$ is a Y-premouse.*

Proof. Consider the case that $\mathcal{M}$ is type 3. Let $\psi : \mathrm{Ult}(\mathcal{N}^{\mathrm{sq}}, E^{\mathcal{N}}) \to \mathcal{R} = \mathrm{Ult}(\mathcal{M}^{\mathrm{sq}}, E^{\mathcal{M}})$ be the map induced by π. Let $\psi' = \psi\restriction(\mathcal{N}||\mathrm{o}(\mathcal{N}))$. Then $\psi' : \mathcal{N}||\mathrm{o}(\mathcal{N}) \to \mathcal{R}|\psi(\mathrm{o}(\mathcal{N}))$ is fully elementary. Now apply 2.24 and 2.6. $\square$

Definition 2.26. Let $\mathcal{M}, \mathcal{N}$ be k-sound $\mathcal{J}$-models over a, b and $\pi : \mathcal{M} \to \mathcal{N}$. Then π is **(weakly, nearly)** ***k*-good** iff $\pi\restriction a \cup \{a\} = \mathrm{id}$ and π is a (weak, near) k-embedding.

If $\pi\colon \mathcal{M} \to \mathcal{N}$ is a weak 0-embedding then π is ***ν*-preserving** iff, if $\mathcal{M}, \mathcal{N}$ are type 3 (so literally $\pi\colon \mathcal{M}^{\mathrm{sq}} \to \mathcal{N}^{\mathrm{sq}}$) and $a, f \in \mathcal{M}^{\mathrm{sq}}$ are such that $\nu(E^{\mathcal{M}}) = [a, f]^{\mathcal{M}}_{E^{\mathcal{M}}}$, then $\nu(E^{\mathcal{N}}) = [\pi(a), \pi(f)]^{\mathcal{N}}_{E^{\mathcal{N}}}$. ⊣

Remark 2.27. We use the following in place of the notion of *condenses well* (see [23, 2.1.10]). We explain why we made this replacement in Appendix A.

Definition 2.28. Let Y be a projecting, uniformly Σ_1 operator. We say that Y **condenses finely** iff Y condenses coarsely and we have the following. Let $k < \omega$. Let $\mathcal{M}^*$ be a Y-premouse over a, with a largest whole proper segment $\mathcal{M}$, such that $\mathcal{M}^+ = \mathcal{M}^*\downarrow\mathcal{M}$ is sound and $\rho_{k+1}(\mathcal{M}^+) = \mathcal{M}$. Let $\mathcal{P}^*, \bar{a}, \mathcal{P}, \mathcal{P}^+$ be likewise. Let $\mathcal{N}$ be a sound whole Y-premouse over $\bar{a}$. Let $G \subseteq \mathrm{Col}(\omega, \mathcal{P} \cup \mathcal{N})$ be V-generic. Let $\mathcal{N}^+, \pi, \sigma \in V[G]$, with $\mathcal{N}^+$ a sound $\mathcal{J}$-model over $\mathcal{N}$ such that $\mathcal{N}^* = \mathcal{N}^+ \downarrow \bar{a}$ is defined (i.e. acceptable). Suppose $\pi : \mathcal{N}^* \to \mathcal{M}^*$ is such that $\pi(\mathcal{N}) = \mathcal{M}$ and either:

1. $\mathcal{M}^*$ is k-sound and $\mathcal{N}^* = \mathfrak{C}_{k+1}(\mathcal{M}^*)$; or
2. $(\mathcal{N}^*, k+1)$ is in the $\mathcal{N}$-dropdown sequence of $\mathcal{N}^*$, and likewise $(\mathcal{P}^*, k+1), \mathcal{P}$, and either:
   (a) π is k-good, or
   (b) π is fully elementary, or
   (c) π is a weak k-embedding, $\sigma : \mathcal{P}^* \to \mathcal{N}^*$ is k-good, $\sigma(\mathcal{P}) = \mathcal{N}$ and $\pi \circ \sigma \in V$ is a near k-embedding.

Then $\mathcal{N}^+ \trianglelefteq Y_1(\mathcal{N})$.

We say that Y **almost condenses finely** iff $\mathcal{N}^+ \trianglelefteq Y_1(\mathcal{N})$ whenever the hypotheses above hold with $\mathcal{N}^+, \pi, \sigma \in V$. $\dashv$

In the preceding definition, if $\mathcal{N}^*, \mathcal{M}^*$ are type 3, and so $\mathrm{dom}(\pi) = (\mathcal{N}^*)^{\mathrm{sq}}$, then by 2.16, $\mathrm{o}(\mathcal{M}) < \mathrm{crit}(E^{\mathcal{M}^*}) < \nu(E^{\mathcal{M}^*})$, so it is reasonable to say that $\pi(\mathcal{N}) = \mathcal{M}$, for instance.

Lemma 2.29. *Let Y be an operator over $\mathscr{B}$ with base in* HC*. Suppose that Y almost condenses finely. Then Y condenses finely.*

Proof. Suppose not. Let $\mathcal{M}^*, \mathcal{P}^*, \mathcal{N}^+$, etc, as in 2.28, constitute a counterexample. Let $\mathcal{M}^\$ = Y_1(\mathcal{M})$ and $\mathcal{P}^\$, \mathcal{N}^\$$ likewise. Since $\mathcal{M}^*$ has a largest whole proper segment, $\mathcal{M}^*$ and all other relevant objects are in $\mathscr{B}$. Note that $\mathcal{N}^\$ \not\trianglelefteq \mathcal{N}^+$. For if $\mathcal{N}^\$ \triangleleft \mathcal{N}^+$ then $\mathcal{N}^\$ \downarrow a^{\mathcal{N}}$ is a sound Y-premouse and there is a Y-premouse $\mathcal{N}'$ such that $\lfloor \mathcal{N}' \rfloor = \mathcal{J}_1(\mathcal{N}^\$)$. But then because Y is uniformly Σ_1 and using π, $Y_1(\mathcal{M}) \triangleleft \mathcal{M}^+$, contradiction.

Let $\mathbb{P} = \mathrm{Col}(\omega, \mathcal{P} \cup \mathcal{N})$. Let $X \in \mathscr{B}$ be transitive, containing all relevant objects, and such that $X \vDash (\mathsf{ZF}^-)^{-\epsilon}$. (For any $A \in \mathscr{B}$ there is γ such that $L_\gamma(A) \vDash (\mathsf{ZF}^-)^{-\epsilon}$, so there is such a $\gamma < \mathrm{o}(\mathscr{B})$.) In particular, in X we have $\mathcal{M}, \mathcal{N}, \mathcal{P}$, etc, and have $p \in \mathbb{P}$ and $\mathbb{P}$-names $\tilde{\mathcal{N}}^+, \tilde{\pi}, \tilde{\sigma}$ for $\mathcal{N}^+, \pi, \sigma$, and in X, p forces that

$$\text{“}\mathcal{M}^*, \tilde{\mathcal{N}}^+, \mathcal{N}^\$, \text{ etc, satisfy the hypotheses of 2.28 and } \tilde{\mathcal{N}}^+ \not\trianglelefteq \mathcal{N}^\$ \ \&\ \mathcal{N}^\$ \not\trianglelefteq \tilde{\mathcal{N}}^+\text{”}. \tag{2.1}$$

Let $\pi : Z \to X$ be elementary with Z countable, and everything relevant in $\mathrm{rg}(\pi)$. Let $\pi(\mathcal{N}^Z) = \mathcal{N}$, etc. Let $G \subseteq \mathrm{Col}(\omega, \mathcal{P}^Z \cup \mathcal{N}^Z)$ be Z-generic with $p^Z \in G$. Then because

Y condenses coarsely and by 2.24 and 2.25, $(\mathcal{M}^*)^Z, (\tilde{\mathcal{N}}^+)^Z_G$, etc, satisfy the hypotheses of 2.28, and $(\mathcal{N}^\$)^Z \trianglelefteq Y_1(\mathcal{N}^Z)$. But then because Y almost condenses finely, $(\tilde{\mathcal{N}}^+)^Z_G \trianglelefteq Y_1(\mathcal{N}^Z)$. Therefore either $(\mathcal{N}^\$)^Z \trianglelefteq (\tilde{\mathcal{N}}^+)^Z_G$ or $(\tilde{\mathcal{N}}^+)^Z_G \trianglelefteq (\mathcal{N}^\$)^Z$, contradicting line 2.1. □

Definition 2.30. An $\mathcal{F}$**-putative iteration tree** is a putative $\mathcal{F}$-iteration tree. (That is, every model of $\mathcal{T}$ except the last, if there is one, is an $\mathcal{F}$-premouse, and every extender used in $\mathcal{T}$ is non-$\mathcal{F}$).

An $\mathcal{F}$**-putative iteration strategy** for a $\mathcal{J}$-model $\mathcal{N}$ is a function Σ such that for each limit length $\mathcal{F}$-tree $\mathcal{T}$ on $\mathcal{N}$, via Σ, $\Sigma(\mathcal{T})$ is a $\mathcal{T}$-cofinal branch b. (Thus, player II *wins* any round of the iteration game which has a last model which is not an $\mathcal{F}$-premouse, and in particular, wins by default if $\mathcal{N}$ is not an $\mathcal{F}$-premouse.) ⊣

Lemma 2.31. *Let $Y, \mathcal{F}$ be uniformly Σ_1 operators with bases in* HC*. Suppose that Y condenses finely. Suppose that $\mathcal{F}$ is a whole continuing Y-mouse operator. Then (a) $\mathcal{F}$ condenses finely. Moreover, (b) let $\mathcal{M}$ be an $\mathcal{F}$-whole $\mathcal{F}$-premouse. Let $\pi\colon \mathcal{N} \to \mathcal{M}$ be fully elementary with $a^{\mathcal{N}} \in C_{\mathcal{F}}$. Then $\mathcal{N}$ is an $\mathcal{F}$-whole $\mathcal{F}$-premouse. So regarding $\mathcal{F}$, the conclusion of 2.23 may be modified by replacing "$\trianglelefteq$" with "$=$".*

Proof Sketch. Let $\mathcal{F}, Y$ be over $\mathscr{B}$. Consider (a). By 2.29 it suffices to see that $\mathcal{F}$ almost condenses finely. We just consider the case of this proof when (2c) of 2.28 holds (omitting the proof that $\mathcal{F}$ condenses coarsely), since this illustrates the main points. So suppose that $\mathcal{M}^*$, etc, are as in (2c) of 2.28.

Let us first observe that $\mathcal{N}^*$ is a Y-premouse. This is easy if $\mathcal{P}^*$ has no largest Y-whole proper segment, so suppose otherwise, and let $\mathcal{P}_Y$ be the largest. Since $\mathcal{P}$ is $\mathcal{F}$-whole and $\mathcal{F}$ is whole, therefore $\mathcal{P} \trianglelefteq \mathcal{P}_Y \triangleleft \mathcal{P}^*$. Then $\mathcal{M}_Y = \pi(\sigma(\mathcal{P}_Y))$ is the largest Y-whole proper segment of $\mathcal{M}^*$, so by 2.24 and 2.25 and using π, $\mathcal{N}_Y = \sigma(\mathcal{P}_Y)$ is a sound Y-whole Y-premouse. Also, $(\mathcal{P}^*, k+1)$ is on the $\mathcal{P}_{\mathcal{F}}$-dropdown sequence of $\mathcal{P}^*$, and so on the $\mathcal{P}_Y$-dropdown sequence of $\mathcal{P}^*$. Likewise $\mathcal{N}^*, \mathcal{N}_Y$. Since Y condenses finely, this implies that $\mathcal{N}^+ \trianglelefteq Y_1(\mathcal{N}_Y)$, so $\mathcal{N}^*$ is a Y-premouse.

So $\mathcal{N}^+$ is a sound continuing Y-premouse (over $\mathcal{N}$) and $\rho_{k+1}(\mathcal{N}^+) = \mathcal{N}$. We claim that $\mathcal{N}^+$ is countably k-Y-iterable. Given this, $\mathcal{N}^+ \trianglelefteq \mathrm{Lp}^Y_+(\mathcal{N})$, so either $\mathcal{N}^+ \trianglelefteq \mathcal{F}_1(\mathcal{N})$ or $\mathcal{F}_1(\mathcal{N}) \trianglelefteq \mathcal{N}^+$. But then $\mathcal{N}^+ \trianglelefteq \mathcal{F}_1(\mathcal{N})$ because if $\mathcal{F}_1(\mathcal{N}) \triangleleft \mathcal{N}^+$ then the usual argument shows that $\mathcal{F}_1(\mathcal{M}) \triangleleft \mathcal{M}^+$, a contradiction. So it suffices to prove this claim.

Let $X \in \mathscr{B}$ be transitive and containing all relevant objects. Let $\tau : Z \to X$ be elementary, with Z countable, and such that $\tau^{-1}(\mathcal{M}^*, \mathcal{P}^*, \mathcal{N})$ are Y-premice and $\tau^{-1}(\mathcal{M}^+)$ is k-Y-iterable. Using $\tau^{-1}(\pi)$ we can lift (above-$\tau^{-1}(\mathcal{N})$) Y-putative trees on $\tau^{-1}(\mathcal{N}^+)$ to Y-trees on $\tau^{-1}(\mathcal{M}^+)$. Let $\mathcal{T}$ on $\tau^{-1}(\mathcal{N}^+)$ be via this strategy, of length $\alpha + 1$. Then using that Y condenses finely and standard fine structure, one can show that $M^{\mathcal{T}}_\alpha$ is a Y-premouse.

(One extra point here is the following. Suppose $M^{\pi\mathcal{T}}_\alpha$ is type 3. Then literally the copy map $\pi_\alpha : (M^{\mathcal{T}}_\alpha)^{\mathrm{sq}} \to (M^{\pi\mathcal{T}}_\alpha)^{\mathrm{sq}}$, so it is not immediate that $M^{\mathcal{T}}_\alpha$ is a Y-premouse. Let

$$\psi : \mathrm{Ult}(M^{\mathcal{T}}_\alpha, E(M^{\mathcal{T}}_\alpha)) \to \mathrm{Ult}(M^{\pi\mathcal{T}}_\alpha, E(M^{\pi\mathcal{T}}_\alpha))$$

be the map induced by π_α. Then using ψ and π_α together one can show that $M^{\mathcal{T}}_\alpha$ is wellfounded and is a Y-premouse.)

Part (b) follows from 2.24 and 2.25, and the observation that if $\mathcal{N}$ has a largest $\mathcal{F}$-whole proper segment $\mathcal{N}_{\mathcal{F}}$ and $\mathcal{N}$ is unsound then $\mathcal{N}{\downarrow}\mathcal{N}_{\mathcal{F}} = \mathrm{Lp}^Y_+(\mathcal{N}_{\mathcal{F}})$, and so $\mathcal{N}{\downarrow}\mathcal{N}_{\mathcal{F}} = \mathcal{F}_1(\mathcal{N}_{\mathcal{F}})$.

This completes the sketch of the proof. □

Definition 2.32. For $\mathcal{T}$ an iteration tree and $\alpha < \mathrm{lh}(\mathcal{T})$ let $\mathrm{base}^{\mathcal{T}}(\alpha)$ denote the least $\beta \leq_{\mathcal{T}} \alpha$ such that $(\beta, \alpha]_{\mathcal{T}}$ does not drop in model or degree. (Therefore either $\beta = 0$ or β is a successor.) Also let $M^{*\mathcal{T}}_0 = M^{\mathcal{T}}_0$ and $i^{*\mathcal{T}}_0 = \mathrm{id}$. ⊣

Definition 2.33. Let $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha \leq \lambda}$ be an $L^{\mathcal{F}}[\mathbb{E}, b]$-construction. Let $k \leq \omega$ and suppose that $\mathcal{N}_\lambda$ is k-$\mathcal{F}$-solid. Let $\mathcal{R}$ be a k-sound $\mathcal{F}$-premouse over b and let $\pi : \mathcal{R} \to \mathfrak{C}_k(\mathcal{N}_\lambda)$ be fully elementary. Let $\mathcal{T}$ be an $\mathcal{F}$-putative iteration tree on $\mathcal{R}$, with $\deg^{\mathcal{T}}(0) = k$. We say that $\mathcal{T}$ is $(\pi, \mathbb{C})$**-realizable** iff for every $\alpha < \mathrm{lh}(\mathcal{T})$, letting $\beta = \mathrm{base}^{\mathcal{T}}(\alpha)$ and $m = \deg^{\mathcal{T}}(\alpha)$, there is $\zeta \leq \lambda$ such that:

- if $[0, \alpha]_{\mathcal{T}}$ does not drop in model or degree then $\zeta = \lambda$, and let $\tau = \pi$,
- if $\zeta = \lambda$ then $m \leq k$,
- if $[0, \alpha]_{\mathcal{T}}$ drops in model or degree then there is a ν-preserving near m-embedding $\tau \colon M^{*\mathcal{T}}_\beta \to \mathfrak{C}_m(\mathcal{N}_\zeta)$, and
- if $M^{*\mathcal{T}}_\beta$ is not type 3 then there is a weak m-embedding $\sigma : M^{\mathcal{T}}_\alpha \to \mathfrak{C}_m(\mathcal{N}_\zeta)$ such that $\sigma \circ i^{*\mathcal{T}}_{\beta,\alpha} = \tau$.
- if $M^{*\mathcal{T}}_\beta$ is type 3 then there is a weak m-embedding $\sigma : \mathcal{R} \to \mathfrak{C}_m(N_\zeta)$ such that $i^{*\mathcal{T}}_{\beta,\alpha} = \tau$, where $\mathcal{R}$ is "$(M^{\mathcal{T}}_\alpha)^{\mathrm{sq}}$".[19] ⊣

Lemma 2.34. *Let $\mathcal{F}$ be a projecting, uniformly Σ_1 operator over $\mathscr{B}$, with a base in* HC*, and which condenses finely. Let $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha < \chi}$ be an $L^{\mathcal{F}}[\mathbb{E}, b]$-construction. Suppose that for all $\alpha < \chi$ and all $\mathcal{R}$, if $\mathcal{N}_\alpha, \mathcal{R}$ are $\mathcal{F}$-premice of type 3, $\mathcal{R}$ is $(0, \omega_1 + 1)$-iterable and $\pi : \mathcal{R}^{\mathrm{sq}} \to \mathcal{N}^{\mathrm{sq}}_\alpha$ is Σ_0-elementary, then $\mathcal{R}$ is not superstrong. Then:*

[19] $(M^{\mathcal{T}}_\alpha)^{\mathrm{sq}}$ might not make literal sense, if say $M^{\mathcal{T}}_\alpha$ is not wellfounded. By "$(M^{\mathcal{T}}_\alpha)^{\mathrm{sq}}$" we mean that either $\alpha = \xi + 1$ and $\mathcal{R} = \mathrm{Ult}_m((M^{*\mathcal{T}}_\alpha)^{\mathrm{sq}}, E^{\mathcal{T}}_\xi)$, or α is a limit and $\mathcal{R}$ is the direct limit of the structures $(M^{\mathcal{T}}_\xi)^{\mathrm{sq}}$ for $\xi \in [\beta, \alpha)_{\mathcal{T}}$, under the iteration maps.

(1) If χ is a limit there is a unique $\mathcal{N}_\chi$ such that $\mathbb{C}\frown\langle\mathcal{N}_\chi\rangle$ is an $L^{\mathcal{F}}[\mathbb{E},b]$-construction.

(2) Suppose $\chi=\lambda+1$, $\mathcal{N}_\lambda$ is ω-$\mathcal{F}$-solid and $\lambda\in\mathscr{B}$. Then there is a unique $\mathcal{N}_\chi$ such that $\mathbb{C}\frown\langle\mathcal{N}_\chi\rangle$ is an $L^{\mathcal{F}}[\mathbb{E},b]$-construction and $\mathfrak{C}_\omega(\mathcal{N}_\lambda)\triangleleft\mathcal{N}_\chi$.

(3) Suppose $\chi=\lambda+1$ and $k<\omega$ is such that $\mathcal{N}_\lambda$ is k-$\mathcal{F}$-solid and for a club of countable elementary $\pi:\mathcal{M}\to\mathfrak{C}_k(\mathcal{N}_\lambda)$, there is a Y-putative, (k,ω_1,ω_1+1)-iteration strategy Σ for $\mathcal{M}$, such that every $\mathcal{T}$ via Σ is $(\pi,\mathbb{C})$-realizable. Then $\mathcal{N}_\lambda$ is $(k+1)$-$\mathcal{F}$-solid.

Proof. We will use 2.24 without explicit mention. Consider (1). Let $\mathcal{N}_\chi=\liminf_{\alpha<\lambda}\mathcal{N}_\alpha$. Then $\mathcal{N}_\chi$ is a passive limit of sound whole proper segments, so $\mathcal{N}_\chi$ is an $\mathcal{F}$-premouse.

Now consider (2). Let $\mathcal{M}_\lambda=\mathfrak{C}_\omega(\mathcal{N}_\lambda)$ and let $\mathcal{M}$ be the largest $\mathcal{F}$-whole segment of $\mathcal{M}_\lambda$. We must verify that $\mathcal{N}_{\lambda+1}$, defined as in 2.22(ii) (with $\alpha=\lambda$), is well-defined (i.e. acceptable) and is an $\mathcal{F}$-premouse. Let $\mathcal{N}$ also be as there. Since every segment of $\mathcal{M}_\lambda$ is sound, it suffices to see that for every $\mathcal{R}'\triangleleft\mathcal{N}$, letting $\mathcal{R}=\mathcal{R}'\downarrow b$, $\mathcal{R}$ is sound (by induction, we may assume that $\mathcal{R}$ is acceptable). We may assume that $\mathcal{M}_\lambda\triangleleft\mathcal{R}$. We have $\rho=\rho_\omega(\mathcal{M}_\lambda)\le\rho_\omega^{\mathcal{P}}$, for each $(\mathcal{P},j+1)$ in the $\mathcal{M}$-dropdown sequence of $\mathcal{M}_\lambda$. Therefore $\rho\le\rho_\omega(\mathcal{R})$. If $\rho<\rho_\omega(\mathcal{R})$ then the soundness of $\mathcal{R}'$ implies that of $\mathcal{R}$. So suppose $\rho_\omega(\mathcal{R})=\rho$. Let $k<\omega$ be such that $\rho_{k+1}(\mathcal{R})=\rho<\rho_k(\mathcal{R})$. Then as before, $\mathcal{R}$ is k-sound. Let $p=p_{k+1}(\mathcal{R})$. Then $\mathcal{M}\in H=\mathrm{Hull}_{k+1}^{\mathcal{R}}(\rho\cup p)$ because $\mathcal{F}$ is uniformly Σ_1, and because H is cofinal in $\mathcal{R}$ if $k=0$. Therefore H has every element of $\mathcal{R}$ which is in the $\mathcal{M}$-drop-down sequence of $\mathcal{R}$. It follows that $\mathcal{M}\cup\{\mathcal{M}\}\subseteq H$. Since $\rho_{k+1}(\mathcal{R})\le\mathrm{o}(\mathcal{M})$, $\rho_{k+1}(\mathcal{R}')=\mathcal{M}$. Also, $p_{k+1}(\mathcal{R}')=p_{k+1}(\mathcal{R})\backslash(\mathrm{o}(\mathcal{M})+1)$ because $\mathcal{R}'$ is $(k+1)$-sound (including $(k+1)$-solid). Therefore $H=\mathcal{R}$.

So it remains to verify that $\mathcal{R}$ is $(k+1)$-solid. If $k>0$ or $p_{k+1}(\mathcal{R}')\neq\emptyset$, we have $p_{k+1}(\mathcal{R})=p_{k+1}(\mathcal{R}')$ as before, so we are done. Suppose $k=0$ and $p_1(\mathcal{R}')=\emptyset$. Let let q be $<_{\mathrm{lex}}$-least such that $\mathcal{M}\in H_q=\mathrm{Hull}_1^{\mathcal{R}}(\rho\cup q)$. Then $H_q=\mathcal{R}$, as before. But we claim that q is 1-solid for $\mathcal{R}$. For let us assume that $q=\{\gamma\}$ for some ordinal γ, for simplicity. Then $\mathcal{M}\notin H_\gamma=\mathrm{Hull}_1^{\mathcal{R}}(\gamma)$, and therefore H_γ is bounded in $\mathcal{R}$, and therefore $\mathrm{Th}_1^{\mathcal{R}}(\gamma)\in\mathcal{R}$, as required. But then $p_1^{\mathcal{R}}=q$, so we are done.

Now consider (3). We may assume that $\lambda>1$, as the only extenders of $\mathcal{N}_1$ are $\mathcal{F}$-extenders, so there are no non-trivial iteration trees on it. Let us also assume that $\mathcal{N}_\lambda$ has a largest $\mathcal{F}$-whole proper segment, since the contrary case is similar but easier. Then $\mathcal{M}^*$ has a largest $\mathcal{F}$-whole proper segment $\mathcal{M}_{\mathcal{F}}$; so $\mathcal{M}^+=\mathcal{M}^*\downarrow\mathcal{M}_{\mathcal{F}}\trianglelefteq\mathcal{F}_1(\mathcal{M}_{\mathcal{F}})$. If $(\mathcal{M}^*,k+1)$ is not on the $\mathcal{M}_{\mathcal{F}}$-drop-down sequence of $\mathcal{M}^*$, then the proof of (a) shows that $\mathcal{M}^*$ is $(k+1)$-sound, and therefore $(k+1)$-$\mathcal{F}$-solid (the "$\mathcal{F}$" since $\mathfrak{C}_{k+1}(\mathcal{M}^*)=\mathcal{M}^*$ in this case). So assume otherwise.

If $\mathcal{M}^*$ is whole let $\mathcal{M}'=\mathcal{F}_1(\mathcal{M}^*)$; otherwise let $\mathcal{M}'=\mathcal{F}_1(\mathcal{M}_{\mathcal{F}})$. So $\mathcal{M}^*\in\mathcal{M}'$. Let $\pi':\bar{\mathcal{M}}'\to\mathcal{M}'$ be elementary, with $\bar{\mathcal{M}}'$ countable and $\pi'(\bar{\mathcal{M}})=\mathcal{M}^*$ for some $\bar{\mathcal{M}}$ and also such that $\pi = \pi' \restriction \bar{\mathcal{M}}$ is in the hypothesized club and $\bar{a} = a^{\bar{\mathcal{M}}} \in C_{\mathcal{F}}$. Because $\mathcal{F}$ condenses coarsely (and using 2.24 or 2.25), $\bar{\mathcal{M}}$ is an $\mathcal{F}$-premouse. Now let Σ be an $\mathcal{F}$-putative strategy for $\bar{\mathcal{M}}$ as hypothesized.

Claim 2.35. *Σ is an $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)$-strategy for $\bar{\mathcal{M}}$.*

Proof Sketch. This is basically as in the proof of 2.31 (though here it is more important that *condenses finely* works with respect to weak embeddings as in (2c) of 2.28). One further point arises, however, in verifying that various models are in the right dropdown sequences in order to apply 2.28. For let $\mathcal{T}$ be via Σ, with last model $M^{\mathcal{T}}_{\alpha}$; say we want to apply 2.28 in order to deduce that $\mathcal{Q} = M^{\mathcal{T}}_{\alpha}$ is an $\mathcal{F}$-premouse. Let $m = \deg^{\mathcal{T}}(\alpha)$. Then by [10, Corollary 2.20], $\rho^{\mathcal{Q}}_{m+1} < \rho^{\mathcal{Q}}_{m}$; this helps to ensure that 2.28 applies. (Note that possibly $[0, \alpha]_{\mathcal{T}}$ does not drop in model or degree, so $m = k$, and $\mathrm{crit}(i^{\mathcal{T}}_{0,\alpha}) < \rho^{\bar{\mathcal{M}}}_{k+1}$. In this case, by [10], $\rho^{\mathcal{Q}}_{k+1} = \sup i^{\mathcal{T}}_{0,\alpha}``\rho^{\bar{\mathcal{M}}}_{k+1}$. We also need this observation in other places, because $\mathcal{T}$ need not be normal.) □

Let $\bar{\mathcal{N}} = \mathfrak{C}_{k+1}(\bar{\mathcal{M}})$ and let $\tau : \bar{\mathcal{N}} \to \bar{\mathcal{M}}$ be the core map. Then there is $\bar{\mathcal{N}}_{\mathcal{F}}$ such that $\tau(\bar{\mathcal{N}}_{\mathcal{F}}) = \bar{\mathcal{M}}_{\mathcal{F}}$. Then $\bar{\mathcal{N}}_{\mathcal{F}}$ is a whole $\mathcal{F}$-premouse, and is the largest $\varphi_{\mathcal{F}}$-putatively whole proper segment of $\bar{\mathcal{N}}$. And $\bar{\mathcal{N}}$ is a $\mathcal{F}$-premouse because $\mathcal{F}$ condenses finely.

Claim 2.36. *$(\bar{\mathcal{N}}, k + 1)$ is on the $\bar{\mathcal{N}}_{\mathcal{F}}$-dropdown sequence of $\bar{\mathcal{N}}$.*

Proof. Suppose not. We will show that $\bar{\mathcal{N}}_{\mathcal{F}} \triangleleft \bar{\mathcal{M}}_{\mathcal{F}}$. But then $(\mathcal{F}(\bar{\mathcal{N}}_{\mathcal{F}})\downarrow\bar{a}) \trianglelefteq \bar{\mathcal{M}}_{\mathcal{F}}$, so $\bar{\mathcal{N}} \in \bar{\mathcal{M}}$, a contradiction.

Let $\rho = \rho_{k+1}(\bar{\mathcal{M}})$. Let $(\mathcal{R}, j)$ be the last element of the $\bar{\mathcal{M}}_{\mathcal{F}}$-dropdown sequence of $\bar{\mathcal{M}}$ with $\mathcal{R} \triangleleft \bar{\mathcal{M}}$. So

$$\rho < \rho^{\mathcal{R}}_{j'} = \rho^{\mathcal{R}}_{\omega} = \mathrm{card}^{\bar{\mathcal{M}}}(\bar{\mathcal{M}}_{\mathcal{F}}) \in \mathrm{rg}(\tau).$$

The negation of the claim implies that $\mathrm{rg}(\tau) \cap \rho^{\mathcal{R}}_{\omega} = \rho$, so $\mathrm{crit}(\tau) = \rho$ and $\tau(\rho) = \rho^{\mathcal{R}}_{\omega}$. Let $\tau(\mathcal{S}) = \mathcal{R}$, so $\tau \restriction \mathcal{S} : \mathcal{S} \to \mathcal{R}$ is fully elementary and $\mathrm{crit}(\tau) = \rho^{\mathcal{S}}_{\omega}$. Therefore since $\mathcal{F}$ condenses finely, $\mathcal{S}$ is a $\mathcal{F}$-premouse. We will show that $\mathcal{S} \triangleleft \bar{\mathcal{M}}|\tau(\rho)$, which suffices since $\bar{\mathcal{N}}_{\mathcal{F}} \trianglelefteq \mathcal{S}$.

Let $\xi \leq l(\mathcal{S})$ be the supremum of ρ and all $\alpha \leq l(\mathcal{S})$ such that $\mathcal{S}|\alpha$ is E-active. Then $\rho \leq \xi \leq l(\bar{\mathcal{M}}_{\mathcal{F}})$. Let $(\mathcal{Q}, l')$ be the last element of the $\mathcal{S}|\xi$-dropdown sequence of $\mathcal{S}$; so $\mathcal{S}|\rho \trianglelefteq \mathcal{Q} \trianglelefteq \mathcal{S}$ and $\rho^{\mathcal{Q}}_{\omega} = \rho$. We claim that $\mathcal{Q} \triangleleft \bar{\mathcal{M}}$.

For let $\mathcal{P} = \tau(\mathcal{Q})$. We may assume that $\rho < \rho^{\mathcal{Q}}_{0}$ (by the ISC). So let $l < \omega$ be such that $\rho^{\mathcal{Q}}_{l+1} = \rho < \rho^{\mathcal{Q}}_{l}$. Then $\mathcal{P}$ is $(l, \omega_1, \omega_1 + 1)$-$\mathcal{F}$-iterable, since l-bounded trees $\mathcal{T}$ on $\mathcal{P}$ can be lifted to k-bounded trees $\mathcal{U}$ on $\bar{\mathcal{M}}$, using that $\mathcal{F}$ condenses finely.

Now arguing as in [2] and [11], we obtain a strategy Σ' for $\bar{\mathcal{M}}$ with the variant of the $\mathfrak{m}$-weak Dodd-Jensen property (see [11]) given by replacing all uses of near j-embeddings with nearly j-good embeddings. Then using Σ', the usual proof of condensation works, giving that $\mathcal{Q} \triangleleft \mathcal{P}$, so $\mathcal{Q} \triangleleft \bar{\mathcal{M}}|\tau(\rho)$.

Now $\mathcal{S} \trianglelefteq \mathcal{F}^\alpha(\mathcal{Q})$ for some $\alpha \in \mathrm{Ord}$, and $\mathcal{S} \in \mathcal{R}$, and $\rho_\omega^{\mathcal{S}} = \rho$, and $\tau(\rho)$ is a cardinal of $\bar{\mathcal{M}}$. So $\mathrm{o}(\mathcal{S}) < \tau(\rho)$, and because $\bar{\mathcal{M}}$ is $\mathcal{F}$-iterable, therefore $\mathcal{S} \triangleleft \bar{\mathcal{M}}|\tau(\rho)$.

This completes the proof of the claim. □

Claim 2.37. *$\bar{\mathcal{M}}$ is $(k+1)$-universal.*

Proof. Since $\mathcal{F}$ condenses finely, and using Claim 2.36, the phalanx $(\bar{\mathcal{M}}, \bar{\mathcal{N}}, \rho_{k+1}(\bar{\mathcal{M}}))$ is $\mathcal{F}$-iterable, via lifting to $\bar{\mathcal{M}}$ using the maps (id, τ).

Now we can adapt the usual proof of universality in the same manner that we adapted the proof of condensation above. □

Claim 2.38. *$\bar{\mathcal{N}} = \mathfrak{C}_{k+1}(\bar{\mathcal{M}})$ is $(k+1)$-solid.*

Proof. This is proved similarly to the previous claim, given a couple of observations. Let $p = p_{k+1}(\bar{\mathcal{N}})$. Let $\alpha \in p$ and $q = p \backslash (\alpha+1)$. Let H be the transitive collapse of $\mathrm{Hull}_{k+1}^{\bar{\mathcal{N}}}(\alpha \cup q)$; we need to see that $H \in \bar{\mathcal{N}}$. As in the proof of (a), we may assume that $\alpha < \mathrm{card}^{\bar{\mathcal{N}}}(\bar{\mathcal{N}}_{\mathcal{F}})$. Now let $\sigma : H \to \bar{\mathcal{N}}$ be the uncollapse. So σ is a near k-embedding. If σ fails to be a k-embedding, i.e., if $\mathrm{rg}(\sigma)$ is bounded in $\rho_k(\bar{\mathcal{N}})$, then we easily have $H \in \bar{\mathcal{N}}$. So assume σ is a k-embedding. Also as in the proof of (a), we may assume that $\bar{\mathcal{N}}_{\mathcal{F}} \in \mathrm{rg}(\sigma)$. Then $\rho_{k+1}^H \leq \alpha < \mathrm{card}^{\bar{\mathcal{N}}}(\bar{\mathcal{N}}_{\mathcal{F}})$, and so $(\bar{\mathcal{N}}, k+1)$ is on the $\bar{\mathcal{N}}_{\mathcal{F}}$-dropdown sequence of $\bar{\mathcal{N}}$.

Now since $\mathcal{F}$ condenses finely, H is a $\mathcal{F}$-premouse, and moreover, the phalanx $(\bar{\mathcal{N}}, H, \alpha)$ is $\mathcal{F}$-iterable, via lifting to $\bar{\mathcal{N}}$ (which is $\mathcal{F}$-iterable via lifting to $\bar{\mathcal{M}}$). Now we can adapt the proof of solidity just as for universality. □

By elementarity, it follows that $\mathcal{M}^*$ is $(k+1)$-universal and $\mathcal{N}^* = \mathfrak{C}_{k+1}(\mathcal{M}^*)$ is $(k+1)$-solid. Therefore $\mathcal{N}^*$ is $(k+1)$-sound. Because $\mathcal{F}$ condenses finely, $\mathcal{N}^*$ is an $\mathcal{F}$-premouse. This completes the proof. □

3 Strategy premice

We now proceed to defining Σ-premice, for an iteration strategy Σ. We first define the operator to be used to feed in Σ.

Definition 3.1 ($\mathfrak{B}(a, \mathcal{T}, b)$, $b^{\mathcal{N}}$)**.** Let $a, \mathcal{P}$ be transitive, with $\mathcal{P} \in \mathcal{J}_1(\hat{a})$. Let $\lambda > 0$ and let $\mathcal{T}$ be an iteration tree[20] on $\mathcal{P}$, of length $\omega\lambda$, with $\mathcal{T}\restriction\beta \in a$ for all $\beta \leq \omega\lambda$. Let $b \subseteq \omega\lambda$. We

[20] We formally take an *iteration tree* to include the entire sequence $\langle M_\alpha^{\mathcal{T}} \rangle_{\alpha < \mathrm{lh}(\mathcal{T})}$ of models. So it is $\Sigma_0(\mathcal{T}, \mathfrak{P})$ to assert that "$\mathcal{T}$ is an iteration tree on $\mathfrak{P}$".

define $\mathcal{N} = \mathfrak{B}(a, \mathcal{T}, b)$ recursively on $\mathrm{lh}(\mathcal{T})$, as the $\mathcal{J}$-model $\mathcal{N}$ over a, with parameter $\mathcal{P}$,[21] such that:

1. $l(\mathcal{N}) = \lambda$,
2. for each $\gamma \in (0, \lambda)$, $\mathcal{N}|\gamma = \mathfrak{B}(a, \mathcal{T}\restriction\omega\gamma, [0, \omega\gamma]_{\mathcal{T}})$,
3. $B^{\mathcal{N}}$ is the set of ordinals $\mathrm{o}(a) + \gamma$ such that $\gamma \in b$,
4. $E^{\mathcal{N}} = \emptyset$.

We also write $b^{\mathcal{N}} = b$. ⊣

It is easy to see that every initial segment of $\mathcal{N}$ is sound, so $\mathcal{N}$ is acceptable and is indeed a $\mathcal{J}$-model (not just a $\mathcal{J}$-structure).

Suppose we are building a Σ-premouse $\mathcal{N}$ for an iteration strategy Σ. Suppose we have built some $\mathcal{M} \trianglelefteq \mathcal{N}$, with $\mathcal{M}$ fairly closed, but there is $\mathcal{T} \in \mathcal{M}$ for which $\mathcal{M}$ has not been instructed regarding $\Sigma(\mathcal{T})$. If $\mathcal{T}$ is the tree for which we next feed $\Sigma(\mathcal{T})$ into $\mathcal{N}$ (that is, immediately after $\mathcal{M}$), then we will have already fed $\Sigma(\mathcal{T}\restriction\alpha)$ into $\mathcal{M}$, for all limits $\alpha < \mathrm{lh}(\mathcal{T})$. We will then use $\mathfrak{B}(\mathcal{M}, \mathcal{T}, \Sigma(\mathcal{T}))$ to extend $\mathcal{M}$, thus feeding in $\Sigma(\mathcal{T})$. Therefore if $\mathrm{lh}(\mathcal{T}) > \omega$ then $\mathfrak{B}(\mathcal{M}, \mathcal{T}, \Sigma(\mathcal{T}))$ codes redundant information (the branches $\Sigma(\mathcal{T}\restriction\alpha)$) before coding $\Sigma(\mathcal{T})$. This redundancy seems to allow one to prove slightly stronger condensation properties, given that Σ has nice condensation properties. It also simplifies the definition of *Σ-premouse*.[22] The key facts are given in 3.3 below.

In the next definition and in the sequel we need the notions of *hull embedding*, *hull condensation* and *branch condensation*; see [6, 1.29, 1.30, 2.14].

Definition 3.2. Let Σ be a partial iteration strategy. Let C be a class of iteration trees, closed under initial segment. We say that (Σ, C) is **suitably condensing** iff for every $\mathcal{T} \in C$ such that $\mathcal{T}$ is via Σ and $\mathrm{lh}(\mathcal{T}) = \lambda + 1$ for some limit λ, either (i) Σ has hull condensation with respect to $\mathcal{T}$, or (ii) $b^{\mathcal{T}}$ does not drop and Σ has branch condensation with respect to $\mathcal{T}$. ⊣

Lemma 3.3. *Let $a, \mathcal{T}, b$ be as in 3.1, and let $\mathcal{R} = \mathfrak{B}(a, \mathcal{T}, b)$. Let $\gamma \leq l(\mathcal{R})$. Let $\bar{\mathcal{R}}$ be a $\mathcal{J}$-structure over $\bar{a}$ with parameter $\bar{\mathcal{P}}$. Suppose there is a partial embedding $\pi : \bar{\mathcal{R}} \to \mathcal{R}|\gamma$ such that there is an $\in^{\bar{\mathcal{R}}}$-cofinal set $X \subseteq \bar{\mathcal{R}}$ with*

$$X \cup \mathrm{o}(\bar{\mathcal{R}}) \cup \bar{\mathfrak{P}} \cup \{\bar{\mathcal{T}}\} \subseteq \mathrm{dom}(\pi),$$

[21] $\mathcal{P} = M_0^{\mathcal{T}}$ is determined by $\mathcal{T}$.

[22] Some difficulties that arise if one codes Σ by only feeding $\Sigma(\mathcal{T})$ itself are discussed in Appendix B.

and $\pi(\bar{\mathcal{T}}) = \mathcal{T}$*, and* π *is* Σ_0*-elementary, for* $\mathcal{L}_0$*. Let* $\bar{B} = B^{\bar{\mathcal{R}}}$*. If* $\gamma = l(\mathcal{R})$ *then suppose that* $b^{\mathcal{R}}$ *is a* $\mathcal{T}$*-cofinal branch. Then:*

1. $\bar{\mathcal{R}}$ *is a* $\mathcal{J}$*-model over* $\bar{a}$ *and* $\bar{B} \subseteq [\mathrm{o}(\bar{a}), \mathrm{o}(\bar{\mathcal{R}}))$*. Let* $\bar{\gamma} = l(\bar{\mathcal{R}})$*. Then* $\omega\bar{\gamma} \leq \mathrm{lh}(\bar{\mathcal{T}})$ *and letting* $\bar{b} = b^{\bar{\mathcal{R}}}$ *(i.e.,* $\alpha \in \bar{b}$ *iff* $\mathrm{o}(\bar{\mathcal{M}}) + \alpha \in \bar{B}$*), then* $\bar{\mathcal{R}} = \mathfrak{B}(\bar{\mathcal{M}}, \bar{\mathcal{T}}\restriction\omega\bar{\gamma}, \bar{b})$*.*

2. *If* π *is* Σ_1*-elementary on* X*, with respect to* $\mathcal{L}_0$*, then* $\bar{b}$ *is cofinal in* $\omega\bar{\gamma}$*.*

3. *Suppose* $\bar{b}$ *is cofinal in* $\omega\bar{\gamma}$*. Then* $\bar{b}$ *is a* $\bar{\mathcal{T}}\restriction\omega\bar{\gamma}$*-cofinal branch, and:*

   (*a*) *Suppose that* $\omega\bar{\gamma} < \mathrm{lh}(\bar{\mathcal{T}})$*. Then* $\bar{b} = [0, \omega\bar{\gamma}]_{\bar{\mathcal{T}}}$*, and therefore* $\bar{\mathcal{R}} \triangleleft \mathfrak{B}(\bar{\mathcal{M}}, \bar{\mathcal{T}}, b^*)$ *for any* $b^* \subseteq \mathrm{lh}(\bar{\mathcal{T}})$*.*

   (*b*) *Suppose that* $\omega\bar{\gamma} = \mathrm{lh}(\bar{\mathcal{T}})$*. Let* $\omega\gamma' = \sup \pi``\omega\bar{\gamma}$*. Then* π *induces a hull embedding from* $\bar{\mathcal{T}} \frown \bar{b}$ *to* $\mathcal{T}' = (\mathcal{T} \frown b)\restriction\omega\gamma' + 1$*.*[23]

   (*c*) *Let* C *be the set of initial segments of* $\mathcal{T}$*. Suppose that* $\mathcal{T}$ *is via* Σ*, where* Σ *is some partial strategy for* $\mathfrak{P}$ *such that* (Σ, C) *is suitably condensing. Suppose that* $\bar{\mathfrak{P}} = \mathfrak{P}$ *and* $\pi\restriction\bar{\mathfrak{P}} = \mathrm{id}$*. Then* $(\bar{\mathcal{T}}\restriction\omega\bar{\gamma}) \frown \bar{b}$ *is via* Σ*.*

Proof. We just prove 3(a). We have $\omega\bar{\gamma} < \mathrm{lh}(\bar{\mathcal{T}})$. Let $\omega\gamma' = \sup \pi``\omega\bar{\gamma}$, so $\omega\gamma' < \mathrm{lh}(\mathcal{T})$. We have $c = [0, \omega\bar{\gamma}]_{\bar{\mathcal{T}}} \in \bar{\mathcal{M}}$, and note that we may assume that $c \in X$. We have $\pi``c \subseteq \pi(c) = [0, \pi(\omega\bar{\gamma})]_{\mathcal{T}}$, and $\pi``c$ is cofinal in $\omega\gamma'$. Therefore $\pi``c \subseteq [0, \omega\gamma']_{\mathcal{T}}$. But similarly, $\pi``\bar{b} \subseteq [0, \omega\gamma']_{\mathcal{T}}$, because $\pi``\bar{b} \subseteq b^{\mathcal{R}|\gamma} \cap \omega\gamma'$ and $\pi``\bar{b}$ is cofinal in $\omega\gamma'$. But then $c = \bar{b}$, as required. □

We next describe the overall structure of potential Σ-premice.

Definition 3.4. Let φ be an $\mathcal{L}_0$-formula. Let $\mathcal{P}$ be transitive. Let $\mathcal{M}$ be a $\mathcal{J}$-model (over some a), with parameter $\mathcal{P}$. Let $\mathcal{T} \in \mathcal{M}$. We say that φ **selects** $\mathcal{T}$ **for** $\mathcal{M}$, and write $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}}$, iff

(a) $\mathcal{T}$ is the unique $x \in \mathcal{M}$ such that $\mathcal{M} \models \varphi(x)$,

(b) $\mathcal{T}$ is an iteration tree on $\mathcal{P}$ of limit length,

(c) for every $\mathcal{N} \triangleleft \mathcal{M}$, we have $\mathcal{N} \not\models \varphi(\mathcal{T})$, and

(d) for every limit $\lambda < \mathrm{lh}(\mathcal{T})$, there is $\mathcal{N} \triangleleft \mathcal{M}$ such that $\mathcal{N} \models \varphi(\mathcal{T}\restriction\lambda)$. ⊣

The generality in the indexing device *type* φ in the following definition was probably motivated by Sargsyan's [6, Definition 1.1].

[23] By our assumptions, if $\gamma' = \mathrm{lh}(\mathcal{T})$ then b is $\mathcal{T}$-cofinal.

Definition 3.5 (Potential $\mathcal{P}$-strategy-premouse, $\Sigma^{\mathcal{M}}$)**.** Let $\varphi \in \mathcal{L}_0$. Let $\mathcal{P}, a$ be transitive with $\mathcal{P} \in \mathcal{J}_1(\hat{a})$. A **potential $\mathcal{P}$-strategy-premouse (over a, of type φ)** is a $\mathcal{J}$-model $\mathcal{M}$ over a, with parameter $\mathcal{P}$, such that the $\mathfrak{B}$ operator is used to feed in an iteration strategy for trees on $\mathcal{P}$, using the sequence of trees naturally determined by $S^{\mathcal{M}}$ and selection by φ. We let $\Sigma^{\mathcal{M}}$ denote the partial strategy coded by the predicates $B^{\mathcal{M}|\eta}$, for $\eta \leq l(\mathcal{M})$.

In more detail, there is an increasing, closed sequence of ordinals $\langle \eta_\alpha \rangle_{\alpha \leq \iota}$ with the following properties. We will also define $\Sigma^{\mathcal{M}|\eta}$ for all $\eta \in [1, l(\mathcal{M})]$ and $\mathcal{T}_\eta = \mathcal{T}_\eta^{\mathcal{M}}$ for all $\eta \in [1, l(\mathcal{M}))$.

1. $1 = \eta_0$ and $\mathcal{M}|1 = \mathcal{J}_1^{\mathsf{m}}(a; \mathcal{P})$ and $\Sigma^{\mathcal{M}|1} = \emptyset$.

2. $l(\mathcal{M}) = \eta_\iota$, so $\mathcal{M}|\eta_\iota = \mathcal{M}$.

3. Given $\eta \leq l(\mathcal{M})$ such that $B^{\mathcal{M}|\eta} = \emptyset$, we set $\Sigma^{\mathcal{M}|\eta} = \bigcup_{\eta' < \eta} \Sigma^{\mathcal{M}|\eta'}$.

Let $\eta \in [1, l(\mathcal{M})]$. Suppose there is $\gamma \in [1, \eta]$ and $\mathcal{T} \in \mathcal{M}|\gamma$ such that $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}|\gamma}$, and $\mathcal{T}$ is via $\Sigma^{\mathcal{M}|\eta}$, but no proper extension of $\mathcal{T}$ is via $\Sigma^{\mathcal{M}|\eta}$. Taking γ minimal such, let $\mathcal{T}_\eta = \mathcal{T}_\varphi^{\mathcal{M}|\gamma}$. Otherwise let $\mathcal{T}_\eta = \emptyset$.

4. Let $\alpha + 1 \leq \iota$. Suppose $\mathcal{T}_{\eta_\alpha} = \emptyset$. Then $\eta_{\alpha+1} = \eta_\alpha + 1$ and $\mathcal{M}|\eta_{\alpha+1} = \mathcal{J}_1^{\mathsf{m}}(\mathcal{M}|\eta_\alpha; \mathcal{P}){\downarrow}a$.

5. Let $\alpha + 1 \leq \iota$. Suppose $\mathcal{T} = \mathcal{T}_{\eta_\alpha} \neq \emptyset$. Let $\omega\lambda = \mathrm{lh}(\mathcal{T})$. Then for some $b \subseteq \omega\lambda$, and $\mathcal{S} = \mathfrak{B}(\mathcal{M}|\eta_\alpha, \mathcal{T}, b)$, we have:

   (a) $\mathcal{M}|\eta_{\alpha+1} \trianglelefteq \mathcal{S}$.

   (b) If $\alpha + 1 < \iota$ then $\mathcal{M}|\eta_{\alpha+1} = \mathcal{S}$.

   (c) If $\mathcal{S} \trianglelefteq \mathcal{M}$ then b is a $\mathcal{T}$-cofinal branch.[24]

   (d) For $\eta \in [\eta_\alpha, l(\mathcal{M})]$ such that $\eta < l(\mathcal{S})$, $\Sigma^{\mathcal{M}|\eta} = \Sigma^{\mathcal{M}|\eta_\alpha}$.

   (e) If $\mathcal{S} \trianglelefteq \mathcal{M}$ then then $\Sigma^{\mathcal{S}} = \Sigma^{\mathcal{M}|\eta_\alpha} \cup \{(\mathcal{T}, b^{\mathcal{S}})\}$.

6. For each limit $\alpha \leq \iota$, $B^{\mathcal{M}|\eta_\alpha} = \emptyset$. ⊣

Definition 3.6 (Whole)**.** Let $\mathcal{M}$ be a potential $\mathcal{P}$-strategy-premouse of type φ. We say $\mathcal{M}$ is **φ-branch-whole** (or just **branch-whole** if φ is fixed) iff for every $\eta < l(\mathcal{M})$, if $\mathcal{T}_\eta \neq \emptyset$ and $\mathcal{T}_\eta \neq \mathcal{T}_{\eta'}$ for all $\eta' < \eta$, then for some b, $\mathfrak{B}(\mathcal{M}|\eta, \mathcal{T}_\eta, b) \trianglelefteq \mathcal{M}$.[25] ⊣

[24]We allow $\mathcal{M}_b^{\mathcal{T}}$ to be illfounded, but then $\mathcal{T}\,{}^\frown b$ is not an iteration tree, so is not continued by $\Sigma^{\mathcal{M}}$.

[25]*φ-whole* depends on φ as the definition of $\mathcal{T}_\eta$ does.

Definition 3.7 (Potential Σ-premouse)**.** Let Σ be a (partial) iteration strategy for a transitive structure $\mathcal{P}$. A **potential Σ-premouse (over a, of type φ)** is a potential $\mathcal{P}$-strategy premouse $\mathcal{M}$ (over a, of type φ) such that $\Sigma^{\mathcal{M}} \subseteq \Sigma$.[26] $\dashv$

Definition 3.8. Let $\mathcal{R}$ be an amenable $\mathcal{J}$-structure for $\mathcal{L}_0$. Let $\beta < l(\mathcal{R})$ and let $n < \omega$. Let $H = \mathcal{S}_{\beta+n}^{S^{\mathcal{R}}}(\hat{a}^{\mathcal{R}})$ (the "$\mathcal{S}$" is in the sense of "$\mathcal{S}$-hierarchy"). Then we define

$$\mathcal{R} \wr (\beta, n) = (H, E, B, S, a^{\mathcal{R}}, \mathfrak{P}^{\mathcal{R}})$$

(an $\mathcal{L}_0$-structure), where $E = E^{\mathcal{R}} \cap H$, $B = B^{\mathcal{R}} \cap H$ and $S = S^{\mathcal{R}} \cap H$. $\dashv$

Note that if $\mathcal{R}$ is a $\mathcal{J}$-model and $\beta < l(\mathcal{R})$ then $\lfloor \mathcal{R}|\beta \rfloor = \lfloor \mathcal{R} \wr (\beta, 0) \rfloor$, but the active predicates of $\mathcal{R}|\beta$ and $\mathcal{R} \wr (\beta, 0)$ can differ.

Definition 3.9. Let $\mathcal{R}, \mathcal{M}$ be $\mathcal{J}$-structures for $\mathcal{L}_0$. Let $\pi : \mathcal{R} \to \mathcal{M}$ be a partial map. Then π is a **very weak 0-embedding** iff π is Σ_0-elementary on its domain (with respect to $\mathcal{L}_0$), there is $X \subseteq \mathcal{R}$, with X cofinal in $\mathrm{o}(\mathcal{R})$, and

$$\mathrm{o}(\mathcal{R}) \cup \mathfrak{P}^{\mathcal{R}} \cup \{\mathcal{R} \wr (\beta, n) \parallel \mathrm{o}(\mathcal{R} \wr (\beta, n)) \in X\} \subseteq \mathrm{dom}(\pi),$$

and π is Σ_1-elementary on parameters in X.

A class C of premice is **very condensing** iff for all $\mathcal{M} \in C$ with $E^{\mathcal{M}} = \emptyset$, and all $\mathcal{J}$-structures $\mathcal{R}$, if there is a very weak 0-embedding $\pi : \mathcal{R} \to \mathcal{M}$ then $\mathcal{R} \in C$. $\dashv$

Lemma 3.10. *Let $\mathcal{M}$ be a $\mathcal{P}$-strategy premouse over a, of type φ. Let $\mathcal{R}$ be a $\mathcal{J}$-structure for $\mathcal{L}_0$.*

(*1*) *Suppose $\mathcal{M}$ is not type 3. Let $\pi : \mathcal{R} \to \mathcal{M}$ be a partial map such that either:*

- (a) *π is a weak 0-embedding, or*
- (b) *π is a very weak 0-embedding, and if $E^{\mathcal{R}} \neq \emptyset$ and $\mathcal{M}$ is not type 3 then item 4 of 2.1 holds for $E^{\mathcal{R}}$.*

Then $\mathcal{R}$ is a $\mathfrak{P}^{\mathcal{R}}$-strategy premouse of type φ. Moreover, if $\mathfrak{P}^{\mathcal{R}} = \mathcal{P}$ and $\pi \restriction \mathcal{P} = \mathrm{id}$ and $\mathcal{M}$ is a Σ-premouse, where $(\Sigma, \mathrm{dom}(\Sigma^{\mathcal{M}}))$ is suitably condensing, then $\mathcal{R}$ is also a Σ-premouse.

[26]If $\mathcal{M}$ is a model all of whose proper segments are potential Σ-premice, and the rules for potential $\mathcal{P}$-strategy premice require that $B^{\mathcal{M}}$ code a $\mathcal{T}$-cofinal branch, but $\Sigma(\mathcal{T})$ is not defined, then $\mathcal{M}$ is not a potential Σ-premouse, whatever its predicates are.

(2) *Suppose* $\mathcal{M}$ *is type 3. Let* $\pi : \mathcal{R} \to \mathcal{M}^{\mathrm{sq}}$ *be a very weak* 0*-embedding. (It follows that* $E^{\mathcal{R}}$ *is an extender over* $\mathcal{R}$*.) Let* $\mu = \mathrm{crit}(E^{\mathcal{R}})$*. If* $\mathcal{U} = \mathrm{Ult}(\mathcal{R}|(\mu^+)^{\mathcal{R}}, E^{\mathcal{R}})$ *is wellfounded then* $\mathcal{R} = \mathcal{Q}^{\mathrm{sq}}$ *for some type 3,* $\mathfrak{P}^{\mathcal{Q}}$*-strategy premouse of type* φ*. Moreover, let* $\kappa = \mathrm{crit}(E^{\mathcal{M}})$ *and suppose that* $\mathcal{V} = \mathrm{Ult}(\mathcal{M}|(\kappa^+)^{\mathcal{M}}, E^{\mathcal{M}})$ *is wellfounded. Then* $\mathcal{U}$ *is wellfounded; let* $\mathcal{R} = \mathcal{Q}^{\mathrm{sq}}$*. Suppose further that* $\mathcal{V}$ *is a* Σ*-premouse, where* $(\Sigma, \mathrm{dom}(\Sigma^{\mathcal{V}}))$ *is suitably condensing. If* $\mathfrak{P}^{\mathcal{Q}} = \mathcal{P}$ *and* $\pi{\restriction}\mathcal{P} = \mathrm{id}$ *then* $\mathcal{Q}$ *is a* Σ*-premouse.*

(3) *Suppose* $\mathcal{M}$ *is not type 3 and there is* $\pi\colon \mathcal{M} \to \mathcal{R}$ *such that either (a)* π *is* Σ_2*-elementary or (b)* π *is cofinal and* Σ_1*-elementary and* $B^{\mathcal{M}} = \emptyset$*.*

Then $\mathcal{R}$ *is a* $\mathfrak{P}^{\mathcal{R}}$*-strategy premouse of type* φ*, and* $\mathcal{R}$ *is branch-whole iff* $\mathcal{M}$ *is branch-whole.*

(4) *Suppose* $\mathcal{M}$ *is type 3 and there is* $\pi\colon \mathcal{M}^{\mathrm{sq}} \to \mathcal{R}$ *such that either (a)* π *is* Σ_2*-elementary; or (b)* π *is cofinal and* Σ_1*-elementary. Let* $\mu = \mathrm{crit}(E^{\mathcal{R}})$ *and suppose that* $\mathrm{Ult}(\mathcal{R}|(\mu^+)^{\mathcal{R}}, E^{\mathcal{R}})$ *is wellfounded.*

Then $\mathcal{R} = \mathcal{Q}^{\mathrm{sq}}$ *for some type 3,* $\mathfrak{P}^{\mathcal{Q}}$*-strategy premouse of type* φ*.*

(5) *Suppose* $B^{\mathcal{M}} \neq \emptyset$*. Let* $\mathcal{T} = \mathcal{T}^{\mathcal{M}}_{\eta}$ *where* $\eta < l(\mathcal{M})$ *is largest such that* $\mathcal{M}|\eta$ *is branch-whole. Let* $b = b^{\mathcal{M}}$ *and* $\omega\gamma = \bigcup b$*. So* $\mathcal{M} \trianglelefteq \mathfrak{B}(\mathcal{M}|\eta, \mathcal{T}, b)$*. Suppose there is* $\pi : \mathcal{M} \to \mathcal{R}$ *such that* π *is cofinal and* Σ_1*-elementary. Let* $\omega\gamma' = \sup \pi``\omega\gamma$*.*

   (a) $\mathcal{R}$ *is a* $\mathfrak{P}^{\mathcal{R}}$*-strategy premouse of type* φ *iff we have either (i)* $\omega\gamma' = \mathrm{lh}(\pi(\mathcal{T}))$*, or (ii)* $\omega\gamma' < \mathrm{lh}(\pi(\mathcal{T}))$ *and* $b^{\mathcal{R}} = [0, \omega\gamma']_{\pi(\mathcal{T})}$*.*

   (b) *If either* $b^{\mathcal{M}} \in \mathcal{M}$ *or* π *is continuous at* $\mathrm{lh}(\mathcal{T})$ *then* $\mathcal{R}$ *is a* $\mathfrak{P}^{\mathcal{R}}$*-strategy premouse of type* φ*.*

Proof. We first consider (1), just proving (1)(b), focusing on the proof that $\mathcal{R}$ is a $\mathfrak{P}^{\mathcal{R}}$-strategy premouse of type φ. So let $\pi : \mathcal{R} \to \mathcal{M}$ be a very weak 0-embedding, as witnessed by X. Using 3.3, it is easy to see that for all $\eta < l(\mathcal{R})$, $\mathcal{R}|\eta$ is a $\mathcal{P}'$-strategy premouse of type φ, and moreover, that $\pi(\eta) < l(\mathcal{M})$, and $\mathcal{R}|\eta$ is branch-whole iff $\mathcal{M}|\pi(\eta)$ is branch-whole, and we may assume that $\mathcal{T}^{\mathcal{R}}_{\eta} \in \mathrm{dom}(\pi)$, and $\pi(\mathcal{T}^{\mathcal{R}}_{\eta}) = \mathcal{T}^{\mathcal{M}}_{\pi(\eta)}$. So we just need to see that the top predicates of $\mathcal{R}$ are valid. Clearly we may assume that $E^{\mathcal{M}} = \emptyset$. Because π is Σ_1-elementary on an $\mathrm{o}(\mathcal{R})$-cofinal set, π is also Σ_1-elementary on an $l(\mathcal{R})$-cofinal set.

Suppose $\mathcal{R}$ is a limit of branch-whole proper segments. Then letting $\eta = \sup \pi``l(\mathcal{R})$, $\mathcal{M}|\eta$ is a limit of branch-whole proper segments, and it follows that for all $\eta' > \eta$, $B^{\mathcal{M}|\eta'} \cap \mathrm{rg}(\pi) = \emptyset$. So $B^{\mathcal{R}} = \emptyset$, as desired.

Now suppose that $\eta < l(\mathcal{R})$ and $\mathcal{R}|\eta$ is the largest branch-whole proper segment of $\mathcal{R}$. Let $\mathcal{T} = \mathcal{T}^{\mathcal{R}}_{\eta}$. If $\mathcal{T} = \emptyset$ then argue like in the previous paragraph. Suppose $\mathcal{T} \neq \emptyset$. Because

$\mathcal{R}|\eta$ is the largest branch-whole proper segment of $\mathcal{R}$, we may assume that $\eta \in X$, and so $\mathcal{M}|\pi(\eta)$ is the largest branch-whole proper segment of $\mathcal{M}$. So the validity of $B^{\mathcal{R}}$ follows from 3.3.

The "moreover" clause of (1) follows from the above argument and 3.3.

For the proof of (2) argue like in the proof of 2.35. For (5)(b), in the case that $\omega\gamma' < \mathrm{lh}(\pi(\mathcal{T}))$, use the hypothesis that $b^{\mathcal{M}} \in \mathcal{M}$ to see that $\pi``b^{\mathcal{M}} \subseteq [0, \omega\gamma']_{\pi(\mathcal{T})}$, and so $b^{\mathcal{R}} = [0, \omega\gamma']_{\pi(\mathcal{T})}$. We omit further detail. □

Remark 3.11. The preceding proof left open the possibility that $\mathcal{R}$ fails to be a $\mathcal{P}$-strategy premouse under certain circumstances (because $B^{\mathcal{R}}$ should be coding a branch that has in fact already been coded at some proper segment of $\mathcal{R}$, but codes some other branch instead). In the main circumstance we are interested in, this does not arise, for a couple of reasons. Suppose that Σ is an iteration strategy for $\mathcal{P}$ with hull condensation, $\mathcal{M}$ is a Σ-premouse, and Λ is a strategy for $\mathcal{M}$. Suppose $\pi : \mathcal{M} \to \mathcal{R}$ is a degree 0 iteration embedding and $B^{\mathcal{M}} \neq \emptyset$ and π is discontinuous at $\mathrm{lh}(\mathcal{T})$. Then we claim that $b^{\mathcal{M}} \in \mathcal{M}$. (It's not relevant whether π itself is via Λ.)

To see this, note that the discontinuity implies that $\mathcal{M} \models$"There is $E \in \mathbb{E}$ which is a total measure and $\mathrm{lh}(\mathcal{T}^{\mathcal{M}})$ has cofinality $\kappa = \mathrm{crit}(E)$". Let $C \in \mathcal{M}$, $C \subseteq \mathrm{lh}(\mathcal{T})$ be a club of ordertype κ. $i_E : \mathcal{M} \to \mathrm{Ult}_0(\mathcal{M}, E)$ is continuous at all points of C. Let $\lambda = \sup i_E``\mathrm{lh}(\mathcal{T})$. Then $i_E``C = i_E(C) \cap \lambda$ is club in λ. But $\mathrm{Ult}_0(\mathcal{M}, E) \models$"$\lambda < \mathrm{lh}(i_E(\mathcal{T}))$ and $\mathrm{cof}(\lambda) = \kappa$ is uncountable". So $[0, \lambda]_{i_E(\mathcal{T})} \cap i_E``C$ is club in $\lambda$, and $C' \in \mathcal{M}$ where $C'$ is (the club) $C \cap i_E^{-1}``[0, \lambda]_{i_E(\mathcal{T})}$. By hull condensation, $\Sigma(\mathcal{T})$ is the downward $\leq_{\mathcal{T}}$-closure of C'.

The other reason is that, supposing $\pi : \mathcal{M} \to \mathcal{R}$ is via Λ, then trivially, $B^{\mathcal{R}}$ must code branches according to Σ. By part (a), we can obtain such a Λ given that we can realize iterates of $\mathcal{M}$ back into a fixed Σ-premouse (with $\mathcal{P}$-weak 0-embeddings as realization maps).

Definition 3.12. Let $\mathcal{P}$ be transitive and Σ a partial iteration strategy for $\mathcal{P}$. Let $\varphi \in \mathcal{L}_0$. Let $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi}$ be the operator such that:

1. $\mathcal{F}_0(a) = \mathcal{J}_1^{\mathsf{m}}(a; \mathcal{P})$, for all transitive a such that $\mathcal{P} \in \mathcal{J}_1(\hat{a})$;

2. Let $\mathcal{M}$ be a sound branch-whole Σ-premouse of type φ. Let $\lambda = l(\mathcal{M})$ and with notation as in 3.5, let $\mathcal{T} = \mathcal{T}_\lambda$. If $\mathcal{T} = \emptyset$ then $\mathcal{F}_1(\mathcal{M}) = \mathcal{J}_1^{\mathsf{m}}(\mathcal{M}; \mathcal{P})$. If $\mathcal{T} \neq \emptyset$ then $\mathcal{F}_1(\mathcal{M}) = \mathfrak{B}(\mathcal{M}, \mathcal{T}, b)$ where $b = \Sigma(\mathcal{T})$.

We say that $\mathcal{F}$ is a **strategy operator**. ⊣

Clearly, with the notation above, if Σ is a strategy for $\mathcal{P}$ which is sufficiently total over an operator background $\mathscr{B}$ and $\mathcal{M} \in \mathscr{B}$, then $\mathcal{M}$ is an $\mathcal{F}_{\Sigma,\varphi}$-premouse iff $\mathcal{M}$ is a Σ-premouse of type φ.

Lemma 3.13. *Let $\mathcal{P}$ be countable and transitive. Let φ be a formula of $\mathcal{L}_0$. Let Σ be a partial strategy for $\mathcal{P}$. Let D_φ be the class of iteration trees $\mathcal{T}$ on $\mathcal{P}$ such that for some $\mathcal{J}$-model $\mathcal{M}$, with parameter $\mathcal{P}$, we have $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}}$. Suppose that (Σ, D_φ) is suitably condensing. Then the class E of Σ-premice of type φ is very condensing; and $\mathcal{F}_{\Sigma,\varphi}$ condenses finely.*

Proof. E is very condensing by 3.10. Clearly $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi}$ is uniformly Σ_1 and projecting. It follows that $\mathcal{F}$ condenses finely. □

Definition 3.14. Let a be transitive and let $\mathcal{F}$ be an operator (with parameter $\mathcal{P}$). We say that $\mathcal{M}_1^{\mathcal{F},\#}(a)$ **exists** iff there is a $(0, \omega_1 + 1)$-$\mathcal{F}$-iterable, non-1-small $\mathcal{F}$-premouse over a (with parameter $\mathcal{P}$). We write $\mathcal{M}_1^{\mathcal{F},\#}(a)$ for the least such sound structure. For $\Sigma, \mathcal{P}, a, \varphi$ as in 3.12, we write $\mathcal{M}_1^{\Sigma,\varphi,\#}(a)$ for $\mathcal{M}_1^{\mathcal{F}_{\Sigma,\varphi},\#}(a)$.

Let $\mathcal{L}_0^+$ be the language $\mathcal{L}_0 \cup \{\dot{\prec}, \dot{\Sigma}\}$, where $\dot{\prec}$ is the binary relation defined by "$\dot{a}$ is self-wellordered, with ordering $\prec_{\dot{a}}$, and $\dot{\prec}$ is the canonical wellorder of the universe extending $\prec_{\dot{a}}$", and $\dot{\Sigma}$ is the partial function defined "$\dot{\mathfrak{P}}$ is a transitive structure and the universe is a potential $\dot{\mathfrak{P}}$-strategy premouse over $\dot{a}$ and $\dot{\Sigma}$ is the associated partial putative iteration strategy for $\dot{\mathfrak{P}}$". Let $\varphi_{\text{all}}(\mathcal{T})$ be the $\mathcal{L}_0$-formula "$\mathcal{T}$ is the $\dot{\prec}$-least limit length iteration tree $\mathcal{U}$ on $\dot{\mathfrak{P}}$ such that $\mathcal{U}$ is via $\dot{\Sigma}$, but no proper extension of $\mathcal{U}$ is via $\dot{\Sigma}$". Then for $\Sigma, \mathcal{P}, a$ as in 3.12, we write $\mathcal{M}_1^{\Sigma,\#}(a)$ for $\mathcal{M}_1^{\Sigma,\varphi_{\text{all}},\#}(a)$.[27]

Let κ be a cardinal and suppose that $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\#}(a)$ exists and is $(0, \kappa^+ + 1)$-iterable. We write $\Lambda_{\mathfrak{M}}$ for the unique $(0, \kappa^+ + 1)$-iteration strategy for $\mathfrak{M}$ (given that κ is fixed). ⊣

Definition 3.15. We say that $(\mathcal{F}, \Sigma, \varphi, D, a)$ is **suitable** iff $a \in \mathrm{HC}$ and a is transitive and $\mathcal{M}_1^{\mathcal{F},\#}(a)$ exists, where either

(i) $\mathcal{F}$ is a projecting, uniformly Σ_1 operator which condenses finely, $C_{\mathcal{F}}$ is the (possibly swo'd) cone above a, D is the set of pairs $(i, X) \in \mathrm{dom}(\mathcal{F})$ such that either $i = 0$ or X is a sound whole $\mathcal{F}$-premouse, and $\Sigma = \varphi = 0$, or

(ii) $\mathcal{P}, \Sigma, \varphi, D_\varphi$ are as in 3.13, $Y = \Sigma$, $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi}$, $D_\varphi \subseteq D$, D is a class of limit length iteration trees on $\mathcal{P}$, via Σ, $\Sigma(\mathcal{T})$ is defined for all $\mathcal{T} \in D$, (Σ, D) is suitably condensing and $\mathcal{P} \in \mathcal{J}_1(\hat{a})$.

We write $\mathcal{G}_{\mathcal{F}}$ for the function with domain C, such that $x \mapsto \Sigma(x)$ in case (ii), and in case (i), $\mathcal{G}_{\mathcal{F}}(0, X) = \mathcal{F}(0, X)$ and $\mathcal{G}_{\mathcal{F}}(1, X) = \mathcal{R}{\downarrow}a^X$ for the least $\mathcal{R} \trianglelefteq \mathcal{F}_1(X)$ such that either $\mathcal{R} = \mathcal{F}_1(X)$ or $\mathcal{R}{\downarrow}a^X$ is unsound. ⊣

Lemma 3.16. *Let $\mathcal{F}$ be as in 3.15 and $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\#}$. Then $\Lambda_{\mathfrak{M}}$ has branch condensation and hull condensation.*

Proof. See 2.34 for related calculations. □

[27] We are only interested in the case that a is self-wellordered. Otherwise, note that $\mathcal{M}_1^{\Sigma,\#}(a) = \mathcal{M}_1^{\#}(a)$.

4 G-organized $\mathcal{F}$-premice

In this section we implement some ideas of Sargsyan within the framework of the previous sections, defining *g-organized $\mathcal{F}$-premice*, assuming that $\mathcal{F}$ has the following absoluteness property. If $\mathcal{F}$ is a strategy operator for a nice enough iteration strategy, then the property does hold. In the following, $\delta^{\mathfrak{M}}$ denotes the Woodin cardinal of $\mathfrak{M}$.

Definition 4.1. Let $(\mathcal{F}, \Sigma, \varphi, C, a)$ be suitable. We say that $(\mathcal{F}, \Sigma, \varphi, C, a)$ (or just $\mathcal{F}$) **determines itself on generic extensions** iff, writing $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\#}(a)$, there are formulas Φ, Ψ in $\mathcal{L}_0$ such that there is some $\gamma > \delta^{\mathfrak{M}}$ such that $\mathfrak{M}|\gamma \models \Phi$ and for any non-dropping $\Sigma_{\mathfrak{M}}$-iterate $\mathcal{N}$ of $\mathfrak{M}$, via a countable iteration tree $\mathcal{T}$, any $\mathcal{N}$-cardinal δ, any $\gamma \in \mathrm{Ord}$ such that $\mathcal{N}|\gamma \models \Phi$ & "δ is Woodin", and any g which is set-generic over $\mathcal{N}|\gamma$ (with $g \in V$), then $(\mathcal{N}|\gamma)[g]$ is closed under $\mathcal{G}_{\mathcal{F}}$, and $\mathcal{G}_{\mathcal{F}}\restriction(\mathcal{N}|\gamma)[g]$ is defined over $(\mathcal{N}|\gamma)[g]$ by Ψ. We say such a pair (Φ, Ψ) **generically determines** $(\mathcal{F}, \Sigma, \varphi, C, a)$ (or just $\mathcal{F}$).

We say an operator $\mathcal{F}$ is **nice** iff for some Σ, φ, C, a, $(\mathcal{F}, \Sigma, \varphi, C, a)$ is suitable and determines itself on generic extensions.

Let $\mathcal{P} \in \mathrm{HC}$, let Σ be an iteration strategy for $\mathcal{P}$ and let C be the class of all limit length trees via Σ. We say that Σ **determines itself on generic extensions** iff $\mathcal{M}_1^{\Sigma,\#}(\mathcal{P})$ exists, (Σ, C) is suitably condensing, and some (Φ, Ψ) generically determines $(\mathcal{F}_{\Sigma,\varphi_{\mathrm{all}}}, \Sigma, \varphi_{\mathrm{all}}, C, \mathcal{P})$. (Note then that the latter is suitable.) $\dashv$

Lemma 4.2. *Let $\mathcal{N}, \delta$, etc, be as in 4.1, except that we allow $\mathcal{T}$ to have uncountable length, and allow g to be in a set-generic extension of V. Then $(\mathcal{N}|\gamma)[g]$ is closed under $\mathcal{G}_{\mathcal{F}}$ and letting $\mathcal{G}'$ be the interpretation of Ψ over $(\mathcal{N}|\gamma)[g]$, $\mathcal{G}'\restriction C = \mathcal{G}_{\mathcal{F}}\restriction(\mathcal{N}|\gamma)[g]$.*

Proof. Suppose not. Let $x \in (\mathcal{N}|\gamma)[g]$ be a counterexample to the claimed agreement between $\mathcal{G}_{\mathcal{F}}, \mathcal{G}'$. So $x \in C \subseteq V$. Let $\mathbb{P}$ be some forcing, and $G \subseteq \mathbb{P}$ be V-generic, such that $g \in V[G]$. Let $\dot{g}$ be a $\mathbb{P}$-name for g. Let $\dot{x} \in \mathcal{N}|\gamma$ be such that $\dot{x}^g = x$. We may assume that $\mathbb{P}$ forces that $\dot{g}$ is $\check{\mathcal{N}}|\check{\gamma}$-generic and $\check{\dot{x}}^{\dot{g}} = \check{x}$. Let α be large and let $\pi : M \preccurlyeq V_\alpha$ with M countable and all relevant objects in $\mathrm{rg}(\pi)$. Write $\pi(\bar{\mathcal{T}}) = \mathcal{T}$, etc. Then $\bar{x} \in C$ and by 3.16, $\bar{\mathcal{T}}$ is via $\Sigma_{\mathfrak{M}}$. For any G^* which is $\bar{\mathbb{P}}$-generic over M, letting $g^* = \bar{\dot{g}}^{G^*}$, we then have $\bar{x} \in \overline{\mathcal{N}|\gamma}[g^*]$, and letting $\mathcal{G}^*$ be the interpretation of Ψ over $\overline{\mathcal{N}|\gamma}[g^*]$, by 4.1 we have

$$\mathcal{G}_{\mathcal{F}}(\bar{x}) = \mathcal{G}^*(\bar{x}) \in \overline{\mathcal{N}|\gamma}[g^*]. \tag{4.1}$$

So $x \in \mathrm{dom}(\mathcal{G}')$ (by the above, this is forced by $\mathbb{P}$), and so $\mathcal{G}'(x) \neq \mathcal{G}_{\mathcal{F}}(x)$, by choice of x. By suitability, $\mathcal{G}_{\mathcal{F}}(x)$ is determined by its theory t over parameters in $\hat{x}$, and $\mathcal{G}'(x)$ is determined by its theory t' in such parameters (the latter is forced). So let φ be some formula and $z \in \hat{x}^{<\omega}$

such that $\varphi(z) \in t$ but $\neg\varphi(z) \in t'$. Fixing a $\mathbb{P}$-name $\dot{t}'$ for t', we may assume that $z, \dot{t}' \in \operatorname{rg}(\pi)$ and that $\mathbb{P}$ forces that $\neg\varphi(\check{z}) \in \dot{t}'$. So with G^*, etc, as above, $\mathcal{G}^*(\bar{x}) \neq \overline{\mathcal{G}_{\mathcal{F}}(x)}$. Therefore by line (4.1), $\overline{\mathcal{G}_{\mathcal{F}}(x)} \neq \mathcal{G}_{\mathcal{F}}(\bar{x})$. This easily implies that we are in case (i) of suitability. Suppose for example that $x = (1, X)$ for some sound whole $\mathcal{F}$-premouse X. Because $\mathcal{F}$ condenses finely, $\overline{\mathcal{G}_{\mathcal{F}}(x)} \trianglelefteq \mathcal{G}_{\mathcal{F}}(\bar{x})$, and so by line (4.1), $\overline{\mathcal{G}_{\mathcal{F}}(x)} \triangleleft \mathcal{G}_{\mathcal{F}}(\bar{x}) = \mathcal{G}^*(\bar{x})$. So over M, $\bar{\mathbb{P}}$ forces that $\overline{\mathcal{G}_{\mathcal{F}}(x)} \triangleleft \mathcal{G}^*(\bar{x})$ and therefore that $\overline{\mathcal{G}_{\mathcal{F}}(x)}$ is sound. Therefore $\mathcal{G}_{\mathcal{F}}(x)$ is sound and $\mathbb{P}$ forces that $\mathcal{G}_{\mathcal{F}}(x) \triangleleft \mathcal{G}'(x)$. Therefore $\mathbb{P}$ forces that $\mathcal{G}'(x){\downarrow}a^x \vDash$"I have a proper segment $\mathcal{R}$ such that $\varphi_{\mathcal{F}}(\mathcal{R})$ and $x \in \mathcal{R}$". Reflecting this to M, $\mathcal{G}^*(\bar{x}) \neq \mathcal{G}_{\mathcal{F}}(\bar{x})$, contradiction. $\square$

In the sequel, we need the notions of *hod premice* and *hod pairs*, and related definitions; see [6].[28]

Definition 4.3. A (hod) premouse P is **reasonable** iff P is super-small and satisfies the first-order consequences of $(\omega, \omega_1, \omega_1 + 1)$-iterability.

A hod pair (Σ, P) is **within scope** iff Σ is fullness preserving (relative to some inductive-like, determined pointclass) and has branch condensation and hull condensation.[29] $\dashv$

For a premouse P, an important consequence of reasonableness is condensation; for a hod premouse, condensation in intervals of the form $[\delta, \gamma)$, where P has no Woodins in (δ, γ).

The following lemma, related to [7, §2], is due to Steel. However, the standard proof seems to have a gap (in the proof of Claim 4.6 below). A correct proof of what is essentially the lemma appeared in [12, §5], but that proof is somewhat buried in another context, so we give a proof here as service to the reader. We state the lemma only for pure $L[\mathbb{E}]$-constructions and mice, but the relativization to $L^{\mathcal{F}}[\mathbb{E}]$-constructions and $\mathcal{F}$-mice is routine.

Lemma 4.4 (Stationarity of $L[\mathbb{E}]$ constructions)**.** *Let γ be an uncountable cardinal. Let (P, Σ) and $\mathbb{C} = \langle N_\alpha \rangle_{\alpha \leq \gamma}$ be such that either (i) P is a k-sound premouse and Σ is a $(k, \gamma+1)$-strategy for P and $\mathbb{C}$ is a fully backgrounded $L[\mathbb{E}]$-construction; or (ii) (P, Σ) is a hod pair, is within scope, Σ is a $\gamma + 1$-strategy, and $\mathbb{C}$ is a hod pair construction (cf. [6]). Suppose that P is reasonable and* $\operatorname{card}(P) < \gamma$.

Suppose that for each active $N_{\alpha+1} = (N_\alpha, E)$, there is an extender E^* such that: *(a)* $\operatorname{card}(P) < \operatorname{crit}(E^*)$; *(b)* $F{\restriction}\nu(E) \subseteq E^*$; *(c) if P is non-tame then $i_{E^*}(\Sigma){\restriction}V_\eta \subseteq \Sigma$ where η is the sup of all $\delta + 1$ such that δ is Woodin in N_α.*

Then there is $\xi \leq \gamma + 1$ such that:

(1) for each $\alpha < \xi$, we have $N_\alpha \trianglelefteq P'$ for some Σ-iterate P' of P, and

[28] See footnote 5.

[29] For hod pairs up to *lsa*-type, branch condensation implies hull condensation.

(2) if $\xi \leq \gamma$ then there is a tree $\mathcal{T}$ via Σ, of successor length, $N_\xi = \mathcal{N}^{\mathcal{T}}$ and $b^{\mathcal{T}}$ does not drop in model.

Proof. It suffices to prove that if (1) holds at ξ, but (2) does not, then (1) holds at $\xi + 1$. This is easy in all cases except when $\xi = \alpha + 1$ and $N_{\alpha+1} = (N_\alpha, E)$ for some E, so suppose this is the case. Let E^* be a background extender for E and let $j = i_{E^*}$. Let $\mathcal{T}$ be the tree witnessing the lemma's conclusion for α. We assume that $\mathcal{T}$ has minimal possible length. We must show that E is used in $\mathcal{T}$. Let $\nu = \nu(E)$ and $\kappa = \text{crit}(E)$. The main point is the following claim:

Claim 4.5. *There is $\beta < \text{lh}(j(\mathcal{T}))$ such that $\nu \leq \nu(E^{\mathcal{T}}_\beta)$ and $E{\restriction}\nu \subseteq E^{\mathcal{T}}_\beta$.*

Proof. As in the proof that comparison of premice terminates, we have $M^{j(\mathcal{T})}_\kappa = M^{\mathcal{T}}_\kappa$ and $\kappa <_{j(\mathcal{T})} j(\kappa)$ and $i^{j(\mathcal{T})}_{\kappa,j(\kappa)}$ exists and

$$i^{\mathcal{T}}_{\kappa,j(\kappa)}{\restriction}M^{\mathcal{T}}_\kappa = j{\restriction}M^{\mathcal{T}}_\kappa. \tag{4.2}$$

So let $\beta + 1 <_{\mathcal{T}} j(\kappa)$ be such that $\text{pred}^{\mathcal{T}}(\beta + 1) = \kappa$. We claim that β works. For let

$$k : \text{Ult}(N_\alpha, E) \to j(N_\alpha)$$

be the factor embedding. Then $\text{crit}(k) \geq \nu(E)$, and if E is type 2 then $\text{crit}(k) \geq \text{lh}(E)$. So N_α, $M^{\mathcal{T}}_\kappa$, $M^{\mathcal{T}}_\beta$ and $M^{j(\mathcal{T})}_{j(\kappa)}$ agree below $(\kappa^+)^{N_\alpha}$. So $E^{\mathcal{T}}_\beta$ measures all sets measured by E and by line (4.2) we have that $E{\restriction}\nu' \subseteq E^{\mathcal{T}}_\beta{\restriction}\nu'$, where $\nu' = \min(\nu, \nu(E^{\mathcal{T}}_\beta))$. Now if $(\kappa^+)^{N_\alpha} < (\kappa^+)^{M^{\mathcal{T}}_\kappa}$ then $\text{crit}(k) = (\kappa^+)^{N_\alpha}$, so E is type 1 and $\nu = (\kappa^+)^{N_\alpha}$, so we are done. So assume $(\kappa^+)^{N_\alpha} = (\kappa^+)^{M^{\mathcal{T}}_\kappa}$, and assume $\nu' < \nu$. Since also $(\kappa^+)^{M^{\mathcal{T}}_\kappa} \leq \nu'$, the ISC applies to $E{\restriction}\nu'$. So $E{\restriction}\nu' \in N_\alpha$, although $E{\restriction}\nu' \notin j(N_\alpha)$. So E is not type 2. So E is type 3, but then $\text{lh}(E^{\mathcal{T}}_\beta) < \nu$, contradicting the fact that $N_\alpha||\nu = j(N_\alpha)||\nu$. □

Claim 4.6. *Either:*

- *E is on $\mathbb{E}_+(M^{\mathcal{T}}_\beta)$, or*
- *$M^{\mathcal{T}}_\beta|\nu(E)$ is active with extender F and E is on $\mathbb{E}_+(\text{Ult}(M^{\mathcal{T}}_\beta|\nu(E), F))$.*

Proof. If $(\kappa^+)^{N_\alpha} = (\kappa^+)^{M^{\mathcal{T}}_\beta}$ this is just by the ISC. So suppose $(\kappa^+)^{N_\alpha} < (\kappa^+)^{M^{\mathcal{T}}_\beta}$. Then E is type 1 and E is a submeasure of $E^{\mathcal{T}}_\beta$ and $M^{j(\mathcal{T})}_\beta||\nu(E) = N_\alpha||\nu(E)$. Thus, we can use [12, 4.11, 4.12, 4.15] (because P is reasonable). The only thing to check here is that if $M^{j(\mathcal{T})}_\beta|\nu$ is active with a type 3 extender F then

$$\text{Ult}(M^{j(\mathcal{T})}_\beta|\nu, F)||\text{lh}(E) = N_\alpha. \tag{4.3}$$

But this is true. For $\mathcal{T}\restriction(\kappa+1) = j(\mathcal{T})\restriction\kappa+1$, and note that $\mathcal{T}$ uses no extenders with index in the interval (κ,ν), and $j(\mathcal{T})$ uses no extender with index in the interval $(\kappa,(\kappa^+)^{M^{\mathcal{T}}_\kappa})$. So $M^{\mathcal{T}}_\kappa|\nu = M^{j(\mathcal{T})}_\beta|\nu$ is active, but since $N_\alpha|\nu$ is passive, we have $E^{\mathcal{T}}_\kappa = F$. But then $\mathcal{T}$ uses no extender with index in the interval $(\nu, \mathrm{lh}(E))$, and so line (4.3) is true. □

Now let λ be least such that $\mathrm{lh}(E^{j(\mathcal{T})}_\lambda) \geq \mathrm{lh}(E)$, and let ξ be the largest limit ordinal such that $\xi \leq \lambda$. By the following claim, we clearly have that $j(\mathcal{T})\restriction\lambda+1$ is via Σ, which completes the proof.

Claim 4.7. $j(\mathcal{T})\restriction\xi+1 = \mathcal{T}\restriction\xi+1$.

Proof. We have $N_\alpha = \mathcal{N}^{\mathcal{T}}$ and $j(N_\alpha) = \mathcal{N}^{j(\mathcal{T})}$. Let χ be the largest cardinal of N_α. Then letting ϵ be the largest limit cardinal of $j(N_\alpha)||\mathrm{lh}(E)$, we have $\epsilon \leq \chi$ and $N_\alpha||(\epsilon^+)^{N_\alpha} = j(N_\alpha)||(\epsilon^+)^{N_\alpha}$. (Though possibly $(\epsilon^+)^{N_\alpha} < (\epsilon^+)^{j(N_\alpha)}$.) Also $\lfloor N_\alpha\rfloor \subseteq j(N_\alpha)$. These things follow from condensation, considering the factor embedding k. Now let $\delta = \delta(j(\mathcal{T})\restriction\xi)$; it follows that $\delta \leq \epsilon$. So $N_\alpha|\delta = j(N_\alpha)|\delta$, and it suffices to see that for each $\xi' \leq \xi$, we have $[0,\xi']_{j(\mathcal{T})} = [0,\xi]_{\mathcal{T}}$. We prove this by induction on ξ'. So assume $\mathcal{T}\restriction\xi' = j(\mathcal{T})\restriction\xi'$. We may assume $\xi' \geq \kappa$, so $\delta' = \delta(\mathcal{T}\restriction\xi') \geq \kappa$ also. Now if $N_\alpha \vDash$"δ' is not Woodin" then let $Q \triangleleft M^{\mathcal{T}}_{\xi'}$ be the Q-structure for δ'. Then $Q \triangleleft N_\alpha$, so $Q \triangleleft j(N_\alpha)$, so $Q \triangleleft M^{j(\mathcal{T})}_{\xi'}$. Therefore $[0,\xi']_{\mathcal{T}} = [0,\xi']_{j(\mathcal{T})}$, as required. So suppose $N_\alpha \vDash$"δ' is Woodin". Since $\kappa \leq \delta' < \mathrm{lh}(E)$, and so by Claim 4.6, P is non-tame. So by our hypothesis, $j(\Sigma)\restriction V_{\delta'+1} \subseteq \Sigma$. Therefore $[0,\xi']_{j(\mathcal{T})} = [0,\xi']_{\mathcal{T}}$ again. □

The next lemma is similar to a result of Sargsyan (cf. [6, Lemma 3.35]).

Lemma 4.8. *Let (P,Σ) be such that P is a countable reasonable (hod) premouse and either (i) P is a premouse and Σ is the unique normal* Ord*-iteration strategy for P; or (ii) (P,Σ) is a hod pair, within scope. Suppose that $\mathcal{M}^{\Sigma,\#}_1(P)$ exists. Then Σ determines itself on generic extensions.*

Proof. We describe a process by which $\mathcal{N}[g]$ can compute $\Sigma\restriction\mathcal{N}[g]$ whenever $\mathcal{N}$ is a correct iterate of $\mathfrak{N} = \mathcal{M}^{\Sigma}_1(P)$. The theorem will then be a straightforward corollary. Let $\mathcal{N}$ be such an iterate of $\mathfrak{N}$ and let $\delta = \delta^{\mathcal{N}}$. Let Λ be the iteration strategy for $\mathcal{N}$.

Consider case (a). Let $\mathbb{C} = \langle N_\alpha\rangle_{\alpha\leq\delta}$ be the maximal $L[\mathbb{E}]$-construction of $\mathcal{N}|\delta$, where background extenders are required to be in $\mathbb{E}^{\mathcal{N}}$. Note that the hypotheses of 4.4 hold in $\mathcal{N}$ with respect to $P, \delta, \Sigma\restriction\mathcal{N}, \mathbb{C}$.

There is $\alpha < \delta$ such that clause (ii) of 4.4 attains. For in $\mathcal{N}$, δ is Woodin, and P is super-small, so we can apply the universality of N_δ (see [19, Lemma 11.1]). Note that $\alpha < \kappa$ where κ is the least strong of $\mathcal{N}$. Fix a successor cardinal cutpoint θ of $\mathcal{N}$ such that $\alpha < \theta < \kappa$. Then via copying/resurrection, both N_α and P are iterable in V via lifting to nowhere-dropping iteration trees on $\mathcal{N}$ based on $\mathcal{N}|\theta$. Let Σ_P be the resulting strategy for P. By the uniqueness of Σ we have $\Sigma_P = \Sigma$.

In case (b), we proceed similarly, but form the hod pair construction $\mathbb{C}$ inside $\mathcal{N}$, instead of the $L[\mathbb{E}]$-construction. As in [6, 2.2.2] and with notation as there, we have $\alpha < \delta$ and a tree $\mathcal{T}$ via Σ with last model $\mathcal{R}$ such that $b^{\mathcal{T}}$ does not drop, $\mathcal{R} = \mathcal{R}_\alpha$ and $\Sigma_\alpha = \Sigma_{\mathcal{R},\mathcal{T}}$. But by branch condensation and the uniqueness of choices of dropping branches, Σ has pullback consistency. So again letting Σ_P be the pullback strategy, we have $\Sigma_P = \Sigma$.

So it suffices to see that $\Lambda{\restriction}X$ is sufficiently definable over $\mathcal{N}[g]$, where X is the class of trees $\mathcal{T} \in \mathcal{N}[g]$ such that $\mathcal{T}$ is based on $\mathcal{N}|\theta$ and is nowhere-dropping. Iterating $\mathcal{N}$ for $\mathcal{N}|\theta$-based trees just requires computing the correct Q-structures, which requires sufficient ordinals and knowledge of Σ. But we don't yet know that $\Sigma``\mathcal{N}[g] \subseteq \mathcal{N}[g]$. We will compute the Q-structures indirectly, by such trees $\mathcal{T}$ to trees in $\mathcal{N}$.

Let $\mathbb{P} \in \mathcal{N}$ be a partial order and let $\dot{\mathcal{T}} \in \mathcal{N}$ be a $\mathbb{P}$-name such that $\mathbb{P}$ forces that $\dot{\mathcal{T}}$ is a nowhere dropping, $\mathcal{N}|\theta$-based tree on $\mathcal{N}$, of limit length, via the strategy to be described; it will follow that $\dot{\mathcal{T}}^g$ is a correct tree on $\mathcal{N}$.

Claim 4.9. *Let g be $\mathbb{P}$-generic over $\mathcal{N}$. Let $Q = Q(\dot{\mathcal{T}}^g)$. Then $Q \in \mathcal{N}[g]$.*

In fact, let λ be the maximum of δ, $(\mathrm{lh}(\dot{\mathcal{T}}^g)^{++})^{\mathcal{N}[g]}$, and $(\mathrm{card}(\mathbb{P})^{++})^{\mathcal{N}}$. Then there is a short tree $\mathcal{V} \in \mathcal{N}|\lambda$, $\mathcal{V}$ on $\mathcal{N}$, according to Λ, of successor length, such that for some $\alpha \leq \mathrm{o}(\mathcal{N}^{\mathcal{V}})$, if G is $\mathrm{Col}(\omega, \lambda)$ generic over $\mathcal{N}[g]$, then in $\mathcal{N}[g][G]$, there is a $\mathcal{P}$-strategy-premouse Q which is a Q-structure for $\mathcal{M}(\dot{\mathcal{T}}^g)$, and a Σ_1-elementary embedding $\pi : Q \to \mathcal{N}^{\mathcal{V}}|\alpha$. So Q is unique with these properties and $Q(\dot{\mathcal{T}}^g) = Q \in \mathcal{N}[g]$.

Proof. Suppose not. Let $p \in \mathbb{P}$ force the failure. We may assume $p = 1_{\mathbb{P}}$. In $\mathcal{N}$, we first form a Boolean valued comparison of $M(\dot{\mathcal{T}})$ with $\mathcal{N}$, forming a $\mathbb{P}$-name for a tree $\dot{\mathcal{U}}$ on $M(\dot{\mathcal{T}})$ and a tree $\mathcal{V}$ on $\mathcal{N}$. Since $\mathcal{N}$ is a proper class Σ-premouse, it correctly computes Q-structures as far as they exist during this comparison. Suppose we have a limit stage $(\mathcal{V},\dot{\mathcal{U}}){\restriction}\lambda$ of this comparison. If a condition q forces that $\dot{\mathcal{U}}{\restriction}\lambda$ is eventually only padding then below q, nothing need be done for $\dot{\mathcal{U}}$ at stage λ. Now suppose q forces otherwise. Suppose $p \leq q$ forces that here is a cofinal branch b of $\dot{\mathcal{U}}$ such that $Q(M(\mathcal{V}{\restriction}\lambda)) \trianglelefteq M_b^{\dot{\mathcal{U}}}$. Then below p, we set $[0, \lambda]_{\dot{\mathcal{U}}} = b$. If $p \leq q$ forces otherwise, then below p, we declare that $\dot{\mathcal{U}}$ is *uncontinuable*, and terminate the comparison. (In the latter case p forces that $\dot{\mathcal{U}}$ has limit length; we deal with this later.) For each stage α of the comparison, let lh_α be the index of any extender (forced by some p to be) used at that stage. For limit λ, let $M((\mathcal{V},\dot{\mathcal{U}}){\restriction}\lambda)$ be the lined up part of that stage, of height $\sup_{\alpha<\lambda} \mathrm{lh}_\alpha$.

Subclaim 4.10. *We have:*

(a) $\mathcal{V}$ is based on $\mathcal{N}|\theta$;

(*b*) *if* α *is such that* $[0,\alpha]_{\mathcal{V}}$ *does not drop and* $\mathbb{P}$ *forces that* $M^{\dot{\mathcal{U}}}_\alpha|\theta' = M^{\mathcal{V}}_\alpha|\theta'$, *where* $\theta' = i^{\mathcal{V}}_{0,\alpha}(\theta)$, *then the comparison terminates at stage* α, *and in fact,* $\mathbb{P}$ *forces that* $M^{\dot{\mathcal{U}}}_\alpha \trianglelefteq M^{\mathcal{V}}_\alpha|\theta'$;

(*c*) *at every limit stage* λ, *a Q-structure for* $M((\mathcal{V},\dot{\mathcal{U}})\restriction\lambda)$ *exists;*

(*d*) *the comparison terminates* (*i.e. there is* α *such that* $\mathbb{P}$ *forces that either* $\dot{\mathcal{U}}$ *is uncontinuable, or* $M^{\mathcal{V}}_\alpha \trianglelefteq M^{\dot{\mathcal{U}}}_\alpha$, *or* $M^{\dot{\mathcal{U}}}_\alpha \trianglelefteq M^{\mathcal{V}}_\alpha$);

(*e*) *there is* $p \in \mathbb{P}$ *forcing that if* $\dot{\mathcal{U}}$ *has a final model, then* $\mathcal{N}^{\dot{\mathcal{U}}} \triangleleft \mathcal{N}^{\mathcal{V}}$.

Proof. Part (b) implies (a) and (c). Suppose (b) fails. Let α be the least failure, and let p be a condition forcing this failure. Let $g \subseteq \mathbb{P}$ be generic with $p \in g$. Let $\mathcal{T}'$ be the tree on $\mathcal{N}$ which uses the same extenders as does $\mathcal{T} = \dot{\mathcal{T}}^g$, and let $W_0 = \mathcal{N}^{\mathcal{T}'}$. So W_0 is proper class (as $\mathcal{T}$ was nowhere dropping). Let $\mathcal{U}'$ be the tree on W_0 using the same extenders as $\mathcal{U}^g$. Let $W = M^{\mathcal{U}'}_\alpha$. So $\theta' < \mathrm{o}(W)$. We can compare $(M^{\mathcal{V}}_\alpha, W)$, producing trees $(\mathcal{T}_1,\mathcal{T}_2)$. The comparison begins above θ', a cardinal of $M^{\mathcal{V}}_\alpha$. Suppose $b^{\mathcal{U}'}$ drops. So $\rho_\omega(W) < \theta'$. Also then, $b^{\mathcal{T}_1}$ drops, whereas $b^{\mathcal{T}_2}$ does not, and $\mathcal{T}_1,\mathcal{T}_2$ have the same last model. But the last model Z of $\mathcal{T}_1$ has $\rho_\omega(Z) \geq \theta'$, contradiction. So $b^{\mathcal{U}'}$ does not drop, and so neither do $b^{\mathcal{T}_1}, b^{\mathcal{T}_2}$, and $j = k$ where $j = i^{\mathcal{V} \frown \mathcal{T}_1}$ and $k = i^{\mathcal{T}' \frown \mathcal{U}' \frown \mathcal{T}_2}$. But $j(\theta) = \theta'$ and $k(\theta) > \theta'$, contradiction. This gives (b).

The usual proof that boolean-valued comparisons terminate gives (d).

So if (e) fails, then $b^{\mathcal{V}}$ drops, so $\mathcal{N}^{\mathcal{V}}$ is unsound, and $\mathbb{P}$ forces that $\mathcal{N}^{\dot{\mathcal{U}}} = \mathcal{N}^{\mathcal{V}}$. But then again the usual methods yield a contradiction. □

Now let p be as in part (e), and let $g \subseteq \mathbb{P}$ be $\mathcal{N}$-generic, with $p \in g$. Let $\mathcal{T} = \dot{\mathcal{T}}^g$ and $\mathcal{U} = \dot{\mathcal{U}}^g$. Let $Q = Q(M(\mathcal{T}))$. Let $W_0,\mathcal{U}'$ be as before, and let $\mathcal{U}_Q$ be the 0-maximal tree on Q given by $\mathcal{U}$ (with the same extenders and branches).

Suppose that $\mathcal{U}$ has a last model R. So we have $R \triangleleft \mathcal{N}^{\mathcal{V}}$ and $b^{\mathcal{U}}$ does not drop,and so neither do $b^{\mathcal{U}'}$ or $b^{\mathcal{U}_Q}$. Let $\pi : \mathcal{N}^{\mathcal{U}_Q} \to i^{\mathcal{U}'}(Q)$ be the factor map. Then π is a weak 0-embedding. So by 3.10, $\mathcal{N}^{\mathcal{U}_Q}$ is a Σ-premouse. Also, $i^{\mathcal{U}_Q} : Q \to \mathcal{N}^{\mathcal{U}_Q}$ is continuous at $\delta = \delta(\dot{\mathcal{T}}^g)$, and $\mathcal{N}^{\mathcal{U}_Q}$ has no E-active levels above $i^{\mathcal{U}_Q}(\delta) = \rho_\omega(\mathcal{N}^{\mathcal{U}_Q})$. It follows that $\mathcal{N}^{\mathcal{U}_Q} \trianglelefteq \mathcal{N}^{\mathcal{V}}$. Also, $i^{\mathcal{U}_Q}$ is Σ_1-elementary. So Q, $\mathcal{V}$, $\mathcal{N}^{\mathcal{U}_Q}$ and $i^{\mathcal{U}_Q}$ witness the truth of the claim, a contradiction.[30]

Suppose now that $\dot{\mathcal{U}}^g$ is uncontinuable, so has limit length. Let $b = \Lambda(\dot{\mathcal{U}}^g)$. It follows that b does not drop, and with $\mathcal{U}'$ as above, $i^{\mathcal{U}'}(\delta) = \delta(\dot{\mathcal{U}}^g)$. We have $M(\mathcal{U}) \triangleleft \mathcal{N}^{\mathcal{V}}$, since

[30]Ostensibly $\mathcal{N}^{\mathcal{U}_Q}$ might be a strict segment of the Q-structure for $\mathcal{N}^{\mathcal{V}}|i^{\mathcal{U}_Q}(\delta)$, but this is not relevant. If one chooses $n < \omega$ appropriately, and takes $\mathcal{U}_Q$ to be n-maximal instead of 0-maximal, then one can arrange that $\mathcal{N}^{\mathcal{U}_Q}$ is the Q-structure.

$M(\mathcal{U})$ has no largest cardinal and is sound. Therefore $i^{\mathcal{U}'}(Q) \trianglelefteq \mathcal{N}^{\mathcal{V}}$, which again gives a contradiction. $\square$

This completes the proof that $\mathcal{N}[g]$ computes $\Sigma\restriction\mathcal{N}[g]$. Now let Φ be the formula "There is no largest cardinal, there is a Woodin cardinal δ, $\mathcal{P}$ is absorbed by the $L[\mathbb{E}]$-construction (or hod pair construction) at some stage $< \delta$, and every partial order $\mathbb{P}$ forces that the process described above always succeeds". Let Ψ be the formula defining $\Sigma\restriction\mathcal{N}[g]$ through the above process. Note that if $\mathcal{N}' \trianglelefteq \mathcal{N}$ and $\mathcal{N}' \models \Phi$ and g is set generic over $\mathcal{N}'$, then $\mathcal{N}'[g]$ is indeed closed under Σ, and $\Sigma\restriction\mathcal{N}'[g]$ is defined over $\mathcal{N}'[g]$ by Ψ. So (Φ, Ψ) generically determines Σ, as required. (We don't actually need that the Woodin of $\mathcal{N}'$ is a cardinal of $\mathcal{N}$.) $\square$

Remark 4.11. In the above lemma, we can replace the Ord-iterability of $\mathcal{M}_1^{\Sigma}$ by $\kappa^+ + 1$-iterability. In this case, by $\mathcal{M}_1^{\Sigma}$, we mean $\mathcal{M}|\kappa^+$, where $\mathcal{M}$ is the $(\kappa^+)^{\text{th}}$ iterate of $\mathcal{M}_1^{\Sigma,\sharp}$ via its top extender.

Notation 4.12. Let $\mathcal{F}$ be a nice operator (see 4.1) over $\mathscr{B}$. Let $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\sharp}$ and let $\Lambda_{\mathfrak{M}}$ be the $(0, \mathrm{o}(\mathscr{B})+1)$-strategy for $\mathfrak{M}$. Let (Φ, Ψ) be a pair that generically determines $\mathcal{F}$. These objects are fixed for the remainder of this section.

In order to define g-organization, we need the following notion due to Sargsyan:

Definition 4.13 (Sargsyan, [6])**.** Let M be a transitive structure. Let $\dot{G}$ be the name for the generic $G \subseteq \mathrm{Col}(\omega, M)$ and let $\dot{x}_{\dot{G}}$ be the canonical name for the real coding $\{(n,m) \mid G(n) \in G(m)\}$, where we identify G with $\bigcup G$. The **tree $\mathcal{T}_M$ for making M generically generic**, is the iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of maximal length such that:

1. $\mathcal{T}$ is via $\Lambda_{\mathfrak{M}}$ and is everywhere non-dropping.

2. $\mathcal{T}\restriction\mathrm{o}(M)+1$ is the tree given by linearly iterating the first total measure of $\mathfrak{M}$ and its images.

3. Suppose $\mathrm{lh}(\mathcal{T}) \geq \mathrm{o}(M)+2$ and let $\alpha+1 \in (\mathrm{o}(M), \mathrm{lh}(\mathcal{T}))$. Let $\delta = \delta(\mathcal{M}^{\mathcal{T}}_{\alpha})$ and let $\mathbb{B} = \mathbb{B}(M^{\mathcal{T}}_{\alpha})$ be the extender algebra of $M^{\mathcal{T}}_{\alpha}$ at δ. Then $E^{\mathcal{T}}_{\alpha}$ is the extender E with least index in $M^{\mathcal{T}}_{\alpha}$ such that for some condition $p \in \mathrm{Col}(\omega, M)$, $p \Vdash$"There is a $\mathbb{B}$-axiom induced by E which fails for $\dot{x}_{\dot{G}}$".

Assuming that $\mathfrak{M}$ is sufficiently iterable, then $\mathcal{T}_M$ exists and has successor length. $\dashv$

Definition 4.14. Given a successor length, nowhere dropping tree $\mathcal{T}$ on $\mathfrak{M}$, let $P^{\Phi}(\mathcal{T})$ be the least $P \trianglelefteq \mathcal{N}^{\mathcal{T}}$ such that for some cardinal δ' of $\mathcal{N}^{\mathcal{T}}$, we have $\delta' < \mathrm{o}(P)$ and $P \models \Phi+$"δ' is Woodin". Let $\lambda = \lambda^{\Phi}(\mathcal{T})$ be least such that $P^{\Phi}(\mathcal{T}) \trianglelefteq M^{\mathcal{T}}_{\lambda}$. Then δ' is a cardinal of $M^{\mathcal{T}}_{\lambda}$. Let $I^{\Phi} = I^{\Phi}(\mathcal{T})$ be the set of limit ordinals $\leq \lambda$. $\dashv$

Sargsyan is responsible for the main point of the following definition, the central notion of this section (cf. [6, Definition 3.37]). He noticed that one can feed $\mathcal{F}$ into a structure $\mathcal{N}$ indirectly, by feeding in the branches for $\mathcal{T}_{\mathcal{M}}$, for various $\mathcal{M} \trianglelefteq \mathcal{N}$. The operator ${}^{\mathrm{g}}\mathcal{F}$, defined below, and used in building g-organized $\mathcal{F}$-premice, uses this idea. We will also ensure that being such a structure is first-order - other than wellfoundedness and the correctness of the branches - by allowing sufficient spacing between these branches.

Definition 4.15 (${}^{\mathrm{g}}\mathcal{F}$)**.** We define the forgetful operator ${}^{\mathrm{g}}\mathcal{F}$, over $\mathscr{B}$. Let b be a transitive structure with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$. [31] We define $\mathcal{M} = {}^{\mathrm{g}}\mathcal{F}(b)$, a $\mathcal{J}$-model over b, with parameter $\mathfrak{M}$, as follows.

For each $\alpha \leq l(\mathcal{M})$, $E^{\mathcal{M}|\alpha} = \emptyset$.

Let α_0 be the least α such that $\mathcal{J}_\alpha(b) \vDash \mathsf{ZF}$. Then $\mathcal{M}|\alpha_0 = \mathcal{J}^{\mathsf{m}}_{\alpha_0}(b; \mathfrak{M})$.

Let $\mathcal{T} = \mathcal{T}_{\mathcal{M}|\alpha_0}$. We use the notation $P^\Phi = P^\Phi(\mathcal{T})$, $\lambda = \lambda^\Phi(\mathcal{T})$, etc, as in 4.14. The predicates $B^{\mathcal{M}|\gamma}$ for $\alpha_0 < \gamma \leq l(\mathcal{M})$ will be used to feed in branches for $\mathcal{T}\restriction\lambda + 1$, and therefore P^Φ itself, into $\mathcal{M}$. Let $\langle \xi_\alpha \rangle_{\alpha < \iota}$ enumerate $I^\Phi \cup \{0\}$.

There is a closed, increasing sequence of ordinals $\langle \eta_\alpha \rangle_{\alpha \leq \iota}$ and an increasing sequence of ordinals $\langle \gamma_\alpha \rangle_{\alpha \leq \iota}$ such that:

1. $\eta_1 = \gamma_0 = \eta_0 = \alpha_0$.

2. For each $\alpha < \iota$, $\eta_\alpha \leq \gamma_\alpha \leq \eta_{\alpha+1}$, and if $\alpha > 0$ then $\gamma_\alpha < \eta_{\alpha+1}$.

3. $\gamma_\iota = l(\mathcal{M})$, so $\mathcal{M} = \mathcal{M}|\gamma_\iota$.

4. Let $\alpha \in (0, \iota)$. Then γ_α is the least ordinal of the form $\eta_\alpha + \tau$ such that $\mathcal{T}\restriction\xi_\alpha \in \mathcal{J}_\tau(\mathcal{M}|\eta_\alpha)$ and if $\alpha > \alpha_0$ then $\delta(\mathcal{T}\restriction\xi_\alpha) < \tau$. (We explain below why such τ exists.) And $\mathcal{M}|\gamma_\alpha = \mathcal{J}^{\mathsf{m}}_\tau(\mathcal{M}|\eta_\alpha; \mathfrak{M})\downarrow b$.

[31] G-organized premice identify $\mathfrak{M}$ explicitly. For our intended application, i.e. the analysis of scales in $\mathrm{Lp}^{{}^{\mathrm{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F}\restriction\mathrm{HC})$, this is not of importance, because anyway, $\mathcal{M}_1^{\mathcal{F},\#}$ is analytical in $\mathcal{F}\restriction\mathrm{HC}$. However, it seems that one might want to consider a hierarchy of premice $\mathcal{M}$ over $\mathbb{R}$, similar to $\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}}(\mathbb{R})$, but in which $\mathfrak{M}$ is not identified explicitly. It seems we might have achieved this by, in some initial segment of $\mathcal{M}$, feeding in $\mathcal{F}(X)$ for enough sets $X \in \mathrm{HOD}^{\mathcal{M}}$, until $\mathfrak{M}$ can be identified, as in the following sketch. Suppose we have defined $\mathcal{M}|\alpha$; let $\tilde{\mathcal{F}} = \mathcal{F}^{\mathcal{M}|\alpha}$ be the partial operator which is computed naturally from the fragment of $\mathcal{F}$ already fed in to $\mathcal{M}|\alpha$. Working in $\mathcal{M}|\alpha$, let $\tilde{Q}$ be the function defined as follows. Let H be a transitive set. Suppose there is $\gamma \in \mathrm{Ord}$ such that $Q = \mathcal{J}^{\tilde{\mathcal{F}}}_\gamma(H)$ is defined (i.e., $\tilde{\mathcal{F}}$ computes this), and Q is a Q-structure for H. Then set $\tilde{Q}(H) = Q$. Otherwise $\tilde{Q}(H)$ is undefined. Over $\mathcal{M}|\alpha$, consider the set $\mathfrak{M}^{\mathcal{M}_\alpha}$ of countable $\mathcal{M}_1^{\mathcal{F},\#}$*-like* $\mathcal{J}$-models $\mathcal{N}$ which are $\tilde{\mathcal{F}}$*-consistently* $\tilde{Q}$*-short tree iterable*; we omit any precise definitions of these notions. Then $\mathfrak{M} \in \mathfrak{M}^{\mathcal{M}|\alpha}$, and $\tilde{Q}$-guided trees on $\mathfrak{M}$ will be via $\Lambda_{\mathfrak{M}}$. Over $\mathcal{M}|\beta$ for $\beta \geq \alpha$, attempt to compare all such $\mathcal{N}$, and simultaneously iterate to make $\mathcal{M}|\alpha$ generically generic. If at some stage the least disagreement, between say $\mathcal{N}_1$ and $\mathcal{N}_2$, is due to the fact that say $\mathcal{F}^{\mathcal{N}_1}(x) \neq \mathcal{F}^{\mathcal{N}_2}(x)$, then we can feed in $\mathcal{F}(x)$ over some later $\mathcal{M}|\gamma$. Then if $\mathcal{N}_i$ is an iterate of $\mathcal{M}_i$, we will have $\{\mathcal{M}_1, \mathcal{M}_2\} \not\subseteq \mathfrak{M}^{\mathcal{M}|\gamma}$, and we start over with γ replacing α. If we reach a $\tilde{Q}$-maximal stage of the comparison, which is in fact not maximal (for $\mathfrak{M}$) then we can feed in the corresponding Q-structure. This process will eventually produce an iterate of $\mathfrak{M}$ over which $\mathbb{R}$ is generic, and therefore, over which $\mathcal{F}\restriction\mathrm{HC}$ and $\mathfrak{M}$ are definable.

5. Let $\alpha \in (0, \iota)$. Then $\mathcal{M}|\eta_{\alpha+1} = \mathfrak{B}(\mathcal{M}|\gamma_\alpha, \mathcal{T}\restriction\xi_\alpha, \Lambda(\mathcal{T}\restriction\xi_\alpha))\downarrow b$.

6. Let $\alpha < \iota$ be a limit. Then $\mathcal{M}|\eta_\alpha$ is passive.

7. γ_ι is the least ordinal of the form $\eta_\iota + \tau$ such that $\mathcal{T}\restriction\lambda+1 \in \mathcal{J}_{\eta_\iota+\tau}(\mathcal{M}|\eta_\iota)$ and $\tau > \mathrm{o}(M^{\mathcal{T}}_\lambda)$; with this τ, $\mathcal{M} = \mathcal{J}^{\mathsf{m}}_\tau(\mathcal{M}|\eta_\iota; \mathfrak{M})\downarrow b$. ⊣

Remark 4.16. We have $P^\Phi \lhd M^{\mathcal{T}}_\lambda \in \mathcal{M} = {}^{\mathrm{g}}\mathcal{F}(b)$. In fact, $\{P^\Phi\}$ is $\Sigma_1^{\mathcal{M}}$, in $\mathcal{L}_0^-$, uniformly in b. We leave the proof of this to the reader, but just note that this uses the fact that the relevant part of the $\mathrm{Col}(\omega, \mathcal{M}|\alpha_0)$ forcing relation for ${}^{\mathrm{g}}\mathcal{F}(b)$ is sufficiently locally definable. For given $p \in \mathrm{Col}(\omega, \mathcal{M}|\alpha_0)$, and $\alpha \leq \lambda$, and an extender $E \in \mathbb{E}(M^{\mathcal{T}}_\alpha)$ such that $\nu(E)$ is inaccessible in $M^{\mathcal{T}}_\alpha$, the question of whether $p \Vdash$ "E induces an extender algebra axiom not satisfied by $\dot{x}_{\dot{G}}$" is computed over $\mathcal{M}|(\eta_\iota + \nu(E))$. (Such an axiom has form

$$\bigvee_{\gamma < \mathrm{crit}(E)} \varphi_\gamma \iff \bigvee_{\gamma < \nu(E)} \varphi_\gamma,$$

where for each $\gamma < \nu(E)$, $\varphi_\gamma \in M^{\mathcal{T}}_\alpha|\nu(E)$, so the forcing relation below p regarding the truth of φ_γ is computed somewhere below $\mathcal{M}|(\eta_\iota + \nu(E))$.)

Likewise, in item 4 of 4.15, τ exists. Also, for $\bar{\mathcal{M}} \trianglelefteq \mathcal{M} = {}^{\mathrm{g}}\mathcal{F}(b)$, the sequences $\langle \mathcal{M}|\eta_\alpha \rangle_{\alpha \leq \iota} \cap \bar{\mathcal{M}}$ and $\langle \mathcal{M}|\gamma_\alpha \rangle_{\alpha \leq \iota} \cap \bar{\mathcal{M}}$ and $\langle \mathcal{T}\restriction\alpha \rangle_{\alpha \leq \lambda+1} \cap \bar{\mathcal{M}}$ are $\Sigma_1^{\bar{\mathcal{M}}}$ in $\mathcal{L}_0^-$, uniformly in b and $\bar{\mathcal{M}}$.

To see that ${}^{\mathrm{g}}\mathcal{F}(b)$ is acceptable, it suffices to see that every initial segment of ${}^{\mathrm{g}}\mathcal{F}(b)$ is sound. By the above remarks, there is a formula φ of $\mathcal{L}_0$, and a Σ_1 formula ψ of $\mathcal{L}_0^-$, such that ${}^{\mathrm{g}}\mathcal{F}(b) \models \neg\psi$, and for any $\mathcal{J}$-structure $\mathcal{N}$, $\mathcal{N}$ is an acceptable initial segment of ${}^{\mathrm{g}}\mathcal{F}(b)$ iff $\mathcal{N}$ is a $\Lambda_{\mathfrak{M}}$-premouse of type φ and $\mathcal{N} \models \neg\psi$. (Here ψ asserts that "some proper segment has the form of ${}^{\mathrm{g}}\mathcal{F}(b)$".) But therefore if $\mathcal{N}$ is such, then $\mathcal{N}$ is sound, by 3.16 and 3.10 and the proof that initial segments of L are sound.

Definition 4.17. Let b be transitive with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$. A **potential g-organized $\mathcal{F}$-premouse over** b is a potential ${}^{\mathrm{g}}\mathcal{F}$-premouse over b, with parameter $\mathfrak{M}$. ⊣

Note that because we only feed in branches for non-maximal trees on $\mathfrak{M}$, the only non-extender information being fed into a g-organized $\mathcal{F}$-premouse can be computed by $\mathcal{F}$-construction. The following lemma is an easy corollary to 4.16.

Lemma 4.18. *There is a formula φ_{g} in $\mathcal{L}_0$, such that for any transitive b with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$, and any $\mathcal{J}$-structure $\mathcal{M}$ over b, $\mathcal{M}$ is a potential g-organized $\mathcal{F}$-premouse over b iff $\mathcal{M}$ is a potential $\Lambda_{\mathfrak{M}}$-premouse over b, of type φ_{g}.*

Lemma 4.19. *${}^{\mathrm{g}}\mathcal{F}$ is basic and condenses finely. Moreover, the class of g-organized $\mathcal{F}$-premice is very condensing.*

Proof. The "moreover" clause follows 3.13, and implies that ${}^{\mathrm{g}}\mathcal{F}$ condenses finely (since it is clear that ${}^{\mathrm{g}}\mathcal{F}$ is projecting and uniformly Σ_1). □

Definition 4.20. Let $\mathcal{M}$ be a g-organized $\mathcal{F}$-premouse over b. We say $\mathcal{M}$ is **$\mathcal{F}$-closed** iff $\mathcal{M}$ is a limit of ${}^{\mathrm{g}}\mathcal{F}$-whole proper segments. ⊣

As in [6], the main point of g-organization is the following. Because $\mathcal{F}$ determines itself on generic extensions, $\mathcal{F}$-closure ensures closure under $\mathcal{G}_{\mathcal{F}}$:

Lemma 4.21. *Let $\mathcal{M}$ be an $\mathcal{F}$-closed g-organized $\mathcal{F}$-premouse over b. Then $\mathcal{M}$ is closed under $\mathcal{G}_{\mathcal{F}}$. In fact, for any set generic extension $\mathcal{M}[g]$ of $\mathcal{M}$, with $g \in V$, $\mathcal{M}[g]$ is closed under $\mathcal{G}_{\mathcal{F}}$ and $\mathcal{G}_{\mathcal{F}}\restriction\mathcal{M}[g]$ is definable over $\mathcal{M}[g]$, via a formula in $\mathcal{L}_0^-$, uniformly in $\mathcal{M}, g$.*

Proof sketch. We show that $\mathcal{M}$ is closed under $\mathcal{G}_{\mathcal{F}}$; the generalization to generic extensions of $\mathcal{M}$ and the definability of $\mathcal{G}_{\mathcal{F}}$ is similar.[32]

Let $z \in \lfloor\mathcal{M}\rfloor$; we want to see that $\mathcal{G}_{\mathcal{F}}(z) \in \lfloor\mathcal{M}\rfloor$. Let $\kappa < l(\mathcal{M})$ be such that $z \in \mathcal{M}|\kappa$ and $\mathcal{M}|\kappa$ is ${}^{\mathrm{g}}\mathcal{F}$-whole. Let $R = {}^{\mathrm{g}}\mathcal{F}(\mathcal{M}|\kappa)$, so $R \trianglelefteq \mathcal{M}$. Let α_0 be the least $\alpha > \kappa$ such that $R|\alpha \vDash \mathsf{ZF}^-$. Let $P^\Phi = P^\Phi(\mathcal{T}_{R|\alpha_0})$. Let $\mathbb{P} = \mathrm{Col}(\omega, R|\alpha_0)$. Let $\dot{x}$ be the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic real coding $R|\alpha_0$. Let $\dot{z}$ be the canonical $\mathbb{P}$-name for z. Now $R \vDash$"$\mathbb{P}$ forces that $\dot{x}$ is extender algebra generic over P^Φ". Let t be the theory of $\mathcal{G}_{\mathcal{F}}(z)$, in parameters in $\hat{z}^{<\omega}$. Then for all $\vec{w} \in \hat{z}^{<\omega}$ and formulas φ, $\varphi(\vec{w}) \in t$ iff, letting $\dot{\vec{w}}$ be the canonical $\mathbb{P}$-name for $\vec{w}$, then in R, $\mathbb{P}$ forces that $P^\Phi[\dot{x}] \vDash$"There is y such that $\Psi(\dot{z}, y)$ and $\varphi(\dot{\vec{w}})$ is in the theory of y". This follows from 4.2. □

The analysis of scales in $\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}}(\mathbb{R})$ runs into a problem (see footnote 37). Therefore we will analyze scales in a slightly different hierarchy.

Definition 4.22. Fix a natural coding of elements of HC by reals. Let $X \subseteq \mathrm{HC}$. Given a set $X \subseteq \mathrm{HC}$, X^{cd} denotes the set of codes for elements of X in this coding. We say that X is **self-scaled** iff there are scales on X^{cd} and $\mathbb{R}\backslash X^{\mathrm{cd}}$ which are analytical (i.e., Σ^1_n for some $n < \omega$) in X^{cd}. ⊣

Note that for any $\mathcal{J}$-model $\mathcal{M}$ such that $\mathrm{HC}^{\mathcal{M}} \in \mathcal{M}$, the decoding function (for the above codes), restricted to $\mathbb{R}^{\mathcal{M}}$, is definable over $\mathrm{HC}^{\mathcal{M}}$, so if $X \subseteq \mathrm{HC}^{\mathcal{M}}$ then $(X^{\mathrm{cd}})^{\mathcal{M}} = X^{\mathrm{cd}} \cap \mathcal{M}$.

[32]Without the assumption that $g \in V$, it seems that the domain of $\mathcal{G}_{\mathcal{F}}\restriction\mathcal{M}[g]$ might not be definable over $\mathcal{M}[g]$.

Definition 4.23. Let b be transitive with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$.

Then $^{\mathsf{G}}\mathcal{F}(b)$ denotes the least $\mathcal{N} \trianglelefteq {}^{\mathrm{g}}\mathcal{F}(b)$ such that either $\mathcal{N} = {}^{\mathrm{g}}\mathcal{F}(b)$ or $\mathcal{J}_1(\mathcal{N}) \vDash$"$\Theta$ does not exist". (Therefore $\mathcal{J}_1^{\mathsf{m}}(b; \mathfrak{M}) \trianglelefteq {}^{\mathsf{G}}\mathcal{F}(b)$.)

We say that $\mathcal{M}$ is a **potential Θ-g-organized $\mathcal{F}$-premouse over** X iff $\mathfrak{M} \in \mathrm{HC}^{\mathcal{M}}$ and for some $X \subseteq \mathrm{HC}^{\mathcal{M}}$, $\mathcal{M}$ is a potential $^{\mathsf{G}}\mathcal{F}$-premouse over $(\mathrm{HC}^{\mathcal{M}}, X)$ with parameter $\mathfrak{M}$ and $\mathcal{M} \vDash$"X is self-scaled". We write $X^{\mathcal{M}} = X$. ⊣

In our application to core model induction, we will be most interested in the cases that either $X = \emptyset$ or $X = \mathcal{F}{\restriction}\mathrm{HC}^{\mathcal{M}}$. Clearly Θ-g-organized $\mathcal{F}$-premousehood is not first order. Certain aspects of the definition, however, are:

Definition 4.24. Let "I am a Θ-g-organized premouse over X" be the $\mathcal{L}_0$ formula ψ such that for all $\mathcal{J}$-structures $\mathcal{M}$ and $X \in \mathcal{M}$ we have $\mathcal{M} \vDash \psi(X)$ iff (i) $X \subseteq \mathrm{HC}^{\mathcal{M}}$; (ii) $\mathcal{M}$ is a $\mathcal{J}$-model over $(\mathrm{HC}^{\mathcal{M}}, X)$; (iii) $\mathcal{M}|1 \vDash$"X is self-scaled"; (iv) every proper segment of $\mathcal{M}$ is sound; and (v) for every $\mathcal{N} \trianglelefteq \mathcal{M}$:

– if $\mathcal{N} \vDash$"Θ exists" then $\mathcal{N}{\downarrow}(\mathcal{N}|\Theta^{\mathcal{N}})$ is a $\mathfrak{P}^{\mathcal{N}}$-strategy premouse of type φ_{g};

– if $\mathcal{N} \vDash$"Θ does not exist" then $\mathcal{N}$ is passive. ⊣

Lemma 4.25. *Let $\mathcal{M}$ be a $\mathcal{J}$-structure and $X \in \mathcal{M}$. Then the following are equivalent: (i) $\mathcal{M}$ is a Θ-g-organized $\mathcal{F}$-premouse over X; (ii) $\mathcal{M} \vDash$"I am a Θ-g-organized premouse over X" and $\mathfrak{P}^{\mathcal{M}} = \mathfrak{M}$ and $\Sigma^{\mathcal{M}} \subseteq \Lambda_{\mathfrak{M}}$; (iii) $\mathcal{M}|1$ is a Θ-g-organized premouse over X and every proper segment of $\mathcal{M}$ is sound and for every $\mathcal{N} \trianglelefteq \mathcal{M}$,*

– *if $\mathcal{N} \vDash$"Θ exists" then $\mathcal{N}{\downarrow}(\mathcal{N}|\Theta^{\mathcal{N}})$ is a g-organized $\mathcal{F}$-premouse;*

– *if $\mathcal{N} \vDash$"Θ does not exist" then $\mathcal{N}$ is passive.*

Lemma 4.26. *$^{\mathsf{G}}\mathcal{F}$ is basic and condenses finely. Moreover, the class of Θ-g-organized $\mathcal{F}$-premice is very condensing.*

Proof. We prove the "moreover" clause, using the equivalence of (i) and (iii) in 4.25. Let $\pi : \mathcal{R} \to \mathcal{M}$ be a very weak 0-embedding where $\mathcal{M}$ is a Θ-g-organized $\mathcal{F}$-premouse. Because of the elementarity of π with respect to $\dot{a}$, $\mathcal{R}|1$ is a Θ-g-organized premouse. If $\mathcal{R}$ is active then $\mathcal{M}$ is active, so $\mathcal{M} \vDash$"Θ exists" and $(B^{\mathcal{M}} \cup E^{\mathcal{M}}) \cap \mathcal{M}|\Theta^{\mathcal{M}} = \emptyset$, so $\Theta^{\mathcal{M}} \in \mathrm{rg}(\pi)$ and $\pi(\Theta^{\mathcal{R}}) = \Theta^{\mathcal{M}}$. So if $\mathcal{R} \vDash$"Θ does not exist" then $\mathcal{R}$ is passive. If $\mathcal{R} \vDash$"Θ exists" then $\mathcal{M} \vDash$"Θ exists" and $\pi(\Theta^{\mathcal{R}}) = \Theta^{\mathcal{M}}$, and letting X witness that π is a very weak 0-embedding, we may assume that $\mathcal{R}|\Theta^{\mathcal{R}} \in X$. Therefore $\pi : \mathcal{R}' \to \mathcal{M}'$ is a very weak 0-embedding, where $\mathcal{R}' = \mathcal{R}{\downarrow}(\mathcal{R}|\Theta^{\mathcal{R}})$ and $\mathcal{M}' = \mathcal{M}{\downarrow}(\mathcal{M}|\Theta^{\mathcal{M}})$. So by 4.19, $\mathcal{R}'$ is a g-organized $\mathcal{F}$-premouse. □

Corollary 4.27. *Let $\mathcal{M}$ be an n-sound Θ-g-organized $\mathcal{F}$-premouse and let $\pi : \mathcal{N} \to \mathcal{M}$ be a weak n-embedding. If $\mathcal{M}$ is n-maximally iterable then so is $\mathcal{N}$.*

5 Local $\mathrm{HOD}_{\mathcal{F}}$ analysis

Let $\mathcal{F}$ be a nice operator. Let $\mathcal{M}$ be a Θ-g-organized $\mathcal{F}$-premouse.

Suppose $\mathcal{M} \models$ "Θ exists". Set $\theta = \Theta^{\mathcal{M}}$. Fix $n_0 < \omega$ such that $\mathcal{M}$ is n_0-sound and $\rho_{n_0}(\mathcal{M}) \geq \theta$. Letting $l(\mathcal{M}) = \gamma_0$, we assume that for all $\langle \xi, k \rangle <_{\text{lex}} \langle \gamma_0, n_0 \rangle$, $\mathcal{M}|\xi$ is countably k-iterable. It's clear that if $a \in \mathcal{M}|\xi$, then

$$\mathrm{Hull}_{k+1}^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup \{a\}) \cong H$$

for some $H \in \mathcal{M}|\theta$. The following, however, is less clear.

Lemma 5.1.

$$(\lfloor \mathcal{M}|\theta \rfloor, \in, \dot{S}^{\mathcal{M}|\theta}) \prec_{\Sigma_1} (\lfloor \mathcal{M} \rfloor, \in, \dot{S}^{\mathcal{M}}).$$

Moreover, for every $a \in \mathcal{M}|\theta$ and $\langle \xi, n \rangle <_{\text{lex}} \langle \gamma_0, n_0 \rangle$, if $\theta \leq \xi$, then for some $\tau < \theta$,

$$\mathrm{Hull}_{n+1}^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup \{a\}) \cong \mathcal{M}|\tau.$$

Proof. Assuming the second clause, let us deduce the first. Let φ be in $\mathcal{L}_0^-$-Σ_1 and $a \in \mathcal{M}|\theta$. Suppose $\mathcal{M} \models \varphi(a)$. We must show that $\mathcal{M}|\theta \models \varphi(a)$. Let $\xi < \gamma_0$ be least such that $\mathcal{M}|(\xi + 1) \models \varphi(a)$. Fix $n < \omega$ and an $\mathrm{r}\Sigma_{n+1}$ formula ψ such that $\mathcal{M}|\xi \models \psi(a)$, and for any $\mathcal{J}$-model $\mathcal{N}$ and $a' \in \mathcal{N}$, if $\mathcal{N} \models \psi(a')$ then $\mathcal{J}_1(\mathcal{N}) \models \varphi(a')$. Let H be the transitive collapse of $\mathrm{Hull}_{n+1}^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup \{a\})$. Let $\pi : H \to \mathcal{M}|\xi$ be the uncollapse. Then $\mathrm{crit}(\pi) < \theta$, since $\rho_{n+1}^{\mathcal{M}|\xi} \neq \mathbb{R}$. Moreover, $\mathrm{crit}(\pi) = \Theta^H$, and $a \in H|\Theta^H$, so $\mathcal{J}_1(H) \models \varphi(a)$. By the second clause, $H \triangleleft \mathcal{M}|\theta$, so we are done.

Now we prove the second clause. For each $\eta < \theta$, let H_η be the transitive collapse of $\mathrm{Hull}_{n+1}^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup \eta)$, and let $\pi_\eta : H_\eta \to \mathcal{M}|\xi$ be the uncollapse. For each $\eta < \theta$, we have $H_\eta \in \mathcal{M}|\theta$ and $\mathrm{crit}(\pi_\eta) < \theta$, since $\rho_{n+1}^{\mathcal{M}|\xi} \neq \mathbb{R}^{\mathcal{M}}$. We say η is a *generator* iff $\eta = \mathrm{crit}(\pi_\eta)$. Note that the generators form a club in θ, and if η is a generator then $\eta = \Theta^{H_\eta}$. Also let H'_η be the least $H \triangleleft \mathcal{M}|\theta$ such that $\eta \leq o(H)$ and H projects to $\mathbb{R}^{\mathcal{M}}$. Now $\mathrm{Hull}_{n+1}^{\mathcal{M}|\xi}(\mathbb{R} \cup \{a\}) \cong H_\eta$ for some generator η. So part (a) of the following claim finishes the proof.

Claim 5.2. *Let $\eta < \theta$ be a generator. Then:*

(*a*) *$H_\eta \triangleleft \mathcal{M}|\theta$, and in fact, $H_\eta \trianglelefteq H'_\eta$.*

(*b*) *If η is the least generator then $\rho_{n+1}^{H_\eta} = \mathbb{R}^{\mathcal{M}}$ and $p_{n+1}^{H_\eta} = \emptyset$.*

(*c*) *If $\zeta < \eta$ is the largest generator $< \eta$, then $\rho_{n+1}^{H_\eta} = \mathbb{R}^{\mathcal{M}}$ and $p_{n+1}^{H_\eta} = \{\zeta\}$.*

(*d*) *If η is a limit of generators then $\rho_{n+1}^{H_\eta} = \eta$ and $p_{n+1}^{H_\eta} = \emptyset$.*

Proof. The proof is by induction on η.

Suppose η is the least generator. Then $H_\eta = \mathrm{Hull}_{n+1}^{H_\eta}(\mathbb{R}^{\mathcal{M}})$, which gives (b), and gives that H_η is a fully sound Θ-g-organized $\mathcal{F}$-premouse; clearly $a^{H_\eta} = a^{\mathcal{M}}$. So by countable n-iterability and 4.27, $H_\eta \triangleleft \mathcal{M}|\theta$, and $H_\eta = H'_\eta$ since $\eta = \Theta^{H_\eta}$.

Now suppose ζ is the largest generator $< \eta$. Then $\eta \subseteq X = \mathrm{Hull}_{n+1}^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup \{\zeta\})$, so $\rho_{n+1}^{H_\eta} = \mathbb{R}^{\mathcal{M}}$ and $p_{n+1}^{H_\eta} \leq \{\zeta\}$. But $H'_\zeta \in X$, so $H'_\zeta \subseteq X$ and $H_\zeta \in X$. Therefore $p_{n+1}^{H_\eta} = \{\zeta\}$ and H_η is $(n+1)$-solid, and $(n+1)$-sound, so fully sound. The rest is as in the previous case; again we get $H'_\eta = H_\eta$.

Suppose η is a limit of generators. The $\mathrm{r}\Sigma_{n+1}$ facts about H_η follow readily by induction. Since $\rho_{n+1}^{H_\eta} = \Theta^{H_\eta}$ and H_η is $(n+1)$-sound, and H_η cannot have extenders overlapping η, comparison gives $H_\eta \trianglelefteq H'_\eta$, as required. □

We say that $\mathcal{M}$ is **relevant** iff $\mathcal{M} \vDash$"Θ exists" and there is $\lambda \in (\Theta^{\mathcal{M}}, l(\mathcal{M}))$ such that $\mathcal{M}|\lambda \vDash \mathsf{ZF}$.

Suppose that $\mathcal{M}$ is relevant. Let $T^{\mathcal{M}}$ denote the following $\mathcal{L}_0^-$ theory:

$$T^{\mathcal{M}} = \mathrm{Th}_{\Sigma_1, \mathcal{L}_0^-}^{\mathcal{M}|\theta}(\theta) = \mathrm{Th}_{\Sigma_1, \mathcal{L}_0^-}^{\mathcal{M}}(\theta).$$

(The second equality is by 5.1.) Note then that $\mathfrak{M}, U, U'$ are coded into $T^{\mathcal{M}}$, where U, U' are the trees of scales as in 4.22. (In fact, they are coded into $T^{\mathcal{M}} \cap \gamma^{<\omega}$, for some γ such that $\mathcal{M}|\xi$ is not relevant for any $\xi \leq \gamma$.) More generally, we say that a set of ordinals A is $\mathrm{OD}_{\mathcal{F}}^{\mathcal{M}}$ iff $A \in \mathcal{M}$ and there is $\xi < l(\mathcal{M})$ such that A is $\mathcal{L}_0$-definable from ordinal parameters over $\mathcal{M}|\xi$. By 5.1,

$$\mathrm{OD}_{\mathcal{F}}^{\mathcal{M}} \cap \mathcal{P}(<\theta) = \mathcal{J}_1(\widehat{T^{\mathcal{M}}}) \cap \mathcal{P}(<\theta) = \mathcal{P}(<\theta) \cap \bigcup_{\gamma<\theta} \mathcal{J}_1(\widehat{T^{\mathcal{M}} \cap \gamma^{<\omega}}).$$

We now define a g-organized $\mathcal{F}$-premouse $\mathcal{H}$ over $\mathcal{T}^{\mathcal{M}}$, by *S-construction*, as in [17]. Let $\lambda > \theta$ be least such that $\mathcal{M}|\lambda \vDash \mathsf{ZF}^-$. For $\alpha \in [1, \lambda]$ let

$$\mathcal{H}_{\theta+\alpha} = \mathcal{H}|\alpha = \mathcal{J}_\alpha^{\mathsf{m}}(T^{\mathcal{M}}; \mathfrak{M}).$$

Note $\mathfrak{M}, U, U' \in \mathcal{H}|1$ and the Vopenka algebra $\mathbb{P}$ defined over $\mathcal{M}|\theta$ as in [17] is in $\mathcal{H}|2$. Also, $\mathcal{M}|\theta$ is ${}^{\mathsf{G}}\mathcal{F}$-whole, so $\mathcal{M}|\lambda = \mathcal{J}_\lambda^{\mathsf{m}}(\mathcal{M}|\theta; \mathfrak{M})\downarrow a^{\mathcal{M}}$. For $\alpha \geq 1$ we will have $l(\mathcal{H}_{\theta+\alpha}) = \alpha$, and so $\mathrm{o}(\mathcal{H}_{\theta+\alpha}) = \mathrm{o}(\mathcal{M}|(\theta+\alpha))$. For $\alpha \geq \lambda$ we will have $\mathcal{H}_\alpha = \mathcal{H}|\alpha$, and so $\mathrm{o}(\mathcal{H}|\alpha) = \mathrm{o}(\mathcal{M}|\alpha)$.

Now $\mathcal{M}\downarrow(\mathcal{M}|\theta)$ is g-organized. Above $\mathcal{H}|\lambda$, we do a level-by-level restriction of branches and extenders from $\mathcal{M}$ to $\mathcal{H}$, setting, for $\alpha > \lambda$, (i) $B^{\mathcal{H}|\alpha} = B^{\mathcal{M}|\alpha}$ and (ii) $E^{\mathcal{H}|\alpha} = E^{\mathcal{M}|\alpha} \cap \mathcal{H}|\alpha$. Condition (i) will be reasonable because we maintain that for each $\alpha \geq \lambda$, $\mathcal{M}|\alpha$ is a symmetric submodel of a generic extension of $\mathcal{H}|\alpha$ (via $\mathbb{P}$), and this will give that if $\mathcal{H}|\alpha, \mathcal{M}|\alpha$ are whole then the genericity iterations used to define ${}^{\mathrm{g}}\mathcal{F}(\mathcal{H}|\alpha)$ and ${}^{\mathrm{g}}\mathcal{F}(\mathcal{M}|\alpha)$ are identical.

The translation of fine structure between $\mathcal{H}$ and $\mathcal{M}$ is mostly as in [17], so we omit most of the details. Here is a summary. For $\alpha \geq 1$ we define $\mathcal{H}_\alpha(\mathbb{R}^{\mathcal{M}})$ as the $\mathcal{L}_0$-structure

$$\mathcal{H}_\alpha(\mathbb{R}^{\mathcal{M}}) = (\mathcal{J}_{\theta+\alpha}^{T^{\mathcal{M}}, S^{\mathcal{H}_\alpha}}(\mathrm{HC}^{\mathcal{M}}); E^{\mathcal{H}_\alpha}, B^{\mathcal{H}_\alpha}, S^{\mathcal{H}_\alpha}, (\mathrm{HC}^{\mathcal{M}}, T^{\mathcal{M}}), \mathfrak{M}).$$

(This is not a $\mathcal{J}$-model.) Truth in $\mathcal{H}(\mathbb{R}^{\mathcal{M}})$ can be reduced to truth in $\mathcal{H}$ via the forcing relation for $\mathbb{P}$. And $\mathcal{H}(\mathbb{R}^{\mathcal{M}})$ determines $\mathcal{M}$: given that $\mathcal{M}|\theta \in \mathcal{H}_\lambda(\mathbb{R}^{\mathcal{M}})$, the extender sequence of $\mathcal{H}$ determines that of $\mathcal{M}$ above θ by the local definability of the forcing; because $\mathfrak{M}, U, U' \in \mathcal{H}|1$ and by induction applied to relevant initial segments of $\mathcal{M}|\theta$, we do indeed have that $\mathcal{M}|\theta \in \mathcal{H}_\lambda(\mathbb{R}^{\mathcal{M}})$. The local definability of the forcing is also used to show that the reduction of $\mathcal{M}$-truth to $\mathcal{H}$-truth is local. The main theorem, which generalizes [17, 3.9], is the following.

Lemma 5.3. *We have:*

(*1*) *For $\xi \leq l(\mathcal{M})$ such that $\mathcal{M}|\xi$ is relevant, $\mathcal{M}||\xi$ is $\mathcal{L}_0^-$-Σ_1 over $\mathcal{H}^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}})$, and $\mathcal{M}|\xi$ is $\mathcal{L}_0$-Σ_1 over $\mathcal{H}^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}})$, uniformly in ξ.*

(*2*) *$\mathcal{H}$ is an n_0-sound g-organized $\mathcal{F}$-premouse over $T^{\mathcal{M}}$.*

(*3*) *For all $(\beta, k) \leq_{\mathrm{lex}} (l(\mathcal{M}), n_0)$ with $\lambda \leq \beta$, we have $\rho_k(\mathcal{H}|\beta) = \rho_k(\mathcal{M}|\beta)$, and $p_k(\mathcal{H}|\beta) = p_k(\mathcal{M}|\beta)\backslash\{\theta\}$.*

(*4*) *For all $\beta \in [\theta, l(\mathcal{M})]$, $\mathcal{M}|\beta$ is ${}^{\mathsf{G}}\mathcal{F}$-whole iff either $\beta = \theta$, or $\beta > \lambda$ and $\mathcal{H}_\beta = \mathcal{H}|\beta$ is ${}^{\mathrm{g}}\mathcal{F}$-whole.*

Proof sketch. For most of the details, see the proof of [17, 3.9]. We just give enough of a sketch to describe the new features.

As usual, (1) will follow from the proof, and by induction, we may assume that (1) holds for $\xi \leq \theta$. This implies $\mathcal{M}|\theta \in \mathcal{H}_\lambda(\mathbb{R}^{\mathcal{M}})$, unless there is no relevant $\xi < \theta$ (a fact regarding which $T^{\mathcal{M}}$ informs us). In the latter case, $\mathcal{M}|\theta = \mathcal{J}_\theta^{\mathsf{m}}(a^{\mathcal{M}}; \mathfrak{M})$. But there is an $\mathcal{L}_0^-$-Σ_1 formula defining $(U, \mathfrak{M})$ over $\mathcal{H}_1(\mathbb{R}^{\mathcal{M}})$ (by referring to $T^{\mathcal{M}}$), and $\mathcal{H}_1(\mathbb{R}^{\mathcal{M}}) \models$"$X = p[U]$", where $X = X^{\mathcal{M}}$, which suffices.

We prove the remaining items by induction. We claim that for $\eta \in [\lambda, l(\mathcal{M})]$, $\mathcal{H}_\eta$ is a g-organized $\mathcal{F}$-premouse over $T^{\mathcal{M}}$, and the models $\mathcal{M}|\eta, \mathcal{H}_\eta$ are *related*. That is, (4) holds for all $\beta \leq \eta$; below any $p \in \mathbb{P}$, $\mathcal{H}_\eta(\mathbb{R}^{\mathcal{M}})$ is a symmetric inner model of a $\mathbb{P}$-forcing extension of $\mathcal{H}_\eta$; $\mathcal{M}|\eta$ is defined over $\mathcal{H}_\eta(\mathbb{R}^{\mathcal{M}})$ as described above; (3) holds for $\beta \leq \eta$. Moreover, everything is uniform in η. These facts are proved by induction on η.

The fact that λ is least such that $\mathcal{H}_\lambda \models \mathsf{ZF}$, and that the claim holds at $\eta = \lambda$, follows the proof of [17, 3.9]. Suppose $\beta < l(\mathcal{M})$, $\beta \geq \theta$, $\mathcal{M}|\beta$ is ${}^{\mathsf{G}}\mathcal{F}$-whole, we have proved the claim for $\eta \leq \beta$, and (4) holds at β. Let $\mathcal{N} = {}^{\mathrm{g}}\mathcal{F}(\mathcal{M}|\beta)$ and $\mathcal{I} = {}^{\mathrm{g}}\mathcal{F}(\mathcal{H}_\beta)$, and suppose that $(\mathcal{N}\downarrow a^{\mathcal{M}}) \trianglelefteq \mathcal{M}$. We want to prove the claim for $\eta \leq l(\mathcal{N}\downarrow a^{\mathcal{M}})$. This is done as for [17, 3.9], except that we also need to see that

$$l(\mathcal{I}) = l(\mathcal{N}) \tag{5.1}$$

and that for each $\alpha < l(\mathcal{I})$,

$$B^{\mathcal{I}|\alpha} = B^{\mathcal{N}|\alpha}. \tag{5.2}$$

So, clearly $\alpha = \alpha'$, where α (resp., α') is the least $> \beta$ such that $\mathcal{J}_\alpha(\mathcal{M}|\beta) \models \mathsf{ZF}$ (resp., $\mathcal{J}_{\alpha'}(\mathcal{H}_\beta) \models \mathsf{ZF}$), and that $\mathcal{M}|\alpha, \mathcal{H}_\alpha$ are related. Let $\mathcal{T} = \mathcal{T}_{\mathcal{H}_\alpha}$ and $\mathcal{U} = \mathcal{T}_{\mathcal{M}|\alpha}$. We now prove by induction on γ that for all $\gamma \leq \epsilon = \max(\lambda^\Phi(\mathcal{T}) + 1, \lambda^\Phi(\mathcal{U}) + 1)$,

$$\mathcal{T}\restriction\gamma = \mathcal{U}\restriction\gamma. \tag{5.3}$$

Clearly then $\lambda^\Phi(\mathcal{T}) = \lambda^\Phi(\mathcal{U})$; with an inspection of 4.15, lines (5.1) and (5.2) follow.

So suppose that line (5.3) holds at γ and $\gamma < \epsilon$; we need to see that $E^{\mathcal{T}}_\gamma = E^{\mathcal{U}}_\gamma$. Suppose $\gamma < \lambda^\Phi(\mathcal{T})$. Let $\dot{x}_{\mathcal{H}_\alpha}$ be the canonical name for the $\mathrm{Col}(\omega, \mathcal{H}_\alpha)$-generic real coding $\mathcal{H}_\alpha$. Let δ be least such that $\mathcal{M}^{\mathcal{U}}_\gamma \in \mathcal{M}|\delta$, so then $\mathcal{M}^{\mathcal{T}}_\gamma = \mathcal{M}^{\mathcal{U}}_\gamma \in \mathcal{H}_\delta$, and by induction, $\mathcal{M}|\delta$ and $\mathcal{H}_\delta$ are related. Let $\dot{x}_{\mathcal{M}|\alpha}$ be likewise. Let $p \in \mathrm{Col}(\omega, \mathcal{H}_\alpha)$ be such that p forces, over[33] $\mathcal{H}_\delta$, that $E^{\mathcal{T}}_\gamma$ induces an axiom which fails for $\dot{x}_{\mathcal{H}_\alpha}$. Now $\mathrm{Col}(\omega, \mathcal{M}|\alpha)$ factors as $\mathrm{Col}(\omega, \mathcal{H}_\alpha) \times \mathrm{Col}(\omega, \mathcal{M}|\alpha)$. Let $\dot{G}_0, \dot{G}_1$ be the canonical names for the corresponding generics, and let $\dot{x}_{0,\mathcal{M}|\alpha}$ and $\dot{x}_{1,\mathcal{M}|\alpha}$ be the corresponding generic reals coding $\mathcal{H}_\alpha$ and $\mathcal{M}|\alpha$ respectively. Then letting $p' \in \mathrm{Col}(\omega, \mathcal{M}|\alpha)$ force that $p \in \dot{G}_0$, we have that p' forces that $E^{\mathcal{T}}_\gamma$ induces an axiom which fails for $\dot{x}_{0,\mathcal{M}|\alpha}$. But using the natural definitions, $\dot{x}_{0,\mathcal{M}|\alpha}$ is arithmetic in $\dot{x}_{\mathcal{M}|\alpha}$, and so it is easy to see that p' forces that $E^{\mathcal{T}}_\gamma$ induces an axiom which fails for $\dot{x}_{\mathcal{M}|\alpha}$, as required.

The converse is similar, but we need to use the fact that $\mathcal{M}|\delta$ can be realized as a symmetric submodel of a $\mathbb{P}$-generic extension of $\mathcal{H}_\delta$. (It doesn't suffice that this holds for $\mathcal{M}|\alpha$ and $\mathcal{H}_\alpha$, since the forcing relation which demonstrates the fact that $E^{\mathcal{U}}_\gamma$ induces a bad axiom need not be in $\mathcal{M}|\alpha$.) We omit further detail.

The case that $\mathcal{M}\downarrow(\mathcal{M}|\beta) \triangleleft \mathcal{N}$ is handled mostly in the same manner, though in this case it can be that line (5.3) fails for $\gamma = \lambda^\Phi(\mathcal{U}) + 1$, for example. We need to see that $l(\mathcal{H}\downarrow\mathcal{H}_\beta) < l(\mathcal{I})$, and that for each $\alpha < l(\mathcal{H}\downarrow\mathcal{H}_\beta)$, line (5.2) holds. But if $\gamma < \lambda^\Phi(\mathcal{T})$ and

[33]This forcing is absolute, but the point is that the relevant forcing relation is in $\mathcal{H}_\delta$.

$M^{\mathcal{T}}_\gamma \in \mathcal{M}$, then $\gamma < \lambda^\Phi(\mathcal{U})$ and $\mathcal{U}\restriction\gamma+1 = \mathcal{T}\restriction\gamma+1$ and $E^{\mathcal{U}}_\gamma = E^{\mathcal{T}}_\gamma$; and vice versa. This is enough. □

The next theorem relates the iterability of $\mathcal{H}$ and $\mathcal{M}$. The proof of 5.4 uses 5.3 and is just like that in [17, 3.18].

Theorem 5.4. *Let $\mathcal{M}$ be an n_0-sound Θ-g-organized $\mathcal{F}$-premouse. Suppose $\mathcal{M}$ is relevant, $\rho_{n_0}(\mathcal{M}) \geq \Theta^{\mathcal{M}}$ and $\mathcal{M}|\xi$ is countably k-iterable for all $\langle \xi, k\rangle <_{\text{lex}} \langle l(\mathcal{M}), n_0\rangle$. Then*

$$\mathcal{H}^{\mathcal{M}} \textit{ is countably } n_0\textit{-iterable} \iff \mathcal{M} \textit{ is countably } n_0\textit{-iterable above } \Theta^{\mathcal{M}},$$

and for all $\gamma \in \mathrm{Ord}$,

$$\mathcal{H}^{\mathcal{M}} \textit{ is } (n_0,\gamma)\textit{-iterable} \iff \mathcal{M} \textit{ is } (n_0,\gamma)\textit{-iterable above } \Theta^{\mathcal{M}}.$$

Remark 5.5. In the sequel, we will also need S-construction, performed mostly as above, for example, in the following context. Let $\mathcal{M}$ be a g-organized $\mathcal{F}$-premouse. Let $\eta < l(\mathcal{M})$ be such that $\mathcal{M}|\eta$ is a ${}^{\mathrm{g}}\mathcal{F}$-whole strong cutpoint of $\mathcal{M}$ (see 6.22). Let $g \subseteq \mathrm{Col}(\omega, \mathcal{M}|\eta)$ be $\mathcal{M}$-generic. Then $\mathcal{M}[g]$ can be reorganized as a g-organized $\mathcal{F}$-premouse $\mathcal{M}[g]^*$ over $(\mathcal{M}|\eta, g)$. Moreover, the fine structure and iterability of $\mathcal{M}[g]^*$ corresponds to the fine structure and iterability of $\mathcal{M}$ above η, in a manner similar to 5.3 and 5.4. We leave the precise formulation and proofs of these facts to the reader.

Using related arguments, we also get that $\mathcal{M} = \mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}}(\mathbb{R})$ and $\mathcal{N} = \mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R})$ have the same $\mathcal{P}(\mathbb{R})$. Moreover, if $(\mathcal{F}\restriction\mathrm{HC})^{\mathrm{cd}}$ is self-scaled then $\mathcal{P} = \mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathrm{HC}, \mathcal{F}\restriction\mathrm{HC})$ also has the same $\mathcal{P}(\mathbb{R})$. Likewise $\mathcal{Q} = \mathrm{Lp}^{\mathcal{F}}(\mathbb{R})$, if it is well-defined. In fact, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{Q}$ have literally the same extender sequences and for all α such that $\mathcal{M}|\alpha$ is active, there is a straightforward translation between $\mathcal{M}|\alpha$, $\mathcal{N}|\alpha$, $\mathcal{P}|\alpha$ and $\mathcal{Q}|\alpha$. (We use here that B-predicates in both the ${}^{\mathrm{g}}\mathcal{F}$ and ${}^{\mathsf{G}}\mathcal{F}$ hierarchies code a branch $b = \Lambda_{\mathfrak{M}}(\mathcal{T})$ computable from the Q-structure for $M(\mathcal{T})$, which is a segment of $L^{\mathcal{F}}(M(\mathcal{T}))$.)

6 Scales

Let $\mathcal{F}$ be a nice operator and let $X \subseteq \mathrm{HC}$ be self-scaled. We now give the scales analysis of $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, X)$. In the context of our application to the core model induction, the analysis will proceed from optimal determinacy hypotheses; such optimality is important in that context, as explained in [20].[34]

[34] Let Σ be the unique iteration strategy for $\mathcal{M}_1^\sharp$. Suppose $\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R}) \models \mathsf{AD}^+ + \mathsf{MC}$. Then in fact $\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \mathrm{Lp}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$. This is because in $L(\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R}))$, $L(\mathcal{P}(\mathbb{R})) \models \mathsf{AD}^+ + \Theta = \theta_0 + \mathsf{MC}$ and hence by [4],

When $\mathcal{M}$ is a $\mathcal{J}$-model and we talk about, for example, $\Sigma_1^{\mathcal{M}}$, as a pointclass (for $\mathcal{M}$), we mean the collection of all subsets of $\mathbb{R}^{\mathcal{M}}$ which are $\mathcal{L}_0$-$\Sigma_1^{\mathcal{M}}$-definable over $\mathcal{M}$.

6.1 Scales on $\Sigma_1^{\mathcal{M}}$ sets for passive $\mathcal{M}$

Theorem 6.1. *Let $\mathcal{M}$ be a countably iterable passive Θ-g-organized $\mathcal{F}$-premouse such that $\mathcal{M} \models \mathsf{AD}$. Then $\mathcal{M} \models$ "$\Sigma_1^{\mathcal{M}}$ has the scale property".*

Proof. For simplicity we assume that $l(\mathcal{M})$ is a limit ordinal; for the contrary case make the usual modifications using the $\mathcal{S}$-hierarchy. We work with $\mathrm{HC} = \mathrm{HC}^{\mathcal{M}}$ (possibly $\mathrm{HC} \subsetneq \mathrm{HC}^V$). Let $\Phi \in \mathcal{L}_0^-$ be Σ_1. For $x \in \mathbb{R}$, let $P(x) \Leftrightarrow \mathcal{M} \models \Phi(x)$. We will show that there is a $\Sigma_1^{\mathcal{M}}$-scale on P.

For $x \in \mathbb{R}$ and $\beta < l(\mathcal{M})$ let $P^\beta(x) \Leftrightarrow \mathcal{M}|\beta \models \Phi(x)$. Then $P = \bigcup_{\beta < l(\mathcal{M})} P^\beta$. For each $\beta < l(\mathcal{M})$, we construct a closed game representation $x \mapsto G_x^\beta$ for P^β, such that G_x^β is continuously associated to x. Let

$$P_k^\beta(x,u) \;\Leftrightarrow\; u \text{ is a position of length } k \text{ from which player I has a winning quasi-strategy in } G_x^\beta.$$

We will define G_x^β in such a way that $P_k^\beta \in \mathcal{M}$ and the map $\langle \beta, k \rangle \mapsto P_k^\beta$ is $\Sigma_1^{\mathcal{M}}$. This will give us the desired $\Sigma_1^{\mathcal{M}}$ scale essentially by the argument in [18]. (If $X \neq \emptyset$ there will be moves in G_x^β which are sets of reals, coding ordinals, via a coding in $\mathcal{M}$. This, however, does not affect the construction described in [18] in any significant manner.)

Let $X = X^{\mathcal{M}}$. Then $\{(X, X^{\mathrm{cd}})\}$ is $\Delta_1^{\mathcal{M}}$. Let $\vec{\leq} = \langle \leq_n \rangle_{n<\omega}$ and $\vec{\leq}' = \langle \leq_n' \rangle_{n<\omega}$ be scales on X^{cd} and $\mathbb{R} \backslash X^{\mathrm{cd}}$ as in 4.22. Let U and U' be the trees of these scales, respectively. Possibly $U, U' \notin \mathcal{M}$ (because $\mathcal{M}$ might not have enough ordinals), but U, U' are "in $\mathcal{M}$" in the codes (given by the norms of the scales).

Fix $\beta \in [1, l(\mathcal{M}))$ and $x \in \mathbb{R}$. Before defining G_x^β we give an outline. Player II will play reals. Player I will (attempt to) build a countable, iterable, passive, Θ-g-organized $\mathcal{F}$-premouse $\mathcal{P}$ over $X \cap \mathcal{P}$, containing all reals played by player II, such that $\mathcal{P} \models \Phi(x)$, but for all $\gamma < l(\mathcal{P})$, $\mathcal{P} \models \neg\Phi(x)$. To enforce that player I indeed plays an iterable Θ-g-organized $\mathcal{F}$-premouse over $X \cap \mathcal{P}$, he must simultaneously build a very weak 0-embedding $\pi : \mathcal{P} \to \mathcal{M}|\gamma$ for some $\gamma \leq \beta$ [35] and build branches through U and U' (in the codes).

in $L(\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R}))$, $\mathcal{P}(\mathbb{R}) \subseteq \mathrm{Lp}(\mathbb{R})$. Therefore, even though the hierarchies $\mathrm{Lp}(\mathbb{R})$ and $\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R})$ are different, as far as sets of reals are concerned, we don't lose any information by analyzing the scales pattern in $\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R})$ instead of that in $\mathrm{Lp}(\mathbb{R})$.

[35] One could have instead used an approach more like that used in [17].

We now proceed to the details. Player I will describe his model using the language

$$\mathcal{L}^* =_{\text{def}} \mathcal{L}_0 \cup \{\dot{x}_i \mid i < \omega\} \cup \{\dot{X}\}.$$

The constant symbol $\dot{x}_i$ will denote the i^{th} real played in the game. Fix recursive maps

$$m, n : \{\sigma \mid \sigma \text{ is an } \mathcal{L}^*\text{-formula}\} \to \{2n \mid 1 \leq n < \omega\}$$

which are one-to-one, have disjoint recursive ranges, and are such that whenever $\dot{x}_i$ occurs in σ, then $i < \min(m(\sigma), n(\sigma))$.

Fix an $\mathcal{L}_0^-$-Σ_1 formula $\sigma_0(v_0, v_1, v_2)$ that defines over each $\mathcal{M}|\gamma$, the graph of a surjection

$$h_\gamma \colon [\mathrm{o}(\mathcal{M}|\gamma)]^{<\omega} \times \mathbb{R} \twoheadrightarrow \mathcal{M}|\gamma.$$

Let T be the following $\mathcal{L}^*$ theory:

(1) Extensionality

(2) "I am a $\mathcal{J}$-model"

$(3)_i$ $\dot{x}_i \in \mathbb{R}$

(4) $\Phi(\dot{x}_0) \wedge \forall \gamma > 0 \left[\dot{S}_\gamma \not\models \Phi(\dot{x}_0)\right]$

(5) $\forall u, v, y, z\, [\sigma_0(u, v, y) \wedge \sigma_0(u, v, z) \Rightarrow y = z]$

$(6)_\varphi$ $\exists v \varphi(v) \Rightarrow \exists v \exists F \in \mathrm{Ord}^{<\omega} \left[\varphi(v) \wedge \sigma_0(F, \dot{x}_{m(\varphi)}, v)\right]$

$(7)_\varphi$ $\exists v\, [\varphi(v) \wedge v \in \mathbb{R}] \Rightarrow \varphi(\dot{x}_{n(\varphi)})$

(8) $\dot{a} = (\mathrm{HC}, \dot{X})$

For each $n < \omega$, let e_n be the set of pairs (n, E) where E is a $\leq_n$-equivalence class of elements of X^{cd}. Let $e = \bigcup_{n<\omega} e_n$. Let W be the tree of the scale $\vec{\leq}$, in the codes given by e. (In particular, W is a set of finite sequences σ, where for each $i < \mathrm{lh}(\sigma)$, $\sigma_i \in e_i$.) Let W' be defined likewise from $\vec{\leq}'$. For $\sigma = ((a_0, b_0), \ldots, (a_{n-1}, b_{n-1}))$ let $p_0[\sigma] = (a_0, \ldots, a_{n-1})$ and $p_1[\sigma] = (b_0, \ldots, b_{n-1})$.

A run of the game G_x^β is of length ω. For each n, player I plays $i_n, x_{2n}, \eta_n, \Lambda_n$ where $i_n \in \{0, 1\}$, $x_{2n} \in \mathbb{R}$, $\eta_n < \mathrm{o}(\mathcal{M}|\beta)$ and $\Lambda_n \in (W \cup W')^n$. Player II plays $x_{2n+1} \in \mathbb{R}$. If $u = \langle (i_k, x_{2k}, \eta_k, x_{2k+1}) \mid k < n \rangle$ is a partial play of length n, we let

$$T^*(u) = \{(\neg)^i \sigma \mid \sigma \text{ is an } \mathcal{L}^*\text{-sentence} \wedge n(\sigma) < n \wedge i = i_{n(\sigma)}\},$$

where $(\neg)^0\sigma = \sigma$ and $(\neg)^1\sigma = \neg\sigma$. If p is a full run of G^β_x let

$$T^*(p) = \bigcup_{n<\omega} T^*(p\restriction n).$$

Let "$\iota v\varphi(v)$" stand for "the unique v such that $\varphi(v)$".

We next describe the payoff conditions for player I. These are mostly analogous to those in [17]. Conditions (f) and (g) ensure that for each $i < \omega$, if player I asserts, for example, that "$\dot{x}_i \in \dot{X}^{\mathrm{cd}}$" then $\langle \Lambda_{n,i}\rangle_{n\in(i,\omega)}$ is an infinite branch through W witnessing that $x_i \in X^{\mathrm{cd}}$.

A full run $p = \langle (i_k, x_{2k}, \eta_k, \Lambda_k, x_{2k+1}) \mid k < \omega\rangle$ of G^β_x is a win for player I iff

(a) $T^*(p)$ is a complete consistent extension of T,

(b) $x_0 = x$,

(c) for all $i, m, n < \omega$, "$\dot{x}_i(n) = m$" $\in T^*(p)$ iff $x_i(n) = m$,

(d) if φ and ψ are $\mathcal{L}^*$-formulae with one free variable and

$$\text{“}\iota v\varphi(v) \in \mathrm{Ord}\ \&\ \iota v\psi(v) \in \mathrm{Ord}\text{”} \in T^*(p),$$

then "$\iota v\varphi(v) \leq \iota v\psi(v)$"$\in T^*(p)$ iff $\eta_{n(\varphi)} \leq \eta_{n(\psi)}$,

(e) if $\psi, \sigma_0, \ldots, \sigma_{n-1}$ are $\mathcal{L}^*$-formulas with one free variable and

$$\text{“}\iota v\psi(v) \in \mathrm{Ord}\ \&\ \dot{S}_{\iota v\psi(v)} \text{ exists ”} \in T^*(p),$$

and for all $k < n$,

$$\text{“}\iota v\sigma_k(v) \in \mathrm{o}(\dot{S}_{\iota v\psi(v)})\text{”} \in T^*(p)$$

then $\eta_{n(\psi)} < \beta$ and for any $\mathcal{L}_0$-formula $\theta(v_1, \ldots, v_n)$,

$$\text{“}\dot{S}_{\iota v\psi(v)} \models \theta[\iota v\sigma_0(v), \ldots, \iota v\sigma_{n-1}(v)]\text{”} \in T^*(p)$$

if and only if

$$\dot{S}^{\mathcal{M}}_{\eta_{n(\psi)}} \models \theta[\eta_{n(\sigma_0)}, \ldots, \eta_{n(\sigma_{n-1})}],$$

(f) for all $i < m \leq n < \omega$, $\Lambda_{m,i} \trianglelefteq \Lambda_{n,i}$ and $p_0[\Lambda_{n,i}] = x_i\restriction n$,

(g) for all $i < m < \omega$, if "$\dot{x}_i \in \dot{X}^{\mathrm{cd}}$" $\in T^*(p)$ then $\Lambda_{m,i} \in W$, and otherwise $\Lambda_{m,i} \in W'$.

In condition (e), we allow $\eta_{n(\psi)} = 0$ (where $\dot{S}^{\mathcal{N}}_0 = a^{\mathcal{N}}$ for any $\mathcal{J}$-structure $\mathcal{N}$). Because of the payoff conditions, we could have added a sentence like "$\dot{\mathfrak{P}}$ is a premouse (of some kind)" to T (or any other sentences satisfied by all initial segments of $\mathcal{M}$), without any significant effect.

We next define the notion of *honesty* and show that the only winning strategy for player I is to be honest. Note here that if $\mathcal{M}|\gamma \vDash \Phi(x)$, and γ is least such, then $\gamma = \alpha + 1$, where $\mathcal{M}|\alpha$ projects to $\mathbb{R}$, and therefore, since $\mathcal{M}$ is Θ-g-organized, $\mathcal{M}|\gamma$ is passive.

We say a position $u = \langle (i_k, x_{2k}, \eta_k, \Lambda_k, x_{2k+1}) \mid k < n \rangle$ is (β, x)-**honest** iff $\mathcal{M}|\beta \vDash \Phi(x)$ and letting $\gamma = \alpha + 1 \leq \beta$ be the least such that $\mathcal{M}|\gamma \vDash \Phi(x)$, we have

(i) $n > 0 \Rightarrow x_0 = x$,

(ii) letting I_u be the interpretation of $\mathcal{L}^*$ in which $\dot{x}_i^{I_u} = x_i$ for $0 < i < 2n$ and $\dot{X}^{I_u} = X$, all formulas in $T^*(u)$ are true of $(\mathcal{M}|\gamma, I_u)$, and

(iii) if $\sigma_0, \ldots, \sigma_{m-1}$ enumerate all $\mathcal{L}^*$-formulae σ of one free variable such that $n(\sigma) < n$ and

$$(\mathcal{M}|\gamma, I_u) \vDash \iota v \sigma(v) \in \mathrm{Ord},$$

and if for each $k < m$, $\delta_k < \mathrm{o}(\mathcal{M}|\gamma)$ is such that

$$(\mathcal{M}|\gamma, I_u) \vDash \delta_k = \iota v \sigma_k(v),$$

then, in $V^{\mathrm{Col}(\omega, \mathcal{M}|\beta)}$, there is an order-preserving map

$$\pi : \mathrm{o}(\mathcal{M}|\gamma) \to \mathrm{o}(\mathcal{M}|\beta)$$

such that for each $k < m$, we have $\pi(\delta_k) = \eta_{n(\sigma_k)}$, and the partial embedding

$$\pi \restriction \mathrm{o}(\mathcal{M}|\alpha) : \mathcal{M}|\alpha \to \mathcal{M}|\pi(\alpha)$$

is fully elementary, with respect to $\mathcal{L}_0$, on its domain,

(iv) for each $i < m < n$, $\Lambda_{m,i} \trianglelefteq \Lambda_{n-1,i}$ and $x_i \restriction m = p_0[\Lambda_{m,i}]$, and if $x_i \in X^{\mathrm{cd}}$ then there is $f \in W_{x_i}$ such that $f \restriction m = p_1[\Lambda_{m,i}]$, and if $x_i \notin X^{\mathrm{cd}}$ then there is $f \in W'_{x_i}$ such that $f \restriction m = p_1[\Lambda_{m,i}]$.

Let $Q_k^\beta(x, u)$ iff u is a (β, x)-honest position of length k.

The following two claims complete our proof of Theorem 6.1. Their proofs are similar to those of [17, Claims 4.2, 4.3].

Claim 6.2. *For all β, k we have $Q_k^\beta \in \mathcal{M}$, and the map $(\beta, k) \mapsto Q_k^\beta$ is $\Sigma_1^{\mathcal{M}}$.*

Proof Sketch. The truth of condition (iv) of honesty is easily computed.[36]

Regarding the other conditions, the proof is basically like that of [17, Claim 4.2], except that we modify some details and give a complete proof. Let $\gamma = \mathrm{o}(\mathcal{M}|\beta)$, $A = \mathrm{Th}_1^{\mathcal{M}|\beta}(\gamma)$ and $A' = \gamma \cup \{A\}$. Let $\lambda \in \mathrm{Ord}$ be least such that $\mathcal{J}_\lambda(A')$ is admissible. The "embedding game" $\mathcal{G}$ (see [17, Claim 4.2]) is definable from A and is fully analysed in $\mathcal{J}_\alpha(A')$ for some $\alpha < \lambda$. Now we claim that for each $\alpha < \lambda$,

$$t_\alpha = \mathrm{Th}_1^{\mathcal{J}_\alpha(A')}(A') \in \mathcal{M}.$$

This suffices. For if N is any structure with $A' \subseteq N$ and satisfying "$V = L[A']$, I see a full analysis of $\mathcal{G}$ but no proper segment of me does", then N is wellfounded and so $N = \mathcal{J}_\alpha(A')$ for some α (since otherwise the wellfounded part of N is admissible, contradicting the minimality of N). Therefore $\mathcal{M}$ can identify the theory of the unique such N, allowing the rest of the proof of [17, Claim 4.2] to go through.

So we show that $t_\alpha \in \mathcal{M}$. Let $\leq$ be a prewellorder of $\mathbb{R}^{\mathcal{M}}$ of length $\geq \gamma$, with $\leq$ in $\mathcal{M}$. Say that a structure N (possibly illfounded) is *good* iff N extends A' and $N \models$"$V = L[A']$" and $N = \mathrm{Hull}_1^N(A')$ and $\mathrm{Th}_1^N(A')$ is $(\undertilde{\Sigma}_1^1(\leq))^{\mathcal{M}}$ (in the codes given by $\leq$). We claim that for every $\alpha < \lambda$, $\mathcal{J}_\alpha(A')$ is good (and therefore $t_\alpha \in \mathcal{M}$). All requirements are clear other than the fact that t_α is $(\undertilde{\Sigma}_1^1(\leq))^{\mathcal{M}}$.

Now if there is any illfounded good N, then the wellfounded part of N is admissible, and therefore $\mathcal{J}_\alpha(A') \triangleleft N$ for each $\alpha < \lambda$, which easily gives the claim. So suppose all good structures are wellfounded.

We claim that there is a largest good structure. For suppose not. Let S be the set of all Σ_1 theories of good structures. Clearly $S \in \mathcal{M}$. Now for each $N \in S$ let $t_N = \mathrm{Th}_1^N(A')$. Let $t = \bigcup S$. Then $t \in \mathcal{M}$, and $t = \mathrm{Th}_1^N(A')$ for $N = \mathcal{J}_\xi(A')$, for some ordinal ξ. Moreover, $N = \mathrm{Hull}_1^N(A')$. But then by the coding lemma applied in $\mathcal{M}$, N is good, contradiction.

So let N be the largest good structure. Let $N = \mathcal{J}_\xi(A')$ and $N' = \mathcal{J}_{\xi+1}(A')$. We claim that $N \preccurlyeq_1 N'$, and therefore that N is admissible, completing the proof. So suppose otherwise. We claim that N' is good, for a contradiction. Clearly $N' = \mathrm{Hull}_1^{N'}(A')$, so we just need to see that $t' = \mathrm{Th}_1^{N'}(A')$ is $(\undertilde{\Sigma}_1^1(\leq))^{\mathcal{M}}$. By the coding lemma, it suffices to see that $t' \in \mathcal{M}$. Now t' is recursively equivalent to $\oplus_{n<\omega} T_n$ where $T_n = \mathrm{Th}_n^N(A')$. But each of these theories are in $\mathcal{M}$ since $T_1 = t_N \in \mathcal{M}$. Therefore, by the coding lemma, each T_n is $(\undertilde{\Sigma}_1^1(\leq))^{\mathcal{M}}$. Let T be the set of parameters $x \in \mathbb{R}$ coding (relative to $(\undertilde{\Sigma}_1^1(\leq))^{\mathcal{M}}$) one of the theories T_n, for some $n < \omega$. Then $T \in \mathcal{M}$ because in fact, T is $(\undertilde{\Sigma}_{10}^1(\leq))^{\mathcal{M}}$. Therefore

[36] One does not need to consider the rank analysis of trees here, and there may not be enough ordinals in $\mathcal{M}$ to do so. Instead, directly use the fact that W, W' are the trees of scales, which are analytical in (X, z), to compute the truth of (iv), essentially inside $\mathcal{M}|1$.

$\oplus_{n<\omega} T_n \in \mathcal{M}$, as required. $\square$

Claim 6.3. *For all β, k and all length k partial plays u in G_x^β, player* I *has a winning quasi-strategy starting from u iff u is (β, x)-honest. That is, $P_k^\beta(x,u) \Leftrightarrow Q_k^\beta(x,u)$.*

Proof Sketch. This is mostly like the proof of [17, Claim 4.3]. But consider the proof that every strategic position u is (β, x)-honest; we adopt the notation from the proof of [17, Claim 4.3]. Certainly $\mathcal{N}$ is a $\mathcal{J}$-model, and by payoff conditions (e)–(g), every proper segment of $\mathcal{N}$ is fully sound and $\mathcal{N}$ is a $\mathcal{J}$-model over $(\mathrm{HC}^{\mathcal{M}}, X)$ with parameter $\mathfrak{M}$. (The fact that $\dot{\mathfrak{P}}^{\mathcal{N}} = \mathfrak{M}$ is by payoff condition (e), since $\dot{\mathfrak{P}} \in \mathcal{L}_0$. The fact that $\dot{X}^{\mathcal{N}} = X$ is because player I built witnessing branches through W, W'.) Because

$$\mathcal{N} \vDash \exists y \in \mathbb{R}\left[\Phi(y) \wedge \forall \gamma > 0\left[\dot{S}_\gamma \nvDash \Phi(y)\right]\right],$$

we have $l(\mathcal{N}) = \alpha+1$ for some $\alpha \in \mathrm{Ord}$, and note that $\mathcal{M}|\pi(\alpha)+1$ satisfies the same formula. So $\mathcal{M}|\pi(\alpha)$ and $\mathcal{N}$ project to $\mathbb{R}$, so $\mathcal{M}|\pi(\alpha)+1$ is passive (because $\mathcal{M}$ is Θ-g-organized). But then because $\pi{\restriction}\mathrm{o}(\mathcal{N}|\alpha) : \mathcal{N}|\alpha \to \mathcal{M}|\pi(\alpha)$ is fully elementary on its domain, there is a unique very weak 0-embedding $\pi' : \mathcal{N} \to \mathcal{M}|\pi(\alpha)+1$ such that $\pi'{\restriction}\alpha+1 = \pi{\restriction}\alpha+1$. Therefore by 4.26, $\mathcal{N}$ is a Θ-g-organized $\mathcal{F}$-premouse. Now arguing as in the proof of [17, Claim 4.3], using the results of §5, $\mathcal{N}|\alpha$ (and so $\mathcal{N}$) is iterable, etc. $\square$

This completes our sketch of the proof. $\square$

6.2 Σ_1 gaps

Definition 6.4. Let $\mathcal{M}$ be a $\mathcal{J}$-model such that $\mathrm{HC}^{\mathcal{M}} \in \mathcal{M}|1$.

We write $\mathcal{N} \prec_1 \mathcal{M}$ iff $\mathcal{N} \trianglelefteq \mathcal{M}$ and whenever ψ is an $\mathcal{L}_0^-$-Σ_1 formula then for any $a_1, ..., a_n \in \mathbb{R}^{\mathcal{M}}$,

$$\mathcal{M} \vDash \psi[a_1, ..., a_n] \Rightarrow \mathcal{N} \vDash \psi[a_1, ..., a_n].$$

Let $\alpha \le \beta \le l(\mathcal{M})$. We call the interval $[\alpha, \beta]$ a **Σ_1-gap** iff (i) $\mathcal{M}|\alpha \prec_1 \mathcal{M}|\beta$; (ii) for all $\alpha' \in [1, \alpha)$, $\mathcal{M}|\alpha' \not\prec_1 \mathcal{M}|\alpha$; (iii) for all $\beta' \in (\beta, l(\mathcal{M})]$, $\mathcal{M}|\beta \not\prec_1 \mathcal{M}|\beta'$; (iv) if $\beta = l(\mathcal{M})$ then $\mathcal{M}$ is fully sound and $\mathrm{HC}^{\mathcal{J}_1(\mathcal{M})} = \mathrm{HC}^{\mathcal{M}}$ and $\mathcal{M} \not\prec_1 \mathcal{J}_1^{\mathsf{m}}(\mathcal{M}; \mathfrak{P}^{\mathcal{M}}){\downarrow}a^{\mathcal{M}}$. $\dashv$

Definition 6.5. Let $\mathcal{M}$ be an n-sound Θ-g-organized $\mathcal{F}$-premouse. Let $n > 0$ and $b \in \mathfrak{C}_0(\mathcal{M})$. The $\mathrm{r}\Sigma_n$ **type realized by b over $\mathcal{M}$**, denoted $\mathrm{r}\Sigma_{n,b}^{\mathcal{M}}$, is

$$\{\varphi(v) \in \mathcal{L}_0 \mid \varphi \text{ is either } \mathrm{r}\Sigma_n \text{ or } \mathrm{r}\Pi_n \text{ and } \mathfrak{C}_0(\mathcal{M}) \vDash \varphi[b]\}.$$

Let $[\alpha,\beta]$ be a Σ_1-gap of $\mathcal{M}$. We say the gap is **admissible** iff $\mathcal{M}|\alpha$ is admissible. We say the gap is **strong** iff it is admissible and letting $n<\omega$ be the least such that $\rho_n(\mathcal{M}|\beta)=\mathbb{R}^{\mathcal{M}}$, then every $\mathrm{r}\Sigma_n$-type realized over $\mathcal{M}|\beta$ is realized over $\mathcal{M}|\gamma$ for some $\gamma<\beta$. We say the gap is **weak** iff it is admissible but not strong. ⊣

Inside a Σ_1-gap there are no new scales. The proof of the following theorems are routine generalizations of the corresponding proofs in [18].

Theorem 6.6 (Kechris-Solovay)**.** *Let $\mathcal{M}$ be a Θ-g-organized $\mathcal{F}$-premouse which is countably 0-iterable. Suppose $[\alpha,\beta]$ is a Σ_1-gap of $\mathcal{M}$ and $\mathcal{M}|\alpha\models \mathsf{AD}$. Then:*

1. *There is a $\Pi_1^{\mathcal{M}|\alpha}$ relation on $\mathbb{R}^{\mathcal{M}}$ with no uniformizing function $f\in\mathcal{M}|\beta$.*
2. *For $\alpha\le\gamma<\beta$ and all $n\in[1,\omega)$, $\mathcal{M}\models$"The pointclasses $\underset{\sim}{\mathrm{r}\Sigma}_n^{\mathcal{M}|\gamma}$ and $\underset{\sim}{\mathrm{r}\Pi}_n^{\mathcal{M}|\gamma}$ do not have the scale property."*

A relation witnesing item 1 of Theorem 6.6 is $(\mathbb{R}^{\mathcal{M}})^2\backslash\mathcal{C}^{\mathcal{M}|\alpha}$ where $\mathcal{C}^{\mathcal{M}|\alpha}(x,y)$ iff $x,y\in\mathbb{R}^{\mathcal{M}}$ and there is $\gamma<\alpha$ such that y is $\mathcal{L}_0$-definable over $\mathcal{M}|\gamma$ from parameters in $\mathrm{Ord}\cup\{x\}$. The same relation witnesses that there is no new scale definable over the end of a strong gap.

Theorem 6.7 (Martin)**.** *Let $\mathcal{M}$ be a Θ-g-organized $\mathcal{F}$-premouse such that $\mathcal{M}$ is countably 0-iterable. Suppose $\mathcal{M}\models\mathsf{AD}$. Let $[\alpha,\beta]$ be a strong Σ_1-gap of $\mathcal{M}$ such that $\beta<l(\mathcal{M})$. Then:*

1. *There is a $\Pi_1^{\mathcal{M}|\alpha}$ relation on $\mathbb{R}^{\mathcal{M}}$ which has no uniformization definable over $\mathcal{M}|\beta$.*
2. *For all $n<\omega$, $\mathcal{M}\models$"The pointclasses $\underset{\sim}{\mathrm{r}\Sigma}_n^{\mathcal{M}|\beta}$ and $\underset{\sim}{\mathrm{r}\Pi}_n^{\mathcal{M}|\beta}$ do not have the scale property".*

Remark 6.8. The only case remaining in the analysis of scales in $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},X)$ is at the end of a weak gap. For let $\mathcal{M}$ be a Θ-g-organized $\mathcal{F}$-premouse and let $[\alpha,\beta]$ be a gap of $\mathcal{M}$. If $[\alpha,\beta]$ is inadmissible then $\alpha=\beta$ and $\mathcal{M}|\alpha\models$"$\Theta$ does not exist", and therefore $\mathcal{M}|\alpha$ is passive. So 6.1, combined with the argument in [18], suffices to cover all pointclasses in $\mathcal{J}_1(\mathcal{M}|\alpha)$ (given determinacy there). This is the main reason that we analyze scales in $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},X)$ instead of in $\mathrm{Lp}^{\mathrm{g}\mathcal{F}}(\mathbb{R},X)$. The analysis of scales in the latter runs into difficulties in the preceding case.[37] So we are left with strong and weak gaps, and strong gaps are dealt with as usual. We deal with weak gaps in three cases, as described in the introduction.

[37]Let $\mathcal{M}$ be a g-organized $\mathcal{F}$-premouse over $\mathrm{HC}^{\mathcal{M}}$. Suppose $\alpha=l(\mathcal{M})$ and $[\alpha,\alpha]$ is an inadmissible gap of $\mathcal{M}$, and $B^{\mathcal{M}}\neq\emptyset$. We would like to prove that $\Sigma_1^{\mathcal{M}}$, or at least $\underset{\sim}{\Sigma}_1^{\mathcal{M}}$, has the scale property. One might try to mimic the proof of 6.1; but we need to have player I build a B-active structure $\mathcal{N}$. Aiming for the scale property for $\underset{\sim}{\Sigma}_1^{\mathcal{M}}$, one can ensure that player I builds a g-organized $\mathcal{F}$-premouse $\mathcal{N}$, and in the proof that every strategic position is honest, can arrange that the resulting generic run produces a structure $\mathcal{N}$ such that $\mathcal{N}\trianglelefteq\mathcal{M}$. But this does not give that $B^{\mathcal{N}}=B^{\mathcal{M}}\cap\mathcal{N}$, and the latter is needed to verify honesty.

6.3 Scales at the end of a weak gap from strong determinacy

The first scale construction for weak gaps proceeds from a strong determinacy assumption. It is most useful for weak gaps $[\alpha, \beta]$ of $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$ where $\mathcal{F}\restriction\mathrm{HC} \notin \mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)|\alpha$.

Theorem 6.9. *Let $\mathcal{R}$ be a Θ-g-organized $\mathcal{F}$-mouse. Suppose $\mathcal{R} \models \mathsf{AD}$ and $[\alpha, \beta]$ is a weak gap in $\mathcal{R}$ with $\beta < l(\mathcal{R})$. Let $n < \omega$ be least such that $\rho_n(\mathcal{R}|\beta) = \mathbb{R}^{\mathcal{R}}$. Then $\mathcal{R} \models$ "$\underset{\sim}{\Upsilon\Sigma}{}_n^{\mathcal{R}|\beta}$ has the scale property".*

Proof Sketch. Since the proof is almost the same as that of [17, Theorem 4.16], we only sketch it here. However, our approach is a little different from that used in [17].[38] For simplicity, we assume that $X^{\mathcal{R}} = \emptyset$ and $n = 1$ and β is a limit ordinal. (If $X^{\mathcal{R}} \neq \emptyset$ make changes as in the proof of 6.1.) Let $\mathcal{M} = \mathcal{R}|\beta$.

Let $p = p_1^{\mathcal{M}}$ and let $w_1 \in \mathbb{R}^{\mathcal{M}}$ be such that the solidity witness(es) W for p is in $\mathrm{Hull}_1^{\mathcal{M}}(p, w_1)$ and such that $\Sigma = \Sigma^{1,\mathcal{M}}_{\langle p, w_1\rangle}$ is a non-reflecting type.

We now define a sequence $\langle \beta_i, Y_i, \psi_i\rangle_{i<\omega}$. There are two cases to consider. We write $\mathcal{M}^{\wr}_{\gamma}$ for $\mathcal{M} \wr (\gamma, 0)$[39].

Case 6.10. $\mathcal{M}$ is either E-passive or E-active type 3.

Let β_0 be the least $\gamma < \beta$ such that

$$\max(p) < \mathrm{o}(\mathcal{M}^{\wr}_{\gamma}). \tag{6.1}$$

Now suppose $\beta_i < \beta$ is defined. Then we define Y_i, ψ_i and β_{i+1} as follows:

$$Y_i = \mathrm{Hull}_{\omega}^{\mathcal{M}^{\wr}_{\beta_i}}(\mathbb{R}^{\mathcal{M}} \cup \{p\}), \tag{6.2}$$

$$\psi_i = \text{ least } \psi \in \Sigma \text{ such that } \mathcal{M}^{\wr}_{\beta_i} \models \neg\psi[\langle p, w_1\rangle], \tag{6.3}$$

$$\beta_{i+1} = \text{ least } \gamma \text{ such that } \mathcal{M}^{\wr}_{\gamma} \models \psi_i[\langle p, w_1\rangle]. \tag{6.4}$$

Case 6.11. $\mathcal{M}$ is E-active type 1 or 2.

We make the following changes to the construction from the previous case. Let $E = E^{\mathcal{M}}$ and $\kappa = \mathrm{crit}(E)$.

Let β_0 be the least γ such that $\nu(E^{\mathcal{M}}) < \mathrm{o}(\mathcal{M}^{\wr}_{\gamma})$ and line (6.1) holds. Given β_i, we define Y_i by line (6.2), then let

$$\xi_i = \sup(Y_i \cap (\kappa^+)^{\mathcal{M}}),$$

[38] This is because the proof of [17, Claim 4.18] is incomplete (at least, the authors do not see why, in the notation of that proof, $\mathcal{N} = \mathcal{M}$, because it is not clear that $\mathcal{N}$ is sound). Our approach gets around this problem, and also simplifies the proof, because it eliminates the need for the "bounding integers" m_k and n_k played by player I in the game G^i_x of [17].

[39] In [17], this is denoted $\mathcal{M}||\gamma$.

define ψ_i by line (6.3), and then let[40]

$$\beta_{i+1} = \text{ least } \gamma \text{ such that } \mathcal{M}^{\wr}_{\gamma} \models \psi_i[\langle p, w_1\rangle] \text{ and } E \cap \mathcal{M}^{\wr}_{\gamma} \text{ measures all sets in } \mathcal{M}|\xi_i.$$

Claim 6.12. *$\bigcup_{i<\omega} Y_i = \mathcal{M}$. In particular, the β_i's are cofinal in β.*

Proof. Let $\mathcal{N}$ be the transitive collapse of $\bigcup_{i<\omega} Y_i$ and let $\pi\colon \mathcal{N} \to \bigcup_{i<\omega} Y_i$ be the uncollapse map. Let $\beta_\omega = \sup_{i<\omega} \beta_i$. Note that $\mathcal{M}^{\wr}_{\beta_\omega} \models \Sigma$ and so $\mathrm{Hull}_1^{\mathcal{M}}(\langle p, w_1\rangle) \subseteq \mathrm{rg}(\pi)$. Therefore $W, \beta_i \in \mathrm{rg}(\pi)$. In fact, $\beta_i \in Y_j$ for $i < j$.[41] So $\mathrm{Th}_1^{\mathcal{M}}(\{\beta_0, \beta_1, \ldots\})$ is recorded in Σ. So letting $\pi(\beta_i^*) = \beta_i$, we have that π is Σ_1-elementary on $\{\beta_i^* \mid i < \omega\}$, which is cofinal in $\mathrm{o}(\mathcal{N})$. So π is a weak 0-embedding. Clearly $\mathcal{N}$ is a $\mathcal{J}$-structure. So by 4.26, $\mathcal{N}$ is a Θ-g-organized $\mathcal{F}$-premouse, and clearly $\mathrm{HC}^{\mathcal{N}} = \mathrm{HC}^{\mathcal{M}}$.

Let $\pi(p^*) = p$. It is easy to see that $\mathcal{N} = \mathrm{Hull}_1^{\mathcal{N}}(\mathbb{R}^{\mathcal{N}} \cup \{p^*\})$. But p^* is 1-solid for $\mathcal{N}$ since $W \in \mathrm{rg}(\pi)$ (so $\pi^{-1}(W)$ is a generalized solidity witness for p^*).[42] Therefore $\mathcal{N}$ is 1-sound and $p^* = p_1^{\mathcal{N}}$. Since trees on $\mathcal{N}$ can be lifted to trees on $\mathcal{M}$ via π, $\mathcal{N}$ is countably 0-${}^{\mathsf{G}}\mathcal{F}$-iterable. Since $\mathcal{N}$ is also minimal realizing Σ, therefore $\mathcal{N} = \mathcal{M}$.

The fact that $\pi = \mathrm{id}$ now follows as usual, using the fact that $p^* = p$. □

Using notation mostly as in [17] (i.e., the proof of [17, Theorem 4.16]), we proceed to define the game G^i_x as there, making some modifications. Player I describes his model using the language $\mathcal{L} = \mathcal{L}_0 \cup \{\dot{x}_i, \dot{\beta}_i, \dot{\mathcal{M}}_i\}_{i<\omega} \cup \{\dot{G}, \dot{p}, \dot{W}\}$; each of the symbols in $\mathcal{L}\backslash\mathcal{L}_0$ are constants. Let B_0 be defined from $\mathcal{L}$ as in [17]. Let S_0 be the set of sentences in B_0 which involve no constants of the form $\dot{x}_i$ for $i \notin \{1, 2\}$ and are true in $\mathfrak{C}_0(\mathcal{M})$ when $(\dot{x}_1, \dot{x}_2, \dot{G}, \dot{p}, \dot{W}, \dot{\beta}_k, \dot{\mathcal{M}}_k)$ are interpreted as $(w_1, w_2, p, p, W, \beta_k, \mathcal{M}^{\wr}_{\beta_k})$. A run of G^i_x has the form

I	T_0, s_0, η_0	T_1, s_1, η_1	$\cdots$
II	s_1	s_3	$\cdots$

where T_i, s_i are as in [17] and $\eta_i \in \mathrm{o}(\mathcal{M})$. The winning conditions for player I are, verbatim, the winning conditions (1)–(6) as stated in [17].[43]

We define the term x**-honest** exactly as in [17] except that we drop condition (iv) from there. The rest of the proof is mostly a routine adaptation of the proof in [17]; we just mention the main changes.

[40] Recall that E is the $\mathcal{M}$-amenable predicate coding the active extender of $\mathcal{M}$.

[41] So it would not have made any difference to add the parameters $\beta_0, \ldots, \beta_{i-1}$ to the hull defining Y_i.

[42] *Generalized solidity witness* is defined in [8]. Since π is only a weak 0-embedding, we do not yet know that $\pi^{-1}(W)$ is the (standard) solidity witness.

[43] We have no need for the integer moves m_k, nor any version of condition (8) used in [17].

Claim 6.13. *For any position u of G^i_x, player I wins G^i_x from u if and only if u is x-honest.*

Proof sketch. Consider the proof that every strategic position is honest. We use notation mostly as in the proof of [17, Claim 4.19], with a couple of changes. Let $\mathcal{N}$ be the reduct of $\mathcal{A}$ to an $\mathcal{L}_0$-structure. Let $\mathcal{N}_k$ be (the $\mathcal{L}_0$-structure) $\dot{\mathcal{M}}_k^{\mathcal{A}}$. So, because $\mathcal{A} \vDash S_0$, $\mathcal{N}_k = \mathcal{N}^{\wr}_{\beta^*_k}$ and $\mathcal{N}$ is the "union" of the $\mathcal{N}_k$. Let $p^* = \dot{p}^{\mathcal{A}} = G^*$. As in the proof of [17, Claim 4.19] we get that $\mathcal{N}$ is a countably iterable Θ-g-organized $\mathcal{F}$-premouse which is minimal for realizing Σ. Clearly $X^{\mathcal{N}} = \emptyset = X^{\mathcal{M}}$. Also, $\mathcal{N}$ is sound with $\rho_1^{\mathcal{N}} = \mathbb{R}^{\mathcal{N}}$ and $p_1^{\mathcal{N}} = p^*$. For let $H = \mathrm{Hull}_1^{\mathcal{N}}(\mathbb{R}^{\mathcal{N}} \cup p^*)$. Then because $\mathcal{A} \vDash S_0$, we have $\mathcal{N}_k \in H$ for each $k < \omega$; it follows that $H = \lfloor \mathcal{N} \rfloor$. And W^* is a generalized solidity witness for p^*, because this is enforced by formulas in S_0 regarding $\dot{W}$ and $\dot{p}$. So $\mathcal{N} = \mathcal{M}$ and $p^* = p$. Because $\mathcal{A}$ satisfies S_0, this implies that $W^* = W$, $\beta^*_k = \beta_k$ and $\mathcal{N}_k = \mathcal{M}^{\wr}_{\beta_k}$ for all $k < \omega$. This completes our sketch. □

Claim 6.14. *Let $k < \omega$. Then $\{u \mid u$ is an x-honest position of G^i_x of length $k\} \in \mathcal{M}$.*

Proof sketch. The proof is the same as that of [17, Claim 4.20] (except that condition (iv) of [17] is not involved, so the use of the Coding Lemma regarding this condition is avoided). In the computation of the definability of (v) we still use the Coding Lemma; it is here that we use our assumption that $\mathcal{J}_1(\mathcal{M}) \vDash \mathsf{AD}$ (beyond that $\mathcal{M} \vDash \mathsf{AD}$). □

The remaining details are as in [17]. This completes the proof of Theorem 6.9. □

6.4 Scales at the end of a weak gap from optimal determinacy

As described in [20], typically in the core model induction, one does not have the stronger determinacy hypothesis required to apply 6.9. So we need generalizations of [17, Theorem 4.17] and [20, Theorem 0.1], which are the second and third cases of our scale constructions for weak gaps, respectively.

Definition 6.15. Let $\mathcal{M}$ be a Θ-g-organized $\mathcal{F}$-premouse.

We say that $\mathcal{M}$ is **mandatory** iff either $\mathcal{M}$ is active or there is some $E \in \mathbb{E}^{\mathcal{M}}$ such that E is total over $\mathcal{M}$.

We say that $\mathcal{M}$ is **self-analysed** iff for every mandatory $\mathcal{N} \trianglelefteq \mathcal{M}$ there is $\mathcal{P} \trianglelefteq \mathcal{M}$ such that $\mathcal{N} \triangleleft \mathcal{P}$ and $\mathcal{P}$ is admissible.

We say that $\mathcal{M}$ is **self-coded** iff $\mathcal{M}$ is not self-analysed but for every mandatory $\mathcal{N} \trianglelefteq \mathcal{M}$ there is $\mathcal{P} \triangleleft \mathcal{M}$ such that $\mathcal{N} \trianglelefteq \mathcal{P}$ and $\rho_\omega^{\mathcal{P}} = \mathbb{R}^{\mathcal{M}}$. ⊣

Note that if $\mathcal{M} \vDash$"Θ does not exist" or $\mathcal{M}$ has no active segment above $\Theta^{\mathcal{M}}$ then $\mathcal{M}$ is either self-analysed or self-coded.

Theorem 6.16. *Let $\mathcal{M}$ be a sound Θ-g-organized $\mathcal{F}$-mouse such that $\mathcal{M} \vDash \mathsf{AD}$ and $\mathcal{M}$ is either self-analysed or self-coded. Suppose that $\mathcal{M}$ ends a weak gap of $\mathcal{M}$. Let $n < \omega$ be least such that $\rho_n^{\mathcal{M}} = \mathbb{R}^{\mathcal{M}}$. Then $\mathcal{M} \vDash$ "$\underset{\sim}{\mathrm{r}\Sigma}_n^{\mathcal{M}}$ has the scale property".*

Proof Sketch. The proof is similar to that of 6.9, but we use the fact that $\mathcal{M}$ is either self-analysed or self-coded to reduce the reliance on determinacy.[44] Note that $\mathcal{M}$ is passive. Suppose for simplicity that $X^{\mathcal{M}} = \emptyset$, β is a limit ordinal and $n = 1$.

We define most things, including Y_k and B_k, as in the proof of 6.9. Fix $x \in \mathbb{R}$ and $i < \omega$; we want to define the game G_x^i. Let $m : B_0 \times B_0 \to \omega$ and $n : B_0 \to \omega$ be recursive and injective with disjoint ranges, and such that for all $\varphi, \psi \in B_0$, φ, ψ have support $m(\varphi, \psi)$ and φ has support $n(\varphi)$ and if $\varphi \neq \psi$ then $m(\varphi, \varphi) < m(\varphi, \psi)$. A run of G_x^i consists of the same types of objects as in the proof of 6.9, except that we also require that $\eta_k \in Y_k$. The rules of G_x^i are (1)–(5) as stated in [17], along with rule (6) below, which requires player I to play a wellfounded model, and rule (7) below, which requires player I to build, for each mandatory initial segment $\mathcal{P}$ of his model, a partial embedding $\mathcal{P} \to \mathcal{R}$ for some $\mathcal{R} \trianglelefteq \mathcal{M}$, which is elementary on ordinal parameters (but these embeddings need not agree with one another):

(6) if $\varphi, \psi \in B_0$ each have one free variable and

$$\text{“}\iota v \varphi(v) \in \mathrm{Ord} \;\&\; \iota v \psi(v) \in \mathrm{Ord}\text{”} \in T^*,$$

then "$\iota v \varphi(v) \leq \iota v \psi(v)$"$\in T^*$ iff $\eta_{n(\varphi)} \leq \eta_{n(\psi)}$,

(7) if $\psi, \sigma_0, \ldots, \sigma_{j-1} \in B_0$ each have one free variable and $k < \omega$ and

$$\text{“}\iota v \psi(v) < l(\dot{\mathcal{M}}_k) \;\&\; \dot{\mathcal{M}}_k|(\iota v \psi(v)) \text{ is mandatory”} \in T^*$$

and for all $i < j$,

$$\text{“}\iota v \sigma_i(v) \in \mathrm{o}(\dot{\mathcal{M}}_k|(\iota v \psi(v)))\text{”} \in T^*$$

then $\eta_{m(\psi,\psi)} < l(\mathcal{M}_k)$ and for any $\mathcal{L}_0$-formula $\theta(v_1, \ldots, v_n)$,

$$\text{“}\dot{\mathcal{M}}_k|(\iota v \psi(v)) \vDash \theta[\iota v \sigma_0(v), \ldots, \iota v \sigma_{j-1}(v)]\text{”} \in T^*$$

if and only if

$$\mathcal{M}|\eta_{m(\psi,\psi)} \vDash \theta[\eta_{m(\psi,\sigma_0)}, \ldots, \eta_{m(\psi,\sigma_{j-1})}].$$

[44] Of course determinacy is still required in the, supressed, norm propagation part of the argument.

We leave to the reader most of the remaining details, including the precise formulation of x-**honesty** (of a position in G^i_x). The analysis of commitments made pertaining to rule (6) are dealt with as in [18]. Consider rule (7). If $\mathcal{M}$ is self-analysed then the analogue of condition (v) of x-*honest* from [17] can be computed in some admissible proper segment of $\mathcal{M}$ (we don't need the Coding Lemma for this). If $\mathcal{M}$ is not self-analysed but is self-coded then there is $\gamma < \beta$ such that $\rho_\omega(\mathcal{M}|\gamma) = \mathbb{R}^{\mathcal{M}}$ and every mandatory initial segment $\mathcal{P}$ of $\mathcal{M}$ is such that $\mathcal{P} \trianglelefteq \mathcal{M}|\gamma$. One can therefore use the Coding Lemma as in the proof of Claim 6.2 to compute the analogue of condition (v) over $\mathcal{M}|\gamma$.

If $n > 1$ then we do not require the Coding Lemma for computing honesty. For in this case there are arbitrarily large $\mathcal{P} \triangleleft \mathcal{M}$ such that $\mathcal{P}$ is admissible, and so $\mathcal{M}$ is self-analysed, and there will be cofinally many admissible $\mathcal{P} \in Y_k$ such that $\mathcal{P} \triangleleft \mathcal{M}$.

This completes our sketch. □

We now proceed to the generalization of [20, Theorem 0.1], the final scale construction of the paper. While it uses only the weaker determinacy assumption, it requires a mouse capturing hypothesis, as in [20].

Remark 6.17. Suppose V is a $\mathcal{J}$-model and HC exists. Let Γ be a pointclass of the form $\Sigma_1^{V|\alpha}$ for some $\alpha < l(V)$. Recall that (in this setting) for $x \in \mathbb{R}$, $C_\Gamma(x)$ denotes the set of all $y \in \mathbb{R}$ such that for some ordinal $\gamma < \omega_1$, x (as a subset of ω) is $\Delta_\Gamma(\{\gamma\})$.

Let $x \in \mathrm{HC}$ be such that x is transitive and $f : \omega \to x$ a surjection. Then $c_f \in \mathbb{R}$ denotes the code for $(x, \in)$ determined by f. And $C_\Gamma(x)$ denotes the set of all $y \in \mathrm{HC} \cap \mathcal{P}(x)$ such that for all surjections $f : \omega \to x$ we have $f^{-1}(y) \in C_\Gamma(c_f)$.

Lemma 6.18. *Let $\mathcal{P}$ be a Θ-g-organized $\mathcal{F}$-premouse satisfying* AD *and let $\mathcal{Q} \triangleleft \mathcal{P}$ be such that $\mathcal{Q}$ is passive and admissible. Let Γ be the pointclass $\Sigma_1^{\mathcal{Q}}$. Let $x \in \mathrm{HC}^{\mathcal{P}}$ with x transitive and infinite. Then working in $\mathcal{P}$, for all $y \in \mathrm{HC}$, the following are equivalent:*

(1) $y \in C_\Gamma(x)$,

(2) there is $\mathcal{R} \triangleleft \mathcal{Q}$ such that y is definable over $\mathcal{R}$ from parameters in $\mathrm{Ord} \cup x \cup \{x\}$,

(3) for comeager many bijections $f : \omega \to x$, $f^{-1}(y) \in C_\Gamma(c_f)$.

Proof. The proof is mostly like that of [13, Theorem 3.4]; we just mention a couple of points. For $x \in \mathbb{R}$, the equivalence of (1) and (2) follows because $\mathcal{Q} \models \mathsf{AD} + \mathsf{KP}$. Now consider the proof that (3) implies (2). If $\mathcal{P}$ satisfies (3), then we may take the witnessing comeager set C to be a countable intersection of dense sets, and then $C \in \mathcal{Q}$. So by KP there is $\mathcal{R} \triangleleft \mathcal{Q}$ such that for every $f \in C$, $f^{-1}(y)$ is definable over $\mathcal{R}$ from parameters in $\mathrm{Ord} \cup \{c_f\}$. As in [13], there is then some $\alpha < \omega_1^{\mathcal{P}}$ and $n < \omega$ and injection $\sigma : n \to x$ such that for comeager many bijections $f : \omega \to x$ extending σ, $f^{-1}(y)$ is the α^{th} real which is definable over $\mathcal{R}$ from parameters in $\mathrm{Ord} \cup \{c_f\}$, in the natural ordering. Letting $\delta = l(\mathcal{R})$, this defines y over $\mathcal{Q}|(\delta + 2)$ from parameters in $\{\delta, x\} \cup \mathrm{rg}(\sigma)$. □

Definition 6.19. Let $\mathcal{P}$ be a Θ-g-organized $\mathcal{F}$-premouse satisfying AD. Let $\mathcal{Q} \triangleleft \mathcal{P}$ be passive and admissible and let Γ be the pointclass $\Sigma_1^{\mathcal{Q}}$. Suppose that $\mathcal{F}^* = \mathcal{F}\restriction\mathrm{HC}^{\mathcal{P}} \in \mathcal{Q}$.

Now work in $\mathcal{P}$. Let $x \in \mathrm{HC}$ be transitive. Then $\mathrm{Lp}^{\Gamma, {}^{\mathrm{g}}\mathcal{F}^*}(x)$ denotes $(\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}^*}(x))^{\mathcal{Q}}$. (So the relevant iteration strategies must be inside $\mathcal{Q}$.)

Still inside $\mathcal{P}$, we say that **super-small ${}^{\mathrm{g}}\mathcal{F}^*$-mouse capturing for Γ holds on a cone** iff there is $z \in \mathbb{R}$ such that for all transitive $x \in \mathrm{HC}$ with $z \in \mathcal{J}_1(\hat{x})$, we have $C_\Gamma(x) = \mathrm{Lp}^{\Gamma, {}^{\mathrm{g}}\mathcal{F}^*}(x) \cap \mathcal{P}(x)$ and $\mathrm{Lp}^{\Gamma, {}^{\mathrm{g}}\mathcal{F}^*}(x)$ is super-small. ⊣

Theorem 6.20. *Let $\mathcal{M}$ be a Θ-g-organized $\mathcal{F}$-mouse such that $\mathcal{M} \models \mathsf{AD}$. Let $[\alpha, \beta]$ be a weak gap of $\mathcal{M}$. Suppose there is a transitive rud-closed set $\mathcal{M}_{\mathsf{DC}}$ such that $\mathcal{M}|\beta \in \mathcal{M}_{\mathsf{DC}}$ and $\mathbb{R}^{\mathcal{M}} = \mathbb{R}^{\mathcal{M}_{\mathsf{DC}}}$ and $\mathcal{M}_{\mathsf{DC}} \models \mathsf{DC}_{\mathbb{R}}$.*[45] *Let Γ be the pointclass $\Sigma_1^{\mathcal{M}|\alpha}$. Suppose that $\mathcal{F}^* = \mathcal{F}\restriction\mathrm{HC}^{\mathcal{M}} \in \mathcal{M}|\alpha$ and that $\mathcal{M} \models$ "super-small ${}^{\mathrm{g}}\mathcal{F}^*$-mouse capturing for Γ holds on a cone". Let $n < \omega$ be least such that $\rho_n(\mathcal{M}|\beta) = \mathbb{R}^{\mathcal{M}}$. Then $\mathcal{M} \models$ "$\mathrm{r}\undertilde{\Sigma}_n^{\mathcal{M}|\beta}$ has the scale property".*

Remark 6.21. Recall that if $\beta = l(\mathcal{M})$ then by 6.4 we are assuming that $\mathcal{M}$ is sound. If $\mathbb{R}^{\mathcal{M}} = \mathbb{R}$ and $\mathsf{DC}_{\mathbb{R}}$ holds then V suffices as $\mathcal{M}_{\mathsf{DC}}$.

Proof. We follow the proof of [20], making some modifications. By 6.16 we may assume that $\mathcal{M} \models$"Θ exists" and there is some $\xi + 1 \in (\Theta^{\mathcal{M}}, l(\mathcal{M}))$ such that $\mathcal{M}|\xi \models \mathsf{ZF}$. Therefore $\mathcal{P}(\mathbb{R}) \cap \mathcal{M} \subseteq \mathcal{M}|\xi$ and $\mathcal{M}|\xi \models \mathsf{ZF} + \mathsf{AD}$. We work mostly inside $\mathcal{M}$ or $\mathcal{M}_{\mathsf{DC}}$, and so with $\mathbb{R} = \mathbb{R}^{\mathcal{M}}$. We write $\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}}(x)$ for $(\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}^*}(x))^{\mathcal{M}}$, and likewise for restrictions like $\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F},\Gamma}(x)$. (We will not be interested in $(\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}}(x))^V$ if it disagrees with $(\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}^*}(x))^{\mathcal{M}}$.) Let $z_0 \in \mathbb{R}$ be a base for the mouse capturing cone. Let us assume for notational simplicity that $z_0 = \emptyset$; the relativization above a non-trivial z_0 is immediate.[46]

Remark 6.22. For the rest of the proof, except where mentioned otherwise, *premouse* abbreviates *g-organized $\mathcal{F}$-premouse*, and likewise all related terminology (such as *iteration tree*, Lp, etc).

Let $\mathcal{P}$ be a $\mathcal{J}$-model and $\eta \leq \mathrm{o}(\mathcal{P})$. Recall that η is a *cutpoint* of $\mathcal{P}$ iff whenever $E \in \mathbb{E}_+^P$ and $\mathrm{crit}(E) < \eta$, we have $\mathrm{lh}(E) \leq \eta$. And η is a *strong cutpoint* of $\mathcal{P}$ iff whenever $E \in \mathbb{E}_+^P$ and $\mathrm{crit}(E) \leq \eta$, we have $\mathrm{lh}(E) \leq \eta$. We will also say that $\mathcal{P}|\eta$ is a (*strong*) *cutpoint* iff

[45] $\mathcal{M}_{\mathsf{DC}}$ provides a universe in which we can execute certain arguments in the proof of [20, Theorem 0.1] without introducing new reals. The authors believe that [20, Theorem 0.1] should also have adopted a hypothesis along these lines.

[46] In fact, in the typical setting, if $\mathcal{M}$ is far enough past $\mathcal{M}_{\mathcal{F}}$ (for example, if $\mathcal{M}$ has any extender on its sequence) then $z_0 = \emptyset$ does suffice.

$o(\mathcal{P}|\eta)$ is a (strong) cutpoint (which is iff η is a (strong) cutpoint). Recall also that $\mathcal{P}$ is *η-sound* iff for every $n < \omega$, if $\eta < \rho_n^{\mathcal{P}}$ then $\mathcal{P}$ is n-sound, and if $\rho_{n+1}^{\mathcal{P}} \leq \eta$ then letting $p = p_{n+1}^{\mathcal{P}}$, $p \backslash \eta$ is $(n+1)$-solid for $\mathcal{P}$, and $\mathcal{P} = \mathrm{Hull}_{n+1}^{\mathcal{P}}(\eta \cup p)$.

Definition 6.23. Let $t \in \mathrm{HC}$ with $\mathfrak{M} \in \mathcal{J}_1(\hat{t})$. Let $1 \leq k \leq \omega$. A premouse $\mathcal{N}$ over t is *k-suitable* iff there is a strictly increasing sequence $\langle \delta_i \rangle_{i<k}$ such that

(a) $\forall \delta \in \mathcal{N}$, $\mathcal{N} \vDash$"δ is Woodin" if and only if $\exists i < k(\delta = \delta_i)$.

(b) If $k = \omega$ then $o(\mathcal{N}) = \sup_{i<\omega} \delta_i$, and if $k < \omega$ then $o(\mathcal{N}) = \sup_{i<\omega} (\delta_{k-1}^{+i})^{\mathcal{N}}$.

(c) If $\mathcal{N}|\eta$ is a ${}^{\mathrm{g}}\mathcal{F}$-whole strong cutpoint of $\mathcal{N}$ then $\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathrm{Lp}^{\Gamma}(\mathcal{N}|\eta)$.[47]

(d) Let $\xi < o(\mathcal{N})$, where $\mathcal{N} \vDash$"ξ is not Woodin". Then $C_{\Gamma}(\mathcal{N}|\xi) \vDash$"$\xi$ is not Woodin". [48]

We write $\delta_i^{\mathcal{N}} = \delta_i$; also let $\delta_{-1}^{\mathcal{N}} = 0$ and $\delta_k^{\mathcal{N}} = o(\mathcal{N})$. ⊣

In the context of k-suitability, we omit the phrase "over t", but all relevant premice will implicitly be over t for some fixed t.

It is an easy consequence of (c) that if $\mathcal{N}|\eta$ is any strong cutpoint of $\mathcal{N}$ then $\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathrm{Lp}_+^{\Gamma}(\mathcal{N}|\eta)$ (just apply (c) to the largest ${}^{\mathrm{g}}\mathcal{F}$-whole segment of $\mathcal{N}|\eta$).

Let $\mathcal{N}$ be k-suitable and let $\xi \in o(\mathcal{N})$ be a limit ordinal, such that $\mathcal{N} \vDash$"ξ isn't Woodin". Let $Q \triangleleft \mathcal{N}$ be the Q-structure for ξ. Let α be such that $\xi = o(\mathcal{N}|\alpha)$. Clearly if $\alpha < \xi$ or $\mathcal{N}|\xi$ is not ${}^{\mathrm{g}}\mathcal{F}$-whole then $Q = \mathcal{N}|\xi$. So suppose $o(\mathcal{N}|\xi) = \xi$ and $\mathcal{N}|\xi$ is ${}^{\mathrm{g}}\mathcal{F}$-whole. If ξ is a strong cutpoint of $\mathcal{N}$ then $Q \triangleleft \mathrm{Lp}(\mathcal{N}|\xi)$ by (c). Assume now that $\mathcal{N}$ is reasonably iterable. If ξ is a strong cutpoint of Q, our mouse capturing hypothesis combined with (d) gives that $Q \triangleleft \mathrm{Lp}^{\Gamma}(\mathcal{N}|\xi)$. If ξ is an $\mathcal{N}$-cardinal then indeed ξ is a strong cutpoint of Q, since $\mathcal{N}$ has only finitely many Woodins. If ξ is not a strong cutpoint of Q, then by definition, we do not have $Q \triangleleft \mathrm{Lp}^{\Gamma}(\mathcal{N}|\xi)$. However, using $*$-translation (see [19]), one can find a level of $\mathrm{Lp}^{\Gamma}(\mathcal{N}|\xi)$ which corresponds to Q.

Let $\mathcal{Q}$ be a premouse and $\delta < o(\mathcal{Q})$, such that $\mathcal{Q}$ is a Q-structure for $\mathcal{Q}|\delta$. Note that if δ is a cutpoint of $\mathcal{Q}$ then δ is a strong cutpoint of $\mathcal{Q}$. For if $\delta = \mathrm{crit}(F)$ for some $F \in \mathbb{E}_+(\mathcal{Q})$, then since there is $\mu < \delta$ such that $\mathcal{Q} \vDash$"μ is $< \delta$-strong, as witnessed by $\mathbb{E}^{\mathcal{Q}|\delta}$", then by coherence and the ISC, δ is in fact not a cutpoint, contradiciton. We will use this observation later without explicit mention.

Definition 6.24 (Γ-guided)**.** Let $\mathcal{P}$ be k-suitable and $\mathcal{T} \in \mathrm{HC}$ be a normal iteration tree on $\mathcal{P}$. We say $\mathcal{T}$ is **$\mathcal{Q}$-guided** iff for each limit $\lambda < \mathrm{lh}(\mathcal{T})$, $\mathcal{Q} = \mathcal{Q}(\mathcal{T}\restriction\lambda, [0,\lambda]_{\mathcal{T}})$ exists and

[47] Literally we should write "$\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathrm{Lp}^{\Gamma}(\mathcal{N}|\eta){\downarrow}t$", but we will be lax about this from now on.

[48] We could also define a suitable pre mouse $\mathcal{N}$ as a Θ-g-organized $\mathcal{F}$-premouse and the proof given below would work the same.

$\Phi(\mathcal{T}\restriction\lambda)\,^\frown\,(\mathcal{Q},\delta(\mathcal{T}))$ is (ω,ω_1)-iterable. We say that $\mathcal{T}$ is **Γ-guided** iff it is $\mathcal{Q}$-guided and there are iteration strategies in Γ for the phalanxes above. $\dashv$

Remark 6.25. Let $\mathcal{P}$ be k-suitable. For a normal tree $\mathcal{T}$ on $\mathcal{P}$ of limit length there is at most one $\mathcal{T}$-cofinal branch b such that $\mathcal{T}\,^\frown\,b$ is $\mathcal{Q}$-guided. (Let b_0, b_1 be distinct such branches; we can successfully compare the phalanxes $\Phi(\mathcal{T}\,^\frown\,b_0)$ and $\Phi(\mathcal{T}\,^\frown\,b_1)$. Standard fine structure and the fact that $\mathcal{P}$ has at most ω-many Woodins then leads to contradiction.) Therefore if $\mathcal{T}\,^\frown\,b$ is normal, via an ω_1-iteration strategy for $\mathcal{P}$, is based on $[\delta^{\mathcal{P}}_{i-1}, \delta^{\mathcal{P}}_i)$ and $\mathcal{Q}(\mathcal{T}, b)$ exists then $\mathcal{T}\,^\frown\,b$ is $\mathcal{Q}$-guided.

Definition 6.26. Let $\mathcal{N}$ be a ${}^{\mathrm{g}}\mathcal{F}$-whole premouse. We write $\mathcal{Q}^{\Gamma}_{\mathrm{t}}(\mathcal{N})$ for the unique $\mathcal{Q} \trianglelefteq \mathrm{Lp}^{\Gamma}_{+}$ such that $\mathcal{Q}$ is a Q-structure for $\mathcal{N}$, if such exists.[49]

Let $k \leq \omega$, $\mathcal{P}$ be k-suitable and $\mathcal{T}$ a normal, limit length, Γ-guided tree on $\mathcal{P}$. We say that $\mathcal{T}$ is **short** iff $\mathcal{Q}^{\Gamma}_{\mathrm{t}}(M(\mathcal{T}))$ exists; otherwise that $\mathcal{T}$ is **maximal**. $\dashv$

Definition 6.27. Let $\mathcal{P}$ be k-suitable. Let $\mathcal{T}$ be an iteration tree on $\mathcal{P}$. We say that $\mathcal{T}$ is **suitability strict** iff for every $\alpha < \mathrm{lh}(\mathcal{T})$:

(1) If $[0,\alpha]_{\mathcal{T}}$ does not drop then $M^{\mathcal{T}}_\alpha$ is k-suitable.

(2) If $[0,\alpha]_{\mathcal{T}}$ drops and there are trees $\mathcal{U}, \mathcal{V}$ such that $\mathcal{T}\restriction\alpha+1 = \mathcal{U}\,^\frown\,\mathcal{V}$, where $\mathcal{U}$ has last model $\mathcal{R}$, $b^{\mathcal{U}}$ does not drop, and there is $i \in [0,k)$ such that $\mathcal{V}$ is based on $[\delta^{\mathcal{R}}_{i-1}, (\delta^{+\omega}_i)^{\mathcal{R}})$, then no $\mathcal{Q} \trianglelefteq M^{\mathcal{T}}_\alpha$ is $(i+1)$-suitable.

Let Σ be a (partial) iteration strategy for $\mathcal{P}$. We say that Σ is **suitability strict** iff every tree $\mathcal{T}$ via Σ is suitability strict. $\dashv$

Definition 6.28. Let $\mathcal{P}$ be k-suitable. We say that $\mathcal{P}$ is **short tree iterable** iff for every normal Γ-guided tree $\mathcal{T}$ on $\mathcal{P}$, we have:

(1) $\mathcal{T}$ is suitability strict.

(2) If $\mathcal{T}$ has limit length and is short then there is b such that $\mathcal{T}\,^\frown\,b$ is a Γ-guided tree.[50]

(3) If $\mathcal{T}$ has successor length then every one-step putative normal extension of $\mathcal{T}$ is an iteration tree.

Let $\mathcal{P}$ be short tree iterable. The **short tree strategy** $\Psi_{\mathcal{P}}$ for $\mathcal{P}$ is the partial iteration strategy Ψ for $\mathcal{P}$, such that $\Psi(\mathcal{T}) = b$ iff $\mathcal{T}$ is normal and short and $\mathcal{T}\,^\frown\,b$ is Γ-guided. (By 6.25 this specifies $\Psi_{\mathcal{P}}$ uniquely.) $\dashv$

[49] The "t" is for *tame*. While $\mathcal{Q}$ might not be tame, $\mathrm{o}(\mathcal{N})$ is a strong cutpoint of $\mathcal{Q}$.

[50] Recall that *tree* now abbreviates ${}^{\mathrm{g}}\mathcal{F}$*-tree*.

Lemma 6.29. *Let $\mathcal{N}$ be k-suitable.*

(1) Suppose $\mathcal{N}$ is short tree iterable. Then $\Psi_{\mathcal{N}}$ is $\Gamma(\{\mathcal{N}\})$-definable, and so $\Psi_{\mathcal{N}} \in \mathcal{M}$.[51]

(2) Suppose there is a suitability strict normal (ω, ω_1)-strategy Σ for $\mathcal{N}$. Then $\mathcal{N}$ is short tree iterable and $\Psi_{\mathcal{N}} \subseteq \Sigma$. Moreover, for any $\mathcal{T}$ via Σ, $\mathcal{T}$ is via $\Psi_{\mathcal{N}}$ iff for every limit $\lambda < \mathrm{lh}(\mathcal{T})$, $\mathcal{Q}(\mathcal{T}, b)$ exists where $b = [0, \lambda]_{\mathcal{T}}$.

Proof. Part (1) follows from the admissibility of $\mathcal{M}|\alpha$.

Consider (2). Let $\mathcal{T}$ on $\mathcal{N}$ be normal, of limit length, via both Σ and $\Psi_{\mathcal{N}}$. Let $b = \Sigma(\mathcal{T})$. It suffices to show that (a) if $\mathcal{Q}(\mathcal{T}, b)$ exists then $\mathcal{T}$ is short, and (b) if $\mathcal{T}$ is short then $b = \Psi_{\mathcal{N}}(\mathcal{T})$. (Note that if $\mathcal{Q}(\mathcal{T}, b)$ does not exist then $M_b^{\mathcal{T}}$ is k-suitable so $\mathcal{T}$ is maximal.)

Consider (a); suppose $\mathcal{Q} = \mathcal{Q}(\mathcal{T}, b)$ exists. If b does not drop then $M_b^{\mathcal{T}}$ is suitable and $\delta \neq \delta_i(M_b^{\mathcal{T}})$ for any $i < k$. So $C_\Gamma(M(\mathcal{T})) \vDash$"$\delta$ is not Woodin", so our mouse capturing hypothesis implies that $\mathcal{T}$ is short. So suppose that b drops. We can't have $C_\Gamma(M(\mathcal{T})) \subseteq \mathcal{Q}$, by suitability strictness. If δ is a cutpoint of $\mathcal{Q}$ (and so a strong cutpoint) we can then compare $\mathcal{Q}$ with $\mathrm{Lp}^\Gamma(M(\mathcal{T}))$; since the comparison is above δ, we get that $\mathcal{Q} \trianglelefteq \mathrm{Lp}^\Gamma(M(\mathcal{T}))$, so $\mathcal{T}$ is short. So suppose δ is not a cutpoint of $\mathcal{Q}$. Let $E \in \mathbb{E}_+(\mathcal{Q})$ be least such that $\kappa = \mathrm{crit}(E) < \delta$ and let $\mathcal{T}'$ be the normal tree given by $\mathcal{T} \,\widehat{}\, \langle b, E\rangle$. Then $\mathcal{N}^{\mathcal{T}'} \vDash$"$\kappa$ is a limit of Woodins", so $b^{\mathcal{T}'}$ drops and $C_\Gamma(M(\mathcal{T})) \not\subseteq \mathcal{N}^{\mathcal{T}'}$ (by suitability strictness). Also $\mathcal{N}^{\mathcal{T}'} \vDash$"$\delta$ is Woodin" and δ is a cutpoint of $\mathcal{N}^{\mathcal{T}'}$. So $\mathcal{N}^{\mathcal{T}'} = \mathcal{Q}_{\mathrm{t}}^\Gamma(M(\mathcal{T}))$ exists, so $\mathcal{T}$ is short.

Consider (b). Since $\mathcal{T}$ is short, $\mathcal{Q} = \mathcal{Q}(\mathcal{T}, b)$ exists. We claim that $\mathcal{T} \,\widehat{}\, b$ is Γ-guided, which suffices. For it's easy to reduce to the case that δ is not a cutpoint of $\mathcal{Q}$. Let $\mathcal{T}'$ be as above, let $\lambda = \mathrm{lh}(\mathcal{T})$ and $\alpha = \mathrm{pred}^{\mathcal{T}'}(\lambda + 1)$. Let $\mathcal{M}_{\lambda+1}^{*\mathcal{T}'} = \mathcal{M}_\alpha^{\mathcal{T}}|\gamma$. Then $\mathcal{M}_\alpha^{\mathcal{T}}|\gamma \vDash$"$\kappa$ is a limit of cutpoints". It follows that $\mathcal{T}\restriction[\alpha, \mathrm{lh}(\mathcal{T}))$ can be considered an above-κ, normal tree on $M_\alpha^{\mathcal{T}}|\gamma$, and the iterability of the phalanx $\Phi(\mathcal{T}) \,\widehat{}\, (\mathcal{Q}, \delta)$ reduces to the above-κ iterability of $M_\alpha^{\mathcal{T}}|\gamma$, which reduces to the above-δ iterability of $\mathcal{N}^{\mathcal{T}'}$ (because of the existence of $i_{\alpha,\lambda+1}^{\mathcal{T}'}$). But $\mathcal{N}^{\mathcal{T}'} \trianglelefteq \mathrm{Lp}^\Gamma(M(\mathcal{T}))$, so we are done. □

Definition 6.30. Let $A \in \mathcal{P}(\mathbb{R}) \cap \mathcal{M}$. We define the phrase $\mathcal{T}$ **respects** A as in [20], except that we also require that $\mathcal{T}$ be suitability strict (and making any obvious adaptations to our setting). We define $\mathcal{N}$ **is normally** A**-iterable** as in [20], except that we also require that $\mathcal{N}$ be short tree iterable. Using these definitions, we then define **(almost, locally)** A**-iterable** as in [20]. ⊣

Lemma 6.31. *The analogue of* [20, Lemma 1.9.1] *holds.*

[51] But it seems that we might have $\Psi_{\mathcal{N}} \notin \mathcal{M}|\alpha$.

Proof. This is mostly an immediate generalization. The proof in [20] can be run inside $\mathcal{M}_{\mathsf{DC}}$ (in fact, inside $\mathcal{M}$, since $\mathcal{M} \models \mathsf{DC}_{\mathbb{R}}$). Use suitability strictness to see that, for example, in the comparison of $\mathcal{R}|0$ with $\mathcal{N}|0$ (notation as in [20]), no tree drops on its main branch. □

Remark 6.32. We make a further observation on the comparison above. Let $(\mathcal{T},\mathcal{U})$ be the Γ-guided portion of the comparison of, for example, $(\mathcal{R}|0,\mathcal{N}|0)$. Let $\lambda < \mathrm{lh}(\mathcal{T},\mathcal{U})$ be a limit; suppose $\mathcal{T}\restriction\lambda$ is cofinally non-padded. So $\mathcal{Q} = \mathcal{Q}(\mathcal{T}\restriction\lambda, [0,\lambda]_{\mathcal{T}})$ exists. Then in fact, $\delta(\mathcal{T}\restriction\lambda)$ is a strong cutpoint of $\mathcal{Q}$. For otherwise, by the proof of 6.29, $[0,\lambda]_{\mathcal{T}}$ drops in a manner which cannot be undone; i.e., for all $\alpha \geq \lambda$, $[0,\alpha]_{\mathcal{T}}$ drops, a contradiction. Similar remarks pertain to genericity iterations on k-suitable models.

Lemma 6.33. *Let $A \in \mathcal{M} \cap \mathcal{P}(\mathbb{R})$. Then for a cone of $s \in \mathbb{R}$ there is an ω-suitable, A-iterable premouse over s.*

Proof. The following proof is based on the sketch given in [20, 1.12.1].[52] We give a full account here, since the proof is rather involved (it will take several pages) and the possibility of non-tame mice was not covered explicitly in [20]. Moreover, comparing our proof with the remarks in [20, Footnote 12], we will not manage to establish the full Dodd-Jensen property for the iteration strategy we construct, but we will obtain a version of the Dodd-Jensen property which suffices for our purposes.

Say that a set of reals constituting a counterexample to the theorem is **Γ-bad**. Suppose there is a Γ-bad set. For other pointclasses $\bar{\Gamma}$ we define **$\bar{\Gamma}$-bad** analogously.

Let $\zeta_0 < \alpha$ and $z_0 \in \mathbb{R}$ and $\psi_{\mathcal{F}}$ be a Σ_1 formula of $\mathcal{L}_0^-$ such that $\mathcal{F}\restriction\mathrm{HC}$ is definable over $\mathcal{M}|\zeta_0$ from z_0 and $\mathcal{M}|(\zeta_0+1) \models \psi_{\mathcal{F}}(z_0)$ but $\mathcal{M}|\zeta_0 \models \neg\psi_{\mathcal{F}}(z_0)$. Since there is $\xi+1 \in (\theta, l(\mathcal{M}))$ such that $\mathcal{M}|\xi \models \mathsf{ZF}$, by 5.1 there are $\bar{\alpha}, \bar{\xi}, \bar{\beta}$ such that $\zeta_0 < \bar{\alpha} < \bar{\xi} < \bar{\beta} < \alpha$ and $[\bar{\alpha}, \bar{\beta}]$ is a gap of $\mathcal{M}$ and $\Theta^{\mathcal{M}|\bar{\beta}} < \bar{\xi}$ and letting $\bar{\Gamma} = \Sigma_1^{\mathcal{M}|\bar{\alpha}}$, $\mathcal{M}|\bar{\xi} \models \mathsf{ZF}+$"There is a $\bar{\Gamma}$-bad set $A \subseteq \mathbb{R}$". Fixing such a set A, note that A really is $\bar{\Gamma}$-bad. We may assume that $\bar{\beta}$ is least such that there are $\bar{\alpha}, \bar{\xi}$ as above. Then note that $\bar{\beta} = \bar{\xi}+1$, $\rho_1^{\mathcal{M}|\bar{\beta}} = \mathbb{R}$, $p_1^{\mathcal{M}|\bar{\beta}} = \{\bar{\xi}\}$ and $\bar{\beta}$ ends a weak gap of $\mathcal{M}$ (the Σ_1 type of $(\{\bar{\xi}\}, z_0)$ does not reflect, using the choice of ζ_0, z_0). We will show that A is *not* $\bar{\Gamma}$-bad, a contradiction. Let $\langle A_i \rangle_{i<\omega}$ be a self-justifying system at the end of the gap $\mathcal{M}|\bar{\beta}$, with $A_0 = A$. Since $\mathcal{M} \models \mathsf{AD}$, in $\mathcal{M}|\bar{\xi}$ there is a cone of reals s such that there is no ω-suitable, A-iterable premouse over s. Let $z_1 \geq_T z_0$ be a base for this cone, and

[52] We are using g-organized $\mathcal{F}$-mice as our mice over reals. The authors believe that, had we used a hierarchy Z of mice over reals more closely related to the hierarchy of Θ-g-organized mice, then the proof in [16, §7] could be adapted to work in the present context. (One needs to define Z such that Θ-g-organized mice can be realized as derived models of Z-mice, in a reasonably level-by-level manner.) Such a proof would have the advantage of providing some extra information. However, one would need to define and use the relevant Prikry forcing, so it seems to be more work overall, and our approach also has the advantage that it is less dependent on the precise hierarchy of mice over reals that is used. There is also a third approach, which starts out like [16, §7], and, instead of using Prikry forcing, finishes more like our present proof.

such that for every $i < \omega$ there is $\zeta < \Theta^{\mathcal{M}|\bar{\beta}}$ such that A_i is definable over $\mathcal{M}|\zeta$ from z_1. We write $\overline{\mathrm{Lp}}$ for $\mathrm{Lp}^{\bar{\Gamma}}$.

Let $\mathcal{P} \triangleleft \mathrm{Lp}(z_1)$ be least such that $\mathcal{P}$ projects to ω and $\Sigma_{\mathcal{P}}$ is not a $\bar{\Gamma}$ strategy, where $\Sigma_{\mathcal{P}}$ is the (ω, ω_1+1)-iteration strategy for $\mathcal{P}$; by our mouse capturing hypothesis $\mathcal{P}$ exists and is super-small.

We say that a pointclass Λ is **lovely** iff $\Lambda = \Sigma_1^{\mathcal{M}|\zeta}$ for some $\zeta < \alpha$. Let $\langle \Gamma_i \rangle_{i \in [0,9]}$ be lovely pointclasses such that $\bar{\Gamma} \subseteq \Delta_{\Gamma_9}$ and $\Sigma_{\mathcal{P}}$ is $\Delta_{\Gamma_9}(z_1)$ and for each $i \in [1,9]$, $\Gamma_i \subseteq \Delta_{\Gamma_{i-1}}$. Let T_0 be the tree of a scale for a universal Γ_0 set, with $T_0 \in \mathcal{M}$. By Woodin [24] there is $z_2 \in \mathbb{R}$ such that $z_1 \leq_T z_2$ and $H^* = \mathrm{HOD}_{T_0, z_1}^{L_\xi[T_0, z_2]} \models$ "Δ_0 is Woodin", where $\Delta_0 = \omega_2^{L_\xi[T_0, z_2]}$. (We use here that $\mathcal{M}|\xi \models \mathsf{ZF}$.)

Let $T_i, U_i \in H^*$ be trees projecting respectively to a universal Γ_i set and its complement. Let Δ_i be least such that $V_{\Delta_i}^{H^*}$ is Γ_i-Woodin. Let $\lambda < \xi$ be large and such that $(V_\lambda^{H^*}, \Delta_9)$ is a coarse premouse. Let $\pi_H : (H, \Delta) \to (V_\lambda^{H^*}, \Delta_9)$ be elementary, with H countable, $\pi_H, H \in H^*$, and $z_1, T_i, U_i \in \mathrm{rg}(\pi)$ for each $i \leq 9$ (let $U_0 = \emptyset$). Let $\pi_H(T_i^H, U_i^H) = (T_i, U_i)$. Then by arguments in [13] (using $\mathcal{M}|\xi$ as a background $\mathsf{ZF} + \mathsf{AD}$ model):

Fact 6.34. *In $\mathcal{M}|\alpha$ there is a unique (ω_1, ω_1+1)-iteration strategy Λ_H for (H, Δ) such that for each countable successor length tree $\mathcal{T}$ via Λ_H, letting $j = i^{\mathcal{T}}$ and $J = \mathcal{N}^{\mathcal{T}}$, then*

$$p[j(T_8^H)] \subseteq p[T_8] \ \& \ p[j(U_8^H)] \subseteq p[U_8].$$

Moreover, the restriction of Λ_H to HC^{H^*} is the unique π_H-realization strategy in H^*. Further, for $i \geq 1$, $J \models$ "$j(T_i^H), j(U_i^H)$ are $\mathrm{Col}(\omega, j(\Delta))$-absolutely complementing". Moreover,

$$C^H = C_{\bar{\Gamma}} \restriction V_\Delta^H \in H \ \& \ j(C^H) = C_{\bar{\Gamma}} \restriction V_{j(\Delta)}^J;$$

$$\mathcal{F}^H = \mathcal{F} \restriction V_\Delta^H \in H \ \& \ j(\mathcal{F}^H) = \mathcal{F} \restriction i^{\mathcal{T}}(\mathcal{F}^H).$$

Let $\mathbb{C} = \langle N_\alpha \rangle_{\alpha \leq \Delta}$ be the maximal fully backgrounded $L^{\mathrm{g}\mathcal{F}}[\mathbb{E}, z_1]$-construction as computed in H. The fact that this construction does not break down follows from 2.34 and 6.34. (For Λ_H agrees with the π_H-realization strategy. Also, let R, N_α be type 3 and $\pi : R^{\mathrm{sq}} \to N_\alpha^{\mathrm{sq}}$ be Σ_0-elementary. We may assume that π is cofinal in $\nu(N_\alpha)$, by the ISC. It follows that R, and likewise $R' = \mathrm{Hull}_1^R(\emptyset)$, are iterable in $\mathcal{M}|\alpha$. So $R' \triangleleft \mathrm{Lp}(z_1)$, so R' is not superstrong by our mouse capturing hypothesis, so R is not superstrong. So 2.34 applies.) Also by 6.34, for every $\alpha \leq \Delta$ and $n < \omega$, the $(n, \omega_1, \omega_1+1)$-strategy for $\mathfrak{C}_n(N_\alpha)$ given by resurrection and lifting to Λ_H, is a $({}^{\mathrm{g}}\mathcal{F}$-$)$strategy.

Claim 6.35. *There is $\gamma < \Delta$ and $k < \omega$ such that $\rho_{k+1}(N_\gamma) = \omega$ and $\mathfrak{C}_\omega(N_\gamma)$ is not (k, ω_1+1)-iterable in $\mathcal{M}|\bar{\alpha}$.*

Proof. It suffices to see that $\mathbb{C}$ reaches $\mathcal{P}$. By the definability of $\mathcal{P}$, $\mathcal{P} \in H^*$ and $\mathcal{P} \in H$, and letting $\Sigma^H_{\mathcal{P}} = \Sigma_{\mathcal{P}} \restriction V^H_\Delta$, we have $\Sigma^H_{\mathcal{P}} \in H$, and $\Sigma^H_{\mathcal{P}}$ is moved correctly by Λ_H. It follows that the background extenders used in $\mathbb{C}$ all cohere $\Sigma^H_{\mathcal{P}}$, and so we can apply 4.4 (the stationarity of $\mathbb{C}$ with respect to $\mathcal{P}$). So we just need to rule out the possibility that for some normal tree $\mathcal{T}$ on $\mathcal{P}$ via $\Sigma_{\mathcal{P}}$, with last model $\mathcal{P}'$, $N_\Delta \trianglelefteq \mathcal{P}'$. But because $\Sigma_{\mathcal{P}}$ is a Γ_9 strategy and N_Δ is definable over V^H_Δ, we have $\mathcal{T} \in C_{\Gamma_9}(V^H_\Delta)$. But $C_{\Gamma_9}(V^H_\Delta) \models$"$\Delta$ is Woodin", so by the universality of N_Δ (see [19, Lemma 11.1]), $\mathcal{T} \notin C_{\Gamma_9}(V^H_\delta)$, contradiction. □

By the previous claim, we may let $(\gamma, m, \eta) \in \mathrm{Ord}^3$ be lexicographically least such that, letting $\mathcal{P} = \mathfrak{C}_m(N_\gamma)$, η is a ${}^{\mathrm{g}}\mathcal{F}$-whole cutpoint of $\mathcal{P}$ and $\mathcal{R} = \mathrm{Hull}^{\mathcal{P}}_{m+1}(\eta \cup p^{\mathcal{P}}_{m+1})$ is η-sound, and $\mathcal{R}$ is not above-η, $(m, \omega_1 + 1)$-iterable in $\mathcal{M}|\bar{\alpha}$. Let $\Sigma_{\mathcal{R}}$ be the $(m, \omega_1, \omega_1 + 1)$-iteration strategy for $\mathcal{R}$ given by resurrection and lifting to Λ_H. We take $\pi_0 : \mathcal{R} \to \mathcal{P}$ to be the base lifting map. Let $\mathcal{T}$ be on $\mathcal{R}$ via $\Sigma_{\mathcal{R}}$ and $\lambda < \mathrm{lh}(\mathcal{T})$, and let $\mathcal{U}$ be the lifted tree on H. Write $\mathbb{C}_\lambda = i^{\mathcal{U}}_{0,\lambda}(\mathbb{C})$. Let $n = \deg^{\mathcal{T}}(\lambda)$. Write $\pi_\lambda : M^{\mathcal{T}}_\lambda \to \mathcal{P}_\lambda$ for the lifting map; here π_λ is a weak n-embedding and $\mathcal{P}_\lambda = \mathfrak{C}_n(N^{\mathbb{C}_\lambda}_\xi)$ for some $\xi \leq i^{\mathcal{U}}_{0,\lambda}(\gamma)$, with $\xi = i^{\mathcal{U}}_{0,\lambda}(\gamma)$ iff $[0, \lambda]_{\mathcal{T}}$ does not drop in model. (Note that the codomain is $i^{\mathcal{U}}_{0,\lambda}(\mathcal{P})$, not $i^{\mathcal{U}}_{0,\lambda}(\mathcal{R})$, when $[0, \lambda]_{\mathcal{T}}$ does not drop in model.)

Given a premouse $\mathcal{N}$ and $\zeta \in \mathrm{o}(\mathcal{N})$, we say that $\mathcal{N}$ is $(\bar{\Gamma}, k, \zeta)$**-iterable** iff there is an above-ζ, $(k, \omega_1 + 1)$-iteration strategy for $\mathcal{N}$ in $\mathcal{M}|\bar{\alpha}$. We say $\mathcal{N}$ is $(\bar{\Gamma}, \zeta)$-iterable iff $\mathcal{N}$ is $(\bar{\Gamma}, m, \zeta)$-iterable.

Claim 6.36. *Let $\mathcal{T}$ be an above-η normal tree on $\mathcal{R}$ via $\Sigma_{\mathcal{R}}$, of length $\lambda + 1$ for a limit λ. Let $b = b^{\mathcal{T}}$ and $\mathcal{Q} = \mathcal{Q}(\mathcal{T} \restriction \lambda, b)$. Let $k = \omega$ if $\mathcal{Q} \triangleleft M^{\mathcal{T}}_\lambda$ and $k = \deg^{\mathcal{T}}(\lambda)$ otherwise. Suppose that the phalanx $\mathfrak{P} = \Phi(\mathcal{T} \restriction \lambda) \frown \langle \mathcal{Q} \rangle$ is not normally $(k, \omega_1 + 1)$-iterable in $\mathcal{M}|\bar{\alpha}$ (here k indicates the degree for $\mathcal{Q}$). Let $\delta = \delta(\mathcal{T} \restriction \lambda)$ and $M_{\mathcal{T}} = M(\mathcal{T} \restriction \lambda)$. Then either:*

(*a*) *δ is a strong cutpoint of $\mathcal{Q}$, $\mathcal{Q} = M^{\mathcal{T}}_\lambda$, $b^{\mathcal{T}}$ does not drop in model or degree and $\mathcal{Q}||(\delta^+)^{\mathcal{Q}} = \overline{\mathrm{Lp}}(M_{\mathcal{T}})$; or*

(*b*) *δ is not a cutpoint of $\mathcal{Q}$, and letting $E \in \mathbb{E}^{\mathcal{Q}}_+$ be such that $\mathrm{crit}(E) < \delta < \mathrm{lh}(E)$, with $\mathrm{lh}(E)$ minimal, and letting $\mathcal{T}^+$ be the normal tree $\mathcal{T} \frown \langle E \rangle$, then $b^{\mathcal{T}^+}$ does not drop in model or degree, and $\mathcal{Q}||\mathrm{lh}(E) = \overline{\mathrm{Lp}}(M_{\mathcal{T}})$.*

Proof. Suppose δ is a cutpoint (hence strong cutpoint) of $\mathcal{Q}$. Because δ is a cutpoint, the difficulty in iterating $\mathfrak{P}$ gives that $\mathcal{Q}$ is not $(\bar{\Gamma}, k, \delta)$-iterable. Because δ is a strong cutpoint and by standard fine structure, $\mathcal{Q} \triangleleft \mathrm{Lp}(M_{\mathcal{T}})$.

We leave the proof that $\mathcal{Q} = M^{\mathcal{T}}_\lambda$ to the reader; assume this. We show that b does not drop in model or degree; suppose otherwise. Let $m' = \deg^{\mathcal{T}}(\lambda)$, so $\mathcal{Q} = \mathrm{Hull}^{\mathcal{Q}}_{m'+1}(\delta \cup p^{\mathcal{Q}}_{m'+1})$. We have $(\gamma', m') <_{\mathrm{lex}} (i^{\mathcal{U}}_{0,\lambda}(\gamma), m)$ where $\mathcal{P}_\lambda = \mathfrak{C}_{m'}(N^{\mathbb{C}_\lambda}_{\gamma'})$. We have $p^{\mathcal{P}_\lambda}_{m'+1} = \pi_\lambda(p^{\mathcal{Q}}_{m'+1})$ and the

$m'+1$-solidity witnesses for $(\mathcal{P}_\lambda, p^{\mathcal{P}_\lambda}_{m'+1})$ are in $\mathrm{rg}(\pi_\lambda)$. (The latter is by the commutativity between the copy and iteration maps.) But

$$\mathrm{rg}(\pi_\lambda) \subseteq \bar{\mathcal{P}} = \mathrm{Hull}^{\mathcal{P}_\lambda}_{m'+1}(\pi_\lambda(\delta) \cup p^{\mathcal{P}_\lambda}_{m'+1}).$$

Therefore $\bar{\mathcal{P}}$ is $\pi_\lambda(\delta)$-sound. Moreover, we have a weak m'-embedding $\sigma : \mathcal{Q} \to \bar{\mathcal{P}}$ such that $\sigma(\delta) = \pi_\lambda(\delta)$. So σ lifts above-δ trees on $\mathcal{Q}$ to above-$\sigma(\delta)$ trees on $\bar{\mathcal{P}}$. Therefore $\bar{\mathcal{P}}$ is not $(\bar{\Gamma}, m', \pi_\lambda(\delta))$-iterable. This contradicts the minimality of $(i^{\mathcal{U}}_{0,\lambda}(\gamma), m)$ in $M^{\mathcal{U}}_\lambda$.

So $b^{\mathcal{T}}$ does not drop. An argument similar to the preceding one gives that $\mathcal{Q}||(\delta^+)^{\mathcal{Q}} \subseteq \overline{\mathrm{Lp}}(M_{\mathcal{T}})$. Suppose that $\mathcal{Q}||(\delta^+)^{\mathcal{Q}} \in \overline{\mathrm{Lp}}(M_{\mathcal{T}})$. Let $\mathcal{Q}' \triangleleft \overline{\mathrm{Lp}}(M_{\mathcal{T}})$ be such that $\mathcal{Q}'||(\delta^+)^{\mathcal{Q}'} = \mathcal{Q}||(\delta^+)^{\mathcal{Q}}$ and $\mathcal{Q}'$ projects to δ. Now $\mathcal{Q}'{\downarrow}z_1$ is δ-sound. For let $n < \omega$ be such that $\rho^{\mathcal{Q}'}_{n+1} = M_{\mathcal{T}} \neq \rho^{\mathcal{Q}'}_n$. Then $\mathcal{Q}'{\downarrow}z_1$ is n-sound, and $p^{\mathcal{Q}'}_{n+1}$ is $(n+1)$-solid for $\mathcal{Q}'{\downarrow}z_1$, and

$$Q' = \mathrm{Hull}^{Q'}_{n+1}(p^{Q'}_{n+1}), \tag{6.5}$$

and so it suffices to see that $\mathcal{Q}' = \mathcal{K}$ where

$$\mathcal{K} = \mathrm{Hull}^{\mathcal{Q}'\downarrow z_1}_{n+1}(\delta \cup p^{\mathcal{Q}'}_{n+1}).$$

By line (6.5), it suffices to see that $\delta \in \mathcal{K}$. But if not then $\delta = \mathrm{crit}(\pi)$ where π is the uncollapse embedding, but since δ is Woodin in $\mathcal{Q}$, this implies that δ is not a cutpoint of $\mathcal{Q}$, a contradiction. So comparing $\mathcal{Q}$ with $\mathcal{Q}'{\downarrow}z_1$, we get $\mathcal{Q} = \mathcal{Q}'{\downarrow}z_1$. So $\mathcal{Q}$ is $(\bar{\Gamma}, \delta)$-iterable, a contradiction.

Now suppose δ is not a cutpoint of $\mathcal{Q}$. Suppose that $b^{\mathcal{T}^+}$ drops in model or degree. Since δ is a strong cutpoint of $\mathcal{N}^{\mathcal{T}^+}$, then as before, by choice of (γ, m), $\mathcal{N}^{\mathcal{T}^+}$ is $(\bar{\Gamma}, j, \delta)$-iterable, where $j = \deg^{\mathcal{T}^+}(\mathcal{N}^{\mathcal{T}^+})$. Therefore, letting $\kappa = \mathrm{crit}(E)$ and $\mathrm{lh}(\mathcal{T}^+) = \xi+1$, $M^{*\mathcal{T}^+}_\xi$ is $(\bar{\Gamma}, j, \kappa)$-iterable (we can copy trees using i_E). But κ is a cutpoint of $M^{*\mathcal{T}^+}_\xi$. So $\mathcal{T}^+ = (\mathcal{T}\restriction\chi+1)\,{}^\frown\,\mathcal{T}'$, where $\chi = \mathrm{pred}^{\mathcal{T}}(\xi)$ and $\mathcal{T}'$ is an above-κ, j-maximal tree on $M^{*\mathcal{T}^+}_\xi$. Thus, the iterability of $\mathfrak{P}$ can be reduced to that of $M^{*\mathcal{T}^+}_\xi$ above κ. Therefore $\mathfrak{P}$ is iterable in $\mathcal{M}|\bar{\alpha}$, a contradiction. So $b^{\mathcal{T}^+}$ does not drop. We then get $\mathcal{Q}||\mathrm{lh}(E) = \overline{\mathrm{Lp}}(M_{\mathcal{T}})$ by the arguments just given. $\square$

Let $\mathcal{T}$ be an above-η normal tree on $\mathcal{R}$, of limit length. Let b be a $\mathcal{T}$-cofinal branch. We say that b is **$\bar{\Gamma}$-verified for $\mathcal{T}$** iff $\Phi(\mathcal{T})\,{}^\frown\,\langle Q\rangle$ is normally (k, ω_1+1)-iterable in $\mathcal{M}|\bar{\alpha}$, where $Q = Q(\mathcal{T}, b)$ and if $Q \triangleleft M^{\mathcal{T}}_b$ then $k = \omega$ and if $Q = M^{\mathcal{T}}_b$ then $k = \deg^{\mathcal{T}}(b)$.

Claim 6.37. *Let $\mathcal{T}$ be as above. Then there is at most one branch $\bar{\Gamma}$-verified for $\mathcal{T}$. However, the following partial strategy Ψ is not an above-η, (m, ω_1+1)-strategy for $\mathcal{R}$: Given $\mathcal{T}$, let $\Psi(\mathcal{T})$ be the unique branch which is $\bar{\Gamma}$-verified for $\mathcal{T}$.*

Proof. Uniqueness follows from the usual comparison and fine structural arguments, using the η-soundness of $\mathcal{R}$. Suppose existence holds. Then by uniqueness and because $\mathcal{M}|\bar{\alpha}$ is admissible, $\mathcal{R}$ is $(\bar{\Gamma}, \eta)$-iterable, contradiction. $\square$

Definition 6.38. We define the term $\bar{\Gamma}$-k**-suitable** analogously to *k-suitable* (cf. 6.23), but with $\bar{\Gamma}$ replacing Γ. We likewise define $\bar{\Gamma}$-A**-iterable** and $\bar{\Gamma}$**-suitability strict**. Let R be $\bar{\Gamma}$-ω-suitable with $z_1 \in R$. Then σ_i^R denotes the $\mathrm{Col}(\omega, \delta_i^R)$-term capturing A_i over R (see [13]). Let Q be a structure and $\pi : Q \to P$. We say that π is an $\vec{A}$**-embedding** iff π is Σ_1-elementary and $\sigma_i^R \in \mathrm{rg}(\pi)$ for all $i < \omega$. $\dashv$

Claim 6.39. *(i) N_γ has infinitely many Woodins in the interval $(\eta, \rho_m(N_\gamma))$. Let δ_ω be the supremum of the first ω-many and let $N = (N_\gamma|\delta_\omega)\downarrow(N_\gamma|\eta)$. Then (ii) N is $\bar{\Gamma}$-ω-suitable.*

Proof. We will construct a $\bar{\Gamma}$-ω-suitable premouse which is an initial segment of a $\Sigma_{\mathcal{R}}$-iterate of $\mathcal{R}$. This is by applying Claim 6.37 and an obvious generalization thereof, in tandem with Claim 6.36, up to ω many times. So let $\mathcal{T}_0$ on $\mathcal{R}_0 = \mathcal{R}$ be via $\Sigma_{\mathcal{R}}$ (so above $\delta_{-1} = \eta$), witnessing the failure of "existence" in Claim 6.37, with $\mathcal{T}_0$ of minimal length. Let $\delta_0 = \delta(\mathcal{T}_0)$. Let $b = \Sigma(\mathcal{T}_0)$. So Claim 6.36 applies to $\Phi(\mathcal{T}_0) \frown \langle Q(\mathcal{T}_0, b)\rangle$. We use notation as there, so write $\mathcal{T} = \mathcal{T}_0 \frown b$ and $\delta = \delta_0$.

Suppose first that conclusion (b) of Claim 6.36 holds. Let $\kappa = \mathrm{crit}(E)$. Since E overlaps δ and $b^{\mathcal{T}^+}$ does not drop in model or degree, $\mathcal{N}^{\mathcal{T}^+}$ has at least κ-many Woodins $< \delta$, and $\delta < \rho_m(\mathcal{N}^{\mathcal{T}^+})$. And $\mathcal{N}^{\mathcal{T}^+}$ is not $(\bar{\Gamma}, \delta)$-iterable. Now let δ^*_ω be the supremum of the first ω-many Woodins of $\mathcal{N}^{\mathcal{T}^+}$ above η. Let ζ be least such that $\delta^*_\omega < \mathrm{lh}(E^{\mathcal{T}}_\zeta)$. So $\mathcal{N}^{\mathcal{T}^+}|\delta^*_\omega = M^{\mathcal{T}}_\zeta|\delta^*_\omega$. Note that δ^*_ω is a strong cutpoint of $M^{\mathcal{T}}_\zeta$ and $\zeta \in b^{\mathcal{T}^+}$, and so $[0, \zeta]_{\mathcal{T}}$ does not drop in model or degree. Therefore $M^{\mathcal{T}}_\zeta$ is not $(\bar{\Gamma}, \delta^*_\omega)$-iterable. Now let $\mathcal{U}$ be the lifted tree, via Σ_H, on H. We have $\mathcal{P}_\zeta = i^{\mathcal{U}}_{0,\zeta}(\mathfrak{C}_m(N_\gamma))$ and $\pi_\zeta(\delta^*_\omega) < \rho_m(\mathcal{P}_\zeta)$ and $\pi_\zeta(\delta^*_\omega)$ is the sup of the first ω Woodins of $\mathcal{P}_\zeta$ above η, and $\mathcal{P}_\zeta$ is not $(\bar{\Gamma}, \pi_\zeta(\delta^*_\omega))$-iterable. By the elementarity of $i^{\mathcal{U}}_{0,\zeta}$, this gives (i), and $(*)$ $\mathcal{P} = \mathfrak{C}_m(N_\gamma)$ is not $(\bar{\Gamma}, \delta_\omega)$-iterable.

We now verify condition (c) of $\bar{\Gamma}$-ω-suitability (cf. 6.23). Let κ be a cutpoint of $\mathcal{P}|\delta_\omega$ with $\eta \leq \kappa$. Let $\mathcal{C}_\kappa$ be the κ-core of $\mathcal{P}$. We claim that $(**)$ $\mathcal{C}_\kappa$ is not $(\bar{\Gamma}, \kappa)$-iterable. For we have $\pi_0 : \mathcal{R} \to \mathcal{P}$ is the core map. Let $\bar{\kappa} \in \mathrm{o}(\mathcal{R})$ be least such that $\pi_0(\bar{\kappa}) \geq \kappa$, and let $\pi_0(\bar{\delta}_\omega) = \delta_\omega$.

Suppose $\pi_0(\bar{\kappa}) = \kappa$. Let ξ be least such that $i^{\mathcal{T}^+}(\bar{\kappa}) < \mathrm{lh}(E^{\mathcal{T}}_\xi)$. Then $M^{\mathcal{T}}_\xi$ is not $(\bar{\Gamma}, i^{\mathcal{T}}_{0,\xi}(\bar{\kappa}))$-iterable because $i^{\mathcal{T}}_{0,\xi}(\bar{\kappa})$ is a cutpoint of $M^{\mathcal{T}}_\xi$, and $M^{\mathcal{T}}_\zeta$ is not $(\bar{\Gamma}, \delta^*_\omega)$-iterable. But then since $M^{\mathcal{T}}_\xi$ is $i^{\mathcal{T}}_{0,\xi}(\bar{\kappa})$-sound, $i^{\mathcal{U}}_{0,\xi}(\mathcal{C}_\kappa)$ is not $(\bar{\Gamma}, i^{\mathcal{U}}_{0,\xi}(\kappa))$-iterable, which gives $(**)$.

Now suppose $\pi_0(\bar{\kappa}) > \kappa$. Let ξ be least such that $\kappa' = \sup i^{\mathcal{T}^+}``\bar{\kappa} < \mathrm{lh}(E^{\mathcal{T}}_\xi)$. Then $\xi \in b^{\mathcal{T}^+}$ and $\kappa' \leq \mathrm{crit}(i^{\mathcal{T}^+}_{\xi, b^{\mathcal{T}^+}})$. One can show that $\pi_\xi(\kappa') > i^{\mathcal{U}}_{0,\xi}(\kappa)$; and $\pi_\xi \circ i^{\mathcal{T}}_{0,\xi}``\bar{\kappa} \subseteq i^{\mathcal{U}}_{0,\xi}(\kappa)$. Therefore κ' is a cutpoint of $M^{\mathcal{T}^+}_\xi$; and $M^{\mathcal{T}^+}_\xi$ is κ'-sound. Now argue much as before, giving $(**)$.

Now let κ be a ${}^{\mathrm{g}}\mathcal{F}$-whole strong cutpoint of $\mathcal{P}|\delta_\omega$. Let $\mathcal{C}_{\kappa+1}$ be the $(\kappa+1)$-core of $\mathcal{P}$. By $(**)$, the choice of γ and universality for premice over $\mathcal{P}|\kappa$, we have

$$\mathcal{P}|(\kappa^+)^{\mathcal{P}} = \mathcal{P}_{\kappa+1}|(\kappa^+)^{\mathcal{P}_{\kappa+1}} = \overline{\mathrm{Lp}}(\mathcal{P}|\kappa).$$

This gives condition (c) of $\bar{\Gamma}$-ω-suitability.

It remains to verify condition (d). So let $\xi < \delta_\omega$ with $\xi \geq \eta$ and ξ not Woodin in $\mathcal{P}$; we must show that $C_{\bar{\Gamma}}(\mathcal{P}|\xi) \models$"$\xi$ is not Woodin". We may assume that $\mathcal{P}|\xi$ is ${}^{\mathrm{g}}\mathcal{F}$-whole, and by condition (c), also that ξ is not a strong cutpoint of $\mathcal{P}$. Let $F \in \mathbb{E}^{\mathcal{P}}$ be least such that $\mu = \mathrm{crit}(F) \leq \xi < \mathrm{lh}(F)$. Note that by coherence and the ISC, μ is a limit of cutpoints of $\mathcal{P}|\xi$. So if $\mu = \xi$ then $\mathcal{P}|\xi$ is the Q-structure for ξ, so we are done. So suppose $\mu < \xi$. We may assume that $\mathcal{P}||\mathrm{lh}(F) \models$"$\xi$ is Woodin", because otherwise there is $\mathcal{Q} \triangleleft \mathcal{P}||\mathrm{lh}(F)$ such that $\mathcal{Q}$ is a Q-structure for ξ and ξ is a strong cutpoint of $\mathcal{Q}$, and so $\mathcal{Q} \triangleleft \overline{\mathrm{Lp}}(\mathcal{P}|\xi)$ (by resurrection and the choice of γ). Therefore μ is not a cardinal of $\mathcal{P}$. Let $\mathcal{Q} \triangleleft \mathcal{P}$ be least such that $\mathrm{lh}(F) \leq \mathrm{o}(\mathcal{Q})$ and $\rho_\omega^{\mathcal{Q}} < \mu$. Then $\mathcal{Q}$ collapses ξ. Let $\zeta \in [\rho_\omega^{\mathcal{Q}}, \mu)$ be a ${}^{\mathrm{g}}\mathcal{F}$-whole strong cutpoint of $\mathcal{Q}$. Then $\mathcal{Q} \trianglelefteq \overline{\mathrm{Lp}}(\mathcal{P}|\zeta)$, so $\mathcal{Q} \in C_{\bar{\Gamma}}(\mathcal{P}|\xi)$, which suffices. This completes the proof that $\mathcal{P}|\delta_\omega$ is $\bar{\Gamma}$-ω-suitable in this case.

Now suppose that conclusion (a) of Claim 6.36 holds. Let $\mathcal{T}_0^+ = \mathcal{T}_0 \frown \langle b \rangle$ and let $\mathcal{R}_1 = \mathcal{N}^{\mathcal{T}_0^+}$. Then $b^{\mathcal{T}_0^+}$ does not drop in model or degree. And δ_0 is a strong cutpoint of $\mathcal{R}_1$, $\mathcal{R}_1$ is δ_0-sound, projects $< \delta_0$, and is not $(\bar{\Gamma}, \delta_0)$-iterable. So the obvious modification of Claim 6.37 applies to $\mathcal{R}_1$ above δ_0. Pick $\mathcal{T}_1$ on $\mathcal{R}_1$, above δ_0, like before. Again apply Claim 6.36. If its conclusion (b) holds proceed as before, and otherwise let $\mathcal{R}_1 = \mathcal{N}^{\mathcal{T}_1^+}$ and pick $\mathcal{T}_2$ on $\mathcal{R}_1$, etc.

If the above process produces $\mathcal{R}_n$ and $\mathcal{T}_n$ for all $n < \omega$, then we get (i) much as before, and note that, letting δ_n be the n^{th} Woodin of $\mathcal{P} = \mathfrak{C}_m(N_\gamma)$ above η, then $\mathcal{P}$ is not $(\bar{\Gamma}, \delta_n)$-iterable. Part (ii) follows much like before. □

Claim 6.40. *Let P be $\bar{\Gamma}$-ω-suitable and let $\pi : Q \to P$ be an $\vec{A}$-embedding. Then (i) Q is $\bar{\Gamma}$-ω-suitable and for each $i < \omega$, (ii) $\pi(\sigma_i^Q) = \sigma_i^P$, and (iii) $\mathrm{rg}(\pi)$ is cofinal in δ_i^N.*

Proof. Parts (i) and (ii) are by condensation of term relations for self-justifying-systems; see [13]. Consider (iii). If $\mathrm{rg}(\pi) \cap \delta_i^P$ is bounded in δ_i^P, then we may assume that $\mathrm{crit}(\pi) = \delta_i^Q$, by taking the appropriate hull (cf. the first part of the proof of [20, Lemma 1.16.2]). But then $Q|\delta_i^Q = P|\delta_i^Q$, and $P|\delta_i^Q$ is not $\bar{\Gamma}$-Woodin, but $Q \models$"δ_i^Q is Woodin", so Q is not $\bar{\Gamma}$-ω-suitable, contradiction. □

Definition 6.41. Let $\mathcal{T} = \langle \mathcal{T}_\alpha \rangle_{\alpha \leq \gamma}$ be a stack of normal iteration trees. We say that $\mathcal{T}$ is **relevant** iff for every $\alpha < \gamma$, $b^{\mathcal{T}_\alpha}$ does not drop. (Here we allow $\mathcal{T}_\gamma$ to be trivial, and it might drop.) The term **relevantly-$(\omega, \omega_1, \omega_1 + 1)$-iteration strategy** is defined as is $(\omega, \omega_1, \omega_1 + 1)$-*iteration strategy*, except that the former only acts on relevant trees. $\dashv$

From now on we fix N as defined in Claim 6.39. Let Σ_N be the relevantly-$(\omega, \omega_1, \omega_1 + 1)$ strategy for N given by resurrection and lifting to Λ_H. The next claim follows from 6.34.

Claim 6.42. *For any successor length tree $\mathcal{U}$ on H via Λ_H, $i^{\mathcal{U}}(N)$ is $\bar{\Gamma}$-ω-suitable and $i^{\mathcal{U}}\restriction N : N \to i^{\mathcal{U}}(N)$ is an $\vec{A}$-embedding.*

Claim 6.43. *Σ_N is $\bar{\Gamma}$-suitability strict. Moreover, let $\mathcal{T}$ be via Σ_N, of successor length, such that $b^{\mathcal{T}}$ does not drop. Then $i^{\mathcal{T}}$ is an $\vec{A}$-embedding.*

Proof. Let $\mathcal{T}$ be via Σ_N, of successor length. If $b^{\mathcal{T}}$ does not drop, then the lemma's conclusions regarding $\mathcal{N}^{\mathcal{T}}$ and $i^{\mathcal{T}}$ follow from 6.40 and 6.42.

Suppose $b^{\mathcal{T}}$ drops and that $i < \omega$ is as in 6.27(2), but some $R \trianglelefteq \mathcal{N}^{\mathcal{T}}$ is $\bar{\Gamma}$-$(i+1)$-suitable. For simplicity assume that $\mathcal{T}$ consists of just one normal tree and that $\mathcal{T}$ has minimal possible length. It follows that for every extender E used in $\mathcal{T}$, $\nu(E) < \delta = \delta_i^R$. Let $n = \deg^{\mathcal{T}}(b^{\mathcal{T}})$. Then $\rho_{n+1}(\mathcal{N}^{\mathcal{T}}) < \mathrm{o}(R)$ and $\mathcal{N}^{\mathcal{T}}$ is δ-sound. So let $Q \trianglelefteq \mathcal{N}^{\mathcal{T}}$ be least such that $R \trianglelefteq Q$ and $\rho_\omega^Q \leq \delta$. So $R|(\delta^+)^R = \mathrm{Lp}^{\bar{\Gamma}}(R|\delta) = Q|(\delta^+)^Q$. Also $Q \models$"δ is Woodin" and Q is δ-sound and δ is a strong cutpoint of Q (because η is a strong cutpoint of N). So letting $j < \omega$ be such that $\rho_{j+1}^Q \leq \delta < \rho_j^Q$, Q is not $(\bar{\Gamma}, j, \delta)$-iterable. Let $\mathcal{U}$ be the Λ_H-tree on H given by lifting $\mathcal{T}$. Let J be the last model of $\mathcal{U}$. Let $\alpha \in \mathrm{o}(J)$ and $\pi : \mathcal{N}^{\mathcal{T}} \to \mathfrak{C}_n(N_\alpha^{i^{\mathcal{U}}(\mathbb{C})})$ be the lifting map. Then using π and resurrection in J, and by choice of γ, we get that Q is $(\bar{\Gamma}, j, \delta)$-iterable, a contradiction. (Suppose $\mathcal{N}^{\mathcal{T}}$ is type 3. If $\nu(E(\mathcal{N}^{\mathcal{T}})) < \mathrm{o}(Q) < \mathrm{o}(\mathcal{N}^{\mathcal{T}})$ then let $E^* \in J$ be a background extender for $N_\alpha^{i^{\mathcal{U}}(\mathbb{C})}$ and lift Q to a model in $\mathrm{Ult}(J, E^*)$. If $Q = \mathcal{N}^{\mathcal{T}}$ then $\delta < \mathrm{crit}(E^Q)$ so there is no problem.) $\square$

Definition 6.44. Let Q be $\bar{\Gamma}$-ω-suitable. Let Σ be a relevantly-$(\omega, \omega_1, \omega_1)$ iteration strategy for Q. We say that $(\mathcal{T}, P)$ is a **Σ-pair** iff $\mathcal{T}$ is a countable tree on Q via Σ, with last model P. We say that a Σ-pair $(\mathcal{T}, P)$ is **non-dropping** iff $b^{\mathcal{T}}$ does not drop. We say that Σ is **$\vec{A}$-good** iff for every non-dropping Σ-pair $(\mathcal{T}, P)$, P is $\bar{\Gamma}$-ω-suitable and $i^{\mathcal{T}}$ is an $\vec{A}$-embedding. If $(\mathcal{T}, P)$ is a non-dropping Σ-pair, we write $\Sigma_P^{\mathcal{T}}$ for the $(\mathcal{T}, P)$-tail of Σ (that is, $\Sigma_P^{\mathcal{T}}$ is the relevantly-$(\omega, \omega_1, \omega_1 + 1)$ iteration strategy Λ for P where $\Lambda(\mathcal{U}) = \Sigma(\mathcal{T}, \mathcal{U})$). $\dashv$

The following claim is immediate:

Claim 6.45. *Let Σ be a relevantly-$(\omega, \omega_1, \omega_1 + 1)$-iteration strategy for Q. Let $(\mathcal{T}, P)$ be a non-dropping Σ-pair. If Σ is suitability strict then $\Sigma_P^{\mathcal{T}}$ is suitability strict. If Σ is $\vec{A}$-good then $\Sigma_P^{\mathcal{T}}$ is $\vec{A}$-good.*

Claim 6.46. *Let Q be $\bar{\Gamma}$-ω-suitable. Then there is at most one suitability strict $\vec{A}$-good relevantly-$(\omega, \omega_1, \omega_1+1)$ iteration strategy for Q.*

Proof. Let Σ, Λ be two such strategies, and let $\mathcal{T}$ be of limit length, via Σ, Λ, such that $b = \Sigma(\mathcal{T}) \neq \Lambda(\mathcal{T}) = c$. We may assume that $\mathcal{T}$ is normal. We can compare the phalanx $\Phi(\mathcal{T}) \frown b$ with the phalanx $\Phi(\mathcal{T}) \frown c$, forming trees $\mathcal{U}, \mathcal{V}$, using Σ, Λ, respectively. The comparison is successful. By suitability strictness, we have $\mathcal{N}^{\mathcal{U}} = P = \mathcal{N}^{\mathcal{V}}$. By standard fine structure, $b^{\mathcal{U}}$ and $b^{\mathcal{V}}$ do not drop and $\mathcal{N}^{\mathcal{U}} \models$"$\delta(\mathcal{T})$ is Woodin". In particular, $\delta(\mathcal{T}) = \delta_k^P$ for some $k < \omega$. Because Σ, Λ are $\vec{A}$-strategies and by 6.40, therefore $\mathrm{rg}(i^{\mathcal{U}}) \cap \mathrm{rg}(i^{\mathcal{V}})$ is unbounded in δ_k^P. But then $\mathrm{rg}(i_b^{\mathcal{T}}) \cap \mathrm{rg}(i_c^{\mathcal{T}})$ is unbounded in δ_k^P, so $b = c$. Contradiction. □

We are now in a position to establish a version of the Dodd-Jensen property.

Claim 6.47. *Let Σ be an $\vec{A}$-good, suitability strict strategy for Q. Let $(\mathcal{T}, P)$ be a non-dropping Σ-pair.*

(*1*) *Let $\pi : R \to P$ be an $\vec{A}$-embedding. Then the π-pullback Λ of $\Sigma_P^{\mathcal{T}}$ is $\vec{A}$-good and suitability strict. Therefore if $R = Q$ then $\Lambda = \Sigma$.*

(*2*) *Let $\pi : Q \to P$ be an $\vec{A}$-embedding. Then for all $\alpha < \mathrm{o}(Q)$, $i^{\mathcal{T}}(\alpha) \leq \pi(\alpha)$.*

Proof. The first clause of (1) is proven like 6.43. This together with 6.46 yields the second clause. For (2), the standard proof of the Dodd-Jensen property applies; the copying does not break down by (1). □

One can now deduce that N is $\bar{\Gamma}$-A-iterable, because 6.45 and 6.47 apply to N and Σ_N, which is enough of the Dodd-Jensen property for Σ_N to apply the proof of [14, Theorem 4.6]. Let $g \subseteq \mathrm{Col}(\omega, \mathcal{N}|\eta)$ be $\mathcal{N}$-generic. Let $x \in \mathbb{R} \cap \mathcal{N}|(\eta+1)[g]$ code $(\mathcal{N}|\eta, g)$. Then we can reorganize $N[x]$ as a premouse N^* over x, and N^* is $\bar{\Gamma}$-ω-suitable and $\bar{\Gamma}$-A-iterable; these facts all follow by S-construction.[53] But $x \geq_T z_1$, contradicting the choice of z_1. This completes the proof of 6.33. □

Now for simplicity assume $n = 1$ and $\beta = l(\mathcal{M})$ is a limit ordinal; we allow that $X^{\mathcal{M}} \neq \emptyset$. Let $p, w_1, W, \Sigma, \langle \beta_i, Y_i, \psi_i \rangle_{i<\omega}$ be as in the proof of 6.9. Claim 6.12 holds. Let $z = w_1$, $G = p$, $X = X^{\mathcal{M}}$, and U, U' the trees of the scales as in 4.22. Define the language

$$\mathcal{L} = \mathcal{L}_0 \cup \{\dot{\beta}_i, \dot{\mathcal{M}}_i\}_{i<\omega} \cup \{\dot{G}, \dot{p}, \dot{W}, \dot{z}, \dot{X}, \dot{U}, \dot{U'}\};$$

[53] S-construction for g-organized $\mathcal{F}$-premice; cf. 5.5. Now $N{\downarrow}(N|\eta)$ is a premouse over $N|\eta$. Using S-construction we can translate back and forth between premice P over $N|\eta$ and premice P^* over x, where P^* is a reorganization of $P[x]$, and iterates of P correspond to iterates of P^*, with iteration maps agreeing over P.

each symbol in $\mathcal{L}\backslash\mathcal{L}_0$ is a constant symbol. Relative to these definitions, let B_0, $\langle B_0^i\rangle_{i<\omega}$ and $\vec{S}=\langle S_i\rangle_{i<\omega}$ be as in [20]. The analogue of [20, Corollary 1.14] holds (since $\langle S_i\rangle_{i<\omega}\in\mathcal{J}_1(\mathcal{M})$, its proof works in $\mathcal{M}_{\mathsf{DC}}$; thus, the resulting iterate $\mathcal{N}$ is in $\mathcal{M}$). Regarding [20, Lemma 1.15.1], see [9] for details on the process of interleaving comparison with genericity iteration.[54] Also, in the proof of [20, Lemma 1.15.1], with notation as there, instead of demanding $\pi : H\to V_\gamma$ we can make do with $\pi : H\to Z$ where $Z\in\mathscr{B}$ is transitive and sufficiently large, where $\mathcal{F}$ is over $\mathscr{B}$ (and thus we can find such π, H). We need to prove the following:

Lemma 6.48. *Let $\mathcal{N}$ be ω-suitable and $\vec{S}$-iterable. Let $\pi\colon\mathcal{Q}\to\mathcal{N}$ be Σ_1-elementary with $\tau^{\mathcal{N}}_{i,j}\in\operatorname{rg}(\pi)$ for all $i,j<\omega$. Then there is some $m<\omega$ such that for all $n\geq m$, $\operatorname{rg}(\pi)$ is cofinal in $\delta^{\mathcal{N}}_n$.*

Proof. The proof mostly follows that of [20, 1.16.2]. But consider the proof of its Claim; we adopt the same notation. Within that proof, consider the proof that $\mathcal{M}^*=\bar{\mathcal{M}}$. We prove this, as things are different here. Let X^*, U^*, etc, be $\dot{X}^{\mathcal{M}^*}$, $\dot{U}^{\mathcal{M}^*}$, etc. Let $\bar{X}$ be $X^{\bar{\mathcal{M}}}$, etc. Let $X^-=X^{\mathcal{M}^-}$, etc.

First note that $X^*=X\cap\mathcal{M}^*=\bar{X}$, for $\rho^-\circ\psi^*$ yields order-preserving maps $U^*\to U$ and $U'^*\to U'$. Therefore $a^{\mathcal{M}^*}=a^{\bar{\mathcal{M}}}$. So essentially as in the proof of 6.9, $\mathcal{M}^*$ is a 1-sound $\mathcal{J}$-model over $a^{\bar{\mathcal{M}}}$ with $\rho_1(\mathcal{M}^*)=\mathbb{R}^{\mathcal{M}^*}$ and $p_1^{\mathcal{M}^*}=p$.

Because $\rho^*\circ\psi^*\colon\mathcal{H}^*\to\mathcal{H}$ is Σ_1-elementary, and by 4.19, $\mathcal{H}^*$ is a $(0,\omega_1+1)$-iterable g-organized $\mathcal{F}$-premouse over $T^{\mathcal{M}^*}$ (in V). Likewise for $\mathcal{H}^{\mathcal{M}^*|\eta}$ for every η such that $\mathcal{M}^*|\eta$ is relevant. So $\mathcal{M}^*$ is a $(0,\omega_1+1)$-iterable Θ-g-organized $\mathcal{F}$-premouse over $X^{\bar{\mathcal{M}}}$.

So we can compare $\mathcal{M}^*$ with $\bar{\mathcal{M}}$. Because they are both 1-sound and minimal for realizing Σ, they are equal. $\square$

We modify the statement of [20, Lemma 1.20.1] as follows: Let $\mathcal{Q}$ be ω-suitable, j-sound and j-realizable. We claim that with respect to trees above $\delta^{\mathcal{Q}}_{j-1}$, $\mathcal{Q}$ is short tree iterable, and the conclusions of [20, Lemma 1.20.1] hold, except with (a)(ii) replaced by "$\mathcal{Q}$-to-$\mathcal{P}$ drops", and (b)(ii) replaced by "b drops and $\mathcal{T}\,^\frown b$ is Γ-guided". Let us argue that $\mathcal{Q}$ is short tree iterable above $\delta^{\mathcal{Q}}_{j-1}$. Assume $j=0$ for simplicity. First note that whenever $\pi:\mathcal{Q}\to\mathcal{N}$ is a 0-realization, the π-pullback $(\Psi_{\mathcal{N}})^\pi$ of the short tree strategy $\Psi_{\mathcal{N}}$ for $\mathcal{N}$ is suitability strict. To see this argue like in the proof of 6.43. Then, as in the proof of 6.29, it follows that $(\Psi_{\mathcal{N}})^\pi$ is precisely the short tree strategy for $\mathcal{Q}$. This suffices. Now consider the uniqueness of the branch b described in [20, Lemma 1.20.1](b) (as modified above). Given two such branches b,c, we compare the phalanxes $\Phi(\mathcal{T}\,^\frown b),\Phi(\mathcal{T}\,^\frown c)$, producing trees $\mathcal{U},\mathcal{V}$. If $\mathcal{T}$ is short then

[54]The issue is as follows. Let $\mathcal{T}$ be one of the trees involved in the comparison. Let $\alpha<\operatorname{lh}(\mathcal{T})$; it might be that $[0,\alpha]_{\mathcal{T}}$ drops. But then the usual procedure for choosing the least extender on $\mathbb{E}_+(\mathcal{M}^{\mathcal{T}}_\alpha)$ producing a bad extender algebra axiom need not make sense, because in fact, the relevant extender algebra is not even in $M^{\mathcal{T}}_\alpha$.

note that both $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ are Γ-guided, so $b = c$. If $\mathcal{T}$ is maximal then b, c cannot drop; rule out the possibility that, for example, $\mathcal{N}^{\mathcal{U}} \triangleleft \mathcal{N}^{\mathcal{V}}$ and $b^{\mathcal{V}}$ drops, by using suitability strictness.

Let $\Sigma, \mathcal{Q}, (F, \prec^*), \mathcal{Q}_\infty$ be defined as in [20, §2].[55] Note that $\Sigma, (F, \prec^*) \in \mathcal{M}_{\mathsf{DC}}$. We have the analogue of [20, Lemma 2.1.2], but we mention some points. First, we don't quite need that $\mathcal{Q}_\infty$ is fully wellfounded for the proof; it suffices that $\mathcal{M}_{\mathsf{DC}} \models$"$\mathcal{Q}_\infty$ is wellfounded in the codes". But because $\mathcal{M}_{\mathsf{DC}}$ need not have many ordinals beyond $\mathcal{M}$, it seems possible that $\mathcal{Q}_\infty$ be illfounded. However, standard arguments show that $\mathcal{Q}_\infty|\delta_0^{\mathcal{Q}_\infty}$ is wellfounded (in fact $\delta_0^{\mathcal{Q}_\infty} \leq \Theta^{\mathcal{M}}$). The latter is enough for the scale construction to go through. The rest of the argument is essentially as in [20]. This completes the proof. $\square$

6.5 Scales analysis within core model induction

We finish by explaining how we use the scale existence theorems in application to the core model induction. In such application, $\mathcal{F}$ will not just be nice, but *very* nice.

Definition 6.49. Let $\mathcal{F}$ be an operator over $\mathscr{B}$. We say that $\mathcal{F}$ is **very nice** iff $\mathcal{F}$ is nice and $\mathbb{R} \in \mathscr{B}$ and letting $\mathcal{N} = \mathcal{J}_1(\mathrm{HC}, \mathcal{F}{\restriction}\mathrm{HC})$, $\mathcal{N} \models \mathsf{AD}$ and every set of reals in $\mathcal{N}$ has a scale in N. $\dashv$

Remark 6.50. Let $\mathcal{F}$ be very nice. Let $z \in \mathbb{R}$ be such that there are scales on $\mathcal{F}^{\mathrm{cd}}$ and $\mathbb{R}\backslash\mathcal{F}^{\mathrm{cd}}$ which are analytical in $(\mathcal{F}^{\mathrm{cd}}, z)$. Let $X = \mathcal{F} \cup \{z\}$. Then using the scales existence theorems 6.1, 6.16, 6.20 together with 6.8, we get the scales analysis for $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$ from optimal determinacy and super-small mouse capturing hypotheses. This gives the scales analysis for $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F}{\restriction}\mathrm{HC})$, as required. (Note that at passive segments the scales are $\Sigma_1(z)$, maybe not Σ_1.)

We have dealt with $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F}{\restriction}\mathrm{HC})$ instead of $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R})$, because we seem to need extra assumptions to obtain the scales analysis from optimal assumptions in the latter. We now discuss what we need for this. In application, *if* there are no divergent AD pointclasses, $\mathcal{F}$ will in fact be *extremely* nice.

Definition 6.51. Let $\utilde{\Gamma}$ be a boldface pointclass and $X \subseteq \mathbb{R}$. We say that $\utilde{\Gamma}$ is an **AD-pointclass** iff AD holds with respect to all sets in $\utilde{\Gamma}$. We say that $\utilde{\Gamma}, X$ are **Wadge compatible** iff A, X are Wadge compatible for every $A \in \utilde{\Gamma}$.

Let $\mathcal{F}$ be an operator. We say that $\mathcal{F}$ is **extremely nice** iff there is $X \subseteq \mathbb{R}$ $\mathcal{F}$ is very nice, $\mathcal{F}{\restriction}\mathrm{HC}$ is projectively equivalent to X, and for every AD-pointclass $\utilde{\Gamma}$, $\utilde{\Gamma}, X$ are Wadge compatible. $\dashv$

[55] We use "F" where [20] uses "$\mathcal{F}$" to avoid conflicts of notation.

Remark 6.52. Let $\mathcal{F}$ be an extremely nice operator. We want to see that the scales analysis in $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R})$ proceeds from optimal determinacy assumptions. Let $\mathcal{N} \triangleleft \mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R})$ be such that $\mathcal{N} \models \mathsf{AD}$ and $\mathcal{N}$ ends a gap $[\alpha, \beta]$ of $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R})$, such that $[\alpha, \beta]$ is not strong. Suppose that if $[\alpha, \beta]$ is weak and $\mathcal{F}{\restriction}\mathrm{HC} \in \mathcal{N}|\alpha$ then super-small mouse capturing for $\Gamma = \Sigma_1^{\mathcal{N}|\alpha}$ holds on a cone. We claim that one of the scale existence theorems 6.1, 6.9, or 6.20 applies.

For by 6.8 and the mouse capturing hypothesis, we may assume that the gap is admissible, and so weak, and that $\mathcal{F}{\restriction}\mathrm{HC} \notin \mathcal{N}|\alpha$, so $X \notin \mathcal{N}|\alpha$. We claim that then $\mathcal{J}_1(\mathcal{N}) \models \mathsf{AD}$, so 6.9 applies. If every set of reals in $\mathcal{J}_1(\mathcal{N})$ is Wadge below X, this is because $\mathcal{J}_1(\mathrm{HC}, \mathcal{F}{\restriction}\mathrm{HC}) \models \mathsf{AD}$. So suppose otherwise. Let $\mathcal{P} \trianglelefteq \mathcal{N}$ be least such that there is a set $Z \in \mathcal{J}_1(\mathcal{P})$ such that $Z \not\leq_W X$. If $\mathcal{P} \triangleleft \mathcal{N}$ then $\mathcal{J}_1(\mathcal{P}) \models \mathsf{AD}$, so by the Wadge compatibility given by 6.51, we have $\mathcal{F}{\restriction}\mathrm{HC} \in \mathcal{J}_1(\mathcal{P})$, so $\alpha \leq l(\mathcal{P})$. We claim that $\mathcal{F}{\restriction}\mathrm{HC} \notin \mathcal{N}|\beta$. Because $\mathcal{F}$ is extremely nice and by 6.6, this is clear if $\mathrm{Th}^{\mathcal{N}|\alpha}_{\mathrm{r}\Pi_1} \leq_W X$ or $\mathrm{Th}^{\mathcal{N}|\alpha}_{\mathrm{r}\Sigma_1} \leq_W X$. Otherwise, by Wadge compatibility, $X <_W \mathrm{Th}^{\mathcal{N}|\alpha}_{\mathrm{r}\Sigma_1}$. But then because $\mathcal{N}|\alpha$ is admissible, $X \in \mathcal{N}|\alpha$, so $\mathcal{F}{\restriction}\mathrm{HC} \in \mathcal{N}|\alpha$, contradiction. So $\mathcal{P} = \mathcal{N}$. Since $\mathcal{N}$ ends a weak gap, there are sets $X_i \in \mathcal{P}(\mathbb{R}) \cap \mathcal{N}$ such that $\mathcal{P}(\mathbb{R}) \cap \mathcal{J}_1(\mathcal{N})$ is exactly the sets which are projective in $\oplus_{i<\omega} X_i$. It follows that $\mathcal{P}(\mathbb{R}) \cap \mathcal{J}_1(\mathcal{N}) \subseteq \mathcal{P}(\mathbb{R}) \cap \mathcal{J}_1(\mathbb{R}, X)$, so $\mathcal{J}_1(\mathcal{N}) \models \mathsf{AD}$ (and so $X \in \mathcal{J}_1(\mathcal{N})$).

A Operator condensation

Our use of 2.28 (i.e., *condenses finely*) overcomes a problem which arises with the notion of *condenses well* from [23, 2.1.10] when it is used in concert with other definitions in [23]. (*Condenses well* also appeared in early versions of [15], in the same form.) In this appendix we illustrate this problem. All definitions and notation here are following [23, §2].

Let J be the function $x \mapsto \mathcal{J}_2(x)$. Clearly J is a mouse operator (see [23, 2.1.7]). Let $F = F_J$ (see [23, 2.1.8]). Then we claim that F does not condense well (contrary to [23, 2.1.12]). We verify this.

Clearly regular premice $\mathcal{M}$ whose ordinals are closed under "$+\omega$" can be arranged as models $\tilde{\mathcal{M}}$ with parameter $\emptyset$ (see [23, 2.1.1]), such that for each $\alpha < l(\tilde{\mathcal{M}})$, $\tilde{\mathcal{M}}|\alpha + 1 = F(\tilde{\mathcal{M}}|\alpha)$.

Now let $\mathcal{M}$ be a premouse such that for some $\kappa < \mathrm{o}(\mathcal{M})$, κ is measurable in $\mathcal{M}$, via some measure on $\mathbb{E} = \mathbb{E}^{\mathcal{M}}$, and $\mathcal{M} \models$"$\lambda = \kappa^{+\kappa}$ exists", $\rho_\omega^{\mathcal{M}} = \lambda$, and $\mathcal{M} = \mathcal{J}_1(\mathcal{M}_0)$ where $\mathcal{M}_0 = \mathcal{J}^{\mathbb{E}}_\lambda$. Let $\mathcal{M}^* = \mathcal{J}_1(\tilde{\mathcal{M}}_0)$, arranged as a model with parameter $\emptyset$ extending $\tilde{\mathcal{M}}_0$. Note that because $\rho_\omega^{\mathcal{M}} = \lambda = \rho(\mathcal{M}_0)$, we have $\tilde{\mathcal{M}}_0 \in \mathcal{M}^* \in F(\tilde{\mathcal{M}}_0)$. Also, $l(\mathcal{M}^*) = \lambda + 1$ and $(\mathcal{M}^*)^- = \tilde{\mathcal{M}}_0$ (see [23, 2.1.3]). (Thus, we can't say $\mathcal{M}^* = \tilde{\mathcal{M}}$, because $\tilde{\mathcal{M}}$ is not defined.)

Let $E \in \mathbb{E}^{\mathcal{M}}$ be $\mathcal{M}$-total with $\mathrm{crit}(E) = \kappa$. Let $\mathcal{N} = \mathrm{Ult}_0(\mathcal{M}, E)$ and $\pi = i_E$. Then $\rho_1^{\mathcal{N}} = \sup \pi``\lambda < \pi(\lambda)$. Let $\mathcal{N}_0 = \pi(\mathcal{M}_0)$ and $\mathcal{N}^* = \mathcal{J}_1(\tilde{\mathcal{N}}_0)$, arranged as a model with parameter $\emptyset$ extending $\tilde{\mathcal{N}_0}$. Then $\rho_1(\mathcal{N}^*) < \pi(\lambda) = \rho(\tilde{\mathcal{N}_0})$, and therefore $\mathcal{N}^* = F(\tilde{\mathcal{N}_0})$. But $\pi : \mathcal{M}^* \to \mathcal{N}^*$ is a 0-embedding (and $\pi(\tilde{\mathcal{M}_0}) = \tilde{\mathcal{N}_0}$). Since $\mathcal{M}^* \neq F(\tilde{\mathcal{M}_0})$, F does not condense well (see [23, 2.1.10(1)]). (Note also that by using $\mathrm{Ult}_1(\mathcal{M}, E)$ in place of $\mathrm{Ult}_0(\mathcal{M}, E)$, we would get that π is *both* a 0-embedding and Σ_2-elementary, so even this hypothesis is consistent with having $\mathcal{M}^* \neq F(\tilde{\mathcal{M}_0})$.)

The preceding example seems to extend to any (first-order) mouse operator J such that for all x, $\mathcal{J}_1(x) \in J(x)$.

B Strategy premice

Our definition of Σ-premouse (for a strategy Σ with hull condensation) differed a little from the standard one. The standard one is along the lines of: given $\mathcal{M}|\alpha$, letting $\mathcal{T} \in \mathcal{M}|\alpha$ be the $<_{\mathcal{M}|\alpha}$-least tree for which $\mathcal{M}|\alpha$ does not know $\Sigma(\mathcal{T})$, and $\omega\lambda = \mathrm{lh}(\mathcal{T})$, let $\mathcal{M}|(\alpha + \lambda) = (\mathcal{J}_\lambda(\mathcal{M}|\alpha), B)$, where $B \subseteq \omega\alpha + \omega\lambda$ codes $\Sigma(\mathcal{T})$ amenably.

We need that an ultrapower of a Σ-premouse is also a Σ-premouse. As has been observed by others, this is not true of the hierarchy described above. For suppose $\mathcal{M}|\alpha$, $\mathcal{T}$ and λ are as above, and $\mathrm{lh}(\mathcal{T})$ has measurable cofinality κ in $\mathcal{M}|(\alpha + \lambda)$, and E is an extender over $\mathcal{M} = \mathcal{M}|(\alpha + \lambda)$ with $\mathrm{crit}(E) = \kappa$. Then $U = \mathrm{Ult}_0(\mathcal{M}, E)$ is not in the hierarchy. For i_E is discontinuous at $\mathrm{lh}(E)$, but $\mathrm{o}(U) = \sup i_E``\mathrm{o}(\mathcal{M})$.

There seem to be two natural attempts to correct this problem. One is to feed in all initial segments of $\Sigma(\mathcal{T})$ (even though they have been fed in earlier), immediately prior to feeding in $\Sigma(\mathcal{T})$ itself. But this approach seems flawed. For $(*)$ let $\mathcal{M}'$ be a structure in this hierarchy, with $B^{\mathcal{M}'} \neq \emptyset$, but $B^{\mathcal{M}'}$ coding a branch which is not $\mathcal{T}'$-cofinal (for the relevant tree $\mathcal{T}'$). So $B^{\mathcal{M}'}$ codes $[0, \omega\gamma']_{\mathcal{T}'}$ for some $\omega\gamma' < \mathrm{lh}(\mathcal{T}')$. Let $\pi : \mathcal{M} \to \mathcal{M}'$ be fully elementary. Then clearly $B^{\mathcal{M}}$ codes $[0, \omega\gamma]_{\mathcal{T}}$ where $\pi(\mathcal{T}) = \mathcal{T}'$ and $\pi(\gamma) = \gamma'$, and $\omega\gamma < \mathrm{lh}(\mathcal{T})$. But we need that $[0, \omega\gamma]_{\mathcal{T}} \subseteq \Sigma(\mathcal{T})$, and this is not clear (even though Σ has hull condensation).

The other correction, which is better, is to simply not feed in $\Sigma(\mathcal{T})$ in the case that $\mathrm{lh}(\mathcal{T})$ has measurable cofinality in $\mathcal{M}|(\alpha + \lambda)$ (as witnessed by some measure on $\mathbb{E}^{\mathcal{M}}$). For by the argument in 3.11, $\mathcal{M}$ already has $\Sigma(\mathcal{T})$ as an element, and there is a uniform procedure which $\mathcal{M}$ can use to determine it.

Thus, one must show that the relevant ultrapowers and substructures of models in the resulting hierarchy are also in the hierachy. It is easy to see that ultrapowers preserve the relevant first-order properties. Given that we also have a weak 0-embedding realizing the ultrapower into some structure in the hierarchy, then Σ itself will also be preserved (by hull condensation).

So let $\mathcal{M}'$ be a Σ-premouse and let $\pi : \mathcal{M} \to \mathcal{M}'$ be a weak 0-embedding. We want to know that $\mathcal{M}$ is a Σ-premouse. We just need to verify the first-order properties.

We need to rule out the possibility that $B^{\mathcal{M}} = \emptyset$ (and therefore $B^{\mathcal{M}'} = \emptyset$), but there is some $B \neq \emptyset$ such that $(\mathcal{M}, B)$ is a Σ-premouse. Let $\mathcal{T} \in \mathcal{M}$ be the relevant tree (with B coding $\Sigma(\mathcal{T})$). Because π is a weak 0-embedding, this implies that $\mathcal{T}' = \pi(\mathcal{T})$ is the least tree for which $\mathcal{M}'$ does not know $\Sigma(\mathcal{T}')$, and π is discontinuous at $\mathrm{lh}(\mathcal{T})$. Suppose also that $\mathcal{M} = \mathfrak{C}_1(\mathcal{M}')$ and π is the core map, and $\mathcal{M}'$ is $(0, \omega_1, \omega_1 + 1)$-iterable. Then by the usual proof of solidity (with a little extra argument to deal with the possibility that $\mathcal{M}$ is not a Σ-premouse), $\mathcal{M}$ and $\mathcal{M}'$ are 1-solid and $\pi(p_1^{\mathcal{M}}) = p_1^{\mathcal{M}'}$, and then using the comparison argument in the proof of universality, and the commutativity of π with the resulting iteration embeddings, one gets that $\mathrm{lh}(\mathcal{T})$ has measurable cofinality in $\mathcal{M}$, and therefore $\mathcal{M}$ is in fact a Σ-premouse, contradiction. (For the higher degree core maps, the present situation cannot arise, just by elementarity.)

Now suppose that $B^{\mathcal{M}'} \neq \emptyset$. It is easy to see that $B^{\mathcal{M}}$ codes some branch b through $\mathcal{T}$, and that $B^{\mathcal{M}} \cap \mathcal{M}$ is cofinal in $\mathrm{o}(\mathcal{M})$ (by the Σ_1-elementarity of π on a set cofinal in $\mathrm{o}(\mathcal{M})$). But b need not be $\mathcal{T}$-cofinal. (For example, if $\mathrm{o}(\mathcal{M}')$ has uncountable cofinality, it is easy to find $\mathcal{N} \triangleleft \mathcal{M}$ such that letting $\mathcal{M} = (\mathcal{N}, B^{\mathcal{M}'} \cap \mathcal{N})$ and $\pi = \mathrm{id}$, then $\pi : \mathcal{M} \to \mathcal{M}'$ is a weak 0-embedding, and $\mathcal{T} = \mathcal{T}'$.) If we have that π is Σ_1-elementary on a set $X \subseteq \mathrm{o}(\mathcal{M})$ which is both cofinal in $\mathrm{o}(\mathcal{M})$ and cofinal in $\mathrm{lh}(\mathcal{T})$, then b will be cofinal in $\mathcal{T}$.

These arguments give that the models produced in an $L[\mathbb{E}, \Sigma]$-construction will all be Σ-mice, as long as iterates of countable elementary substructures are realizable back into models of the construction, in the usual manner. But we opted for the hierarchy for Σ-premice defined in §3 because it has stronger condensation properties, and without assuming any iterability.

We make one more remark regarding strategy premice. It seems that one might try to define strategy premice over non-wellordered sets a by feeding in branches b_x for multiple trees $\mathcal{T}_x$ simultaneously, thus avoiding the need to select a single tree $\mathcal{T}$. However, we do not see how to arrange this in such a manner that the branch predicate B is always amenable. For example, suppose our supposed strategy premouse is a $\mathcal{J}$-model $\mathcal{N}$ over $\mathbb{R}$, and $\mathcal{N}|\eta$ is given, and we have identified, for each $x \in \mathbb{R}$, a tree $\mathcal{T}_x \in \mathcal{N}|\eta$, and now we want to feed in $b_x = \Sigma(\mathcal{T}_x)$, simultaneously. Let's say we have arranged that $\lambda = \mathrm{lh}(\mathcal{T}_x)$ is independent of x. Then we can easily knit together the predicates used to define $\mathfrak{B}(\mathcal{N}|\eta, \mathcal{T}_x, b_x)$, as x ranges over $\mathbb{R}$. Let $\mathcal{M}$ be the resulting structure and let $B = B^{\mathcal{M}}$. For B to be amenable, for each $\alpha < \lambda$, we must have that the function B_α is in $\mathcal{M}$, where $B_\alpha(x) = b_x \cap \alpha$. But it seems that even B_2 could contain non-trivial information, and maybe $B_2 \notin \mathcal{M}$; note that essentially, $B_2 \subseteq \mathbb{R}$. Even if the sets B_α could be added amenably, it seems that the problems described in (∗) above would be an obstacle to proving that the resulting hierarchy has nice condensation.

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